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PARAMETRIC EXCITATION OF TRANSVERSE WAVES IN A PLASMA*

by

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ABSTRACT 19539

It is shown that an externally-imposed, oscillating electric field excites transverse electromagnetic waves propagating perpendicularly to it, in a cold plasma. The mechanism is closely related to the parametric excitation of longitudinal plasma oscillations recently predicted by Aliev and Silin. The problem provides an application of non-secular perturbation methods, when the equations of motion are expanded in powers of the external electric field. Arbitrarily small perturbations which arise spontaneously in the plasma are amplified by the action of the electric field, for a certain range of the driving frequency. The growth rate of the oscillations is calculated.

Author

I. INTRODUCTION

If some parameter of a system capable of oscillating at frequency Ω is forced to vary with frequency 2Ω, it is known that the system will often spontaneously break into oscillation at the frequency Ω. The name usually attached to this effect is "parametric excitation" or "parametric resonance" (see, e.g., Minorsky[1] or Bogolyubov and Mitropolskii[2]).

Recently Aliev and Silin[3] and Silin[4] have demonstrated the possibility of parametric excitation of longitudinal electrostatic waves in a collisionless plasma. The exciting mechanism was a spatially-uniform, externally-driven, electric field. The wave vectors of the excited waves necessarily had non-vanishing components along the direction of the applied electric field.

The purpose of the present calculation is to show the existence of a mechanism for parametric excitation of <u>transverse</u> electromagnetic waves, in the same situation considered by Aliev and Silin.[3,4] These waves propagate <u>perpendicularly</u> to the applied electric field, and represent an effect which will be competitive with that discussed in references 3 and 4.

In any application of the parametric excitation technique to, say, the turbulent heating of a plasma, the excitation of transverse waves must be considered as potentially a source of energy loss. The transverse oscillations will be excited at a frequency generally above ω_p, the plasma frequency, and will be free to leave the plasma.

Equations governing the time development of the field variables are derived in Sec. II, but without the electrostatic assumption. They are then specialized to the case in which the disturbances propagate perpendicularly to the applied electric field. In Sec. III, a perturbation treatment, in which the expansion parameter measures the strength of the externally-applied electric field, is given for these equations. The perturbation theory is an application of the Krylov-Bogoliubov-Mitropolskii-Frieman methods[2,5] which have been applied elsewhere[6] to cold-plasma problems. The two "time scales" which characterize the present problem are: (1) the "fast" time scale, measured by the frequencies of the oscillations excited; and (2) the "slow" scale which measures the rate at which energy is fed into the oscillations, and which depends upon the strength of the external electric field. The conditions under which the two time scales will be quite different can be explicitly given.

II. EQUATIONS FOR THE PERTURBED FIELD QUANTITIES

We have in mind an axially-symmetric plasma extending to infinity in the $\pm$ z directions. An electric field which does not vary in space, and which is understood to be externally-driven, is given by

$$\vec{E}^{(o)} = \vec{E}_{ext.} \sin \omega_o t = \hat{e}_z E_{ext.} \sin \omega_o t. \qquad (1)$$

The plasma is understood to be cold, spatially-uniform, and to contain no d.c. magnetic field. The particles of species i (charge e_i, mass m_i) oscillate about their equilibrium positions with velocities

$$\vec{v}_i^{(o)} = - \frac{e_i}{m_i} \frac{\vec{E}_{ext.}}{\omega_o} \cos \omega_o t = -\hat{e}_z V_{oi} \cos \omega_o t \ . \qquad (2)$$

Finally, we complete the description of the "equilibrium" state by specifying

$n_i^{(o)}$ = number density of the ith species of particle = constant,

$\vec{B}^{(o)}$ = magnetic field = 0,

$\Sigma_i e_i n_i^{(o)} = 0$.

Such a state is not an exact solution of the full set of cold plasma equations and Maxwell's equations. It is, however, a good approximation for an axially-symmetric plasma out to radius r from the axis of symmetry, where

$$r^2 \ll c^2/[\omega_o^{\,2} - \Sigma_i\ \omega_{pi}^2]\ ,$$

$$r\ (e_i\ E_{ext.}/m_i c^2)\ (1 - \Sigma_i\ \omega_{pi}^2/\omega_o^2) \ll 1\ , \tag{3}$$

with $\omega_{pi}^2 = 4\pi\ n_i^{(o)} e_i^2/m_i$. Hereafter, our remarks apply only to the region defined by (3). The same limitation applies also to the work of Aliev and Silin.

We now linearize the cold plasma equations about this (time-dependent) steady state. The perturbations on the field quantities are written without superscripts or subscripts:

$$\frac{\partial n(i)}{\partial t} + n_i^{(o)} \frac{\partial}{\partial \vec{x}} \cdot \vec{v}(i) + \vec{v}_i^{(o)} \cdot \frac{\partial}{\partial \vec{x}}\ n(i) = 0\ , \tag{4a}$$

$$\frac{\partial \vec{v}(i)}{\partial t} + v_i^{(o)} \cdot \frac{\partial}{\partial \vec{x}}\ \vec{v}(i) = \frac{e_i}{m_i} \left[\vec{E} + \frac{\vec{v}_i^{(o)}}{c} \times \vec{B} \right]\ , \tag{4b}$$

$$\frac{\partial}{\partial \vec{x}} \cdot \vec{E} = 4\pi\Sigma_i \rho(i) = 4\pi\Sigma_i e_i n(i)\ , \tag{4c}$$

$$\frac{\partial}{\partial \vec{x}} \cdot \vec{B} = 0 \ , \tag{4d}$$

$$\frac{\partial}{\partial \vec{x}} \times \vec{E} = -1/c \, \frac{\partial \vec{B}}{\partial t} \ , \tag{4e}$$

$$\frac{\partial}{\partial \vec{x}} \times \vec{B} = 1/c \, \frac{\partial \vec{E}}{\partial t} + \frac{4\pi}{c} \, \Sigma_i e_i \left[n_i^{(o)} \vec{v}(i) + n(i) \vec{v}_i^{(o)} \right] . \tag{4f}$$

Specializing Eqs. (4) to the case $\frac{\partial}{\partial \vec{x}} \times \vec{E} = 0$ and letting $c \to \infty$ leads to the equations of Aliev and Silin.

Eqs. (4) are linear, so we may assume that all the field quantities $n(i)$, $\vec{v}(i)$, $\vec{E}$, $\vec{B}$ [which are, respectively, the perturbations on the number density and velocity of the ith species, and the electric and magnetic fields] have a spatial dependence $\exp\,[i\vec{k} \cdot \vec{x}]$, and freely superpose solutions.

Eqs. (4) differ from the usual linearized cold-plasma equations only by the presence of the time-dependent terms $\vec{v}_i^{(o)}$. We now specialize to the case of propagation perpendicular to $\vec{E}_{ext.}$, or $\vec{k} \cdot \vec{v}_i^{(o)} = 0$. Furthermore, we assume that the external field $E_{ext.}$ is "weak", and represent this by a formal expansion parameter (ultimately, $\epsilon \to 1$) in front of each $\vec{v}_i^{(o)}$, getting:

$$\frac{\partial n(i)}{\partial t} + n_i^{(o)} i\vec{k} \cdot \vec{v}(i) = 0 \tag{5a}$$

$$\frac{\partial \vec{v}(i)}{\partial t} - \frac{e_i}{m_i} \vec{E} = \epsilon \frac{e_i}{m_i} \frac{\vec{v}_i^{(o)}}{c} \times \vec{B} \tag{5b}$$

$$i\vec{k} \cdot \vec{E} - 4\pi\Sigma_i e_i n(i) = 0 \tag{5c}$$

$$i\vec{k} \cdot \vec{B} = 0 \tag{5d}$$

$$i\vec{k} \times \vec{E} + 1/c \frac{\partial \vec{B}}{\partial t} = 0 \tag{5e}$$

$$i\vec{k} \times \vec{B} - 1/c \frac{\partial \vec{E}}{\partial t} - \frac{4\pi}{c} \Sigma_i e_i n_i^{(o)} \vec{v}(i) = \epsilon\, 4\pi\Sigma_i e_i n(i) \vec{v}_i^{(0)} \tag{5f}$$

From this point on, the problem is the technical one of doing a perturbation expansion in ϵ on Eqs. (5).

III. PERTURBATION EXPANSION OF EQS. (5)

By setting $\epsilon = 0$ in Eqs. (5), we recover the usual set of field-free cold-plasma normal modes. If we were to make a naive perturbation expansion in ϵ,

$$\begin{pmatrix} n(i) \\ \vec{v}(i) \\ \vec{E} \\ \vec{B} \end{pmatrix} = \begin{pmatrix} n_0(i) \\ v_0(i) \\ \vec{E}_0 \\ \vec{B}_0 \end{pmatrix} + \epsilon \begin{pmatrix} n_1(i) \\ \vec{v}_1(i) \\ \vec{E}_1 \\ \vec{B}_1 \end{pmatrix} + \epsilon^2 \begin{pmatrix} n_2(i) \\ \vec{v}_2(i) \\ \vec{E}_2 \\ \vec{B}_2 \end{pmatrix} + \ldots \tag{6}$$

(the subscript now indicates the order in ϵ), substituting (6) into (5) and equating coefficients of equal powers of ϵ, the perturbation series would be poorly-behaved. By this, we mean that the "corrections" to the normal mode solutions ($n_1(i)$, $\vec{v}_1(i)$, $\vec{E}_1$, $\vec{B}_1$, etc.) would soon become larger than $n_0(i)$, $\vec{v}_0(i)$, $\vec{E}_0$, $\vec{B}_0$, for certain critical values of the frequency ω_0.

This is a breakdown of the standard perturbation techniques which has become familiar, and there is a by-now standard procedure for avoiding it. For the pure initial-value problem, which we shall be concerned with, the most useful formulation is that of Frieman.[5] One seeks an expansion of the form (6), but with the assumption that the variables $n_o(i)$, $\vec{v}_o(i)$, $\vec{E}_o$, $\vec{B}_o$ depend on time through explicit functional dependences on the arguments t, ϵt, $\epsilon^2 t$, ... As long as $\frac{\partial^n}{(\epsilon^n t)}$ is treated as of $O(1)$, the various dependences on ϵt, $\epsilon^2 t$, ... are not involved, to lowest order in ϵ. It turns out that this arbitrariness can be eliminated, and the dependences determined, by the requirement that the higher-order corrections in Eq. (6) really shall remain small compared to the terms which are formally of lower order in ϵ. This procedure has been discussed in detail elsewhere.[2,5,6] We need, here, results only through $O(\epsilon)$.

Modes in which $\vec{k} \cdot \vec{E} = 0$ and $\vec{k} \times \vec{E} = 0$ turn out to be coupled by Eqs. (5), so that the waves which are driven are of a mixed type, neither purely longitudinal nor purely transverse. It is convenient to pick a direction for $\vec{k}$ and call it the x-direction. Then it may be readily shown from Eqs. (5) that waves which have $\vec{E}$ perpendicular to the xz plane are unaffected by the $\vec{v}_i^{(o)}$ driving terms, and so we may limit consideration to waves which have only x and z components of $\vec{E}$. $\vec{B}$ then has only a y-component, and we may reduce Eqs. (5) to a set of scalar equations, by defining

$$\vec{E} = \vec{E}_T + \vec{E}_L = \hat{e}_z E_T + \hat{e}_x E_L , \tag{7a}$$

$$\vec{B} = \vec{B}_T = B_T \hat{e}_y , \tag{7b}$$

$$\vec{v}(i) = \vec{v}_L + \vec{v}_T = v_L\hat{e}_x + v_T\hat{e}_z , \tag{7c}$$

$$n(i) = n_L , \tag{7d}$$

(with the ith species understood in (7c) and (7d)). Now, the non-trivial members of Eqs. (5) become:

$$\begin{aligned}
&\frac{\partial n_L}{\partial t} + n_i^{(o)} ikv_L = 0 , \\
&\frac{\partial v_L}{\partial t} - (e_i/m_i)\, E_L = \epsilon \, \frac{e_i V_{oi} B_T}{m_i c} \cos \omega_o t , \\
&\frac{\partial v_T}{\partial t} - (e_i/m_i)\, E_T = 0 , \\
&ikE_L - 4\pi\Sigma_i e_i n_L = 0 , \\
&-ikE_T + 1/c \, \frac{\partial B_T}{\partial t} = 0 , \\
&ikB_T - 1/c \, \frac{\partial E_T}{\partial t} - 4\pi/c \; \Sigma_i e_i n_i^{(o)} v_T \\
&\qquad\qquad = -\epsilon \, 4\pi\Sigma_i e_i n_L \, \frac{V_{oi}}{c} \cos \omega_o t , \\
&- 1/c \, \frac{\partial E_L}{\partial t} - 4\pi/c \; \Sigma_i e_i n_i^{(o)} v_L = 0 .
\end{aligned} \tag{8}$$

We now seek a solution of (8) of the form (6):

$$\begin{pmatrix} n_L \\ v_T, v_L \\ E_T, E_L \\ B_T \end{pmatrix} = \begin{pmatrix} n_{Lo} \\ v_{To}, v_{Lo} \\ E_{To}, E_{Lo} \\ B_{To} \end{pmatrix} + \epsilon \begin{pmatrix} n_{L1} \\ v_{T1}, v_{L1} \\ E_{T1}, E_{L1} \\ B_{T1} \end{pmatrix} + \epsilon^2 \begin{pmatrix} n_{L2} \\ v_{T2}, V_{L2} \\ E_{T2}, E_{L2} \\ B_{T2} \end{pmatrix} + \dots , \tag{9}$$

where the zeroth-order values are

$$E_{Lo} = \xi_L^+ e^{i\omega_p t} + \xi_L^- e^{-i\omega_p t} ,$$

$$E_{To} = \xi_T^+ e^{i\Omega_k t} + \xi_T^- e^{-i\Omega_k t} ,$$

$$B_{To} = \frac{ck}{\Omega_k} (\xi_T^+ e^{i\Omega_k t} - \xi_T^- e^{-i\Omega_k t}) ,$$

$$v_{To} = \frac{e_i}{im_i\Omega_k} (\xi_T^+ e^{i\Omega_k t} - \xi_T^- e^{-i\Omega_k t}) ,$$

$$v_{Lo} = \frac{e_i}{im_i\omega p} (\xi_L^+ e^{i\omega_p t} - \xi_L^- e^{-i\omega_p t}) ,$$

$$e_i n_{Lo} = \frac{ik}{4\pi} \frac{\omega_{pi}^2}{\omega_p^2} (\xi_L^+ e^{i\omega_p t} + \xi_L^- e^{-i\omega_p t}) , \tag{10}$$

with

$$\omega_{pi}^2 = 4\pi\, n_i^{(o)} e_i^2/m_i \;,$$

$$\omega_p^2 = \Sigma_i\, \omega_{pi}^2 \;,$$

$$\Omega_k^2 = c^2 k^2 + \omega_p^2 \;.$$

The only difference (so far) from a classica perturbation expansion is that we consider $\xi_L^{\pm}$ and $\xi_T^{\pm}$ to be at this stage arbitrary functions of ϵt, ..., known only at $\epsilon t = 0$. Their dependence upon ϵt will emerge, in the usual way, as a consequence of the requirement that the $O(\epsilon)$ parts of Eq. (9) shall remain small compared to the $O(1)$ parts. Now, noting that in Eqs. (7), $\frac{\partial}{\partial t}$ must be interpreted as

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t} + \epsilon\,\frac{\partial}{\partial(\epsilon t)} + \epsilon^2\,\frac{\partial}{\partial(\epsilon^2 t)} + \cdots \;, \qquad (11)$$

we may pass on to the $O(\epsilon)$ part of the expansion.

A small amount of algebraic juggling with Eqs. (8) shows that

$$\left(\frac{\partial^2}{\partial t^2} + \Omega_k^2\right) E_T = \epsilon\, 4\pi\, \Sigma_i \frac{\partial}{\partial t}\, e_i n_L V_{oi} \cos \omega_o t$$

$$\left(\frac{\partial^2}{\partial t^2} + \omega_p^2\right) E_L = -\epsilon\, \Sigma_i \frac{\omega_{pi}^2}{c}\, B_T V_{oi} \cos \omega_o t. \tag{12}$$

We substitute (9) and (10) into (12), and make use of (11), getting

$$\left(\frac{\partial^2}{\partial t^2} + \Omega_k^2\right) E_{T1} + 2i\Omega_k \left(\frac{\partial \xi_T^+}{\partial(\epsilon t)}\, e^{i\Omega_k t} - \frac{\partial \xi_T^-}{\partial(\epsilon t)}\, e^{-i\Omega_k t}\right)$$

$$= ik \left(\Sigma_i \frac{\omega_{pi}^2}{\omega_p^2} \frac{V_{oi}}{2}\right) \frac{\partial}{\partial t} \left\{ (\xi_L^+ e^{i\omega_p t} + \xi_L^- e^{-i\omega_p t}) (e^{i\omega_o t} + e^{-i\omega_o t}) \right\} \tag{13}$$

and

$$\left(\frac{\partial^2}{\partial t^2} + \omega_p^2\right) E_{L1} + 2i\omega_p \left(\frac{\partial \xi_L^+}{\partial(\epsilon t)}\, e^{i\omega_p t} - \frac{\partial \xi_L^-}{\partial(\epsilon t)}\, e^{-i\omega_p t}\right)$$

$$= -\frac{k}{2\Omega_k} (\Sigma_i\, \omega_{pi}^2\, V_{oi})\, (\xi_T^+ e^{i\Omega_k t} - \xi_T^- e^{-i\Omega_k t})\, (e^{i\omega_o t} + e^{-i\omega_o t}) , \tag{14}$$

upon equating coefficients of ϵ.

Most of the solutions to Eqs. (13) and (14) have the property that E_{T1} and E_{L1} grow quite large, even if we start them from zero, for certain critical values of ω_o. They soon grow to dominate E_{To} and E_{Lo}. In detail, the terms on the right of (13) which contain $\exp[\pm i(\omega_o - \omega_p)t]$ drive E_{T1} to large values when $\omega_o - \omega_p \approx \Omega_k$, and those on the right of (14) containing $\exp[\pm i(\omega_o - \Omega_k)t]$ drive E_{L1} to large values when the same condition is met.

The only way we can avoid this catastrophe is to use the (as yet arbitrary) terms $\partial\xi_L^{\pm}/\partial(\epsilon t)$ and $\partial\xi_T^{\pm}/\partial(\epsilon t)$ on the left of Eqs. (13) and (14) to <u>cancel</u> the trouble-causing terms on the right. One can then readily solve for a well-behaved E_{T1} and E_{L1}, once this has been done.

Suppose we define

$$\omega_o = \omega_p + \Omega_k + \epsilon\Delta_k \ ,$$

where Δ_k (we assume it is "small", and so write an ϵ in front of it) measures the departure from perfect resonance. Note that for (15) to be satisfied, we must have $\omega_o \gtrsim 2\,\omega_p$. The cancelation of the aforementioned resonant terms gives us:

$$\pm 2i\Omega_k \frac{\partial \xi_T^{\pm}}{\partial(\epsilon t)} = \mp k\ (\omega_o - \omega_p) \left[\Sigma_i \ \frac{\omega_{pi}^2 V_{oi}}{2\omega_p^2} \right] \ \xi_L^{\mp}\ e^{\pm i\Delta_k \epsilon t} \qquad (16)$$

and

$$\pm 2i\omega_p \frac{\partial \xi_L^{\pm}}{\partial(\epsilon t)} = \pm \frac{k}{2\Omega_k} \left[\Sigma_i\ \omega_{pi}^2\ V_{oi} \right] \xi_T^{\mp}\ \ e^{\pm i\ \Delta_k \epsilon t} \ . \qquad (17)$$

Eqs. (16) and (17) are an autonomous pair of differential equations (with periodic coefficients) for the amplitudes $\xi_T^{\pm}$, $\xi_L^{\pm}$, as functions of the "slow" time variable ϵt. On the question of whether or not they have growing solutions hinges the question of whether or not the waves can be parametrically excited by this mechanism.

That growing solutions do exist can be readily demonstrated; eliminating $\xi_L^{\pm}$ between Eqs. (16) and (17) leads to

$$\frac{\partial^2 \xi_T^{\pm}}{\partial(\epsilon t)^2} \mp i\ \Delta_k \frac{\partial \xi_T^{\pm}}{\partial(\epsilon t)} - \lambda^2 \xi_T^{\pm} = 0\ , \qquad (18)$$

where

$$\lambda^2 = \frac{k^2}{8\omega_p \Omega_k} \left(\Sigma_i \frac{\omega_{pi}^2 V_{oi}}{2\omega_p^2} \right) \left(\Sigma_i\ \omega_{pi}^2\ V_{oi} \right).$$

λ^2 is real and positive, and in any plasma, the Σ_i's will be dominated by the electron contribution. Therefore

$$\lambda^2 \approx (k/4)^2 \; \frac{\omega_p}{\Omega_k} \left(\frac{e\, E_{ext.}}{m_e \omega_o} \right)^2 \quad ,$$

or, picking $\lambda > 0$ for definiteness,

$$\approx 1/4 \left(\frac{e\, E_{ext.}}{m \omega_o c} \right) \sqrt{\frac{\omega_p}{\Omega_k} (\Omega_k^2 - \omega_p^2)} \quad . \qquad (19)$$

All the solutions of (18) are of the form

$$\xi_{\mp}^{\pm} \sim \exp\,[\gamma_{\pm}\, \epsilon t]$$

where

$$\gamma_+ = (i\Delta_k \pm \sqrt{-\Delta_k^2 + 4\lambda^2})\;/2 \;,$$

$$\gamma_- = (-i\Delta_k \pm \sqrt{-\Delta_k^2 + 4\lambda^2})\;/2 \;. \qquad (20)$$

Eqs. (20) imply that there always exist exponentially growing solutions which occur whenever $\Delta_k^2 < 4\lambda^2$. For the case of perfect matching, we have $\Delta_k = 0$, and a maximum growth rate which is just λ.

IV. DISCUSSION

For any $\omega_o \gtrsim 2\,\omega_p$, there will always exist a range of wave numbers and frequencies for which the parametric resonance condition (15) is met, and for which Eqs. (20) predict growing oscillations.

The growth rate for the most rapidly growing waves, λ, must be $\ll$ the other frequencies of the problem, for the multiple time scale approach to be applicable. For this, it suffices that

$$\frac{\lambda}{\omega_p} \approx \frac{1}{4} \left(\frac{e\,E_{ext.}}{m\,\omega_o c}\right) \sqrt{\frac{\Omega_k^2 - \omega_p^2}{\omega_p \Omega_k}} \ll 1. \tag{21}$$

The waves will be oscillating at frequencies Ω_k (transverse part) and ω_p (longitudinal part), both of which are much larger than the maximum growth rate λ.

The foregoing theory does not provide an expression for the limiting amplitude of the oscillations, since no such expression can come from a linear theory, such as that of Eqs. (4), or of Aliev and Silin.[7]

In a laboratory plasma, excitation in this manner can be expected to lead to transverse electromagnetic waves which leave the plasma, perpendicularly to the applied electric field. They should be peaked in frequency about $\omega_o - \omega_p$, and occupy a bandwidth which is roughly proportional to the strength of the applied electric field. Their appearance should become less rapid as ω_o increases. Thus, some fairly simple and straightforward experimental predictions can be drawn from the foregoing theory.

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[7]One other significant difference between this case and that of Aliev and Silin is worth noting at this point. We are regarding k as fixed, and thus find only a narrow band of frequencies ω_0 for which transverse waves of wavelength $2\pi/k$ are parametrically excited. However, for $\omega_0 > 2\,\omega_p$, there will always exist some k's for which $\gamma_{\pm}$ have positive real parts. This is in contrast to the integer-multiple relationship that must obtain between ω_0 and ω_p in the longitudinal case of Refs. 3 and 4. It is only a reflection of the fact that for longitudinal waves, all k's oscillate at ω_p, in the cold-plasma approximation.