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ON THE EXISTENCE OF LIAPUNOV FUNCTIONS

FOR THE PROBLEM OF LURIE

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ON THE EXISTENCE OF LIAPUNOV FUNCTIONS
FOR THE PROBLEM OF LURIE

INTRODUCTION This paper is an extension of the work of Yacubovich and Kalman on the existence of Liapunov functions for the problem of Lurie. The primary result of this paper is the removal of the unnecessary hypothesis of complete controllability and complete observability from the theorem of Kalman. These hypotheses have been used either explicitly or implicitly by many authors working in this field. Indeed, the change of coordinates introduced by Lurie, the so called Lurie transformations, can be made only if the system is completely controllable.

The first section contains a collection of elementary results from the theory of linear algebra and control theory. None of these results are new, but since one cannot give a single reference or even a short list of references where the proofs can be found, they have been included. The papers [1], [2], [3] contain most of the results. Several of the lemmas and proofs have been taken directly from the forthcoming monograph by S. Lefschetz on Stability of Nonlinear Control Systems [4].

The second sections contains the extensions of the lemma of Kalman-Yacubovich. The proof of the first lemma follows very closely the proof as given by Kalman in [2].

The third section contains a few applications of the lemmas developed in the second section.

Since section 1 contains a series of preliminary results that are used to prove the main result, lemma 2, it is recommended that the reader first read section 2 and refer back when necessary.

1. PRELIMINARIES Let A be a real $n \times n$ matrix and b, c two real n-vectors (column). Let E^n be Euclidean n-space. Denote by $A(z)$ the characteristic matrix of A, that is $A(z) = zI - A$ where I is the identity

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matrix and z is a scalar complex variable and let $A(z)^{-1} = \{A(z)\}^{-1}$. Let $'$ denote the transpose, $*$ the conjugate transpose and $|\;|$ the determinate. Thus $|A(z)|$ is the characteristic polynomial of A. The subspaces of E^n generated by the vectors $b, Ab, \ldots,$ is called the <u>cyclic subspace generated by</u> b <u>relative to</u> A and will be denoted by $[A, b]$. The orthogonal complement of $[A, b]$ in E^n will be denoted by $[A, b]^o$. Let the dimension of $[A, b]$ be p.

By definition $[A, b]^o = \{x \epsilon E^n\colon x'A^k b = 0,\ k = 0, 1, \ldots,\}$ and so if $x \epsilon [A, b]^o$ then $x'(\exp At)b = x'\{\sum_{k=0}^{\infty} (k!)^{-1} A^k t^k\} b = \sum_{k=0}^{\infty} (k!)^{-1} x' A^k b t^k = 0$. If y is such that $y'(\exp At)b = 0$ for all t then by differentiating k times and setting $t = 0$ one obtains $x'A^k b = 0$. Thus, $[A, b]^o = \{x \epsilon E^n\colon x'(\exp At)b = 0,$ for all $t.\}$ Since the Laplace transform of $x'(\exp At)b$ is $x'A(z)^{-1}b$ it follows that $[A, b]^o = \{x \epsilon E^n\colon x'A(z)^{-1}b = 0$ for any set of z having a finite limit point.$\}$

Now assume that all the characteristic roots of A have negative real parts. In this case the rational function $x'A(z)^{-1}b$ is either zero or has at least one pole in the left hand plane since the degree of the denominator is at least one greater than the degree of the numerator. If $x'A(i\omega)^{-1}b$ is pure imaginary for all real ω then the poles and zeroes of $x'A(i\omega)^{-1}b$ must be symmetric about the imaginary axis. Thus $\mathrm{Re}\, x'A(i\omega)^{-1}b \equiv 0$ for all real ω implies $x'A(z)^{-1}b = 0$ for all z. Thus:

<u>In general</u>

$$[A, b]^o = \{x \epsilon E^n\colon\ x'A^k b = 0,\ \ k = 0, 1, 2, \ldots\}$$
$$= \{x \epsilon E^n\colon\ x'(\exp At)b \equiv 0 \text{ for all } t \epsilon (-\infty, \infty)\}$$
$$= \{x \epsilon E^n\colon\ x'A(z)^{-1}b \equiv 0 \text{ for any set of } z \text{ having a finite limit point}\}$$

[redacted]

and if all the characteristic roots of A have negative real parts then

$$[A, b]^{\circ} = \{x \epsilon E^n: \ \text{Re } x' A(i\omega)^{-1} b \equiv 0 \ \text{ for all real } \omega\}$$

One says the pair (A, b) is completely controllable provided $[A, b] = E^n$ and the pair (A, c') is completely observable if (A', c) is completely controllable.

Since $b \epsilon [A, b]$ and A maps $[A, b]$ into itself, it follows that if we choose a basis $e_1, \ldots, e_n$ for E^n such that $e_1, \ldots, e_p$ is a basis for $[A, b]$ and $e_{p+1}, \ldots, e_n$ is a basis for $[A, b]^{\circ}$ then the matrix A and the vector b have the following form

$$A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}, \qquad b = \begin{pmatrix} b_1 \\ 0 \end{pmatrix}$$

where A_1, A_2, A_3 are $p \times p$, $p \times (n-p)$, $(n-p) \times (n-p)$ matrices, b_1 is a p vector and (A_1, b_1) is completely controllable.

Now let us assume that (A, b) is completely controllable. The characteristic polynomial $|A(z)| = z^n + a_n z + \ldots + a_1$ is the minimal polynomial because if $g(z)$ is the minimal polynomial and it is of degree lower than $|A(z)|$ then $g(A)b = 0$ is a nontrivial linear combination of $b, Ab, \ldots, A^{n-1}b$ and thus contradicts the fact that (A, b) is completely controllable.

The following vectors form a basis for E^n

$$e_n = b$$

$$e_{n-1} = (A + a_n I)b$$

$$\vdots$$

$$e_1 = (A^{n-1} + a_n A^{n-1} + \ldots + a_2 I)b$$

and if we choose $e_1, \ldots, e_n$ as a basis for E^n the matrix A and the vectors b, $A(z)^{-1}b$ have the following simple form

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdot\cdot & \cdot & & & & \\ -a_1 & -a_2 & -a_3 & & -a_{n-1} & -a_n \end{pmatrix} \quad b = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}, \quad A(z)^{-1}b = \frac{1}{|A(z)|} \begin{pmatrix} 1 \\ z \\ \vdots \\ z^{n-1} \end{pmatrix}$$

Thus, if $\tilde{g}(z) = g_1 + g_2 z + \ldots + g_n z^{n-1}$ is any real polynomial of degree less than n, then $g'A(z)^{-1}b = \tilde{g}(z)\{|A(z)|\}^{-1}$, where g is the real n-vector with components g_i. The vector g is chosen so that $g' = (g_1, g_2, \ldots, g_n)$.

The last preliminary result is the following: <u>Let (A, b) be completely controllable and k any real n-vector. Let $k'A(z)^{-1}b = p(z)\{|A(z)|\}^{-1}$. Then the degree of the greatest common devisor of $p(z)$ and $|A(z)|$ is equal to the dimension of $[A', k]^0$.</u>

We can choose a basis for E^n such that $A = \text{diag}(C_{\lambda_1}, C_{\lambda_2}, \ldots, C_{\lambda_n})$, $k' = (k_1', k_2', \ldots, k_r')$, $b' = (b_1', b_2', \ldots, b_r')$ where C_{λ_i} is the $n_i \times n_i$ matrix

$$C_{\lambda_i} = \begin{pmatrix} \lambda_i & 1 & 0 & \\ 0 & \lambda_i & 1 & \\ \cdot & \cdot & \cdot & \\ 0 & 0 & 0 & \lambda_i \end{pmatrix}$$

and b_i and k_i are n_i-vectors and $\sum_1^r n_i = n$. It is easy to see that the general result follows at once if it is true for any such block matrix.

Consider then the $n \times n$ matrix C_λ and the two n-vectors b and c. There exists a change of coordinates that leaves C_λ unchabged and reduces b to the simple form $b' = (0, 0, \ldots, 1)$. The transformation of coordinates

$$\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ 0 & \alpha_1 & \cdots & \alpha_{n-1} \\ \cdot & \cdot & \cdot & \\ 0 & 0 & & \alpha_1 \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

is nonsingular provided $\alpha_1 \neq 0$ and it preserves the form of C_λ. One can easily verify that if (C_λ, b) is completely controllable and $b' = (b_1, \ldots, b_n)$ then $b_n \neq 0$. Thus the following system of equations has a solution for $\alpha_1, \ldots, \alpha_n$ with $\alpha_1 = b_n^{-1} \neq 0$ since the determinant is $b_n^n \neq 0$

$$\begin{aligned} \alpha_1 b_1 + \alpha_2 b_2 + \cdots + \alpha_n b_n &= 0 \\ \alpha_1 b_2 + \cdots + \alpha_{n-1} b_n &= 0 \\ \cdot \quad \cdot \quad \cdot & \\ \alpha_1 b_n &= 1 \end{aligned}$$

In this coordinate system, if $k' = (k_1, \ldots, k_n)$ then

$$k' C_\lambda(z)^{-1} b = \frac{(-1)^{n+1} k_1}{(z - \lambda)^n} + \cdots + \frac{p(z)}{(z - \lambda)^n}$$

If the degree of the greatest common devisor of $(p(z), (z - \lambda)^n)$ is s, then $k_1 = k_2 = \cdots = k_s = 0$ and $k_{s+1} \neq 0$.

Now also in this coordinate system $k'(\exp C_\lambda t)x =$

$$\left(k_1,\ k_1 t + k_2,\ \ldots,\ \frac{k_1 t^{n-1}}{(n-1)!} + \ldots + k_n\right)(x_1, \ldots, x_n)' e^{\lambda t}$$

If again $k_1 = k_2 = \ldots = k_s = 0$ and $k_{s+1} = 0$ then the number of linearly independent x such that $k'(\exp C_\lambda t)x \equiv 0$ is equal to s.

2. THE MAIN LEMMAS. The extension of the Kalman-Yacubovich lemma will require several steps. The first lemma is a slight extension of the lemma as given by Kalman [2] and the proof of this lemma follows very closely his proof. It will give enough information to remove the complete observability assumption required by Kalman. We obtain the additional information that B is positive definite and that (A, q') is completely observable.

Lemma 1. Let A be an $n \times n$ real matrix all of whose characteristic roots have negative real parts, let τ be a nonnegative real number and let b, k be two real n-vectors. Assume (A, b) is completely controllable. If

(1.1) $\tau + 2 \ \mathrm{Re}\ k'A(i\omega)^{-1}b \geq 0$

for all real ω then there exist two $n \times n$ real symmetric matrices B and D and a real n-vector q such that

a) $A'B + BA = -qq' - D$

b) $Bb - k = \sqrt{\tau}q$

c) (A, q') is completely observable.

d) B is positive definite and D is positive semi definite.

e) <u>if</u> $i\omega_o$, ω_o <u>real is a root of</u> $-q'A(z)^{-1}b + \sqrt{\tau}$ <u>then it is a root of</u> $b'A(-z)^{-1}DA(z)^{-1}b$.

f) <u>all the roots of</u> $-q'A(z)^{-1}b + \sqrt{\tau}$ <u>are in the closed left hand plane</u>

<u>Proof:</u> Let $m(z) = A(z)^{-1}b$ and $\psi(z) = |A(z)|$. Then (1.1) can be written

$$(1.2) \qquad 0 \leq \tau + m\,(i\omega)k + k'm(i\omega) = \frac{\eta(i\omega)}{\psi(i\omega)\psi(-i\omega)}$$

Clearly $\eta(z)$ is an even polynomial with real coefficients and hence its zeroes are symmetric about both the real and imaginary axis and its zeroes on the imaginary axis are of even multiplicity. Thus we can write $\eta(i\omega) = \theta(i\omega)\theta(-i\omega)$ where $\theta(z)$ is a real polynomial with all roots in the closed left hand plane.

We can factor $\theta(z) = \theta_1(z)\ \theta_2(z)$ such that all the zeroes of $\theta_1(z)$ are in the open left hand plane and all the zeroes of $\theta_2(z)$ are on the imaginary axis. Let the degree of θ_1 and θ_2 be n_1 and n_2 respectively.

At this point we wish to add to both sides of (1.2) a term that does not destroy the inequality and at the same time makes the rational function on the right hand side irreducible. If $n_1 = 0$ we have nothing to do. If $n_1 \neq 0$ then $n_1 < n - 1$. We now define a polynomial $\tilde{g}(z)$ so that

i) $\tilde{g}(z)$ has real coefficients and is of degree less than or equal to $n - 1$

ii) $\Gamma(i\omega) = \theta(i\omega)\theta(-i\omega) - \tilde{g}(i\omega)\tilde{g}(-i\omega) \geq 0$ for all real ω

iii) The greatest common divisor of $\Gamma(z)$ and $\psi(z)\psi(-z)$ is one.

Let $g_2(z)$ be any real polynomial of degree $n - n_2 - 1$ with zeroes different from those of $\psi(z)$ and $\theta_1(z)$. Then clearly $\tilde{g}(z)$ can be chosen as $\tilde{g}(z) = \alpha\theta_2(z)g_2(z)$ where α is sufficiently small and positive. We then define the vector g so that $\tilde{g}(z)\{|A(z)|\}^{-1} = g'A(z)^{-1}b$.

Since $\theta(i\omega)\ \theta(-i\omega) - g(i\omega)g(-i\omega) \geq 0$ we can by the same reasoning as in the above write $\theta(i\omega)\ \theta(-i\omega) - g(i\omega)g(-i\omega) = \nu(i\omega)\ \nu(-i\omega)$ where $\nu(z)$ is a real polynomial all of whose roots are in the left half plane.

Thus

$$(1.3) \qquad 0 \leq \tau + m(i\omega)^*k + k'm(i\omega) - m^*(i\omega)gg'm(i\omega) = \frac{\nu(i\omega)\ \nu(-i\omega)}{\psi(i\omega)\ \psi(-i\omega)}$$

In general the formal degree of $\nu(z)$ is n and the leading coefficient is $\sqrt{\tau}$ so we can write

$$\frac{\nu(z)}{\psi(z)} = -\frac{\mu(z)}{\psi(z)} + \sqrt{\tau}$$

where μ is real and of degree $n-1$. The vector q is then defined by $\mu(z)\{\psi(z)\}^{-1} = q'm(z)$. Since the greatest common division of μ and ψ is 1, (A, q') is completely observable. The property (f) then holds. Define $D = gg'$ and since by construction the zeroes on the imaginary axis of $g'm(z)$ and $-q'm(z) + \sqrt{\tau}$ are the same property (e) holds.

Now define B by

$$B = \int_0^\infty e^{A't}\{qq' + D\}e^{At}dt$$

and so $A'B + BA = -qq' - D$. Since (A, q') is completely observable, B is positive definite. From (1.3) it follows that

$$\begin{aligned} m^*(i\omega)k + k'm(i\omega) &= m^*(i\omega)Dm(i\omega) + (-q'm(i\omega) + \sqrt{\tau})(-m^*(i\omega) + \sqrt{\tau}) - \tau \\ &= m^*(i\omega)\{qq' + D\}m(i\omega) - \sqrt{\tau}\ (q'm(i\omega) + m^*(i\omega)q) \\ &= b'Bm(i\omega) + m^*(i\omega)Bb - \sqrt{\tau}\ (q'm(i\omega) + m^*(i\omega)q) \end{aligned}$$

and hence $\mathrm{Re}\{Bb - k - \sqrt{\tau}q\}'m(i\omega) = 0$ and so $Bb - k = \sqrt{\tau}q$.

The next step is the removal of the assumption that (A, b) be completely controllable. This is done with the following lemma.

Lemma 2. Let A be a real $n \times n$ matrix all of whose characteristic roots have negative real parts; let τ be a real nonnegative number and let b, k be two real n-vectors. If

$$\tau + 2\,\mathrm{Re}\, k'A(i\omega)^{-1}b \geq 0$$

for all real ω then there exists two $n \times n$ real symmetric matrices B, D and a real n-vector q such that

(a) $A'B + BA = -qq' - D$

(b) $Bb - k = \sqrt{\tau}q$

(c) D is positive semi definite and B is positive definite

(d) $\{x \in E^n : x'Dx = 0\} \cap [A', q]^{\circ} = \{0\}$

(e) $q \notin [A, b]^{\circ}$

(f) if $i\omega$, ω real, is a root of $-q'A(z)^{-1}b + \sqrt{\tau}$ then it is a root of $b'A(-z)^{-1}DA(z)^{-1}b$.

Choose a coordinate system for E^n such that

$$A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ 0 \end{pmatrix}, \quad k = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$$

where A_1, A_2, A_3 are $p \times p$, $p \times (n - p)$, $(n - p) \times (n - p)$ matrices respectively; b_1, k_1 are p vectors; k_2 is an $(n - p)$ vector and such that (A_1, b_1) is completely controllable. Clearly if A has all characteristic roots with negative real parts then so do A_1 and A_3. If we partition B, D and q in the same way, ie

$$B = \begin{pmatrix} B_1 & B_2 \\ B_2' & B_3 \end{pmatrix}, \quad D = \begin{pmatrix} D_1 & 0 \\ 0 & D_3 \end{pmatrix}, \quad q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$$

we find that we must solve the following set of matrix equations

1) $A_1'B_1 + B_1A_1 = -q_1q_1' - D_1$

2) $A_2'B_1 + A_3'B_2' + B_2'A_1 = -q_2q_1'$

3) $A_2'B_2 + A_3'B_3 + B_2'A_2 + B_3A_3 = -q_2q_2' - D_3$

4) $B_1b_1 - k_1 = \sqrt{\tau}q_1$

5) $B_2'b_1 - k_2 = \sqrt{\tau}q_2$

By hypothesis $\tau + 2\,\mathrm{Re}\,k_1'\,A_1(i\omega)^{-1}b_1 \geq 0$ for all real ω and so by lemma 1 there exists a solution to the equations 1 and 4 and by (c) of lemma 1 the condition (e) of lemma 2 is satisfied. Also by (e) of lemma 1 the condition (f) of lemma 2 is satisfied. Now let us consider the equations 2) and 5). Since B_1 and q_1 are known by lemma 1 these two equations have only B_2' and q_2 as unknowns. We can solve 2) for B_2' in terms of q_2 by the formula

$$B_2' = \int_0^\infty e^{A_3't}\{q_2q_1 - A_2'B_1\}e^{A_1t}\,dt$$

and then substitute this into (4) to obtain

$$Rq_2 = \{\int_0^\infty e^{A_3' t} q_1 e^{A_1 t} b dt - \sqrt{\tau} I\} q_2 = k + \int_0^\infty e^{A_3' t} A_2' B_1 e^{A_1 t} dt .$$

Since the right hand side of the above is known, we can solve for q_2 provided the matrix in the bracket, R, is non singular. There is no loss in generality in assuming that A_3' is in triangular form and so $e^{A_3' t}$ is in triangular form. A typical term from the diagonal of R is then

$$\int_0^\infty e^{\lambda_i t} q_1 e^{A_1 t} b_1 dt - \sqrt{\tau} = q_1(-\lambda_i I - A_1)^{-1} b_1 - \sqrt{\tau} = q_1' A_1(-\lambda_1)^{-1} b_1 - \sqrt{\tau}$$

But this term is not zero since $-\lambda_i$ is in the open right hand plane and by condition (f) of lemma 1, we know that the zeroes of $q_1 A_1(z)^{-1} b_1 - \sqrt{\tau}$ are in the closed left hand plane. Thus R is non singular and q_2 and B_2 are determined.

Now choose D_3 to be any positive definite matrix. It is clear then then equation 5) has a solution and that (d) is satisfied.

Since B satisfies $A'B + BA = -qq' - D$ it must be of the form

$$B = \int_0^\infty e^{A' t} qq' e^{At} dt + \int_0^\infty e^{A' t} D e^{At} dt.$$

If x_o is such that $x_o B x_o = 0$ then $x_o e^{A' t} q \equiv 0$ and $x_o D x_o = 0$ and thus by (d), $x_o = 0$. Hence B is positive definite.

In some critical cases the following lemma is useful. This lemma is in essence due to Yacubovich [5] and was implicitly used by Meyer in [6].

Lemma 3. _Let_ A _be an_ n x n _real matrix all of whose characteristic roots have zero real parts and are simple. If the residues of_ $k'A(z)^{-1}b$ _are all positive then there exist a positive definite matrix_ B _such that_

$$A'B + BA = 0 \quad \underline{\text{and}} \quad Bb - k = 0.$$

Proof: This lemma follows at once by making a change of coordinates so that A is diagonal. In this coordinate system B is chosen to be diagonal also.

3. APPLICATIONS. The lemmas developed in Section 2 can be applied to many different systems that have been considered in the literature. Let us consider the so-called direct control system. The equations are

$$(3.1) \qquad \begin{aligned} \dot{x} &= Ax - b\phi(\sigma) \\ \sigma &= c'x \end{aligned}$$

where A is a real $n \times n$ matrix; b, x and c are real n-vectors and $\phi(\sigma)$ is a continuous scalar function of the scalar σ such that $\sigma\phi(\sigma) > 0$ for all $\sigma \neq 0$. The vector x and the scalar σ are functions of the real variable t, time, and $\dot{x}$ is the derivative of x with respect to t. Let us assume also that through each point in E^n there exists a unique trajectory of (3.1). We wish to prove

If all the characteristic roots of A have negative real parts and if there exist two nonnegative constants α and β such that

$$(3.2) \qquad \alpha + \beta > 0 \quad \text{and} \quad \mathrm{Re}(\alpha + i\omega\beta)c'A(i\omega)^{-1}b \geqq 0$$

for all real ω, then all solutions of (3.1) are bounded, the trivial solution $x = 0$ is stable and moreover if $\alpha \neq 0$ the trivial solution is asymptotically stable in the large.

If the trivial solution, $x = 0$, of the linear system $\dot{x} = \{A - \mu bc'\}x$ is asymptotically stable for all $\mu > 0$ when $\alpha = 0$ then all solutions of (3.1) are asymptotically stable in the large also.

Proof: Using the relation $i\omega I = A(i\omega) + A$ in (3.2) we obtain

$$\beta c'b + 2\mathrm{Re}\left(\frac{\alpha c + \beta A'c}{2}\right)' A(i\omega)^{-1}b \geqq 0 \qquad \text{for all real } \omega$$

and thus by lemma 2 there exists a real n-vector q and two positive symmetric matrices B and D such that

$$A'B + BA = -qq' - D \quad , \quad Bb - \left(\frac{\alpha c + \beta A'c}{2}\right) = \sqrt{\beta}\, c'bq$$

and moreover B is definite. Thus

$$(3.3) \qquad V = x'Bx + \beta \int_0^\sigma \phi(\sigma)d\sigma$$

is a positive definite function and tends to ∞ as $|x| \to \infty$. The derivative $\dot{V}$ of V along the trajectories of (3.1) is given by

$$-\dot{V} = -x'(A'B + BA)x + 2(Bb - \frac{\alpha}{2} c + \frac{\beta}{2} A'c)'x\Phi(\sigma) + \beta c'b\Phi(\sigma) + \alpha\sigma\Phi(\sigma)$$

(3.4)

$$= x'Dx + (\sqrt{\tau}\ \Phi\ (\sigma) + q'x)^2 + \alpha\sigma\Phi(\sigma)$$

Note that $\alpha\sigma\Phi(\sigma)$ has been added and subtracted from $\dot{V}$.

Clearly $-\dot{V}$ is also positive and hence, by the well known theorems of Liapunov Theory all solutions are bounded and the origin is stable. In order to prove asymptotically stable, we must show that no solution remains in the set where $-\dot{V} = 0$. Let $\alpha \neq 0$ and assume there exists a solution $x(t)$ of (3.1) such that $x(0) = x_o$ and $x(t)$ remains in the set where $-\dot{V} = 0$. But if $\dot{V} = 0$ then $\sigma = 0$, and thus, such a solution is a solution of $\dot{x} = Ax$. Hence $x(t) = (\exp At)x_o$. From the second term we obtain $q'(\exp At)x_o \equiv 0$. Also, $x_oDx_o = 0$ and so by part (d) of lemma 2, $x_o = 0$.

In general we cannot conclude more than stability in the use when $\alpha = 0$, but if the linear system $\dot{x} = \{A - \mu bc'\}x$ is asymptotically stable for all $\mu > 0$ then the system (3.1) is asymptotically stable in the large also.

In order to rule out solutions that remain in the set where $-\dot{V} = 0$, we must be sure that there is no solution such that $\sqrt{\tau}\Phi(\sigma(t)) = -q'x(t)$.

If $\tau \neq 0$ then a solution of (3.1) remains in the set where $-\dot{V} = 0$ must satisfy the linear equation $\dot{x} = \{A + \tau^{-\frac{1}{2}} bq'\}x$. By condition (e) of lemma 2 there exists a nonnegative integer m such that $q'b = q'Ab = \dots =$ $= q'A^{m-1} = 0$ and $q'A^m b \neq 0$. Hence if $\tau = 0$ there exists an m such that a solution of (3.1) that remains in the set where $-\dot{V} = 0$ must satisfy $\dot{x} = \{A - (q'A^m b)^{-1} bq'A^{m+1}\}\, x$.

As we have seen, a solution that remains in the set where $-\dot{V} = 0$ is a solution of a linear constant coefficient differential equation. Let us assume that there exists a non trivial solution $x(t)$ of (3.1) that remains in the set where $-\dot{V} = 0$. We can assume $\sigma(t) \not\equiv 0$ since if $\sigma \equiv 0$ we could repeat the previous argument. Since $x(t)$ is a solution of a linear equation and is bounded for all t then $x(t)$ must be of the form

$$x(t) = \sum_{j=-N}^{N} v_j \ \{\exp i\omega_j t\}$$

where the v_j are n-vectors such that $v_{-j} = \bar{v}_j$ and ω_j are real scalars such that $\omega_{-j} = -\omega_j$. Clearly $\Phi(\sigma(t))$ must be of the form

$$\Phi(\sigma(t)) = \sum_{j=-N}^{N} a_j \ \exp i\omega_j t$$

where the a_j are scalars such that $a_j = -a_{-j}$. By substituting these forms into (3.1) one obtains

$$v_j = -a_j A(i\omega_j)^{-1} b$$

Thus, by the well known formula from the theory of almost periodic functions

$$\lim_{t\to\infty} \frac{1}{T}\int_0^T \sigma(t)\Phi(\sigma(t))dt = -\sum_{j=-N}^{N} |a_j|^2 c'A(i\omega\ )^{-1}b > 0$$

We shall have a contradiction once we prove

Lemma 4. Let the system $\dot{x} = \{A - \nu bc'\}x$ be asymptotically stable for all $\nu > 0$. If $i\omega_j$ is a characteristic root of $A + \tau^{-\frac{1}{2}} bq'$ if $\tau \neq 0$ or if $A - (q'A^m b)^{-1}bq'A^{m+1}$ if $\tau = q'b = \ldots = q'A^{m-1}b = 0$ and $q'A^m b \neq 0$ then $\text{Im } c'A(i\omega_j)^{-1}b = 0$ and $c'A(i\omega_j)^{-1}b \geq 0$.

We shall consider only the case when $\tau \neq 0$, since the other case is very similar. Since $\alpha = 0$ we may take $\beta = 1$. Then

$$qq' + D = -(A'B + BA) = A^*(i\omega_j)B + BA(i\omega_j)$$

$$|q'A(i\omega_j)^{-1}b|^2 + b'A^*(i\omega_j)^{-1}DA(i\omega_j)^{-1}b = 2\text{Re}\, b'BA(i\omega_j)^{-1}b$$

Now the characteristic polynomial of $A + \tau^{-\frac{1}{2}} bq'$ is $|A(z)|\ \{1 - \tau^{-\frac{1}{2}}q'A(z)^{-1}b\}$ and so

$$\tau = \tau^{-\frac{1}{2}}q'A(i\omega_j)^{-1}b = b'BA(i\omega)^{-1}b - \frac{1}{2}\, c'AA(i\omega_j)^{-1}b$$

Since $\sqrt{\tau} + q'A(i\omega_j)^{-1}b = 0$ by lemma 2 part (f) $b'A(i\omega_j)^{-1}DA(i\omega_j)^{-1}b = 0$. Thus

$$\tau + 2\text{Re}\ c'AA(i\omega_j)^{-1}b = \text{Re}\ i\omega_j c'A(i\omega_j)^{-1}b = 0$$

or

$$\text{Im}\ c'A(i\omega_j)^{-1}b = 0.$$

Since the linear system $\dot{x} = \{A - \nu bc'\}x$ is asymptotically stable for all $\nu > 0$ the theorem of Nyquist gives $c'A(i\omega_j)b \geq 0$.

The above theorem can be modified several ways.

(i) if the matrix A has some characteristic roots on the imaginary axis then the lemmas 2 and 3 can be used to prove asymptotic stability in a manner similar to that found in [5] and [6]. In particular <u>if A has 2s simple, distinct, nonzero pure imaginary characteristic roots, the characteristic root zero of multiplicity p where $p = 0, 1, 2$ all the other characteristic roots have negative real parts then (3.1) is asymptotically stable in the large, provided</u>

1) <u>there exist two nonnegative constants α and β, $\alpha + \beta > 0$ such that $\mathrm{Re}(\alpha + i\omega\beta)c'A(i\omega)^{-1}b \geq 0$ for all real ω and if $i\omega$, ω real, is a characteristic root of A then the residue of $(\alpha + \lambda\beta)c'A(\lambda)^{-1}b$ at $i\omega$ is positive</u>

2) <u>when $\alpha = 0$ the linear equation $\dot{x} = \{A - \mu bc'\}x$ is asymptotically stable for all $\mu > 0$</u>

3) <u>If A is singular and $\alpha = 0$ then $\int_0^\sigma \phi(\tau)d\tau \to \infty$ as $|\sigma| \to \infty$.</u>

In order to prove this theorem one first changes coordinates such that the system (3.1) takes the form

$$\dot{x}_1 = A_1 x_1 - b_1\phi(\sigma)$$

$$\dot{x}_2 = A_2 x_2 - b_2\phi(\sigma)$$

$$\dot{x}_3 = A_3 x_3 - b_3\phi(\sigma)$$

$$\sigma = c_1' x_1 + c_2 x_2 + c_3 x_3$$

where x_1, b_1, c_1 are r-vectors; x_2, b_2, c_2 are 2s vectors and A_1, A_2 are $r \times r$, $2s \times 2s$ matrices respectively. The vectors x_3 and b_3 are p-vectors and A_3 is a $p \times p$ matrix where $p = 0, 1, 2$. The characteristic roots of A_1 all have negative real parts, the characteristic roots of A_2 are all simple nonzero pure imaginary numbers and the characteristic root of A_3 is zero. The matrix $A_3 = (0)$ if $p = 1$ and $A_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ if $p = 2$. Let

$$V = x_1 B_1 x + x_2 B_2 x_2 + x_3 B_3 x_3 + \beta \int_0^{\sigma} \phi(\tau) d\tau$$

where B_1 is given by lemma 2 as in the above and B_2 is given by lemma 3 and $B_3 = 0$ if $\dot{p} = 0$, $B_3 = \alpha$ if $p = 1$, $B_3 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ if $\dot{p} = 2$.

Thus, B_1, B_2 and B_3 are $r \times r$, $2s \times 2s$ and $p \times p$ symmetric matrices respectively and V is positive definite. One can proceed as before with only very minor changes in the argument.

(ii) If $\phi(\sigma)$ is restricted so that $0 < \sigma\phi(\sigma) < k\sigma^2$ for $\sigma \neq 0$ then instead of adding and subtracting $\alpha\sigma\,\phi(\sigma)$ from $-\dot{V}$ one can subtract $\alpha\phi(\sigma)(\sigma - k^{-1}\phi(\sigma))$. The proof carries over and the theorem remains the same except that $c'A(i\omega)^{-1}b$ is replaced by $c'A(i\omega)^{-1}b + k^{-1}$.

(iii) Let us make the change of variables $y(t) = e^{-\lambda t}x(t)$ where $x(t)$ is a solution of (3.1) and λ is any real number such that $\lambda > \text{Re}\,\lambda_i$, $i = 1, \ldots, n$ and λ_i, $i = 1, \ldots, n$ are the characteristic roots of A. Note that λ may be positive or negative and the characteristic roots of A may have positive or negative real parts. Then $y(t)$ satisfies the equation

$$(3.5) \qquad \dot{y} = (A - \lambda I)y - be^{-\lambda t}\phi(e^{\lambda t}c'y)$$

Let $V = y'By$ and then the derivative of V along the trajectories of (3.5) is

$$-\dot{V} = -y'\{(A - \lambda I)'B + B(A - \lambda I)\}y + 2\{Bb - \frac{1}{2}c\}'ye^{-\lambda t}\phi(e^{\lambda t}c'y)$$
$$+ c'ye^{-\lambda t}\phi(e^{\lambda t}c'y).$$

As before there exists a B such that V is positive definite and $-\dot{V} \geqq 0$ for all y provided

$$\text{Re } c'(A - \lambda I)(i\omega)^{-1}b = \text{Re } c'A(i\omega + \lambda)^{-1}b \geqq 0$$

for all real ω. Thus $y(t)$ is bounded and the bound depends only on $\|y_o\|$. Therefore there exists a positive scalar function K such that $\|y(t)\| \leqq K(\|y(o)\|)$ for all $t \geqq 0$ or $\|x(t)\| \leqq e^{\lambda t}K(\|x(o)\|)$.

(iv) Lefschetz [4] proves that if you replace $\geqq 0$ by > 0 in (1.1) you can replace D in lemma 1 part a) by $\epsilon D'$ where ϵ is sufficiently small and D' is positive definite. Using the same method of proof as used in lemma 2 one obtains the following lemma

<u>Let</u> A <u>be a real</u> $n \times n$ <u>matrix all of whose characteristic roots have negative real parts; let</u> τ <u>be a real nonnegative number and</u> b, k <u>be any two real</u> n-<u>vectors. If</u>

$$\tau + 2\text{Re } k'A^{-1}(i\omega)b > 0$$

<u>for all real</u> ω <u>then there exists two real positive definite matrices</u> B <u>and</u> D <u>and a real</u> n-<u>vector</u> q <u>such that</u>

(a) $A'B + BA = -qq' - D$

(b) $Bb - k = \sqrt{\tau} q$.

In general D can be taken as $\epsilon D'$ where D' is an arbitrary positive definite matrix. This lemma is almost the same as the lemma given by Yacubovich [7].

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