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A survey of some current research in functional-differential equations

Jack K. Hale

1. Introduction. Functional-differential equations provide a mathematical model for a physical system in which the rate of change of the system may depend upon its past history; that is, the future state of the system depends not only on the present but also a part of its past history. A special case of such an equation is a differential-difference equation

$$\dot{x}(t) = f(t,x(t),x(t-r))$$

where r is a nonnegative constant. For $r = 0$, this is an ordinary differential equation. A more general equation, which we choose to call a functional-differential equation, is one of the form

$$\dot{x}(t) = f(t,x_t) \tag{1}$$

where x is an n-vector and the symbol x_t is defined as follows. If x is a function defined on $[-r, \infty)$, then for each fixed t in $[0, \infty)$, x_t is a function defined on the interval $[-r,o]$, r finite, whose values are given by $x_t(\theta) = x(t+\theta)$, $-r \leq \theta \leq 0$. In other words, the graph of x_t is the graph of x on $[t-r,t]$ shifted to the interval $[-r,0]$. To obtain a solution of (1) for $t \geq t_o$, one specifies an initial function on the interval $[t_o - r, t_o]$ and then extends the function to $t \geq t_o$ by the relation (1).

Functional-differential equations arise in various applications. The importance of such equations has been amply emphasized by Volterra [1,2] in the discussion of visco-elastic materials and the interaction of biological species. Such equations also occur in other aspects of biology, econometrics, number theory and problems of feedback control. It is also hard to visualize an adaptive control system which would not use in a significant manner a part of its past history.

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It may even be possible to formulate such equations as functional-differential equations.

Although functional-differential equations have been investigated for many years, they have received more intensive study in the past few years, probably due to the diverse applications of such equations and especially due to the present interest in control problems. As a consequence some books are now available on the subject (Mishkis [3], Pinney [4], Krasovskii [5], Bellman and Cooke [6], Halanay [7]). The book of Minorsky [8] also contains material on differential-difference equations and an excellent discussion of specific applications. Hahn [9] includes a section on stability by Lyapunov functions.

In this short report, we attempt to indicate some of the areas of investigation that are presently being discussed in the literature. Naturally, the discussion will be biased by the viewpoint of the author and is not in any way to be understood as a criticism of topics not included below. Also it is impossible to even mention all areas of research. We only attempt to present enough topics to stimulate the reader to consider the above books as well as some of the literature for details.

Throughout the presentation we will emphasize a geometric approach for the discussion of equation (1). This approach has certainly proved to be advantageous in ordinary differential equations. To the author's knowledge, it was Krasovskii who first pointed out that the natural concept of a state for a system described by (1) is not the value of x at time t but the restriction of x to the interval $[t-r,t]$; or, equivalently, the function x_t defined above. This is natural since the state of a system at any particular time should be that part of the system which determines its behavior in the future. Of course, this implies that the orbits of trajectories of the system will take place in a function space rather than in Euclidean space. This introduces some complications but, on the other hand, it indicates the direction for the development of a qualitative or geometric theory for functional-differential equations.

To be more specific, let C be the space of continuous vector functions on the interval $[-r,0]$ and, for any φ in C, let the norm of φ, designated by $\|\varphi\|$, be defined by

$$\|\varphi\| = \max_{-r \leq \theta \leq 0} |\varphi(\theta)|$$

where $|x|$ is say the Euclidean norm of a vector x.

Suppose $f(t,\varphi)$ is defined for $t \geq 0$, φ in C, $\|\varphi\| < H$. If σ is a given real number and φ is a continuous function defined on $[\sigma-r,\sigma]$ with $\|\varphi_\sigma\| < H$, then we say $x = x(\sigma, \varphi)$ is a solution of (1) with initial value φ at σ if x is defined and continuous on $[\sigma-r, \sigma+A)$ for some $A>0$, coincides with φ on $[\sigma-r, \sigma]$, $\|x_t\| < H$ and x satisfies (1) for $\sigma \leq t < \sigma + A$. Throughout the remaining discussion, we assume existence and uniqueness of solutions of (1) for any σ, φ and the solution is defined on $[\sigma, \infty)$ If (t,φ) in (1) is continuous in t,φ and locally Lipschitzian in φ, then existence and uniqueness is proved in a manner similar to ordinary differential equations. Uniqueness theorems under conditions as general as the ones for ordinary equations do not seem to be available.

If σ, φ are as above, then we define a trajectory of (1) through (σ, φ_σ) as the set of points in $[\sigma,\infty) \times C$ given by $\{t, x_t(\sigma,\varphi), t \geq \sigma\}$. If (1) is autonomous; that is, $f(t,\varphi)$ is independent of φ, then we may take $\sigma = 0$ and designate the solution of (1) by $x(\varphi)$. In the autonomous case, the orbit of (1) through φ is the set of points in C given by $\cup_{t \geq 0} x_t(\varphi)$.

2. Liapunov stability. With the above interpretation of a solution of (1) as defining a trajectory in the space $[\sigma,\infty) \times C$, it is almost obvious how to define Lyapunov stability. In fact, if $f(t,0) = 0$, the solution $x = 0$ of (1)

is called <u>uniformly stable</u> if the following conditions are satisfied for <u>every</u> $\sigma \geq 0$:

i) there is a $b = b(\sigma) > 0$ such that φ in C, $\|\varphi\| < b$ implies the solution $x(\sigma,\varphi)$ of (1) exists for $t \geq \sigma$ and $\| x_t(\sigma,\varphi) \| < H$ for $t \geq \sigma$;

ii) for every $\epsilon > 0$, there is a $\delta = \delta(\epsilon) > 0$ such that φ in C, $\|\varphi\| < \delta$ implies the solution $x(\sigma,\varphi)$ of (1) satisfies $\| x_t(\sigma,\varphi)\| < \epsilon$ for $t \geq \sigma$.

The solution $x = 0$ of (1) is called <u>asymptotically stable</u> if it is stable and in addition for every $\sigma \geq 0$, there is an $H_o = H_o(\sigma) > 0$ such that $\|\varphi\| < H_o$ implies $\| x_t(\sigma,\varphi)\| \to 0$ as $t \to \infty$.

This is the same definition of uniform stability as for ordinary differential equations except for the fact that we assume properties i) and ii) for every $\sigma \geq 0$. In the case of ordinary differential equations this is not necessary, but, for functional-differential equations, a system can be uniformly stable at σ and not uniformly stable for $\sigma_1 > \sigma$. For examples of this property see Zverkin [10].

We have always assumed our retardation r is finite. If r is infinite, then one can also discuss stability and obtain the results below, but we cannot use the uniform norm in $C((-\infty, 0])$. If $C((-\infty, 0])$ is given the compact open topology (uniform convergence on compact subsets), then our space becomes a metric space and everything is repeated with the metric rather than norm. Driver [11] has also discussed infinite retardations, but the results seem to be weaker due to his topology.

Following Krasovskii [5], we say a scalar function $V(t,\varphi)$ defined and continuous for $t \geq 0$, φ in C, $\|\varphi\| < H$ is <u>positive definite</u> if there exists a continuous positive definite function $w(s)$, $0 \leq s < H$, such that

$V(t, \varphi) \geq w(\|\varphi\|)$ for all $t \geq 0$, φ in C, $\|\varphi\| < H$. The function $V(t, \varphi)$ has an <u>infinitely small upper bound</u> if there is a continuous function $w(s)$, $0 \leq s < H$, $w(0) = 0$, such that $V(t, \varphi) \leq w(\|\varphi\|)$ for $t \geq 0$, φ in C, $\|\varphi\| < H$. The derivative of V along the solutions of (1) is denoted by $\dot{V}_{(1)}$ and is defined by

$$\dot{V}_{(1)}(t, x_t) = \overline{\lim_{h \to 0^+}} \frac{1}{h} [V(t + h, x_{t+h}) - V(t, x_t)].$$

With these definitions, one can prove the usual theorems of Liapunov for stability and asymptotic stability. More specifically, if there is a $V(t, \varphi)$ which is positive definite, has an infinitely small upper bound and $\dot{V}_{(1)} \leq 0$, then the solution $x=0$ of (1) is uniformly stable. If, in addition, $-\dot{V}_{(1)}$ is positive definite, then $x = 0$ is asymptotically stable. Furthermore, if the solution $x = 0$ of (1) is assumed to be asymptotically stable, then one can construct a $V(t, \varphi)$ with the above properties such that $-\dot{V}$ is positive definite (the converse theorem of asymptotic stability). The importance of the converse theorems is to deduce properties concerning the implications of stability; for example, stability with respect to the first approximation, stability under constantly acting disturbances, etc. For details of this type of investigation, see Krasovskii [5], Halanay [7].

Given a particular equation, one would hope to construct a Liapunov functional from which sufficient conditions for stability could be deduced. Unfortunately, it seems to be almost impossible for specific equations to find Liapunov functionals which are positive definite in the sense described above. We give the following simple example to illustrate the properties that are more easily satisfied in applications.

Consider the equation

(2) $$\dot{x}(t) = -ax(t) - bx(t - r)$$

where x is a scalar and a, b, r are constants, $a > 0$, $r \geq 0$. If

$$V(\varphi) = \frac{1}{2a}\varphi^2(0) + \frac{1}{2}\int_{-r}^{0}\varphi^2(\theta)d\theta,$$

then

$$\dot{V}_{(2)}(x_t) = -\frac{1}{2}[x^2(t) + \frac{b}{a}x(t)x(t-r) + x^2(t-r)]$$

and it is clear that neither V nor $-\dot{V}_{(2)}$ is positive definite in the above sense regardless of the values of a and b. On the other hand, $V(\varphi) \geqq \varphi^2(0)/2a$ and $-\dot{V}_{(2)}(\varphi) \geqq k\varphi^2(0)$ if $|b| < a$, which is certainly a type of positive definiteness. One can show that conditions of this type on V and $-\dot{V}$ are sufficient for asymptotic stability. We do not state the result any more precisely, but refer the reader to Krasovskii [5] and Driver [11] for the theory and examples. Much more research is needed in the area of determining practical conditions on V and $\dot{V}$ which will ensure stability.

Another possible attempt to obtain sufficient conditions for the stability of (2) would be to take the function V as only a function of the vector $x(t)$ and not include any of its past history. In particular, if

$$V(x(t)) = x^2(t)/2a,$$

then

$$\dot{V}_{(2)} = -[x^2(t) + \frac{b}{a}x(t)x(t-r)]$$

which does not even have fixed sign. On the other hand if $|b| < a$ and $|x(t-r)| < |x(t)|$ then $\dot{V}_{(2)} < -\delta x^2(t)$ where δ is a positive constant. It is rather remarkable that these weak conditions on V and $\dot{V}$ imply asymptotic stability of the solution $x = 0$ of (2). General results along this line were first given by Razumikhin [12, 13]. See, also, Krasovskii [5] and Driver [11].

In the particular case of equation (2) one can actually obtain the exact region of stability of the zero solution as a function of a, b, r. The region $|b| < a$, $a > 0$, is the maximal region which yields stability for all values of r. One would hope to be able to obtain a more realistic approximation of the stability region by using a more clever choice of the function V. If the function V is chosen as

$$V(\varphi) = \varphi^2(0) + \alpha\left[\int_{-r}^{0} \varphi(\theta)d\theta\right]\varphi(0) + \int_{-r}^{0} \beta(\theta)\varphi^2(\theta)d\theta$$

then it was shown by Hale [14] that α and $\beta(\theta)$ can be chosen as functions of r in such a way that the application of the previous type of stability theorem yields a region of stability which approaches the region $a + b > 0$ as $r \to 0$. This is the exact region of stability of (2) for $r = 0$.

Much more research is needed in the area of determining sufficient conditions for stability by use of Liapunov functionals and also many more examples need to be constructed to show the types of functionals that occur in applications.

For autonomous ordinary differential equations, the importance of relaxing the conditions on $\dot{V}$ was pointed out by LaSalle [15]. He gave many applications in which it was not too difficult to construct positive definite Liapunov functions but $\dot{V}$ would be only ≤ 0. The limiting behavior of the solutions was then shown to be determined by the largest invariant set contained in the set where $\dot{V} = 0$. In particular, if this set contained only the origin, then solutions will approach the origin with increasing time.

For autonomous functional-differential equations, the concepts of ω-limit set and invariant set can be introduced. One can then obtain a generalization of the theorem of LaSalle to functional equations which is a practical tool. The reader can consult Hale [14, 16] for the details of this theory as well as applications. One of the applications is an interesting problem in the stability of nuclear reactors considered in a beautiful paper of Levin and Nohel [17], who also were using Liapunov functionals and essentially the concept of invariant set. Another interesting application is a model of the interaction of biological species considered by Volterra [2] who, by the way, also used a type of Liapunov functional. The paper [16] also contains some results on instability.

Many papers on functional-differential equations and control theory have appeared in recent years in the journal Applied Mathematics and Mechanics (Prikl. Mat. Mek.). The reader may consult this journal for the general flavor of the research, but we think the paper of Krasovskii [18] deserves special attention. Krasovskii studies the problem of the stabilization of a system by indirect control; that is, the control parameters are determined through a differential equation and, in particular, a functional-differential equation. He then gives as an example the problem of trying to stabilize, by means of a linear control variable, a pendulum at its unstable equilibrium position when only the deviation from the vertical can be measured. Krasovskii shows that the system can be stabilized if the control variable satisfies an appropriate linear functional-differential equation, but it can never be stabilized by a control variable which satisfies a linear autonomous ordinary differential equation.

3. Behavior near equilibrium points and cycles. One of the basic problems in ordinary differential equations is to understand the behavior of solutions near invariant sets. This theory is fairly complete near those invariant sets for which it is possible to introduce a local coordinate system. It would be desirable to obtain the same type of information for functional-differential equations. Some results along this line have been obtained for equilibrium points and cycles and these are briefly described below.

Consider the autonomous equation

(3) $$\dot{x}(t) = f(x_t).$$

An equilibrium point of (3) is a constant function which satisfies (3); that is, a constant function b for which $f(b) = 0$. Without loss in generality, we can assume $b = 0$ and if f has continuous Frechet derivatives of order two, then (3) can be written in the form

(4) $$\dot{x}(t) = \int_{-r}^{0}[d\eta(\theta)]x(t+\theta) + F(x_t)$$

where η is a matrix whose elements are functions of bounded variation on $[-r, 0]$ and $|F(\varphi)|/\|\varphi\| \to 0$ as $\|\varphi\| \to 0$.

A basic understanding of the solutions of (4) in a neighborhood of zero requires a detailed investigation of the linear system

(5) $$\dot{x}(t) = \int_{-r}^{0}[d\eta(\theta)]x(t+\theta).$$

As expected, the characteristic equation

(6) $$\det[\lambda I - \int_{-r}^{0}[d\eta(\theta)]e^{\lambda\theta}] = 0$$

and the characteristic values (solutions of this equation) will play a fundamental role. To any solution of (6), there are a finite number of linearly independent solutions of (5) of the form $p(t)e^{\lambda t}$ where $p(t)$ is a polynomial in t. Solutions of this type are called characteristic functions. It is actually the case that the characteristic functions serve as a basis for the solutions of (5) in the sense that any solution of (5) with initial value φ at 0 can be expanded in a uniformly convergent infinite series of characteristic functions on an interval $[\sigma, T]$, $\sigma > 0$. For the investigation of (5) along this line, see Pinney [4] and Bellman and Cooke [6].

To understand the geometric properties of the solutions of (4) near zero, it is advantageous to interpret the solutions of (5) as orbits in C. In this approach, the expansions of solutions in terms of characteristic exponents is not needed. If $x = x(\varphi)$ denotes the solution of (5) with initial value φ at 0, then $x_t(\varphi)$ is a bounded linear operator, taking C into C for each fixed $t \geqq 0$. If we designate this operator by $T(t)$, that is $T(t)\varphi = x_t(\varphi)$, then $T(t + \tau) = T(t)T(\tau)$ for all t, $\tau \geqq 0$; that is, $T(t)$ is a semigroup of operators. Furthermore, $T(t)$ is compact for $t \geqq r$. One can now borrow results from the theory of functional analysis to analyze the behavior of the orbits of (5) in C, an orbit through φ being defined as before as $\cup_{t \geq 0} T(t)\varphi$. To any solution λ of (6), there corresponds a finite dimensional subspace of C with basis $\Phi = (\varphi_1, \dots, \varphi_p)$ which is invariant under the operator $T(t)$ for each $t \geqq 0$ and

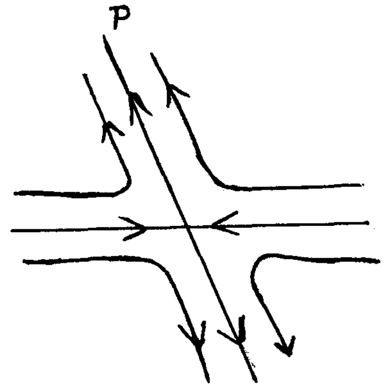
$$T(t)\Phi = \Phi e^{Bt}$$

where B is a square matrix of dimension p whose only eigenvalue is λ; that is, on this subspace the solutions of (5) interpreted in C behave essentially as an ordinary differential equation. If it is assumed that no solution of (6) lies on the imaginary axis and $\lambda_1, \dots, \lambda_k$ are the solutions of (6) with positive real parts, then there exist two subspaces P, Q of C, which are both invariant under $T(t)$, $t \geqq 0$, such that every φ in C can be uniquely decomposed as $\varphi = \varphi_P + \varphi_Q$, φ_P in P, φ_Q in Q and

(7) $$\|x_t(\varphi) = T(t)\varphi\| \leqq Ke^{\alpha t}, \quad t \leqq 0, \quad \varphi \text{ in } P$$

$$\|x_t(\varphi) = T(t)\varphi\| \leqq Ke^{-\alpha t}, \quad t \geqq 0, \quad \varphi \text{ in } Q$$

where K, α are positive constants. The subspace P is finite dimensional and Q is infinite dimensional. Once the estimate (7) is obtained, it is natural to call the equilibrium point a <u>saddle point</u> and the orbits of (5) are essentially as in the accompanying diagram where the arrows



designate the direction of the motion with increasing time. One can also give an explicit procedure for computing the subspaces P, Q from system (5) and a system adjoint to system (5). The explicit form of the subspaces P, Q is important in the applications, especially in the theory of perturbations discussed in the next section. Shimanov [19] has also used this method to discuss stability of a nonlinear system when the linear part has some characteristic values with zero real parts. For details of this theory see Shimanov [19, 20] and Hale [21].

Once this geometric picture of the orbits of (5) is obtained, it is natural to ask the following question: is the saddle point property of system (5) preserved for system (4)? More specifically, do there exist sets P^*, Q^* which are homeomorphic near zero to P, Q respectively such that the solutions of (4) with initial value on P^* remain on P^* for all t in $(-\infty, 0]$ and approach zero as $t \to -\infty$ and the solutions of (4) with initial values on Q^* remain on Q^* for t in $[0, \infty)$ and approach zero as $t \to \infty$? The affirmative answer to this question as well as more detailed information is given by Hale and Perelló [22].

Now suppose that system (3) has a nonconstant periodic solution $x^0(t)$ of period 2π. In the space C, this periodic solution generates a closed curve Γ. If the concept of asymptotically orbital stability with asymptotic phase is defined as in ordinary equations, then the following question can be posed: what conditions on f in (3) will ensure that the curve Γ is asymptotically orbitally stable with asymptotic phase ?

To answer this question, we proceed as in ordinary differential equations to discuss the linear variational equation of the periodic solution $x^0(t)$. This will be a linear functional-differential of equation of the form

$$\dot{x}(t) = \int_{-r}^{0} [d\eta(t, \theta)]x(t + \theta) \tag{8}$$

where $\eta(t, \theta)$ is a matrix which is periodic in t of period 2π. Hahn [23], Stokes [24], Halanay [7] and Shimanov [25] have discussed in detail systems of the form (8). In particular, if $x(\varphi)$ is the solution of (8) with initial value φ at 0, then $x_t(\varphi)$ again defines a continuous, linear mapping of C into C for each fixed $t \geqq 0$. If the operator $U(t)$, $t \geq 0$, is defined on C by $U(t)\varphi = x_t(\varphi)$, then the characteristic multipliers of (8) can be defined as the elements of the point spectrum of the operator $U(2\pi)$ (the monodromy operator). With this definition of the multipliers, one can then discuss in what sense the Floquet theory is applicable to (8). It is true that the behavior of solutions of (8) for large values of t is determined by the characteristic multipliers. On the other hand there may be only a finite number of characteristic multipliers and the expansion of solutions in terms of characteristic functions is impossible. Hahn [23] has given some conditions on the measure $\eta(t, \theta)$ for which such an expansion theorem is true. See the above works of Hahn, Stokes, Halanay and Shimanov for the details of this theory.

Using the above theory for systems of the type (8), Stokes [26] has proved the following interesting result for system (3): If the linear variational equation associated with a nonconstant periodic solution of (3) has all characteristic multipliers with modulus less than one except for the obvious multiplier which is equal to one, then the curve Γ in C generated by this periodic solution is asymptotically orbitally stable with asymptotic phase.

This result is a direct generalization of the known property of ordinary differential equations and can actually be used to determine stability of periodic solutions which arise in the perturbation theory of linear systems described in the next section.

The proof employed by Stokes is a nontrivial generalization of the one given in Coddington and Levinson [27] for ordinary differential equations, and, therefore, a local coordinate system in the neighborhood of Γ is not necessary. In order to go further in this direction of a qualitative theory, it seems to be essential to have local coordinate systems near the simple invariant sets in C. If the form of the new equations could be obtained, many important results of ordinary differential equations could be extended to functional-differential equations. A simple case of the possible new equations was considered by Hale [28].

4. Theory of oscillations. Consider the homogeneous linear equation

$$\dot{x}(t) = \int_{-r}^{0}[d\eta(t, \theta)]x(t + \theta) \tag{9}$$

where η is a function which is sufficiently smooth so that $\int_{-r}^{0}[d\eta(t, \theta)]\varphi(\theta)$ is a continuous function of t for all φ in C, and also consider the nonhomogeneous equation

$$\dot{x}(t) = \int_{-r}^{0}[d\eta(t, \theta)]x(t + \theta) + f(t) \tag{10}$$

where f is a continuous function on $(-\infty, \infty)$. Halanay [7] has proved that a necessary and sufficient condition that all solutions of (10) be bounded on $[0, \infty)$ for every function $f(t)$ bounded on $[0, \infty)$ is that the zero solution of the homogeneous equation be uniformly asymptotically stable (the Perron problem). Furthermore, uniform asymptotic stability of the zero solution of (9) implies exponential asymptotic stability. With a more detailed analysis one can show that this type of stability and η, f almost periodic implies there is a unique almost periodic solution of (10). These results can then be used along with successive approximations to obtain the existence of almost periodic solutions of nonlinear equations of the type

$$\dot{x}(t) = \int_{-r}^{0} [d\eta(t, \theta)]x(t + \theta) + \epsilon f(t, x_t)$$

where ϵ is a small parameter (see Halanay [7]). General results along this line have been obtained for small perturbations of nonlinear equations by Yoshizawa [29] by using the converse of the stability theorems of Lyapunov. Reference [29] also contains other references on this same subject.

If the function η in equation (9) does not depend upon t, then one can show easily from the general theory of linear autonomous systems mentioned in section 3 that a necessary and sufficient condition that the nonhomogeneous system have a bounded solution in $(-\infty, \infty)$ for every forcing function f bounded in $(-\infty, \infty)$ is that no characteristic values of (9) lie on the imaginary axis. In this case, the bounded solution is also unique. What happens when η does depend upon t? This problem seems to be unanswered at the present time.

More interesting questions in the theory of nonlinear oscillations is the case in which the homogeneous equation has solutions which do not tend to zero as either $t \to \infty$ or $-\infty$; for example, a periodic solution. For simplicity we restrict our attention to the case in which η does not depend upon t and f is periodic of period 2π; that is, the equation

(11) $$\dot{x}(t) = \int_{-r}^{0} [d\eta(\theta)]x(t + \theta) + f(t), \quad f(t + 2\pi) = f(t).$$

One can then prove the following: a necessary and sufficient condition that (11) has a periodic solution of period 2π is that

$$\int_0^{2\pi} y(t) \cdot f(t)dt = 0$$

for all periodic solutions $y(t)$ of period 2π of the "adjoint" equation

$$\dot{y}(s) = -\int_{-r}^{0} y(s - \theta)d\eta(\theta). \tag{12}$$

In particular, if the homogeneous part of equation (11) has no periodic solutions of period 2π, then (12) has no periodic solutions of period 2π and, thus, (11) has a unique periodic solution of period 2π (see Halanay [7]).

As is well known in ordinary differential equations, this is a basic result for discussing periodic solutions of perturbed linear systems (see Coddington and Levinson [27], Cesari [30], Hale [31]). It is also true that one can use this result to discuss functional-differential equations of the form

$$\dot{x}(t) = \int_{-r}^{0} [d\eta(\theta)]x(t + \theta) + \epsilon f(t, x_t) \tag{13}$$

where ϵ is a small parameter and $f(t, \varphi)$ is periodic in t of period 2π. Some results in this direction may be found in Halanay [7]. A more complete discussion extending the method of Cesari and Hale will appear in the forthcoming Ph.D. thesis of Perelló from Brown University. Perelló exploits the general theory of linear systems mentioned in section 3 to derive the bifurcation or determining equations for the periodic solutions of period 2π of (13) and, thereby, reduces the problem to the solution of a finite number of transcendental equations.

Research is also being devoted to the extension of the perturbation methods of solving (13) when $f(t, \varphi)$ is more general than a periodic function of t. In particular, the method of averaging of Krylov-Bogoliubov-Mitropolski-Diliberto (see [32, 33]) has been extended to functional-differential equations. For the case when r, the retardation parameter, is $\epsilon\tau$, this was done by Halanay [7], and the case for arbitrary r by Hale [34]. This theory is too complicated to describe here, but we mention one simple consequence of the theory. Consider the system

$$(14) \qquad \dot{x}(t) = \epsilon f(t, x_t)$$

where $f(t, \varphi)$ is almost periodic in t uniformly with respect to φ on every compact subset of $\|\varphi\| < H$. If

$$f_0(\varphi) = \lim_{T\to\infty} \frac{1}{T} \int_0^T f(t, \varphi)d\varphi$$

and there is an equilibrium solution ρ_0 of the ordinary differential equation

$$\dot{y} = f_0(y)$$

such that the characteristic exponents of the linear variational equation have negative real parts, then, for ϵ sufficiently small, system (14) has an asymptotically stable almost periodic solution which reduces to ρ_0 for $\epsilon = 0$. Other interesting examples are discussed in [34].

In the theory of autonomous ordinary differential equations which do not contain small parameters, one of the basic methods for determining existence of limit cycles is to determine a subset of the Euclidean space which is homeomorphic to a cell such that any solution of the equation with initial value on the subset returns to the subset at a future time. One can then use the Brouwer fixed point theorem to assert the existence of a limit cycle. In a series of papers, Jones [35, 36] has shown that the application of similar arguments (but, of course, in the function space C) lead to existence of nonconstant periodic solutions of the equations

$$\dot{x}(t) = -\alpha\, x(t-1)(1+x(t)), \qquad \alpha > \pi/2,$$
$$\dot{x}(t) = -\alpha\, x(t-1)(1-x^2(t)), \qquad \alpha > \pi/2,$$

as well as much more general equations. The periodic solutions of the second equation above are related to the elliptic functions. If many more examples of this type were available, then it seems feasible that one could begin to formulate energy principles for functional-differential equations.

Much of the theory of ordinary differential equations is devoted to boundary value problems. This theory for functional-differential equations is still in its infancy and this is probably due to the fact that not enough applications have been discovered to dictate the proper manner in which to formulate the problems. Some results have been obtained for a few special problems and the reader is referred to Norkin [37] where additional references may also be obtained.

5. Other problems. In the previous discussion, many areas of investigation have not been mentioned. In this section, we refer to two other important areas. First of all, there is the interesting class of functional-differential equations known as equations of neutral type; that is, those equations in which the derivative of x in (1) also appears on the right hand side of (1). The system

$$\dot{x}(t) = ax(t) + b\dot{x}(t-1)$$

is of neutral type.

Certain types of problems in the theory of transmission lines can be reduced to the study of equations of neutral type (see Miranker [38]). The general theory of these equations is contained in Bellman and Cooke [6] and, in some respects, is formally very similar to the systems discussed in the previous pages. On the other hand, the problems are much more complicated and not too well understood. For example, if all the roots of the characteristic equation of a linear autonomous system of neutral type are in the left half of the complex plane, it is not always true that all solutions approach zero as $t \to \infty$. The reason for the difficulty is that the characteristic roots in such a situation are not necessarily bounded away from the imaginary axis. Even if the solutions do approach zero, the rate of decrease depends very strongly upon the smoothness of the initial data. The papers of Hahn [39] and Snow [40] are good introductions to this fascinating subject.

Another interesting area of investigation for functional-differential equations is singular perturbations; that is, systems of equations in which a small parameter is multiplying some of the highest derivatives. Cooke [41] (see this paper for additional references) has given a detailed presentation of this question for linear n^{th} order scalar equations. It turns out that the introduction of retardations in singular perturbations leads to considerable difficulty, but Cooke has managed to obtain criteria for regular degeneracy of the solutions which generalize known criteria for ordinary differential equations.

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