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# AN ASYMPTOTIC SOLUTION OF CONICAL SHELLS OF CONSTANT THICKNESS

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Prepared by

Chin Hao Chang and Han Yun Chu

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University, Alabama

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# AN ASYMPTOTIC SOLUTION OF CONICAL SHELLS OF CONSTANT THICKNESS

by

Chin Hao Chang[1]

ABSTRACT 20072.

A solution of truncated conical shells of constant thickness is obtained as the ratio of the thickness to the radius at the larger end goes to zero asymptotically by separating the solution into two parts: membrane and bending. These two parts are coupled by the lateral displacement. A particular solution due to lateral normal loads is also given and two numerical examples are presented. One numerical example considers a semicircular shell segment with the smaller end fixed, the other end free and the two generator edges simply supported. The shell is subjected to a lateral normal load which is constant in the meridional direction and varies sinusoidally in the circumferential direction. The other numerical example considers a cantilevered complete cone with the larger end free. A rigid plate is attached to the free end and a moment is applied. Comparisons with other available results are given in both examples.

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[1] Associate Professor in Engineering Mechanics, University of Alabama.

## Introduction

Conical shells of constant thickness have been studied by a number of investigators. The axial symmetrical solutions of such a shell have been well established [1, 2][2]; while for asymmetrical cases the solutions have been approached by two different ways. One approach uses the method of power series [3, 4, & 5], while the other treats the membrane and bending solutions separately. It has been found that, by keeping the first order terms only, the bending solutions are in forms of Bessel functions [6]. In reference [6], by recognizing the rapid decay of the bending solutions near edges, an edge-zone solution was also presented to replace the solutions of Bessel functions. The power series approach was not recommended by several researchers [7, 8] because of slow convergence.

It was found in Progress Report 1 [9] that, for conical shells of linearly varying thickness, the solution consists of two parts: membrane and bending effect. Both parts are expressed as polynomial functions of $y^{\lambda}$ as far as the y-function is concerned, where y is a dimensionless variable of length measured in the meridional direction and the $\lambda$'s are real constants for the membrane solutions and complex numbers for the solutions of bending effect. Furthermore, the $\lambda$'s of the membrane solutions will approach finite values while those of the bending solutions will become infinite as the ratio of the thickness of shell to the radius at a section approaches zero. These different characteristics of the two parts of the solutions enable one to treat them separately.

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[2] Numbers in brackets designate references at end of report.

Since conical shells of linearly varying thickness and those of constant thickness will behave alike when the ratio of thickness to radius is very small, in this report an asymptotic solution of conical shells of constant thickness is obtained by assuming that the solution possesses characteristics similar to the solution for conical shells of linearly varying thickness. The asymptotic solution obtained includes the particular solution due to a lateral normal load. Two numerical examples are also given. One is for semi-circular cone frustum similar to the one given in Progress Report 1. This example is designed to compare the results for the same shell with different types of thickness. The other is a complete cone frustum with the smaller circular end fixed and the other end free. At the free end a rigid plate is attached and a moment is applied. A solution for the latter example is available in [6] so that a comparison can be made between the two solutions.

## Basic Equations

A set of exact equations for shells of revolution of isotropic and elastic material within the framework of generalized plane stresses of linear theory of elasticity is given in explicit form in Reference [1]. For thin shells exploiting the approximations $r_1 + z \approx r_1$ and $r_2 + z \approx r_2$ where $r_1$ and $r_2$ are two principal radii of the middle surface of the shells and z is the normal distance measured from the middle surface to a generic point, the elastic relations between stress resultants, couples and displacements are simplified considerably. For a conical shell these relations are:

$$N_s = \frac{D}{L}\frac{1}{y^2}\left[\frac{1}{2} y\dot{v} + \nu(u'\sec\alpha + v + w\tan\alpha)\right]$$

$$N_\theta = \frac{D}{L}\frac{1}{y^2}\left[u'\sec\alpha + v + w\tan\alpha + \frac{\nu}{2} y\dot{v}\right]$$

$$N_{s\theta} = N_{\theta s} = \frac{D}{L}\frac{1-\nu}{2}\frac{1}{y^2}\left[\frac{1}{2} y\dot{u} - u + v'\sec\alpha\right]$$

$$M_s = D\,k\,\frac{1}{y^4}\left[\frac{1}{4}(y^2\ddot{w} - y\dot{w}) + \nu\left(w''\sec^2\alpha + \frac{1}{2} y\dot{w}\right)\right] \tag{1}$$

$$M_\theta = D\,k\,\frac{1}{y^4}\left[w''\sec^2\alpha + \frac{1}{2} y\dot{w} + \frac{\nu}{4}(y^2\ddot{w} - y\dot{w})\right]$$

$$M_{s\theta} = M_{\theta s} = D\,k\,(1-\nu)\frac{1}{y^4}\left[\frac{1}{2} y\dot{w}'\sec\alpha - w'\sec\alpha\right]$$

in which $N_s, \ldots, M_{s\theta}$ are normal and shearing stress resultants and couples in the respective directions indicated by the subscripts and $y = \sqrt{\frac{s}{L}}$. The s and $\theta$ are meridional and circumferential coordinates of the middle surface of the shell; u, v, and w are circumferential, meridional and normal displacements, respectively. Outward w is positive. D and k are defined as follows:

$$D = \frac{Et}{1-\nu^2} \quad \text{and} \quad k = \frac{1}{12}\left(\frac{t}{L}\right)^2 = \frac{1}{12}\left(\frac{t}{R}\right)^2\cos^2\alpha \tag{2}$$

where E is Young's modulus of elasticity, $\nu$ is Poisson's ratio, t is thickness, L is the length from the apex to the larger end of the shell, $R = L\cos\alpha$ is the radius of the shell at $y = 1$, and the $\alpha$ is the base angle of the shell.

The dots indicate partial differentiation with respect to y and primes partial differentiation with respect to θ.

When the equation of equilibrium of moments about the normal of a surface element is overlooked, the other five equations are:

$$\frac{1}{2} y N_s^{\cdot} + N_s + N_{\theta s}' \sec \alpha - N_\theta = - P_s L y^2$$

$$\frac{1}{2} y N_{s\theta}^{\cdot} + 2N_{s\theta} + N_\theta' \sec \alpha - Q_\theta \tan \alpha = - P_\theta L y^2$$

$$\frac{1}{2} y Q_s^{\cdot} + Q_s + Q_\theta' \sec \alpha + N_\theta \tan \alpha = P_r L y^2 \qquad (3)$$

$$\frac{1}{2} y M_s^{\cdot} + M_s + M_{\theta s}' \sec \alpha - M_\theta = L y^2 Q_s$$

$$\frac{1}{2} y M_{s\theta}^{\cdot} + 2M_{s\theta} + M_\theta' \sec \alpha = L y^2 Q_\theta$$

where $Q_s$ and $Q_\theta$ are the transverse shear forces per unit length acting on sections perpendicular to the s- and θ- directions; $P_r$, $P_s$, and $P_\theta$ are surface loads per unit area in normal, meridional, and circumferential directions, respectively.

The eleven equations in (1) and (3) govern the eleven unknowns involved. When the last two moment equations of (3) are used to eliminate the transverse shearing forces $Q_s$ and $Q_\theta$ in the second and third equations of (3), the first three equations of (3) become

## Membrane Solutions

The membrane solutions of conical shells of constant thickness are well known. However, available solutions are presented in forms of stress only. In what follows, the displacements will be sought.

When equations (1) with k = 0 are substituted into equations (4), the three equations become

$$\frac{1-\upsilon}{8} y^2 u^{\cdot\cdot} + \frac{1-\upsilon}{8} yu^{\cdot} - \frac{1-\upsilon}{2} u + u'' \sec^2 \alpha + \frac{1+\upsilon}{4} yv^{\cdot\prime} \sec \alpha$$

$$+ \frac{3-\upsilon}{2} v' \sec \alpha + w' \tan \alpha \sec \alpha = -\frac{1}{D} L^2 y^4 P_\theta$$

$$\frac{1+\upsilon}{4} yu^{\cdot\prime} \sec \alpha - \frac{3-\upsilon}{2} u' \sec \alpha + \frac{1}{4} [y^2 v^{\cdot\cdot} + yv^{\cdot}] + \frac{1-\upsilon}{2} v'' \sec^2 \alpha \qquad (6)$$

$$- v + \frac{\upsilon}{2} w^{\cdot} \tan \alpha - w \tan \alpha = -\frac{1}{D} L^2 y^4 P_s$$

$$u' \sec \alpha + v + w \tan \alpha + \frac{\upsilon}{2} yv^{\cdot} = \frac{1}{D} L^2 y^4 P_r$$

Assume

$$u = Ay^\lambda \frac{\sin n\pi\theta}{\cos \theta_1}$$

$$v = By^\lambda \frac{\cos n\pi\theta}{\sin \theta_1} \qquad (7)$$

$$w = Cy^\lambda \frac{\cos n\pi\theta}{\sin \theta_1}$$

where $\lambda$ is an unknown constant; $\theta_1$ is the central angle between two

$$\frac{1}{2} yN^{\cdot}_{s} + N_{s} + N'_{\theta s} \sec \alpha - N_{\theta} = - P_{s} Ly^{2}$$

$$\frac{1}{2} yN^{\cdot}_{s\theta} + 2N_{s\theta} + N'_{\theta} \sec \alpha - \frac{1}{Ly^{2}} [\frac{1}{2} y M^{\cdot}_{s\theta} + M_{s\theta} + M_{\theta s} \tan \alpha$$

$$+ M'_{\theta} \tan \alpha \sec \alpha ] = - P_{\theta} Ly^{2} \qquad (4)$$

$$N_{\theta} \tan \alpha + \frac{1}{Ly^{2}} [\frac{1}{4} y^{2} M^{\cdot\cdot}_{s} + \frac{3}{4} yM^{\cdot}_{s} + (yM^{\cdot\prime}_{s\theta} + 2 M'_{s\theta}) \sec \alpha$$

$$+ M''_{\theta} \sec^{2} \alpha - \frac{1}{2} y M^{\cdot}_{\theta}] = P_{r} Ly^{2}$$

Substituting equations (1) into (4), three equations for three unknown displacements are obtained. In what follows, instead of dealing with these three displacements, each displacement will be divided into three parts: the first is due to membrane action, the second is due to the bending effects and the third, the last part, is for the particular solutions due to lateral normal loads. Denoting these three parts by superscripts I, II, and P, respectively, the displacements may be expressed as

$$u = u^{I} + u^{II} + u^{P}$$

$$v = v^{I} + v^{II} + v^{P} \qquad (5)$$

$$w = w^{I} + w^{II} + w^{P}$$

These three parts of solution will be discussed in the following sections.

generators. Substitution of equations (7) into the homogeneous part of equations (6) and cancelling out the y and sinusoidal functions, one has three homogeneous algebraic equations for three unknown constants A, B, and C. Letting the determinant of the equations vanish in order to have nontrival solutions, results in the following characteristic equation for $\lambda$

$$\lambda^2(\lambda^2 - 4) = 0 \tag{8}$$

When the determined values of $\lambda$ are substituted back into the algebraic equations one may express the constants A and B in terms of C. This gives the first part of the solutions of the displacements as follows:

$$u^I = \mp \frac{1}{m}\left\{\frac{m^2}{m^2-1}\left[C_1 - \left(\frac{1-\upsilon}{2(m^2-1)} - \ell ny\right) C_2\right] + C_3 y^2 + \frac{m^2-2(1+\upsilon)}{m^2-4} C_4 y^{-2}\right\} \begin{matrix}\sin\\ \cos\end{matrix} \frac{n\pi\theta}{\theta_1}$$

$$v^I = \left\{\frac{1}{m^2-1}\left[C_1 - \left(\frac{m^2-\upsilon}{2(m^2-1)} - \ell ny\right)C_2\right] + \frac{2}{m^2-4} C_4 y^{-2}\right\} \begin{matrix}\cos\\ \sin\end{matrix} \frac{n\pi\theta}{\theta_1} \tag{9}$$

$$w^I = \frac{1}{\tan \alpha}\left\{C_1 + C_2 \ell ny + C_3 y^2 + C_4 y^{-2}\right\} \begin{matrix}\cos\\ \sin\end{matrix} \frac{n\pi\theta}{\theta_1}$$

where

$$m = \frac{n\pi}{\theta_1} \sec \alpha$$

The corresponding stresses may be obtained from (1) as

$$N_s^I = \frac{Et}{L}\left\{\frac{1}{2(m^2-1)} C_2 y^{-2} - \frac{2}{m^2-4} C_4 y^{-4}\right\} \begin{matrix}\cos\\ \sin\end{matrix} \frac{n\pi\theta}{\theta_1} \tag{10}$$

$$N_{s\theta}^I = \mp \frac{Et}{L} \frac{2}{m(m^2-4)} C_4 y^{-4} \begin{matrix}\sin\\ \cos\end{matrix} \frac{n\pi\theta}{\theta_1}$$

$N_\theta^I$ vanishes identically.

## Solutions of The Bending Effect

It was learned from [9] that the displacement functions due to bending may be assumed in the following form:

$$u^{II} = \frac{1}{\gamma^2} U$$

$$v^{II} = \frac{1}{\gamma} V \tag{11}$$

and

$$w^{II} = W$$

where

$$\gamma^4 = \frac{16}{k} \tag{12}$$

Thus $\gamma \to \infty$ as $t/R \to 0$. Furthermore, the y-function of U, V, and W may be expressed in forms of $y^{c\gamma}$ where c is a finite constant. Thus the differentiations with respect to y will change the orders of magnitude of the functions concerned. In order to avoid this, a new variable $\eta$ is introduced such that

$$\eta = y^{\gamma} \tag{13}$$

When expressions (11) and (13) are substituted into the elastic relations (1), retaining only the terms of the lowest order of $\frac{1}{\gamma}$, yields

$$N_s = \frac{D}{L}\left[\frac{1}{2}\eta V,_{\eta} + \upsilon W \tan\alpha\right]\eta^{-\frac{2}{\gamma}}$$

$$N_\theta = \frac{D}{L}\left[W \tan\alpha + \frac{1}{2}\upsilon\eta V,_{\eta}\right]\eta^{-\frac{2}{\gamma}}$$

$$N_{s\theta} = N_{\theta s} = \frac{D}{L}\frac{1-\upsilon}{2}\frac{1}{\gamma}\left[\frac{1}{2}\eta U,_{\eta} + V,_{\theta}\right]\sec\alpha \cdot \eta^{-\frac{2}{\gamma}}$$

$$M_s = D\frac{4}{\gamma^2}\left[\eta^2 W,_{\eta\eta} + \eta W,_{\eta}\right]\eta^{-\frac{4}{\gamma}} \tag{14}$$

$$M_\theta = \upsilon M_s$$

$$M_{s\theta} = M_{\theta s} = D\frac{8}{\gamma^3}(1-\upsilon)W,_{\theta\eta}\sec\alpha \cdot \eta^{-\frac{4}{\gamma}}$$

Transforming the variable y to $\eta$ and making use of the asymptotic expressions (14), the homogeneous part of equations (4), when only the terms of the lowest order of $\frac{1}{\gamma}$ are retained, the following three equations are obtained:

$$\eta^2 V,_{\eta\eta} + \eta V,_{\eta} + 2\upsilon\eta W,_{\eta}\tan\alpha = 0 \tag{15a}$$

$$\frac{1}{4}\frac{1-\upsilon}{2}\,[\eta^2 U,_{\eta\eta} + \eta U,_{\eta} + 2\eta V,_{\theta\eta} \sec\alpha]$$

$$+ [W,_{\theta} \tan\alpha + \frac{1}{2}\upsilon\eta V,_{\theta\eta}] \sec\alpha = 0 \qquad (15b)$$

and

$$\eta^4 W,_{\eta\eta\eta\eta} + 6\eta^3 W,_{\eta\eta\eta} + 7\eta^2 W,_{\eta\eta} + \eta W,_{\eta}$$

$$+ \eta^{\frac{4}{\gamma}} [W \tan^2\alpha + \frac{1}{2}\upsilon\eta V,_{\eta} \tan\alpha] = 0$$

Since

$$\eta^{\frac{4}{\gamma}} \rightarrow 1 \qquad \text{as} \qquad \gamma \rightarrow \infty$$

the last equation becomes

$$\eta^4 W,_{\eta\eta\eta\eta} + 6y^3 W,_{\eta\eta\eta} + 7\eta^2 W,_{\eta\eta} + \eta W,_{\eta} + W \tan^2\eta$$

$$+ \frac{1}{2}\upsilon\eta V,_{\eta} \tan\alpha = 0 \qquad (15c)$$

The integration of equation (15a) with respect to $\eta$ results in

$$\eta V,_{\eta} = -2\upsilon W \tan\alpha \qquad (16)$$

in which, without loss of generality, an integration constant has been dropped. Substitution of (16) into equation (15c) yields

$$\eta^4 W,_{\eta\eta\eta\eta} + 6\eta^3 W,_{\eta\eta\eta} + 7\eta^2 W,_{\eta\eta} + \eta W,_{\eta}$$

$$+ W \tan^2\alpha\,(1-\upsilon^2) = 0 \qquad (17)$$

Assuming

$$W = \frac{1}{\tan \alpha} \overline{C} \, \eta^{\lambda} \, \frac{\cos}{\sin} \, \overline{n\pi\theta} \over \theta_1 \tag{18}$$

equation (17) results in a characteristic equation

$$\lambda^4 + (1-\upsilon^2) \tan^2 \alpha = 0 \tag{19}$$

which gives

$$\lambda = \pm q \, (1 \pm i) \tag{20}$$

where

$$q \equiv \left| [(1-\upsilon^2) \tan^2 \alpha]^{\frac{1}{4}} \frac{\sqrt{2}}{2} \right| \tag{21}$$

Letting

$$\begin{aligned} V &= \overline{B} \, \eta^{\lambda} \, \frac{\cos}{\sin} \, \frac{n\pi\theta}{\theta_1} \\ u &= \overline{A} \, \eta^{\lambda} \, \frac{\sin}{\cos} \, \frac{n\pi\theta}{\theta_1} \end{aligned} \tag{22}$$

and making use of equations (16) and (15b)

$$\begin{aligned} \overline{B} &= - \frac{2\upsilon}{\lambda} \overline{C} \\ \overline{A} &= \pm \frac{4}{\lambda^2} (2+\upsilon) \, m \, \overline{C} \end{aligned} \tag{23}$$

where $\overline{A}$, $\overline{B}$, and $\overline{C}$ are complex numbers. When the identity

$$\eta^{i} = \cos(\ell n\, \eta) + i \sin(\ell n \eta)$$

is used and the complex numbers are transformed to real numbers, one has

$$W = \frac{1}{\tan\alpha}\Big\{\eta^{q}[C_5 \cos(q\,\ell n\eta) + C_6 \sin(q\ell n\eta)]$$

$$+ \eta^{-q}[C_7 \cos(q\ell n\eta) + C_8 \sin(q\,\ell n\eta)]\Big\} \begin{matrix}\cos\\ \sin\end{matrix} \frac{n\pi\theta}{\theta_1}$$

$$V = -\frac{\upsilon}{q}\Big\{\eta^{q}[(C_5 - C_6)\cos(q\ell n\eta) + (C_5 + C_6)\sin(q\,\ell n\eta)]$$

$$- \eta^{-q}[(C_7 + C_8)\cos(q\ell n\eta) + (C_8 - C_7)\sin(q\ell n\eta)]\Big\} \begin{matrix}\cos\\ \sin\end{matrix} \frac{n\pi\theta}{\theta_1} \qquad (24)$$

$$U = \mp\frac{2(2+\upsilon)m}{q^2}\Big\{\eta^{q}[C_6 \cos(q\,\ell n\eta) - C_5 \sin(q\,\ell n\eta)]$$

$$-\eta^{-q}[C_8 \cos(q\,\ell n\eta) - C_7 \sin(\ell n\eta)]\Big\} \begin{matrix}\sin\\ \cos\end{matrix} \frac{n\pi\theta}{\theta_1}$$

Expressing in terms of the variable y and denoting

$$\gamma q = \rho \qquad (25)$$

the solutions given in (11) and the induced stress forces and couples obtained from (14) assume the following final forms:

$$W^{II} = \frac{1}{\tan\alpha}\Big\{y^{\rho}[C_5 \cos(\rho\,\ell ny) + C_6 \sin(\rho\,\ell ny)]$$

$$+ y^{-\rho}[C_7 \cos(\rho\,\ell ny) + C_8 \sin(\rho\,\ell ny)]\Big\} \begin{matrix}\cos\\ \sin\end{matrix} \frac{n\pi\theta}{\theta_1}$$

$$v^{II} = \frac{\upsilon}{\rho} Y_1$$

$$u^{II} = \mp \frac{2(2+\upsilon)m}{\rho^2} Y_2$$

$$N_s^{II} = 0$$

$$N_\theta^{II} = \frac{Et}{L} \tan \alpha \, w^{II} \frac{1}{y^2} \tag{26}$$

$$M_s^{II} = \frac{2Et}{\rho^2} \tan \alpha \frac{1}{y^4} Y_2$$

$$M_\theta^{II} = \upsilon M_s^{II}$$

$$M_{s\theta}^{II} = \mp \frac{2(1-\upsilon)}{\rho^3} m \tan \alpha \frac{1}{y^4} \Big\{ y^{\rho} [(C_5+C_6) \cos(\rho \, \ell ny)$$

$$+ (C_6-C_5)\sin(\rho \, \ell ny)] - y^{-\rho} [(C_7-C_8)\cos(\rho \, \ell ny)$$

$$+ (C_8 + C_7)\sin(\rho \, \ell ny)] \Big\} \begin{matrix}\cos \\ \sin\end{matrix} \frac{n\pi\theta}{\theta_1}$$

$$S_s^{II} = \frac{Et}{L} \tan \alpha \frac{1}{\rho} \frac{1}{y^4} Y_1$$

$$S_\theta^{II} = \mp \frac{Et}{L} 2 (2-\upsilon) \frac{1}{\rho^2} m \tan \alpha \frac{1}{y^4} Y_2$$

where

$$Y_1 = y^{\rho} [(C_6-C_5) \cos(\rho \, \ell ny) - (C_5+C_6)\sin(\rho \, \ell ny)]$$
$$+ y^{-\rho} [(C_8+C_7)\cos(\rho \, \ell ny) + (C_8-C_7)\sin(\rho \, \ell ny)]$$

$$Y_2 = y^{\rho} [C_6 \cos(\rho \, \ell ny) - C_5 \sin(\rho \, \ell ny)]$$
$$- y^{-\rho} [C_8 \cos(\rho \, \ell ny) - C_7 \sin(\rho \, \ell ny)]$$

## A Particular Solution

Consider a conical shell subjected to a lateral normal load which is constant along the meredians and has a sinusoidal distribution in the circumferential direction. This was the case treated in [9]. The set of equations (6) of membrane theory may be used for the particular solution.

Let

$$P_\theta = P_s = 0$$

$$P_r = p_n \begin{matrix}\cos \\ \sin\end{matrix} \frac{n\pi\theta}{\theta_1} \tag{27}$$

and assume

$$u^p = d_1 y^4 \begin{matrix}\sin \\ \cos\end{matrix} \frac{n\pi\theta}{\theta_1}$$

$$v^p = d_2 y^4 \begin{matrix}\cos \\ \sin\end{matrix} \frac{n\pi\theta}{\theta_1} \tag{28}$$

$$w^p = d_3 y^4 \begin{matrix}\cos \\ \sin\end{matrix} \frac{n\pi\theta}{\theta_1}$$

where $d_1$, $d_2$, and $d_3$ are coefficients to be determined by the substitution of expressions (27) and (28) into equations (6). When this is done the results are

$$u^p = \pm \frac{p_n L^2}{Eh} \frac{1}{12} \frac{m}{\tan \alpha} [11 + 2\upsilon - m^2] y^4 \begin{matrix}\sin \\ \cos\end{matrix} \frac{n\pi\theta}{\theta_1}$$

$$v^p = \frac{p_n L^2}{Eh} \frac{1}{12 \tan \alpha} [3(1-2\nu) - m^2] y^4 \frac{\cos}{\sin} \frac{n\pi\theta}{\theta_1}$$

$$w^p = \frac{p_n L^2}{Eh} \frac{1}{12 \tan \alpha} (m^2-1)(m^2-9) y^4 \frac{\cos}{\sin} \frac{n\pi\theta}{\theta_1} \tag{29}$$

The corresponding stresses obtained from (1) with k = 0 are:

$$N_s^{\ p} = \frac{p_n L}{6 \tan \alpha} (3-m^2) y^2 \frac{\cos}{\sin} \frac{n\pi\theta}{\theta_1}$$

$$N_\theta^{\ p} = \frac{p_n L}{\tan \alpha} y^2 \frac{\cos}{\sin} \frac{n\pi\theta}{\theta_1} \tag{30}$$

$$N_{\theta s}^{\ p} = \pm \frac{p_n L}{3 \tan \alpha} m\, y^2 \frac{\sin}{\cos} \frac{n\pi\theta}{\theta_1}$$

By retaining the solutions of the lowest order of $\frac{1}{\rho}$, one finally has the complete solutions for a shell subjected to the lateral load (7)

$$\begin{aligned}
u &= u^I + u^p, \quad v = v^I + v^p, \\
w &= w^I + w^{II} + w^p; \\
N_s &= N_s^{\ I} + N_s^{\ p}, \quad N_\theta = N_\theta^{\ II} + N_s^{\ p}, \\
N_{s\theta} &= N_{\theta s} = T_s = N_{s\theta}^{\ I} + N_{s\theta}^{\ p} \\
M_s &= M_s^{\ II} \quad M_\theta = M_\theta^{\ II} \\
M_{\theta s} &= M_{s\theta} = M_{s\theta}^{\ II}, \quad S_\theta = S_\theta^{\ II} \\
S_s &= S_s^{\ II}
\end{aligned} \tag{31}$$

In what follows, two numerical examples are given. One is the engine shroud discussed in [9]. The other is a cantilevered complete cone frustum for which the numerical solutions are available in [5] and [6]. Comparisons of the present solution with those given in [6] will be made.

Example 1

The engine shroud considered is a semicircular truncated conical shell segment which has two generators simply supported with the small end fixed and the other end free. Thus the lower set of sinusoidal functions of the solutions is used with the following boundary conditions for the solution.

$$u = v = w = \frac{\partial w}{\partial s} = 0 \qquad \text{at } y = \sqrt{\frac{L_1}{L}}$$
$$N_s = T_s = M_s = S_s = 0 \quad \text{at } y = 1 \tag{32}$$

The same material and geometrical constants as used in [9] are used here, i.e.,

$$\upsilon = \frac{1}{3}, \; \alpha = 75^{o} \text{ and } \frac{L_1}{L} = 0.90 \tag{33}$$

Numerical results for $\frac{t}{R} = 0.006$ and $n = 1, 2$ are computed. The results are given in the form

$$F_n(y, \theta) = f_n(y) \begin{matrix} \sin \\ \cos \end{matrix} \frac{n\pi\theta}{\theta_1} \qquad n = 1 \text{ and } 2 \tag{34}$$

The functions $f_n(y)$ are shown as the solid lines in Figs. 1 to 7. The respective functions obtained in [9] are also shown in these figures by dotted lines if there are some differences.

Example 2

In this example, a cantilevered complete cone frustum fixed at the smaller end is considered. At the larger free end, a rigid plate is attached and a moment, M, is applied about a horizontal axis. Thus the solutions are symmetrical about the vertical axis through the center of the cone. For such a complete cone, the upper set of sinusoidal functions of the solutions is used with the angle $\theta$ measured from the vertical line taking $n = 1$ and $\theta_1 = \pi$.

The boundary conditions at the free end, referring to [5], can be given as follows:

$$\pi R_1 [\overline{T}_s + \overline{S}_s \sin \alpha - \overline{N}_s \cos \alpha] = 0$$

$$\pi R_1 [\overline{M}_s - R_1 (\overline{N}_s \sin \alpha + \overline{S}_s \cos \alpha)] = -M$$

$$\overline{u} \sec \alpha + \overline{v} + \overline{w} \tan \alpha = 0 \qquad (35)$$

$$\frac{\partial \overline{w}}{\partial s} + \frac{1}{R_1} [\overline{v} \sin \alpha - \overline{w} \cos \alpha] = 0$$

where $R_1 = \dfrac{R}{\sin \alpha}$ and a function with a bar indicates that the function is of function of y only. When the asymptotic solutions are used and the terms of the lowest order of $\frac{1}{\rho}$ are retained, the conditions (35) become

$$\overline{N}_{s\theta}{}^{I} - \overline{N}_s{}^{I} \cos \alpha = 0$$

$$\bar{N}_s^{I} = \frac{M}{\pi R_1^2} \frac{1}{\sin \alpha}$$

$$\bar{u}^{I} \sec \alpha + \bar{v}^{I} + (\bar{w}^{I} + \bar{w}^{II}) \tan \alpha = 0 \qquad \text{at } y = 1 \qquad (36)$$

$$\frac{\partial \bar{w}^{II}}{\partial s} = 0$$

The other four boundary conditions at the fixed end are:

$$u^{I} = v^{I} = w^{I} + w^{II} = \frac{\partial w^{II}}{\partial s} = 0 \qquad \text{at } y = \sqrt{\frac{L_1}{L}} \qquad (37)$$

The following material and geometrical constants are used:

$$\upsilon = 0.3, \ \frac{t}{R} = \frac{1}{40}, \ \tan \alpha = \frac{4}{3}, \text{ and } \frac{L_1}{L} = \frac{5}{8} \qquad (38)$$

Two sets of stress ratios, $\sigma_2/\sigma_{1Mmax}$ and $\sigma_m/\sigma_{1Mmax}$, were computed, and are given in Fig. 9, where

$$\sigma_2 = \frac{N_\theta}{t}$$

$$\sigma_{1Mmax} = \frac{N_s}{t}\Big]_{max} \qquad (39)$$

$$\sigma_m = \frac{6M_s}{h^2}$$

## Closing Remarks

The asymptotic solutions obtained are relatively simple when compared to other available solutions for conical shells of constant thickness. The results given by the first example show that the difference between the solutions for linearly varying thickness and constant thickness is relatively small. This indicates that the assumption that these two types of shell will behave alike when the ratio of thickness to radius is very small is acceptable.

The difference between the present and other available solutions shown in the second example may be attributed to the relatively large ratio of t/R which is $\frac{1}{40}$. Such a shell is relatively thick for the application of the asymptotic solutions. Nevertheless, the results may still be valuable for preliminary design purposes assuming that the other solutions are better than the present solutions. This assumption, however, needs further verification which can probably be obtained by an experimental study.

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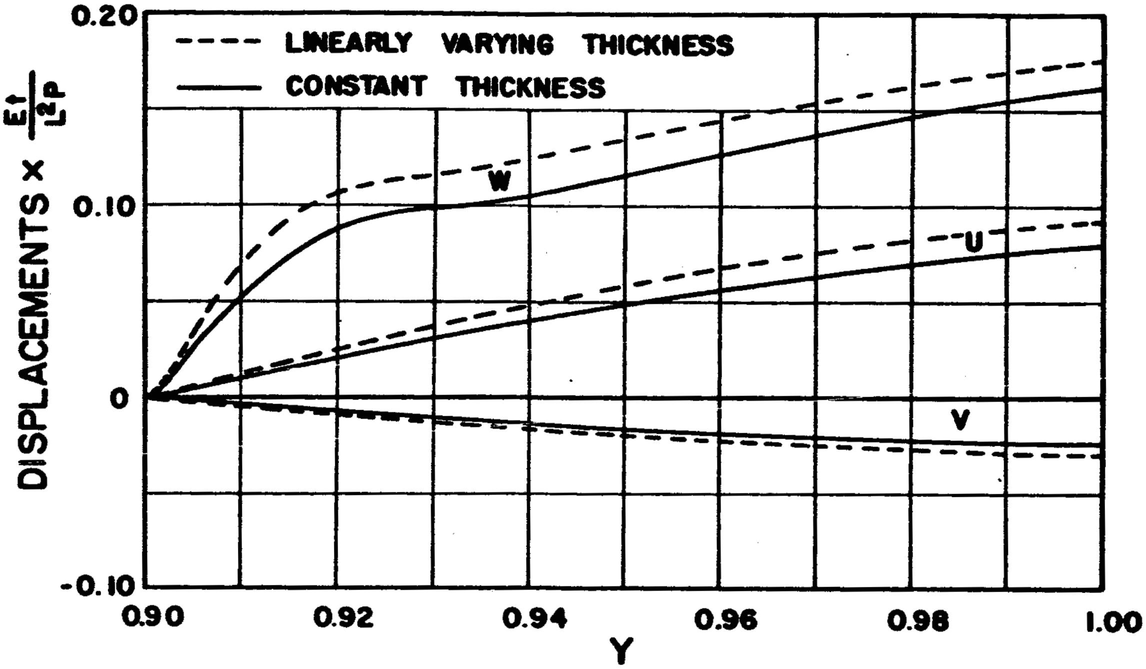


FIG. 1 DISPLACEMENTS U, V AND W (n=1)

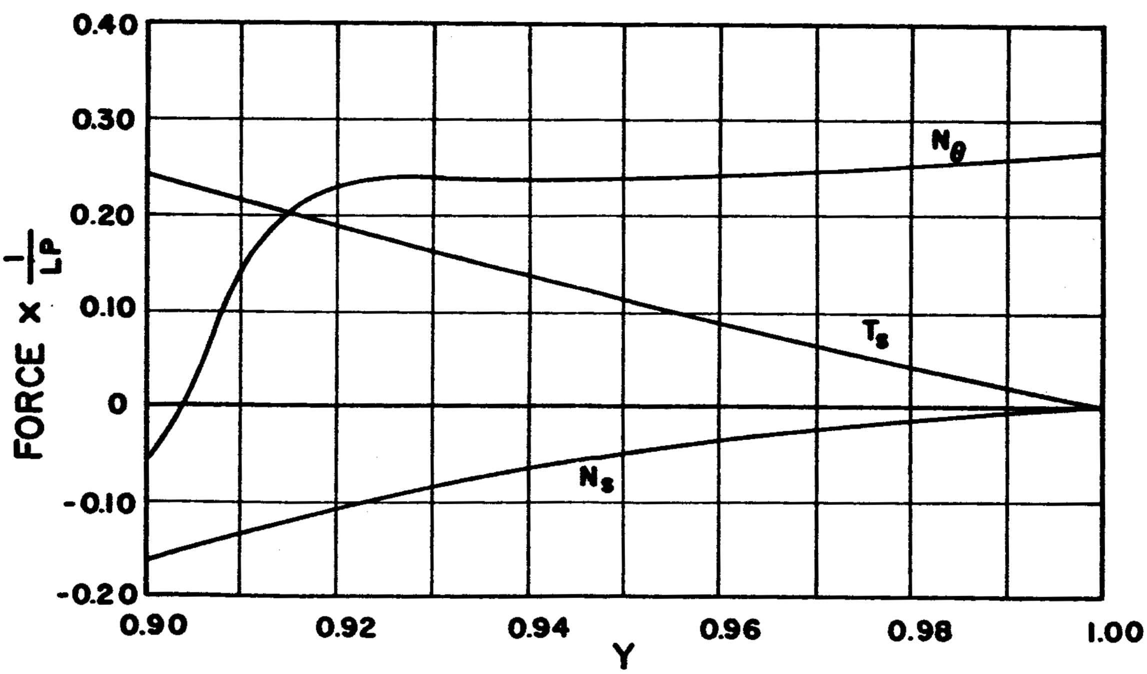


FIG. 2 MEMBRANE FORCES $N_\theta$ $T_s$ AND $N_s$ (n=1)

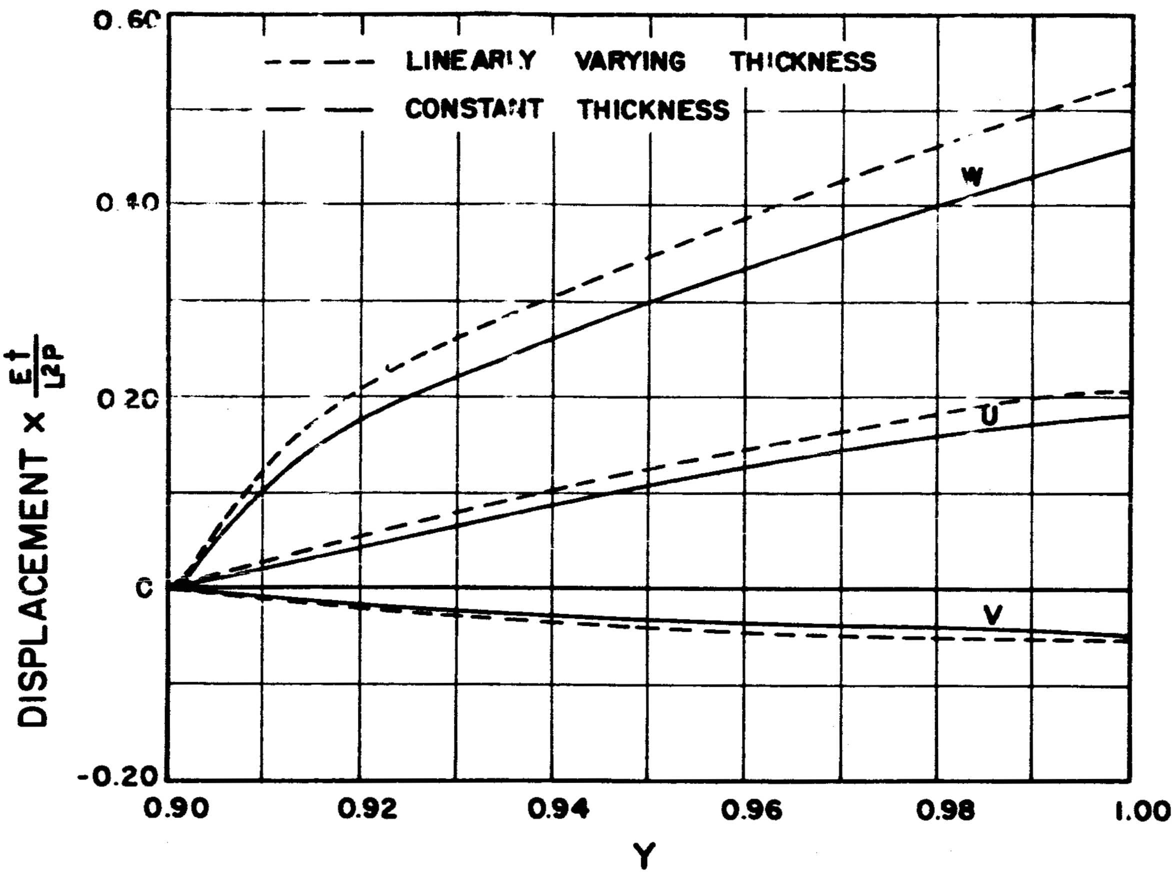


FIG. 3 DISPLACEMENTS U, V AND W (n=2)

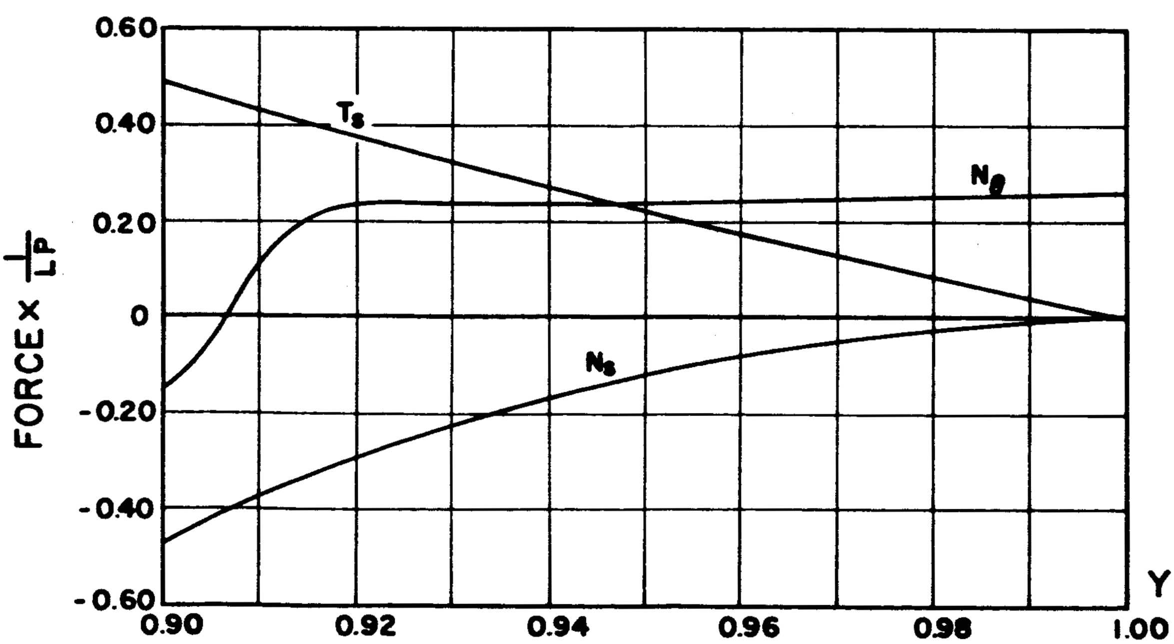


FIG. 4 MEMBRANE FORCES $N_\theta$, $T_S$ AND $N_S$ (n=2)

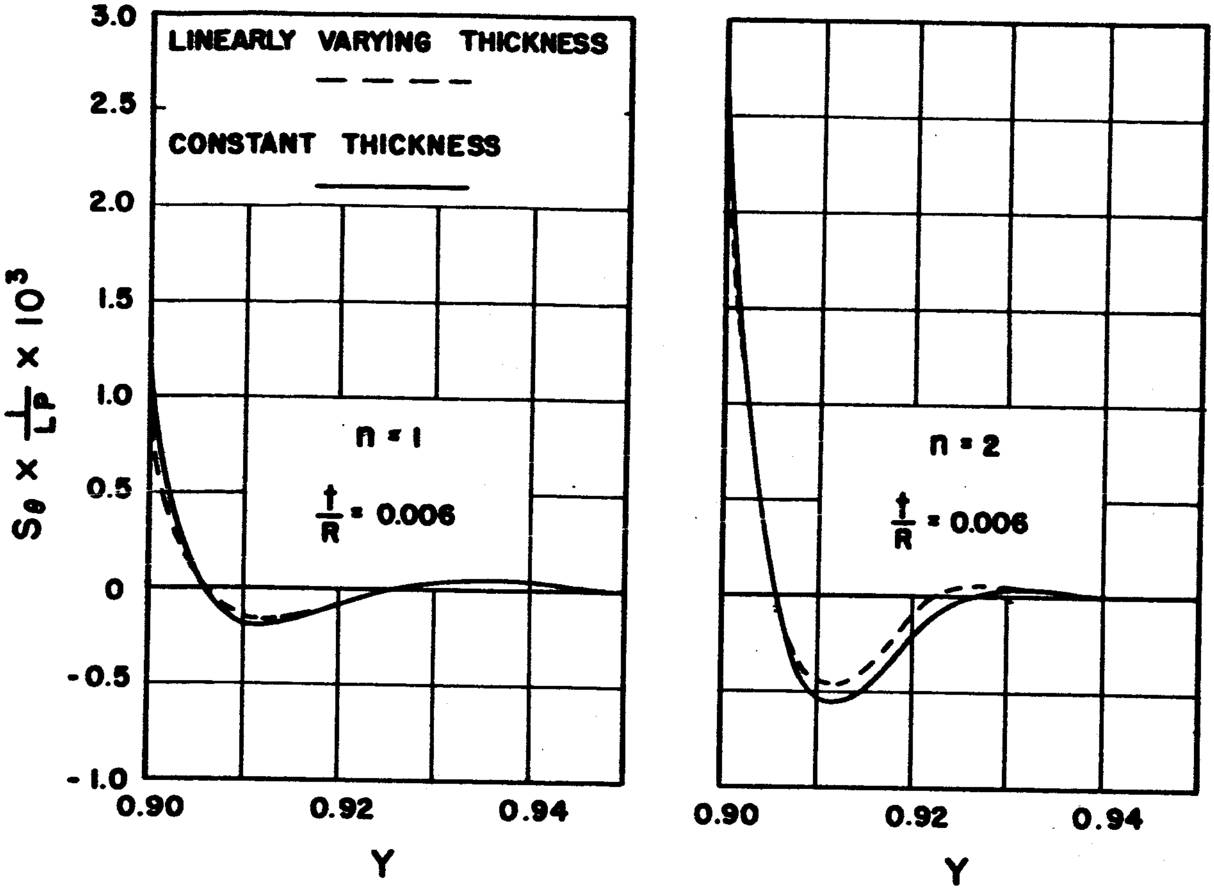


FIG. 5 TRANSVERSE SHEARING FORCE $S_\theta$

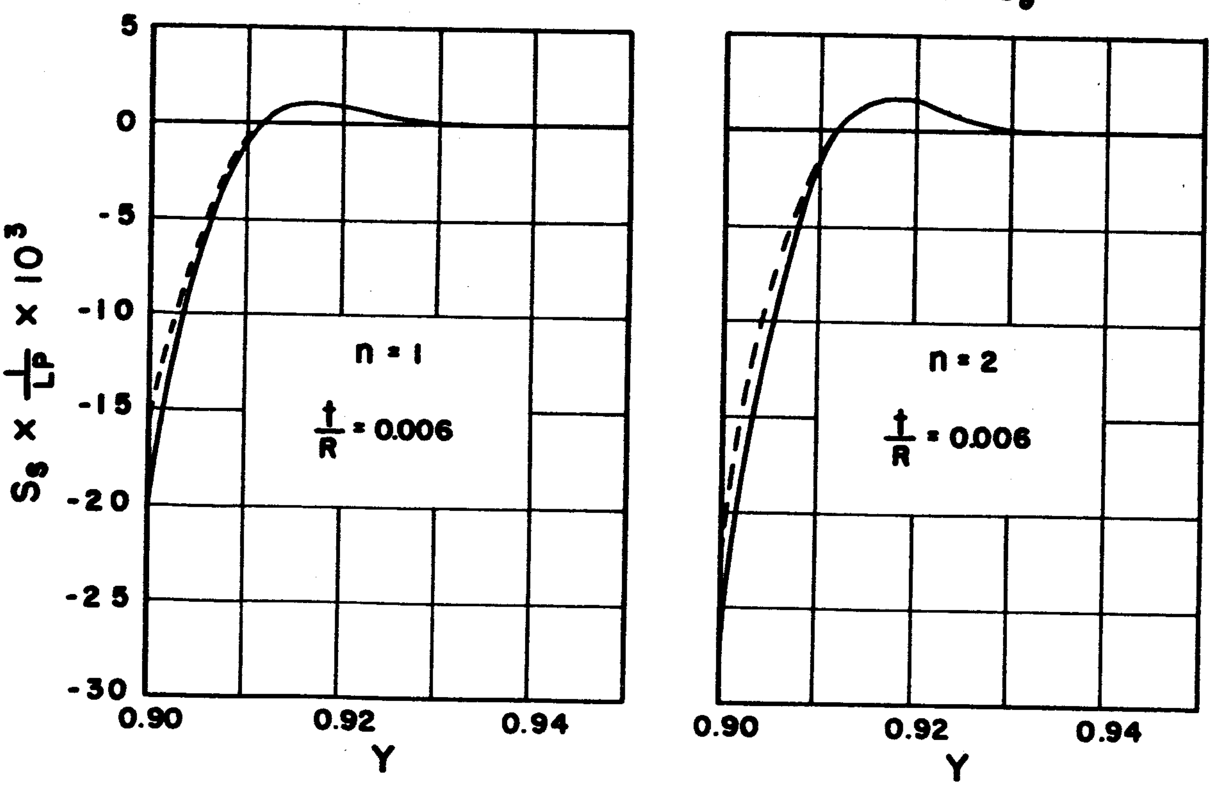


FIG. 6 TRANSVERSE SHEARING FORCE $S_s$

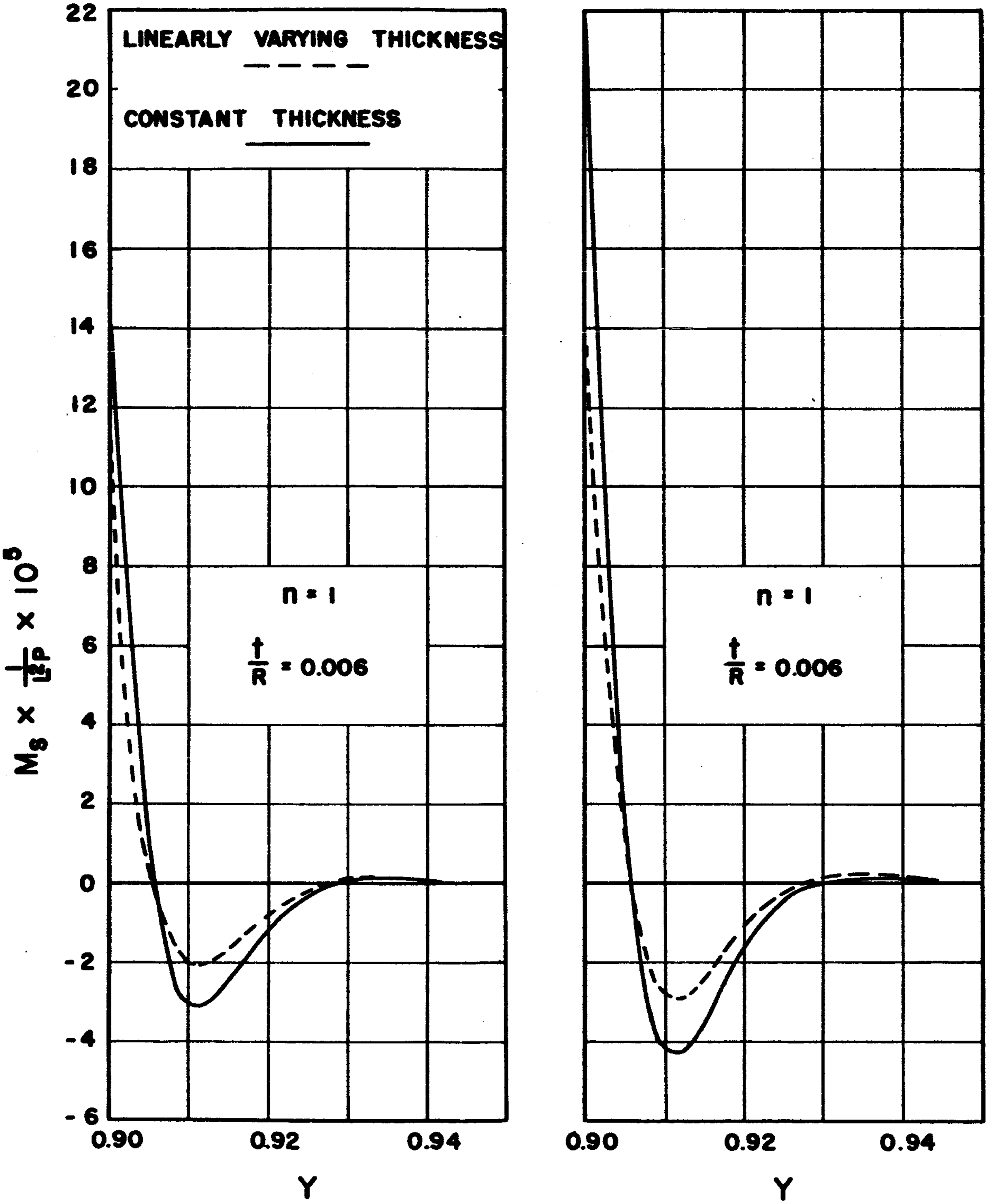


FIG. 7 NORMAL MOMENT $M_s$

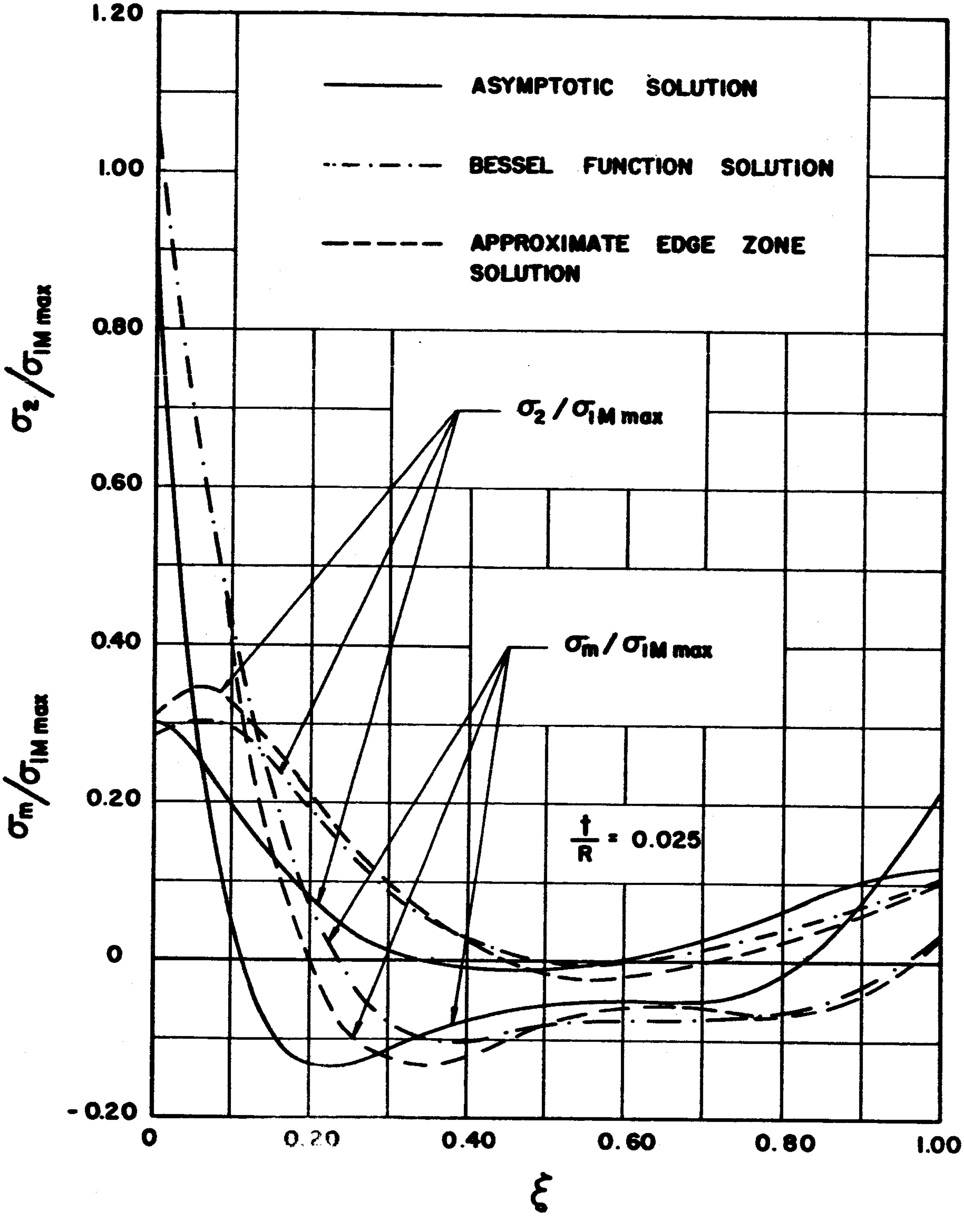


FIG. 8 STRESSES FOR EXAMPLE 2

# COMPUTER PROGRAMS

# FOR PROGRESS REPORTS 1, 2 AND 6

Prepared by

Han Yun Chu*

The Univac Solid-State Fortran II language is used for the computer programs included.

The symbols used in the computer programs are different from those in the equations. The following table shows the correspondences:

| In equations | $\frac{Et}{pL^2} d_1$ | $\frac{Et}{pL^2} d_2$ | $\frac{Et}{pL^2} b_1$ | $\frac{Et}{pL^2} C_i$ | $\frac{Et}{pL^2} w$ | $\frac{1}{p\ L^2} M_s$ |
|---|---|---|---|---|---|---|
| In programs | D(1) | D(2) | D(3) | C(i) | W | Y |

| $\frac{1}{pL} S_s$ | $\frac{Et}{pL^2} u$ | $\frac{Et}{pL^2} v$ | $\frac{Et}{pL} \frac{\partial w}{\partial s}$ | $\frac{1}{pL} N_s$ | $\frac{1}{pL} N_\theta$ | $\frac{1}{pL} T_s$ | $\frac{1}{pL} S_\theta$ |
|---|---|---|---|---|---|---|---|
| Z | E | F | G | H | A | B | X |

| $\tau$ | $\alpha$ | $\nu$ | L | $\delta$ | k | $\rho$ | y | $\xi$ | m | $\beta$ | $\sigma_{2m}/\sigma_{1m_{max}}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| TS | S | P | FLI | DB | FKS | Q | T | R | U | CT | RN |

| $\sigma_m/\sigma_{1m_{max}}$ |
|---|
| RM |

*Research Associate

1. Computer program 1 is for the example given in Progress Report 1 for a semicircular truncated conical shell of linearly varying thickness with the smaller end fixed and the other end free, subjected to lateral normal loads which are constant in the meridional direction and with sinusoidal variation in the circumferential direction. The generator edges of the shell are simply supported.

Computer Program 1.

```
      DIMENSION D(3), C(8)
2     READ 1, TS, S, P, R, U
1     FORMAT (5E15.7)
      FL1=373.312/0.96593
      DB=TS/FL1
      FKS=DB**2/12.0
      Q=(SQRT(2.0)/2.0)*((16.0*(1.0-P**2)*(TAN(S))**2)/FKS)**(1.0/4.0)
      Q=ABS(Q)
      PRINT 4, S, P, R, FL1, DB, FKS, Q
4     FORMAT (6HS    =,E15.7,/,6HP    =,E15.7,/,6HR    =,E15.7,/,
     1        6HFL1  =,E15.7,/,6HDB   =,E15.7,/,6HFKS  =,E15.7,/,
     2        6HQ    =,E15.7,/)
      V=Q*LN(R)
      D(1)=U/(TAN(S)*3.0)*(U**2-3.0*(5.0-P)/2.0-3.0*(1.0+P)/(2.0*U**2))
      D(2)=U/TAN(S)/3.0*(U**2-7.0+2.0*P)
      D(3)=1.0/(6.0*TAN(S))*(3.0*(1.0-2.0*P)-U**2)
      C(1)=(U**2-1.0)/TAN(S)*(-D(3)*R**2-((1.0+P)*D(3)-U*P*(D(1)-D(2))+
     1(1.0-P)/4.0*U*(D(2)+2.0*U*D(3)))/((1.0-P**2)*R**2)+U*(D(2)+2.0*U*
     2D(3))/(8.0*(1.0+P)*R**4))
      C(2)=(U**2-2.0*(1.0-P))/(2.0*(1.0-P**2)*TAN(S))*((1.0+P)*D(3)-U*P
     1*(D(1)-D(2))+(1.0-P)*U*(D(2)+2.0*U*D(3))/4.0)
      C(3)=U/TAN(S)*((U*D(3)-D(1)-D(2)*LN(R))+U/(2.0*(1.0-P**2)*R**4)
     1*((1.0+P)*D(3)-U*P*(D(1)-D(2))+(1.0-P)/4.0*U*(D(2)+2.0*U*D(3)))
     2-(D(2)+2.0*U*D(3))/(12.0*(1.0+P)*R**6)*(U**2+2.0*(1.0+P)))
      C(4)=-(U*(D(2)+2.0*U*D(3))*(U**2-7.0+2.0*P))/(24.0*(1.0+P)*TAN(S))
      CT=(R**Q*(3.0*COS(V)-SIN(V))+R**(-Q)*(COS(V)-SIN(V)))/(R**Q*(-COS(
     1V)+SIN(V))+R**(-Q)*(COS(V)+SIN(V)))
      C(6)=-(C(1)*R+C(2)/R+C(3)*R**3+C(4)/R**3+U/TAN(S)*D(2)*(1.0+LN(R))
     1*R**3)/(R**Q*((2.0+CT)*COS(V)+SIN(V))+R**(-Q)*(CT*COS(V)+SIN(V)))
      C(5)=(2.0+CT)*C(6)
      C(7)=CT*C(6)
      C(8)=C(6)
```

```
      PRINT 7, (J, C(J), J=1, 8)
7     FORMAT (2HC(, I2, 4H) = , E15.7, /)
5     READ 6, T
6     FORMAT (E15.7)
      W=(C(1)+C(2)/T**2+C(3)*T**2+C(4)/T**4+U/TAN(S)*D(2)*(1.0+LN(T))*T*
     1*2+T**(-1)*(T**Q*(C(5)*COS(Q*LN(T))+C(6)*SIN(Q*LN(T)))+T**(-Q)*(C(
     27)*COS(Q*LN(T))+C(8)*SIN(Q*LN(T)))))
      Y=(2.0/Q**2*T*(T**Q*(C(6)*COS(Q*LN(T))-C(5)*SIN(Q*LN(T)))+T**(-Q)
     1*(-C(8)*COS(Q*LN(T))+C(7)*SIN(Q*LN(T)))))*(TAN(S))**2
      Z=(1.0/(Q*T)*(T**Q*((-C(5)+C(6))*COS(Q*LN(T))+(-C(5)-C(6))
     1*SIN(Q*LN(T)))+T**(-Q)*((C(7)+C(8))*COS(Q*LN(T))+(-C(7)+C(8))
     2*SIN(Q*LN(T)))))*(TAN(S))**2
      E=U*TAN(S)*(C(1)/(U**2-1.0)+C(2)/((U**2-2.0*(1.0-P))*T**2)+C(3)
     1*T**2/U**2+((U**2-4.0*(1.0+P))*C(4))/(U**2*(U**2-7.0+2.0*P)*T**4))
     2+(D(1)+D(2)*LN(T))*T**2
      F=TAN(S)*(C(1)/(U**2-1.0)+(2.0*C(2))/(U**2-2.0*(1.0=P))
     1*T**(-2)+3.0*C(4)/((U**2-7.0+2.0*P)*T**4))+D(3)*T**2
      G=Q/(2.0*T**3)*(T**Q*((C(5)+C(6))*COS(Q*LN(T))+(-C(5)+C(6))
     1*SIN(Q*LN(T)))+T**(-Q)*((-C(7)+C(8))*COS(Q*LN(T))
     2+(-C(7)-C(8))*SIN(Q*LN(T))))
      H=-2.0*TAN(S)*(C(2)/((U**2-2.0*(1.0-P))*T**2)+3.0*C(4)/((U**2-7.0+
     12.0*P)*T**4))+T**2/(1.0-P**2)*((1.0+P)*D(3)-U*P*(D(1)-D(2)))
      A=1.0/(1.0-P**2)*(D(3)*(1.0+P)-U*(D(1)-D(2)))*T**2
     1+T**(-1)*(T**Q*(C(5)*COS(Q*LN(T))+C(6)*SIN(Q*LN(T)))
     2+T**(-Q)*(C(7)*COS(Q*LN(T))+C(8)*SIN(Q*LN(T))))*TAN(S)
      B=((6.0*TAN(S)*C(4))/(U*(U**2-7.0+2.0*P)*T**4)
     1+((D(2)+2.0*U*D(3))*T**2)/(4.0*(1.0+P)))
      X=2.0*U*(2.0-P)/(Q**2*T)*(T**Q*(C(6)*COS(Q*LN(T))-C(5)*SIN(Q*
     1LN(T)))+T**(-Q)*(-C(8)*COS(Q*LN(T))+C(7)*SIN(Q*LN(T))))
     2*(TAN(S))**2
      PRINT 8, W, TS, Y, Z, E, F, U, G, H, A, B, T, X
8     FORMAT (4HW = , E19.7, 15X, 4HTS= , E15.7, /, 4HY = , E19.7, /,
     1        4HZ = , E19.7, /,   4HE = , E19.7, /, 4HF = , E19.7, 15X,
     2        4HU = , E15.7, /,   4HG = , E19.7, /, 4HH = , E19.7, /,
     3        4HA = , E19.7, /,   4HB = , E19.7, 15x, 4HT = , E15.7, /,
     4        4HX = , E19.7, 3/)
      GO TO 5
      END
DATA CARDS:
0.40       1.309       0.33       0.90       3.86
0.40       1.309       0.33       0.90       7.72
0.60       1.309       0.33       0.90       3.86
0.60       1.309       0.33       0.90       7.72
0.80       1.309       0.33       0.90       3.86
0.80       1.309       0.33       0.90       7.72

0.900
0.902
 ---
```

0.912
0.916
---
0.940
0.950
---
1.000

2. Computer program 2 is for the example given in Progress Report 2 for the same shell as given in 1 but subjected to thermal loads instead of lateral normal loads.

Computer Program 2.

The same program as Computer Program 1 is used except cards D(1), D(2) and D(3) are replaced by the following cards respectively.

```
 D(1)=U/(6.0*(1.0-P))*(3.0*(1.0+3.0*P)-3.0*(5.0-P)*TAN(S)
1-(2.0*U**2-3.0*(1.0+P)/U**2)*(1.0-TAN(S)))
 D(2)=U/(3.0*(1.0-P))*((1.0+4.0*P-U**2)-TAN(S)*(7.0-2.0*P-U**2))
 D(3)=U**2/(6.0*(1.0-P))*(1.0-TAN(S)+3.0/U**2*((1.0-2.0*P)
1*TAN(S)+1.0))
```

3. Computer Program 3 is for the first example given in Progress Report 6 for the same shell as given in 1, but of constant thickness.

Computer Program 3.

```
      DIMENSION D(3), C(8)
      READ 1, TS, S, P, R
1     FORMAT (4E15.7)
      FL1=373.312/0.96593
      Q=(48.0*(1.0-P**2))**(0.25)*(FL1*TAN(S)/TS)**(0.5)
      Q=ABS(Q)
      PRINT 2, TS, S, P, R, FL1, Q
2     FORMAT (6HTS =, E15.7, /, 6HS =, E15.7, /, 6HP =, E15.7,
     1/, 6HR  =, E15.7, /, 6HFL1 =, E15.7, /, 6HQ  =, E15.7, 3/)
      V=Q*LN(R)
      BT=12.0*TAN(S)
3     READ 4, T
4     FORMAT (E15.7)
      DO 5, UR=1.0, 2.0.1.0
      U=3.86*UR
      D(1)=-U*(11.0+2.0*P-U**2)/BT
      D(2)=(3.0*(1.0-2.0*P)-U**2)/BT
      D(3)=(U**2-1.0)*(U**2-9.0)/BT
      C(4)=2.0*U**2*(U**2-4.0)/BT
      C(2)=12.0*((U**2-1.0)**2)/BT
      C(1)=((U**2-P)/(2.0*(U**2-1.0))-LN(R))*C(2)-(U**2-1.0)
     1*(2.0*C(4)/(R**2*(U**2-4.0))+D(2)*R**4)
      C(3)=-U*D(1)*R**2-(U**2-2.0-2.0*P)*C(4)/(R**4*(U**2-4.0))
     1-U**2/(R**2*(U**2-1.0))*(C(1)-((1.0-P)/(2.0*(U**2-1.0))
     2-LN(R))*C(2))
      CK=(R**Q*(3.0*COS(V)-SIN(V))+R**(-Q)*(COS(V)-SIN(V)))
     1/(R**Q*(-COS(V)+SIN(V))+R**(-Q)*(CK*COS(V)+SIN(V)))
      C(6)=-(C(1)+C(2)*LN(R)+C(3)*R**2+C(4)*R**(-2)+D(3)*R**4)
     1/(R**Q*((CK+2.0)*COS(V)+SIN(V))+R**(-Q)*(CK*COS(V)+SIN(V)))
      C(7)=CK*C(6)
      C(5)=C(7)+2.0*C(6)
      C(8)=C(6)
      VT=Q*LN(T)
      W=1.0/TAN(S)*(C(1)+C(2)*LN(T)+C(3)*T**2+C(4)*T**(-2)+D(3)*T**4
     1+T**Q*(C(5)*COS(VT)+C(6)*SIN(VT))+T**(-Q)*
     2(C(7)*COS(VT)+C(8)*SIN(VT)))
      Y=2.0*TAN(S)/(Q**2*T**4)*(T**Q*(C(6)*COS(VT)-C(5)*SIN(VT))
     1+T**(-Q)*(-C(8)*COS(VT)+C(7)*SIN(VT)))
      Z=TAN(S)/(Q*T**4)*(T**Q*((C(6)-C(5))*COS(VT)-(C(5)+C(6))*SIN(VT))
     1+T**(-Q)*((C(8)+C(7))*COS(VT)-(C(7)-C(8))*SIN(VT)))
      E=1.0/U*(U**2/(U**2-1.0)*(C(1)-((1.0-P)/(2.0*(U**2-1.0))
```

```
      1-LN(T))*C(2))+C(3)*T**2+(U**2-2.0-2.0*P)*C(4)
      2/(T**2*(U**2-4.0)))+D(1)*T**4
       F=1.0/(U**2-1.0)*(C(1)-((U**2-P)/(2.0*(U**2-1.0))-LN(T))*C(2))
      1+2.0*C(4)/(T**2*(U**2-4.0))+D(2)*T**4
       G=Q/(2.0*TAN(S)*T**2)*(T**Q*((C(5)+C(6))*COS(VT)
      1+(C(6)-C(5))*SIN(VT))-T**(-Q)*((C(7)-C(8))*COS(VT)
      2+(C(8)+C(7))*SIN(VT)))
       H=C(2)/(2.0*(U**2-1.0)*T**2)-2.0*C(4)/((U**2-4.0)*T**4)
      1+T**2/(1.0-P**2)*((2.0+P)*D(2)-P*U*D(1)+P*D(3))
       A=T**2/(1.0-P**2)*((2.0*P+1.0)*D(2)-U*D(1)+D(3))+1.0/T**2
      1*(T**Q*(C(5)*COS(VT)+C(6)*SIN(VT))+T**(-Q)*
      2(C(7)*COS(VT)+C(8)*SIN(VT)))
       B=2.0*C(4)/(U*(U**2-4.0)*T**4)+T**2/(2.0*(1.0+P))
      1*(D(1)+U*D(2))
       X=2.0*TAN(S)*(2.0-P)*U/(Q**2*T**4)*(T**Q*(C(6)*COS(VT)
      1-C(5)*SIN(VT))+T**(-Q)*(-C(8)*COS(VT)+C(7)*SIN(VT)))
       PRINT 6, W, T, Y, Z, E, F, U, G, H, A, B, X
6      FORMAT (5H W = , E19.7, 15X, 4HT = , E15.7, /, 5H Y = , E19.7, /,
      15H Z = , E19.7, /, 5H E = , E19.7, /, 5H F = , E19.7, 15X,
      24HU = , E15.7, /, 5H G = , E19.7, /, 5H H = , E19.7, /,
      35H A = , E19.7, /, 5H B = , E19.7, /, 5H X = , E19.7, 3/)
5      CONTINUE
       GO TO 3
       END
```

DATA CARDS:

```
0.6          1.309          0.333          0.90
0.900
0.902
 ---
0.912
0.916
 ---
0.940
0.950
 ---
1.000
```

4. Computer Program 4 is for the second example given in Progress Report 6 for a complete truncated conical cantilever shell; at the larger free end a rigid plate is attached and a moment is applied.

Computer Program 4.

```
      DIMENSION C(8)
      READ 1, TS, S, P, R
1     FORMAT (4E15.7)
      FL1=13.333
      PI=3.1416
      BJ=1.0/(PI*(COS(S))**2*SIN(S))
      Q=ABS((48.0*(1.0-P**2))**(0.25)*(FL1*TAN(S)/TS)**(0.5))
      U=1.0/COS(S)
      PRINT 4, TS, S, P, Q, R, U
4     FORMAT (6H TS = , E15.7, /, 6H S  = , E15.7, /, 6H P  = , E15.7, /,
     16H Q  = , E15.7, /, 6H R  = , E15.7, /, 6H U  = , E15.7, 3/)
      V=Q*LN(R)
      C(1)=(U**2-1.0)*BJ/ R**2
      C(2)=0
      C(3)=-BJ*(1.0+P+U**2/2.0)/R**4
      C(4)=-(U**2-4.0)*BJ/2.0
      DET=(R**Q-R**(-Q))**2-4.0*(SIN(V))**2
      BET=C(1)+C(2)*LN(R)+C(3)*R**2+C(4)/R**2
      C(5)=-1.0/DET*(BET*(R**Q*(COS(V)+SIN(V))-R**(-Q)*(COS(V)-SIN(V)))
     1+P*BJ*(1.0-R**(-2.0*Q)+2.0*(SIN(V))**2+SIN(2.0*V)))
      C(6)=1.0/DET*(BET*((R**Q-R**(-Q))*COS(V)-(R**Q-3.0*R**(-Q))*
     1SIN(V))+P*BJ*(SIN(2.0*V)+COS(2.0*V)-R**(-2.0*Q)))
      C(7)=P*BJ-C(5)
      C(8)=P*BJ-2.0*C(5)-C(6)
      PRINT 7, (J, C(J), J=1, 8)
7     FORMAT (2HC(, I2, 4H) = , E15.7, /)
5     READ 6, XC
6     FORMAT (E15.7)
      T=SQRT((5.0*XC+8.333)/13.333)
      VT=Q*LN(T)
      HM=-2.0*C(4)/(R**4*(U**2-4.0))
      A=1.0/T**2*(T**Q*(C(5)*COS(VT)+C(6)*SIN(VT))
     1+T**(-Q)*(C(7)*COS(VT)+C(8)*SIN(VT)))
      Y=2.0*TAN(S)/(Q**2*T**4)*(T**Q*(C(6)*COS(VT)-C(5)*SIN(VT))
     1-T**(-Q)*(C(8)*COS(VT)-C(7)*SIN(VT)))
      RM=6.0*Y*FL1/(TS*HM)
      RN=A/HM
      PRINT 9, HM, T, Y, A, RM, XC, RN
```

```
9      FORMAT (4HHM = , E15.7, 15X, 4HT  =, E15.7, /, 4HY  = , E15.7, /,
     14HA = , E15.7, /, 4HRM =, E15.7, 15X, 4HXC =, E15.7, /, 4HRN =, E15.7, 3/)
       GO TO 5
       END
```

```
DATA CARDS
0.2             0.927           0.3             0.7900
0.00
0.02
- --
0.20
0.25
1.00
```