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A PICARD ITERATIVE METHOD FOR ESTIMATING SOLUTIONS TO CERTAIN DIFFERENTIAL EQUATIONS*

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Introduction

The method for estimating solutions to differential equations discussed here is a numerical implementation of Picard iteration. It is similar to a method given by Axelsson,[1] and can also be regarded as an implicit Runge-Kutta method described by Butcher.[2]

The justification, given here, of the scheme, establishes no more than the fact that it delivers Runge-Kutta accuracy at the cost of iteration. Experience, however, indicates that in many instances the proposed method is more effective and efficient than standard Runge-Kutta techniques. Examples to this effect and data, for use in implementing the method, are given in the appendix.

Description of the Method

For the purpose of exposition a first order initial value problem will be used. Higher order equations are solved by repeated use of the first order method.

$$\text{Let } y'(x) = F(x,y), \ (x_0<x<x_0+h), \text{ and } y(x_0) = a_0.$$

If $y'(x)$ can be represented by a convergent Fourier series of Legendre polynomials and if $P_0(u)$, $P_1(u)$, $P_2(u)$ $\cdots$ are the Legendre

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polynomials orthogonal over $0 \le u \le 1$, then

$$y'(x_0+hu) = \sum_{k=0}^{\infty} (2k+1)P_k(u)\int_0^1 F(x_0+hv,\ y(x_0+hv))P_k(v)dv$$

and

$$\begin{aligned} y(x_0+hu) = y(x_0) + h\Big\{\tfrac{1}{2}\,[P_0(u)-P_1(u)]\int_0^1 F(x_0+hv,\ y(x_0+hv))dv \\ + \sum_{k=1}^{\infty} \tfrac{1}{2}[P_{k-1}(u)-P_{k+1}(u)]\int_0^1 F(x_0+hv,\ y(x_0+hv))P_k(v)dv\Big\} \end{aligned} \tag{1}$$

If $F(x,y)$ satisfies a Lipschitz condition then a convergent Picard iterative process is given by

$$\begin{aligned} y_{\ell+1}(x_0+hu) = y(x_0) + h\Big\{\tfrac{1}{2}[P_0(u)-P_1(u)]\int_0^1 F(x_0+hv,\ y_\ell(x_0+hv))dv \\ + \sum_{k=1}^{\infty} \tfrac{1}{2}\,[P_{k-1}(u) - P_{k+1}(u)]\int_0^1 F(x_0+hv,\ y_\ell(x_0+hv))P_k(v)dv\Big\} \end{aligned} \tag{2}$$

where $y_1(x_0 + hv) = a_0$.

A numerical estimate to the solution of the differential equation is obtained by truncating the above series and approximating the integrals by quadratures. Two quadratures, Gauss and Gauss-Lobatto, have been tried. If Gaussian quadrature is used, the series (2) truncated, and $P_{k-1}(u)-P_{k+1}(u)$ replaced by $-\frac{2(2k+1)}{k(k+1)}$ $(u)(1-u)P_k'(u)$ the the result is

$$y_{\ell+1}(x_0+hu) = y(x_0) + h\{u \sum_{i=1}^{m} w_i F(x_0+hv_i,\ y_\ell(x_0+hv_i))$$

$$- \sum_{k=1}^{n} \frac{2k+1}{k(k+1)} u(1-u)P_k'(u) \sum_{i=1}^{m} w_i F(x_0+hv_i, y_\ell(x_0+hv_i))P_k(v_i)\}$$

where $w_1 \cdots w_m$ and $v_1 \cdots v_m$ are the Gaussian weights and the zeros of $P_m(v)$, respectively.

A change in the order of summation yields

$$y_{\ell+1}(x_0+hu) = y(x_0) + h \sum_{i=1}^{m} Z_i(u)F(x_0+hv_i,\ y_\ell(x_0+hv_i)) \tag{3}$$

where

$$Z_i(u) = w_i[u - \sum_{k=1}^{n} \frac{2k+1}{k(k+1)} u(1-u)\ P_k'(u)P_k(v_i)]$$

$$= w_i \int_0^u \sum_{j=1}^{n} (2j+1)P_j(t)P_j(v_i)dt = w_i \int_0^u K(t,v_i)dt.$$

If $m = n+1$ the indefinite integral $\int_0^x f(u)du$ is given exactly by $\sum_{i=1}^{m} Z_i(x)f(v_i)$ when $f(u)$ is a polynomial of degree less than or equal to n. A least square approximation results if f is a polynomial of degree $n+1$.

Also $Z_i'(v_j) = \delta_{ij}$. Hence $Z_i'(u)$ is the polynomial $l_i(u) = \frac{P_m(u)}{(u-v_i)P_m'(v_i)}$ associated with Lagrangian interpolation. This fact could have been used to generate the iterative process (3) above.

If (3) converges to a function $y_c(x_0+hu)$ and $\dot{y}$ is used to denote differentiation with respect to (x_0+hu) it is seen that $y_c(x_0+hu)$ is a polynomial such that

$$\dot{y}_c(x_0+hv_i) = F(x_0+hv_i, y_c(x_0+hv_i)), \quad 1 \leq i \leq m.$$

If h is sufficiently small and F(x,y) satisfies a Lipschitz condition the process (3) will converge because if $a_{ij} = Z_j(v_i)$ and A denotes the m X m matrix $\{a_{ij}\}$ it is seen that $||A||$ is bounded.

Now if

$$\dot{y}(x_0 + hu) = F(x_0 + hu, y(x_0 + hu)),$$

then

$$\begin{aligned}\dot{y}(x_0+hu) - \dot{y}_c(x_0+hu) &= F(x_0+hu, y(x_0+hu)) \\ &\quad - F(x_0+hu, y_c(x_0+hu)) \qquad (4) \\ &\quad + F(x_0+hu, y_c(x_0+hu)) - \dot{y}_c(x_0+hu).\end{aligned}$$

Let

$$\begin{aligned}F(x_0+hu, y(x_0+hu)) &= F(x_0+hu, y_c(x_0+hu)) \\ &\quad + [y(x_0+hu) - y_c(x_0+hu)]F_2(x_0+hu, \theta(x_0+hu))\end{aligned}$$

where F_2 denotes partial differentiation with respect to the second variable.

Substituting in (4) yields

$$y(x_0+hx) - y_c(x_0+hx) = h\int_0^x \exp\,(g(x,v))\phi(v)dv$$

where $g(x,v) = h\int_v^x F_2(x_0+ht,\theta(x_0+ht))dt$ and

$\phi(v) = \bar{F}(x_0+hv,y_c(x_0+hv)) - \dot{y}_c(x_0+hv)$.

If $\phi(v)$ has m bounded derivatives then

$$|\phi(v)| = \left|\frac{\phi^{(m)}(\bar{v})}{m!}\right| |P_m(v)| \binom{2m}{m}^{-1} \leq \frac{h^m}{m!}\binom{2m}{m}^{-1} M_1$$

where M_1 is the maximum of $\left|\dfrac{d^m F(x_0+hv,y_c(x_0+hv)}{d(x_0+hv)^m}\right|$.

Thus

$$|y(x_0+hx) - y_c(x_0+hx)| \leq \frac{h^{m+1}}{m!}\binom{2m}{m}^{-1} M_1 \int_0^x \exp(g(x,v))dv.$$

for $0 < x < 1$.

A higher order result can be obtained for $x = 1$ if $F(x,y(x))$ is sufficiently well-behaved (has partial derivatives of sufficiently high order). Let $\phi_1(x_0+hv) = \exp[g(1,v)] = \exp\left(\int_{x_0+hv}^1 F_2(t,\theta(t))dt\right)$ then

$$y(x_0+h) - y_c(x_0+h) = \int_0^1 \phi_1(x_0+hv)\phi(v)dv$$

$$= \int_0^1 \phi_2(x_0+hv)dv$$

Since $\phi_2(x_0+h\ v_i) = 0 \quad , \quad 1 \le i \le m$

$$|y(x_0+h) - y_c(x_0+h)| \le \frac{h^{2m}}{(2m-1)!} \binom{2m}{m}^{-2} M_2$$

where M_2 is a maximum of $\left|\dfrac{d^{2m}\ \phi_2(x_0+hv)}{d(x_0+hv)^{2m}}\right|$.

If Lobatto quadrature is used the series (2) is rearranged in the form

$$y(x_0+hu) = y(x_0) + h\{\int_0^1 (1-v)\ F(x_0+hv,\ y(x_0+hv))dv$$

$$+ \sum_{h-v}^{\infty} \frac{2k+1}{k(k+1)} P_k(u) \int_0^1 v(1-v)\ P_k'(v)\ F(x_0+hv,\ y(x_0+hv))dv\}.$$

before truncation.

Truncation and replacement of the integrals by quadrature results in the iterative process

$$y_{\ell+1}(x_0+hu) = y(x_0) + h\{w_0\ F(x_0,y(x_0)) + \sum_{j=1}^{m-1} W_j(u)\ F(x_0+hv_j,\ y_\ell(x_0+hv_j))\}$$

where $W_j(u) = w_j(1-v_j) + w_j \sum_{k=1}^{m-1} \frac{(2k+1)}{k(k+1)} v_j(1-v_j)\ P_k'(v_j)\ P_k(u)$,

$w_0 \cdots w_m$ and $v_0 \cdots v_m$ are Lobatto weights and the zeros of $v(1-v)\ P_m'(v)$ respectively.

This is the previously referenced result of Axelsson. In this case $W_j'(v_i) = \delta_{ij}$ and the iterative process converges to a polynomial $y_c(x_0+hv)$ such that

$$\dot{y}_c(x_0+hv_j) = F(x_0+hv_j, y_c(x_0+hv_j)) \text{ where } v_j \text{ is a zero of } P_m'(v).$$

However, in this instance $y_c(x_0) \neq y(x_0)$ and order of error terms are computed for $|y(x_0+hu) - y_c(x_0+hu) - y(x_0) + y_c(x_0)|$. For m reasonably large this discrepancy between $y(x_0)$ and $y_c(x_0)$ seems to be an advantage rather than a disadvantage. This may be due to the fact that exact fitting of initial conditions results in more accumulation error.

Conclusion and Results.

The particular methods used here were selected so that classical forms for interpolation error would apply. Computer trials have given satisfactory results when Gaussian quadrature was used to evaluate the integrals involved in the truncation suggested for use with Lobatto quadrature and conversely. This suggests that error analysis should be based on truncation error associated with Fourier series. Efforts in this direction have not, as yet, yielded results more satisfactory than those given.

In applying iterative schemes of the nature discussed here it is desirable to have a set of reasonably good starting values. For the first step x_0 to x_0+h the starting values used were $y(x_i) = y(x_0)$. For subsequent steps starting values were obtained by extrapolating the polynomial $y_c(x)$ to the right. In most cases this worked rather well, but it is possible that other methods of prediction would serve better.

The Lobatto quadrature scheme was used to obtain the results reported here. The equations

$$y(x_0 + hv_j) = y(x_0) + w_0\ F(x_0) + h \sum_{i=1}^{m-1} W_i(v_j)\ F(x_0+hv_j, y(x_0+hv_j))$$

were solved by Gauss Siedel iteration. For the purpose of comparison the differential equations used were solved by the iterative technique, based on a 9 point Lobatto quadrature, with a step size h, and a given convergence criterion. The same differential equations were then solved using a fifth order Runge-Kutta method with several step sizes. Of the several Runge-Kutta solutions the one judged to be most nearly equivalent (in accuracy and efficiency) to the iterative solution was selected for comparison.

The programs were written in Fortran and run in double precision on the IBM 7094. Run times that are given were obtained through a call clock command and do not include program compile time. The true answer as well as the estimated answer was computed and compared for each point used in the computational routines.

Error values are given at selected points. The errors given are the computed value of the known solution minus the estimated solution. The step size h, the run time t, in seconds, and the convergence criterion ε are given.

The iterative process was discontinued when at the rightmost point v_j

$$|F(x_0+hv_j,\ y_{\ell+1}(x_0 + hv_j)) - F(x_0 + hv_j, y_\ell(x_0+hv_j))|$$

became less than ε.

In each instance reported a nine point Lobatto quadrature was used with the iterative process.

The numbers $Z_i(x_j)$ and $W_i(x_j)$ are given for 5, 9, and 13 point Gauss and Gauss-Lobatto quadrature. They were computed in double precision on the IBM 7094 and can serve as input data for computer programs designed for using these iterative methods for solving differential equations.

REFERENCES

1. Axelsson, O. BIT 4 1964, 69-86.

2. Butcher, J. C. Math Comp. 18 (1964) 50-64, 233-244.

APPENDIX I

The following tables give the errors in estimates to the solution of sample differential equations when obtained by a fifth order Runge-Kutta method (R K Error) as compared to the errors obtained when an iterative process (I Error) was used.

Table 1

Differential equation

$$y' = \frac{1}{x^3}\,[y^3+3xy^2+4x^2y+x^3] \qquad \frac{1}{e} < x \leq \frac{1}{e} + 6.5$$

$$y = \frac{x}{\sqrt{4-2\,\log_e x}} - x.$$

Runge-Kutta

Step size h = .005

Run time t = 20.00 sec.

Iterative Method

Step size h = 0.05

Run time t = 15.13 sec.

Convergence constant $\varepsilon = 1.0 \times 10^{-11}$

Table 2

Differential equation

$$y' = -50y + y \sin x + e^{-8x}(42-\sin x), \quad 0 < x \leq 1$$

$$y = e^{-8x}$$

Runge-Kutta:

Step size h = 0.001

Run time t = 31.93 sec.

Iterative method:

Step size, h = 0.05

Run time t = 35.22 sec.

Convergence constant $\varepsilon = 1.0 \times 10^{-13}$

Table 3

$$y'' = 9y - 20 \sin x, \quad 0 < x \leq 3.0$$

$$y = e^{-3x} + 2 \sin x$$

Runge-Kutta:

Step size h = 0.01

Run time t = 7.77 sec.

Iterative method:

Step size h = 0.25

Run time t = 7.63 sec.

Convergence constant $\varepsilon = 1.0 \times 10^{-11}$

TABLE 1

X	R K Error	I Error
1/e + 0.1	-0.202865D-12	-0.174860D-14
0.2	-0.356825	-0.280331
0.3	-0.503402	-0.352495
0.4	-0.655614	-0.413558
0.5	-0.819205	-0.610622
0.6	-0.101588D-11	-0.785482
0.7	-0.119160	-0.960342
0.8	-0.140382	-0.112687D-13
0.9	-0.163546	-0.128785
1.0	-0.188765	-0.146549
1.1	-0.216204	-0.164868
1.2	-0.245992	-0.184297
1.3	-0.278271	-0.204281
1.4	-0.313260	-0.227040
1.5	-0.351146	-0.251465
1.6	-0.932108	-0.277555
1.7	-0.436362	-0.307531
1.8	-0.484184	-0.336397
1.9	-0.535821	-0.369704
2.0	-0.591576	-0.401900
2.1	-0.651761	-0.439648
2.2	-0.716826	-0.482391
2.3	-0.787120	-0.532351
2.4	-0.863048	-0.587307
2.5	-0.945066	-0.647815
2.6	-0.102912D-10	-0.713318
2.7	-0.112962	-0.779376
2.8	-0.123338	-0.856537
2.9	-0.134575	-0.939248
3.0	-0.146752	-0.102695D-12
3.1	-0.159964	-0.111854
3.2	-0.174312	-0.122790

X	R K Error	I Error
3.3	-0.189910	-0.133726
3.4	-0.206829	-0.146216
3.5	-0.225404	-0.159150
3.6	-0.245602	-0.173111
3.7	-0.267706	-0.187960
3.8	-0.291904	-0.204530
3.9	-0.318488	-0.223931
4.0	-0.347707	-0.243347
4.1	-0.379920	-0.265633
4.2	-0.415536	-0.289809
4.3	-0.454994	-0.316482
4.4	-0.498867	-0.346639
4.5	-0.547771	-0.379501
4.6	-0.602495	-0.417998
4.7	-0.663981	-0.461797
4.8	-0.733312	-0.510480
4.9	-0.811897	-0.563882
5.0	-0.901365	-0.627164
5.1	-0.100384D-09	-0.698330
5.2	-0.112194	-0.779820
5.3	-0.125900	-0.875299
5.4	-0.141933	-0.992761
5.5	-0.160842	-0.112532D-11
5.6	-0.183375	-0.128763
5.7	-0.210539	-0.147837
5.8	-0.243708	-0.171374
5.9	-0.284853	-0.200550
6.0	-0.336834	-0.238786
6.1	-0.403955	-0.288924
6.2	-0.492985	-0.359445
6.3	-0.615131	-0.464961
6.4	-0.790129	-0.655564
6.5	-0.105585D-08	-0.111120D-10

TABLE 2

X	R K Error	I Error
0.05	-0.430929D-10	-0.260902D-14
0.10	-0.323417	-0.205391
0.15	-0.219075	-0.133226
0.20	-0.146663	-0.929811D-15
0.25	-0.980460D-11	-0.596744
0.30	-0.655379	-0.402455
0.35	-0.438074	-0.270616
0.40	-0.292833	-0.121430
0.45	-0.197339	-0.780625D-16
0.50	-0.130868	-0.503069
0.55	-0.875007D-12	-0.390312
0.60	-0.585052	-0.197758D-15
0.65	-0.391213	-0.149186
0.70	-0.261617	-0.100397
0.75	-0.174968	-0.678710D-16
0.80	-0.117029	-0.448859
0.85	-0.782897D-13	-0.219117D-15
0.90	-0.523726	0.167509
0.95	-0.350413	0.113841
1.00	-0.234483	0.764362D-16

TABLE 3

X	R K Error	I Error
0.25	-0.248001D-10	-0.657918D-12
0.50	-0.193078D-11	-0.577315
0.75	0.541772D-10	-0.325961
1.00	0.162107D-09	-0.247579D-13
1.25	0.377101	0.376143D-12
1.50	0.820256	0.104871D-11
1.75	0.174841D-08	0.232425
2.00	0.370493	0.496280
2.25	0.783976	0.105229D-10
2.50	0.165871D-07	0.222832
2.75	0.361689	0.471761
3.00	0.742893	0.998713

Quadrature points and weights for the 5,9, and 13 point Gauss integration formulas

X(1) =	0.4691007703066802D-01	W(1) =	0.1184634425280945D-00
X(2) =	0.2307653449471585D-00	W(2) =	0.2393143352496830D-00
X(3) =	0.5000000000000000D 00	W(3) =	0.2844444444444444D-00
X(4) =	0.7692346550528415D 00	W(4) =	0.2393143352496830D-00
X(5) =	0.9530899229693319D 00	W(5) =	0.1184634425280945D-00

X(1) =	0.1591988024618701D-01	W(1) =	0.4063719418078700D-01
X(2) =	0.8198444633668200D-01	W(2) =	0.9032408034742850D-01
X(3) =	0.1933142836497050D-00	W(3) =	0.1303053482014675D-00
X(4) =	0.3378732882980955D-00	W(4) =	0.1561735385200015D-00
X(5) =	0.5000000000000000D 00	W(5) =	0.1651196775006300D-00
X(6) =	0.6621267117019044D 00	W(6) =	0.1561735385200015D-00
X(7) =	0.8066857163502950D 00	W(7) =	0.1303053482014675D-00
X(8) =	0.9180155536633180D 00	W(8) =	0.9032408034742850D-01
X(9) =	0.9840801197538129D 00	W(9) =	0.4063719418078700D-01

X(1) =	0.7908472640705988D-02	W(1) =	0.2024200238265800D-01
X(2) =	0.4120080038851098D-01	W(2) =	0.4606074991886400D-01
X(3) =	0.9921095463334501D-01	W(3) =	0.6943675510989350D-01
X(4) =	0.1788253302798300D-00	W(4) =	0.8907299038097299D-01
X(5) =	0.2757536244817765D-00	W(5) =	0.1039080237684445D-00
X(6) =	0.3847708420224325D-00	W(6) =	0.1131415901314485D-00
X(7) =	0.5000000000000000D 00	W(7) =	0.1162757766154370D-00
X(8) =	0.6152291579775676D 00	W(8) =	0.1131415901314485D-00
X(9) =	0.7242463755182235D 00	W(9) =	0.1039080237684445D-00
X(10) =	0.8211746697201699D 00	W(10) =	0.8907299038097299D-01
X(11) =	0.9007890453666549D 00	W(11) =	0.6943675510989350D-01
X(12) =	0.9587991996114890D 00	W(12) =	0.4606074991886400D-01
X(13) =	0.9920915273592940D 00	W(13) =	0.2024200238265800D-01

Constants associated with Gauss quadrature for the 5,9, and 13 point formulas respectively

	$Z_1(x_i)$	$Z_2(x_i)$	$Z_3(x_i)$
X(1)=	0.5193904511595521D-01	0.1083143016873644D-02	-0.1546726163707234D-01
X(2)=	0.1342261416863164D-00	0.1024518421758591D-00	-0.2331735754219876D-02
X(3)=	0.1137762880042245D-00	0.2600046516806412D-00	0.1422222222222222D-00
X(4)=	0.1151573009105928D-00	0.2462013800279819D-00	0.2867761801986642D-00
X(5)=	0.1241680057083205D-00	0.2242546215345454D-00	0.2999117060815167D-00
X(1)=	0.1952276506997044D-01	-0.4797308121617143D-02	0.7258648848675012D-03
X(2)=	0.4497585504801112D-01	0.4186056636877524D-01	-0.5627508176694419D-02
X(3)=	0.3812031695894941D-01	0.1010843514298415D-00	0.6013392172781163D-01
X(4)=	0.4215886942842090D-01	0.8457657301086466D-01	0.1446978850484127D-00
X(5)=	0.3994037286983455D-01	0.9294337227271687D-01	0.1242571104107755D-00
X(6)=	0.4063398525461718D-01	0.9015029390824517D-01	0.1313355191092380D-00
X(7)=	0.4114299850520293D-01	0.8878291286050957D-01	0.1325532716816294D-00
X(8)=	0.3990714059694486D-01	0.9263838204870172D-01	0.1265617981057803D-00
X(9)=	0.4125524773846499D-01	0.8834648399454648D-01	0.1335748771198865D-00
X(1)=	0.9930620765386001D-02	-0.3128671573329502D-02	0.1595581533971721D-02
X(2)=	0.2216415922660012D-01	0.2213668265818997D-01	-0.4392909517101449D-02
X(3)=	0.1914739892433289D-01	0.5102594721954042D-01	0.3295221522933987D-01
X(4)=	0.2100236885868644D-01	0.4311266596464565D-01	0.7716821173728725D-01
X(5)=	0.1969756245740481D-01	0.4806335008237965D-01	0.6498629817168876D-01
X(6)=	0.2060754999701716D-01	0.4474042818430168D-01	0.7222236635140502D-01
X(7)=	0.2003793958110275D-01	0.4680042908513713D-01	0.6787869345449576D-01
X(8)=	0.2030187597830581D-01	0.4582652909136529D-01	0.6998835979915766D-01
X(9)=	0.2030219662272786D-01	0.4587867985394867D-01	0.6969323430524976D-01
X(10)=	0.2009433354638438D-01	0.4654503674219237D-01	0.6859866702017138D-01
X(11)=	0.2043710693970312D-01	0.4541173769430608D-01	0.7059456237779322D-01
X(12)=	0.2004502313519012D-01	0.4671930313158011D-01	0.6824969034192294D-01
X(13)=	0.2038882234858161D-01	0.4556892620608015D-01	0.7032682670673343D-01

	$Z_4(x_i)$	$Z_5(x_i)$	$Z_6(x_i)$
X(1)=	0.150597137151374 6D-01	-0.570456318022603 3D-02	
X(2)=	-0.688704477829895 5D-02	0.330614161750155 6D-02	
X(3)=	-0.206903164309582 0D-01	0.468715452386993 1D-02	
X(4)=	0.136862493073823 7D-00	-0.157626991582218 5D-01	
X(5)=	0.238231192232809 3D-00	0.665243974121392 4D-01	

	$Z_4(x_i)$	$Z_5(x_i)$	$Z_6(x_i)$
X(1)=	0.202211644293743 8D-02	-0.357694545304708 0D-02	0.393337354628650 7D-02
X(2)=	-0.449797720588380 7D-03	0.334723320925989 6D-02	-0.428120437033596 2D-02
X(3)=	-0.718998840224841 3D-02	0.510983136448853 7D-03	0.186725911656384 5D-02
X(4)=	0.742806407443964 6D-01	-0.104818041381582 2D-01	0.349429974656969 7D-02
X(5)=	0.170000445255253 2D-00	0.825598387503150 0D-01	-0.138269067352453 6D-01
X(6)=	0.152679238773425 9D-00	0.175601481638788 3D-00	0.818928977756000 6D-01
X(7)=	0.154306279403440 7D-00	0.164608694364182 0D-00	0.163363526922250 0D-00
X(8)=	0.160454742890339 2D-00	0.161772444291370 3D-00	0.156623336240591 8D-00
X(9)=	0.152240164973710 6D-00	0.168696622953677 2D-00	0.154151422077059 8D-00

	$Z_4(x_i)$	$Z_5(x_i)$	$Z_6(x_i)$
X(1)=	-0.490529197399089 1D-03	-0.373149821596111 0D-03	0.100803022153751 6D-02
X(2)=	0.162754241628691 7D-02	-0.243908781679607 5D-04	-0.104672059371921 2D-02
X(3)=	-0.529884413421455 6D-02	0.162296670053023 9D-02	0.185229036161650 3D-03
X(4)=	0.422108796768699 6D-01	-0.629163653666353 2D-02	0.202680442831504 2D-02
X(5)=	0.987306720425799 1D-01	0.497456764473650 8D-01	-0.755449923996093 5D-02
X(6)=	0.839174765450176 1D-01	0.114334001071425 4D-00	0.552257603259493 1D-01
X(7)=	0.918730747240789 4D-01	0.989622717612285 2D-01	0.123214025135076 4D-00
X(8)=	0.879625586396184 5D-01	0.106016707489140 9D-00	0.109092620971472 9D-00
X(9)=	0.888987723735378 5D-01	0.103707412930484 8D-00	0.114218002420833 7D-00
X(10)=	0.901539977730776 8D-01	0.102820753356562 7D-00	0.113865007587067 1D-00
X(11)=	0.874885359792365 0D-01	0.105714508140469 9D-00	0.111426078526088 4D-00
X(12)=	0.907282724881505 9D-01	0.101954342367243 2D-00	0.115133871063991 4D-00
X(13)=	0.878230960196282 9D-01	0.105401530506223 6D-00	0.111582856651346 7D-00

	$Z_7(x_i)$	$Z_8(x_i)$	$Z_9(x_i)$
X(1)=	-0.3269528918417616D-02	0.1977596352881684D-02	-0.6180535576785867D-03
X(2)=	0.3743550095685014D-02	-0.2314301701271591D-02	0.7300535838423765D-03
X(3)=	-0.2247923480163348D-02	0.1541167486920333D-02	-0.5058043244164042D-03
X(4)=	-0.1030170907767974D-02	0.1737864391817276D-03	0.3208926168441256D-05
X(5)=	0.6048237790686522D-02	-0.2619291925284954D-02	0.6968213109533487D-03
X(6)=	-0.1439253684693946D-01	0.5747507336561018D-02	-0.1521675247633463D-02
X(7)=	0.7017142647365413D-01	-0.1076027108241271D-01	0.2516877221836513D-02
X(8)=	0.1359328563781599D-00	0.4846351397865458D-01	-0.4338660867223821D-02
X(9)=	0.1295794833166042D-00	0.9512138846904347D-01	0.2111442911081716D-01

	$Z_7(x_i)$	$Z_8(x_i)$	$Z_9(x_i)$
X(1)=	-0.1403462541525888D-02	0.1558733480261906D-02	-0.1493506737535639D-02
X(2)=	0.1704827912482847D-02	-0.1992280933390860D-02	0.1953681401908337D-02
X(3)=	-0.1213540376726331D-02	0.1715511604342134D-02	-0.1806484370570707D-02
X(4)=	-0.1882795163713604D-03	-0.7234174547625405D-03	0.1087270411482209D-02
X(5)=	0.2920751334819008D-02	-0.1076412289969134D-02	0.2006108381117409D-03
X(6)=	-0.8950200175866587D-02	0.4048969160279629D-02	-0.2108683720484488D-02
X(7)=	0.5813788830771849D-01	-0.1007243500377368D-01	0.4945752007179485D-02
X(8)=	0.1252259767913835D-00	0.5791582980536463D-01	-0.1042597730307978D-01
X(9)=	0.1133550252804524D-00	0.1206960893720130D-00	0.5416234732071715D-01
X(10)=	0.1164640561315507D-00	0.1111147857027928D-00	0.1101996603055120D-00
X(11)=	0.1174893169922765D-00	0.1129563610961925D-00	0.1022850570664783D-00
X(12)=	0.1145709487032313D-00	0.1141883107254122D-00	0.1039324146458969D-00
X(13)=	0.1176792391569990D-00	0.1121335599098900D-00	0.1042811735899208D-00

	$Z_{10}(x_i)$	$Z_{11}(x_i)$	$Z_{12}(x_i)$
X(1)=	0.124989436032949 8D-02	-0.890071598085273 0D-03	0.491823712849540 5D-03
X(2)=	-0.165528210671859 8D-02	0.118706476984238 3D-02	-0.658553213377111 4D-03
X(3)=	0.158445439918594 6D-02	-0.115780726707867 2D-02	0.649012224476329 7D-03
X(4)=	-0.108100739155166 2D-02	0.838088088646905 7D-03	-0.484286824012215 2D-03
X(5)=	0.174218008069314 8D-03	-0.256479192829687 7D-03	0.182070064913180 6D-03
X(6)=	0.111043173897100 8D-02	-0.551604691813050 3D-03	0.234220828020292 1D-03
X(7)=	-0.280008434018228 5D-02	0.155806165777242 7D-02	-0.739679167809574 0D-03
X(8)=	0.515551383344898 3D-02	-0.278561124325557 1D-02	0.132032173679967 2D-02
X(9)=	-0.965768165964280 8D-02	0.445045693857763 0D-02	-0.200260016537103 5D-02
X(10)=	0.468621107031921 2D-01	-0.773145662651105 0D-02	0.294808395485525 0D-02
X(11)=	0.943718345160640 1D-01	0.364845398782880 0D-01	-0.496519729960483 5D-02
X(12)=	0.874454479669707 4D-01	0.738296646270133 6D-01	0.239240672602196 1D-01
X(13)=	0.895635195774005 7D-01	0.678411735750806 3D-01	0.491894214934045 1D-01

$Z_{13}(x_i)$

```
X(1)=  -0.1468199649862095D-03
X(2)=   0.1969792464338922D-03
X(3)=  -0.1951045559783830D-03
X(4)=   0.1476688361511866D-03
X(5)=  -0.6019424085263412D-04
X(6)=  -0.5987359369799953D-04
X(7)=   0.2040627988693346D-03
X(8)=  -0.3655476115985035D-03
X(9)=   0.5444399231101236D-03
X(10)= -0.7603664750684390D-03
X(11)=  0.1094603458614414D-02
X(12)= -0.1922156845614520D-02
X(13)=  0.1031138161868777D-01
```

Quadrature points and weights for the 5,9, and 13 point Gauss - Lobatto integration formulas

X(1)	=	0.	W(1)	=	0.4999999999999999D-01
X(2)	=	0.1726731646460115D-00	W(2)	=	0.2722222222222222D-00
X(3)	=	0.5000000000000000D 00	W(3)	=	0.3555555555555555D-00
X(4)	=	0.8273268353539885D 00	W(4)	=	0.2722222222222222D-00
X(5)	=	1.0000000000000000D 00	W(5)	=	0.4999999999999999D-01
X(1)	=	0.	W(1)	=	0.1388888888888888D-01
X(2)	=	0.5012100229426986D-01	W(2)	=	0.8274768078040274D-01
X(3)	=	0.1614068602446311D-00	W(3)	=	0.1372693562500809D-00
X(4)	=	0.3184412680869109D-00	W(4)	=	0.1732142554865231D-00
X(5)	=	0.5000000000000000D 00	W(5)	=	0.1857596371882086D-00
X(6)	=	0.6815587319130890D 00	W(6)	=	0.1732142554865231D-00
X(7)	=	0.8385931397553689D 00	W(7)	=	0.1372693562500809D-00
X(8)	=	0.9498789977057301D 00	W(8)	=	0.8274768078040274D-01
X(9)	=	1.0000000000000000D 00	W(9)	=	0.1388888888888888D-01
X(1)	=	0.	W(1)	=	0.6410256410256410D-02
X(2)	=	0.2334507667891805D-01	W(2)	=	0.3890084337340946D-01
X(3)	=	0.7682621767406389D-01	W(3)	=	0.6749096334480414D-01
X(4)	=	0.1569057654591213D-00	W(4)	=	0.9182343260177502D-01
X(5)	=	0.2585450894543319D-00	W(5)	=	0.1103838967830550D-00
X(6)	=	0.3753565349468800D-00	W(6)	=	0.1220078951533382D-00
X(7)	=	0.5000000000000000D 00	W(7)	=	0.1259654246667233D-00
X(8)	=	0.6246434650531200D 00	W(8)	=	0.1220078951533382D-00
X(9)	=	0.7414549105456680D 00	W(9)	=	0.1103838967830550D-00
X(10)	=	0.8430942345408786D 00	W(10)	=	0.9182343260177502D-01
X(11)	=	0.9231737823259361D 00	W(11)	=	0.6749096334480414D-01
X(12)	=	0.9766549233210818D 00	W(12)	=	0.3890084337340946D-01
X(13)	=	1.0000000000000000D 00	W(13)	=	0.6410256410256410D-02

Constants associated with Gauss - Lobatto quadrature for the 5,9, and 13 point formulas respectively

	$W_1(x_i)$	$W_2(x_i)$	$W_3(x_i)$
X(1)=	0.1524774528788104D-00	-0.4044011245821461D-01	0.1063582422541544D-01
X(2)=	0.2888636342763056D-00	0.1777777777777778D-00	-0.1664141205408352D-01
X(3)=	0.2615863979968066D-00	0.3959956680137700D-00	0.1197447693434116D-00
X(4)=	0.2923674247642566D-00	0.2888888888888888D-00	0.3687436863468540D-00
X(1)=	0.4762215980278096D-01	-0.2046876619410781D-01	0.1602232535614350D-01
X(2)=	0.8658404112901086D-01	0.7333736062164268D-01	-0.1970099596570858D-01
X(3)=	0.8115507322904750D-01	0.1464508516034278D-00	0.8912877679760983D-01
X(4)=	0.8369955136739388D-01	0.1327247851468220D-00	0.1873230701258434D-00
X(5)=	0.8203382619747552D-01	0.1404240930909589D-00	0.1652136519161402D-00
X(6)=	0.8339513797201167D-01	0.1345140321606501D-00	0.1796450883685010D-00
X(7)=	0.8201390811386579D-01	0.1403341653028478D-00	0.1663281426026987D-00
X(8)=	0.8444682958457459D-01	0.1302217352692886D-00	0.1888333161687236D-00
X(1)=	0.2250590196440397D-01	-0.1056219508653100D-01	0.9630383818578735D-02
X(2)=	0.4060957291353151D-01	0.3645813412320934D-01	-0.1135851653983146D-01
X(3)=	0.3824120745200046D-01	0.7162840061216503D-01	0.4811103831717491D-01
X(4)=	0.3925552286702953D-01	0.6564994013786940D-01	0.9839923700183464D-01
X(5)=	0.3867158443833413D-01	0.6859007170291548D-01	0.8862500305997267D-01
X(6)=	0.3906898856442434D-01	0.6671750788320492D-01	0.9388288378423614D-01
X(7)=	0.3876468098695678D-01	0.6810238717563348D-01	0.9027059966516704D-01
X(8)=	0.3902111225698279D-01	0.6695892367940613D-01	0.9313754480500402D-01
X(9)=	0.3878507982073752D-01	0.6799821459132483D-01	0.9059189434479433D-01
X(10)=	0.3902402180350362D-01	0.6695444418613220D-01	0.9311232027511560D-01
X(11)=	0.3874757881543321D-01	0.6815623565001856D-01	0.9023479849702715D-01
X(12)=	0.3926949190984054D-01	0.6589298839537678D-01	0.9563017390854064D-01

	$W_4(x_i)$	$W_5(x_i)$	$W_6(x_i)$
X(1)=	-0.1149868205703775D-01	0.6886112883838056D-02	-0.3064809052772344D-02
X(2)=	0.1162053155494231D-01	-0.6430832881957887D-02	0.2755324089417813D-02
X(3)=	-0.1774204374450480D-01	0.8000603570398158D-02	-0.3154736840887400D-02
X(4)=	0.9287981859410430D-01	-0.1410881463931891D-01	0.4544571103258888D-02
X(5)=	0.2035016809326954D-00	0.8408547868892599D-01	-0.9181495353354941D-02
X(6)=	0.1741391056332445D-00	0.1929152514522473D-00	0.6393199562843250D-01
X(7)=	0.1972583192452336D-00	0.1571919301303900D-00	0.1577381224441807D-00
X(8)=	0.1603628117913832D-00	0.2066436764161251D-00	0.1006532756378319D-00

X(1)=	-0.8760675121219126D-02	0.7545257559339001D-02	-0.6042738377735769D-02
X(2)=	0.8459654734337807D-02	-0.6708455749633218D-02	0.5149508957944579D-02
X(3)=	-0.1201338381768474D-01	0.7705995585255798D-02	-0.5401152711501094D-02
X(4)=	0.5673973627964141D-01	-0.1213211873405804D-01	0.6851040163539482D-02
X(5)=	0.1191279632932073D-00	0.6180294414753740D-01	-0.1161521413802020D-01
X(6)=	0.1058292095123678D-00	0.1324718421698889D-00	0.6298271233336168D-01
X(7)=	0.1135131888948562D-00	0.1162122485010693D-00	0.1375806388047412D-00
X(8)=	0.1078640266974486D-00	0.1262539484904354D-00	0.1191143845010360D-00
X(9)=	0.1126779869057236D-00	0.1183304519841301D-00	0.1313665773744765D-00
X(10)=	0.1080240039316659D-00	0.1256861782339282D-00	0.1208159157130564D-00
X(11)=	0.1132648897501973D-00	0.1175846430308481D-00	0.1320081630470009D-00
X(12)=	0.1035064560705869D-00	0.1325051477927818D-00	0.1117574104587091D-00

	$W_7(x_i)$
X(1)=	0.7337726665385972D-03
X(2)=	-0.6474571916055687D-03
X(3)=	0.7138545829301656D-03
X(4)=	-0.9518705869914940D-03
X(5)=	0.1592607551357494D-02
X(6)=	-0.3836360348606863D-02
X(7)=	0.3512552097762362D-01
X(8)=	0.1149494662431836D-00

	$W_7(x_i)$	$W_8(x_i)$	$W_9(x_i)$
X(1)=	0.4423252120331013D-02	-0.2880992966853821D-02	0.1588634104255761D-02
X(2)=	-0.3678283082958297D-02	0.2359892852916177D-02	-0.1288887673611363D-02
X(3)=	0.3677443172014303D-02	-0.2294090124729455D-02	0.1231538258024383D-02
X(4)=	-0.4246053334830466D-02	0.2519870083844098D-02	-0.1314112202235829D-02
X(5)=	0.5795646652129312D-02	-0.3129292111641543D-02	0.1552832936795423D-02
X(6)=	-0.1046394701654116D-01	0.4554687270604282D-02	-0.2059451182757377D-02
X(7)=	0.6020495100593010D-01	-0.8744066510194582D-02	0.3198429542178954D-02
X(8)=	0.1341400138891274D-00	0.5364416050198016D-01	-0.6575804399420828D-02
X(9)=	0.1143018995719641D-00	0.1223972805980247D-00	0.4371239428628549D-01
X(10)=	0.1287163508974140D-00	0.1019242420534665D-00	0.1031819491382807D-00
X(11)=	0.1144626375913912D-00	0.1191445719077605D-00	0.8219304878024176D-01
X(12)=	0.1394767271857962D-00	0.9066079084878427D-01	0.1122780133418735D-00

	$W_{10}(x_i)$	$W_{11}(x_i)$
X(1)=	-0.665272304199158 3D-03	0.153264558512191 5D-03
X(2)=	0.536519157929199 0D-03	-0.123178430405980 3D-03
X(3)=	-0.507251246275968 5D-03	0.115763552960661 8D-03
X(4)=	0.532039664343036 6D-03	-0.120268883544214 2D-03
X(5)=	-0.611423830445136 5D-03	0.136162386475647 4D-03
X(6)=	0.773455461433917 1D-03	-0.168145190977295 6D-03
X(7)=	-0.109910835823569 0D-02	0.229258934996075 6D-03
X(8)=	0.184102320689726 3D-02	-0.354679493381403 3D-03
X(9)=	-0.413743726867630 4D-02	0.659635921355918 3D-03
X(10)=	0.310328292242936 6D-01	-0.170872954028082 3D-02
X(11)=	0.780531584311908 7D-01	0.163949414082475 4D-01
X(12)=	0.482890733852443 4D-01	0.543234702961277 1D-01