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THOMPSON RAMO WOOLDRIDGE INC.

ONE SPACE PARK • REDONDO BEACH, CALIFORNIA

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A UNIVERSAL FORMULATION FOR CONIC TRAJECTORIES

BASIC VARIABLES AND RELATIONSHIPS

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by

Jack E. Brooks

and

William W. Lemmon

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TRW SPACE TECHNOLOGY LABORATORIES
THOMPSON RAMO WOOLDRIDGE INC.
ONE SPACE PARK · REDONDO BEACH, CALIFORNIA

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INTRODUCTION

The current writeup is believed to present some new results in the formulation of conic trajectory relations for spaceflight application. Such results are by no means unique, however, as considerable work along similar lines has been done in many places in recent years. In particular, the work of S. Herrick (References 1, 2) and R. Battin (Reference 3) has come to the attention of the writers. It is natural that the special considerations of current spaceflight trajectory and guidance problems should lead to such an interest in new approaches and formulations as compared with those of classical celestial mechanics. For example, initial position and velocity values are more useful as defining quantities for a conic in such problems than the classical orbital parameters. Also, to facilitate automatic computation there has been a need for a universal conic trajectory formulation that avoids indeterminancies associated with eccentricities near zero or unity and achieves a single set of equations for all conic types. Both of these considerations are embodied in the formulation discussed herein.

Fundamental to the present approach is the use of a particular independent variable. This variable is proportional to the eccentric anomaly or its hyperbolic counterpart but is defined universally and remains determinant for all values of the eccentricity. The conic relations are developed in terms of a special set of functions which replace the standard circular and hyperbolic trig functions. These can be characterized as "truncated" trig functions, for their series expansions correspond to those for trig functions with the early terms eliminated to remove indeterminancies. Because of their close relation to the trig functions one is not surprised that they exhibit many useful identities and relationships. The resulting universal formulation is not only valid for all values of the eccentricity but also is valid for the case of rectilinear motion as well.

The results presented are far from exhaustive, as they have been developed only in conjunction with certain particular investigations. Considerable extension will no doubt be realized as more applications are developed. A general formulation is presented in Sections 1, 2, 3 and Appendices A-D that is self-contained, in that it is developed from the basic inverse-square equation of motion. The remaining material represents a compendium of results specifically developed in conjunction with recent investigations into certain perturbation techniques and results. These investigations will be documented separately, with the current writeup meant to serve as a general supporting reference for defining the many quantities and for deriving their associated relations. Nevertheless, the results are felt to represent a reasonably general framework.

1. REVIEW OF DERIVATION FOR CONIC TRAJECTORIES

Starting with the basic law of force we shall derive some of the fundamental relations for inverse-square motion. Let $\hat{r} = \hat{r}(t)$ denote the radius vector relative to the center of force. The equation of motion is then given by

$$\ddot{\hat{r}} = -\frac{\mu}{r^3}\hat{r} \qquad (1.1)$$

where $r = |\hat{r}|$ and μ denotes the force field constant. In keeping with (1.1) we never have $r = 0$; that is, $r > 0$. Let $\hat{v}$ denote the velocity, where

$$\hat{v} = \hat{v}(t) = \dot{\hat{r}}(t) \qquad (1.2)$$

Let $\hat{L}$ denote the angular momentum per unit mass, where

$$\hat{L} \equiv \hat{r} \times \hat{v} = \hat{r} \times \dot{\hat{r}} \qquad (1.3)$$

Then differentiating (1.3) and using (1.1) we obtain

$$\dot{\hat{L}} = \hat{r} \times \ddot{\hat{r}} + \dot{\hat{r}} \times \dot{\hat{r}} = 0$$

so that

$$\hat{L} = \text{constant} \qquad (1.4)$$

We introduce the unit reference axis $\hat{i}$ where

$$\hat{i} \equiv \frac{\hat{r}}{r} \qquad (1.5)$$

We note that $\hat{i} \cdot \hat{L} = \hat{r} \cdot \hat{L} = \hat{v} \cdot \hat{L} = 0$

Also, we introduce the important quantity B, where

$$B \equiv \hat{v} \cdot \hat{r} \qquad (1.6a)$$

$$= r\dot{r} \qquad (1.6b)$$

From the vector identity

$$\hat{r} \times \hat{L} = \hat{r} \times (\hat{r} \times \hat{v}) = (\hat{r} \cdot \hat{v})\hat{r} - r^2\hat{v}$$

we write

$$\hat{v} = \frac{1}{r^2}(B\hat{r} + \hat{L} \times \hat{r}) \tag{1.7a}$$

$$= \frac{1}{r}(B\hat{i} + \hat{L} \times \hat{i}) \tag{1.7b}$$

$$= \dot{r}\hat{i} + \frac{\hat{L}}{r} \times \hat{i} \tag{1.7c}$$

The following derivatives are of interest:

$$\frac{d\hat{i}}{dt} = \frac{\hat{v}}{r} - \frac{\dot{r}\hat{r}}{r^2} = \frac{1}{r^2}\hat{L} \times \hat{i} \tag{1.8a}$$

$$\frac{d}{dt}(\hat{L} \times \hat{i}) = \frac{1}{r^2}\hat{L} \times (\hat{L} \times \hat{i}) = -\frac{L^2}{r^2}\hat{i} \tag{1.8b}$$

Let α denote twice the energy per unit mass, so that

$$\alpha = v^2 - \frac{2\mu}{r} \tag{1.9}$$

Then differentiating (1.9) and using (1.6), (1.1) we obtain

$$\dot{\alpha} = 2\hat{v} \cdot \left(\ddot{\hat{r}} + \frac{\mu\hat{r}}{r^3}\right) = 0$$

Thus

$$\alpha = \text{constant} \tag{1.10}$$

Utilizing (1.7c), (1.9) with α constant we obtain the differential equation for r as follows:

$$v^2 = (\dot{r})^2 + \frac{L^2}{r^2} = \alpha + \frac{2\mu}{r} \geq 0 \tag{1.11}$$

We note that taking the modulus of (1. 7b) yields

$$v = \frac{1}{r} \sqrt{B^2 + L^2} \tag{1.12}$$

Another quantity of interest is $\hat{\epsilon}$ defined by

$$\hat{\epsilon} \equiv \frac{1}{\mu} \hat{v} \times \hat{L} - \frac{\hat{r}}{r} \tag{1.13a}$$

Substituting (1. 7b) into (1. 13a) yields

$$\hat{\epsilon} = \left(\frac{L^2}{\mu r} - 1\right) \hat{i} - \frac{B}{\mu r} \hat{L} \times \hat{i} \tag{1.13b}$$

Differentiating (1. 13a) and using (1. 8a), (1. 1), (1. 5) we obtain

$$\dot{\hat{\epsilon}} = \frac{1}{\mu} \ddot{\hat{r}} \times \hat{L} - \frac{1}{r^2} \hat{L} \times \hat{i} = 0$$

so that

$$\hat{\epsilon} = \text{constant} \tag{1.14}$$

The quantity $\epsilon = |\hat{\epsilon}|$ is called the eccentricity and is obtained from (1. 13a) using (1. 11)

$$\epsilon = \left(1 + \frac{L^2}{\mu^2} \alpha\right)^{1/2} \geq 0 \tag{1.15}$$

Also from (1. 13a)

$$\hat{r} \cdot \hat{\epsilon} = \frac{1}{\mu} L^2 - r$$

so that

$$r = \frac{L^2/\mu}{1 + \hat{\epsilon} \cdot \hat{i}} \tag{1.16}$$

Equation (1. 16) is the well-known equation for a conic in polar form with $\hat{\epsilon} \cdot \hat{i}$ equal to the cosine of the true anomaly. Thus, we see by inspection that $\hat{\epsilon}$ is in the direction from the focus to the peri-apsis point, at which r takes on a minimum value for the conic. Other conventional orbital parameters are as follows

$$\text{semi-latus rectum} \equiv p \equiv \frac{L^2}{\mu} \qquad (1.17)$$

$$\text{energy parameter} \equiv \Lambda \equiv \frac{rv^2}{\mu} = 2 + \frac{r\alpha}{\mu} \qquad (1.18)$$

$$\text{semi-major axis} \equiv a = \frac{r}{2 - \Lambda} = -\frac{\mu}{\alpha} \qquad (1.19)$$

$$\text{peri-apsis distance} \equiv r_p = \frac{p}{1 + \epsilon} \qquad (1.20)$$

2. UNIVERSAL CONIC FORMULATION

We now introduce a particular independent variable ψ defined relative to some initial time t_o as follows:

$$\psi \equiv \int_{t_o}^{t} \frac{dt}{r} \, ^{*} \tag{2.1}$$

Thus

$$\frac{dt}{d\psi} = r \tag{2.2}$$

and

$$\dot{r} = \frac{1}{r} \frac{dr}{d\psi} \tag{2.3}$$

From (1.6b), (2.3) we note that

$$\frac{dr}{d\psi} = B \tag{2.4}$$

Thus the differential equation (1.11) can be written as

$$\left(\frac{dr}{d\psi}\right)^2 = B^2 = \alpha r^2 + 2\mu r - L^2 \tag{2.5}$$

where we have utilized (1.12). We write the various time derivatives below for reference

$$\dot{r} = \frac{B}{r} \tag{2.6}$$

$$\dot{B} = \alpha + \frac{\mu}{r} \tag{2.7}$$

$$\ddot{r} = \frac{1}{r^3} (L^2 - \mu r) \tag{2.8}$$

$$\ddot{B} = -\frac{\mu B}{r^3} \tag{2.9}$$

$$\dot{\psi} = \frac{1}{r} \tag{2.10}$$

---

*For $\epsilon < 1$, $\psi = E - E_o / |\alpha|^{1/2}$ where E is the eccentric anomaly. For $\epsilon > 1$, $\psi = F - F_o / |\alpha|^{1/2}$ where F is the counterpart of the eccentric anomaly for hyperbolic trajectories. The subscript o will relate to $t = t_o$.

To obtain a solution to (2.5) we first differentiate with respect to ψ and use (2.4) to obtain

$$\frac{dB}{d\psi} = \alpha r + \mu \tag{2.11}$$

and similarly

$$\frac{d^2B}{d\psi^2} = \alpha B \tag{2.12}$$

Thus <u>for $\alpha \neq 0$</u>

$$B = K_1 e^{\sqrt{z}} + K_2 e^{-\sqrt{z}} \tag{2.13}$$

where K_1, K_2 are constants of integration and

$$\sqrt{z} \equiv \sqrt{\alpha}\, \psi \tag{2.14a}$$

$$z = \alpha\psi^2 \tag{2.14b}$$

From (2.11)

$$r = \frac{1}{\alpha}\left(\frac{dB}{d\psi} - \mu\right)$$

$$= \frac{1}{\sqrt{\alpha}}\left(K_1 e^{\sqrt{z}} - K_2 e^{-\sqrt{z}}\right) - \frac{\mu}{\alpha} \tag{2.15}$$

Let r_o, B_o denote values for r, B at $t = t_o$, i.e., for $\psi = z = 0$. Then

$$K_1 = \frac{1}{2\sqrt{\alpha}}\left[\sqrt{\alpha}\, B_o + (\alpha r_o + \mu)\right] \tag{2.16a}$$

$$K_2 = \frac{1}{2\sqrt{\alpha}}\left[\sqrt{\alpha}\, B_o - (\alpha r_o + \mu)\right] \tag{2.16b}$$

For $\alpha = 0$, ($\epsilon = 1$), the solution to (2.11) is

$$B = B_o + \mu\psi \tag{2.17}$$

and

$$r = r_o + B_o\psi + \frac{\mu\psi^2}{2} \tag{2.18}$$

We now determine expressions for the various conic trajectory quantities in terms of the Herrick-Lemmon functions discussed in Appendix A. Thus with

$$C = \frac{1}{2}\left(e^{\sqrt{z}} + e^{-\sqrt{z}}\right) \quad , \quad \sqrt{a}\, S = \frac{1}{2}\left(e^{\sqrt{z}} - e^{-\sqrt{z}}\right)$$

we solve to obtain

$$e^{\sqrt{z}} = C(\psi, a) + \sqrt{a}\, S(\psi, a) = C + \sqrt{a}\, S$$

$$e^{-\sqrt{z}} = C(\psi, a) - \sqrt{a}\, S(\psi, a) = C - \sqrt{a}\, S$$

Then using (2. 16), (C. 1a) and the above, we write (2. 13), (2. 15) as follows

$$r = r_o C(\psi) + B_o S(\psi) + \mu ⊄(\psi) \tag{2. 19a}$$

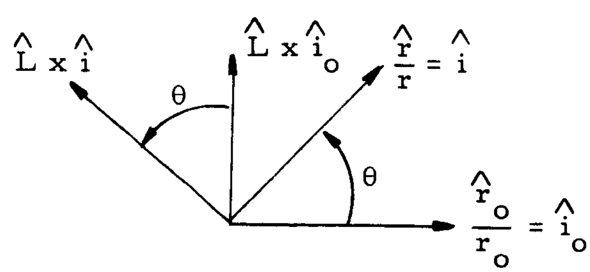
$$= r_o + B_o S(\psi) + (\mu + a r_o) ⊄(\psi) \tag{2. 19b}$$

$$B = B_o C(\psi) + (\mu + a r_o) S(\psi) \tag{2. 20a}$$

$$= B_o + (\mu + a r_o) S(\psi) + a B_o ⊄ \tag{2. 20b}$$

We note the important result that (2. 19), (2. 20) reduce to (2. 18), (2. 17) respectively by simply setting $\epsilon = 1$ (i. e., $a = 0$) and utilizing $C(\psi, 0) = 1$, $S(\psi, 0) = \psi$, $⊄(\psi, 0) = (1/2)\psi^2$. Thus we may consider (2. 19), (2. 20) as valid for all ϵ. The indeterminacy associated with $\epsilon = 1$ has been absorbed into the Herrick-Lemmon functions. It is also noteworthy that the solutions (2. 19), (2. 20) are valid for rectilinear motion with $L = 0$, $\epsilon = 1$. This is in contrast to the conventional expression (1. 16) which becomes indeterminate.

To specify the direction of $\hat{r}$ we introduce the range angle θ shown in the figure and defined below.



Let $\hat{i}_o$ be defined by

$$\hat{i}_o \equiv \frac{\hat{r}_o}{r_o}$$

Also let θ be defined by

$$\cos\theta \equiv \frac{\hat{r}}{r} \cdot \frac{\hat{r}_o}{r_o} = \hat{i} \cdot \hat{i}_o \qquad (2.21a)$$

and for $\hat{L} \neq 0$

$$\sin\theta \equiv \frac{\hat{L}}{L} \cdot (\hat{i}_o \times \hat{i}) \qquad (2.21b)$$

and for $\hat{L} = 0$, $\hat{i} = \hat{i}_o$ so that $\cos\theta = 1$ and

$$\theta = 0 \qquad (2.21c)$$

Thus for $\hat{L} \neq 0$ we may write

$$\hat{i} = \cos\theta\, \hat{i}_o + \sin\theta \frac{\hat{L}}{L} \times \hat{i}_o \qquad (2.22a)$$

$$\frac{\hat{L}}{L} \times \hat{i} = -\sin\theta\, \hat{i}_o + \cos\theta \frac{\hat{L}}{L} \times \hat{i}_o \qquad (2.22b)$$

Also, from (1.8), (2.2)

$$\frac{d\hat{i}}{d\psi} = r \frac{d\hat{i}}{dt} = \frac{\hat{L} \times \hat{i}}{r} \qquad (2.23a)$$

$$\frac{d}{d\psi}(\hat{L} \times \hat{i}) = \hat{L} \times \frac{d\hat{i}}{d\psi} = \frac{-L^2}{r} \hat{i} \qquad (2.23b)$$

Thus differentiating (2.22) and comparing with (2.23) yields

$$\frac{d\theta}{d\psi} = \frac{L}{r} \qquad (2.24a)$$

and

$$\dot{\theta} = \frac{L}{r^2} \qquad (2.24b)$$

We note that (2.24) is valid for all $\hat{L}$. Recalling from (2.21) that $\theta = 0$ when $t = t_o$, then

$$\theta = L \, \textcircled{H} \qquad (2.25a)$$

$$\dot{\textcircled{H}} = \frac{\dot{\theta}}{L} = \frac{1}{r^2} \qquad (2.25b)$$

where

$$\textcircled{H} \equiv \int_o^{\psi} \frac{d\psi}{r} \qquad (2.26a)$$

$$= \int_{t_o}^{t} \frac{dt}{r^2} \qquad (2.26b)$$

$$= \frac{1}{L} \theta \qquad (2.26c)$$

The expression (2.26a), (2.26b) remains determinate for $L = 0$, which is not the case for (2.26c).

To determine θ in terms of the Herrick-Lemmon functions we proceed as follows. Recalling (1.13b) and utilizing (2.21), (2.22), for $\hat{L} \neq 0$ we obtain

$$\mu(\hat{L} \times \hat{i}_o) \cdot \hat{\epsilon} = \frac{L}{r}\left[(L^2 - \mu r) \sin\theta - LB \cos\theta\right] = \text{constant} = \frac{-L^2 B_o}{r_o}$$

$$\mu \hat{i}_o \cdot \hat{\epsilon} = \frac{1}{r}\left[(L^2 - \mu r) \cos\theta + LB \sin\theta\right] = \text{constant} = \frac{L^2}{r_o} - \mu$$

Solving the above equations for $\sin\theta$, $\cos\theta$ and after some manipulation utilizing (2.19), (2.20), etc., we obtain

$$\sin\theta = \frac{L}{r r_o} (B_o \text{¢} + r_o S) \qquad (2.27a)$$

$$\cos\theta = 1 - \frac{L^2}{r r_o} \text{¢} \qquad (2.27b)$$

The equations (2.27) are valid for all $\hat{L}$ including $\hat{L} = 0$.

We now consider the well-known "f, g" representation as linear combinations of $\hat{r}_o$, $\hat{v}_o$:

$$\hat{r} = \hat{r}(t) = f(t_o, t)\hat{r}(t_o) + g(t_o, t)\hat{v}(t_o) = f\hat{r}_o + g\hat{v}_o \qquad (2.28a)$$

$$\hat{v} = \hat{v}(t) = \dot{f}(t_o, t)\hat{r}(t_o) + \dot{g}(t_o, t)\hat{v}(t_o) = \dot{f}\hat{r}_o + \dot{g}\hat{v}_o \qquad (2.28b)$$

After some manipulation, the coefficient functions are then given as follows:

$$f = f(t_o, t) = \frac{\hat{r}(t)}{L^2} \cdot \left[\hat{v}(t_o) \times \hat{L}\right] = 1 - \frac{\mu}{r_o} ¢ \qquad (2.29a)$$

$$g = g(t_o, t) = \frac{\hat{r}(t)}{L^2} \cdot \left[\hat{L} \times \hat{r}(t_o)\right] = r_o S + B_o ¢ \qquad (2.29b)$$

$$\dot{f} = \dot{f}(t_o, t) = \frac{\hat{v}(t)}{L^2} \cdot \left[\hat{v}(t_o) \times \hat{L}\right] = \frac{-\mu}{r r_o} S \qquad (2.29c)$$

$$\dot{g} = \dot{g}(t_o, t) = \frac{\hat{v}(t)}{L^2} \cdot \left[\hat{L} \times \hat{r}(t_o)\right] = 1 - \frac{\mu}{r} ¢ \qquad (2.29d)$$

We also introduce the quantity h that is related to g as follows:

$$h \equiv r_o S + B ¢ \qquad (2.30)$$

We may also write $\hat{r}, \hat{v}$ as

$$\hat{r} = \frac{1}{r_o}(u\hat{i}_o + g\hat{L} \times \hat{i}_o) \qquad (2.31a)$$

$$\hat{v} = \frac{1}{r_o}(\dot{u}\hat{i}_o + \dot{g}\hat{L} \times \hat{i}_o) \qquad (2.31b)$$

where we have introduced u defined by

$$u \equiv r r_o - L^2 ¢ \qquad (2.32a)$$

$$= r_o^2 f + B_o g \qquad (2.32b)$$

and

$$\dot{u} \equiv r_o^2 \dot{f} + B_o \dot{g} \tag{2.33a}$$

$$= B_o - \frac{\mu}{r} g \tag{2.33b}$$

$$= \frac{1}{r} (r_o B - L^2 S) \tag{2.33c}$$

Substituting (2.28) into (1.3) we obtain

$$f\dot{g} - g\dot{f} = 1 \tag{2.34a}$$

Then

$$u\dot{g} - g\dot{u} = r_o^2 \tag{2.34b}$$

Also using (2.29b), (C.12a), (2.19) we obtain

$$g^2 = (u + rr_o)¢ \tag{2.35}$$

Using (2.31b), (2.33b), (2.29c), (1.7b), we may also express $\hat{v}$ as follows

$$\hat{v} = \hat{v}_o - \frac{\mu}{rr_o} \hat{w} \tag{2.36}$$

where

$$\hat{w} \equiv g\hat{i}_o + ¢\hat{L} \times \hat{i}_o \tag{2.37a}$$

$$= S\hat{r}_o + r_o ¢ \hat{v}_o \tag{2.37b}$$

and

$$\hat{L} \cdot \hat{w} = 0 \tag{2.38}$$

We note that

$$\frac{\hat{w}}{rr_o} = - \frac{1}{\mu} \int_{t_o}^{t} \dot{\hat{v}} \, dt = \int_{t_o}^{t} \frac{\hat{r}}{r^3} dt \tag{2.39a}$$

so that

$$\frac{\partial}{\partial t} \left(\frac{\hat{w}}{rr_o} \right) = \frac{\hat{r}}{r^3} \tag{2.39b}$$

Also, recalling (1.8a), (2.39a)

$$\hat{i} = \hat{i}_o + \int_{t_o}^{t} \dot{\hat{i}}\, dt$$

$$= \hat{i}_o + \frac{\hat{L} \times \hat{w}}{r r_o} \qquad (2.40)$$

Then

$$\hat{r} = r\hat{i}_o + \frac{1}{r_o} \hat{L} \times \hat{w} \qquad (2.41)$$

and

$$\hat{L} \times \hat{r} = r\hat{L} \times \hat{i}_o - \frac{L^2}{r_o} \hat{w} \qquad (2.42)$$

To relate time to ψ we need the counterpart of Kepler's equation. Let τ be defined by

$$\tau = \tau(t_o, t) = t - t_o \qquad (2.43)$$

Then

$$\frac{d\tau}{dt} = 1 \qquad (2.44a)$$

$$\frac{d\tau}{d\psi} = r \qquad (2.44b)$$

Thus

$$\tau = \int_{\psi_1=0}^{\psi} r d\psi_1$$

$$= \int_{\psi_1=0}^{\psi} \left[r_o C(\psi_1) + B_o S(\psi_1) + \mu ¢(\psi_1)\right] d\psi_1$$

$$= r_o S(\psi) + B_o ¢(\psi) + \mu \$(\psi) \qquad (2.45a)$$

$$= g + \mu \$ \qquad (2.45b)$$

where we have used the integration formulas (D.1), (D.2), (D.3), and (2.29b).

It is sometimes convenient to introduce the chord $\hat{d}$, where using (2.31a), (1.7b), (2.32b) we obtain

$$\hat{d} \equiv \hat{r} - \hat{r}_o \tag{2.46a}$$

$$= (f - 1)\hat{r}_o + g\hat{v}_o \tag{2.46b}$$

$$= \frac{1}{r_o}\left[\left(u - r_o^2\right)\hat{i}_o + g\hat{L} \times \hat{i}_o\right] \tag{2.46c}$$

The magnitude of $\hat{d}$ is given by

$$d = \left|\hat{r} - \hat{r}_o\right| \tag{2.47a}$$

$$= \left[r^2 + r_o^2 - 2rr_o \cos\theta\right]^{1/2} \tag{2.47b}$$

$$= \left[\left(r - r_o\right)^2 + 2L^2 ¢\right]^{1/2} \tag{2.47c}$$

$$= \left(r^2 + r_o^2 - 2u\right)^{1/2} \tag{2.47d}$$

3. FUNCTIONAL REPRESENTATIONS

The basic quantities $\hat{r}$, $\hat{v}$, etc., are functions of the time, t. However, in utilizing the reference time t_o we find it convenient to introduce functions of (t_o, t). Thus, primary quantities depending on t alone may be represented by functions of (t_o, t) as in (2.28), and of course the important quantity ψ is a function of (t_o, t) as given by (2.1). In addition, we find it convenient to introduce functions of (t_o, ψ) as in (2.19), (2.20). It will therefore be useful to introduce an explicit notation for such functional representations and to discuss the underlying functional relationships. To do this we consider a quantity F to have the two functional representations $F(t_o, t)$ and $F^\dagger(t_o, \psi)$, where these are related as follows:

$$F = F(t_o, t) = F^\dagger(t_o, \psi) \tag{3.1}$$

or in general

$$F(t_1, t_2) = F^\dagger(t_1, x) \tag{3.2}$$

where

$$x = \int_{\tau=t_1}^{t_2} \frac{d\tau}{r(\tau)} \tag{3.3}$$

Thus, the primary quantities r(t), B(t) have representations $r^\dagger(t_o, \psi)$, $B^\dagger(t_o, \psi)$ given by (2.19), (2.20) respectively. On the other hand, the quantity g in (2.28a) cannot be represented as a function of t alone. Recalling (2.29b) we see that the two functional representations of g can be written explicitly as follows:

$$g = g(t_o, t) = \frac{\hat{r}(t)}{L^2} \cdot \left[\hat{L} \times \hat{r}(t_o)\right] \tag{3.4a}$$

$$= g^\dagger(t_o, \psi) = r(t_o)\, S(\psi) + B(t_o)\, ₵(\psi) \tag{3.4b}$$

The use of explicit functional representations as above allows us to define functional transformations and introduce new quantities that will prove useful. Thus we define a "reverse" on "bar" transformation as follows:

$$\overline{F(t_o,\ t)} \equiv F(t,\ t_o) \tag{3.5a}$$

so that it follows from (3.2), (3.3) that

$$\overline{F^{\dagger}(t_o,\ \psi)} = F^{\dagger}(t,\ -\psi) \tag{3.5b}$$

In general if $F = F(t_o,\ t)$ then we let $\overline{F}$ denote $\overline{F(t_o,\ t)} = F(t,\ t_o)$. The relation between $\overline{F}$ and F will depend on the particular functional form. For a primary quantity that can be expressed as a function of t alone we have the following important result:

$$F = F(t_o,\ t) = F(t) \tag{3.6}$$

$$\overline{F} = F(t,\ t_o) = F(t_o) \tag{3.7}$$

Thus, for r, B we utilize (2.19), (2.20) and (3.5), (3.7) to obtain the following identities:

$$r_o = \overline{r^{\dagger}(t_o,\ \psi)}$$

$$= r(t)\ C(-\psi) + B(t)\ S(-\psi) + \mu\cancel{C}(-\psi)$$

or

$$r_o = rC(\psi) - BS(\psi) + \mu\cancel{C}(\psi) \tag{3.8}$$

and similarly

$$B_o = BC(\psi) - (\mu + ar)\ S(\psi) \tag{3.9}$$

From (2.29) we obtain

$$\overline{f} = f(t,\ t_o) = \frac{\hat{r}(t_o)}{L^2} \cdot \left[\hat{v}(t) \times \hat{L}\right] = \dot{g}(t_o,\ t) = \dot{g} \tag{3.10}$$

$$\overline{g} = g(t,\ t_o) = \frac{\hat{r}(t_o)}{L^2} \cdot \left[\hat{L} \times \hat{v}(t)\right] = -g(t_o,\ t) = -g \tag{3.11}$$

$$\bar{\dot{f}} = \dot{f}(t,\ t_o) = \frac{\hat{v}(t_o)}{L^2} \cdot \left[\hat{v}(t) \times \hat{L}\right] = -\dot{f}(t_o,\ t) = -\dot{f} \tag{3.12}$$

$$\bar{\dot{g}} = \dot{g}(t,\ t_o) = \frac{\hat{v}(t_o)}{L^2} \cdot \left[\hat{L} \times \hat{r}(t)\right] = f(t_o,\ t) = f \tag{3.13}$$

The results (3.10), (3.12), (3.13) also follow directly by applying (3.5b) to (2.29a), (2.29c), (2.29d). Applying (3.5b) to (2.29b) we obtain

$$g = -\bar{g} = -g\dagger(t,\ -\psi) = r\,S - B¢ \tag{3.14}$$

Thus equating (2.29b), (3.14) yields the following important identity:

$$(B + B_o)¢ = (r - r_o)S \tag{3.15a}$$

also

$$r_o(g + B¢) = r(g - B_o\ ¢) = r\ r_o\ S$$

so that

$$(r - r_o)g = (r_o\ B + r\ B_o)¢ \tag{3.15b}$$

Recalling (2.30), we see from (3.15a) that

$$h = rS - B_o\ ¢ \tag{3.16a}$$

$$= -\bar{h} \tag{3.16b}$$

also

$$h = (r + r_o)\ S - g \tag{3.16c}$$

$$= g + (B - B_o)\ ¢ \tag{3.16d}$$

From (2.43), (2.45),

$$\bar{\tau} = -\tau \tag{3.17}$$

and

$$\bar{\tau} = \bar{g} - \mu\$ \tag{3.18a}$$

$$= -rS + B¢ - \mu\$ \tag{3.18b}$$

Thus (3.17), (3.18) also yield (3.15).

Some further illustrations on using the reverse transform to derive relationships are as follows. Applying the reverse transform to (2.28) yields

$$\hat{r}_o = \dot{g}\hat{r} - g\hat{v} \tag{3.19a}$$

$$\hat{v}_o = -\dot{f}\hat{r} + f\hat{v} \tag{3.19b}$$

Recalling (2.32) we note that

$$\bar{u} = u \tag{3.20a}$$

$$= r^2 \dot{g} - Bg \tag{3.20b}$$

Then the transform of (2.31) yields

$$\hat{r}_o = \frac{1}{r} (u\hat{i} - g\hat{L} \times \hat{i}) \tag{3.21a}$$

$$\hat{v}_o = \frac{1}{r} (\bar{\dot{u}}\hat{i} + f\hat{L} \times \hat{i}) \tag{3.21b}$$

where from (2.33),

$$\bar{\dot{u}} = -r^2 \dot{f} + Bf \tag{3.22a}$$

$$= B + \frac{\mu}{r_o} g \tag{3.22b}$$

$$= \frac{1}{r_o} \left(r B_o + L^2 S\right) \tag{3.22c}$$

and from (2.34b),

$$uf + g\bar{\dot{u}} = r^2 \tag{3.23}$$

Note that $\bar{\dot{u}} \neq \dot{\bar{u}} = \dot{u}$.

Recalling (2.37a) (3.11), we write

$$\bar{\hat{w}} = -(g\hat{i} - ¢\hat{L} \times \hat{i}) \tag{3.24}$$

Also, from (2.21), (2.27), (2.29b), (2.32a), (2.35), (2.37):

$$-\overline{\hat{w}} \cdot \hat{i}_o = g \cos\theta + \dot{\phi} L \sin\theta$$

$$= g$$

$$= \hat{w} \cdot \hat{i}_o$$

$$-\overline{\hat{w}} \cdot \hat{L} \times \hat{i}_o = Lg \sin\theta - L^2 \dot{\phi} \cos\theta$$

$$= L^2 \dot{\phi}$$

$$= \hat{w} \cdot \hat{L} \times \hat{i}_o$$

Therefore

$$\hat{w} = -\overline{\hat{w}} \tag{3.25a}$$

$$= g\hat{i} - \dot{\phi}\hat{L} \times \hat{i}$$

In dealing with an interval (t_o, t_f) we shall also find it convenient to introduce ψ^* defined relative to t_f, where

$$\psi^* \equiv \int_t^{t_f} \frac{dt}{r(t)} \tag{3.26}$$

Then

$$\psi_f \equiv \int_{t_o}^{t_f} \frac{dt}{r(t)} = \psi + \psi^* \tag{3.27}$$

For a quantity $F = F(t_o, t)$ we define a related quantity F^*, where

$$F^* \equiv F(t_f, t) \tag{3.28a}$$

$$= F^\dagger (t_f, -\psi^*) \tag{3.28b}$$

The result (3.28b) is not one of definition but rather is obtained by utilizing (3.26) and the general relationship between $F(\tau_1, \tau_2)$ and $F^\dagger(\tau_1, x)$ defined by (3.2), (3.3). The relationships (3.28a), (3.28b) can be considered as defining a corresponding functional transformation, but the explicit introduction of a notation for such a transformation will be avoided by simply utilizing $t_o \rightarrow t_f$, $\psi \rightarrow -\psi^*$ in any relationship. Thus we obtain the following results in terms of ψ^* and with an f subscript to refer to the time t_f.

From (3.28a):

$$r^* \equiv r(t_f,\ t) = r(t) = r$$

From (3.28b):

$$r^* = r^\dagger(t_f,\ -\psi^*)$$

Therefore (2.19) can be written in terms of ψ^* as

$$r = r_f\, C(\psi^*) - B_f\, S(\psi^*) + \mu \cent(\psi^*) \tag{3.29a}$$

$$= r_f\, C^* - B_f\, S^* + \mu \cent^* \tag{3.29b}$$

$$= r_f - B_f\, S^* + (\mu + \alpha r_f)\, \cent^* \tag{3.29c}$$

where we introduce $C^* \equiv C(\psi^*)$, $S^* \equiv S(\psi^*)$ etc. Similarly (2.20) becomes

$$B = B_f\, C^* - (\mu + \alpha r_f)\, S^* \tag{3.30a}$$

$$= B_f - (\mu + \alpha r_f)\, S^* + \alpha B_f\, \cent^* \tag{3.30b}$$

The various identities may also be transformed. Thus (3.8), (3.9), (3.15a) become

$$r_f = rC^* + BS^* + \mu \cent^* \tag{3.31}$$

$$B_f = BC^* + (\mu + \alpha r)\, S^* \tag{3.32}$$

$$(B + B_f)\, \cent^* = (r_f - r)\, S^* \tag{3.33}$$

The relationship

$$\frac{dr}{d\psi} = \frac{\partial}{\partial \psi}\, r^\dagger(t_o, \psi) = B^\dagger(t_o,\ \psi) = B \tag{3.34a}$$

transforms to

$$-\frac{dr}{d\psi^*} = \frac{d}{d(-\psi^*)}\, r^\dagger(t_f,\ -\psi^*) = B^\dagger(t_f,\ -\psi^*) = B \tag{3.34b}$$

The result (3.34b) can of course also be obtained by differentiating (3.29c) and comparing with (3.30a), but the use of transformations such as in (3.34) will often be very useful.

We also introduce the following quantities:

$$f^* \equiv f(t_f, t) = f^\dagger(t_f, -\psi^*) = 1 - \frac{\mu}{r_f} \text{¢}^* \tag{3.35a}$$

$$g^* \equiv g(t_f, t) = g^\dagger(t_f, -\psi^*) = -r_f S^* + D_f \text{¢}^* \tag{3.35b}$$

$$\dot{f}^* \equiv \dot{f}(t_f, t) = \dot{f}^\dagger(t_f, -\psi^*) = \frac{\mu}{r\, r_f} S^* \tag{3.35c}$$

$$\dot{g}^* \equiv \dot{g}(t_f, t) = \dot{g}^\dagger(t_f, -\psi^*) = 1 - \frac{\mu}{r} \text{¢}^* \tag{3.35d}$$

Then (2.28) can be written as

$$\hat{r} = \hat{r}(t) = f(t_f, t)\,\hat{r}(t_f) + g(t_f, t)\,\hat{v}(t_f) = f^*\,\hat{r}_f + g^*\,\hat{v}_f \tag{3.36a}$$

$$\hat{v} = \hat{v}(t) = \dot{f}(t_f, t)\,\hat{r}(t_f) + \dot{g}(t_f, t)\,\hat{v}(t_f) = \dot{f}^*\,\hat{r}_f + \dot{g}^*\,\hat{v}_f \tag{3.36b}$$

We also introduce τ^*, where

$$\tau^* = \tau(t_f, t) = t - t_f \tag{3.37a}$$

$$= \tau^\dagger(t_f, -\psi^*) = g^* - \mu \$^* \tag{3.37b}$$

we note that

$$\tau - \tau^* = t_f - t_o \tag{3.38}$$

4. SOME TIME INTEGRALS AND ASSOCIATED RELATIONS

We shall now derive some additional results that will prove useful in the application of our universal conic formulation to various problems. For example, certain time integrals will be needed. As a preparatory step to the discussion of such integrals we proceed as follows. Consider some general function $F(\psi)$, with the variable $\psi = \psi(t_o, t)$ defined by (2.1). Then using (2.2), (2.19):

$$\int_{\tau=t_o}^{t} F\left[\psi(t_o, \tau)\right] d\tau = \int_o^{\psi} F(\psi)\, r^{\dagger}(t_o, \psi)\, d\psi$$

$$= r_o \int_o^{\psi} CF\, d\psi + B_o \int_o^{\psi} SF\, d\psi + \mu \int_o^{\psi} ¢F\, d\psi \tag{4.1}$$

For $F = 2C$ we utilize the integrals (D.16), (D.17), (D.18) and obtain

$$2\int_{t_o}^{t} C\, dt = r_o \psi + rS - \mu\$ \tag{4.2}$$

For $F = 2S$ we utilize the integrals (D.17), (D.22), (D.23) and obtain

$$2\int_{t_o}^{t} S\, dt = Sg + B_o \$ + \mu ¢^2 \tag{4.3a}$$

$$= \psi g + B\$ + \mu ¢' \tag{4.3b}$$

where we have used the identities (C.2a), (C.9a) and the result

$$B = B_o + \mu S + \alpha(r_o S + B_o ¢)$$

$$= B_o + \mu S + \alpha g \tag{4.4}$$

For $F = 2¢$, using the integrals (D.18), (D.23), (D.26) we obtain

$$2\int_{t_o}^{t} ¢\, dt = \tau ¢ - r_o \$ + \mu \$' \tag{4.5}$$

For $F = 2\$$, using the integrals (D. 19), (D. 24), (D. 27) we obtain

$$2 \int_{t_o}^{t} \$ \, dt = \tau \$ - r_o \, ¢' - B_o \, \$' \qquad (4.6)$$

It is convenient to introduce a quantity Λ, where

$$\Lambda \equiv ¢ \, \$ + \$' \qquad (4.7a)$$

$$= -\overline{\Lambda} \qquad (4.7b)$$

Utilizing (C. 2b), (C. 10a) we obtain

$$\Lambda = \frac{1}{\alpha} (S \, ¢ - 3\$) \qquad (4.7c)$$

We also introduce Υ, where

$$\Upsilon \equiv \tau \, ¢ + \mu \$' \qquad (4.8a)$$

From (2. 45b), (4. 7a) we obtain

$$\Upsilon = g \, ¢ + \mu \Lambda \qquad (4.8b)$$

also,

$$\Upsilon = -\overline{\Upsilon} \qquad (4.8c)$$

We now introduce the quantities ρ, σ, ζ, η defined below.

$$\rho = \rho(t_o, t) \equiv 2 \int_{\tau = t_o}^{t} ¢ \left[\psi(t_o, \tau)\right] d\tau \qquad (4.9a)$$

$$= \Upsilon - r_o \, \$ \qquad (4.9b)$$

$$= g \, ¢ + \mu \Lambda - r_o \, \$ \qquad (4.9c)$$

$$= \frac{r_o}{\mu} (g - f \tau) + \mu \, \$' \qquad (4.9d)$$

$$\zeta = \zeta(t_o, t) \equiv \int_{\tau=t_o}^{t} \left[3 - 2f(t_o, \tau)\right] d\tau \tag{4.10a}$$

$$= \tau + \frac{\mu}{r_o} \rho \tag{4.10b}$$

$$= g + \frac{\mu}{r_o} \Upsilon \tag{4.10c}$$

$$= g(2 - f) + \frac{\mu^2}{r_o} \Lambda \tag{4.10d}$$

$$\eta = \eta(t_o, t) \equiv \int_{\tau=t_o}^{t} \left[-3 + 4f(t_o, \tau)\right] d\tau \tag{4.11a}$$

$$= \tau - \frac{2\mu}{r_o} \rho \tag{4.11b}$$

$$= \tau - \frac{2\mu}{r_o} \Upsilon + 2\mu\$ \tag{4.11c}$$

$$= g - \frac{2\mu}{r_o} \Upsilon + 3\mu\$ \tag{4.11d}$$

$$\sigma = \sigma(t_o, t) \equiv 2 \int_{\tau=t_o}^{t} g(t_o, \tau) \, d\tau \tag{4.12}$$

From (4.10c) we obtain

$$(\zeta - g) r_o = \mu\Upsilon = -\mu\overline{\Upsilon} = -(\overline{\zeta} + g) r$$

or

$$(r_o - r)g = r_o\zeta + r\overline{\zeta} \tag{4.13}$$

Combining (4.10b), (4.11b) we note that

$$\eta = 3\tau - 2\zeta \tag{4.14a}$$

and

$$\zeta = \frac{1}{2} (3\tau - \eta) \tag{4.14b}$$

Also

$$\zeta + \bar{\zeta} = -\frac{1}{2}(\eta + \bar{\eta}) \tag{4.14c}$$

Utilizing (2.29b), (4.3b), (4.5), we evaluate σ as follows:

$$\sigma = 2\int_{t_o}^{t} (r_o S + B_o ¢)\, d\tau$$

$$= \tau B_o ¢ + \mu r_o ¢' + \mu B_o \$' + r_o \left[(B - B_o)\$ + \psi_g\right]$$

$$= \tau g + \mu(r_o ¢' + B_o \$') \tag{4.15}$$

To obtain (4.15) we have made use of the identity given below.

$$S\tau - \psi_g = S(g + \mu\$) - \psi_g$$

$$= \mu S \$ + g(S - \psi)$$

$$= \$ (\mu S + a g)$$

$$= \$ (B - B_o) \tag{4.16}$$

Where in deriving (4.16) we have utilized (2.45), (C.2a), (4.4).

Recalling the "reverse" transformation (3.5) we write

$$\bar{\rho} = \rho(t, t_o) - -2\int_{\tau=t_o}^{t} ¢ \left[\psi(t, \tau)\right] d\tau \tag{4.17a}$$

$$= r\$ - \Upsilon \tag{4.17b}$$

We note that

$$\rho + \bar{\rho} = (r - r_o)\$ \tag{4.17c}$$

$$\bar{\zeta} = \zeta(t, t_o) = -\int_{\tau=t_o}^{t} \left[3 - 2f(t, \tau)\right] d\tau \tag{4.18a}$$

$$= -\tau + \frac{\mu}{r} \bar{\rho} \tag{4.18b}$$

$$= -g - \frac{\mu}{r} \Upsilon \tag{4.18c}$$

$$= -g(2 - \dot{g}) - \frac{\mu^2}{r} \Lambda \tag{4.18d}$$

$$\bar{\eta} = \eta(t, t_o) = -\int_{\tau=t_o}^{t} \left[-3 + 4f(t, \tau)\right] d\tau \tag{4.19a}$$

$$= -\tau - \frac{2\mu}{r} \bar{\rho} \tag{4.19b}$$

$$= -\tau + \frac{2\mu}{r} \Upsilon - 2\mu\$ \tag{4.19c}$$

$$= -g + \frac{2\mu}{r} \Upsilon - 3\mu\$ \tag{4.19d}$$

$$\bar{\sigma} = \sigma(t, t_o) = -2 \int_{\tau=t_o}^{t} g(t, \tau)\, d\tau \tag{4.20a}$$

$$= \tau g + \mu(r\, ₵' - B\, \$') \tag{4.20b}$$

To obtain some additional results we first obtain the following identity. Let B be expressed in terms of f, g as follows:

$$B = B_o + (\mu + a r_o) S + a B_o ₵$$

$$= B_o - \frac{\mu B_o}{r_o} ₵ + B_o \left(a + \frac{\mu}{r_o}\right) ₵ + \left(a + \frac{\mu}{r_o}\right) r_o S$$

$$= B_o f + \left(a + \frac{\mu}{r_o}\right) g \tag{4.21}$$

where we have utilized (2.20b), (2.29a), (2.29b).

Taking the "reverse" transformation of (4.21) we obtain

$$B_o = B\dot{g} - \left(\alpha + \frac{\mu}{r}\right) g \tag{4.22}$$

From (2.33b), (1.11), (4.22) we write

$$\dot{u} = B\dot{g} - v^2 g \tag{4.23}$$

Therefore, using (2.7), (4.22) we obtain

$$\frac{\partial}{\partial t}\left(\frac{g}{B\,B_o}\right) = \frac{1}{B_o}\left[\frac{\dot{g}}{B} - \frac{g}{B^2}\left(\alpha + \frac{\mu}{r}\right)\right]$$

$$= \frac{1}{B^2} \tag{4.24a}$$

and

$$\int_{\tau=t_o}^{t} \frac{d\tau}{B(\tau)^2} = \frac{g(t_o,\ t)}{B(t_o)\,B(t)} = \frac{g}{B_o\,B} \tag{4.24b}$$

We also obtain a related result. Let λ be defined by

$$\lambda = \lambda(t_o,\ t) \equiv r^2 g - B\sigma \tag{4.25}$$

Then using (4.24a), (2.6), (4.12)

$$\frac{\partial}{\partial t}\left(\frac{\lambda}{B_o\,B}\right) = \frac{\partial}{\partial t}\left[r^2\left(\frac{g}{B_o\,B}\right) - \frac{\sigma}{B_o}\right]$$

$$= \frac{r^2}{B^2} + 2r\left(\frac{B}{r}\right)\left(\frac{g}{B_o\,B}\right) - \frac{2g}{B_o}$$

$$= \left(\frac{r}{B}\right)^2 \tag{4.26}$$

Therefore

$$\int_{\tau=t_o}^{t}\left[\frac{r(\tau)}{B(\tau)}\right]^2 d\tau = \frac{\lambda(t_o,\ t)}{B(t_o)\,B(t)} = \frac{\lambda}{B_o\,B} \tag{4.27}$$

Taking the reverse transform of (4.27) we obtain

$$\bar{\lambda} = \lambda(t,\ t_o) = -r_o^2 g - B_o\,\bar{\sigma} \tag{4.28}$$

By inspection of (4.27) we note that

$$\overline{\lambda} = \lambda(t, t_o) = -\lambda(t_o, t) = -\lambda \tag{4.29}$$

so that

$$\lambda = r_o^{\,2} g + B_o \overline{\sigma} \tag{4.30}$$

Substituting (4.25), (4.28) into (4.29) we obtain the following important identity:

$$B\sigma + B_o \overline{\sigma} = \left(r^2 - r_o^{\,2}\right) g \tag{4.31}$$

Utilizing (2.6), (2.10), (3.14), etc., we obtain

$$\frac{\partial}{\partial t}\left[\frac{₵(\psi)}{r}\right] = \frac{1}{r^3}\,(rS - B₵) = \frac{g}{r^3} \tag{4.32}$$

Also from (2.6), (2.7), (4.22), (2.5), (4.32):

$$\frac{\partial}{\partial t}\left(\frac{Bg}{r^2}\right) = \frac{B\dot{g}}{r^2} + \frac{g}{r^2}\left(\alpha + \frac{\mu}{r}\right) - \frac{2B^2 g}{r^4}$$

$$= \frac{B_o}{r^2} + \frac{2g}{r^4}\,(\alpha r^2 + \mu r - B^2)$$

$$= \frac{B_o}{r^2} + \frac{2g}{r^4}\,(L^2 - \mu r) \tag{4.33a}$$

$$= \frac{2g\,L^2}{r^4} + \frac{d}{\partial t}\left(B_o \, Ⓗ - \frac{2\mu\, ₵}{r}\right) \tag{4.33b}$$

where we have recalled (2.25b).

From (4.33) we then write

$$2L^2 \int_{t_o}^{t} \frac{g}{r^4}\, dt = \frac{Bg}{r^2} + \frac{2\mu\, ₵}{r} - B_o \, Ⓗ \tag{4.34a}$$

$$= \frac{\Delta}{r^2} - B_o \, Ⓗ \tag{4.34b}$$

where using (2.29d), (3.20), (2.32)

$$\Delta \equiv Bg + 2\mu r \, ¢ \tag{4.35a}$$

$$= Bg + 2(1 - \dot{g})\, r^2 \tag{4.35b}$$

$$= r^2 (2 - \dot{g}) - u \tag{4.35c}$$

$$= r(r - r_o) + (\mu r + L^2)\, ¢ \tag{4.35d}$$

$$= -B\, g + 2(r^2 - u) \tag{4.35e}$$

Also

$$\overline{\Delta} = -B_o\, g + 2\mu\, r_o\, ¢ \tag{4.36a}$$

$$= -B_o\, g + 2(1 - f)\, r_o^2 \tag{4.36b}$$

$$= r_o^2 (2 - f) - u \tag{4.36c}$$

$$= r_o(r_o - r) + \left(\mu r_o + L^2\right)¢ \tag{4.36d}$$

$$= B_o\, g + 2\left(r_o^2 - u\right) \tag{4.36e}$$

From (4.12), (4.25), (2.6)

$$\frac{\partial}{\partial t}\left(\frac{\sigma}{r^2}\right) = \frac{1}{r^2}(2g) - \frac{2\sigma}{r^3}\left(\frac{B}{r}\right)$$

$$= \frac{2\lambda}{r^4} \tag{4.37}$$

Therefore

$$2\int_{\tau=t_o}^{t} \frac{\lambda(t_o, \tau)}{r(\tau)^2}\, d\tau = \frac{\sigma(t_o, t)}{r(t)^2} = \frac{\sigma}{r^2} \tag{4.38}$$

We also introduce $\mathcal{J}$ where

$$\mathcal{J} \equiv \int_{t_o}^{t} \frac{\cent}{r^3}\, dt \tag{4.39}$$

From (4.32), (2.6), (3.15b) we obtain

$$\frac{\partial}{\partial t}\left[\left(1 + \frac{r_o}{r}\right)\frac{\cent}{r}\right] = \frac{r_o\, g}{r^4} + \frac{1}{r^4}\left(r\, g - r_o\, B\, \cent\right)$$

$$= 2\,\frac{r_o\, g}{r^4} + \frac{B_o\, \cent}{r^3} \tag{4.40}$$

Therefore

$$\int_{t_o}^{t} \frac{2\, g}{r^4}\, dt = \frac{1}{r_o}\left[\left(1 + \frac{r_o}{r}\right)\frac{\cent}{r} - B_o \mathcal{J}\right] \tag{4.41}$$

Also, from (2.6), (3.20) we obtain

$$\frac{\partial}{\partial t}\left(\frac{g}{r}\right) = \frac{\dot{g}}{r} - \frac{B\, g}{r^3} = \frac{u}{r^3} \tag{4.42}$$

Thus

$$\int_{t_o}^{t} \frac{u}{r^3}\, dt = \frac{g}{r} \tag{4.43}$$

Then from (4.39), (2.32a), (2.26b), (4.43):

$$L^2\, \mathcal{J} = \int_{t_o}^{t} \frac{(r\, r_o - u)}{r^3}\, dt = r_o\, \textcircled{H} - \frac{g}{r} \tag{4.44}$$

5. SOME ADDITIONAL QUANTITIES AND RELATIONS

We shall find it useful to introduce some additional quantities and develop related identities as follows. From (4.8a), (4.9b), (4.15), (2.29b) we obtain

$$B_o\rho = B_o\tau\text{¢} - B_o r_o \$ + (\sigma - \tau g - \mu r_o \text{¢}')$$

$$= \sigma - r_o (\tau S + \mu\text{¢}' + B_o\$)$$

$$= \sigma - r_o \left[(r + r_o)\,\text{¢} + B_o\$\right] \tag{5.1}$$

where we have used

$$\tau S + \mu\text{¢}' = Sg + \mu\text{¢}^2 \tag{5.2a}$$

$$= r_o S^2 + \text{¢}(r - r_o C)$$

$$= (r + r_o)\,\text{¢} \tag{5.2b}$$

To prove (5.2) we utilize (2.45b), (C.9c), (2.29b), (2.19a), (C.12a).

Then using (4.9b), (4.10c) we write (5.1) as follows

$$\sigma = B_o\Upsilon + r_o(r + r_o)\,\text{¢} \tag{5.3a}$$

$$= \frac{r_o}{\mu}\left[B_o(\zeta - g) + \mu\,(r + r_o)\,\text{¢}\right] \tag{5.3b}$$

$$= \frac{r_o}{\mu}\left(B_o\zeta + r_o^{\,2} - \Omega\right) \tag{5.3c}$$

where

$$\Omega \equiv r_o^2 - \mu(r + r_o) ¢ + B_o \cdot g \tag{5.4a}$$

$$= r_o^2 + B_o \zeta - \frac{\mu\sigma}{r_o} \tag{5.4b}$$

$$= r\, r_o - (\mu r + L^2) ¢ \tag{5.4c}$$

$$= u - \mu r ¢ \tag{5.4d}$$

$$= r^2 - \Delta \tag{5.4e}$$

$$= r\, r_o f - L^2 ¢ \tag{5.4f}$$

$$= r^2(2\dot{g} - 1) - Bg \tag{5.4g}$$

To obtain the above results we have utilized (2.32) to write

$$B_o g = u - r_o^2 f \tag{5.5a}$$

$$= r_o(r - r_o) + \left(\mu r_o - L^2\right) ¢ \tag{5.5b}$$

Taking the transform of (5.5) we obtain

$$Bg = r^2 \dot{g} - u \tag{5.6a}$$

$$= r(r - r_o) - (\mu r - L^2) ¢ \tag{5.6b}$$

Combining (5.5b), (5.6b) then

$$(B + B_o)g = r^2 - r_o^2 - \mu(r - r_o) ¢ \tag{5.7a}$$

$$(B - B_o)g = (r - r_o)^2 + \left[2L^2 - \mu(r + r_o)\right] ¢ \tag{5.7b}$$

From (C.9a), (C.10a), (2.29b)

$$\alpha(r_o \text{¢}' + B_o \$') = r_o(\psi S - 2\text{¢}) + B_o(\psi \text{¢} - 3\$)$$

$$= \psi g - (2r_o \text{¢} + 3B_o \$) \tag{5.8}$$

Then utilizing (4.15), (5.8), (2.45b), (5.5b), (5.6b), (4.35d)

$$\alpha\sigma = \alpha\left[\tau g + \mu(r_o \text{¢}' - B_o \$')\right]$$

$$= g(\alpha\tau + \mu\psi) - 2\mu r_o \text{¢} - 3B_o(\tau - g)$$

$$= Bg + 2B_o g - 2\mu r_o \text{¢} - 3B_o \tau$$

$$= 2\left(r^2 - r_o^2\right) - \Delta - 3B_o\tau \tag{5.9}$$

where we have used

$$B = B_o + \mu S + \alpha(\tau - \mu\$)$$

$$= B_o + \alpha\tau + \mu\psi \tag{5.10}$$

To prove (5.10) we utilize (4.4), (2.45b), (C.2a). From (1.11), (5.9), (5.3c), (4.14a), (5.4e) we obtain

$$v_o^2\sigma = \left(\alpha + \frac{2\mu}{r_o}\right)\sigma$$

$$= \Delta - B_o\eta \tag{5.11}$$

so that

$$\Delta = B_o\eta + v_o^2\,\sigma \tag{5.12a}$$

$$\overline{\Delta} = B\overline{\eta} + v^2\,\overline{\sigma} \tag{5.12b}$$

We introduce Γ, where

$$\Gamma \equiv B_o \sigma + r_o^2 \eta \tag{5.13}$$

Then using (1.12), (5.11) we obtain

$$B_o \Gamma = r_o^2 \Delta - L^2 \sigma \tag{5.14a}$$

and

$$B \overline{\Gamma} = r^2 \overline{\Delta} - L^2 \overline{\sigma} \tag{5.14b}$$

Recalling (4.11d), (4.20b), (3.14), (5.2), (2.45b), (5.6a), (2.29)

$$r_o B \eta \; - 2\mu\overline{\sigma} = r_o B \left[\left(1 - \frac{2\mu}{r_o} ₵ \right) \tau + 2\mu \$ \right] - 2\mu(\tau g + \mu r ₵')$$

$$= r_o B \,(\tau + 2\mu \$) - 2\mu r(r + r_o)\, ₵$$

$$= 3 r_o B \tau - 2\left[r_o B g + \mu r (r + r_o)\, ₵ \right]$$

$$= 3\, r_o B \tau + 2 r_o \left[u - r^2 (2 - f) \right] \tag{5.15}$$

Then using (4.14b), we write (5.15) as

$$\overline{\sigma} = \frac{r_o}{\mu} \left[r^2 (2 - f) - B\zeta - u \right] \tag{5.16a}$$

$$= \frac{r_o}{\mu} (\overline{\gamma} - u) \tag{5.16b}$$

where using (5.16), (3.20b)

$$\gamma \equiv r_o^2 (2 - \dot{g}) - B_o \overline{\zeta} \tag{5.17a}$$

$$= u + \frac{\mu \sigma}{r} \tag{5.17b}$$

$$= -Bg + r^2 \vartheta \tag{5.17c}$$

with

$$\mathcal{J} \equiv \mathcal{J}(t_o, t) = \dot{g} + \frac{\mu\sigma}{r^3} \tag{5.18a}$$

$$\overline{\mathcal{J}} = \mathcal{J}(t, t_o) = f + \frac{\mu\overline{\sigma}}{r_o^3} \tag{5.18b}$$

and

$$\overline{\gamma} = r^2(2 - f) - B\zeta \tag{5.19a}$$

$$= u + \frac{\mu\overline{\sigma}}{r_o} \tag{5.19b}$$

$$= B_o g + r_o^2 \overline{\mathcal{J}} \tag{5.19c}$$

Taking the transform of (5.9) and using (5.16a), (4.14a), (4.36e) we obtain

$$v_o^2\overline{\sigma} = \left(\alpha + \frac{2\mu}{r_o}\right)\overline{\sigma}$$

$$= B\eta - \overline{\Delta} + 2r^2(1 - f) + 2\left(r_o^2 - u\right)$$

$$= B\eta - B_o g + 2r^2(1 - f) \tag{5.20}$$

Recalling (5.13) and using (5.20), (4.31), (1.12), (4.36b) we obtain

$$B\Gamma = B_o(B\sigma) + r_o^2 B\eta$$

$$= L^2\overline{\sigma} - r^2\,\overline{\Delta} \tag{5.21}$$

Comparing (5.21) with (5.14b) we see that

$$\overline{\Gamma} = B\overline{\sigma} + r^2\overline{\eta} \tag{5.22a}$$

$$= -\Gamma \tag{5.22b}$$

$$= -\left(B_o\sigma + r_o^2\eta\right) \tag{5.22c}$$

Then

$$B\bar{\sigma} + B_o\sigma = -\left(r^2\bar{\eta} + r_o^2\eta\right) \tag{5.23}$$

Using (4.7c), (4.8b) in (4.19d) and recalling (1.11) we obtain

$$\bar{\eta} = -g + \mu(\alpha\Lambda - S\text{¢}) + \frac{2\mu}{r}\left(g\text{¢} + \mu\Lambda\right)$$

$$= \mu v^2\Lambda - g\left(1 - \frac{\mu\text{¢}}{r}\right) + \frac{\mu}{r}\left(g - rS\right)\text{¢} \tag{5.24}$$

Utilizing (2.29d), (3.14) we then write (5.24) as

$$\bar{\eta} = \mu\left(v^2\Lambda - \frac{B\text{¢}^2}{r}\right) - g\,\dot{g} \tag{5.25}$$

From (C.13d), (3.14), (4.7a), (4.20b) we obtain

$$\mu(r\text{¢}^2 - \Lambda B) = \mu r(S\$ + \text{¢}') - \mu B(\text{¢}\$ + \$')$$

$$= \mu(r\text{¢}' - B\$') + \mu\$(rS - B\text{¢})$$

$$= \bar{\sigma} - \tau g + \mu\$ g \tag{5.26}$$

Then using (2.45b) we write (5.26) as

$$\bar{\sigma} = g^2 + \mu(r\text{¢}^2 - \Lambda B) \tag{5.27}$$

Substituting (5.25), (5.27) into (5.22) and using (1.12), (3.20b) we obtain

$$\Gamma = ug - \mu L^2\Lambda \tag{5.28}$$

Another important identity is the following

$$\frac{\mu\lambda}{r_o} + r_o^2B - L^2g = B_o\frac{\mu\bar{\sigma}}{r_o} + r_o^2B + (\mu r_o - L^2)\,g$$

$$= B_o(\bar{\gamma} - u) + r_o^2B_of + B_o^2g$$

$$= B_o\bar{\gamma} \tag{5.29}$$

where we have utilized (4.30), (5.16b), (2.5), (4.4), (2.32b).

A result that is important in many spaceflight applications is Lambert's theorem. This states that, with μ given, τ is a function only of $r + r_o$, d and a, where d is the chord length as in (2.47). We shall discuss this important topic only briefly. For a detailed discussion see Reference 3. To proceed we combine (2.47), (5.7b) to obtain

$$(B - B_o)\, g = d^2 - \mu(r + r_o)\, \not{c} \tag{5.30}$$

Also adding (2.19b) to its reverse transform yields

$$(B - B_o) S = \left[2\mu + a(r + r_o)\right] \not{c} \tag{5.31}$$

Substituting $(r + r_o)\, \not{c}$ from (5.30) into (5.31) and recalling (4.4) result in

$$\left(B - B_o\right)^2 = ad^2 + 2\mu^2 \not{c} \tag{5.32}$$

Thus from (5.10), (5.31), (5.32) we obtain

$$(B - B_o) = \mu\psi + a\tau \qquad = \frac{\mu\sqrt{z}}{\sqrt{a}} + a\tau \tag{5.33a}$$

$$= \left[2\mu + a(r + r_o)\right] \not{c}/S \qquad = \frac{1}{\sqrt{a}} \left[2\mu + a(r + r_o)\right] \frac{\sqrt{z}\, \widehat{\not{c}}(z)}{\widehat{S}(z)} \tag{5.33b}$$

$$= \pm\sqrt{ad^2 + 2\mu^2 \not{c}} \qquad = \pm\sqrt{ad^2 + \frac{2\mu^2}{a} z\, \widehat{\not{c}}(z)} \tag{5.33c}$$

where we have recalled the definitions in Appendix A with $\widehat{S}(z)$, $\widehat{\not{c}}(z)$ given by (A.12) and

$$\sqrt{z} = \sqrt{a}\,\psi \tag{5.34a}$$

$$z = a\psi^2 \tag{5.34b}$$

The relationship between τ, $r + r_o$, α, d is an implicit one involving the quantity z. To determine τ, d, or $r + r_o$ with α and the other two given we equate the two appropriate equations in (5.33) to solve for z. Evaluating $B - B_o$ we then solve the remaining equation in (5.33) for the desired variable. For τ, d, $r + r_o$ given and α required, then (5.33) represents two equations to be solved for the two unknowns α, z. In proceeding as above there is a possibility of more than one solution for z or even none. Thus the removal of any ambiguity will depend on the particular application.

6. ADDITIONAL TIME DERIVATIVES

The time derivatives of ρ, ζ, η, σ are obtained directly from the expressions (4. 9a), (4. 10a), (4. 11a), and (4. 12). The time derivatives of the transformed quantities $\bar{\rho}$, $\bar{\zeta}$, $\bar{\eta}$, $\bar{\sigma}$, are also needed and we give them below. We first obtain $\dot{\Upsilon}$ as follows.

$$\frac{d\Upsilon}{dt} - \frac{d}{dt}\left(\tau \text{₵} + \mu \$'\right)$$

$$= \text{₵} + \frac{\tau S}{r} + \frac{\mu \text{₵}'}{r}$$

$$= \left(2 + \frac{r_o}{r}\right) \text{₵} \tag{6. 1}$$

where we have utilized (5. 2). From (4. 17b) we obtain

$$\frac{d\bar{\rho}}{dt} = \dot{r}\$ + \text{₵} - \dot{\Upsilon}$$

$$= \frac{1}{r}\left[B\$ - (r + r_o)\, \text{₵}\right] \tag{6. 2}$$

Then using (6. 2), (2. 6), (4. 17b), (4.8a), (3. 14), (5. 2), and (4. 20b)

$$\frac{\partial}{\partial t}\left(\frac{\bar{\rho}}{r}\right) = \frac{1}{r^2}\left[B\$ - (r + r_o)\, \text{₵}\right] - \frac{B}{r^3}\left(-\tau \text{₵} + r\$ - \mu \$'\right)$$

$$= \frac{1}{r^3}\left\{\tau(B\text{₵} - rs) + r\left[\tau S - (r + r_o)\, \text{₵}\right] + B\mu \$'\right\}$$

$$= -\frac{\bar{\sigma}}{r^3} \tag{6. 3}$$

Using (6. 3) in conjunction with (4. 18b), (4. 19b) we obtain

$$\frac{\partial \bar{\zeta}}{\partial t} = -\left(\frac{\mu \bar{\sigma}}{r^3} + 1\right) \tag{6. 4a}$$

$$\frac{\partial \bar{\eta}}{\partial t} = \frac{2\mu \bar{\sigma}}{r^3} - 1 \tag{6. 4b}$$

Differentiating (4. 31) and using (4. 12), (2. 6), (2. 7), (4. 22), and (4. 31) we obtain

$$\frac{\partial\overline{\sigma}}{\partial t} = \frac{1}{B_o}\frac{\partial}{\partial t}\left[\left(r^2 - r_o^2\right)g - B\sigma\right]$$

$$= \frac{1}{B_o}\left[\left(r^2 - r_o^2\right)\dot{g} - \left(a + \frac{\mu}{r}\right)\sigma\right] \tag{6. 5a}$$

$$= \frac{1}{B}\left[\left(r^2 - r_o^2\right) + \overline{\sigma}\left(a + \frac{\mu}{r}\right)\right] \tag{6. 5b}$$

We may also obtain $\dot{\overline{\sigma}}$ by differentiating the integral in (4. 20a), and this is shown below to illustrate the process.

$$\frac{\partial\overline{\sigma}}{\partial t} = \frac{\partial}{\partial t}\int_{\tau=t_o}^{t} 2g(\tau, t)\, d\tau$$

$$= 2\int_{\tau=t_o}^{t} \dot{g}(\tau, t)\, d\tau$$

$$= 2\tau - \frac{2\mu}{r(t)}\int_{\tau=t_o}^{t} ¢\left[\psi(t, \tau)\right] d\tau$$

$$= 2\tau + \frac{\mu\overline{\rho}}{r} \tag{6. 6a}$$

$$= -\overline{\zeta} - \overline{\eta} \tag{6. 6b}$$

where we have utilized (3. 11), (2. 29d), (4. 17a), (4. 18b), and (4. 19b). Combining (6. 6b) and (6. 5b) we write

$$B(\overline{\zeta} + \overline{\eta}) = \left(r_o^2 - r^2\right) - \left(a + \frac{\mu}{r}\right)\overline{\sigma}$$

The result (6. 7) could also be obtained from (5. 3c), (5. 4e), (5. 9), and (4. 14a).

From (4. 30) and (6. 5a) we obtain

$$\frac{\partial\lambda}{\partial t} = r_o^2 \dot{g} + B_o \frac{\partial\bar{\sigma}}{\partial t}$$

$$= r^2 \dot{g} - \left(\alpha + \frac{\mu}{r} \right) \sigma \qquad (6.8)$$

Also, using (6. 8), (2. 6), (4. 25), (5. 18a), and (5. 17c)

$$\frac{\partial}{\partial t} \left(\frac{\lambda}{r} \right) = \frac{1}{r} \left[r^2 \dot{g} - \left(\alpha + \frac{\mu}{r} \right) \sigma \right] - \frac{B}{r^3} \left(r^2 g - B\sigma \right)$$

$$= r \vartheta - \frac{B}{r} g - \frac{L^2 \sigma}{r^3} \qquad (6.9a)$$

$$= \frac{\gamma}{r} - \frac{L^2 \sigma}{r^3} \qquad (6.9b)$$

We shall also evaluate the time integrals given below. Recalling (4. 10a), (2. 6), (5. 19a):

$$\frac{\partial}{\partial t} \left[\frac{\zeta(t_o, t)}{r(t)^2} \right] = \frac{1}{r^2} (3 - 2f) - \frac{2B}{r^4} \zeta \qquad (6.10a)$$

$$= \frac{2\bar{\gamma}}{r^4} - \frac{1}{r^2} \qquad (6.10b)$$

Therefore

$$\int_{\tau = t_o}^{t} \frac{1}{r^2} \left[(3 - 2f) - \frac{2B}{r^2} \zeta \right] d\tau = \frac{\zeta}{r^2} \qquad (6.11)$$

Recalling (2. 26a) and using (6. 10b), we write

$$2 \int_{t_o}^{t} \frac{\bar{\gamma}}{r^4} dt = \textcircled{H} + \frac{1}{r^2} \zeta \qquad (6.12)$$

Using (6.5b), (2.7)

$$\frac{\partial}{\partial t}(B\bar{\sigma}) = \left(r^2 - r_o^2\right) + 2\bar{\sigma}\left(a + \frac{\mu}{r}\right) \tag{6.13}$$

and from (6.13), (2.6), (2.5), and (6.4b):

$$\frac{\partial}{\partial t}\left(\frac{B\bar{\sigma}}{r^2}\right) = \frac{1}{r^2}\left[\left(r^2 - r_o^2\right) + 2\bar{\sigma}\left(a + \frac{\mu}{r}\right)\right] - \frac{2\bar{\sigma}}{r^4}B^2$$

$$= 1 - \left(\frac{r_o}{r}\right)^2 + \frac{2\bar{\sigma}}{r^4}(L^2 - \mu r) \tag{6.14a}$$

$$= \frac{2\bar{\sigma}L^2}{r^4} - \left(\frac{r_o}{r}\right)^2 - \frac{\partial\bar{\eta}}{\partial t} \tag{6.14b}$$

Then using (5.22), we write (6.14) as

$$\frac{\partial}{\partial t}\left(\frac{\Gamma}{r^2}\right) = \frac{r_o^2}{r^2} - \frac{2L^2\bar{\sigma}}{r^4} \tag{6.15}$$

Therefore, recalling (2.26b) we obtain

$$L^2 \int_{\tau=t_o}^{t} \frac{2\bar{\sigma}(t_o, \tau)}{r(\tau)^4} d\tau = L^2 \int_{\tau=t_o}^{t} \frac{2\sigma(\tau, t_o)}{r(\tau)^4} d\tau$$

$$= r_o^2 \, Ⓗ - \frac{\Gamma}{r^2} \tag{6.16}$$

APPENDIX A

HERRICK-LEMMON FUNCTIONS

The following discussion of functions and their relationships represents a recapitulation of Reference 4.

Definitions

For $n \geq 0$, the nth order Herrick-Lemmon function U_n is defined as

$$U_n = U_n(\psi, \alpha) = \psi^n u_n(z) \tag{A.1}$$

where

$$\sqrt{z} = \sqrt{\alpha}\,\psi \tag{A.2a}$$

$$z = \alpha\psi^2 \tag{A.2b}$$

and

$$u_n(z) = \sum_{j=o}^{\infty} \frac{z^j}{(n+2j)!} \tag{A.3}$$

The series (A.3) is convergent for all values of z. Using it in (A.1) we obtain

$$U_n(\psi, \alpha) = \psi^n \sum_{j=o}^{\infty} \frac{\alpha^j \psi^{2j}}{(2j+n)!} \tag{A.4}$$

We denote the even functions by $¢_n$ and the odd functions by $\$_n$ where

For $n \geq 0$:

$$¢_n(\psi, \alpha) = ¢_n(-\psi, \alpha) = U_{2n}(\psi, \alpha) = \sum_{j=n}^{\infty} \frac{\alpha^{j-n}\psi^{2j}}{(2j)!} \tag{A.5a}$$

$$\$_n(\psi, \alpha) = -\$_n(-\psi, \alpha) = U_{2n+1}(\psi, \alpha) = \sum_{j=n}^{\infty} \frac{\alpha^{j-n}\psi^{2j+1}}{(2j+1)!} \tag{A.5b}$$

Finite Form

We may express $u_n(z)$ in finite form in terms of truncated trigonometry functions as follows:

For the even numbered functions with $n = 2k$:

k = 0:

$$u_o(z) = \frac{1}{2}\left(e^{\sqrt{z}} + e^{-\sqrt{z}}\right) \quad \text{(A.6a)}$$

$z \geq 0(a \geq 0)$:

$$u_o(z) = \cosh |z|^{1/2} \quad \text{(A.6b)}$$

$z \leq 0(a \leq 0)$:

$$u_o(z) = \cos |z|^{1/2} \quad \text{(A.6c)}$$

k ≥ 1:

$$u_{2k}(z) = \frac{1}{z^k}\left[u_o(z) - \sum_{j=o}^{k-1} \frac{z^j}{(2j)!}\right] \quad \text{(A.7)}$$

For the odd numbered functions with $n = 2k + 1$:

k = 0:

$$u_1(z) = \frac{1}{2\sqrt{z}}\left(e^{\sqrt{z}} - e^{-\sqrt{z}}\right) \quad \text{(A.8a)}$$

$z \geq 0(a \geq 0)$:

$$u_1(z) = \frac{1}{|z|^{1/2}} \sinh |z|^{1/2} \quad \text{(A.8b)}$$

$z \leq 0(a \leq 0)$:

$$u_1(z) = \frac{1}{|z|^{1/2}} \sin |z|^{1/2} \quad \text{(A.8c)}$$

k ≥ 1:

$$u_{2k+1}(z) = \frac{1}{z^k}\left[u_1(z) - \sum_{j=o}^{k-1} \frac{z^j}{(2j+1)!}\right] \quad \text{(A.9a)}$$

For $a > 0$, $\sqrt{z} = |a|^{1/2}\psi$ is real and we may use the expressions in terms of hyperbolic functions. For $a < 0$, $\sqrt{z} = i|a|^{1/2}\psi$ is a pure imaginary number and we may use the expressions in terms of ordinary trigonometry functions. The series expressions (A.3), (A.4) of course are applicable to either case.

For $\alpha = 0$, we note

$$u_n(0) = \frac{1}{n!} \; ; \; U_n(\psi, 0) = \frac{\psi^n}{n!} \qquad \text{(A. 10a)}$$

Also

$$U_0(0, \alpha) = 1 \qquad \text{(A. 10b)}$$

and for $n \geq 1$

$$U_n(0, \alpha) = 0 \qquad \text{(A. 10c)}$$

The function U_0 - U_5 will be utilized more than the others and are given the following special notation:

	$z \leq 0 (\alpha \leq 0)$	$z \geq 0 (\alpha \geq 0)$	
$C = C(\psi, \alpha) = U_0(\psi, \alpha) =$	$\cos y$	$\cosh y$	(A. 11a)
$S = S(\psi, \alpha) = U_1(\psi, \alpha) =$	$\lvert\alpha\rvert^{-1/2} \sin y$	$\alpha^{-1/2} \sinh y$	(A. 11b)
$¢ = ¢(\psi, \alpha) = U_2(\psi, \alpha) =$	$\lvert\alpha\rvert^{-1}(1 - \cos y)$	$\alpha^{-1}(\cosh y - 1)$	(A. 11c)
$\$ = \$(\psi, \alpha) = U_3(\psi, \alpha) =$	$\lvert\alpha\rvert^{-3/2}(y - \sin y)$	$\alpha^{-3/2}(\sinh y - y)$	(A. 11d)
$₡ = ₡(\psi, \alpha) = U_4(\psi, \alpha) =$	$\lvert\alpha\rvert^{-2}\left(\cos y - 1 + \frac{1}{2} y^2\right)$	$\alpha^{-2}\left(\cosh y - 1 - \frac{1}{2} y^2\right)$	(A. 11e)
$\$\!\!\$ = \$\!\!\$(\psi, \alpha) = U_5(\psi, \alpha) =$	$\lvert\alpha\rvert^{-5/2}\left(\sin y - y + \frac{1}{6} y^3\right)$;	$\alpha^{-5/2}\left(\sinh y - y - \frac{1}{6} y^3\right)$	(A. 11f)

where

$$y \equiv \sqrt{|z|} \qquad \text{(A. 12)}$$

The function u_0 - u_5 are also very important and are given the following special notation:

	$z \leq 0 (a \leq 0)$	$z \geq 0 (a \geq 0)$	
$\widehat{C} = \widehat{C}(z) = u_0(z) =$	$\cos y$	$\cosh y$	(A. 13a)
$\widehat{S} = \widehat{S}(z) = u_1(z) =$	$\frac{1}{y}\sin y$	$\frac{1}{y}\sinh y$	(A. 13b)
$\widehat{₵} = \widehat{₵}(z) = u_2(z) =$	$\frac{1}{y^2}(1 - \cos y)$	$\frac{1}{y^2}(\cosh y - 1)$	(A. 13c)
$\widehat{\$} = \widehat{\$}(z) = u_3(z) =$	$\frac{1}{y^3}(y - \sin y)$	$\frac{1}{y^3}(\sinh y - 1)$	(A. 13d)
$\widehat{₵\!\!\!|} = \widehat{₵\!\!\!|}(z) = u_4(z) =$	$\frac{1}{y^4}\left(\cos y - 1 + \frac{1}{2}y^2\right)$;	$\frac{1}{y^4}\left(\cosh y - 1 - \frac{1}{2}y^2\right)$	(A. 13e)
$\widehat{\$\!\!\!|} = \widehat{\$\!\!\!|}(z) = u_5(z) =$	$\frac{1}{y^5}\left(\sin y - y + \frac{1}{6}y^3\right)$;	$\frac{1}{y^5}\left(\sinh y - y - \frac{1}{6}y^3\right)$	(A. 13f)

where y is defined by (A. 12). The functions given by (A. 13) are plotted versus z in Figures A-1 to A-6. The quantity y is equal to the increment in the eccentric anomaly E for $z < 0$ and is equal to the increment in F, the counterpart of E, for hyperbolic trajectories. Thus for $z < 0$ we have complete orbits relative to the initial point when $z = -4n^2\pi^2$, $n = 1, 2, \cdots$. Also, an increment in the eccentric anomaly of 180 degrees requires passage through the next apsis point. This corresponds to $z = -(2n - 1)^2\pi^2$, $n = 1, 2, \cdots$.

Differentiation

$$\frac{\partial}{\partial\psi} U_0(\psi, a) = aU_1(\psi, a) \quad \text{(A. 14a)}$$

and for $n \geq 1$

$$\frac{\partial}{\partial\psi} U_n(\psi, a) = U_{n-1}(\psi, a) \quad \text{(A. 14b)}$$

Recursion Relationship

For $n \geq 0$

$$U_n(\psi, a) = \frac{\psi^n}{n!} + a\, U_{n+2}(\psi, a) \qquad \text{(A. 15)}$$

Addition Formula

$$C(\psi_1 \pm \psi_2) = C(\psi_1)C(\psi_2) \pm aS(\psi_1)S(\psi_2) \qquad \text{(A. 16a)}$$

$$S(\psi_1 \pm \psi_2) = S(\psi_1)C(\psi_2) \pm S(\psi_2)C(\psi_1) \qquad \text{(A. 16b)}$$

$$\begin{aligned} \text{¢}(\psi_1 \pm \psi_2) &= \text{¢}(\psi_1) + C(\psi_1)\text{¢}(\psi_2) \pm S(\psi_1)S(\psi_2) \\ &= \text{¢}(\psi_1) + \text{¢}(\psi_2) + a\text{¢}(\psi_1)\text{¢}(\psi_2) \pm S(\psi_1)S(\psi_2) \end{aligned} \qquad \text{(A. 16c)}$$

$$\begin{aligned} \$(\psi_1 \pm \psi_2) &= \$(\psi_1) \pm \psi_2\text{¢}(\psi_1) \pm C(\psi_1)\$(\psi_2) + S(\psi_1)\text{¢}(\psi_2) \\ &= \$(\psi_1) \pm \$(\psi_2) + S(\psi_1)\text{¢}(\psi_2) \pm S(\psi_2)\text{¢}(\psi_1) \end{aligned} \qquad \text{(A. 16d)}$$

$$\begin{aligned} \text{₵}(\psi_1 \pm \psi_2) &= \text{₵}(\psi_1) \pm \psi_2\$(\psi_1) + \frac{\psi_2^2}{2}\text{¢}(\psi_1) + C(\psi_1)\text{₵}(\psi_2) \pm S(\psi_1)\$(\psi_2) \\ &= \text{₵}(\psi_1) + \text{₵}(\psi_2) + \text{¢}(\psi_1)\text{¢}(\psi_2) \pm \left[\psi_1\$(\psi_2) + \psi_2\$(\psi_1) + a\$(\psi_1)\$(\psi_2)\right] \end{aligned}$$

(A. 16e)

$$\begin{aligned} \text{\$\$}(\psi_1 \pm \psi_2) &= \text{\$\$}(\psi_1) \pm \psi_2\text{₵}(\psi_1) + \frac{\psi_2^2}{2}\$(\psi_1) \pm \frac{\psi_2^3}{6}\text{¢}(\psi_1) \pm C(\psi_1)\text{\$\$}(\psi_2) + S(\psi_1)\text{₵}(\psi_2) \\ &= \text{\$\$}(\psi_1) \pm \text{\$\$}(\psi_2) + \psi_1\text{₵}(\psi_2) \pm \psi_2\text{₵}(\psi_1) + \$(\psi_1)\text{¢}(\psi_2) \pm \$(\psi_2)\text{¢}(\psi_1) \end{aligned}$$

(A. 16f)

In general, for $n \geq 2$:

For $n \geq 2$:

$$U_n(\psi_1 \pm \psi_2) = \sum_{j=o}^{n-2} (\pm 1)^j \frac{\psi_2^j}{j!} U_{n-j}(\psi_1) + (\pm 1)^n \left[U_o(\psi_1)U_n(\psi_2) \pm U_1(\psi_1)U_{n-1}(\psi_2)\right]$$

(A. 17)

Identities

$$C^2 - \alpha S^2 = 1 \tag{A.18a}$$

$$\hat{C}^2 - z\hat{S} = 1 \tag{A.18b}$$

For $m \geq 1$, $n \geq 2$

$$\frac{\partial}{\partial\psi}(U_m U_n - U_{m+1} U_{n-1}) = U_{m-1} U_n - U_{m+1} U_{n-2}$$

$$= \frac{\psi^{m-1}}{(m-1)!} U_n - \frac{\psi^{n-2}}{(n-2)!} U_{m+1} \tag{A.19}$$

for $n = m \geq 2$

$$\frac{\partial}{\partial\psi}\left(U_n^2 - U_{n+1} U_{n-1}\right) = U_{n-1} U_n - U_{n+1} U_{n-2}$$

$$= \frac{\psi^{n-1}}{(n-1)!} U_n - \frac{\psi^{n-2}}{(n-2)!} U_{n+1} \tag{A.20}$$

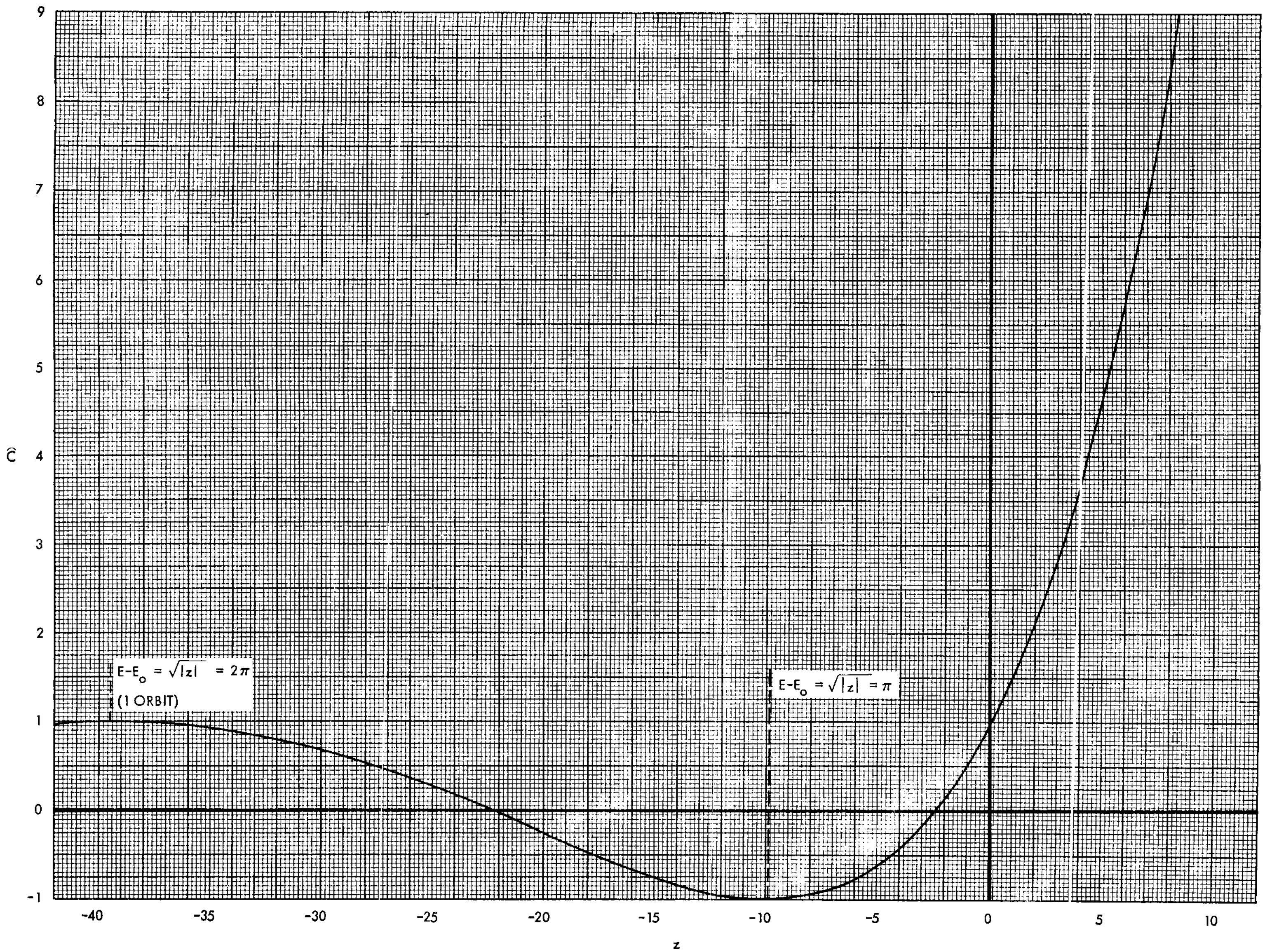


Figure A-1. The Function $\widehat{C}(z)$ Versus z

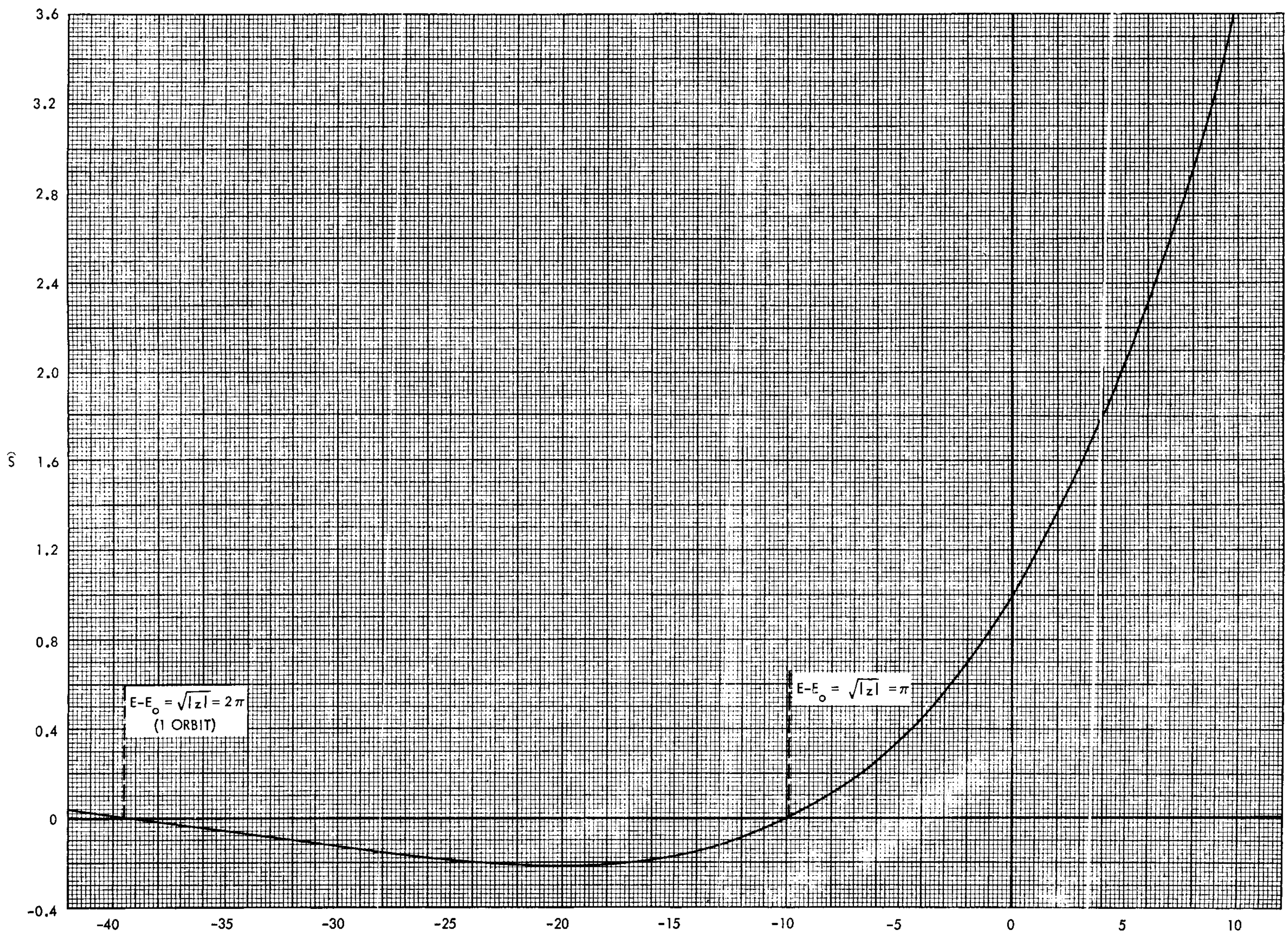


Figure A-2. The Function $\widehat{S}(z) = \widehat{C}'(z)$ Versus z

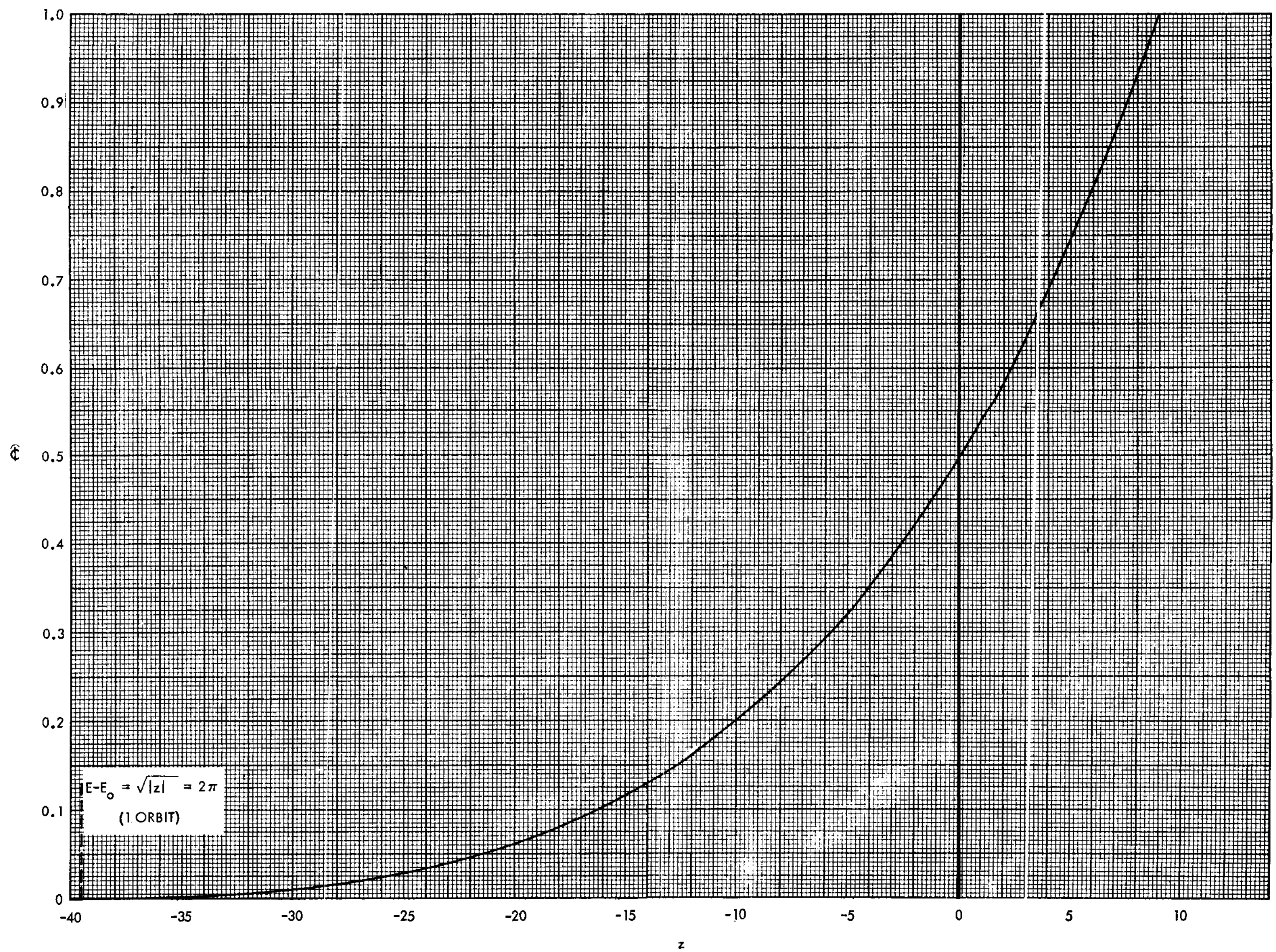


Figure A-3. The Function $\widehat{\mathfrak{C}}(z)$ Versus z

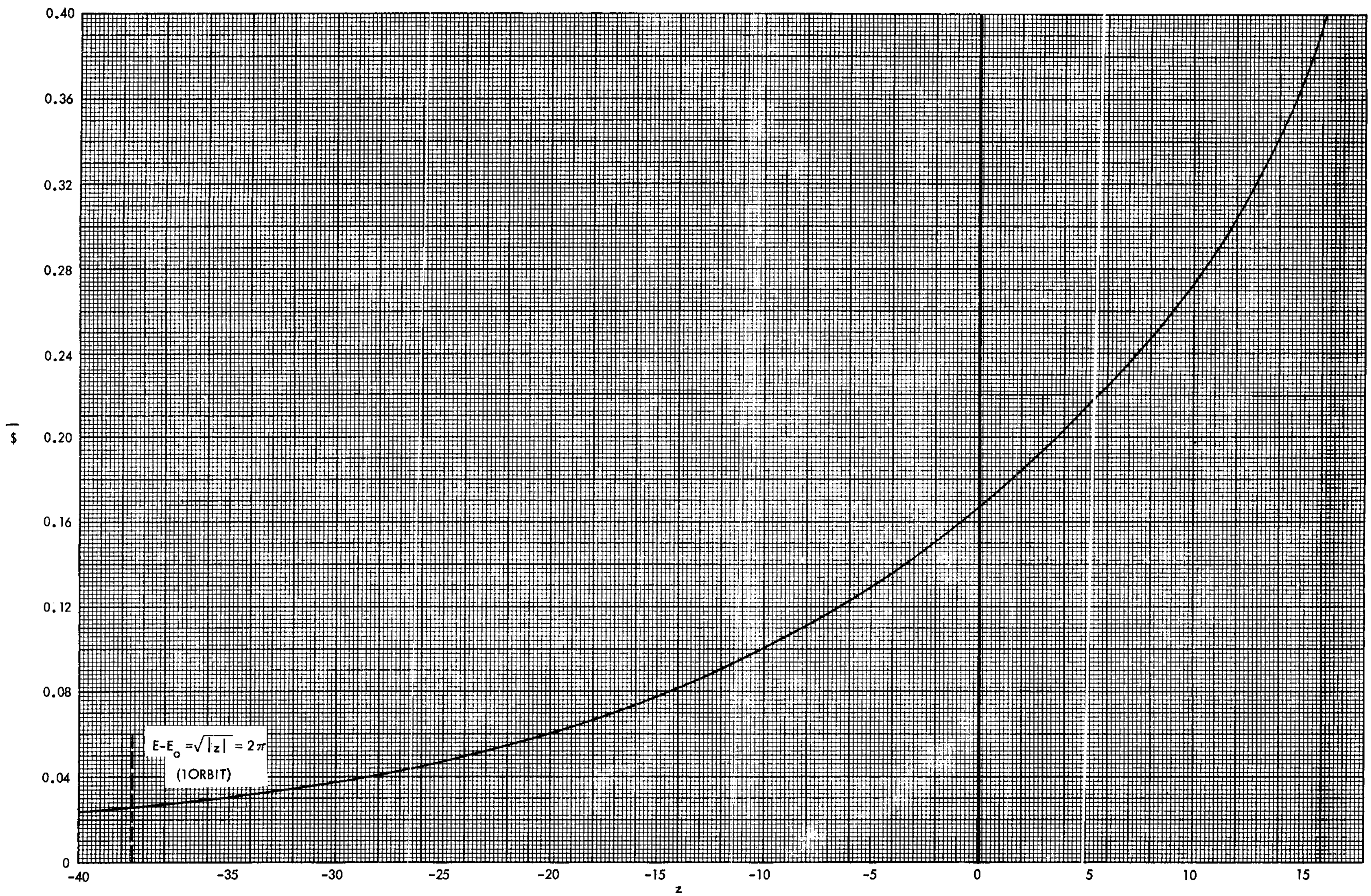


Figure A-4. The Function $\hat{\$}(z)$ Versus z

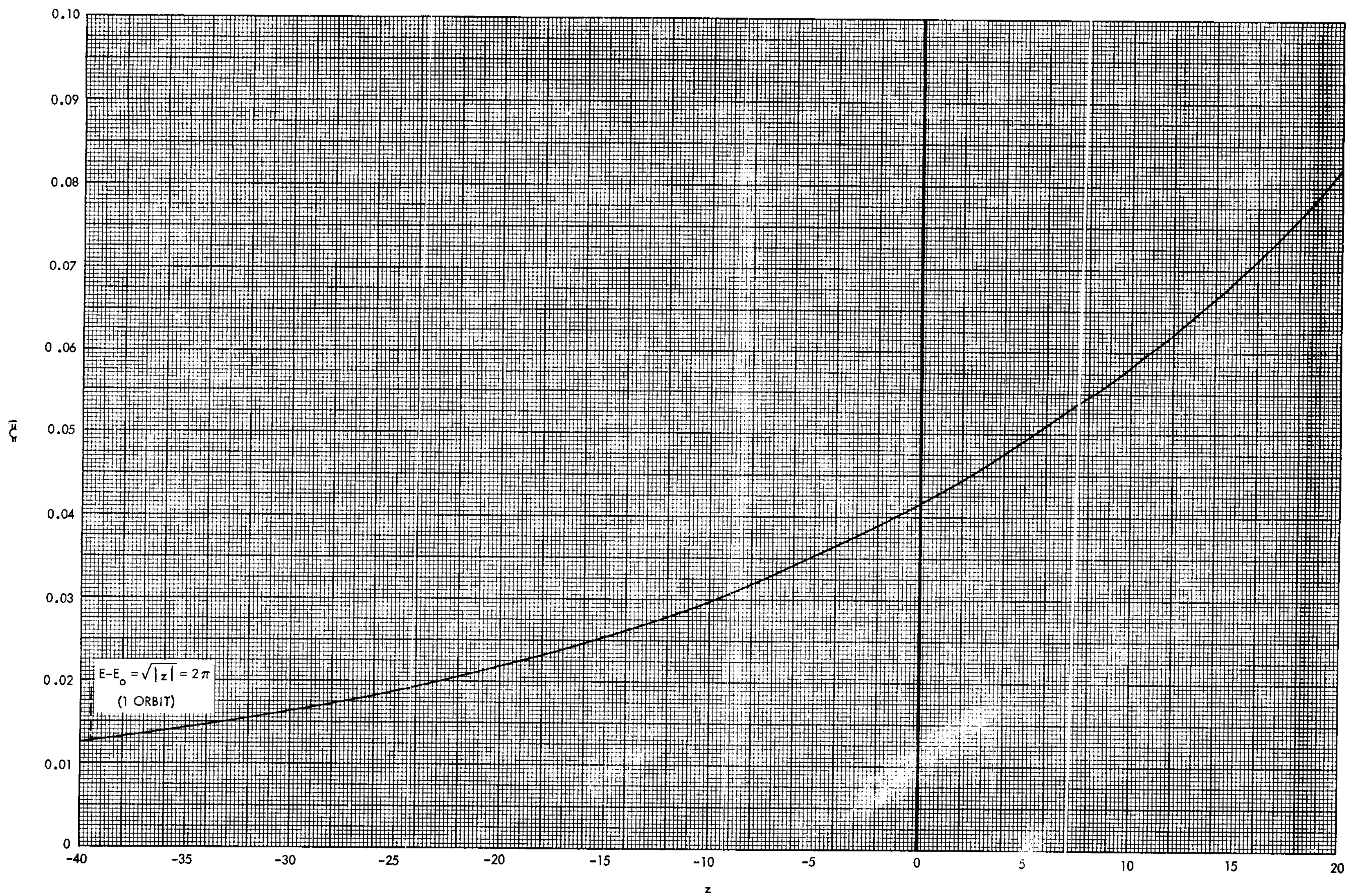


Figure A-5. The Function $\mathbb{C}(z)$ Versus z

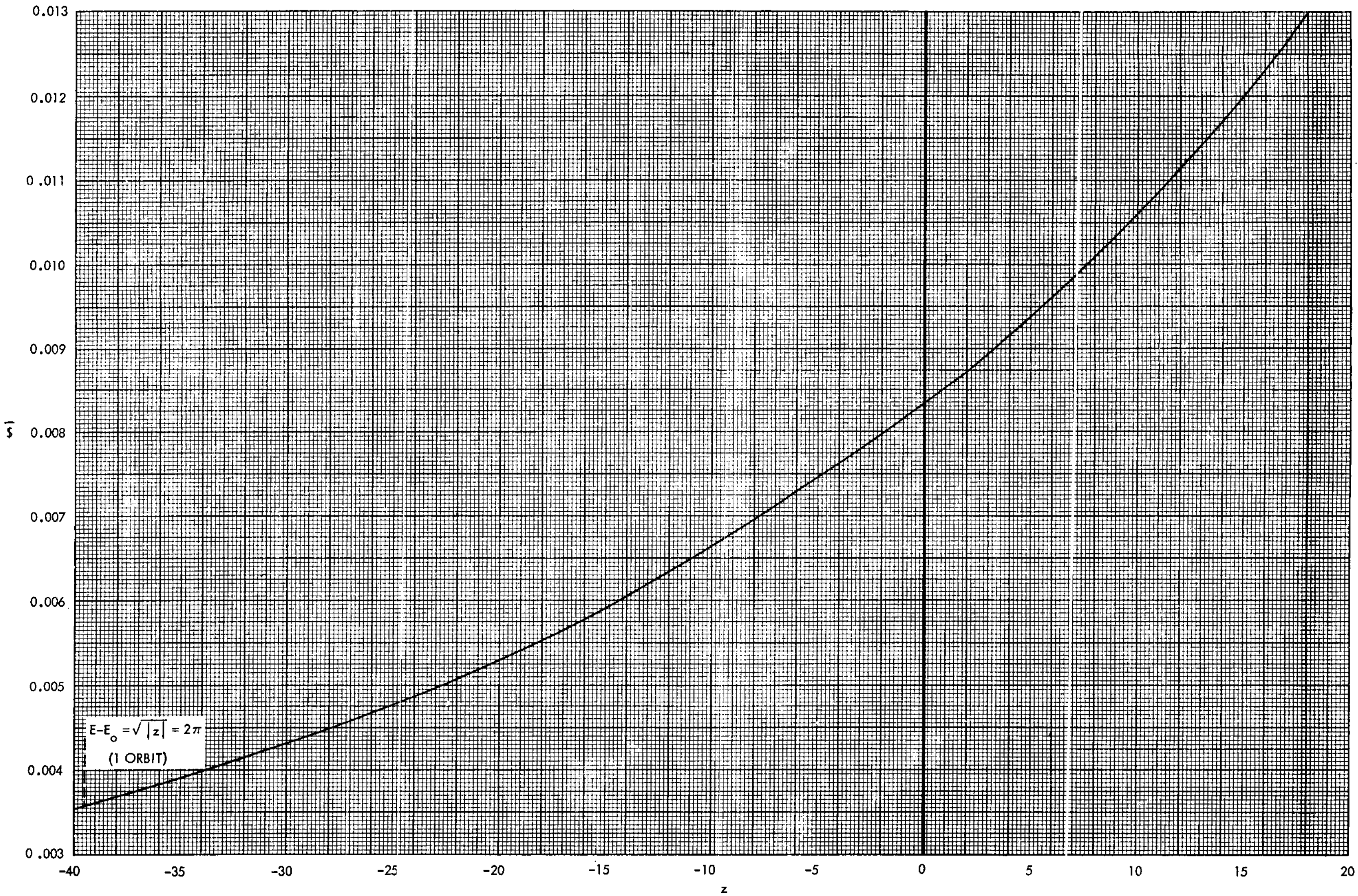


Figure A-6. The Function $\hat{\$}(z)$ Versus z

APPENDIX B

THE ASSOCIATED HERRICK-LEMMON FUNCTIONS

Additional functions from Reference 4 are discussed below.

Definitions

For $n \geq 0$: the nth order function U'_n is defined as

$$U'_n = U'(\psi, \alpha) \equiv 2 \frac{\partial U_n}{\partial \alpha} (\psi, \alpha) \tag{B.1a}$$

$$= \psi^{n+2} u'_n(z) \tag{B.1b}$$

with the function $u'_n(z)$ given by

$$u'_n(z) \equiv 2 \frac{d}{dz} u_n(z) = 2 \sum_{j=0}^{\infty} \frac{(j+1)\, z^j}{(2j+2+n)!} \tag{B.2}$$

in series form

$$U'_n(\psi, \alpha) = 2\psi^{n+2} \sum_{j=0}^{\infty} \frac{(j+1)\, \alpha^j \psi^{2j}}{(2j+2+n)!} \tag{B.3}$$

Recalling (A.5)

$$¢'_n = ¢'_n(\psi, \alpha) = U'_{2n}(\psi, \alpha) = 2 \sum_{j=n+1}^{\infty} \frac{(j-n)\, \alpha^{j-1-n} \psi^{2j}}{(2j)!} \tag{B.4a}$$

$$\$'_n = \$'_n(\psi, \alpha) = U'_{2n+1}(\psi, \alpha) = 2 \sum_{j=n+1}^{\infty} \frac{(j-n)\, \alpha^{j-1-n} \psi^{2j+1}}{(2j+1)!} \tag{B.4b}$$

Finite Form

Utilizing the finite form for u_n given in Appendix A we obtain the following results:

For the even numbered functions with $n = 2k$

$k = 0$:

$$u'_o(z) = u_1(z) \tag{B.5a}$$

$k \geq 1$:

$$u'_{2k}(z) = \frac{1}{z^{k+1}} = \left\{ \left[z\, u_1(z) - 2k u_0(z) \right] + 2 \sum_{j=0}^{k-1} \frac{(k-j)\, z^j}{(2j)!} \right\} \qquad \text{(B.5b)}$$

For the odd numbered functions with $n = 2k + 1$:

$k = 0$:

$$u'_1(z) = \frac{1}{z} \left[u_0(z) - u_1(z) \right] \qquad \text{(B.6a)}$$

$$u'_{2k+1}(z) = \frac{1}{z^{k+1}} \left\{ \left[u_0(z) - (2k+1)\, u_1(z) \right] + 2 \sum_{j=0}^{k-1} \frac{(k-j)\, z^j}{(2j+1)!} \right\} \qquad \text{(B.6b)}$$

for $\alpha > 0$, one makes use of hyperbolic functions and for $\alpha < 0$, ordinary trigonometry functions. For $\alpha = 0$ we note

$$u'_n(0) = \frac{2}{(n+2)!} \quad ; \quad U_n(\psi, 0) = \frac{2\psi^{n+2}}{(n+2)!} \qquad \text{(B.7)}$$

also, for $n \geq 0$

$$U'_n(0, \alpha) = 0 \qquad \text{(B.8)}$$

We utilize the notation C', S', etc., in keeping with (A.11).

Thus

	$z \leq 0$ ($\alpha \leq 0$)	$z \geq 0$ ($\alpha \geq 0$)

$$C' = C'(\psi, \alpha) = U'_0(\psi, \alpha) = |\alpha|^{-1}\, y \sin y \quad ; \quad \alpha^{-1}\, y \sinh y \qquad \text{(B.9a)}$$

$$S' = S'(\psi, \alpha) = U'_1(\psi, \alpha) = |\alpha|^{-3/2} (\sin y - y \cos y) \quad ; \quad \alpha^{-3/2} (y \cosh y - \sinh y) \qquad \text{(B.9b)}$$

$$\phi' = \phi'(\psi, \alpha) = U'_2(\psi, \alpha) = |\alpha|^{-2} \left[2(1 - \cos y) - y \sin y \right] \quad ; \quad \alpha^{-2} \left[y \sinh y - 2(\cosh y - 1) \right] \qquad \text{(B.9c)}$$

$$\$' = \$'(\psi, \alpha) = U_3'(\psi, \alpha)$$

$$= |\alpha|^{-5/2}\left[3(y - \sin y) - y(1 - \cos y)\right] \; ; \; \alpha^{-5/2}\left[y(\cosh y - 1) - 3(\sinh y - y)\right] \tag{B.9d}$$

where as in (A.12),

$$y = \sqrt{|z|} \tag{B.10}$$

We utilize $\widehat{C}'$, $\widehat{S}'$ etc., in keeping with (A.12). Thus

	$z \leq 0 \; (\alpha \leq 0)$	$z \geq 0 \; (\alpha \geq 0)$	
$\widehat{C}' = \widehat{C}'(z) = u_0'(z) =$	$\frac{1}{y} \sin y$	$\frac{1}{y} \sinh y$	(B.11a)
$\widehat{S}' = \widehat{S}'(z) = u_1'(z) =$	$\frac{1}{y^3} (\sin y - y \cos y)$	$\frac{1}{y^3} (y \cosh y - \sinh y)$	(B.11b)
$\widehat{\mathfrak{C}}' = \widehat{\mathfrak{C}}'(z) = u_2'(z) =$	$\frac{1}{y^4}\left[2(1 - \cos y) - y \sin y\right]$	$\frac{1}{y^4}\left[y \sinh y - 2(\cosh y - 1)\right]$	(B.11c)
$\widehat{\$}' = \widehat{\$}'(z) = u_3'(z) =$	$\frac{1}{y^5}\left[3(y - \sin y) - y(1 - \cos y)\right]$	$\frac{1}{y^5}\left[y(\cosh y - 1) - 3(\sinh y - y)\right]$	(B.11d)

The function $\widehat{C}'$ equals $\widehat{S}$ and is plotted in Figure A-2. The functions in (B.11b) - (B.11d) are plotted versus z in Figures B-1 to B-3.

<u>Differentiation</u>

$$\frac{\partial}{\partial \psi} U_0'(\psi, \alpha) = \alpha U_1'(\psi, \alpha) + 2U_1(\psi, \alpha) \tag{B.12a}$$

and for $n \geq 1$:

$$\frac{\partial}{\partial \psi} U_n'(\psi, \alpha) = U_{n-1}'(\psi, \alpha) \tag{B.12b}$$

<u>Identities</u>

For $n \geq 0$:

$$U_n' = -nU_{n+2} + \psi U_{n+1}' \tag{B.13}$$

$$U_n' = 2\,U_{n+2} + \alpha U_{n+2}' \tag{B.14}$$

$$\alpha U'_{n+1} = \psi U_n - (n + 1)\, U_{n+1} \tag{B.15}$$

$$\frac{\psi^{n-1}}{n!}\, U'_n = \frac{d}{d\psi}\left(U^2_{n+1} - U_{n+2} U_n\right) \tag{B.16}$$

<u>Addition Relations</u>

$$\begin{aligned} C'(\psi_1 + \psi_2) = {} & C(\psi_1)\, C'(\psi_2) + C(\psi_2)\, C'(\psi_1) + \alpha S(\psi_1)\, S'(\psi_2) \\ & + \alpha S(\psi_2)\, S'(\psi_1) + 2S(\psi_1)\, S(\psi_2) \end{aligned} \tag{B.17}$$

$$S'(\psi_1 + \psi_2) = C(\psi_1)\, S'(\psi_2) + C(\psi_2)\, S'(\psi_1) + S(\psi_1)\, C'(\psi_2) + S(\psi_2)\, C'(\psi_1) \tag{B.18}$$

$$\begin{aligned} \text{¢}'(\psi_1 + \psi_2) = {} & \text{¢}'(\psi_1) + \text{¢}'(\psi_2) + S(\psi_1)\, S'(\psi_2) + S(\psi_2)\, S'(\psi_1) \\ & + \text{¢}(\psi_1)\, C'(\psi_2) + \text{¢}(\psi_2)\, C'(\psi_1) - 2\text{¢}(\psi_1)\, \text{¢}(\psi_2) \end{aligned} \tag{B.19}$$

$$\begin{aligned} \$'(\psi_1 + \psi_2) = {} & \$'(\psi_1) + \$'(\psi_2) + S(\psi_1)\, \text{¢}'(\psi_2) + S(\psi_2)\, \text{¢}'(\psi_1) \\ & + \text{¢}(\psi_1)\, S'(\psi_2) + \text{¢}(\psi_2)\, S'(\psi_1) \end{aligned} \tag{B.20}$$

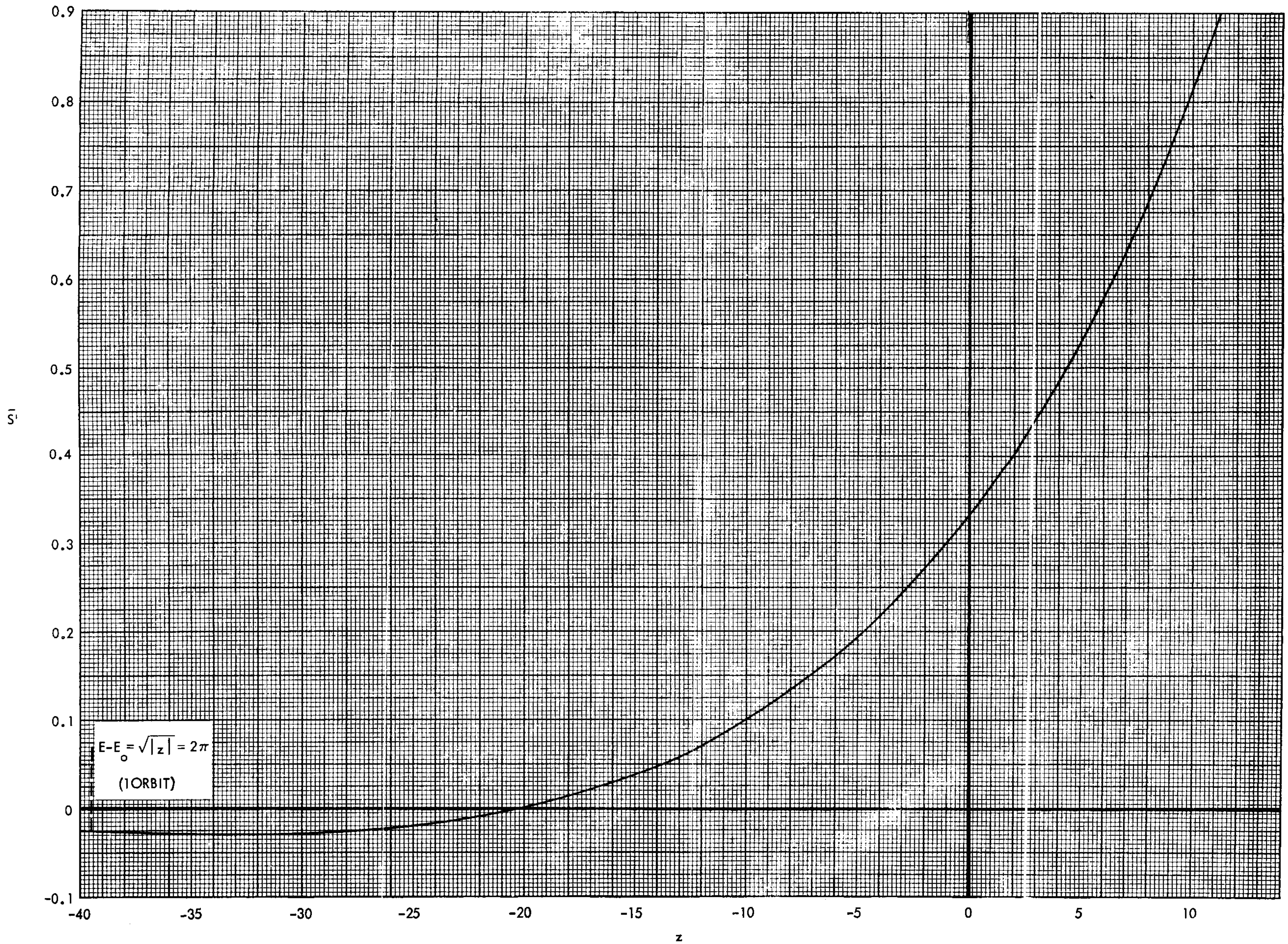


Figure B-1. The Function $\hat{S}'(z)$ Versus z

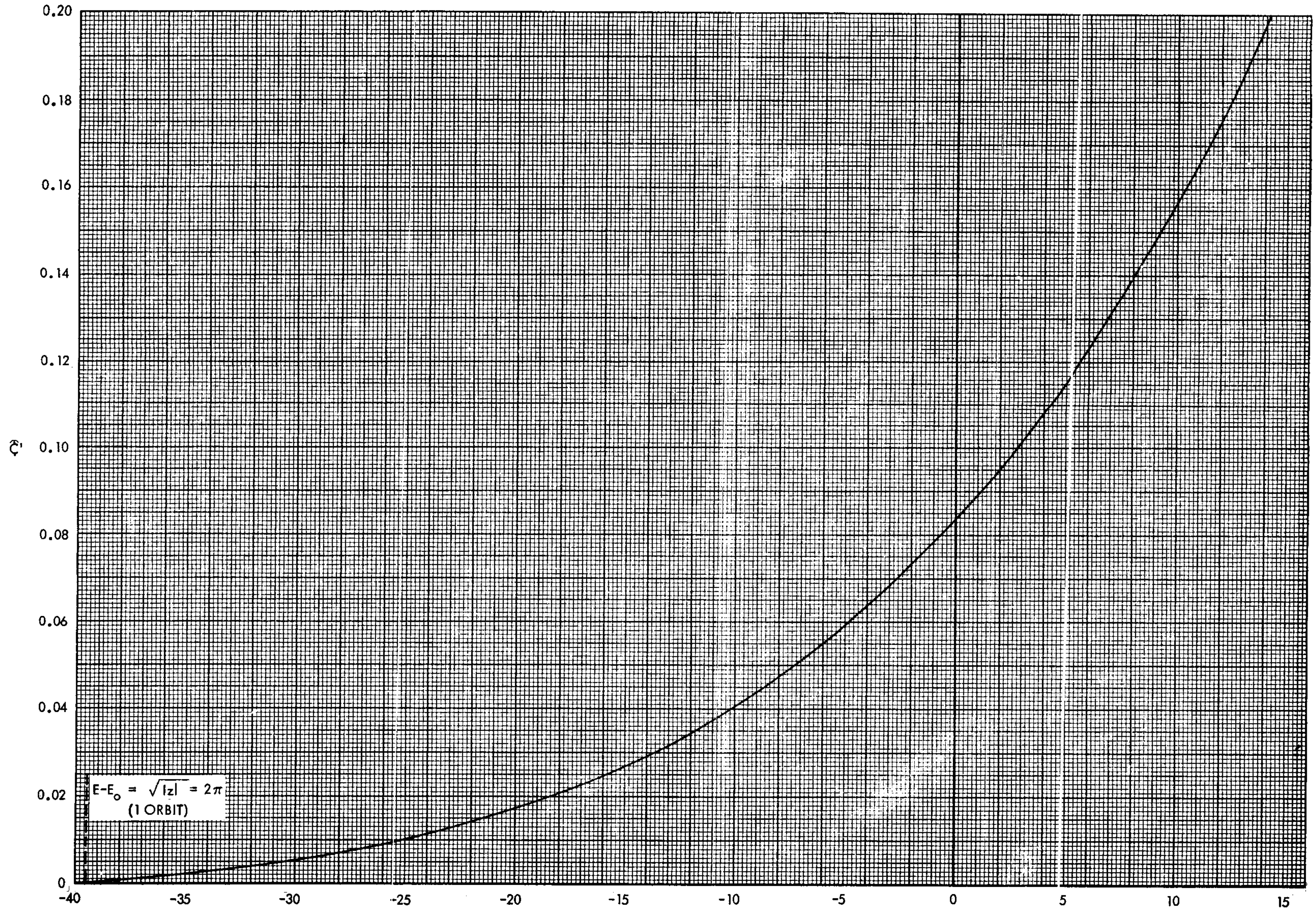


Figure B-2. The Function $\dot{\phi}'(z)$ Versus z

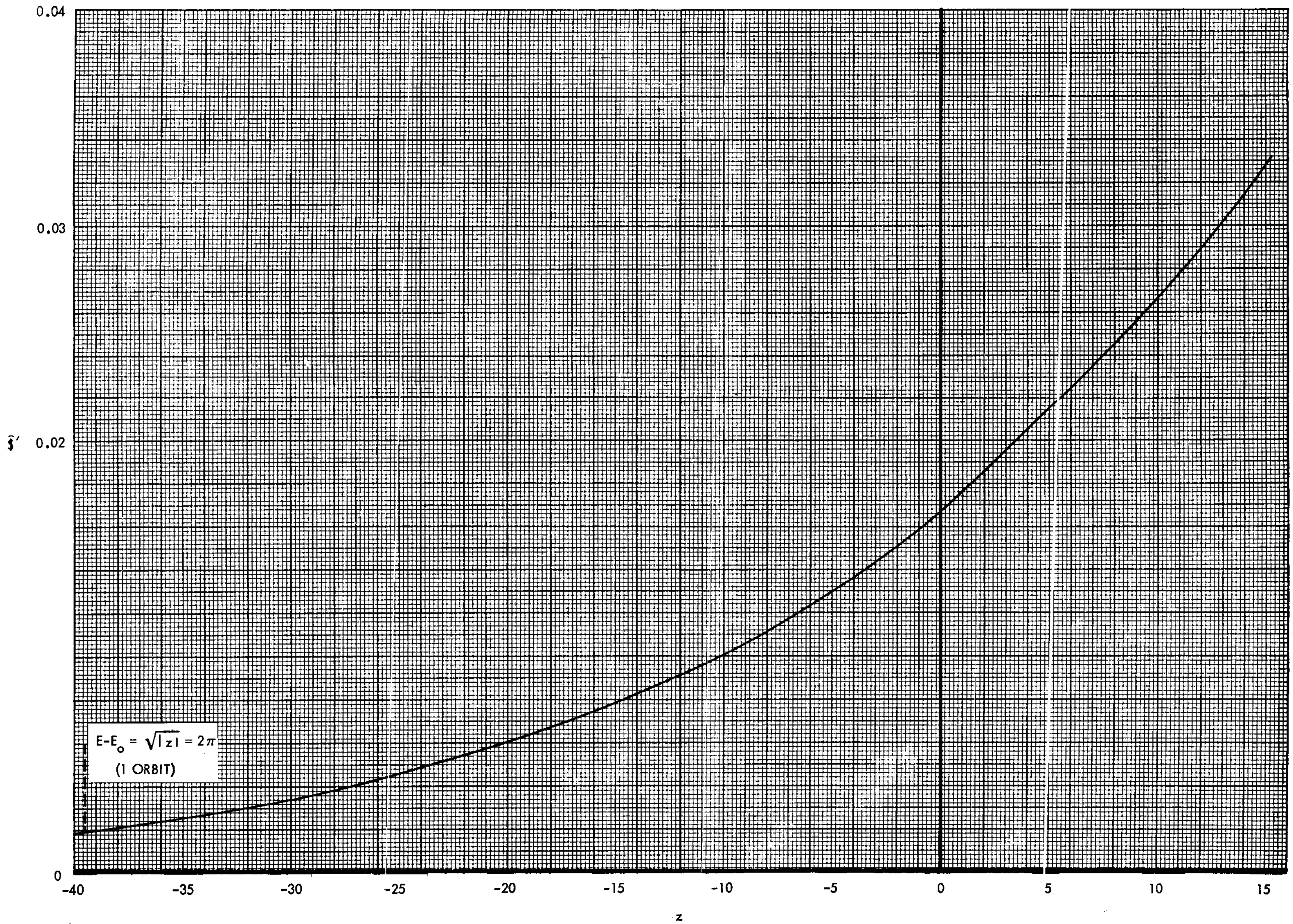


Figure B-3. The Function $\hat{\$}'(z)$ Versus z

APPENDIX C

SUMMARY OF RELATIONS FOR THE HERRICK-LEMMON FUNCTIONS

Identities and Definitions

1a.	$C = 1 + \alpha\text{¢}$	$\widehat{C} = 1 + z\widehat{\text{¢}}$
b.	$= 1 + \frac{\alpha}{2}\psi^2 + \alpha^2\text{₡}$	$= 1 + \frac{1}{2}z + z^2\widehat{\text{₡}}$
c.	$= \frac{1}{\psi}(S + \alpha\psi\text{¢} - \alpha\$)$	$= \widehat{S} + z\widehat{\text{¢}} - z\widehat{\$}$
d.	$= \frac{1}{\psi}(S + \alpha S')$	$= S - z\widehat{S}'$
2a.	$S = \psi + \alpha\$$	$\widehat{S} = 1 + z\widehat{\$}$
b.	$= \psi + \frac{\alpha}{6}\psi^3 + \alpha^2\text{฿}$	$= 1 + \frac{z}{6} + z^2\widehat{\text{฿}}$
c.	$= \psi C - \alpha S'$	$= \widehat{C} + z\widehat{S}'$
d.	$= \frac{1}{\psi}C'$	$= \widehat{C}'$
3a.	$\text{¢} = \frac{1}{\alpha}(C - 1)$	$\widehat{\text{¢}} = \frac{1}{z}(\widehat{C} - 1)$
b.	$= \frac{1}{2}\psi^2 + \alpha\text{₡}$	$= \frac{1}{2} + z\widehat{\text{₡}}$
c.	$= \frac{1}{\psi}\left[S\text{¢} + (1 - C)\$\right]$	$= \widehat{S}\widehat{\text{¢}} + (1 - \widehat{C})\widehat{\$}$
d.	$= \frac{2}{\psi^2}\left[(1 - C)\text{₡} + \text{¢}^2\right]$	$= 2\left[(1 - \widehat{C})\widehat{\text{₡}} + \widehat{\text{¢}}^2\right]$
e.	$= \frac{6}{\psi^3}\left[(1 - C)\text{฿} + \text{¢}\$\right]$	$= 6\left[(1 - \widehat{C})\widehat{\text{฿}} + \widehat{\text{¢}}\widehat{\$}\right]$

4a. $\$ = \frac{1}{a}(S - \psi)$ $\qquad\qquad \widehat{\$} = \frac{1}{z}(\widehat{S} - 1)$

b. $= \frac{1}{6}\psi^3 + a\text{฿}$ $\qquad\qquad = \frac{1}{6} + z\widehat{\text{฿}}$

c. $= \psi\text{¢} - S'$ $\qquad\qquad = \widehat{\text{¢}} - \widehat{S}'$

d. $= \frac{2}{\psi^2}\left[(\psi - S)\,\text{₵} + \$\text{¢}\right]$ $\qquad\qquad = 2\left[(1 - \widehat{S})\,\widehat{\text{₵}} + \widehat{\$}\widehat{\text{¢}}\right]$

5a. $\text{₵} = \frac{1}{a}\left(\text{¢} - \frac{1}{2}\psi^2\right)$ $\qquad\qquad \widehat{\text{₵}} = \frac{1}{z}\left(\widehat{\text{¢}} - \frac{1}{2}\right)$

b. $= \frac{1}{a^2}\left(C - 1 - \frac{1}{2}a\psi^2\right)$ $\qquad\qquad = \frac{1}{z^2}\left(\widehat{C} - 1 - \frac{1}{2}z\right)$

c. $= \frac{1}{2}(\psi\$ - \text{¢}')$ $\qquad\qquad = \frac{1}{2}(\widehat{\$} - \widehat{\text{¢}}')$

6a. $\text{฿} = \frac{1}{a}\left(\$ - \frac{1}{6}\psi^3\right)$ $\qquad\qquad \widehat{\text{฿}} = \frac{1}{z}\left(\widehat{\$} - \frac{1}{6}\right)$

b. $= \frac{1}{a^2}\left(S - \psi - \frac{1}{6}a\psi^3\right)$ $\qquad\qquad = \frac{1}{z^2}\left(\widehat{S} - 1 - \frac{1}{6}z\right)$

$= \frac{1}{3}(\psi\text{₵} - \$')$ $\qquad\qquad = \frac{1}{3}(\widehat{\text{₵}} - \widehat{\$}')$

7. $C' = \psi S$ $\qquad\qquad \widehat{C}' = \widehat{S}$

8a. $S' = \frac{1}{a}(\psi C - S)$ $\qquad\qquad \widehat{S}' = \frac{1}{z}(\widehat{C} - \widehat{S})$

b. $= \psi\text{¢} - \$$ $\qquad\qquad = \widehat{\text{¢}} - \widehat{\$}$

c. $= S\text{¢} - C\$$ $\qquad\qquad = \widehat{S}\widehat{\text{¢}} - \widehat{C}\widehat{\$}$

9a. $\text{¢}' = \frac{1}{a}(\psi S - 2\text{¢})$ $\qquad\qquad \widehat{\text{¢}}' = \frac{1}{z}(\widehat{S} - 2\text{¢})$

b. $= \psi\$ - 2\text{₵}$ $\qquad\qquad = \widehat{\$} - 2\widehat{\text{₵}}$

c. $= \text{¢}^2 - S\$$ $\qquad\qquad = \widehat{\text{¢}}^2 - \widehat{S}\widehat{\$}$

10a. $\text{\$}' = \frac{1}{a}(\psi\text{₵} - 3\text{\$})$ $\qquad$ $\widehat{\text{\$}}' = \frac{1}{z}(\widehat{\text{₵}} - 3\widehat{\text{\$}})$

b. $= \psi\text{₵} - 3\text{฿}$ $\qquad$ $= \widehat{\text{₵}} - 3\widehat{\text{฿}}$

11a. $C^2 = 1 + aS^2$ $\qquad$ $\widehat{C}^2 = 1 + z\widehat{S}^2$

b. $= 1 + a\text{¢} + aC\text{¢}$ $\qquad$ $= 1 + z\widehat{\text{¢}}(1 + \widehat{C})$

c. $= 1 + 2a\text{¢} + a^2\text{¢}^2$ $\qquad$ $= 1 + 2z\widehat{\text{¢}} + z^2\widehat{C}^2$

d. $= C + aC\text{¢}$ $\qquad$ $= \widehat{C} + z\widehat{C}\widehat{\text{¢}}$

12a. $S^2 = \text{¢} + C\text{¢}$ $\qquad$ $\widehat{S}^2 = \widehat{\text{¢}} + \widehat{C}\widehat{\text{¢}}$

b. $= 2\text{₵} + a\text{¢}^2$ $\qquad$ $= 2\widehat{\text{₵}} + z\widehat{\text{¢}}^2$

c. $= \psi^2 + 2a\psi\text{\$} + a^2\text{\$}^2$ $\qquad$ $= 1 + 2z\widehat{\text{\$}} + z^2\widehat{\text{\$}}^2$

d. $= \psi^2 + a(\psi + S)\text{\$}$ $\qquad$ $= 1 + z(1 + \widehat{S})\widehat{\text{\$}}$

13a. $\text{¢}^2 = (S + \psi)\text{\$} - 2\text{₵}$ $\qquad$ $\widehat{\text{¢}}^2 = (1 + S)\widehat{\text{\$}} - 2\widehat{\text{₵}}$

b. $= \frac{1}{2}\psi^2\text{¢} + C\text{₵} - \text{₵}$ $\qquad$ $= \frac{1}{2}\widehat{\text{¢}} + \widehat{C}\widehat{\text{₵}} - \widehat{\text{₵}}$

c. $= \frac{\psi^2}{2}\text{¢} + a\text{¢}\text{₵}$ $\qquad$ $= \frac{1}{2}\widehat{\text{¢}} + z\widehat{\text{₵}}$

d. $= S\text{\$} + \text{¢}'$ $\qquad$ $= \widehat{S}\widehat{\text{\$}} + \widehat{\text{¢}}'$

14. $S\text{₵} - C\text{฿} = \frac{\psi^2}{6}\text{¢} - \frac{\psi^2}{2}\text{\$} + \psi\text{₵} - \text{฿}$; $\widehat{S}\widehat{\text{₵}} - \widehat{C}\widehat{\text{฿}} = \frac{1}{6}\widehat{\text{¢}} - \frac{1}{2}\widehat{\text{\$}} + \widehat{\text{₵}} - \widehat{\text{฿}}$

APPENDIX D

TABLE OF INTEGRALS

1. $$\int C \, d\psi = S$$

2. $$\int S \, d\psi = \text{¢}$$

3. $$\int \text{¢} \, d\psi = \$$$

4. $$\int \$ \, d\psi = \text{₵}$$

5. $$\int \text{₵} \, d\psi = \text{฿}$$

6. $$\int \psi C \, d\psi = \psi S - \text{¢}$$

7. $$\int \psi S \, d\psi = S'$$

8. $$\int \psi \text{¢} \, d\psi = \psi \$ - \text{₵} = \text{¢}' + \text{₵}$$

9. $$\int \psi \$ \, d\psi = \psi \text{₵} - \text{฿}$$

10. $$\int \frac{\psi^2}{2!} C \, d\psi = \frac{\psi^2}{2!} S - \psi \text{¢} + \$$$

11. $$\int \frac{\psi^2}{2!} S \, d\psi = \frac{\psi^2}{2!} \text{¢} - \psi \$ + \text{₵}$$

12. $$\int \frac{\psi^2}{2!} \text{¢} \, d\psi = \frac{\psi^2}{2} \$ - \psi \text{₵} + \text{฿} = \text{฿} + \$\text{¢} - S\text{₵}$$

13. $$\int \frac{\psi^3}{3!} C \, d\psi = \frac{\psi^3}{3!} S - \frac{\psi^3}{2!} \text{¢} + \psi \$ - \text{₵}$$

14. $$\int \frac{\psi^3}{3!} S \, d\psi = \frac{\psi^3}{3!} \text{¢} - \frac{\psi^2}{2!} \$ + \psi \text{₵} - \text{฿} = S\text{₵} - C\text{฿}$$

15. $$\int \frac{\psi^4}{4!} C \, d\psi = \frac{\psi^4}{4!} S - \frac{\psi^3}{3!} \text{¢} + \frac{\psi^2}{2!} \$ - \psi \text{₵} + \text{฿}$$

16. $$\int C^2 \, d\psi = \frac{1}{2} (\psi + CS)$$

17. $$\int CS \, d\psi = \frac{1}{2} S^2$$

18. $$\int C\text{¢} \, d\psi = \frac{1}{2} (S\text{¢} - \$)$$

19. $$\int C\$ \, d\psi = \text{₵} + \frac{a}{2} \$^2 = \frac{1}{2} (S\$ - \text{¢}') = S\$ - \frac{1}{2} \text{¢}^2$$

20. $$\int C\text{₵} \, d\psi = \frac{1}{2} (2S\text{₵} - \text{¢}\$ + \$')$$

21. $$\int C\$\!\!| \, d\psi = \frac{1}{2} \$^2 + S\$\!\!| - \text{¢₵}$$

22. $$\int S^2 \, d\psi = \frac{1}{2} (\$ + S\text{¢})$$

23. $$\int S\text{¢} \, d\psi = \frac{1}{2} \text{¢}^2$$

24. $$\int S\$ \, d\psi = \frac{1}{2} (\text{¢}\$ - \$')$$

25. $$\int S\text{₵} \, d\psi = \text{¢₵} - \frac{1}{2} \$^2$$

26. $$\int \text{¢}^2 \, d\psi = \frac{1}{2} (\$\text{¢} + \$') = \frac{1}{2} \Lambda$$

27. $$\int \text{¢} \cdot \$ \, d\psi = \frac{1}{2} \$^2$$

28. $$\int \$\text{₵} \, d\psi = \frac{1}{2} \text{₵}^2$$

Differentiation

1. $$\frac{dC}{d\psi} = \alpha S$$

2. $$\frac{dS}{d\psi} = C$$

3. $$\frac{d\text{¢}}{d\psi} = S$$

4. $$\frac{d\$}{d\psi} = \text{¢}$$

5. $$\frac{d\text{₡}}{d\psi} = \$$$

6. $$\frac{d\$\!\!\!\Vert}{d\psi} = \text{₡}$$

7. $$\frac{dC'}{d\psi} = \alpha S' + 2S = \psi C + S$$

8. $$\frac{dS'}{d\psi} = C'$$

9. $$\frac{d\text{¢}'}{d\psi} = S'$$

10. $$\frac{d\$'}{d\psi} = C'$$

APPENDIX E

SUMMARY OF EXPRESSIONS FOR VARIOUS QUANTITIES

		Equation Reference
1a.	$\hat{r} = r\hat{i}$	(1.5)
b.	$= f\hat{r}_o + g\hat{v}_o$	(2.28a)
c.	$= \frac{1}{r_o}(u\hat{i}_o + g\hat{L} \times \hat{i}_o)$	(2.31a)
d.	$= r\hat{i}_o + \frac{1}{r_o}\hat{L} \times \hat{w}$	(2.41)
2a.	$\hat{v} = \dot{\hat{r}}$	(1.2)
b.	$= \frac{1}{r^2}(B\hat{r} + \hat{L} \times \hat{r})$	(1.7a)
c.	$= \frac{1}{r}(B\hat{i} + \hat{L} \times \hat{i})$	(1.7b)
d.	$= \dot{r}\hat{i} + \frac{1}{r}\hat{L} \times \hat{i}$	(1.7c)
e.	$= \hat{v}_o - \frac{\mu}{rr_o}\hat{w}$	(2.36)
f.	$= \dot{f}\hat{r}_o + \dot{g}\hat{v}_o$	(2.28b)
g.	$= \frac{1}{r_o}(\dot{u}\hat{i}_o + \dot{g}\hat{L} \times \hat{i}_o)$	(2.31b)
3a.	$\hat{w} = -\overline{\hat{w}}$	(3.25a)
b.	$= \frac{rr_o}{\mu}(\hat{v}_o - \hat{v})$	(2.36)
c.	$= g\hat{i}_o + \text{¢}\hat{L} \times \hat{i}_o$	(2.37a)
d.	$= S\hat{r}_o + r_o\text{¢}\hat{v}_o$	(2.37b)
e.	$= g\hat{i} - \text{¢}\hat{L} \times \hat{i}$	(3.25b)

		Equation Reference
3f.	$\hat{w} = S\hat{r} - r\text{¢}\hat{v}$	(2.37b)
g.	$= r r_o \int_{t_o}^{t} \frac{\hat{r}}{r^3}\, dt$	(2.39a)
h.	$= \frac{r r_o}{L^2} \hat{L} \times (\hat{i}_o - \hat{i})$	(2.42)
4.	$\hat{L} = \hat{r} \times \hat{v}$	(1.3)
5a.	$\hat{i} = \frac{\hat{r}}{r}$	(1.5)
b.	$= \hat{i}_o + \frac{1}{r r_o} \hat{L} \times \hat{w}$	(2.40)
c.	$= \cos\theta\, \hat{i}_o + \sin\theta\, \frac{\hat{L}}{L} \times \hat{i}_o$	(2.22a)
6a.	$\alpha = v^2 - 2\frac{\mu}{r}$	(1.9)
b.	$= \frac{\mu^2}{L^2}(\epsilon^2 - 1)$	(1.15)
c.	$= \frac{z}{\psi^2}$	(2.14b)
7a.	$\psi = \int_{t_o}^{t} \frac{dt}{r}$	(2.1)
b.	$= \sqrt{\frac{z}{\alpha}}$	(2.14a)
8.	$\sqrt{z} = \sqrt{\alpha}\, \psi$	(2.14a)
9a.	$r = \lvert\hat{r}\rvert$	
b.	$= r_o C + B_o S + \mu\text{¢}$	(2.19a)
c.	$= r_o + B_o S + (\mu + \alpha r_o)\text{¢}$	(2.19b)

Equation Reference

9d.	$r = r_o + \frac{B_o}{r_o} g + \left(\frac{L^2}{r_o} - \mu\right) ¢$	(5.5b)
e.	$= \frac{L^2}{\mu} + \frac{B_o}{r_o} g + \left(r_o - \frac{L^2}{\mu}\right) f$	
10a.	$r_o = \lvert \hat{r}_o \rvert$	
b.	$= \bar{r}$	(3.7)
c.	$= rC - BS + \mu ¢$	(3.8)
d.	$= r - BS + (\mu + \alpha r) ¢$	(2.19b)
e.	$= r - \frac{B}{r} g + \left(\frac{L^2}{r} - \mu\right) ¢$	(5.5b)
f.	$= \frac{L^2}{\mu} - \frac{B}{r} g + \left(r - \frac{L^2}{\mu}\right) \dot{g}$	
11a.	$B = \hat{r} \cdot \hat{v}$	(1.6a)
b.	$= r\dot{r}$	(1.6b)
c.	$= B_o C + (\mu + \alpha r_o) S$	(2.20a)
d.	$= B_o + (\mu + \alpha r_o) S + \alpha B_o ¢$	(2.20b)
e.	$= B_o + \mu S + \alpha g$	(4.4)
f.	$= B_o f + \left(\alpha + \frac{\mu}{r_o}\right) g$	(4.21)
g.	$= B_o f + \left(v_o^2 - \frac{\mu}{r_o}\right) g$	(1.11)
h.	$= B_o + \mu\psi + \alpha\tau$	(5.10)
i.	$= B_o \pm \sqrt{\alpha d^2 + 2\mu^2 ¢}$	(5.32)
j.	$= B_o + [2\mu + \alpha(r + r_o)] \frac{¢}{S}$	(5.31)

		Equation Reference
12a.	$B_o = \hat{r}_o \cdot \hat{v}_o$	(1.6a)
b.	$= \overline{B}$	(3.7)
c.	$= BC - (\mu + \alpha r)S$	(3.9)
d.	$= B - (\mu + \alpha r)S + \alpha B \text{¢}$	(2.20b)
e.	$= B\dot{g} - \left(\alpha + \frac{\mu}{r}\right) g$	(4.22)
f.	$= B\dot{g} - \left(v^2 - \frac{\mu}{r}\right) g$	(1.11)
13a.	$f = \frac{\hat{r}}{L^2} \cdot (\hat{v}_o \times \hat{L})$	(2.29a)
b.	$= 1 - \frac{\mu}{r_o} \text{¢}$	(2.29a)
c.	$= \dot{\overline{g}}$	(3.13)
14a.	$g = \frac{\hat{r}}{L^2} \cdot (\hat{L} \times \hat{r}_o)$	(2.29b)
b.	$= r_o S + B_o \text{¢}$	(2.29b)
c.	$= rS - B\text{¢}$	(3.14)
d.	$= \frac{r}{\mu}\left[B_o (1 - \dot{g}) - r_o^{\,2} \dot{f}\right]$	
e.	$= \frac{r_o}{\mu}\left[B(f - 1) - r^2 \dot{f}\right]$	
f.	$= -\overline{g}$	(3.11)
15a.	$\dot{f} = \frac{\hat{v}}{L^2} \cdot (\hat{v}_o \times \hat{L})$	(2.29c)
b.	$= -\frac{\mu S}{r r_o}$	(2.29c)
c.	$= \frac{1}{r r_o}\left[B_o(1 - f) - \frac{\mu}{r_o} g\right]$	

Equation Reference

15d.	$\dot{f} = \frac{1}{r^2}\left[B(f-1) - \frac{\mu}{r_o} g\right]$	
e.	$= -\dot{\bar{f}}$	(3.12)
16a.	$\dot{g} = \frac{\hat{v}}{L^2} \cdot (\hat{L} \times \hat{r}_o)$	(2.29d)
b.	$= 1 - \frac{\mu}{r} ¢$	(2.29d)
c.	$= 1 - \frac{r_o}{r}(1-f)$	
d.	$= \bar{f}$	(3.10)
17a.	$u = r^2 \dot{g} - Bg$	(3.20b)
b.	$= rr_o - L^2 ¢$	(2.32a)
c.	$= r_o^2 f + B_o g$	(2.32b)
d.	$= \Omega + \mu r ¢$	(5.4d)
e.	$= \bar{u}$	(3.20a)
18a.	$\dot{u} = r_o^2 \dot{f} + B_o \dot{g}$	(2.33a)
b.	$= B_o - \frac{\mu}{r} g$	(2.33b)
c.	$= \frac{1}{r}\left(r_o B - L^2 S\right)$	(2.33c)
d.	$= B\dot{g} - v^2 g$	(4.23)
e.	$= \dot{\bar{u}}$	(3.20a)
19a.	$\dot{\bar{u}} = -r^2 \dot{f} + Bf$	(3.22a)
b.	$= B + \frac{\mu}{r_o} g$	(3.22b)

		Equation Reference
19c.	$\bar{\dot{u}} = \frac{1}{r_o}\left(rB_o + L^2 S\right)$	(3.22c)
d.	$= B_o f + v_o^2 g$	(4.23)
20a.	$\tau = t - t_o$	(2.43)
b.	$= r_o S + B_o ¢ + \mu\$$	(2.45a)
c.	$= rS - B¢ + \mu\$$	(3.18b)
d.	$= g + \mu\$$	(2.45b)
e.	$= -\bar{\tau}$	(3.17)
21a.	$\sin\theta = \frac{\hat{L}}{L} \cdot (\hat{i}_o \times \hat{i})$	(2.21b)
b.	$= \frac{L}{rr_o}(r_o S + B_o ¢)$	(2.27a)
c.	$= \frac{L}{rr_o}(rS - B¢)$	
d.	$= \frac{Lg}{rr_o}$	
22a.	$\cos\theta = \hat{i} \cdot \hat{i}_o$	(2.21a)
b.	$= 1 - \frac{L^2}{rr_o} ¢$	(2.27b)
c.	$= \frac{u}{rr_o}$	
23a.	$\textcircled{H} = \int_{t_o}^{t} \frac{dt}{r^2}$	(2.26b)
b.	$= \frac{1}{L}\theta$	(2.26c)
c.	$= \frac{1}{r_o}\left(L^2 \mathfrak{Z} + \frac{g}{r}\right)$	(4.44)

Equation Reference

24a.	$v = \sqrt{a + \frac{2\mu}{r}}$	(1.11)
b.	$= \frac{1}{r}\sqrt{L^2 + B^2}$	(1.12)
25a.	$\sin\beta = \frac{L}{rv}$	
b.	$= \frac{L}{\sqrt{L^2 + B^2}}$	
c.	$= \frac{L}{\sqrt{ar^2 + 2\mu r}}$	
26a.	$\cos\beta = \frac{B}{rv}$	
b.	$= \frac{B}{\sqrt{L^2 + B^2}}$	
c.	$= \frac{B}{\sqrt{ar^2 + 2\mu r}}$	
27.	$\tan\beta = \frac{L}{B}$	
28a.	$d = \lvert \hat{r} - \hat{r}_o \rvert$	(2.47a)
b.	$= \left[(r - r_o)^2 + 2L^2 ¢\right]^{1/2}$	(2.47c)
c.	$= \left(r^2 + r_o^2 - 2u\right)^{1/2}$	(2.47d)
d.	$= \left[(B - B_o)g + \mu(r + r_o) ¢\right]^{1/2}$	(5.30)
e.	$= \left\{\frac{1}{a}\left[(B - B_o)^2 - 2\mu^2 ¢\right]\right\}^{1/2}$	(5.32)

		Equation Reference
29a.	$\rho = 2 \int_{\tau=t_o}^{t} \text{¢} \left[\psi(t_o, \tau)\right] d\tau$	(4. 9a)
b.	$= g\text{¢} + \mu\Lambda - r_o \$$	(4. 9c)
c.	$= \Upsilon - r_o \$$	(4. 9b)
d.	$= \frac{r_o}{\mu} (g - f\tau) + \mu\$'$	(4. 9d)
30a.	$\overline{\rho} = -2 \int_{\tau=t_o}^{t} \text{¢} \left[\psi(t, \tau)\right] d\tau$	(4. 17a)
b.	$= -g\text{¢} - \mu\Lambda + r\$$	(4. 9c)
c.	$= -\Upsilon + r\$$	(4. 17b)
d.	$= \frac{r}{\mu} (-g + \dot{g}\tau) - \mu\$'$	(4. 9d)
31a.	$\zeta = \int_{\tau=t_o}^{t} \left[3 - 2f(t_o, \tau)\right] d\tau$	(4. 10a)
b.	$= \tau + \frac{\mu}{r_o} \rho$	(4. 10b)
c.	$= \frac{1}{2} (3\tau - \eta)$	(4. 14b)
d.	$= g + \frac{\mu}{r_o} \Upsilon$	(4. 10c)
e.	$= g(2 - f) + \frac{\mu^2}{r_o} \Lambda$	(4. 10d)
32a.	$\overline{\zeta} = -\int_{\tau=t_o}^{t} \left[3 - 2f(t, \tau)\right] d\tau$	(4. 18a)
b.	$= -\tau + \frac{\mu}{r} \overline{\rho}$	(4. 18b)

		Equation Reference
32c.	$\overline{\zeta} = -\frac{1}{2}(3\tau + \overline{\eta})$	(4.14b)
d.	$= -g - \frac{\mu}{r}\Upsilon$	(4.18c)
e.	$= -g(2 - \dot{g}) - \frac{\mu^2}{r}\Lambda$	(4.18a)
33a.	$\eta = \int_{\tau=t_o}^{t}\left[-3 + 4f(t_o, \tau)\right]d\tau$	(4.11a)
b.	$= \tau - \frac{2\mu}{r_o}\rho$	(4.11b)
c.	$= 3\tau - 2\zeta$	(4.14a)
d.	$= \tau - \frac{2\mu}{r_o}\Upsilon + 2\mu\$$	(4.11c)
e.	$= g - \frac{2\mu}{r_o}\Upsilon + 3\mu\$$	(4.11d)
f.	$= -\mu\left(v_o^2\Lambda + \frac{B_o}{r_o}\cent^2\right) + gf$	(5.25)
34a.	$\overline{\eta} = -\int_{\tau=t_o}^{t}\left[-3 + 4f(t, \tau)\right]d\tau$	(4.19a)
b.	$= -\tau - \frac{2\mu}{r}\overline{\rho}$	(4.19b)
c.	$= -3\tau - 2\overline{\zeta}$	(4.14a)
d.	$= \frac{2\mu}{r}\Upsilon - \tau - 2\mu\$$	(4.19c)
e.	$= \frac{2\mu}{r}\Upsilon - g - 3\mu\$$	(4.19d)
f.	$= = \mu\left(v^2\Lambda - \frac{B\cent^2}{r}\right) - g\dot{g}$	(5.25)

		Equation Reference
35a.	$\sigma = 2 \int_{\tau=t_o}^{t} g(t_o, \tau)\, d\tau$	(4.12)
b.	$= \tau g + \mu(r_o \text{¢}' + B_o \$')$	(4.15)
c.	$= B_o \Upsilon + r_o(r + r_o)\text{¢}$	(5.3a)
d.	$= \frac{r_o}{\mu} \left[B_o (\zeta - g) + \mu(r + r_o) \text{¢}\right]$	(5.3b)
e.	$= \frac{r_o}{\mu} \left(B_o \zeta + r_o^2 - \Omega\right)$	(5.3c)
f.	$= \frac{r}{\mu} (\gamma - u)$	(5.16b)
g.	$= \frac{1}{v_o^2} (\Delta - B_o \eta)$	(5.11)
h.	$= \frac{1}{v^2} \left[B_o \overline{\eta} + Bg + 2r_o^2 (1 - \dot{g})\right]$	(5.20)
i.	$= g^2 + \mu(r_o \text{¢}^2 + B_o \Lambda)$	(5.27)
36a.	$\overline{\sigma} = -2 \int_{\tau=t_o}^{t} g(t, \tau)\, d\tau$	(4.20a)
b.	$= \tau g + \mu(r\text{¢}' - B\$')$	(4.20b)
c.	$= -B \Upsilon + r(r + r_o)\text{¢}$	(5.3a)
d.	$= \frac{r}{\mu} \left[B(\overline{\zeta} + g) + \mu(r + r_o) \text{¢}\right]$	(5.3b)
e.	$= \frac{r}{\mu} (B\overline{\zeta} + r^2 - \overline{\Omega})$	(5.3c)
f.	$= \frac{r_o}{\mu} (\overline{\gamma} - u)$	(5.16b)
g.	$= \frac{1}{v^2} (\overline{\Delta} - B\overline{\eta})$	(5.11)

		Equation Reference
36h.	$\overline{\sigma} = \frac{1}{v_o^2} \left[B\eta - B_o g + 2r^2 (1 - f) \right]$	(5.20)
i.	$= g^2 + \mu(r\text{¢}^2 - B\Lambda)$	(5.27)
37a.	$\Delta = Bg + 2\mu r\text{¢}$	(4.35a)
b.	$= Bg + 2(1 - \dot{g})\, r^2$	(4.35b)
c.	$= r^2 (2 - g) - u$	(4.35c)
d.	$= r(r - r_o) + (\mu r + L^2)\, \text{¢}$	(4.35d)
e.	$= -Bg + 2\,(r^2 - u)$	(4.35e)
f.	$= B_o\, \eta + v_o^2\, \sigma$	(5.12a)
g.	$= \frac{1}{r_o^2} \left(L^2 \sigma + B_o\, \Gamma \right)$	(5.14a)
h.	$= r^2 - \Omega$	(5.4e)
38a.	$\overline{\Delta} \quad -B_o\, g + 2\mu r_o \text{¢}$	(4.36a)
b.	$= -B_o\, g + 2(1 - f) r_o^2$	(4.36b)
c.	$= r_o^2\, (2 - f) - u$	(4.36c)
d.	$= r_o\, (r_o - r) + \left(\mu r_o + L^2 \right) \text{¢}$	(4.36d)
e.	$= B_o\, g + 2\left(r_o^2 - u \right)$	(4.36e)
f.	$= B\overline{\eta} + v^2\, \overline{\sigma}$	(5.12b)
g.	$= \frac{1}{r^2} \, (L^2\, \overline{\sigma} - B\Gamma)$	(5.21)
h.	$= r_o^2 - \overline{\Omega}$	

Equation Reference

39a.	$\Omega = r_o^2 - \mu(r + r_o)₵ + B_o g$	(5.4a)
b.	$= r_o^2 + B_o \zeta - \frac{\mu\sigma}{r_o}$	(5.4b)
c.	$= rr_o - (\mu r + L^2)₵$	(5.4c)
d.	$= u - \mu r ₵$	(5.4d)
e.	$= r^2 - \Delta$	(5.4e)
f.	$= rr_o f - L^2 ₵$	(5.4f)
g.	$= r^2 (2\dot{g} - 1) - Bg$	(5.4g)
40a.	$\bar{\Omega} = r^2 - \mu(r + r_o)₵ - Bg$	(5.4a)
b.	$= r^2 + B\bar{\zeta} - \frac{\mu\bar{\sigma}}{r}$	(5.4b)
c.	$= rr_o - \left(\mu r_o + L^2\right) ₵$	(5.4c)
d.	$= u - \mu r_o ₵$	(5.4d)
e.	$= r_o^2 - \bar{\Delta}$	(5.4e)
f.	$= rr_o \dot{g} - L^2 ₵$	(5.4f)
g.	$= r_o^2 (2f - 1) + B_o g$	(5.4g)
41a.	$\gamma = r_o^2 (2 - \dot{g}) - B_o \bar{\zeta}$	(5.17a)
b.	$= u + \frac{\mu\sigma}{r}$	(5.17b)
c.	$= -Bg + r^2 \mathfrak{g}$	(5.17c)
d.	$= \frac{1}{B} \left(\frac{-\mu\lambda}{r} + r^2 B_o + L^2 g\right)$	(5.24)

		Equation Reference
42a.	$\overline{\gamma} = r^2 (2 - f) - B\zeta$	(5.19a)
b.	$= u + \frac{\mu\overline{\sigma}}{r_o}$	(5.19b)
c.	$= B_o\, g + r_o^2 \overline{\vartheta}$	(5.19c)
d.	$= \frac{1}{B_o} \left(\frac{\mu\lambda}{r_o} + r_o^2 B - L^2 g \right)$	(5.29)
43.	$\vartheta = \dot{g} + \frac{\mu\sigma}{r^3}$	(5.18a)
44.	$\overline{\vartheta} = f + \frac{\mu\overline{\sigma}}{r_o^3}$	(5.18b)
45a.	$\lambda = r^2 g - B\sigma$	(4.25)
b.	$= r_o^2 + B_o \overline{\sigma}$	(4.30)
c.	$= -\overline{\lambda}$	(4.29)
46a.	$\Gamma = B_o \sigma + r_o^2 \eta$	(5.13)
b.	$= -(B\overline{\sigma} + r^2 \overline{\eta})$	(5.22)
c.	$= gu - \mu L^2 \Lambda$	(5.28)
d.	$= grr_o - L^2 \Upsilon$	(5.28)
e.	$= -\overline{\Gamma}$	(5.22b)
47a.	$\mathfrak{J} = \int_{t_o}^{t} \frac{\cent}{r^3} dt$	(4.39)
b.	$= \frac{1}{L^2} \left(r_o \textcircled{H} - \frac{g}{r} \right)$	(4.44)

		Equation Reference
48a.	$\Lambda = -\bar{\Lambda}$	(4. 7b)
b.	$= ¢\$ + \$'$	(4. 7a)
c.	$= \frac{r_o}{\mu^2} \left[\zeta - (2 - f)g \right]$	(4. 10d)
d.	$= \frac{1}{\mu} (\Upsilon - g¢)$	(4. 8b)
e.	$= = -\frac{r}{\mu^2} \left[\bar{\zeta} + (2 - \dot{g})g \right]$	(4. 18d)
f.	$= \frac{1}{a} (S¢ - 3\$)$	(4. 7c)
49a	$h = rS - B_o ¢$	(3. 16a)
b.	$= r_o S + B¢$	(2. 30)
c.	$= (r + r_o)S - g$	(3. 16c)
d.	$= g + (B - B_o) ¢$	(3. 16d)
e.	$= -\bar{h}$	(3. 16b)
50a.	$\Upsilon = \tau¢ + \mu\$'$	(4. 8a)
b.	$= g¢ + \mu \Lambda$	(4. 8b)
c.	$= \rho + r_o \$$	(4. 9b)
d.	$= r\$ - \bar{\rho}$	(4. 17b)
e.	$= \frac{r_o}{\mu} (\zeta - g)$	(4. 10c)
f.	$= -\frac{r}{\mu} (\bar{\zeta} + g)$	(4. 18c)
g.	$= -\bar{\Upsilon}$	(4. 8c)

APPENDIX F

SUMMARY OF MISCELLANEOUS RELATIONS

		Equation Reference
1.	$B^2 = \alpha r^2 + 2\mu r - L^2$	(2.5)
2.	$f\dot{g} - g\dot{f} = 1$	(2.34a)
3.	$uf + g\bar{u} = r^2$	(3.23)
4.	$u\dot{g} - g\dot{u} = r_o^2$	(2.34b)
5a.	$(r + r_o)\text{¢} = \tau S + \mu \text{¢}'$	(5.2b)
b.	$= Sg + \mu \text{¢}^2$	(5.2a)
6.	$(r - r_o)S = (B + B_o)\text{¢}$	(3.15a)
7a.	$(r - r_o)g = (r_o B + r B_o)\text{¢}$	(3.15b)
b.	$= -(r_o \zeta + r\bar{\zeta})$	(4.13)
8.	$(r - r_o)\$ = \rho + \bar{\rho}$	(4.17c)
9.	$\left(r^2 - r_o^2\right)g = B\sigma + B_o\bar{\sigma}$	(4.31)
10.	$(B + B_o)S = (1 + C)(r - r_o)$	(2.19a)
11a.	$(B + B_o)\dot{f} = \mu\left(\frac{1}{r} - \frac{1}{r_o}\right)(1 + C)$	(2.29c)
b.	$= v^2 f - v_o^2 \dot{g}$	
12.	$(B - B_o)S = \left[2\mu + \alpha(r + r_o)\right]\text{¢}$	(2.19b)

		Equation Reference
13.	$(B - B_o)\$ = S\tau - \psi g$	(4.16)
14a.	$(B - B_o)g = d^2 - \mu(r + r_o)¢$	(5.30)
b.	$= (r - r_o)^2 + \left[2L^2 - \mu(r + r_o)\right]¢$	(5.7b)
15.	$(B + B_o)g = r^2 - r_o^2 - \mu(r - r_o)¢$	(5.7a)
16.	$B_o r - B r_o = \mu g - L^2 S$	
17.	$(B - B_o)^2 = \alpha d^2 + 2\mu^2 ¢$	(5.32)
18a.	$B_o g = r_o(r - r_o) + \left(\mu r_o - L^2\right)¢$	(5.5b)
b.	$= r_o^2(1 - 2f) + \overline{\Omega}$	(5.4g)
c.	$= u - r_o^2 f$	(5.5a)
19a.	$Bg = r(r - r_o) - (\mu r - L^2)¢$	(5.6b)
b.	$= r^2(2\dot{g} - 1) - \Omega$	(5.4g)
c.	$= r^2\dot{g} - u$	(5.6a)
20.	$B_o f = B - \left(\alpha + \frac{\mu}{r_o}\right)g$	(4.21)
21.	$B\dot{g} = B_o + \left(\alpha + \frac{\mu}{r}\right)g$	(4.22)
22.	$g^2 = (u + r r_o)¢$	(2.35)
23.	$B_o \Gamma = r_o^2 \Delta - L^2 \sigma$	(5.14a)

		Equation Reference
24.	$B\Gamma = -r^2\bar{\Delta} + L^2\bar{\sigma}$	(5.21)
25.	$(B + B_o)\Gamma = r_o^2\Delta - r^2\bar{\Delta}$	
26.	$B\bar{\sigma} + B_o\sigma = -\left(r^2\bar{\eta} + r_o^2\eta\right)$	(5.23)
27.	$r_o\Delta - r\bar{\Delta} = (r - r_o)(2rr_o - L^2\not{c})$	(4.35d)
	$r_o\Delta + r\bar{\Delta} = \left[2\mu r_o r + (r + r_o)L^2\not{c}\right]$	(4.35d)
28.	$B_o\Delta + B\bar{\Delta} = 2\mu(rB_o + r_oB)\not{c}$	(4.35a)
	$= 2\mu(r - r_o)g$	(3.15b)
29.	$\zeta + \bar{\zeta} = \frac{1}{2}(\eta + \bar{\eta})$	(4.14c)
30.	$B(\bar{\zeta} + \bar{\eta}) = r_o^2 - r^2 - \left(a + \frac{\mu}{r}\right)\bar{\sigma}$	(6.7)
31.	$B_o(\zeta + \eta) = r^2 - r_o^2 - \left(a + \frac{\mu}{r_o}\right)\sigma$	(6.7)
32a.	$B\zeta + \frac{\mu\bar{\sigma}}{r_o} = r^2(2 - f) - u$	(5.15)
33.	$B\zeta + r_o^2\bar{\mathcal{S}} = r^2(2 - f) - B_og$	(5.19)
34.	$v_o^2\bar{\sigma} = B\eta - B_og + 2r^2(1 - f)$	(5.20)
35.	$\left(B - B_o\right)^2 = ad^2 + 2\mu^2\not{c}$	(5.32)

APPENDIX G

SUMMARY OF DERIVATIVES

		Equation Reference
1.	$\dot{r} = \frac{B}{r}$	(1.6b)
2.	$\dot{B} = \alpha + \frac{\mu}{r}$	(2.7)
3.	$\ddot{r} = \frac{1}{r^3}(L^2 - \mu r)$	(2.8)
4.	$\ddot{B} = -\frac{\mu B}{r^3}$	(2.9)
5.	$\dot{\theta} = \frac{L}{r^2}$	(2.24b)
6.	$\dot{\psi} = \frac{1}{r}$	(2.10)
7.	$\dot{\tau} = 1$	(2.44a)
8.	$\dot{\textcircled{H}} = \frac{1}{r^2}$	(2.25b)
9.	$\frac{d\hat{i}}{dt} = \frac{1}{r^2}\hat{L} \times \hat{i}$	(1.8a)
10.	$\frac{d}{dt}\left(\frac{\hat{w}}{rr_o}\right) = \frac{\hat{r}}{r^3}$	(2.39b)
11.	$\dot{\Upsilon} = \left(2 + \frac{r_o}{r}\right) \text{¢}$	(6.1)
12.	$\dot{\rho} = 2\text{¢}$	(4.9a)
13.	$\dot{\zeta} = 3 - 2f$	(4.10a)
14.	$\dot{\eta} = 4f - 3$	(4.11a)

		Equation Reference
15.	$\dot{\sigma} = 2g$	(4.12)
16.	$\dot{\bar{\rho}} = \frac{1}{r}\left[B\$ - (r + r_o)\text{¢}\right]$	(6.2)
17.	$\dot{\bar{\zeta}} = -\left(1 + \frac{\mu\bar{\sigma}}{r^3}\right)$	(6.4a)
18.	$\dot{\bar{\eta}} = \frac{2\mu\bar{\sigma}}{r^3} - 1$	(6.4b)
19a.	$\dot{\bar{\sigma}} = \frac{1}{B_o}\left[\left(r^2 - r_o^2\right)\dot{g} - \left(a + \frac{\mu}{r}\right)\sigma\right]$	(6.5a)
19b.	$= \frac{1}{B}\left[\left(r^2 - r_o^2\right) + \left(a + \frac{\mu}{r}\right)\bar{\sigma}\right]$	(6.5b)
19c.	$= 2\tau + \frac{\mu\bar{\rho}}{r}$	(6.6a)
19d.	$= -(\bar{\zeta} + \bar{\eta})$	(6.6b)
20.	$\dot{\lambda} = r^2\dot{g} - \left(a + \frac{\mu}{r}\right)\sigma$	(6.8)
21.	$\frac{d}{dt}\left(\frac{\text{¢}}{r}\right) = \frac{g}{r^3}$	(4.32)
22.	$\frac{d}{dt}\left(\frac{g}{r}\right) = \frac{u}{r^3}$	(4.42)
23.	$\frac{d}{dt}\left(\frac{g}{B}\right) = \frac{B_o}{B^2}$	(4.24a)
24.	$\frac{d}{dt}\left(\frac{\lambda}{B}\right) = B_o\left(\frac{r}{B}\right)^2$	(4.26)
25a.	$\frac{d}{dt}\left(\frac{\lambda}{r}\right) = r\wp - \frac{B}{r}g - \frac{L^2\sigma}{r^3}$	(6.9a)

Equation Reference

25b.	$$\frac{d}{dt}\left(\frac{\lambda}{r}\right) = \frac{1}{r^3}(r^2\gamma - L^2\sigma)$$	(6.9b)
26.	$$\frac{d}{dt}\left(\frac{\bar{\rho}}{r}\right) = -\frac{\bar{\sigma}}{r^3}$$	(6.2)
27.	$$\frac{d}{dt}(B\bar{\sigma}) = r^2 - r_o^2 + 2\left(a + \frac{\mu}{r}\right)\bar{\sigma}$$	(6.13)
28.	$$\frac{d}{dt}\left(\frac{\sigma}{r^2}\right) = \frac{2\lambda}{r^4}$$	(4.37)
29a.	$$\frac{d}{dt}\left(\frac{\zeta}{r^2}\right) = \frac{1}{r^2}(3 - 2f) - \frac{2B\zeta}{r^4}$$	(6.10a)
29b.	$$= \frac{2\bar{\gamma}}{r^4} - \frac{1}{r^2}$$	(6.10b)
30.	$$\frac{d}{dt}\left(\frac{\Gamma}{r^2}\right) = \frac{r_o^2}{r^2} - \frac{2L^2}{r^4}\bar{\sigma}$$	(6.15)
31.	$$\frac{d}{dt}\left(\frac{Bg}{r^2}\right) = \frac{B_o}{r^2} + \frac{2g}{r^4}(L^2 - \mu r)$$	(4.33a)
32.	$$\frac{d}{dt}\left(\frac{B\bar{\sigma}}{r^2}\right) = 1 - \left(\frac{r_o}{r}\right)^2 + \frac{2}{r^4}(L^2 - \mu r)\bar{\sigma}$$	(6.14a)
33.	$$\frac{d}{dt}\left(\frac{\Delta}{r^2} - B_o \textcircled{H}\right) = \frac{2L^2 g}{r^4}$$	(4.34)
34.	$$\frac{d}{dt}\left[\left(1 + \frac{r_o}{r}\right)\frac{\cent}{r}\right] = \frac{2r_o g}{r^4} + \frac{B_o \cent}{r^3}$$	(4.40)

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