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The Theory of Long-Term Behavior of Artificial Satellite Orbits Due to Third-Body Perturbations

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Under certain simplifications, the equations describing the long-term behavior of a satellite disturbed by a third body reduce to a third order (nonlinear) system which can be integrated completely. Elliptic integrals are required for the time dependence, but the phase-trajectories can be represented entirely by elementary functions. A complete description of the behavior of solutions to these equations is given in this Report.

Author

I. INTRODUCTION

To determine the exact effects of third-body perturbations on a satellite orbit, taking into account both short-period and long-period terms, it would be necessary to solve the restricted three-body problem. This usually requires numerical integration of the exact equations of motion on a high-speed computer. However, such computations become very costly when continuously applied over a period of two or three years, such as may be necessary in the case of a lunar satellite.

Numerical computations of this type have revealed that the amplitudes of the short-term variations in the elements of the osculating ellipse are in most cases quite small when compared with the values of the elements themselves. It therefore seems reasonable to investigate the equations which govern the long-term behavior of satellite orbits perturbed by third bodies. Such equations may be derived by applying the method of averages (Ref. 1), the averaging being carried out over both the period of the satellite about the central (or primary) body, and over the period of the perturbing body about the central body. The solution of these averaged equations (the validity of which, incidentally, is not restricted to small orbital inclination or eccentricity) is the subject of this Report.

Although the discussion presented here neglects entirely the effects of the oblateness (if any) of the central body, these results should be useful as a first approximation in studies of lunar satellite stability.

II. EQUATIONS OF MOTION

Lorell and Anderson (Ref. 2) approximated the three-body equations of motion by expanding the "disturbing function" in powers of r/r_3, the ratio of the distances r of the satellite, and r_3 of the third body, from the central body, and truncating this series after the first term. Then the resulting equations were averaged wrt (with respect to) mean anomaly over the period of the satellite about the central body, which produced a set of six coupled ode (ordinary differential equations). These averaged equations have been incorporated into the "Lunar Lifetime Program"*, which has been shown to give excellent numerical agreement with numerical integrations of the exact equations of motion.

Lorrel and Anderson made a further approximation by carrying out the averaging process over the period of the third body about the central body, as well as over the period of the satellite. The resulting set of six equations is given below (where the bars over the variables signify that they have been averaged over the two (constant) periods mentioned above):

$$\frac{d\bar{a}}{dt} = 0 \tag{1a}$$

$$\frac{d\bar{\chi}}{dt} = -\frac{3}{4}\frac{n_3^2}{\bar{n}}\left\{\frac{7}{3} + \bar{e}^2 - \sin^2\bar{i}\left[(1-\bar{e}^2)\cos^2\bar{\omega} + 2(3+2\bar{e}^2)\sin^2\bar{\omega}\right]\right\} \tag{1b}$$

$$\frac{d\bar{\Omega}}{dt} = -\frac{3}{4}\frac{n_3^2}{\bar{n}}\frac{\cos\bar{i}}{(1-e^2)^{1/2}}\left[(1-\bar{e}^2)\cos^2\bar{\omega} + (1+4\bar{e}^2)\sin^2\bar{\omega}\right] \tag{1c}$$

$$\frac{d\bar{e}}{dt} = \frac{15}{8}\frac{n_3^2}{\bar{n}}\bar{e}(1-\bar{e}^2)^{1/2}\sin^2\bar{i}\sin 2\bar{\omega} \tag{1d}$$

$$\frac{d\bar{i}}{dt} = -\frac{15}{16}\frac{n_3^2}{\bar{n}}\frac{\bar{e}^2}{(1-\bar{e}^2)^{1/2}}\sin 2\bar{i}\sin 2\bar{\omega} \tag{1e}$$

$$\frac{d\bar{\omega}}{dt} = \frac{3}{2}\frac{n_3^2}{\bar{n}}(1-\bar{e}^2)^{1/2}\left[1 + \frac{5}{2}\sin^2\bar{\omega}\,\frac{(\bar{e}^2-\sin^2\bar{i})}{(1-\bar{e}^2)}\right] \tag{1f}$$

*Coded for the JPL IBM 7090 computer.

Since the series expansion of the "disturbing function" in powers of r/r_3 was truncated after the first term, it is reasonable to apply Eqs. (1a) through (1f) only to satellites whose orbital elements are such that $r/r_3 << 1$ at all points of the orbit.

From Eq. (1a), it is seen that $\bar{a}$ = constant. Since by definition, $\bar{n}^2 = \mu/\bar{a}^3$, the mean angular rate $(\bar{n})$ of the satellite is constant within the meaning of the averaged variables being considered. Also, n_3 is constant. Therefore, the right hand side (rhs) of Eqs. (1d), (1e), and (1f) involve only the three averaged variables $\bar{e}$, $\bar{i}$, and $\bar{\omega}$. It should now be possible to determine $\bar{e}$, $\bar{i}$, and $\bar{\omega}$ as functions of time without having to solve for $\bar{\chi}$ or $\bar{\Omega}$. The remaining two equations may be solved later, if so desired.

For studies of the orbital lifetimes of artificial satellites, the quantity of greatest interest is the averaged radius of closest approach, given by

$$\bar{r}_{min} = \bar{a}(1 - \bar{e})$$

Since the actual instantaneous values (a, e, i, ω, Ω, and χ) of the osculating ellipse are assumed to deviate only slightly from the averaged variables ($\bar{a}$, $\bar{e}$, $\bar{i}$, $\bar{\omega}$, $\bar{\Omega}$, and $\bar{\chi}$), a reasonable criterion for determining the end of the satellite orbit is

$$\bar{r}_{min} = r_{central\ body}$$

i.e., the averaged value of the pericenter radius is equal to the surface radius of the central body, so that impact occurs (in the sense of the present usage). This in turn defines a critical eccentricity given by

$$\bar{e}_{cr} = 1 - \frac{r_{central\ body}}{\bar{a}} < 1 \qquad (2)$$

The lifetime of the orbit is therefore the time interval required for $\bar{e}$ to reach the value $\bar{e}_{cr}$. (The satellite cannot escape from the central body, since $\bar{a}$ = constant, and $\bar{r}_{max} = \bar{a}(1 + \bar{e}_{cr}) < 2\bar{a}$).

Since the satellite would impact on the central body if $\bar{e} > \bar{e}_{cr}$, it is required only that the solutions be valid for $0 \leq \bar{e} \leq \bar{e}_{cr} < 1$.

Thus, the determination of the long-term behavior of the orbit of an artificial satellite about a central body, perturbed by a third body, reduces to the solution of Eqs. (1d), (1e), and (1f), a set of three coupled, first-order, nonlinear differential equations. To solve such a system, three integrals of motion are required. These are given in Eqs. (13), (20), and (26).

III. RANGES OF THE VARIABLES

In order for the variables $\bar{e}$ and $\bar{i}$ to be physically meaningful, they must lie in the range $0 \leq \bar{e} \leq \bar{e}_{cr} < 1$, $0° \leq \bar{i} \leq 180°$. It may be shown that if $\bar{e}_0 \neq 0$, then $\bar{e} > 0$ for all finite values of t. The proof is as follows:

From Eq. (1d),

$$dt = \frac{8}{15} \frac{\bar{n}}{n_3^2} \frac{d\bar{e}}{\bar{e}(1-\bar{e}^2)^{1/2} \sin^2 \bar{i} \sin 2\bar{\omega}}$$

$$t = \frac{8}{15} \frac{\bar{n}}{n_3^2} \int_{\bar{e}_0}^{\bar{e}} \frac{dx}{x(1-x^2)^{1/2} \sin^2 \bar{i} \sin 2\bar{\omega}}$$

where x is a dummy variable, and $t = 0$ when $\bar{e} = \bar{e}_0$. Since the main interest is in $\bar{e} \to 0$, then $d\bar{e}/dt < 0$, which implies $\sin 2\bar{\omega} < 0$, or $\sin 2\bar{\omega} = -|\sin 2\bar{\omega}| < 0$. Hence,

$$t = \frac{8}{15} \frac{\bar{n}}{n_3^2} \int_{\bar{e}}^{\bar{e}_0} \frac{dx}{x(1-x^2)^{1/2} \sin^2 \bar{i}\, |\sin 2\bar{\omega}|} \tag{3}$$

It may be assumed that $\bar{e}_0 > \bar{e}$, so that all terms on the rhs of Eq. (3) are > 0. Since $1/\sin^2 \bar{i}\, |\sin 2\bar{\omega}| \geq 1$, then

$$t \geq \frac{8}{15} \frac{\bar{n}}{n_3^2} \int_{\bar{e}}^{\bar{e}_0} \frac{dx}{x(1-x^2)^{1/2}} = \frac{8}{15} \frac{\bar{n}}{n_3^2} \left[-\log \bar{e} + \log\left(1 + \sqrt{1-\bar{e}^2}\right) - \log \frac{1+\sqrt{1-\bar{e}_0^2}}{\bar{e}_0} \right]$$

Assuming that $\bar{e}_0 \neq 0$, then

$$\lim_{\bar{e} \to 0_+} t \geq \lim_{\bar{e} \to 0_+} \frac{8}{15} \frac{\bar{n}}{n_3^2} \left[-\log \bar{e} + \log\left(1 + \sqrt{1-\bar{e}^2}\right) - \log \frac{1+\sqrt{1-\bar{e}_0^2}}{\bar{e}_0} \right] = +\infty$$

Hence $\bar{e} > 0$ for all finite values of t.

Similarly, it may be shown that $0° < \bar{i}_0 < 90°$ implies $0° < \bar{i} < 90°$ for all finite $t > 0$, and that $90° < \bar{i}_0 < 180°$ implies that $90° < \bar{i} < 180°$ for all finite $t > 0$.

IV. SOLUTIONS FOR SPECIAL VALUES OF INITIAL CONDITIONS

Several special cases will now be treated:

A. Special Case I[**] $\qquad \bar{e}_0 = 0, \quad \sin \bar{i}_0 \neq 0$

From Eqs. (1d) and (1e), it is seen that $\left\{ \begin{array}{l} \bar{e} = \bar{e}_0 = 0 \\ \bar{i} = \bar{i}_0 \end{array} \right.$

Equation (1f) then becomes

$$\frac{d\bar{\omega}}{dt} = \frac{3}{2}\frac{n_3^2}{\bar{n}} - \frac{15}{4}\frac{n_3^2}{\bar{n}} \sin^2 \bar{i}_0 \sin^2 \bar{\omega}$$

Integrating,

$$t = \int_{\bar{\omega}_0}^{\bar{\omega}} \frac{dx}{\ell_1^2 - \ell_2^2 \sin^2 x} \tag{4}$$

where

$$\left\{ \begin{array}{l} \ell_1^2 = \dfrac{3}{2}\dfrac{n_3^2}{\bar{n}} > 0 \\[2ex] \ell_2^2 = \dfrac{15}{4}\dfrac{n_3^2}{\bar{n}} \sin^2 \bar{i}_0 > 0 \end{array} \right.$$

The integral on the rhs of Eq. (4) is No. 436.7 in Dwight (Ref. 3), and may be expressed in terms of elementary functions of $\bar{\omega}$ and $\bar{\omega}_0$.

[**] The special case $\bar{e}_0 = 1$ is not of physical interest.

B. Special Case II† $\bar{i}_0 = 0°$

From Eqs. (1d) and (1e):

$$\begin{cases} \bar{e} = \bar{e}_0 \\ \bar{i} = \bar{i}_0 = 0° \end{cases}$$

Equation (1f) then becomes

$$\frac{d\bar{\omega}}{dt} = \frac{3}{2}\frac{n_3^2}{\bar{n}}(1 - \bar{e}_0^2)^{1/2} + \frac{15}{4}\frac{n_3^2}{\bar{n}}\frac{\bar{e}_0^2}{(1 - \bar{e}_0^2)^{1/2}} \sin^2 \bar{\omega}$$

The solution is

$$t = \int_{\bar{\omega}_0}^{\bar{\omega}} \frac{dx}{\ell_4^2 + \ell_5^2 \sin^2 x} \tag{5}$$

where

$$\begin{cases} \ell_4^2 = \dfrac{3}{2}\dfrac{n_3^2}{\bar{n}}(1 - \bar{e}_0^2)^{1/2} > 0 \\ \ell_5^2 = \dfrac{15}{4}\dfrac{n_3^2}{\bar{n}}\dfrac{\bar{e}_0^2}{(1 - \bar{e}_0^2)^{1/2}} > 0 \end{cases}$$

The integral on the rhs of Eq. (5) is No. 436.5 in Dwight (Ref. 3).

† The case $\bar{i}_0 = 180°$ has the same solution as Special Case II, except that $\bar{i} = \bar{i}_0 = 180°$.

C. Special Case III $\quad \bar{i}_0 = 90°, \quad \bar{e}_0 \neq 0$

The solution of Eq. (1e) in this case is

$$\bar{i} = \bar{i}_0 = 90°$$

Equations (1d) and (1f) then become

$$\frac{d\bar{e}}{dt} = \frac{15}{8}\frac{n_3^2}{\bar{n}}\,\bar{e}\,(1-\bar{e}^2)^{1/2}\sin 2\bar{\omega} \tag{6a}$$

$$\frac{d\bar{\omega}}{dt} = \frac{3}{2}\frac{n_3^2}{\bar{n}}(1-\bar{e}^2)^{1/2}\left(1-\frac{5}{2}\sin^2\bar{\omega}\right) \tag{6b}$$

1. *Subcase* III_A $\quad \sin^2\bar{\omega}_0 = \frac{2}{5}$

The solution of Eq. (6b) in this case is $\bar{\omega} = \bar{\omega}_0$. Equation (6a) becomes

$$\frac{d\bar{e}}{\bar{e}(1-\bar{e}^2)^{1/2}} = \frac{15}{8}\frac{n_3^2}{\bar{n}}(\sin 2\bar{\omega}_0)\,dt,$$

which integrates to

$$-\log\left|\frac{1+\sqrt{1-x^2}}{x}\right|_{\bar{e}_0}^{\bar{e}} = \frac{15}{8}\frac{n_3^2}{\bar{n}}(\sin 2\bar{\omega}_0)\,t$$

Since $0 < \bar{e} < 1$, the absolute value signs may be dropped to obtain the final result.

$$\left[\frac{1+\sqrt{1-\bar{e}^2}}{\bar{e}}\right] = \left[\frac{1+\sqrt{1-\bar{e}_0^2}}{\bar{e}_0}\right]\exp\left[-\frac{15}{8}\frac{n_3^2}{\bar{n}}(\sin 2\bar{\omega}_0)\,t\right] \tag{7}$$

Equation (7) represents an exponential decay of eccentricity toward $\bar{e} = 0$ if $\bar{\omega}_0$ is in the second or fourth quadrant, and an increase of eccentricity toward $\bar{e} = \bar{e}_{cr}$ if $\bar{\omega}_0$ is in the first or third quadrant.

2. *Subcase III_B* $\qquad \sin^2 \bar{\omega}_0 \neq \frac{2}{5}$

For this case, the rhs of Eq. (6b) does not vanish initially. Hence, at least initially, Eq. (6a) may be divided by Eq. (6b) to yield

$$\frac{d\bar{e}}{d\bar{\omega}} = \frac{5}{4} \bar{e} \frac{\sin 2\bar{\omega}}{\left(1 - \frac{5}{2} \sin^2 \bar{\omega}\right)}$$

Integrating,

$$\bar{e}^2 = \bar{e}_0^2 \frac{\left|1 - \frac{5}{2} \sin^2 \bar{\omega}_0\right|}{\left|1 - \frac{5}{2} \sin^2 \bar{\omega}\right|}$$

Since the main interest is in $0 \leq \bar{e} < 1$, then $|(1 - 5/2) \sin^2 \bar{\omega}|$ must remain $\neq 0$. Hence, $(1 - 5/2 \sin^2 \bar{\omega})$ will have the same sign as $(1 - 5/2 \sin^2 \bar{\omega}_0)$. The absolute value signs may then be dropped, i.e.,

$$\bar{e}^2 = \bar{e}_0^2 \frac{\left(1 - \frac{5}{2} \sin^2 \bar{\omega}_0\right)}{\left(1 - \frac{5}{2} \sin^2 \bar{\omega}\right)}$$

or

$$\sin^2 \bar{\omega} = \frac{2}{5} \frac{1}{\bar{e}^2} \left[\bar{e}^2 - \bar{e}_0^2 \left(1 - \frac{5}{2} \sin^2 \bar{\omega}_0\right)\right] \tag{8}$$

It should be noted that even though Eq. (8) was derived by assuming that $\sin^2 \bar{\omega}_0 \neq 2/5$, it is valid in the limiting case $\sin^2 \bar{\omega}_0 = 2/5$. It follows, then, that

$$\sin 2\bar{\omega} = 2 \sin \bar{\omega} \cos \bar{\omega}$$

$$= 2\beta \sqrt{\frac{2}{5}} \sqrt{1 - \frac{\bar{e}_0^2}{\bar{e}^2}\left(1 - \frac{5}{2}\sin^2 \bar{\omega}_0\right)} \cdot \sqrt{1 - \frac{2}{5}\left[1 - \frac{\bar{e}_0^2}{\bar{e}^2}\left(1 - \frac{5}{2}\sin^2 \bar{\omega}_0\right)\right]}$$

where

$$\beta = \begin{cases} +1 \text{ if } \sin 2\bar{\omega} > 0 \\ -1 \text{ if } \sin 2\bar{\omega} < 0 \end{cases}$$

Equation (6a) then becomes

$$\frac{d\bar{e}}{dt} = \frac{15}{4}\sqrt{\frac{2}{5}}\,\frac{n_3^2}{\bar{n}}\,\beta\,\frac{(1-e^2)^{1/2}}{\bar{e}}\sqrt{\bar{e}^2 - \bar{e}_0^2\left(1 - \frac{5}{2}\sin^2 \bar{\omega}_0\right)} \cdot \sqrt{\frac{3}{5}\bar{e}^2 + \frac{2}{5}\bar{e}_0^2\left(1 - \frac{5}{2}\sin^2 \bar{\omega}_0\right)}$$

Assuming that $\sin 2\bar{\omega}_0 \neq 0$, it is possible at least initially to divide through by the rhs of this equation, to obtain

$$dt = \frac{4}{3}\,\frac{1}{\sqrt{6}}\,\frac{\bar{n}}{n_3^2}\,\beta\,\frac{\bar{e}\,d\bar{e}}{\sqrt{-(\bar{e}^2 - 1)\left[\bar{e}^2 - \bar{e}_0^2\left(1 - \frac{5}{2}\sin^2 \bar{\omega}_0\right)\right]\left[\bar{e}^2 + \frac{2}{3}\bar{e}_0^2\left(1 - \frac{5}{2}\sin^2 \bar{\omega}_0\right)\right]}}$$

Letting

$$\begin{cases} \eta = \bar{e}^2 \\ d\eta = +2\,\bar{e}\,d\bar{e} \end{cases}$$

and

$$\begin{cases} \eta_1 = 1 \\ \eta_2 = \bar{e}_0^2 \left(1 - \frac{5}{2} \sin^2 \bar{\omega}_0\right) \\ \eta_3 = -\frac{2}{3} \bar{e}_0^2 \left(1 - \frac{5}{2} \sin^2 \bar{\omega}_0\right) \end{cases}$$

this equation becomes

$$dt = \frac{2}{3} \frac{1}{\sqrt{6}} \frac{\bar{n}}{n_3^2} \beta \frac{d\eta}{\sqrt{-(\eta - \eta_1)(\eta - \eta_2)(\eta - \eta_3)}}$$

Integrating,

$$t = \frac{2}{3} \frac{1}{\sqrt{6}} \frac{\bar{n}}{n_3^2} \beta \int_{\bar{e}_0^2}^{\bar{e}^2} \frac{d\eta}{\sqrt{-(\eta - \eta_1)(\eta - \eta_2)(\eta - \eta_3)}} \qquad \text{ELLIPTIC INTEGRAL OF THE FIRST KIND} \qquad (9)$$

The only problem that remains in Subcase III_B is to determine the value of β. From Eq. (6b), note that

$$\begin{cases} \sin^{-1}\sqrt{\frac{2}{5}} < \bar{\omega} < \pi - \sin^{-1}\sqrt{\frac{2}{5}} \Longrightarrow \frac{d\bar{\omega}}{dt} < 0 \\ \pi - \sin^{-1}\sqrt{\frac{2}{5}} < \bar{\omega} < \pi + \sin^{-1}\sqrt{\frac{2}{5}} \Longrightarrow \frac{d\bar{\omega}}{dt} > 0 \\ \pi + \sin^{-1}\sqrt{\frac{2}{5}} < \bar{\omega} < 2\pi - \sin^{-1}\sqrt{\frac{2}{5}} \Longrightarrow \frac{d\bar{\omega}}{dt} < 0 \\ 2\pi - \sin^{-1}\sqrt{\frac{2}{5}} < \bar{\omega} < 2\pi + \sin^{-1}\sqrt{\frac{2}{5}} \Longrightarrow \frac{d\bar{\omega}}{dt} > 0 \end{cases}$$

If $\sin^2 \bar{\omega}_0 \neq 2/5$, then $\bar{\omega}$ will tend toward the first or third quadrant in the manner shown in Fig. 1.

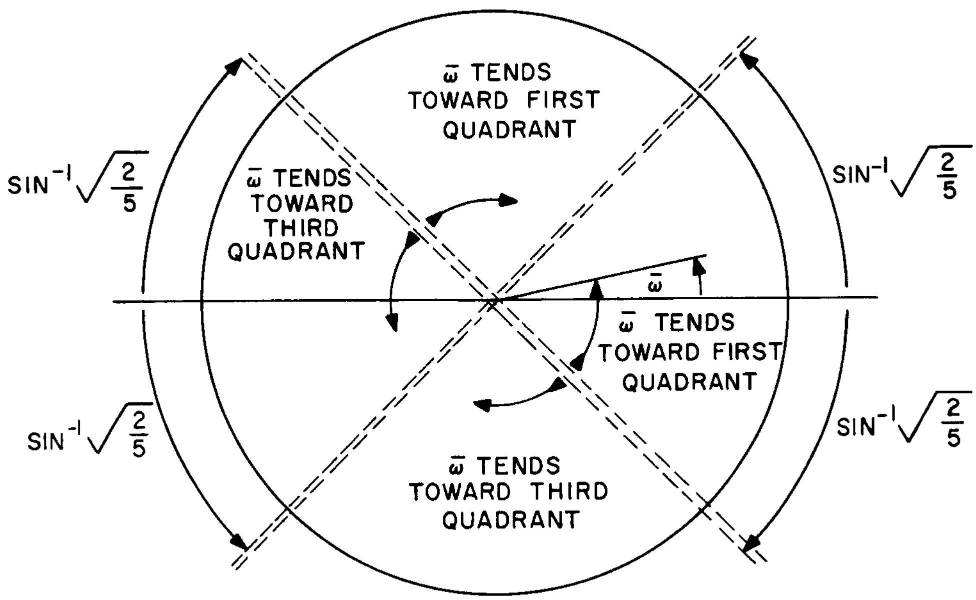

Fig. 1. Range of variation of apse angle for 90° inclination

Since

$$\begin{cases} \bar{\omega} = 0°,\ 180° \Longrightarrow \dfrac{d\bar{\omega}}{dt} > 0 \\ \bar{\omega} = 90°,\ 270° \Longrightarrow \dfrac{d\bar{\omega}}{dt} < 0 \end{cases}$$

then once $\bar{\omega}$ has entered the first or third quadrant, it will remain there for all subsequent times. A finite time, t_1, will be required to enter the first or third quadrant, assuming that $\sin^2 \bar{\omega}_0 \neq 2/5$.

Note that from Eq. (8),

$$\sin^2 \bar{\omega} = 0 \Longrightarrow \bar{e}^2 = \bar{e}_1^2 \equiv \bar{e}_0^2 \left(1 - \frac{5}{2} \sin^2 \bar{\omega}_0\right) \tag{10a}$$

$$\sin^2 \bar{\omega} = 1 \Longrightarrow \bar{e}^2 = \bar{e}_2^2 \equiv -\frac{2}{3} \bar{e}_0^2 \left(1 - \frac{5}{2} \sin^2 \bar{\omega}_0\right) \tag{10b}$$

The value of t_1 may always be found by a procedure similar to the following:

Assume that $\pi/2 < \bar{\omega}_0 < \pi - \sin^{-1}\sqrt{2/5}$. Then $\bar{\omega}$ will move toward the value $\bar{\omega} = 90°$. The value which $\bar{e}^2$ must have when $\bar{\omega} = 90°$ is given by Eq. (10b). The value of t_1 is given by Eq. (9) (using $\beta = -1$, since $\bar{\omega}$ is in the second quadrant for $0 < t < t_1$):

$$t_1 = -\frac{2}{3}\frac{1}{\sqrt{6}}\frac{\bar{n}}{n_3^2}\int_{\bar{e}_0^2}^{\bar{e}_2^2}\frac{d\eta}{\sqrt{-(\eta-\eta_1)(\eta-\eta_2)(\eta-\eta_3)}} \tag{11}$$

where

$$\begin{cases} \eta_1 = 1 \\ \eta_2 = \bar{e}_0^2\left(1 - \dfrac{5}{2}\sin^2\bar{\omega}_0\right) \\ \eta_3 = -\dfrac{2}{3}\bar{e}_0^2\left(1 - \dfrac{5}{2}\sin^2\bar{\omega}_0\right) \end{cases}$$

Since $d\bar{e}/dt < 0$ for $\bar{\omega}$ in the second quadrant, and $d\bar{e}/dt > 0$ for $\bar{\omega}$ in the first quadrant, then $\bar{e}$ reaches its minimum value at $\bar{\omega} = 90°$; i.e., $\bar{e}_{min} = \bar{e}_2$.

For $t > t_1$, $\bar{\omega}$ will remain in the first quadrant, so that $d\bar{e}/dt > 0$. Hence, $\bar{e}$ will increase until the orbit intersects the surface of the central body; i.e., until $\bar{e} = \bar{e}_{cr}$.

The time $(\tau - t_1)$ required for $\bar{e}$ to increase from $\bar{e}_2$ to $\bar{e}_{cr}$ is given by Eq. (9) (using $\beta = +1$) as

$$\tau - t_1 = \frac{2}{3}\frac{1}{\sqrt{6}}\frac{\bar{n}}{n_3^2}\int_{\bar{e}_2^2}^{\bar{e}_{cr}^2}\frac{d\eta}{\sqrt{-(\eta-\eta_1)(\eta-\eta_2)(\eta-\eta_3)}} \tag{12}$$

where

$$\begin{cases} \eta_1 = 1 \\ \eta_2 = \bar{e}_2^2 \left(1 - \frac{5}{2} \sin^2 \bar{\omega}_2\right) = -\frac{3}{2} \bar{e}_2^2 \\ \eta_3 = -\frac{2}{3} \bar{e}_2^2 \left(1 - \frac{5}{2} \sin^2 \bar{\omega}_2\right) = \bar{e}_2^2 \end{cases}$$

The lifetime of the satellite is then equal to τ.

It may be concluded that all polar satellites ($\bar{i}_0 = 90°$) are unstable, except for those having $\bar{\omega}_0 = \pi - \sin^{-1}\sqrt{2/5}$ or $\bar{\omega}_0 = 2\pi - \sin^{-1}\sqrt{2/5}$. From Eq. (1c), it is seen that $\bar{i}_0 = 90° \Longrightarrow \bar{\Omega} =$ constant $= \bar{\Omega}_0$.

Several features of the motion for this special case, such as the two integrals of Eqs. (8) and (9), are analogous to the corresponding features in the case of general initial conditions. Hence, a thorough understanding of the motion for the case $\bar{i}_0 = 90°$ is useful in solving the general case where $\bar{i}_0 \neq 90°$.

V. SOLUTIONS FOR GENERAL VALUES OF INITIAL CONDITIONS

Since the cases $\bar{e}_0 = 0$, $\bar{i}_0 = 0°$, $\bar{i}_0 = 90°$, $\bar{i}_0 = 180°$, have been solved above, and $\bar{e}_0 = 1$ has been ruled out, in all that follows,it may be assumed that $0 < \bar{e}_0 < 1$, $0° < \bar{i}_0 < 90°$. As shown above, it follows that $0 < \bar{e} < 1$, $0° < \bar{i} < 90°$, for all finite $t > 0$. (The case $0 < \bar{e}_0 < 1$, $90° < \bar{i}_0 < 180°$, may be treated in an analogous manner.)

Since it is now assumed that $\sin^2 \bar{i}_0 \neq 0$, $\sin 2\,\bar{i}_0 \neq 0$, Eq. (1d) can at least be initially divided by Eq. (1e), provided that $\sin 2\,\bar{\omega}_0 \neq 0$. (In the case $\sin 2\,\bar{\omega}_0 = 0$, it is possible to think of Eq. (1d) being divided by Eq. (1e) at some point where $\sin 2\,\bar{\omega}_0 \neq 0$. Then let $\sin 2\,\bar{\omega}_0 \to 0$.)

The following equation is obtained:

$$\frac{d\bar{e}}{d\bar{i}} = -\frac{(1-\bar{e}^2)}{\bar{e}} \tan \bar{i}$$

Integrating,

$$|1-\bar{e}^2| = \text{constant} \cdot \frac{1}{|\cos \bar{i}|^2}$$

Applying the initial conditions, and using the facts that $0 < (1-\bar{e}^2) < 1$, $|\cos \bar{i}|^2 = \cos^2 \bar{i}$, the following integral is obtained:

$$(1-\bar{e}^2)\cos^2 \bar{i} = (1-\bar{e}_0^2)\cos^2 \bar{i}_0$$

or

$$\sin^2 \bar{i} = 1 - \frac{(1-\bar{e}_0^2)}{(1-\bar{e}^2)} \cos^2 \bar{i}_0 \qquad \text{FIRST INTEGRAL OF MOTION} \qquad (13)$$

The physical interpretation of Eq. (13) is as follows: "That component of angular momentum of the artificial satellite which is normal to the ecliptic plane defined by the orbits of the two massive bodies is conserved." This follows from the fact that the averaged angular momentum $\bar{h}$, of the satellite about the central body is given by

$$\bar{h} = \mu^{1/2}\,\bar{a}^{1/2}\,(1 - \bar{e}^2)^{1/2}$$

and the fact that $\bar{a}$ is constant.

Equation (13) is one of the three integrals required to solve Eqs. (1d), (1e), and (1f).

Assuming that $\sin 2\bar{\omega}_0 \neq 0$, Eq. (1f) may at least be initially divided by Eq. (1d) to obtain

$$\frac{d\bar{\omega}}{d\bar{e}} = \frac{4}{5}\,\frac{1}{\bar{e}}\,\frac{\left[1 + \frac{5}{2}\sin^2\bar{\omega}\,\frac{(\bar{e}^2 - \sin^2\bar{i})}{(1 - \bar{e}^2)}\right]}{\sin^2\bar{i}\,\sin 2\bar{\omega}}$$

which is equivalent to

$$P(\bar{e},\bar{\omega})\,d\bar{\omega} + Q(\bar{e},\bar{\omega})\,d\bar{e} = 0 \tag{14}$$

where

$$P(\bar{e},\bar{\omega}) = \sin 2\bar{\omega} \tag{15a}$$

$$Q(\bar{e},\bar{\omega}) = -\frac{4}{5}\,\frac{1}{\bar{e}}\,\frac{1}{\sin^2\bar{i}}\left[1 + \frac{5}{2}\sin^2\bar{\omega}\,\frac{(\bar{e}^2 - \sin^2\bar{i})}{(1 - \bar{e}^2)}\right] \tag{15b}$$

One method for solving Eq. (14) is to determine an integrating factor. One such factor is

$$\mu = \bar{e}^2\sin^2\bar{i} = \frac{\bar{e}^2}{(1 - \bar{e}^2)}\left[(1 - \bar{e}^2) - (1 - \bar{e}_0^2)\cos^2\bar{i}_0\right] \tag{16}$$

Multiplying the lhs of Eq. (14) by the integrating factor μ will produce an exact differential.

$$\mu\,(P\,d\bar{\omega} + Q\,d\bar{e}) = d\Phi = 0$$

Hence

$$
\left\{
\begin{aligned}
\frac{\partial \Phi}{\partial \bar{\omega}} &= \mu P = \frac{\bar{e}^2}{(1-\bar{e}^2)} \left[(1-\bar{e}^2) - (1-\bar{e}_0^2)\cos^2 \bar{i}_0\right] \sin 2\bar{\omega} \qquad (17a) \\
\frac{\partial \Phi}{\partial \bar{e}} &= \mu Q = -\frac{4}{5}\,\frac{\bar{e}}{(1-\bar{e}^2)} \left\{(1-\bar{e}^2) + \frac{5}{2}\sin^2\bar{\omega}\left[\bar{e}^2 - 1 + \frac{(1-\bar{e}_0^2)}{(1-\bar{e}^2)}\cos^2 \bar{i}_0\right]\right\} \qquad (17b)
\end{aligned}
\right.
$$

Integrating Eq. (17a) wrt $\bar{\omega}$, obtains

$$
\Phi(\bar{e},\bar{\omega}) = \frac{\bar{e}^2}{(1-\bar{e}^2)} \left[(1-\bar{e}^2) - (1-\bar{e}_0^2)\cos^2 \bar{i}_0\right]\left(-\frac{1}{2} + \sin^2 \bar{\omega}_0\right) + f_1(\bar{e}) \qquad (18)
$$

Differentiating Eq. (18) wrt $\bar{e}$ yields

$$
\frac{\partial \Phi}{\partial \bar{e}} = \left(-\frac{1}{2} + \sin^2\bar{\omega}\right)\left\{\frac{-2\bar{e}^3}{(1-\bar{e}^2)} + \frac{2\bar{e}}{(1-\bar{e}^2)^2}\left[(1-\bar{e}^2) - (1-\bar{e}_0^2)\cos^2 \bar{i}_0\right]\right\} + \frac{df_1(\bar{e})}{d\bar{e}} \qquad (19)
$$

Equating the rhs of Eq. (19) and (17b),

$$
\frac{df_1(\bar{e})}{d\bar{e}} = -\frac{4}{5}\bar{e} - \frac{\bar{e}^3}{(1-\bar{e}^2)} + \frac{\bar{e}}{(1-\bar{e}^2)^2}\left[(1-\bar{e}^2) - (1-\bar{e}_0^2)\cos^2 \bar{i}_0\right]
$$

which may be integrated wrt $\bar{e}$, to yield

$$
f_1(\bar{e}) = \frac{\bar{e}^2}{10} - \frac{1}{2}\,\frac{(1-\bar{e}_0^2)}{(1-\bar{e}^2)}\cos^2 \bar{i}_0 + B_1
$$

where B_1 = constant. Since $d\Phi = 0$, then Φ = constant = Φ_0.

Evaluating Eq. (18) at the initial conditions yields the result

$$\Phi_0 - B_1 = -\frac{2}{5}\,\bar{e}_0^2\left(1 - \frac{5}{2}\sin^2\bar{i}_0\sin^2\bar{\omega}_0\right) - \frac{1}{2}\left(1 - \bar{e}_0^2\right)\cos^2\bar{i}_0$$

Equation (18) is therefore equivalent to

$$\sin^2\bar{\omega} = \frac{2}{5}\,\frac{\left(1-\bar{e}^2\right)\left[\bar{e}^2 - \bar{e}_0^2\left(1 - \frac{5}{2}\sin^2\bar{i}_0\sin^2\bar{\omega}_0\right)\right]}{\bar{e}^2\left[\left(1-\bar{e}^2\right) - \left(1-\bar{e}_0^2\right)\cos^2\bar{i}_0\right]} \qquad \text{SECOND INTEGRAL OF MOTION} \qquad (20)$$

Equation (20) represents the second of the three integrals required to solve Eqs. (1d), (1e), and (1f).

The remaining task is to find $\bar{e}$ as a function of t (or vice versa). This requires the integration of Eq. (1d). Equation (20) may be written as

$$\sin^2\bar{\omega} = \frac{2}{5}\,\frac{1}{\bar{e}^2\sin^2\bar{i}}\left[\bar{e}^2 - \bar{e}_0^2\left(1 - \frac{5}{2}\sin^2\bar{i}_0\sin^2\bar{\omega}_0\right)\right]$$

Hence,

$$\sin 2\bar{\omega} = 2\sin\bar{\omega}\cos\bar{\omega}$$

$$= 2\beta\left\{\frac{2}{5}\,\frac{1}{\bar{e}^2\sin^2\bar{i}}\left[\bar{e}^2 - \bar{e}_0^2\left(1 - \frac{5}{2}\sin^2\bar{i}_0\sin^2\bar{\omega}_0\right)\right]\right\}^{1/2}$$

$$\cdot\left\{1 - \frac{2}{5}\,\frac{1}{\bar{e}^2\sin^2\bar{i}}\left[\bar{e}^2 - \bar{e}_0^2\left(1 - \frac{5}{2}\sin^2\bar{i}_0\sin^2\bar{\omega}_0\right)\right]\right\}^{1/2}$$

where

$$\beta = \begin{cases} +1, & \text{if } \sin 2\bar{\omega} > 0 \\ -1, & \text{if } \sin 2\bar{\omega} < 0 \end{cases} \qquad (21)$$

Defining

$$\eta_1 = \bar{e}_0^2 \left(1 - \frac{5}{2} \sin^2 \bar{i}_0 \sin^2 \bar{\omega}_0\right) \tag{22}$$

Then

$$\sin 2\bar{\omega} = 2 \sqrt{\frac{2}{5}}\, \beta \, \frac{1}{\bar{e}^2 \sin^2 \bar{i}} \, (\bar{e}^2 - \eta_1)^{1/2} \left[\bar{e}^2 \sin^2 \bar{i} - \frac{2}{5} (\bar{e}^2 - \eta_1)\right]^{1/2}$$

Equation (1d) may then be written as

$$\frac{d\bar{e}}{dt} = \frac{3}{4} \sqrt{6}\, \frac{n_3^2}{\bar{n}} \, \frac{\beta}{\bar{e}} \, (\bar{e}^2 - \eta_1)^{1/2} \left\{-\bar{e}^4 - \bar{e}^2 \left[-1 + \frac{5}{3} (1 - \bar{e}_0^2) \cos^2 \bar{i}_0 + \frac{2}{3} \eta_1\right] + \frac{2}{3} \eta_1\right\}^{1/2}$$

Defining

$$A_1 = -1 + \frac{5}{3} (1 - \bar{e}_0^2) \cos^2 \bar{i}_0 + \frac{2}{3} \eta_1$$

$$A_2 = -\frac{2}{3} \eta_1 \tag{23}$$

so that

$$\frac{d\bar{e}}{dt} = \frac{3}{4} \sqrt{6}\, \frac{n_3^2}{\bar{n}} \, \frac{\beta}{\bar{e}} \, (\bar{e}^2 - \eta_1)^{1/2} (-\bar{e}^4 - A_1 \bar{e}^2 - A_2)^{1/2}$$

Defining

$$\eta_2 = \frac{1}{2} \left(-A_1 + \sqrt{A_1^2 - 4A_2}\right)$$

$$\eta_3 = \frac{1}{2} \left(-A_1 - \sqrt{A_1^2 - 4A_2}\right) \tag{24}$$

the previous equation becomes

$$\frac{d\bar{e}}{dt} = \frac{3}{4}\sqrt{6}\,\frac{n_3^2}{\bar{n}}\,\frac{\beta}{\bar{e}}\left[-(\bar{e}^2-\eta_1)(\bar{e}^2-\eta_2)(\bar{e}^2-\eta_3)\right]^{1/2} \tag{25}$$

At some point where $d\bar{e}/dt \neq 0$, divide through by rhs to obtain

$$dt = \frac{4}{3}\,\frac{1}{\sqrt{6}}\,\frac{\bar{n}}{n_3^2}\,\beta\,\frac{\bar{e}\,d\bar{e}}{\left[-(\bar{e}^2-\eta_1)(\bar{e}^2-\eta_2)(\bar{e}^2-\eta_3)\right]^{1/2}}$$

Letting

$$\begin{cases} \eta = \bar{e}^2 \\ d\eta = +2\,\bar{e}\,d\bar{e} \end{cases}$$

the previous equation becomes

$$dt = \frac{2}{3}\,\frac{1}{\sqrt{6}}\,\frac{\bar{n}}{n_3^2}\,\beta\,\frac{d\eta}{\left[-(\eta-\eta_1)(\eta-\eta_2)(\eta-\eta_3)\right]^{1/2}}$$

Integrating, and defining $t = 0$ when $\bar{e} = e_0$, $\bar{i} = \bar{i}_0$, $\bar{\omega} = \omega_0$,

$$t = \frac{2}{3}\,\frac{1}{\sqrt{6}}\,\frac{\bar{n}}{n_3^2}\,\beta\int_{\bar{e}_0^2}^{\bar{e}^2}\frac{d\eta}{\left[-(\eta-\eta_1)(\eta-\eta_2)(\eta-\eta_3)\right]^{1/2}} \qquad \text{THIRD INTEGRAL OF MOTION} \tag{26}$$

The rhs of Eq. (26) is in general an elliptic integral of the first kind, and may be evaluated numerically by use of transformations given in Franklin (Ref. 4).

Equation (26) represents the third and final integral of motion required for the solution of the three coupled first-order equations.

VI. BEHAVIOR OF NODE ANGLE $\bar{\Omega}$

Having solved Eqs. (1d), (1e), and (1f), it is now possible to solve Eq. (1c) for the behavior of the node angle $\bar{\Omega}$. This equation may be written as

$$\frac{d\bar{\Omega}}{dt} = -\frac{3}{4}\frac{n_3^2}{\bar{n}}\frac{\cos \bar{i}}{(1-\bar{e}^2)^{1/2}}\left[(1-\bar{e}^2) + 5\bar{e}^2 \sin^2 \bar{\omega}\right] \tag{27}$$

It was shown above that $\bar{i}_0 = 90° \Longrightarrow \bar{\Omega} = \text{constant} = \bar{\Omega}_0$.

From Eq. (13),

$$\cos \bar{i} = \frac{(1-\bar{e}_0^2)^{1/2}}{(1-\bar{e}^2)^{1/2}} \cos \bar{i}_0$$

since $\bar{i}$ must lie in the same quadrant as $\bar{i}_0$, as shown above. Equation (27) may then be written as

$$\frac{d\bar{\Omega}}{dt} = -\frac{3}{4}\frac{n_3^2}{\bar{n}}(1-\bar{e}_0^2)^{1/2} \cos i_0 \frac{\{\bar{e}^2 + [1 - 2\eta_1 - (1-\bar{e}_0^2)\cos^2 \bar{i}_0]\}}{[(1-\bar{e}^2) - (1-\bar{e}_0^2)\cos^2 \bar{i}_0]} \tag{28}$$

Dividing Eq. (28) by Eq. (25),

$$\frac{d\bar{\Omega}}{d\bar{e}} = -\beta \frac{1}{\sqrt{6}}(1-\bar{e}_0^2)^{1/2} \cos \bar{i}_0 \frac{\bar{e}\,\{\bar{e}^2 + [1 - 2\eta_1 - (1-\bar{e}_0^2)\cos^2 \bar{i}_0]\}}{\{\bar{e}^2 + [-1 + (1-\bar{e}_0^2)\cos^2 \bar{i}_0]\}\,[-(\bar{e}^2 - \eta_1)(\bar{e}^2 - \eta_2)(\bar{e}^2 - \eta_3)]^{1/2}}$$

Letting

$$\left\{\begin{array}{l} \eta = \bar{e}^2 \\ \\ d\eta = +2\,\bar{e}\,d\bar{e} \end{array}\right.$$

the previous equation becomes

$$\frac{d\bar{\Omega}}{d\eta} = \beta \frac{1}{2\sqrt{6}} (1 - \bar{e}_0^2)^{1/2} \cos \bar{i}_0 \cdot \left[1 + \frac{\{\eta + 2[1 - \eta_1 - (1 - \bar{e}_0^2) \cos^2 \bar{i}_0]\}}{\{\eta + [-1 + (1 - \bar{e}_0^2) \cos^2 \bar{i}_0]\}} \right] \cdot \frac{1}{[-(\eta - \eta_1)(\eta - \eta_2)(\eta - \eta_3)]^{1/2}}$$

which may be integrated in the following manner:

$$\bar{\Omega} = \bar{\Omega}_0 + \frac{\beta}{2\sqrt{6}} \cos \bar{i}_0 (1 - \bar{e}_0^2)^{1/2} \left[\int_{\bar{e}_0^2}^{\bar{e}^2} \frac{d\eta}{[-(\eta - \eta_1)(\eta - \eta_2)(\eta - \eta_3)]^{1/2}} + \int_{\bar{e}_0^2}^{\bar{e}^2} \frac{(\eta - \eta_4)\, d\eta}{(\eta - \eta_5)[-(\eta - \eta_1)(\eta - \eta_2)(\eta - \eta_3)]^{1/2}} \right] \tag{29}$$

where

$$\begin{cases} \eta_4 = 2[-1 + \eta_1 + (1 - \bar{e}_0^2) \cos^2 \bar{i}_0] \\ \eta_5 = 1 - (1 - \bar{e}_0^2) \cos^2 \bar{i}_0 \end{cases} \tag{30}$$

The first integral on the rhs of Eq. (29) is an elliptic integral of the first kind, and hence may be numerically evaluated by means of tables. The second integral, however, is more complicated and probably would have to be numerically integrated for each set of values η_1, η_2, η_3, η_4, η_5.

By an analogous procedure, Eq. (1b) may be integrated, and then $\bar{\chi}$ is expressible in a form similar to Eq. (29).

VII. BEHAVIOR OF PERICENTER ANGLE $\bar{\omega}$

In order to utilize the elliptic integral solution, the value of β must be known. Hence it becomes necessary to partially determine the behavior of $\bar{\omega}$ as a function of t.

In particular, note that

$$\frac{d\bar{\omega}}{dt} = 0 \text{ iff (if and only if) } \frac{2}{5}\left(1-\bar{e}^2\right) = \left(\sin^2 \bar{i} - \bar{e}^2\right) \sin^2 \bar{\omega}$$

which is equivalent to

$$\bar{e}^4 \left[\left(1-\bar{e}_0^2\right)\cos^2 \bar{i}_0 + \bar{e}_0^2 \left(1 - \frac{5}{2}\sin^2 \bar{i}_0 \sin^2 \bar{\omega}_0\right)\right] + \bar{e}^2 \left[-2\bar{e}_0^2 \left(1 - \frac{5}{2}\sin^2 \bar{i}_0 \sin^2 \bar{\omega}_0\right)\right]$$

$$+ \bar{e}_0^2 \left[1 - \left(1-\bar{e}_0^2\right)\cos^2 \bar{i}_0\right]\left(1 - \frac{5}{2}\sin^2 \bar{i}_0 \sin^2 \bar{\omega}_0\right) = 0$$

or

$$\frac{3}{5}\left(1 + A_1 + \eta_1\right)\bar{e}^4 - 2\eta_1 \bar{e}^2 + \left[1 - \left(1-\bar{e}_0^2\right)\cos^2 \bar{i}_0\right]\eta_1 = 0$$

Defining

$$B_1 = \frac{3}{5}\left(1 + A_1 + \eta_1\right) = \left(1-\bar{e}_0^2\right)\cos^2 \bar{i}_0 + \bar{e}_0^2 \left(1 - \frac{5}{2}\sin^2 \bar{i}_0 \sin^2 \bar{\omega}_0\right)$$

$$B_2 = -2\eta_1 = 2\bar{e}_0^2 \left(1 - \frac{5}{2}\sin^2 \bar{i}_0 \sin^2 \bar{\omega}_0\right) \tag{31}$$

$$B_3 = \left[1 - \left(1-\bar{e}_0^2\right)\cos^2 \bar{i}_0\right]\eta_1 = \bar{e}_0^2 \left[1 - \left(1-\bar{e}_0^2\right)\cos^2 \bar{i}_0\right]\left(1 - \frac{5}{2}\sin^2 \bar{i}_0 \sin^2 \bar{\omega}_0\right)$$

Then

$$\frac{d\bar{\omega}}{dt} = 0 \text{ iff } B_1 \bar{e}^4 + B_2 \bar{e}^2 + B_3 = 0 \tag{32}$$

The solution of Eq. (32) is

$$\bar{e}^2 = \frac{-B_2 \pm \sqrt{B_2^2 - 4B_1B_3}}{2B_1} \quad ; B_1 \neq 0 \tag{33}$$

A. Case A

For the special case $B_1 = 0$, (which can occur only if $\sin^2 \bar{i}_0 \sin^2 \bar{\omega}_0 > 2/5$) the solution is

$$\bar{e}^2 = \bar{e}_A^2 \equiv -\frac{B_3}{B_2} \quad ; \quad B_1 = 0 \tag{34}$$

Note that $B_1 = 0 \Longrightarrow B_2 = 2(1-\bar{e}_0^2)\cos^2 \bar{i}_0 \neq 0$. Hence, one of the two solutions of Eq. (33) or Eq. (34) will hold in any particular case.

If $d\bar{\omega}/dt$ is ever to be $=0$, then the corresponding value of $\bar{e}^2$ given by Eq. (33) or Eq. (34) must be real-valued, and such that $0 \leq \bar{e}^2 \leq 1$. This is easily seen to be the case for Eq. (34). Hence, for Case A,

$$\frac{d\bar{\omega}}{dt} = 0 \text{ iff } \begin{cases} \bar{e}^2 = \bar{e}_A^2 \equiv \dfrac{1}{2}\left[1 - (1-\bar{e}_0^2)\cos^2 \bar{i}_0\right] \\[2ex] \sin^2 \bar{\omega} = \sin^2 \bar{\omega}_A \equiv \dfrac{2}{5} \dfrac{(1-\bar{e}_A^2)\left[\bar{e}_A^2 - \bar{e}_0^2\left(1 - \dfrac{5}{2}\sin^2 \bar{i}_0 \sin^2 \bar{\omega}_0\right)\right]}{\bar{e}_A^2\left[(1-\bar{e}_A^2) - (1-\bar{e}_0^2)\cos^2 \bar{i}_0\right]} \end{cases} \tag{35}$$

NECESSARY AND SUFFICIENT CONDITION FOR $\dfrac{d\bar{\omega}}{dt} = 0$

$(B_1 = 0)$

Equation (33) must now be investigated. In order for $\bar{e}^2$ to be real-valued in Eq. (33), $(B_2^2 - 4B_1B_3) \geq 0$. It may be shown that

$$(B_2^2 - 4B_1B_3) = 2B_2(1-\bar{e}_0^2)\cos^2\bar{i}_0\left[1 + \frac{1}{2}B_2 - (1-\bar{e}_0^2)\cos^2\bar{i}_0\right] \tag{36}$$

There are four possibilities that would make $\bar{e}^2$ real-valued:

(a) $$B_2 = 0 \Longrightarrow \sin^2\bar{i}_0 \sin^2\bar{\omega}_0 = \frac{2}{5} \Longrightarrow \begin{cases} B_1 = (1-\bar{e}_0^2)\cos^2\bar{i}_0 \neq 0 \\ B_2 = 0 \\ B_3 = 0 \end{cases}$$

This implies that $\left.\bar{e}^2\right|_{\frac{d\bar{\omega}}{dt}=0} = 0.$

But it takes an infinite time for $\bar{e}$ to reach 0. Hence, $\bar{\omega}$ will approach a limiting value asymptotically. From Eq. (20), it is seen that this limiting value must be such that $\sin^2\bar{\omega}_a = (2/5)\,1/[1-(1-\bar{e}_0^2)\cos^2\bar{i}_0]$. In order that $\sin^2\bar{\omega}_a \leq 1$, it must be that $(1-\bar{e}_0^2)\cos^2\bar{i}_0 \leq 3/5$, but this is certainly satisfied, since $\sin^2\bar{i}_0 \geq 2/5$. In order for $\bar{e} \to 0$, $d\bar{e}/dt < 0$, so that the limiting value of $\bar{\omega}$ must lie in the second or fourth quadrant. Hence, this limiting value must be either $\pi - \sin^{-1}\sqrt{\sin^2\bar{\omega}_a}$ or $2\pi - \sin^{-1}\sqrt{\sin^2\omega_a}$.

(b) $$\left[1 + \frac{1}{2}B_2 - (1-\bar{e}_0^2)\cos^2\bar{i}_0\right] = 0$$

$$\Longrightarrow \sin^2\bar{i}_0 = 0$$

But this is Special Case II above, and may therefore be excluded here.

(c) $$B_2 < 0, \left[1 + \frac{1}{2}B_2 - (1-\bar{e}_0^2)\cos 2\,\bar{i}_0\right] < 0$$

$$\Longrightarrow \begin{cases} \left(1 - \frac{5}{2}\sin^2\bar{i}_0\sin^2\bar{\omega}_0\right) > 0 \\ \left[(1-\bar{e}_0^2) + \frac{5}{2}\bar{e}_0^2\sin^2\bar{\omega}_0\right]\sin^2\bar{i}_0 < 0 \end{cases}$$

The second of these inequalities is never satisfied. Hence (c) cannot occur.

(d)
$$B_2 > 0, \quad \left[1 + \frac{1}{2} B_2 - (1 - \bar{e}_0^2) \cos^2 \bar{i}_0\right] > 0$$

$$\Longrightarrow \begin{cases} \left(1 - \dfrac{5}{2} \sin^2 \bar{i}_0 \sin^2 \bar{\omega}_0\right) < 0 \\ \left[(1 - \bar{e}_0^2) + \dfrac{5}{2} \bar{e}_0^2 \sin^2 \bar{\omega}_0\right] \sin^2 \bar{i}_0 > 0 \end{cases}$$

The second of these inequalities is always satisfied (except in Special Case II); the first is satisfied iff $\sin^2 \bar{i}_0 \sin^2 \bar{\omega}_0 > 2/5$. It may therefore be concluded that for the case of general initial conditions discussed in Section V, $d\bar{\omega}/dt$ is not $= 0$ for any finite $t > 0$, provided that $\sin^2 \bar{i}_0 \sin^2 \bar{\omega}_0 < 2/5$.

It must now be determined whether or not those values of $\bar{e}^2$ for which $d\bar{\omega}/dt = 0$ are such that $0 \leq \bar{e}^2 \leq 1$. Only those cases where $\sin^2 \bar{i}_0 \sin^2 \bar{\omega}_0 > 2/5$ need be considered. Also, since the case $B_1 = 0$ was treated in Eq. (35), it is necessary only to consider the cases $B_1 > 0$ and $B_1 < 0$.

B. Case B: $B_1 > 0$

This will occur whenever

$$(1 - \bar{e}_0^2) \cos^2 \bar{i}_0 > -\bar{e}_0^2 \left(1 - \frac{5}{2} \sin^2 \bar{i}_0 \sin^2 \bar{\omega}_0\right) > 0 \tag{37}$$

The second half of this inequality follows from the fact that only those cases where $\sin^2 \bar{i}_0 \sin^2 \bar{\omega}_0 > 2/5$ are being considered. Since $\bar{e}^2$ must be ≥ 0, it is seen from Eq. (33) that the "+" sign must be chosen (since $B_2 > 0$). Also, it must be true that

$$B_2 \leq \sqrt{B_2^2 - 4 B_1 B_3}$$

or $B_1 B_3 \leq 0$, which is satisfied.

Also, $\bar{e}^2 < 1$ iff

$$-B_2 + \sqrt{B_2^2 - 4B_1B_3} < 2B_1$$

$$\sqrt{B_2^2 - 4B_1B_3} < 2B_1 + B_2 = 2(1-\bar{e}_0^2)\cos^2\bar{i}_0$$

$$(B_2^2 - 4B_1B_3) < 4(1-\bar{e}_0^2)^2\cos^4\bar{i}_0 = 4B_1^2 + 4B_1B_2 + B_2^2$$

$$-B_3 < B_1 + B_2$$

$$(1-\bar{e}_0^2)\cos^2\bar{i}_0\left[1-\bar{e}_0^2\left(1-\frac{5}{2}\sin^2\bar{i}_0\sin^2\bar{\omega}_0\right)\right] > 0$$

But this last inequality is certainly satisfied, since $\sin^2\bar{i}_0\sin^2\bar{\omega}_0 > 2/5$. Hence,

$$0 \le \bar{e}^2 = \bar{e}_B^2 \equiv \frac{-B_2 + \sqrt{B_2^2 - 4B_1B_3}}{2B_1} < 1 \tag{38}$$

Whenever Eq. (37) is satisfied, Case B will occur, and $d\bar{\omega}/dt = 0$ when $\bar{e}^2 = \bar{e}_B^2$.

C. Case C: $B_1 < 0$

This will occur whenever

$$0 < (1-\bar{e}_0^2)\cos^2\bar{i}_0 < -\bar{e}_0^2\left(1-\frac{5}{2}\sin^2\bar{i}_0\sin^2\bar{\omega}_0\right) \tag{39}$$

(It should be kept in mind that only $\sin^2\bar{i}_0\sin^2\bar{\omega}_0 > 2/5$ is being considered, so that $B_2>0$.) The requirement that $\bar{e}^2 \ge 0$ in turn, requires that $(-B_2 \pm \sqrt{B_2^2 - 4B_1B_3}) \le 0$. Therefore, there are two subcases of CaseC:

(a) $0 < B_2 < \sqrt{B_2^2 - 4B_1B_3}$, with the "–" sign chosen

or

$B_1B_3 < 0$

This is not satisfied, since $B_3 < 0$, so that (a) cannot occur.

(b) $$B_2 \geq \sqrt{B_2^2 - 4B_1B_3} \geq 0$$

or

$$B_1B_3 \geq 0$$

This condition is satisfied (in fact, B_1B_3 is always > 0 for Case C); either the "+" or "–" sign may be taken in Eq. (33), both signs yielding $\bar{e}^2 \geq 0$. However,

$$\bar{e}^2 < 1 \text{ iff } \left(-B_2 \pm \sqrt{B_2^2 - 4B_1B_3}\right) > 2B_1$$

(The inequality has changed because $B_1 < 0$).

$$\pm\sqrt{B_2^2 - 4B_1B_3} > 2B_1 + B_2 = 2(1 - \bar{e}_0^2)\cos^2 \bar{i}_0 > 0$$

The "–" sign must therefore be discarded. Therefore,

$$\bar{e}^2 < 1 \text{ iff } +\sqrt{B_2^2 - 4B_1B_3} > 2B_1 + B_2 = 2(1 - \bar{e}_0^2)\cos^2 \bar{i}_0 > 0$$

$$B_2^2 - 4B_1B_3 > 4B_1^2 + 4B_1B_2 + B_2^2$$

$$-B_3 < B_1 + B_2$$

(The inequality has changed because $B_1 < 0$.)

$$(1 - \bar{e}_0^2)\cos^2 \bar{i}_0 \left[1 - \bar{e}_0^2 \cdot \left(1 - \frac{5}{2}\sin^2 \bar{i}_0 \sin^2 \bar{\omega}_0\right)\right] > 0$$

This last inequality is certainly satisfied. Hence,

$$0 \leq \bar{e}^2 = \bar{e}_C^2 \equiv \frac{-B_2 + \sqrt{B_2^2 - 4B_1B_3}}{2B_1} < 1 \tag{40}$$

Whenever Eq. (39) is satisfied, Case C will occur and $d\bar{\omega}/dt = 0$ when $\bar{e}^2 = \bar{e}_C^2$.

From Eqs. (38) and (40), it may be concluded that the value of $\bar{e}^2$ which must occur whenever $d\bar{\omega}/dt = 0$ is always given by $1/2B_1 \; (-B_2 + \sqrt{B_2^2 - 4B_1B_3})$, provided that $B_1 \neq 0$. Therefore,

$$\frac{d\bar{\omega}}{dt} = 0 \text{ for a finite } t > 0 \text{ iff} \qquad \bar{e}^2 = \bar{e}_B^2 \equiv \frac{1}{2B_1}\left[-B_2 + \sqrt{B_2^2 - 4B_1B_3}\right]$$

$$\sin^2\bar{\omega} = \sin^2\bar{\omega}_B \equiv \frac{\frac{2}{5}(1-\bar{e}_B^2)(\bar{e}_B^2 - \eta_1)}{\bar{e}_B^2\left[(1-\bar{e}_B^2) - (1-\bar{e}_0^2)\cos^2\bar{i}_0\right]}$$

$$B_1 \neq 0 \qquad \sin^2\bar{i}_0 \sin^2\bar{\omega}_0 > \frac{2}{5} \tag{41}$$

Note that Eq. (41) yields four values of $\bar{\omega}_B$, such that $d\bar{\omega}/dt\,|_{\bar{\omega}=\bar{\omega}_B} = 0$. Hence, if $\sin^2\bar{i}_0 \sin^2\bar{\omega}_0 > 2/5$, then there are exactly four values of $\bar{\omega}$ at which $(d\bar{\omega}/dt) = 0$ (unless $\sin^2\bar{\omega}_B = 1$). It has not yet been determined whether any of the values $\bar{\omega}_B$ are actually attained.

For the special case where $\sin^2\bar{\omega}_B = 1$, it is seen from Eq. (1d) that $d\bar{e}/dt\,|_{\bar{\omega}=\bar{\omega}_B} = 0$. Hence, $\bar{\omega}$ will be constant with time because of Eq. (20).

From Eq. (20),

$$\sin^2\bar{\omega} = 0 \text{ iff } \bar{e}^2 = \bar{e}_3^2 \equiv \bar{e}_0^2\left(1 - \frac{5}{2}\sin^2\bar{i}_0\sin^2\bar{\omega}_0\right) = \eta_1 \qquad \begin{array}{c}\text{NECESSARY AND}\\ \text{SUFFICIENT}\\ \text{CONDITION FOR}\\ \sin^2\bar{\omega} = 0\end{array} \tag{42}$$

If $\sin^2\bar{i}_0 \sin^2\bar{\omega}_0 \leq 2/5$, then $0 \leq \bar{e}_3^2 \leq \bar{e}_0^2$. If $\sin^2\bar{i}_0\sin^2\bar{\omega}_0 > 2/5$, then $\bar{\omega}$ cannot achieve either of the values 0° or 180°, as this would imply $\bar{e}^2 = \bar{e}_3^2 < 0$.

Also from Eq. (20),

$$\sin^2 \bar{\omega} = 1 \text{ iff } \bar{e}^4 + \bar{e}^2 \left[-1 + \frac{5}{3} (1 - \bar{e}_0^2) \cos^2 \bar{i}_0 + \frac{2}{3} \bar{e}_0^2 \left(1 - \frac{5}{2} \sin^2 \bar{i}_0 \sin^2 \bar{\omega}_0 \right) \right]$$

$$- \frac{2}{3} \bar{e}_0^2 \left(1 - \frac{5}{2} \sin^2 \bar{i}_0 \sin^2 \bar{\omega}_0 \right) = 0$$

Using the previous definitions of A_1 and A_2, this becomes

$$\sin^2 \bar{\omega} = 1 \text{ iff } \bar{e}^4 + A_1 \bar{e}^2 + A_2 = 0,$$

the solution of which is

$$\bar{e}^2 = \bar{e}_4^2 \equiv \frac{1}{2} \left(-A_1 \pm \sqrt{A_1^2 - 4A_2} \right)$$
$$= \eta_2 \text{ or } \eta_3 \qquad \text{NECESSARY AND SUFFICIENT CONDITION FOR } \sin^2 \bar{\omega} = 1 \qquad (43)$$

Since $d\bar{e}/dt = 0$ when $\bar{\omega} = 0°, 90°, 180°, 270°$, and nowhere else, then $\bar{e}_3^2$ and $\bar{e}_4^2$ will be extremal values of $\bar{e}^2$, provided that $d^2\bar{e}/dt^2 \, |_{\sin 2\bar{\omega} = 0} \neq 0$. But,

$$\left. \frac{d^2\bar{e}}{dt^2} \right|_{\sin 2\bar{\omega} = 0} = (\ ) \cdot \left. \frac{d\bar{e}}{dt} \right|_{\sin 2\bar{\omega} = 0} \!\!\!\! \nearrow^{0} + (\ ) \cdot \left. \frac{d\bar{i}}{dt} \right|_{\sin 2\bar{\omega} = 0} \!\!\!\! \nearrow^{0}$$

$$+ \frac{15}{4} \frac{n_3^2}{\bar{n}} \bar{e} (1 - \bar{e}^2)^{1/2} \sin^2 \bar{i} (\cos^2 \bar{\omega} - \sin^2 \bar{\omega}) \cdot \left. \frac{d\bar{\omega}}{dt} \right|_{\sin 2\bar{\omega} = 0}$$

Since

$$\sin^2 \bar{\omega} = 0 \Longrightarrow \frac{d\bar{\omega}}{dt} = \frac{3}{2} \frac{n_3^2}{\bar{n}} (1 - \bar{e}^2)^{1/2} > 0,$$

then $d^2\bar{e}/dt^2 \,|_{\sin^2 \bar{\omega} = 0} > 0$, so that $\bar{e}$ will always reach a relative minimum equal to $\bar{e}_3$ whenever $\bar{\omega}$ has the value 0° or 180°.

Also,

$$\sin^2 \bar{\omega} = 1 \Longrightarrow \frac{d\bar{\omega}}{dt} = \frac{9}{4} \frac{n_3^2}{\bar{n}} \frac{1}{\bar{e}_4^2 (1 - \bar{e}_4^2)^{1/2}} \left[\bar{e}_4^4 + \frac{2}{3} \bar{e}_0^2 \left(1 - \frac{5}{2} \sin^2 \bar{i}_0 \sin^2 \bar{\omega}_0 \right) \right]$$

$$= \frac{9}{4} \frac{n_3^2}{\bar{n}} \frac{1}{(1 - \bar{e}_4^2)^{3/2}} \left[-(1 - \bar{e}_4^2)^2 + \frac{5}{3} (1 - \bar{e}_0^2) \cos^2 \bar{i}_0 \right] \tag{44}$$

From Eq. (44), note that if $\sin^2 \bar{i}_0 \sin^2 \bar{\omega}_0 < 2/5$, then $d\bar{\omega}/dt \,|_{\sin^2 \bar{\omega} = 1} > 0$ regardless of the value of $\bar{e}_4^2$. Hence, $d^2\bar{e}/dt^2 \,|_{\sin^2 \bar{\omega} = 1} < 0$, so that for this case, $\bar{e}$ will achieve a relative maximum of $\bar{e}_4$ whenever $\bar{\omega} = 90°$ or 270°. This does not mean that $\bar{\omega}$ will actually attain the values 90° and 270°.

Assume now that the value of $\bar{e}^2$ (if any) which makes $d\bar{\omega}/dt = 0$ has been found. The corresponding value of $\sin^2 \bar{\omega}$ may be computed from Eq. (35) or Eq. (41). The polar plot of $\bar{\omega}$ may then be subdivided into four sectors, within any one of which $d\bar{\omega}/dt$ has a constant sign.

Since

$$\sin^2 \bar{\omega} = 0 \Longrightarrow \frac{d\bar{\omega}}{dt} = \frac{3}{2} \frac{n_3^2}{\bar{n}} (1 - \bar{e}^2)^{1/2} > 0,$$

then $d\bar{\omega}/dt > 0$ in the sectors centered around $\bar{\omega} = 0°$ and 180°.

The sign of $d\bar{\omega}/dt$ in the sectors centered about $\bar{\omega} = 90°$ and $\bar{\omega} = 270°$ will be the same as that of $d\bar{\omega}/dt \,|_{\sin^2 \bar{\omega} = 1}$. Since $\overline{(1 - \bar{e}_4^2)^2} < 1$, it is seen from Eq. (44) that

$$(1 - \bar{e}_0^2) \cos^2 \bar{i}_0 > \frac{3}{5} \Longrightarrow \left. \frac{d\bar{\omega}}{dt} \right|_{\sin^2 \bar{\omega} = 1} > 0$$

From Eq. (43) it is seen that

$$(1-\bar{e}_4^2) = \frac{1}{2}\,(2 + A_1 \mp \sqrt{A_1^2 - 4A_2})$$

$$(1-\bar{e}_4^2)^2 = \frac{1}{2}\,[A_1^2 + 2(1 + A_1 - A_2) \mp (2 + A_1)\sqrt{A_1^2 - 4A_2}]$$

From Eq. (44) it is then seen that

$$\left.\frac{d\bar{\omega}}{dt}\right|_{\sin^2\bar{\omega}=1} < 0 \text{ iff } \frac{1}{2}\left[A_1^2 + 2(1 + A_1 - A_2) \mp (2 + A_1)\sqrt{A_1^2 - 4A_2}\right] > \frac{5}{3}\,(1-\bar{e}_0^2)\cos^2\bar{i}_0 \quad (45a)$$

$$\left.\frac{d\bar{\omega}}{dt}\right|_{\sin^2\bar{\omega}=1} = 0 \text{ iff } \frac{1}{2}\left[A_1^2 + 2(1 + A_1 - A_2) \mp (2 + A_1)\sqrt{A_1^2 - 4A_2}\right] = \frac{5}{3}\,(1-\bar{e}_0^2)\cos^2\bar{i}_0 \quad (45b)$$

$$\left.\frac{d\bar{\omega}}{dt}\right|_{\sin^2\bar{\omega}=1} > 0 \text{ iff } \frac{1}{2}\left[A_1^2 + 2(1 + A_1 - A_2) \mp (2 + A_1)\sqrt{A_1^2 - 4A_2}\right] < \frac{5}{3}\,(1-\bar{e}_0^2)\cos^2\bar{i}_0 \quad (45c)$$

Assume now that the sign of $d\bar{\omega}/dt$ is known in each of the four sectors.

A polar plot is shown in Fig. 2.

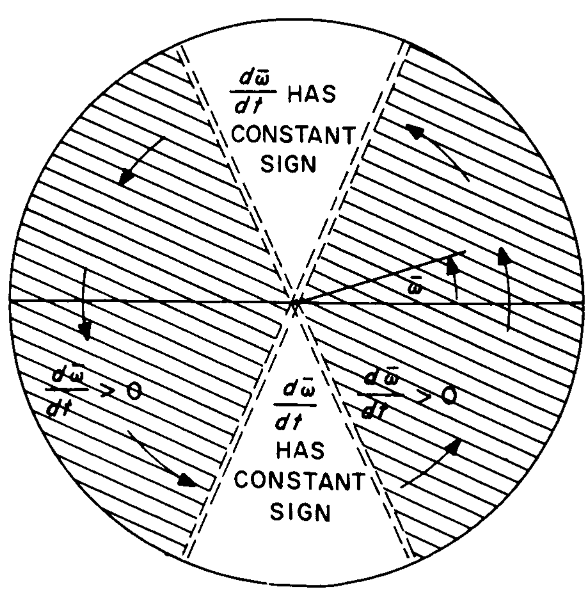

Fig. 2. Range of variation of apse angle, general case

Consider one of the four dotted lines which form the boundaries of the sectors. One of the following three cases must hold:

Case I:
$$\frac{d\bar{\omega}}{dt} < 0 \rightarrow \Vert \leftarrow \frac{d\bar{\omega}}{dt} > 0$$

This requires $d\bar{\omega}/dt \,|_{\sin^2 \bar{\omega} = 1} < 0$, so that inequality Eq. (45a) must hold true. In this case, the line will be a point of stability for $\bar{\omega}$; $\bar{\omega}$ will tend to approach this value and remain there. A finite time interval will probably be required for this approach; also, after some time interval, $\bar{\omega}$ will remain in the same quadrant as the dividing line. Since β will then be known, a complete time history of the motion can be given.

Note that Case I can occur only in the first and third quadrants. Hence, $d\bar{e}/dt > 0$ at all such points.

Case II:
$$\frac{d\bar{\omega}}{dt} > 0 \leftarrow \Vert \rightarrow \frac{d\bar{\omega}}{dt} < 0$$

This again requires $d\bar{\omega}/dt \,|_{\sin^2 \bar{\omega} = 1} < 0$, so that Eq. (45a) must hold true. In this case, the dividing line will be a point of instability for $\bar{\omega}$; $\bar{\omega}$ will tend to move away from this point.

Note that such unstable points can occur only in the second or fourth quadrants; hence, $d\bar{e}/dt < 0$ at all such points.

From Cases I and II, it can be concluded that $d\bar{\omega}/dt \,|_{\sin^2 \bar{\omega} = 1} < 0$ implies that all orbits are unstable, except possibly for the two cases

$$\bar{\omega}_0 = \pi - \sin^{-1}\sqrt{\sin^2 \bar{\omega}_B}, \qquad \bar{\omega}_0 = 2\pi - \sin^{-1}\sqrt{\sin^2 \bar{\omega}_B}$$

Case III: $$\frac{d\bar{\omega}}{dt} > 0 \quad \leftarrow \Vert \leftarrow \quad \frac{d\bar{\omega}}{dt} > 0$$

This requires $d\bar{\omega}/dt\,|_{\sin^2\bar{\omega} = 1} > 0$, so that Eq. (45c) must be satisfied. Dividing lines of this type, if they occur at all, must occur simultaneously in all four quadrants (except for the special case $\sin^2 \bar{\omega}_B = 1$).

VIII. PERIODIC MOTION

In those cases where periodic motion of the averaged variables occurs, the period may be expressed in terms of complete elliptic integrals of the first kind. (By "periodic motion", it is meant here that $\bar{\omega}$ advances from its initial value $\bar{\omega}_0$ successively through the four quadrants, re-attaining the value $\bar{\omega}_0$ after a time interval P. The motion then repeats itself.) Also, it is possible to invert Eq. (26) and write $\bar{e}$ as a function of t.

The equation for $d\bar{\omega}/dt$ involves $\bar{\omega}$ only as $\sin^2 \bar{\omega}$. It may be shown that $d\bar{\omega}/dt$ is symmetrical about the values $\bar{\omega} = 0°, 90°, 180°, 270°$. Hence, the period of motion of $\bar{\omega}$ is four times the length of time required for $\bar{\omega}$ to go from 0° to 90°. Since $\bar{\omega} = 0° \Longrightarrow \bar{e}^2 = \bar{e}_3^2$, and $\bar{\omega} = 90° \Longrightarrow \bar{e}^2 = \bar{e}_4^2$, then the period P of the motion of $\bar{\omega}$ is given by Eq. (26) as (using $\beta = +1$)

$$P = \frac{8}{3\sqrt{6}} \frac{\bar{n}}{n_3^2} \int_{\bar{e}_3^2}^{\bar{e}_4^2} \frac{d\eta}{[-(\eta - \eta_1)(\eta - \eta_2)(\eta - \eta_3)]^{1/2}} \tag{46}$$

Since $\bar{e}^2$ and $\sin^2 \bar{i}$ depend only on $\sin^2 \bar{\omega}$, and $\bar{i}$ remains in the same quadrant, for this case, both $\bar{e}$ and $\bar{i}$ have periods equal to $P/2$.

Before expressing P in terms of the elliptic integral $F(k, \Phi)$, it is necessary to investigate the parameters η_1, η_2, η_3. From Eq. (24), it is easily seen that $\eta_2 \geq \eta_3$ in all cases. Also,

$$(\eta_1 - \eta_3) = \eta_1 + \frac{1}{2} A_1 + \frac{1}{2} \cdot \sqrt{A_1^2 - 4A_2}$$

Since

$$(1 - \bar{e}_0^2) \cos^2 \bar{i}_0 > \frac{3}{5} \Longrightarrow \left\{ \begin{array}{l} \sin^2 \bar{i}_0 \sin^2 \bar{\omega}_0 < \frac{2}{5} \Longrightarrow \eta_1 > 0 \\ \\ A_1 > 0 \end{array} \right\} \Longrightarrow \eta_1 > \eta_3,$$

and since $\bar{e}^2_{min} = \eta_1$ and $\bar{e}^2_{max} = \eta_2$ for periodic motion, it may be concluded that

$$\left.\begin{array}{c} \bar{\omega} \text{ periodic in time} \\ \text{and} \\ (1 - \bar{e}_0^2)\cos^2 \bar{i}_0 > \dfrac{3}{5} \end{array}\right\} \Longrightarrow \left\{\begin{array}{l} \text{(a) } \bar{\omega} \text{ has period: } P;\ \bar{e} \text{ and } \bar{i} \text{ have period } P/2. \\ \text{(b) } \bar{e}^2 \text{ achieves a maximum value } \eta_2 \text{ and a minimum value } \eta_1. \\ \text{(c) } \eta_3 < \eta_1 \leq \eta \leq \eta_2 \text{ within the range of integration.} \end{array}\right. \quad (47)$$

The elliptic integral in Eq. (26) is then Case V on pp. 288 of Franklin (Ref. 4). Hence,

$$\int_{\bar{e}_j^2}^{\bar{e}^2} \frac{d\eta}{[-(\eta - \eta_1)(\eta - \eta_2)(\eta - \eta_3)]^{1/2}} = C_j\,[F(k_j, \Phi) - F(k_j, \Phi_j)]\,; \qquad j = 0, 3, 4$$

where C_j, k_j, Φ are given by the following relations:

$$\sin^2 \Phi = \frac{\eta - \eta_{2_j}}{\eta_{1_j} - \eta_{2_j}} = \frac{\bar{e}^2 - \eta_{2_j}}{\eta_{1_j} - \eta_{2_j}}\,; \qquad 0 \leq \Phi \leq \frac{\pi}{2} \quad \text{(48a)}$$

$$C_j = \frac{-2}{\sqrt{\eta_{2_j} - \eta_{3_j}}} \quad \text{(48b)}$$

$$k_j = +\sqrt{\frac{\eta_{1_j} - \eta_{2_j}}{\eta_{3_j} - \eta_{2_j}}} \quad \text{(48c)}$$

$$\left.\begin{aligned}
\eta_{1_j} &= \bar{e}_j^2 \left(1 - \frac{5}{2} \sin^2 \bar{i}_j \sin^2 \bar{\omega}_j\right) \\
\eta_{2_j} &= \frac{1}{2} \left(-A_{1_j} + \sqrt{A_{1_j}^2 - 4A_{2_j}}\right) \\
\eta_{3_j} &= \frac{1}{2} \left(-A_{1_j} - \sqrt{A_{1_j}^2 - 4A_{2_j}}\right) \\
A_{1_j} &= -1 + \frac{5}{3} (1 - \bar{e}_j^2) \cos^2 \bar{i}_j + \frac{2}{3} \eta_{1_j} \\
A_2 &= -\frac{2}{3} \eta_{1_j}
\end{aligned}\right\} \tag{49}$$

Using these results, the period P may be written as

$$P = \frac{8}{3\sqrt{6}} \frac{\bar{n}}{n_3^2} C_3 \left[F(k_3, 0°) - F(k_3, 90°)\right] \tag{50}$$

where C_3 and k_3 are given by Eqs. (48b) and (48c).

In order to write a time-history of the motion, Eq. (26) must be used, so that the value of β must be known. Since β depends only on the quadrant of $\bar{\omega}$, its initial value is known from $\bar{\omega}_0$. Since $d\bar{\omega}/dt > 0$ for periodic solutions, $\bar{\omega}$ will initially move toward that particular one of the values $\bar{\omega} = 0°, 90°, 180°, 270°$, which is next above $\bar{\omega}_0$. When $\bar{\omega}$ crosses a quadrant boundary for the first time, the value of β will change from $+1$ to -1 (or vice versa), and then remain constant for a time interval of $P/4$. Then β will change back from -1 to $+1$ (or vice versa) and remain constant for the succeeding time interval $P/4$. Continuing in this manner, it is possible to write β as an explicit function of t, for $t > 0$.

For sake of clarity, assume that $0° < \bar{\omega}_0 < 90°$. (For $\bar{\omega}_0$ in any other quadrant, the results are similar.) Then $\beta(t) = +1$ for $0 < t < t_1$.

$$t_1 = \frac{2}{3\sqrt{6}} \frac{\bar{n}}{n_3^2} \int_{\bar{e}_0^2}^{\bar{e}_4^2} \frac{d\eta}{[-(\eta - \eta_1)(\eta - \eta_2)(\eta - \eta_3)]^{1/2}}$$

$$= \frac{2}{3\sqrt{6}} \frac{\bar{n}}{n_3^2} C_0 \left[F(k_0, 0°) - F(k_0, \Phi_0)\right] \tag{51}$$

where C_0, k_0, Φ_0, are given by Eqs. (48). The time-history of β is then given by

$$\beta(t) = \begin{cases} +1; \; 0 < t < t_1 \\ -1; \; t_1 < t < t_1 + P/4 \\ +1; \; t_1 + P/4 < t < t_1 + P/2 \\ -1; \; t_1 + P/2 < t < t_1 + 3P/4 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{cases} \tag{52}$$

The value of t as a function of $\bar{e}$ may then be expressed in the following form:

$$t = \begin{cases} \frac{2}{3\sqrt{6}} \frac{\bar{n}}{n_3^2} C_0 \left[F(k_0, \Phi) - F(k_0, \Phi_0)\right]; \quad 0 < t < t_1 \\ t_1 - \frac{2}{3\sqrt{6}} \frac{\bar{n}}{n_3^2} C_4 \left[F(k_4, \Phi) - F(k_4, \Phi_4)\right]; \quad t_1 < t < t_1 + P/4 \\ t_1 + P/4 + \frac{2}{3\sqrt{6}} \frac{\bar{n}}{n_3^2} C_3 \left[F(k_3, \Phi) - F(k_3, \Phi_3)\right]; \quad t_1 + P/4 < t < t_1 + P/2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{cases} \tag{53}$$

The solution in Eq. (53) is of the general form

$$t - T = D\,[F\,(k_j, \Phi) - F\,(k_j, \Phi_j)]$$

or

$$F\,(k_j, \Phi) = \frac{(t-T)}{D} + F\,(k_j, \Phi_j) \tag{54}$$

where T and D are constants. The general relationship between elliptic integrals of the first kind $F(k, \Phi)$, and the Jacobian elliptic function $sn\ F\,(k, \Phi)$ is as follows:

$$\left\{\begin{aligned} F\,(k, \Phi) &= \int_0^{\Phi} \frac{dy}{\sqrt{1-k^2\sin^2 y}} = \int_0^{\sin\Phi} \frac{dx}{\sqrt{(1-x^2)\,(1-k^2x^2)}} \\ \sin\Phi &= sn\ F\,(k, \Phi) \end{aligned}\right. \tag{55}$$

Using Eq. (54)

$$\sin^2\Phi = sn^2\left[\frac{t-T}{D} + F\,(k_j, \Phi_j)\right]$$

Equation (48a) then becomes

$$\overline{e}^2 = \eta_{2_j} + (\eta_{1_j} - \eta_{2_j})\, sn^2\left[\frac{t-T}{D} + F\,(k_j, \Phi_j)\right] \tag{56}$$

Using the various values of D and T from Eq. (53), it is seen that

$$\bar{e}^2(t) = \begin{cases} \eta_{2_0} + (\eta_{1_0} - \eta_{2_0})\, sn^2 \left[\dfrac{3\sqrt{6}}{2} \dfrac{n_3^2}{\bar{n}} \dfrac{1}{C_0} t + F(k_0, \Phi_0) \right]; & 0 < t < t_1 \\ \eta_{2_4} + (\eta_{1_4} - \eta_{2_4})\, sn^2 \left[-\dfrac{3\sqrt{6}}{2} \dfrac{n_3^2}{\bar{n}} \dfrac{1}{C_4} (t - t_1) + F(k_4, \Phi_4) \right]; & t_1 < t < t_1 + P/4 \\ \eta_{2_3} + (\eta_{1_3} - \eta_{2_3})\, sn^2 \left[\dfrac{3\sqrt{6}}{2} \dfrac{n_3^2}{\bar{n}} \dfrac{1}{C_3} (t - t_1 - P/4) + F(k_3, \Phi_3) \right]; & t_1 + P/4 < t < t_1 + P/2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{cases} \tag{57}$$

Equation (57) is an explicit formula for $\bar{e}^2$ as a function of t. Using this relation for $\bar{e}^2$, it is possible to express both $\sin^2 \bar{i}$ and $\sin^2 \bar{\omega}$ as explicit functions of time.

For those cases where the behavior of the averaged variables is not periodic, it is still possible to write $\bar{e}^2(t)$ in a form similar to Eq. (57).

IX. CONCLUSIONS

Although there are still a few details of the motion of $\bar{\omega}$ which remain to be investigated, the above discussion gives a reasonably good picture of the types of long-term behavior which can occur for close orbits in the restricted 3-body problem[††], within the sense of ode (1a) – (1f). An understanding of the long-term behavior of such systems might in turn be useful in obtaining non-averaged solutions of the restricted 3-body problem.

Also, the results should be useful in choosing initial conditions for numerical studies of orbital lifetimes.

[††] This problem has been treated in an article by the Russian, M. L. Lidov (Ref. 5). Many of the results obtained above are also given by Lidov, although he uses $\bar{\omega}$ instead of $\bar{e}$ as the variable of integration in the third integral of motion.

NOMENCLATURE

$\bar{a}$ averaged value of semimajor axis of the osculating ellipse

$\bar{e}$ averaged eccentricity

e_0, i_0, etc. initial values of the averaged orbital elements

$\bar{i}$ averaged inclination

$\bar{n}$ $\mu/(a)^{3/2}$

n_3 $\dfrac{\mu_3^{1/2}}{(a_3)^{3/2}(1-e_3^2)^{3/2}}$, in which subscript 3 refers to the third (or perturbing) body.

Thus, for the case of an artificial satellite of the Moon, in which the Earth is considered the perturbing body, n_3 is approximately the mean motion of the Earth about the Moon.

t time

μ gravity constant for central body

$\bar{X}$ averaged value of $(-n)$ (time of pericenter passage)

$\bar{\omega}$ averaged pericenter angle

$\bar{\Omega}$ averaged node angle

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