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On a Refinement of the Theory of the Moon's Physical Librations

I. Michelson*

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Abstract

The standard formulation of the dynamics of physical librations of the Moon is re-examined in the light of currently accepted reduced estimates of the mechanical ellipticity of the lunar equator. It is seen that a more complete mathematical model is required which accounts for centrifugal couples and in which the sum of inclinations of lunar orbit (5°9') and equator (1°30') is not regarded as an infinitesimal quantity. Although it remains doubtful whether linearized differential equations can be expected to yield a quantitatively useful theory, a preliminary to more accurate calculation consists in analyzing the motion with fewer restrictions than has been customary. The main features of such a treatment are given which unify the classical analysis by showing how the afore-mentioned inclinations can both be used to estimate the two principal mass parameters that affect physical librations. When accurate short-period libration data become available, the constants in question can be evaluated without recourse to orbital data used in the past.

*Professor of Aerospace Engineering, Illinois Institute of Technology, Chicago, Illinois 60616

Equations of Physical Libration

When the Moon is regarded as a rigid body, its total angular momentum is expressible as a function of its principal-axis inertia moments and angular velocity components:

$$\bar{H} = A\omega_1\bar{i}_1 + B\omega_2\bar{i}_2 + C\omega_3\bar{i}_3 \tag{1}$$

where the principal-axis unit vectors $\bar{i}_1$, $\bar{i}_2$, $\bar{i}_3$ appear with the quantities in each case corresponding to the same axis. The angular velocity components ω_1, ω_2, ω_3 are total quantities, referred to an inertial coordinate frame, and it is emphasized that no point of the Moon is then considered to be fixed in space. Taking account of the time variations of the unit vectors, the rate of change of total angular momentum becomes

$$\frac{d\bar{H}}{dt} = A\frac{d\omega_1}{dt}\bar{i}_1 + B\frac{d\omega_2}{dt}\bar{i}_2 + C\frac{d\omega_3}{dt}\bar{i}_3 + \begin{vmatrix} \bar{i}_1 & \bar{i}_2 & \bar{i}_3 \\ \omega_1 & \omega_2 & \omega_3 \\ A\omega_1 & B\omega_2 & C\omega_3 \end{vmatrix} . \tag{2}$$

If the components of the total moment of external forces acting are denoted M_1, M_2, M_3, these are equated to the components of (2) to obtain the familiar equations of Euler in scalar form

$$\begin{aligned} A\frac{d\omega_1}{dt} - (B - C)\omega_2\omega_3 &= M_1 \\ B\frac{d\omega_2}{dt} - (C - A)\omega_3\omega_1 &= M_2 \\ C\frac{d\omega_3}{dt} - (A - B)\omega_1\omega_2 &= M_3 \end{aligned} \quad . \tag{3}$$

Equations (3) serve as the basis of studies of the physical librations of the Moon, and will be brought to a standard form by evaluating the moments on the right side of (3). For this purpose it is convenient to introduce cartesian axes x, y, z, with origin at the Moon's centroid, Ox passing through the Earth's centroid. Then an element of lunar mass dm situated at (x,y,z) is at distance ρ from Earth centroid:

$$\rho^2 = (r - x)^2 + y^2 + z^2 \quad . \tag{4}$$

The Earth-Moon centroid separation is denoted by r, and the attraction on dm by the Earth mass E considered to act at its centroid is then

$$d\bar{F} = \frac{-Edm}{\rho^2} \frac{(x - r)\bar{i} + y\bar{j} + z\bar{k}}{\rho} \tag{5}$$

the units being chosen so that the universal constant of gravitation is unity. With the usual convention for unit vectors $\bar{i}$, $\bar{j}$, $\bar{k}$, the vector location of the mass element is $x\bar{i} + y\bar{j} + z\bar{k}$ and the moment about lunar centroid 0 of the elementary force (5) is

$$d\bar{M} = \frac{-E\ dm}{\rho^3} \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ x & y & z \\ x-r & y & z \end{vmatrix} = \frac{-E\ dm\ r}{\rho^3} (y\bar{k} - z\bar{j}) \quad . \tag{6}$$

The moment is evaluated by integration of (6), using (4) in the form

$$\frac{1}{\rho^3} = \frac{1}{r^3} [1 + \frac{3x}{r} + \frac{6x^2 - 3/2\ (y^2 + z^2)}{r^2} + 0(\frac{x^3}{r^3})] \tag{7}$$

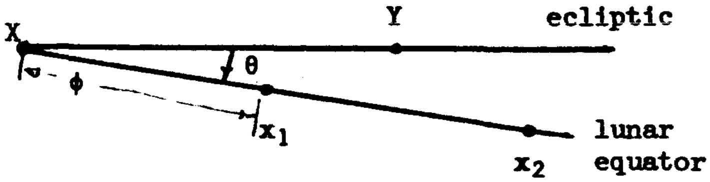
retaining only the first two terms in brackets, so that

$$\bar{M} = \frac{-3E}{r^3} \{\bar{k} \int xy\ dm - \bar{j} \int xz\ dm\} \quad . \tag{8}$$

When (8) is referred to principal inertia axes and x_1, x_2, x_3 are understood to represent the Earth's centroid coordinates in this reference frame, a straightforward reduction gives

$$\bar{M} = \frac{-3E}{r^5}\{(B - C)x_2x_3\bar{i}_1 + (C - A)x_3x_1\bar{i}_2 + (A - B)x_1x_2\bar{i}_3\} \qquad (9)$$

where successive terms in (9) give the right sides of the three equations (3). In addition to the approximation already made in neglecting the third and subsequent terms in (7), a further approximation is introduced in expressing (9) in terms of ecliptic plane coordinates by considering the angle θ subtended by the lunar equator and the ecliptic small enough to justify writing



$$\begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix} = \begin{vmatrix} \cos\phi & \sin\phi & -\theta\sin\phi \\ -\sin\phi & \cos\phi & -\theta\cos\phi \\ 0 & \theta & 1 \end{vmatrix} \begin{vmatrix} X \\ Y \\ Z \end{vmatrix} \quad . \qquad (10)$$

The transformation (10) shows that the descending node of the lunar equator on the ecliptic is taken as the direction of axis X, while Z is Earth's coordinate normal to the ecliptic plane and x_1, x_2 are understood to represent the principal axes normal to the Moon's rotation axis. When the orbit inclination i is regarded as a small angle, the coordinate Z is correspondingly limited, and it is customary to neglect the product of the angle θ with the coordinate Z. In this case it is seen that the products of coordinates (10) required in the moment expression (9) are of the forms shown on the right sides of the equations (11)

$$\begin{aligned} A\frac{dq}{dt} - (B - C)pr &= \frac{-3E}{r^5}(B - C)[\cos\phi\,(YZ + Y^2\theta) - \sin\phi\,(XY\theta + XZ)] \\ B\frac{dr}{dt} - (C - A)qp &= \frac{-3E}{r^5}(C - A)[\cos\phi\,(XZ + ZY\theta) + \sin\phi\,(Y^2\theta + YZ)] \qquad (11) \\ C\frac{dp}{dt} - (A - B)rq &= \frac{-3E}{2r^5}(A - B)[(Y^2 - X^2)\sin 2\phi + 2XY\cos 2\phi] \quad . \end{aligned}$$

By replacing the symbols ω_1, ω_2, ω_3 in (3) by q, r, p respectively, the standard form (11) of the equations of physical librations conforms with the notation of Laplace (with these two exceptions only, that E and r in present equations are written as L and r_1 in the equations identified as (G') in Chapitre II, Livre Cinquième, Première Partie of the Traité de Mecanique Céleste). The rotation of the Moon being principally about the axis, x_3, the corresponding component p is larger than either q or r, and Laplace therefore omitted the second term on the left side in the last one of equations (11), taking the presumed smallness of the principal moment difference (B - A) as further justification for the neglect of this term.

For later reference we note here the form of terms neglected in the square brackets of the last equation of (11). When θ^2 is not neglected, nor the product of θ and Z, it is readily shown that x_3-component of the couple exerted by the Earth on the Moon also contains terms

$$-\theta^2 Y^2 \sin 2\phi - \theta^2 XY \cos 2\phi - \theta Z(2X \cos 2\phi + Y \sin 2\phi)$$

$$- Z^2\theta^2 \sin 2\phi + O(\theta^3). \tag{12}$$

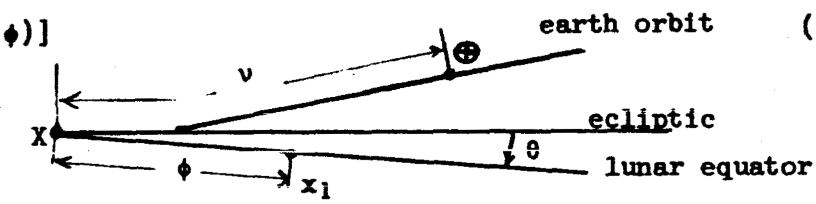
Libration in Longitude; Estimation of Equatorial Principal Inertia Moment Difference (B - A)

When the product rq in the last of equations (11) is neglected, the equation may be regarded as independent of the two companion equations, and its separate solution is investigated for the purpose of estimating the difference B - A of lunar inertia moments. The moment term is evaluated first by expressing X and Y in terms of longitude angle ν measured from descending node of lunar equator:

$$X = r \cos\nu \qquad Y = r \sin\nu \tag{13}$$

It is thus seen that the bracketed term on the right in the last of (11) is given by r^2 multiplied with

$$[\sin 2(\nu - \phi)] \tag{14}$$



where $\nu - \phi$ is the longitude angle subtended by the principal axis x_1 and the Earth-pointing direction. The Earth's orbital motion relative to the Moon is $\dot{\nu} - \dot{\psi}$, and if its mean value is denoted by n, while the periodic inequalities are written as $\frac{d}{dt} \Sigma H \sin\pi$, then

$$\nu - \psi = \int n\, dt + \sum H \sin\pi \quad . \tag{15}$$

Likewise the lunar rotation angle measured from a fixed direction is $\phi - \psi$, and if this is very nearly equal to the mean orbital motion, we can write

$$\phi - \psi = \int n\, dt + u \tag{16}$$

where u is a small quantity taken as the measure of physical libration in longitude. Thus (15) and (16) show that the Earth's couple is determined by (14) as a term

$$\sin 2(-u + \Sigma H \sin\pi) \doteq -2u + 2 \Sigma H \sin\pi \tag{17}$$

provided the terms on the right side of (17) are sufficiently small. Then the libration in longitude is determined by the equation

$$\frac{dp}{dt} + 3n^2 \frac{B - A}{C} u = 3n^2 \frac{(B - A)}{C} \sum H \sin\pi \tag{18}$$

where $\frac{E}{r^3}$ is written as the mean motion term n^2. Since

$$p = \dot{\phi} - \cos\theta\dot{\psi} \doteq \dot{\phi} - \dot{\psi}, \tag{19}$$

and n may be taken as nearly constant, (16) shows that

$$\frac{dp}{dt} \doteq \frac{d^2u}{dt^2} \tag{20}$$

and the libration in longitude is of the same character as the forced oscillator:

$$\frac{d^2u}{dt^2} + 3n^2 \frac{B - A}{C} u = 3n^2 \left(\frac{B - A}{C}\right) \sum H \sin\pi \tag{21}$$

where the inequalities $H \sin\pi$ are regarded as known functions of time. Although the value of the coefficient $3 \frac{B-A}{C}$ on the left side of (21) is not known, the failure to detect librations at frequencies corresponding to the range of values within which it is believed to fall has led to the conclusion that free oscillations are too small to observe. Hence the total libration is attributed to the forcing terms and the associated particular solution of the inhomogeneous equation (21):

$$u = -3n^2 \frac{B - A}{C} \sum \frac{H \sin\pi}{\left(\frac{d\pi}{dt}\right)^2 - 3n^2 \frac{B-A}{C}} \quad . \tag{22}$$

The ellipticity of the Moon's equator being small, and its mechanical measure $\frac{B-A}{C}$, it is seen that appreciable amplitudes can be expected only for great inequalities (large H) and for long-period inequalities ($\frac{d\pi}{dt}$ small). Laplace, having at first no libration measurements with which $\frac{B-A}{C}$ estimates could be inferred, determined that this quantity could be expected to be positive, and less than 0.00284. Later Nicollet's work was taken by Laplace to justify assigning the value 0.000563 in 1819; Franz' corrections reduced this to 0.000315 in 1880; Jeffreys in modern times estimates 0.000118 ± 0000057, indicating an uncertainty possibly as great as 48% of the estimated value itself. For present purposes it is of main interest to note that the 1819 value, itself a small fraction of the upper bound furnished by Laplace, may require a further ten-fold reduction, if current estimates are valid. From (21) and (22) it is seen that downward revision of $\frac{B-A}{C}$ values is accompanied by like reduction in the values of the libration u. In view of the possibly so appreciable reduction of magnitude of terms retained in (21), the question is raised whether terms earlier neglected in obtaining (11) and deriving (21) from it can still be verified to be uniformly very small by comparison with terms retained. Such à postiori verification is necessary to establish the consistency of the analysis and the meaning of its conclusions.

Solar Gravitational Couple

When the Earth-pointing direction and the Moon's x_1 axis are in near coincidence, the forcing couple is very small, as seen from equations (11), (14) and (17); in the limit $\nu - \phi = 0$, it vanishes. At these times it is clear that the Sun's gravitational couple is not small by comparison, and the idealization of physical librations as a two-body phenomenon is inadequate. A peculiarity of the standard analysis as just reviewed is to be noted in this regard. Already not later than 1693 it was known that the Sun plays a role of crucial importance in establishing lunar rotation as it was accurately described by D. Cassini. His characterization of the motion noted the coincidence of equatorial and orbital nodes on the ecliptic; otherwise stated, the fact that the Moon's rotation axis, its orbital axis, and the pole of the ecliptic are permanently coplanar. The complete absence of any dynamic role of the Sun, in the classical libration theory, by contrast, can only be taken as an indication of the incompleteness of the mechanical formulation represented by equations (11).

It is evident that solar couple terms can be introduced into (11) by the simple addition of appropriate terms to the right sides of these equations, containing the Sun's mass S and distance R in place of E and r, and where X, Y represent the Sun's coordinates, Z being zero because the sun is in the ecliptic. Lagrange omitted terms of this type by observing that the ratio $\frac{S}{R} \div \frac{E}{r} = \frac{1}{178.7}$ is small, which represents the fractional value of the constants in the more complete expression for the longitudinal couple

$$-\frac{3}{2}\ \frac{A - B}{C}\left[\frac{E}{r^3}\sin 2(\nu - \phi) + \frac{S}{R^3}\sin 2(\nu' - \phi)\right] \qquad (23)$$

where ν' represents the Sun's longitude. Present indication that the term (14) appearing as a factor multiplying $\frac{E}{r^3}$ is very small, while $\sin 2(\nu' - \phi)$ takes all values in the interval [-1,+1] four times each lunar month, means that a more significant comparison is obtained as a ratio of the two terms in brackets (23). Denoting these respectively by L_E and L_S when the second term is at its peak, the limiting value of the couple ratio is dependent on $(\nu - \phi)$ and indicated for three values of the angle as follows

$2(\nu - \phi)$	$L_S \div L_E$
1"	1150
1'	19
1°	0.33

(24)

Although the angle $(\nu - \phi)$ cannot be considered to be accurately known, it is believed to be of the order of a few minutes of selenocentric arc at most, so that there are four times each month when the Solar couple is dominant; the length of these intervals cannot be specified. There can be little doubt that Cassini's observation has a dynamical explanation to be found in the inclusion of Solar effects in the libration analysis in the manner just indicated.

Equations (11) are arranged to display graviational couple effects on the right sides, and the observation that at least periodically additional solar couple terms must be included provides a convenient opportunity for further physical interpretation. The inertia effects on the left side of each equation are recognized as having distinct meanings for physical librations of small amplitude. Whereas the first term in each case is seen to represent an unsteady acceleration, the second term in each equation is readily verified to represent a quasi-steady centrifugal couple. Thus in idealized motion of the form described by Cassini, the gravitational couple is balanced by centrifugal couple so that the motion is one of relative equilibrium in a rotating frame of reference. Now the centrifugal forces are the consequence of orbital motion, and inclusion of Solar gravitational couple terms raises the question whether the heliocentric orbit curvature gives rise to centrifugal couples which must be considered in addition to those related to the Moon's geocentric orbit. A closer examination confirms that terms are present of the order of 1/13.3 times the geocentric-centrifugal terms, as might be expected. As these do not appear to introduce qualitatively different motion, however, their retention may be deferred until numerically accurate calculations are required.

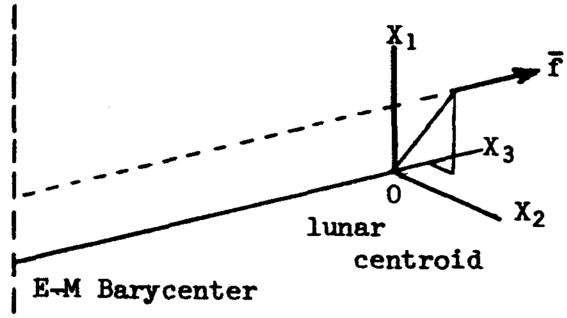
Centrifugal Couple Identification

As a preliminary to the inclusion of the cross-product terms of (11) in the analysis of physical librations, it is useful to clarify their interpretation for the motion under consideration. For the sake of simplicity only the Earth-Moon interaction is included, as the extension will be clear for the heliocentric component of motion and the related forces. When orbit eccentricity is also neglected, each mass element of the Moon is accelerated toward the axis of the orbit plane, so that the effective orbital centrifugal force per unit mass can be written simply as

$$\bar{f} = n^2\{X_2\bar{1}_2 + (\ell + X_3)\bar{1}_3\} \tag{25}$$

where ℓ is the distance from Earth-Moon barycenter to lunar centroid, X_2 and X_3 are orbit-plane cartesian coordinates measured from lunar centroid and oriented as shown, $\bar{1}_2$ and $\bar{1}_3$ are the associated unit vectors. The moment about lunar centroid is obtained by taking the vector product of the position vector

$$X_1\bar{1}_1 + X_2\bar{1}_2 + X_3\bar{1}_3$$



with the force (25) and integrating over all mass points of the Moon; thus

$$\bar{K} = n^2 \int \begin{vmatrix} \bar{1}_1 & \bar{1}_2 & \bar{1}_3 \\ X_1 & X_2 & X_3 \\ 0 & X_2 & \ell+X_3 \end{vmatrix} dm \tag{26}$$

The reduction of (26) is similar to that of the approximate gravitational couple (6), although the corresponding approximations have not been introduced in (26). When orbit plane axes and principal axes are related by the transformation

$$\begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \begin{vmatrix} X_1 \\ X_2 \\ X_3 \end{vmatrix} \tag{27}$$

the couple (26) is seen to be given, after a straightforward calculation, as

$$\bar{K} = -n^2\{(C - B)b_1c_1\bar{i}_1 + (A - C)c_1a_1\bar{i}_2 + (B - A)a_1b_1\bar{i}_3\}. \tag{26'}$$

Furthermore the orbital motion n parallel to X_1 has principal axis components

$$\omega_1 = q = a_1n, \qquad \omega_2 = r = b_1n \qquad \omega_3 = p = c_1n$$

so that, for example, the $\bar{i}_3$ component of centrifugal couple is

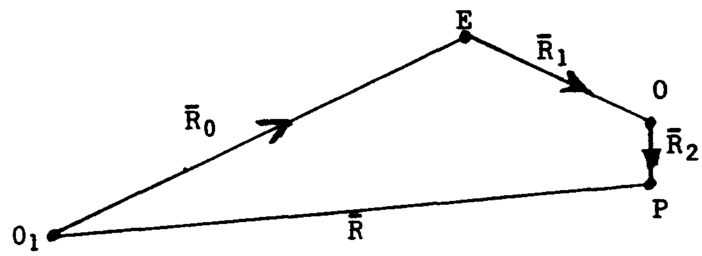
$$-(B - A)qr. \tag{28}$$

Comparison of (28) with the corresponding term in (11), with due regard for sign, then gives the immediate interpretation that the term $\frac{dp}{dt}$, for example, is equal to the sum of x_3-components of gravitational and centrifugal couples. Quasi-steady motion of the Cassini-type represents the dynamic balance of the two couples alone. It is clear, of course, that centrifugal couples must be present in much the same form as the gravitational couples, each representing the effect of nonuniformities of mass distribution while the total of each resultant force simply establishes the orbit conditions. A significant difference between the two couples is to be noted, in that all forces are effectively directed to a single point in the case of the gravitational couple, but only toward a given line (orbit axis) in the case of the centrifugal couple.

In the classical development of physical librations the neglect of the cross-product term in (11) therefore has the meaning that the "pure" inertia term $\frac{dp}{dt}$ is supposed to react alone with the gravitational couple of the libration in longitude. As will be seen below, the presently accepted values of B - A do not justify the neglect of centrifugal couple term, and it may be surmised that the quasi-empirical character of the formulation which insists on the smallness of all dependent variables, has obscured a more fundamentally correct physical interpretation.

Centrifugal Couple, Influence on Libration in Longitude

The principal-axis rotation components q and r in equations (11) can be analyzed by examining the compound rotation determined by orbital motion which produces a curvilinear trajectory of the lunar centroid plus the libration referred to axes fixed within the Moon. A convenient description of compound rotational motion uses the decomposition of position vector $\bar{R}$ of a point P, referred to an inertial center O_1, as



$$\bar{R} = \bar{R}_0 + \bar{R}_1 + \bar{R}_2.$$

The vector $\bar{R}_0$ can, for definiteness, be regarded as locating Earth-Moon barycenter E, while $\bar{R}_1$ extends from E to lunar centroid O, and $\bar{R}_2$ locates the point P from O. Introducing natural coordinate unit vectors such that $\bar{R}_0 = R_0\bar{e}_{R_0}$, $\bar{R}_1 = R_1\bar{e}_{R_1}$, $\bar{R}_2 = R_2\bar{e}_{R_2}$, the velocities $\dot{\bar{R}}_0$, $\dot{\bar{R}}_1$, $\dot{\bar{R}}_2$ determine the motion of P as

$$\bar{V} = \dot{\bar{R}} = \dot{R}_0\bar{e}_{R_0} + \dot{R}_1\bar{e}_{R_1} + \dot{R}_2\bar{e}_{R_2} + \bar{\Omega}_0 \times \bar{R}_0 + (\bar{\Omega}_0 + \bar{\Omega}_1) \times \bar{R}_1$$
$$+ (\bar{\Omega}_0 + \bar{\Omega}_1 + \bar{\Omega}_2) \times \bar{R}_2 \qquad (29)$$

where

$$\bar{\Omega}_0 \equiv \frac{\bar{R}_0 \times \dot{\bar{R}}_0}{R_0{}^2}$$

is the absolute angular velocity of E, while $\bar{\Omega}_0 + \bar{\Omega}_1$ is the absolute angular velocities as seen in reference frames moving with $\bar{e}_{R_1}$ and $\bar{e}_{R_2}$, respectively. The present manner of decomposing rotational motion is well adapted to the description of physical librations since setting $\bar{\Omega}_2 = 0$, for example, corresponds to lunar rotation just equal to orbital motion $\bar{\Omega}_0 + \bar{\Omega}_1$. The extension to rotations more highly compounded is obvious, and the neglect of heliocentric motion is seen to correspond to suppressing $\bar{R}_0$. It is of interest to observe how the familiar centripetal and Coriolis accelerations are generalized even in this simple case of double compound rotation when $\bar{\Omega}_1$ and $\bar{\Omega}_2$ are constants. Direct differentiation of (26), When $\dot{\bar{R}}_0 = 0$, gives acceleration as

$$\bar{A} = \dot{\bar{V}} = \ddot{R}_1\bar{e}_{R_1} + \ddot{R}_2\bar{e}_{R_2} + 2\bar{\Omega}_1 \times \dot{\bar{R}}_1 + 2(\bar{\Omega}_1 + \bar{\Omega}_2) \times \dot{\bar{R}}_2$$

$$+ \bar{\Omega}_1 \times (\bar{\Omega}_1 \times \bar{R}_1) + (\bar{\Omega}_1 + \bar{\Omega}_2) \times [(\bar{\Omega}_1 + \bar{\Omega}_2) \times \bar{R}_2]. \qquad (30)$$

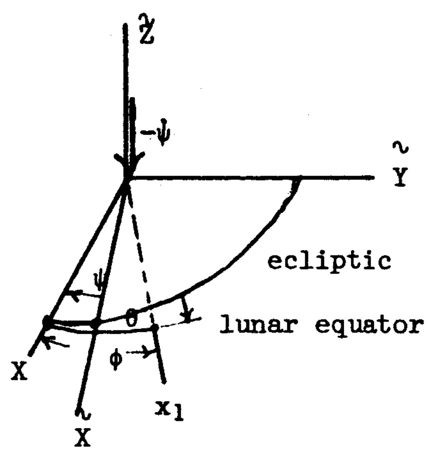
In addition to the familiar-looking terms on the first line of (30), and the conventional centripetal acceleration represented by the first term of the second line, there is finally a term which is expanded by the rules of vector algebra as

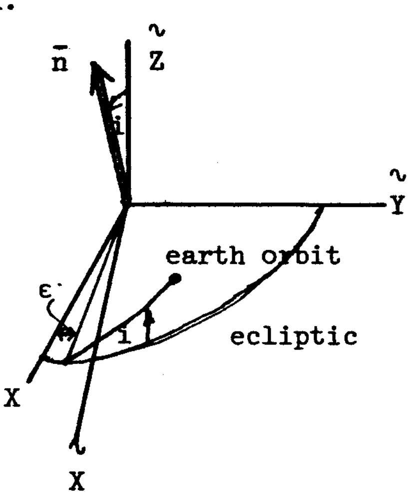
$$-\bar{R}_2(\Omega_1{}^2 + \Omega_2{}^2 + 2\bar{\Omega}_1 \cdot \bar{\Omega}_2) + (\bar{\Omega}_1 + \bar{\Omega}_2)(\bar{R}_2 \cdot \bar{\Omega}_1) \qquad (31)$$

when it is noted that we can take $\bar{\Omega}_2 \cdot \bar{R}_2 = 0$. When the subscript "2" is replaced by "0", the first term shows that the geocentric centripetal acceleration proportional to $\Omega_1{}^2$ is augmented by heliocentric motion not simply by the amount $\Omega_0{}^2$, but also by $2\bar{\Omega}_1 \cdot \bar{\Omega}_0$, which is roughly one seventh of Ω_1^2, and hence much more important than the Ω_0^2 contribution which is roughly $\Omega_1^2/178.7$. The last term in (31), moreover, is nonzero and even the omitted portion $\bar{R}_2\cdot\bar{\Omega}_2$ is nonzero in general for a rigid body, since the vanishing for one point depends on a choice of $\bar{\Omega}_2$ appropriate to that point's location and instantaneous velocity.

The term (28) has been omitted from the libration equations, as already indicated, on account of the smallness of each of its factors. It is now possible to estimate the magnitude of this term and to compare with other terms that are retained in the analysis. Neglecting heliocentric motion, the total angular velocity of the Moon appears as $\bar{\Omega}_1 + \bar{\Omega}_2$ in (29), and the orbital motion Ω_1 can be resolved to find its x_1- and x_2-components q and r. Although these components are not completely evaluated, the libration contribution being neglected still, it seems reasonable to regard the product (28) thus obtained as a first approximation, by considering the librational motion to be small compared with the total orbital motion.

The orbital motion is represented by the angular velocity of magnitude n normal to the orbit plane, and by the vector $-\dot{\psi}$ normal to the ecliptic, representing the motion of the descending node of the lunar equator. A fixed direction in the ecliptic is defined by $\tilde{X}$, so that ψ is measured in the indirect sense from $\tilde{X}$ to the descending node X.





When ϕ measures the longitude of axis x_1 in the lunar equatorial plane from the descending node X, the equator being inclined an angle θ to the ecliptic, the components of $-\dot{\psi}$ in the x_1, x_2 directions are respectively $\sin\phi \sin\theta\dot{\psi}$ and $\cos\phi \sin\theta\dot{\psi}$. The ascending node of Earth's orbit is at longitude ε, known to be a very small angle, and when it is neglected the proper orbital motion, represented by a vector inclined an angle i to the ecliptic normal, has components $-\sin\phi \sin(\theta + i)$ n and $-\cos\phi \sin(\theta + i)$ n. Thus the orbital contributions to q and r are taken as

$$\begin{aligned} q &= -n \sin\phi \sin(\theta + i) + \dot{\psi} \sin\phi \sin\theta \\ r &= -n \cos\phi \sin(\theta + i) + \dot{\psi} \cos\phi \sin\theta \end{aligned} \tag{32}$$

when these are multiplied by (B - A). Taking the values recommended by Commission 17 of the IAU in 1964 as

$$\begin{aligned} \theta &= 1^\circ 30' 54'' \pm 30'' \\ i &= 5^\circ 09' 16'' \pm 23'' \end{aligned} \tag{33}$$

and noting that the $18\frac{2}{3}$-year period of nodal regression gives the ratio

$$\frac{\dot{\psi}}{n} \doteq \frac{1}{248}$$

while

$$\frac{\sin\theta}{\sin(\theta+i)} \doteq \frac{1}{4.28}$$

the first term on the right side of each of equations (32) is more than a thousand times greater than the second. We will accordingly take approximately for the longitudinal centrifugal couple (28)

$$-(B - A)qr = (B - A)\frac{n^2}{2}\sin^2(\theta + i)\sin 2\phi \tag{34}$$

in which form it is clear that the usual neglect of this term is related to the "smallness" of the angles θ and i. Although these are each assuredly small compared with one radian, their sum and its square are to be compared, for example, with the second term on the left in (21) and neglected only if found to be very small by comparison. An equally valid criterion depends on the comparison with the inequality terms on the right side of (21), and this has the further advantage that the terms H are known. When the numerical value of $\sin^2(\theta + i)$ is converted to seconds of angular arc, corresponding to the form in which H is customarily given, and the factor 3 is introduced in numerator and denominator for greater ease of direct comparison the augmented form of (21) becomes

$$\frac{d^2u}{dt^2} + 3n^2 \frac{B - A}{C} u = 3n^2(\frac{B - A}{C})[\Sigma \text{ Hsin}\pi - 3{,}143'' \sin 2\phi] \text{ .} \qquad (35)$$

The values of the inequalities H for the principal elliptic term and annual equation respectively are given (for example by Plummer, p.316) as 22,639" and -668".9 respectively, the latter of the two being retained for purposes of estimating (B - A) from observational data. The centrifugal couple included as the last term in brackets in (35) is evidently appreciably larger than the annual equation forcing term.

A closer inspection of the free oscillations is obtained by using (16) to express the harmonic variation of centrifugal couple in the form

$$\sin 2\phi \doteq \sin 2(nt + \psi) + 2u \cos 2(nt + \psi) \qquad (36)$$

so that the homogeneous equation corresponding to (35) is

$$\frac{d^2u}{dt^2} + 3n^2 \frac{B - A}{C} \{1 + \frac{\sin^2(\theta+i)}{6} 2 \cos 2(nt + \psi)\} u = 0 \qquad (37)$$

recognized as Mathieu's Equation (this modification of the equation of libration in longitude was proposed on different grounds by S.G. Makover in the Bjull. Inst. Teoret. Astr., 8, 249, 1962, and has been the subject of subsequent controversy). In the circumstance that libration amplitudes are small, it thus appears that the centrifugal couple gives rise to free oscillations that are more accurately examined by the methods of Floquet than as represented by harmonic oscillator through (21).

It is of interest to note how the libration in longitude can be expressed in terms of the inclinations θ and i, when the first term on the right side of (36) alone is retained (i.e., when the procedure of questionable consistency is followed that still treats the motion as a harmonic oscillator). Comparing the forms of (21) and (35), it is seen that the particular solution introduced by the centrifugal couple term, analogous to (22), is

$$u_p = +3n^2 \frac{B - A}{C} \frac{\frac{1}{6}\sin^2(\theta+i)}{(2n)^2 - 3n^2 \frac{B-A}{C}} \quad \sin 2(nt + \psi) \tag{38}$$

when $\dot{\psi}$ is neglected as before compared with 2n, a bi-weekly oscillatory frequency.

From (38) it is seen that when appropriate observational data can furnish a value for the amplitude of u_p, the principal inertia moment difference B - A can be expressed in terms of the angles θ and i. It will be recalled that the larger principal moment difference C - A is classically evaluated in terms of these angles, and the two differences B - A and C - A are thus related by the same constants.

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