UNPUBLISHED PRELIMINARY DATA

NSG-107

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Errata sheets attached

July 6, 1964

M-9991-2

TO: Prof. Sheridan

FROM: J. Kreifeldt

SUBJECT: Analysis of Predictor Model

Most feedback control analysis and description of the human operator and other sentient creatures has been limited to a single input - single output treatment. Well known work on human operators by Tustin, Russell, Elkind and Bekey treated the human as a system with a single-input (error between present-input/present-output), and a single output. Kreifeldt[1] constructed a sampled-data model in which the human was considered as a system with two inputs (reference-input, control output) and a single output (control output).

For many situations these assumptions about the inputs to the system are good approximations. For instance, in the laboratory the tracker can actually be limited to one visual input (error presented on an oscilloscope or meter) or two visual inputs (reference input, and system output similarly displayed).

As a model of the real world these assumptions about visual input variables might actually correspond in number or degrees of freedom i.e., aiming a gun at a point target, or constitute a gross simplification and idealization, i.e., as in driving a car down a road.

In the former case, the system input is closely approximated by a single point moving in time. In the latter example, the system input is actually a stretch of road which has spatial as well as time dependence depending on the car's velocity.

Recently, Sheridan[2] has suggested a model of the human operator which attempts to realistically account for inputs which have spatial as well as time dependence. His model is based on previous work by Ziebolz and Paynter[3] and Kelley[4] on predictor systems.

Background for Predictor Model

Basically, Sheridan's suggestion is that a sentient operator such as a driver, forms the command signal to control his vehicle from not only his present error information but also his anticipated future error computed on the basis of what his trajectory will be if he does not alter his present mode of behavior. Since he can see his future reference input "up ahead" he can conduct a thought experiment and anticipate his future error to some degree of certainty depending on his knowledge of his vehicle dynamics and the length of error projection into the future. Ordinarily, the vehicle driver alters not only steering motions but vehicle velocity as well if this is under his control.

In the analysis which follows, the real world is abstracted to the following degree.

1. The view of the path up ahead which the human is to follow although actually spatially continuous is assumed restricted to a view through finite number of slit placed a constant distance apart along the road. The justification for this is that (a) practically the human can only see a finite distance ahead and (b) the path as layed out in space contains only a finite amount of information in the Shannon sense. Thus the road could be reconstructed from samples taken a constant distance apart along it. This distance can be obtained from the sampling theorem for sharply band-limited functions.

$$f_s \geq 2f_H \qquad (1)$$

where

f_s = sampling frequency in cycles/ft.

f_H = the highest frequency (c/ft) in the road

The analysis in general will consider a finite number of slits placed an arbitrary constant distance apart.

2. Assume the operator is forced-pace i.e., his forward velocity is constant. This is assumed for simplicity in analysis.

3. Assume the control signal supplied by the driver to the vehicle is constant over a time period π in which he conducts his thought experiment and this period is constant. This is justified on both physical grounds and simplicity.

The above idealizations are represented in Fig. 1 in a quasi-laboratory situation in which the path is layed out on a strip-chart recorder which moves at a constant velocity. The view through a finite number of space limited windows is indicated by sensors spaced along the chart paper supplying that data to the human. It is important to remember that the view of the chart paper shown is for one instant of time only.

Figure 1 shows that an operator tracking with a preview is a multiple input-single output system.

A model suitable for analog computing of the human's ability to extrapolate the trajectory of his vehicle and estimate and weight future error is a fast-time model of the real vehicle supplied every π seconds with the initial conditions and input of the vehicle and clamped to zero at the end of the π seconds. Thus during the π seconds, the fast time model will describe the same trajectory as the vehicle will over a length of time determined by the scaling factor and if that same input is continually applied. As an example, assume the vehicle impulsive response is

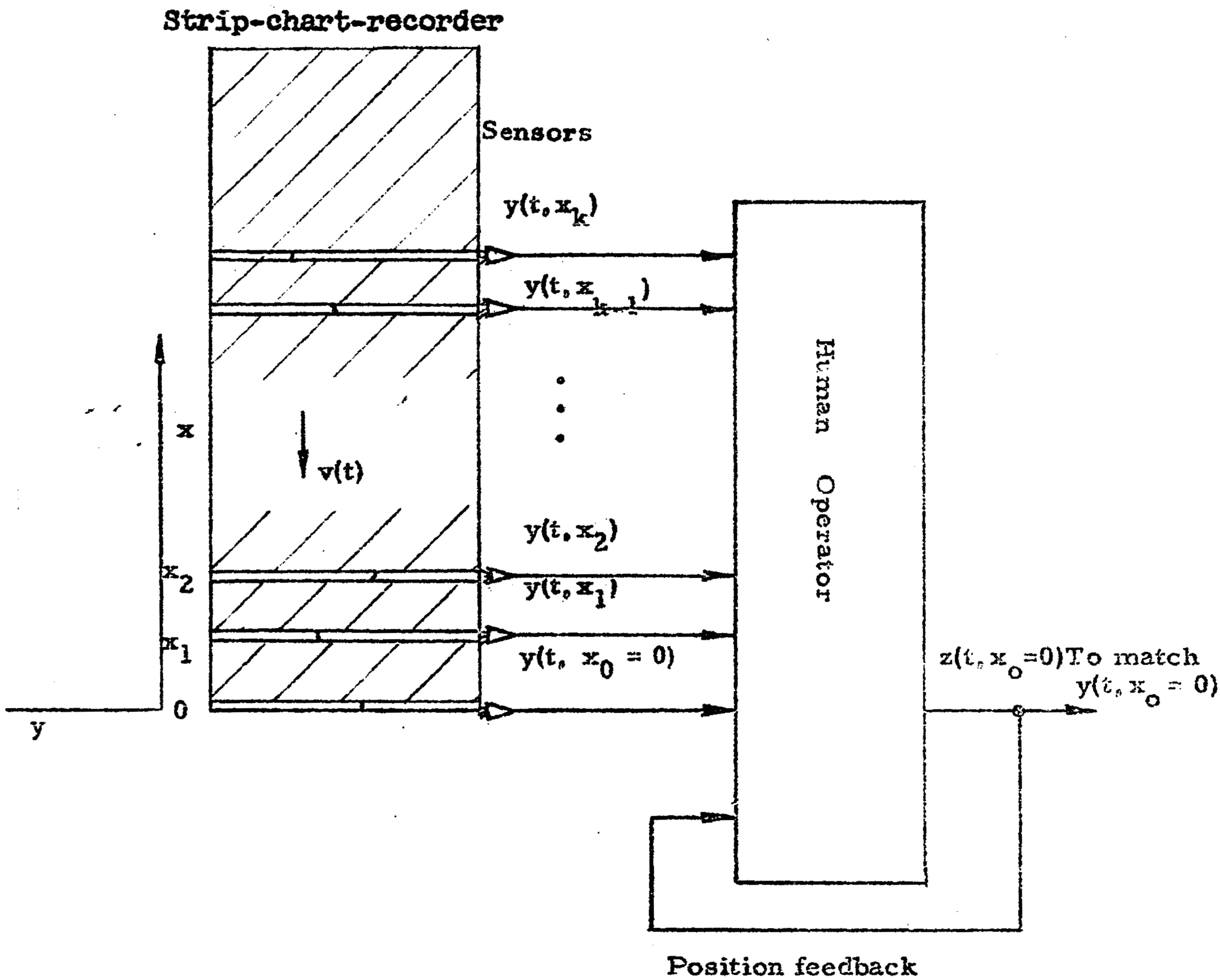


Fig. 1. Representation of the Human Tracking on Moving Strip Paper with Preview Information Discretely Applied.

$$h_v(t) = ae^{-at} \tag{2}$$

assume the fast time model would be chosen as

$$h_m(t) = ake^{-(ak)t} \tag{3}$$

where

$$k > 1$$

Then, for a unity height input, after $t = \pi$ seconds starting from zero time, the model is at $1 - e^{-ak\pi}$ which is where the vehicle will be after $t' = k\pi$ seconds if the unity height input continues for this length of time. Figure 2 indicates the fast time model running in parallel with the real vehicle and zero initial conditions.

After π seconds the fast time model can therefore predict what the error will be at each of the road points (or preview points) if those points are sampled at the beginning of the π seconds and held for the length π seconds. Meanwhile of course, the vehicle has moved a distance $x = V\pi$ down the road and is really seeing a somewhat new picture but if velocity V is slow and/or period π is short, the new picture is not significantly different. Still it is true that the error at the computed points remains the same if the control action remains constant. The error at each of the points computed by the fast time model is then weighted and summed to produce the control signal for the next π seconds.

Figure 3 shows schematically the preview inputs, vehicle fast time model and the error computer consisting of whatever analog indexing switches etc. are needed to compute and form the command input. Initial conditions are supplied to the fast model at the beginning of each command input.

Figure 3 is in some sense an analog computer simulation of the described operation of the preview tracking model. The mathematical analysis of operation must include the following relevant features.

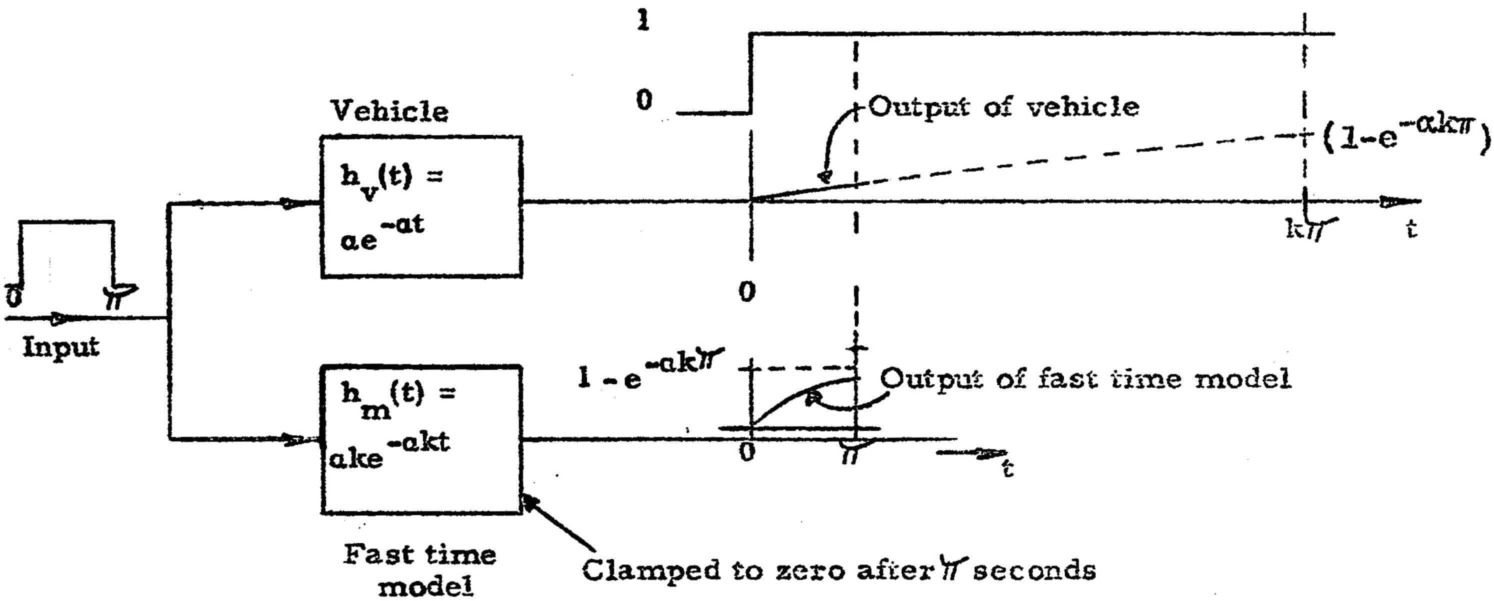


Fig. 2. Comparison Between Fast Time Model and Vehicle Outputs.

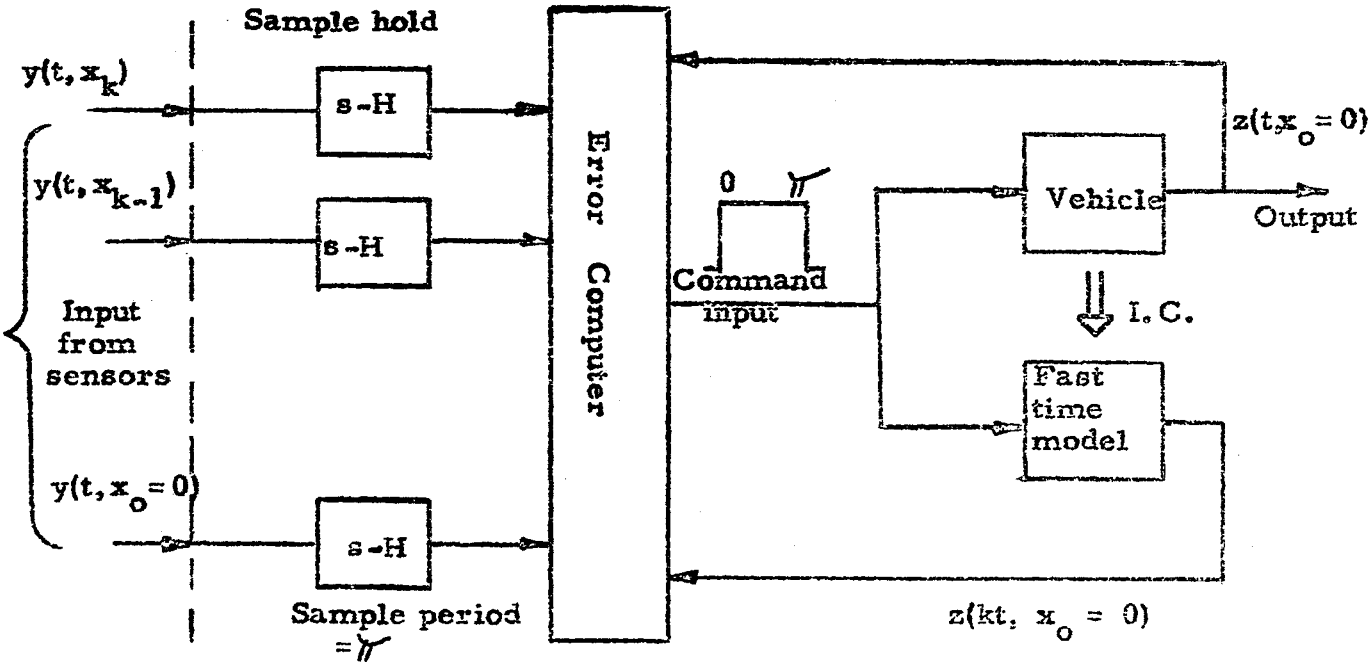


Fig. 3. Processing Preview Inputs, Output of Vehicle and Fast Time Model to Form Command Signal.

Relevant Features of the Model

The relevant features of the model are:

1. There are k + 1 distinct inputs to the model (not including feedback).

2. The command signal to the vehicle is a constant height pulse lasting π seconds.

3. Projected vehicle error at each of the input points is computed and weighted to form the command signal. This process requires π seconds.

4. The vehicle progresses at constant speed.

5. Initial conditions are computed from the vehicle at the beginning of each π seconds interval.

6. The model is suitable for sampled-data analysis.

After a constant height command pulse is applied to the vehicle, the vehicle position is desired at discrete distances $D = 0$, $D = x_1 \ldots D = x_k$ or at times $T_o = 0$, $T_1 = x_1/V \ldots T_k = x_k/V$ for a constant velocity V. The positions at these times (distances) will be due to two factors, the command input and the initial conditions applied as appropriately weighted Dirac delta functions, both applied to a system at rest. That this is so is shown below.

Initial Conditions Considered as an Equivalent Input to a System at Rest

Assume an nth order linear differential equation with initial conditions.

$$a_n \frac{d^n}{dt^n} z(t) + \ldots + a_o z(t) = g(t)$$

$$\text{I.C.} \qquad z(0) = z_o$$

$$\frac{d}{dt} z(t)\Big|_o \triangleq z^{(1)}(0) = z_o^{(1)} \qquad (4)$$

$$\vdots$$

$$\frac{d^{n-1}}{dt^{n-1}} z(t)\Big|_o \triangleq z^{(n-1)}(0) = z_o^{(n-1)}$$

Consider a solution to Eq. (4) which is:

$$z_1(t) = z(t)U(t) \qquad \text{i.e. } z_1(t) = 0 \quad t < 0$$

$$\text{I.C.} = 0 \qquad\qquad = z(t) \quad t > 0$$

where t = 0 is the time origin at which z(t) has the initial conditions, and U(t) is a unit step function.

Since z(t) is a solution for $t \geq 0$ it must satisfy Eq. (4) in some form.

$$a_o z_1(t) = a_o\left[z(t)U(t)\right]$$

$$a_1 z_1^{(1)}(t) = a_1\left[z_o \delta^1(t) + z(t)U(t)\right]$$

$$a_2 z_1^{(2)}(t) = a_2\left[z_o \delta^2(t) + z_o^{(1)}(t)\delta^1(t) + z^{(2)}(t)U(t)\right]$$

$$\vdots \qquad (5)$$

$$a_n z_1^{(n)}(t) = a_n\left[z_o \delta^n(t) + z_o^{(1)} \quad \delta^{n-1}(t) + \dots + \right.$$

$$\left. + z_o^{(n-1)}\delta^1(t) + z^{(n)}(t)U(t)\right]$$

where

$$\delta^n(t) = \frac{d^n}{dt^n} U(t)$$

Adding the above equations yields

$$
\begin{aligned}
&U(t)\left[a_o z(t) + a_1 z^{(1)}(t) + \dots + a_n z^{(n)}(t)\right] \\
&\quad + \delta^1(t)\left[a_1 z_o + a_2 z_o^{(1)} + \dots + a_n z_o^{(n-1)}\right] \\
&\quad + \delta^2(t)\left[0 + a_2 z_o + \dots + a_n z_o^{(n-2)}\right] \\
&\quad \vdots \\
&\quad + \delta^n(t)\left[0 + 0 + \dots + 0 + a_n z_o\right] \\
&\quad = \sum_{i=0}^{n} a_i \frac{d^i z_1(t)}{dt^i} = \sum_{i=0}^{n} a_i \; z_1^{(i)}(t) \quad .
\end{aligned}
\tag{6}
$$

Since

$$a_n z^{(n)}(t) + a_{n-1} z^{(n-1)}(t) + \dots + a_o z(t) = g(t) \; ,$$

$$
\begin{aligned}
a_n \frac{d^n z_1(t)}{dt^n} + a_{n-1} \frac{d^{n-1} a_1(t)}{dt^{n-1}} + \dots + a_o z_1(t) &= g(t)U(t) \\
+ \delta^1(t)\left[a_1 z_o + a_2 z_o^{(1)} + \dots + a_n z_o^{(n-1)}\right]& \\
+ \delta^2(t)\left[a_2 z_o + a_3 z_o^{(1)} + \dots \; a_n z_o^{(n-2)}\right]& \\
\vdots \quad & \\
+ \delta^n(t)\left[a_n z_o\right]& \\
\text{I.C.} = 0 \quad &
\end{aligned}
\tag{7}
$$

Thus the system described by Eq. (4) is equivalent for $t > 0$ to a system dynamically the same but with the initial conditions stated in Eq. (4) appropriately combined and weighted by Dirac delta functions and applied as an input to a relaxed system along with the driving function $f(t)$ for $t > 0$.

This fact allows the analysis of a preview tracking system to proceed fairly smoothly.

<u>Error Prediction</u>

For an nth order system, derivatives to the (n-1)th order of the vehicle output supply analytically, the required initial conditions. In general the vehicle can be considered as excited each π seconds with the constant height command signal and Dirac delta functions of initial conditions to the (n-1) order appropriately weighted. Therefore, to compute where the vehicle will be at the desired points in the future the following technique can be used analytically.

If $h_v(t)$ is the vehicle impulse response and $z(t)$ is vehicle response then let:

$$h_v^{(-1)}(t) \equiv \int_0^t h_v(\tau) d\tau \tag{8}$$

1. Multiply $h_v^{(-1)}(t)$ by the height of the command pulse $g(t)$ and sample or evaluate it at the discrete desired points or times $T_o = 0^+$. $T_1 = x_1/V$, $T_2 = x_2/V$, ..., $T_k = x_k/V$. This represents the contribution to vehicle position caused by the command signal alone.

2. Multiply $h_v(t)$ by $a_1 z_o + a_2 z_o^{(1)} + \ldots a_n z_o^{(n-1)}$ and evaluate as above. This is the contribution due to the impulse in Eq. (7).

3. Define:

$$h_v^{(k)}(t) \equiv \frac{d^k}{dt^k} h_v(t) \tag{9}$$

multiply $h_v^{(k)}(t)$ by $a_k z_o + \ldots a_n z_o^{(n-k)}$ evaluate each as in Eq. (1). This term is the weighting on vehicle position due to kth order Dirac function.

4. Add all the contributions together at each point. This is the total vehicle position at each future point desired.

The above computations are represented schematically in Fig. 4.

Since the various $h_v^{(m)}(t)$ are responses of a linear system, the outputs from each component system in Fig. 4 at time T_i can be found by computing $h_v^{(m)}(T_i)$ and weighting its value at each T_i by the linear combination of initial conditions appropriate for the mth subsystem in Fig. 4.

One of the initial assumptions is that it requires π seconds to obtain the predicted positions and evaluate the error and that this is repeated every π seconds. Figure 4 is represented analytically in Fig. 5. In this figure, $s^j \Leftrightarrow d^j/dt^j$, that is the derivative operation is indicated by the transform symbol "s".

Since the samplers in Fig. 5 operated every π seconds and are to supply the required information to relaxed systems, the systems called $h_v^{(m)}(t)$ are really defined to be zero for $\pi < t < 0$ and during $o < t < \pi$ are the same as the vehicle except on a fast time scale.

During $0 < t < \pi$ the predicted positions and errors can be obtained by sampling each output at a rate of $\pi = \pi/k$ where k was the total number of equally spaced <u>preview</u> points to obtain the (k+1) points (k preview and one present), the π^* sampler would be synchronous with the π sampler. This data could then be processed serially in time in the error computer of Fig. 3.

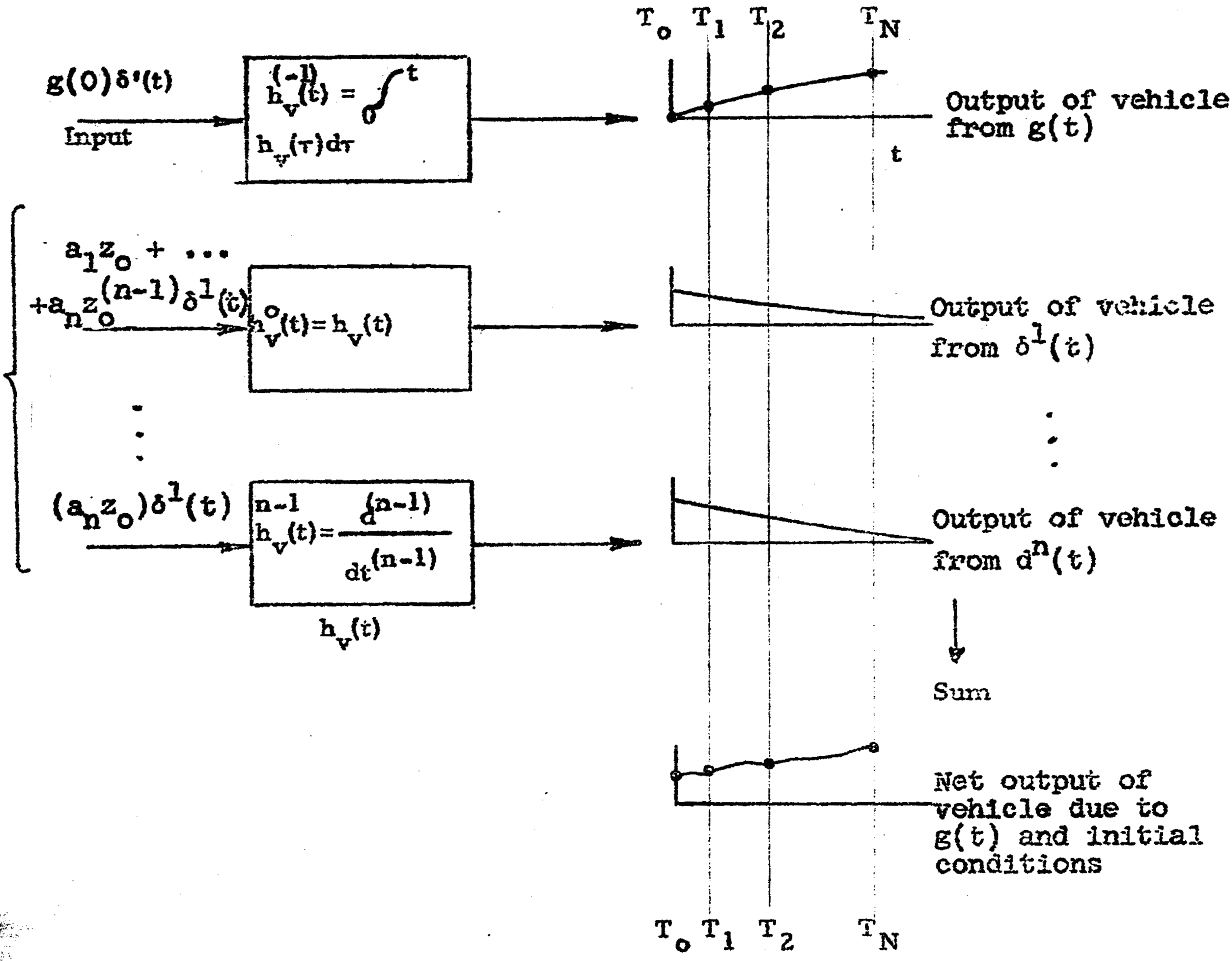


Fig. 4. An Equivalent System Using Weighted Impulses as Inputs Which Exhibits the Same Response as a Real Vehicle Operating With Initial Conditions and A Constant Height Input.

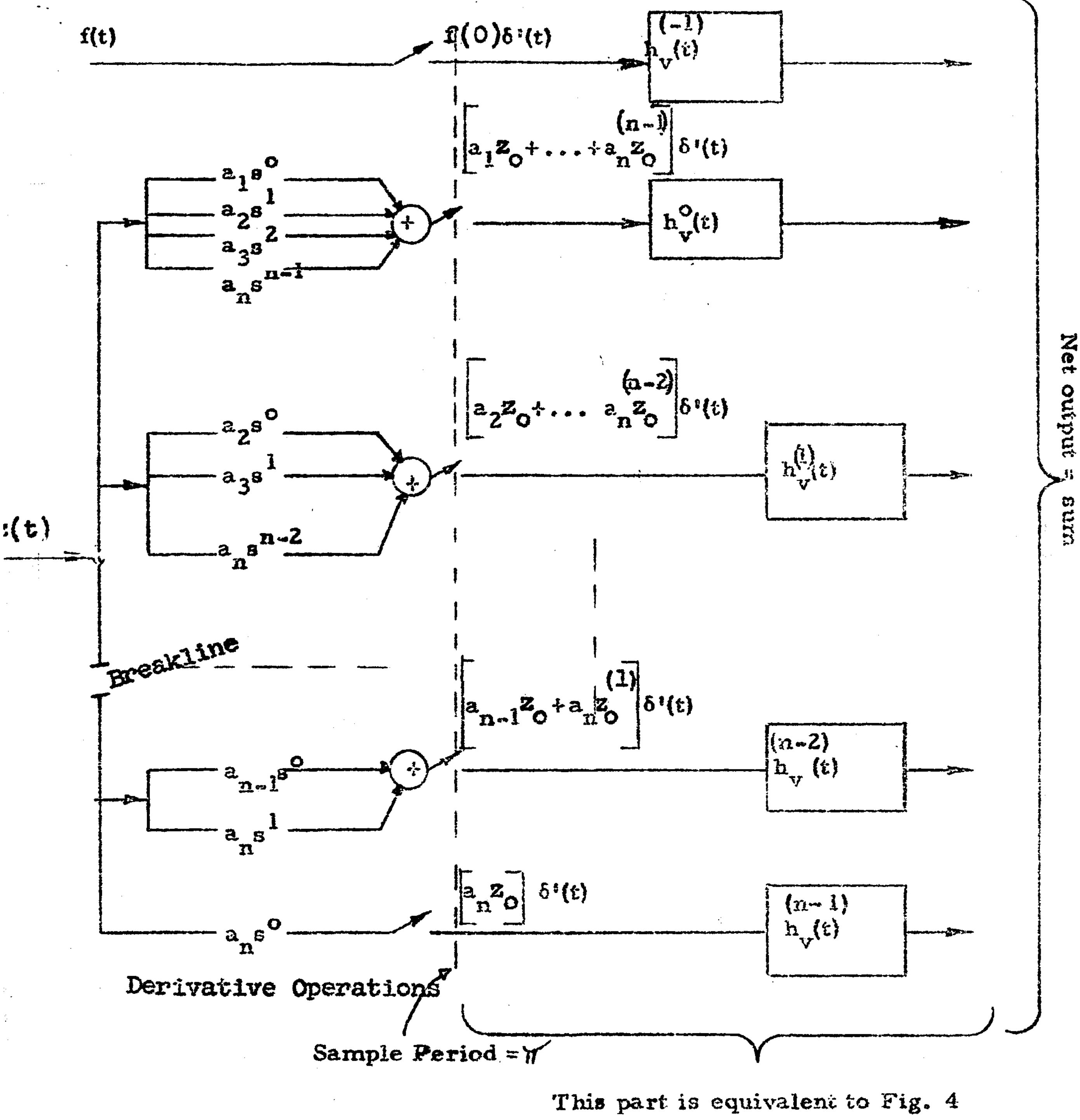


Fig. 5. Analytical Extension to Obtain Initial Conditions for Fig. 4.

Analytically this would require samplers operating at two different periods, and some sequential processing, the analytical description of which is not particularly necessary. An altogether equivalent analytical analysis from the stand point of describing output as a function of the multiple inputs is to consider parallel processing of the error. The method for this is outlined below:

If the $h_v^{(m)}(t)$ are sampled or evaluated at the T_i times corresponding to the preview distances, then the sequence

$$h_v^{(m)}(T_o),\ h_v^{(m)}(T_1),\ h_v^{(m)}(T_2),\ \ldots,\ h_v^{(m)}(T_k) \qquad (10)$$

is obtained. Processed serially this requires π seconds. But since the command signal does not change during this time it is immaterial what method is used to obtain this sequence as long as all the operations are complete by π seconds. Therefore, consider Fig. 6.

The sequences can be added for each $h_v^{(m)}(T_i)$ weighted by appropriate coefficients shown at the left of Fig. 5 and the projected error at each preview point obtained simultaneously instead of sequentially. These errors are then weighted with an additional function (the importance attached by the modelled organism or system to successive times into the future). Then, after adding and delaying for π seconds, we have the equivalent representation desired. Figure 7 shows this analytical representation.

The other alternative (i.e., sequential processing) would have required a linear time-varying filter which might have been more compact in formulation but would lead to the same complexity in evaluation.

The complete analytical block diagram suitable for flow graph analysis is now shown in Fig. 8. The sampler in Fig. 7 has been replaced by a single equivalent one in Fig. 8.

The single sampler in Fig. 8 indicates that sampled-data analysis is applicable. The Z-transform approach will be used where:

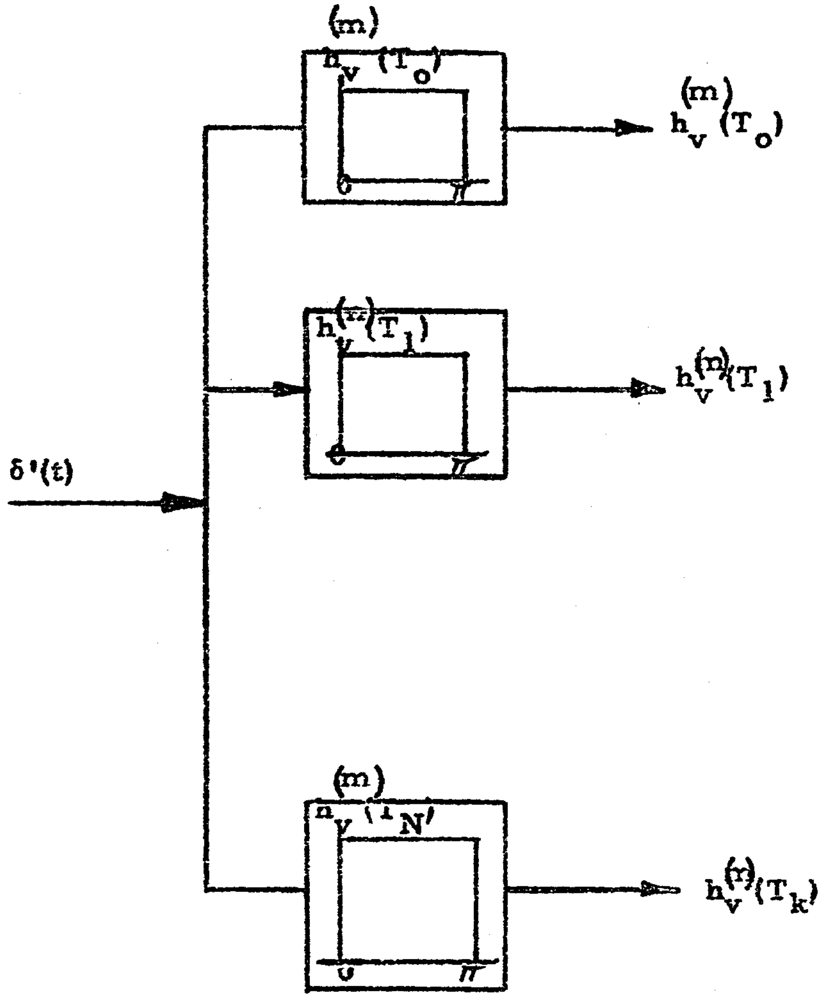
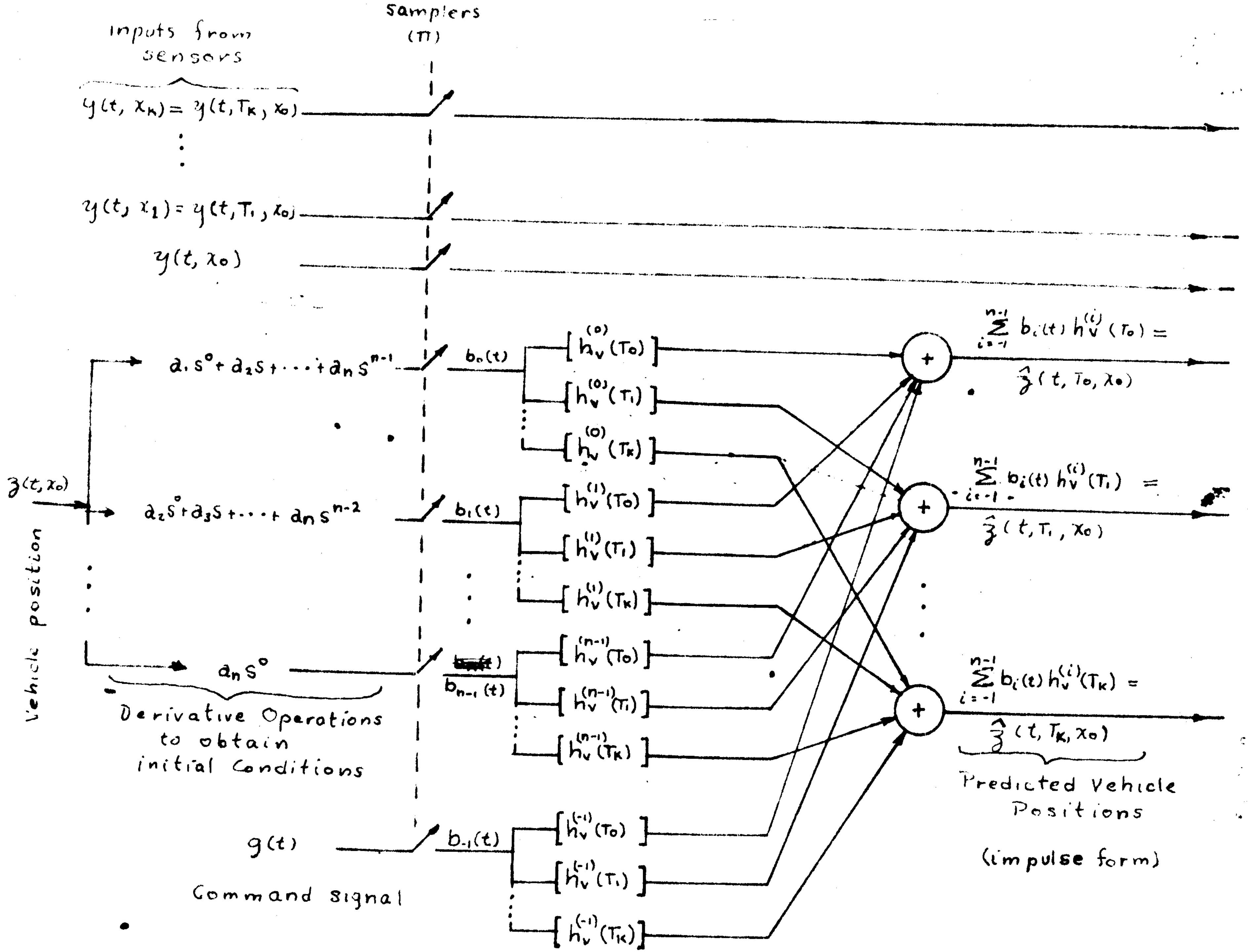
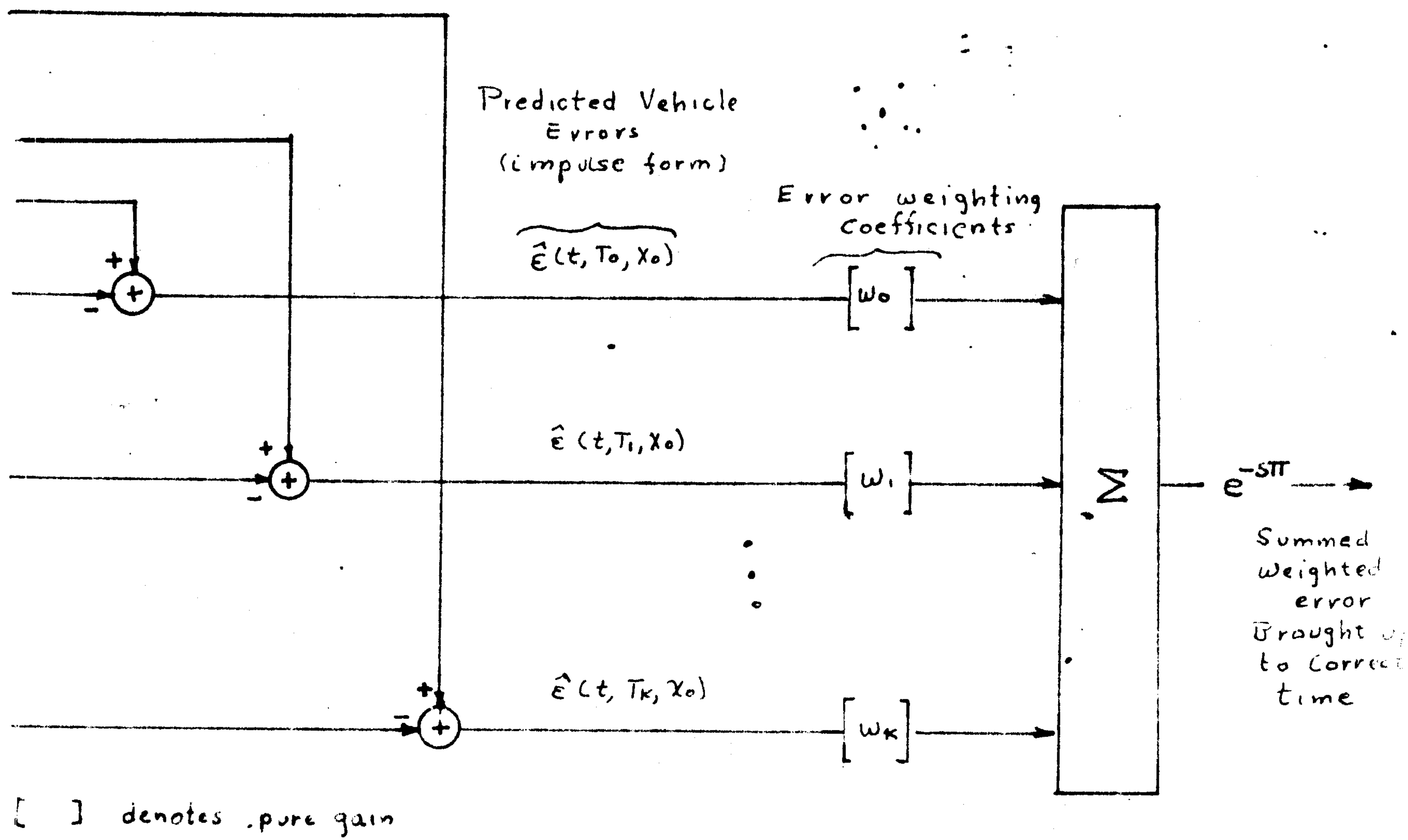
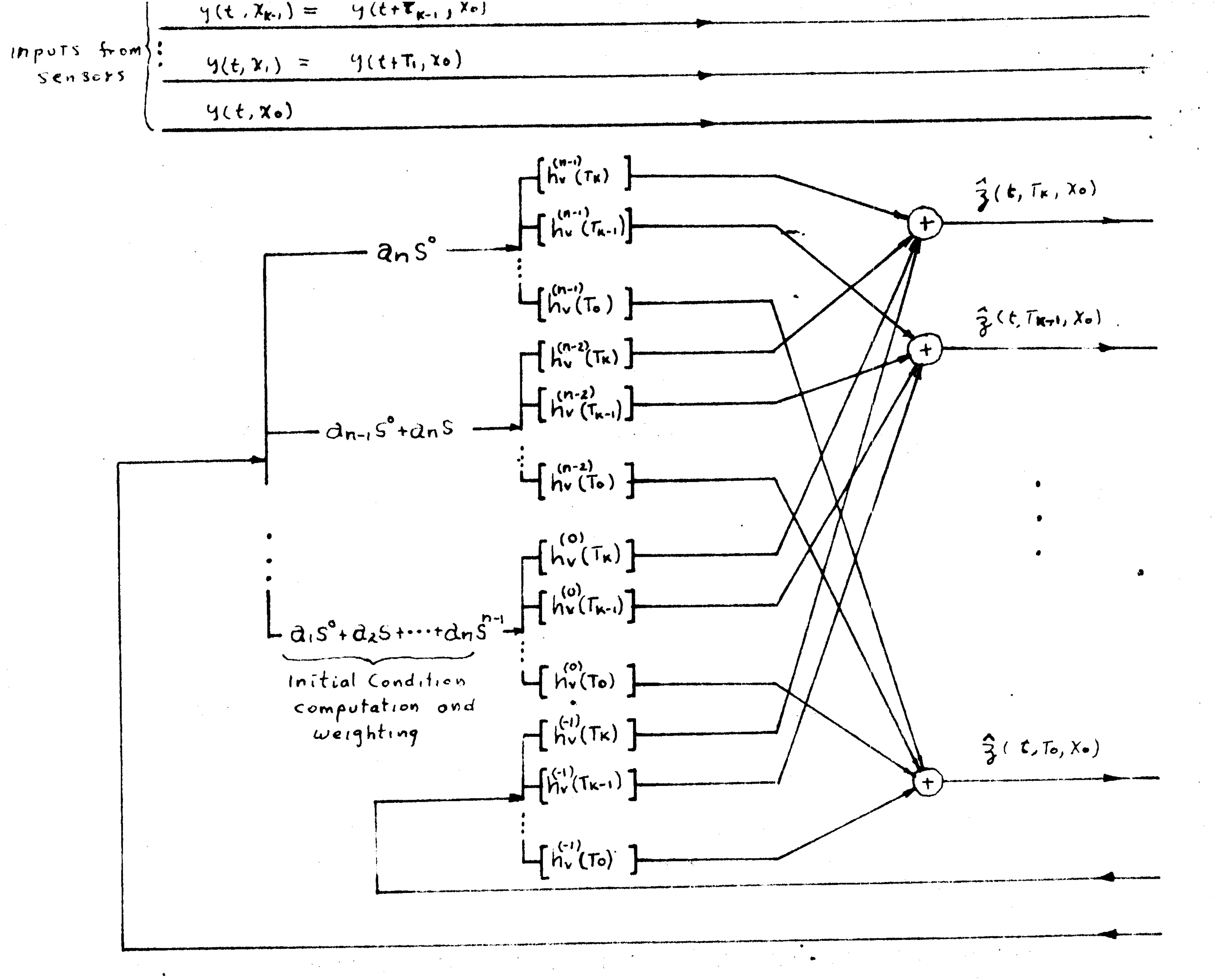


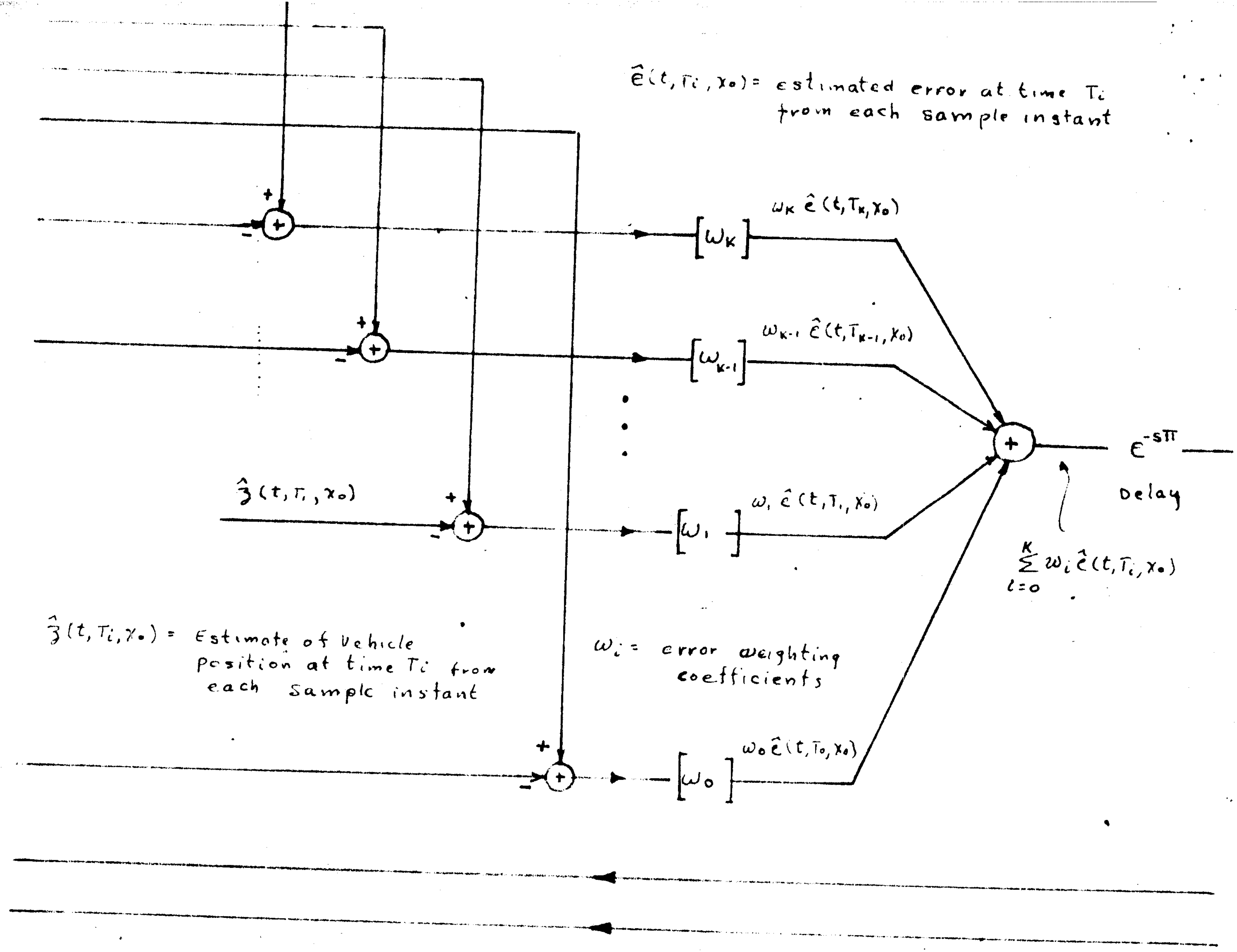
Fig. 6. Parallal Method of Obtaining the Sequence $h_v^{(m)}(T_o) \ldots h_v^{(m)}(T_k)$ Over the Time Period π.

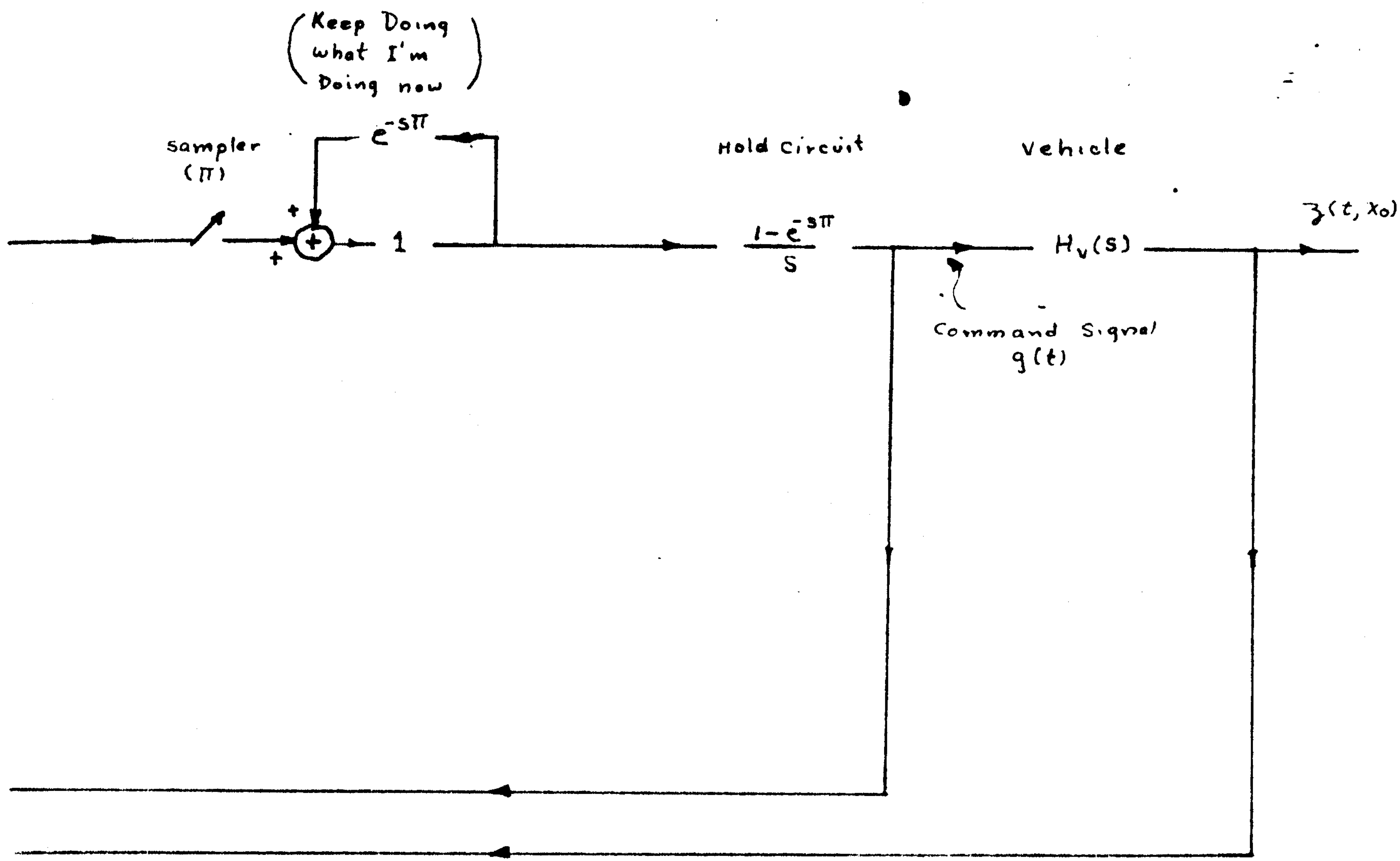


Samplers (T)
Inputs from sensors
$y(t, x_K) = y(t, T_K, x_0)$
$y(t, x_1) = y(t, T_1, x_0)$
$y(t, x_0)$
$a_1 s^0 + a_2 s + \cdots + a_n s^{n-1}$
$a_2 s^0 + a_3 s + \cdots + a_n s^{n-2}$
$a_n s^0$
$z(t, x_0)$
Vehicle position
Derivative Operations to obtain initial Conditions
$g(t)$
Command signal
$b_0(t)$
$b_1(t)$
$b_{n-1}(t)$
$b_{-1}(t)$
$h_v^{(0)}(T_0)$
$h_v^{(0)}(T_1)$
$h_v^{(0)}(T_K)$
$h_v^{(1)}(T_0)$
$h_v^{(1)}(T_1)$
$h_v^{(1)}(T_K)$
$h_v^{(n-1)}(T_0)$
$h_v^{(n-1)}(T_1)$
$h_v^{(n-1)}(T_K)$
$h_v^{(-1)}(T_0)$
$h_v^{(-1)}(T_1)$
$h_v^{(-1)}(T_K)$
$\sum_{i=-1}^{n-1} b_i(t) h_v^{(i)}(T_0) = \hat{z}(t, T_0, x_0)$
$\sum_{i=-1}^{n-1} b_i(t) h_v^{(i)}(T_1) = \hat{z}(t, T_1, x_0)$
$\sum_{i=-1}^{n-1} b_i(t) h_v^{(i)}(T_K) = \hat{z}(t, T_K, x_0)$
Predicted Vehicle Positions
(impulse form)


Predicted Vehicle Errors (impulse form)
Error weighting coefficients
$\hat{\varepsilon}(t, T_0, \chi_0)$
$\hat{\varepsilon}(t, T_1, \chi_0)$
$\hat{\varepsilon}(t, T_K, \chi_0)$
w_0
w_1
w_K
Σ
$e^{-s\pi}$
Summed weighted error Brought up to Correct time
[ ] denotes pure gain


Inputs from sensors
y(t, x_{k-1}) = y(t+τ_{k-1}, x_0)
y(t, x_1) = y(t+T_1, x_0)
y(t, x_0)
a_n s^0
a_{n-1} s^0 + a_n s
a_1 s^0 + a_2 s + ··· + a_n s^{n-1}
Initial Condition computation and weighting
h_v^{(n-1)}(T_K)
h_v^{(n-1)}(T_{K-1})
h_v^{(n-1)}(T_0)
h_v^{(n-2)}(T_K)
h_v^{(n-2)}(T_{K-1})
h_v^{(n-2)}(T_0)
h_v^{(0)}(T_K)
h_v^{(0)}(T_{K-1})
h_v^{(0)}(T_0)
h_v^{(-1)}(T_K)
h_v^{(-1)}(T_{K-1})
h_v^{(-1)}(T_0)
ẑ(t, T_K, x_0)
ẑ(t, T_{K-1}, x_0)
ẑ(t, T_0, x_0)


$\hat{e}(t, T_i, x_0)$ = estimated error at time T_i from each sample instant
$\omega_K \hat{e}(t, T_K, x_0)$
ω_K
$\omega_{K-1} \hat{e}(t, T_{K-1}, x_0)$
ω_{K-1}
$\hat{z}(t, T_1, x_0)$
$\omega_1 \hat{e}(t, T_1, x_0)$
ω_1
ϵ^{-sT}
Delay
$\sum_{i=0}^{K} \omega_i \hat{e}(t, T_i, x_0)$
$\hat{z}(t, T_i, x_0)$ = Estimate of vehicle position at time T_i from each sample instant
ω_i = error weighting coefficients
$\omega_0 \hat{e}(t, T_0, x_0)$
ω_0
+
−



[illegible] Operational Block Diagram Suitable for [illegible] (continued)

$$F(Z) \triangleq \sum_{n=0}^{\infty} f(n)\, Z^{n} = Z\left\{f(t)\right\} \qquad (11)$$

Exponential transforms are also used since the system is a mixture of discrete and continuous elements. The exponential transform is defined:

$$F(s) \triangleq \int_{0}^{\infty} e^{-st} f(t)\, dt = E\left\{f(t)\right\} \qquad (12)$$

The input from the sensors indicated by $y(t, x_i)$ which is the input at position x_i at time t is, because of the constant velocity restriction,

$$y(t, x_i) = y(t + \frac{x_i}{V}, x_o) \qquad (13)$$

or denoting

$$T_i = \frac{x_i}{V} \qquad (14)$$

$$y(t, x_i) = y(t + T_i, x_o) \qquad (15)$$

The exponential transform of this is

$$E\left\{y(t, x_i)\right\} = y(s, x_o)e^{sT_i} \qquad (16)$$

The quantities marked $\hat{z}(t, T_i, x_o)$ are time dependent estimates of vehicle position at the T_i and have the exponential transform

$$E\left\{\hat{z}(t, T_i, x_o)\right\} = \hat{z}(s, T_i, x_o) \qquad (17)$$

From Fig. 8

$$\hat{z}(s, T_i, x_o) = \left[a_1 s^{o} + \dots + a_n s^{n-1}\right] h_n^{o}(T_i) + \left[a_2 s^{o} + \dots + a_n s^{n-2}\right] h_v^{(1)}(T_i)$$

$$+ \dots + a_n s^{o}\, h_n^{(n-1)}(T_i) + F(s)\, h_v^{(-1)}(T_i) \qquad (18)$$

Since the sampler operates every π seconds to produce a discrete quantity, define

$$Z \triangleq e^{-s\pi} \tag{19}$$

This indicates that the output from the sampler has data points spaced π seconds apart.

Figure 8 can be reduced to a signal flow graph of Fig. 9. There are two loops around the sampler in Fig. 8, one after the hold circuit and one after the vehicle. These are denoted as Loop 1 and Loop 2. The symbol "M" denotes the sampling operation.

The transmission around the sampler of Loop 1 is

$$L_1(Z) = -\left[\frac{Z}{1 - Z} \cdot \frac{1 - Z}{s} \cdot \sum_{i=0}^{k} w_i h_v^{(-1)}(T_i)\right]^* = - \sum_{i=0}^{k} w_i h_v^{(-1)}(T_i)\left(\frac{Z}{1 - Z}\right) \tag{20}$$

and the transmission of Loop 2 around the sampler is

$$L_2(Z) = -\left[\frac{Z}{1 - Z} \cdot \frac{1 - Z}{s} H(s)\left[(a_1 s^o + \dots + a_n s^{n-1}) \sum_{i=0}^{k} h_v^{(o)}(T_i) w_i \right.\right.$$
$$+ (a_2 s^o + \dots + a_n s^{n-2}) \sum_{i=0}^{k} h_v^{(1)}(T_i) w_i \tag{21}$$
$$\left.\left. + \dots + a_n s^o \sum_{i=0}^{k} h_v^{(n-1)}(T_i) w_i \right]\right]^*$$

$L_2(Z)$ can be written

$$L_2(Z) = - Z\left[\left\{H(s)\ a_1 s^{-1} + a_2 s^o + a_3 s + \dots a_n s^{n-2}\right] \sum_{i=0}^{k} w_i h_v^{(0)}(T_i)\right\}^* +$$

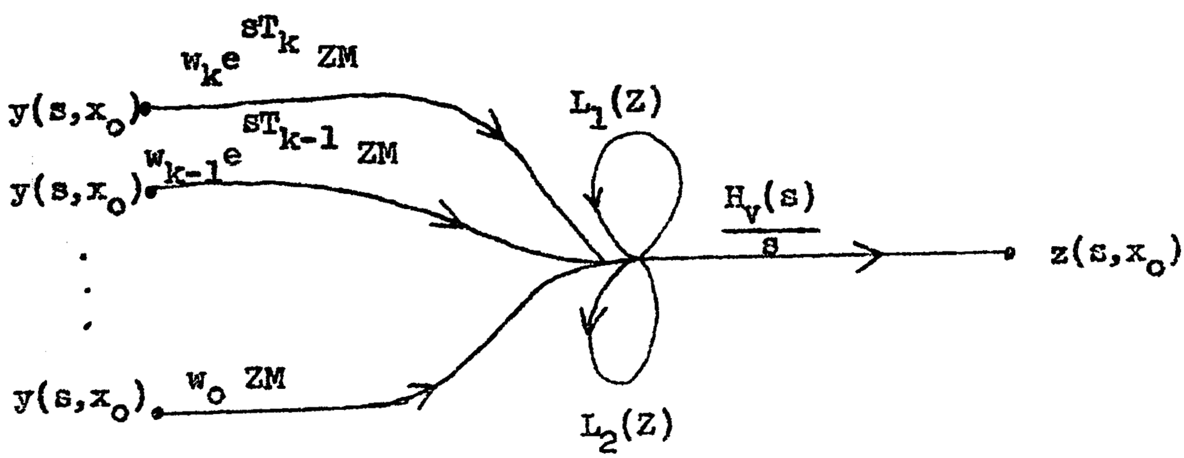


Fig. 9. Reduced Flow Graph of Fig. 8

$$+\left\{H(s)\left[a_2 s^{-1} + a_3 s^{0} + \ldots + a_n s^{n-3}\right]\sum_{i=0}^{k} w_i h_v^{(1)}(T_i)\right\}^{*} + \ldots$$

$$+\left\{H(s)\left[a_{n-1} s^{-1} + a_n\right]\sum_{i=0}^{k} w_i h_v^{(n-2)}(T_i)\right\}^{*} \tag{22}$$

$$+\left\{H(s)\left[a_n s^{-1} \sum_{i=0}^{k} w_i h_v^{(n-1)}(T_i)\right\}^{*}\right]$$

The starred brackets indicate the equivalence

$$\left[F(s)\right]^{*} = \sum_{n=0}^{\infty} f(n\tau)\, Z^{n} = F(Z) \tag{23}$$

Because of the definition of the exponential transform given in Eq. (12)

$$E\left\{\frac{d^k}{dt^k} h_v(t)\right\} = s^k H_v(s) = E\left\{h_v^{(k)}(t)\right\} = H_v^{(k)}(s) \tag{24}$$

$$E\left\{\int_0^t h_v(t)dt\right\} = \frac{1}{s} H_v(s) = H_v^{(-1)}(s) = E\left\{h_v^{(-1)}(t)\right\} \tag{25}$$

therefore

$$L_2(Z) = -Z\left[\left[a_1 H_v^{(-1)}(Z) + a_2 H_v^{(0)}(Z) + a_3 H_v^{(1)}(Z) + \ldots + a_n H_v^{(n-2)}(Z)\right]\sum_{i=0}^{k} w_i h_v^{(0)}(T_i)\right.$$

$$+\left[a_2 H_v^{(-1)}(Z) + a_3 H_v^{(0)}(Z) + \ldots + a_n H_v^{(n-3)}(Z)\right]\sum_{i=0}^{k} w_i h_v^{(1)}(T_i)$$

$$\left. + \ldots + a_n H_v^{(-1)}(Z) \sum_{i=0}^{k} w_i h_v^{(n-1)}(T_i)\right] \tag{26}$$

or

$$L_2(Z) = -Z\left[\left[\sum_{i=1}^{n} a_i H_v^{(i-2)}(Z)\right]\left[\sum_{i=0}^{k} w_i h_v^{(0)}(T_i)\right] + \left[\sum_{i=2}^{n} a_i h_v^{(i-3)}(Z)\right]\right.$$

$$\times\left[\sum_{i=0}^{k} w_i h_v^{(1)}(T_i)\right] \tag{27}$$

$$+ \ldots + \left[\sum_{i=n}^{n} a_i H_v^{(i-(n+1))}(Z)\right]\left[\sum_{i=0}^{k} w_i h_v^{(n-1)}(T_i)\right)$$

In order to make the notation more compact define:

$$\underline{h}_j = \left[h_v^{(j)}(T_o),\ h_v^{(j)}(T_1),\ \ldots,\ h_v^{(j)}(T_k)\right] \quad (1 \times (k+1) \tag{28}$$

$$\underline{W} = \begin{bmatrix} w_o \\ w_1 \\ \vdots \\ w_k \end{bmatrix} \qquad ((k+1) \times 1) \tag{29}$$

$$\underline{a}^j = \left[a_j,\ a_{j+1},\ a_{j+2},\ \ldots,\ a_n,\ 0,\ 0\right] \quad (1xn) \tag{30}$$

$$\underline{H}_{vj}(Z) = \begin{bmatrix} H_v^{(-1)}(Z) \\ H_v^{(0)}(Z) \\ \vdots \\ H_v^{(n-(j+1))}(Z) \\ 0 \\ 0 \end{bmatrix} \qquad (n \times 1) \tag{31}$$

$L_2(Z)$ can now be written

$$L_2(Z) = - Z\Big[(\underline{h}_o \times \underline{w}) \cdot (\underline{a}^1 \times \underline{H}_{v1}(Z) + (\underline{h}_1 \times \underline{w}) \cdot (\underline{a}^2 \times H_{v2}(Z)) + \dots +$$

$$(\underline{h}_{n-1} \times \underline{w}) \cdot (\underline{a}^n \times \underline{H}_{vn}(Z)\Big] \quad (32)$$

or,

$$L_2(Z) = - Z\left[\sum_{i=0}^{n-1} (\underline{h}_i \times \underline{w}) \cdot (\underline{a}^{i+1} \times \underline{H}_{v(i+1)}(Z))\right] \quad (33)$$

To again condense the notation, define:

$$\underline{C} = \Big[(\underline{h}_o \times w),\ (\underline{h}_1 \times \underline{w}),\ \dots,\ (\underline{h}_{n-1} \times \underline{w})\Big]\ (1 \times n) \quad (34)$$

$$\underline{K}(Z) = \begin{bmatrix} a^1 \times \underline{H}_{v1}(Z) \\ a^2 \times \underline{H}_{v2}(Z) \\ \vdots \\ a^n \times \underline{H}_{vn}(Z) \end{bmatrix} \quad (n\text{x}1) \quad (35)$$

Then

$$L_2(Z) = - Z[\underline{C} \times \underline{K}(Z)] \quad (36)$$

Likewise

$$L_1(Z) = - \frac{Z}{1 - Z}\ (\underline{h}_{-1} \times \underline{w}) \quad (37)$$

The expressions for Loops 1 and 2 are complete and the flow graph of Fig. 8 is shown in Figure 9.

The output expressed in terms of the inputs is

$$z(s, x_o) = \sum_{i=0}^{k} \left[y(s,x_o)e^{sT_i}\right]^{*} w_i \frac{Z}{1 - L_1(Z) - L_2(Z)} \frac{H_v(s)}{s} \quad (38)$$

or,

$$z(s,x_o) = \sum_{i=0}^{k} w_i\left[y(s,x_o)e^{sT_i}\right]^{*} \frac{Z}{1 + (\underline{h}-1) \times \underline{w}) \frac{Z}{1 - Z} + \mathcal{Z}(\underline{C} \times \underline{K}(Z))}$$

$$\times \frac{H_v(s)}{s} \quad (39)$$

The above expression Eq. (39) describes completely the output from the predictor model for any input which has space-time dependence and follows the stated assumptions.

The expressions

$$\left[y(s,x_o)e^{sT_i}\right]^{*} = \left[y(s,x_i)\right]^{*} \quad (40)$$

Indicates that $y(t,x_o)$ is to be <u>advanced</u> in time by T_i seconds, sampled at a rate of π seconds and the Z transform taken. x_o is the vehicle's position and is taken equal to zero along x axis.

In order to factor the $y(s,x_o)$ out in some form from Eq. (39) and thus define an impulse response, it is sufficient that

$$y(t,x_o) = 0 \text{ for } t < \frac{x_k}{V} = T_k. \quad (41)$$

This condition implies that the time-space input $y(t,x)$ be

$$y(0,x) = 0 \qquad x < x_k, \; t = 0 \quad (42)$$

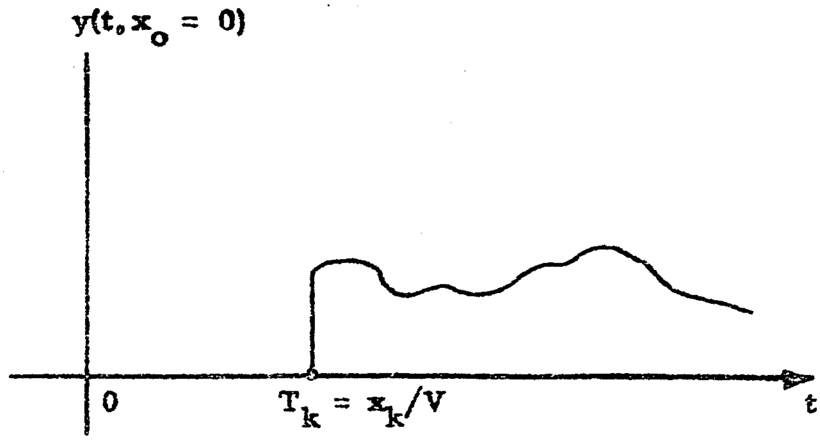


Fig. 10. Graphical Interpretation of Eq. (41) along t axis, $x = 0$.

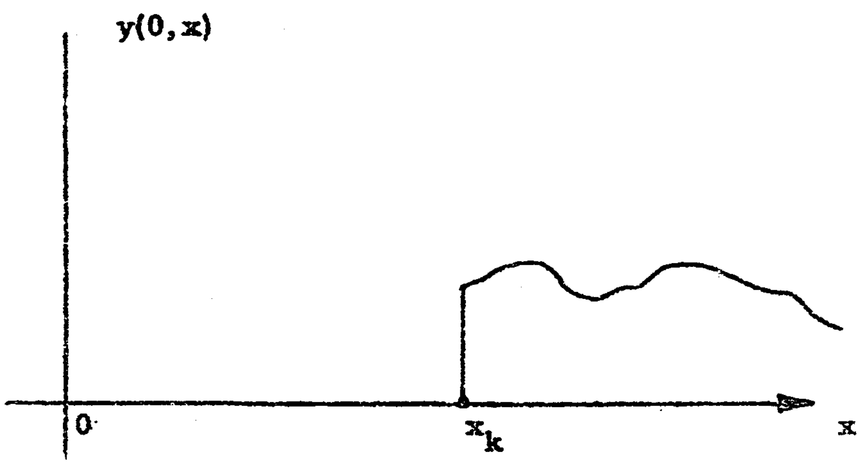


Fig. 11. Graphical Interpretation of Eq. (42) along x axis, $t = 0$.

The graphs of Eq. (41) and Eq. (42) are shown in Figs. 10 and 11.

If we now further assume that T_1 is some multiple of π

$$T_1 = m\pi \tag{43}$$

Then

$$e^{sT_i} = e^{sim\pi} = (e^{-s\pi})^{-im} = Z^{-im} \tag{44}$$

and

$$\left[y(s,x_o)e^{sT_i}\right]^* = \left[y(s,x_o)\right]^* Z^{-im} \tag{45}$$

Now let:

$$y(t,x_o) = f(t - T_k)\, U\,(t - T_k) \tag{46}$$

Then

$$\left[y(s,x_o)\right]^* = Z^{km}\, F(Z) \tag{47}$$

where

$$F(Z) = \sum_{n=0}^{\infty} f(n\pi)Z^n \tag{48}$$

Thus Eq. (45) is

$$\left[y(s,x_o)e^{sT_i}\right]^* = Z^{-im}\, Z^{km}\, F(Z) = F(Z)\, Z^{m(k-i)}$$

and Eq. (39) is:

$$z(s,x_o)=F(Z)\sum_{i=0}^{k} w_i Z^{m(k-i)} \frac{Z}{1 + (\underline{h}-1 \times \underline{w})\frac{Z}{1-Z} + Z(\underline{C} \times \underline{k}(Z))} \times \left[\frac{H_v(s)}{S}\right] \tag{49}$$

On the basis of the above, the impulse response for the whole system (i.e., at $t = 0$ the farthest position up ahead has a unit impulse, all others are zero) is now defined as:

$$P(Z,s) = \sum_{i=0}^{k} w_i Z^{m(k-i)} \frac{Z}{1 + \frac{(\underline{h}_{-1} \times \underline{W})Z}{1 - Z} + Z(\underline{C} \times \underline{k}(Z))} \times \left[\frac{H_v(s)}{s}\right] \quad (50)$$

This form is amenable to the usual sorts of analysis for stability, etc.

<u>Example</u>

Equation (50) will be explicitly stated for a very simple system consisting of a first-order ($n = 1$) vehicle and two points looked at; the present point and one point up ahead; $x_o = 0$, $x_1 = VT_1$,

Specifically:

Let the controlled process have the impulse response

$$h_v(t) = \alpha e^{-\alpha t} \quad (51)$$

then

$$h_v^{-1}(t) = \left[1 - e^{-\alpha t}\right] \quad (52)$$

$$\underline{W} = \begin{bmatrix} w_o \\ w_1 \end{bmatrix} \quad (53)$$

for the two values, one up ahead and one present.

$$\underline{h}_{-1} = \left[0, \ (1 - e^{-\alpha T_1})\right] \quad (54)$$

$$\underline{h}_o = \left[\alpha,\ \alpha e^{-\alpha T_1}\right] \tag{55}$$

$$\underline{a}^1 = \left[\frac{1}{\alpha}\right] \tag{56}$$

$$\underline{H}_{v1}(Z) = \left[H_v^{(-1)}(Z)\right] \tag{57}$$

$$H_v^{(-1)}(Z) = \sum_{n=0}^{\infty} h_v^{(-1)}(n\pi)\ Z^n = \sum_{n=0}^{\infty} (1 - e^{-\alpha n\pi})\ Z^n$$

$$= \frac{1}{1 - Z} - \frac{1}{1 - e^{-\alpha\pi}Z} \tag{58}$$

$$= \frac{1 - e^{-\alpha\pi}Z - (1 + Z)}{(1 - Z)(1 - e^{-\alpha\pi}Z)} = \frac{Z(1 - e^{-\alpha\pi})}{(1 - Z)(1 - e^{-\alpha\pi}Z)}$$

$$\underline{C} = (\alpha w_o + \alpha e^{-\alpha T_1}\ w_1) \tag{59}$$

$$\underline{K}(Z) = \left[\frac{1}{\alpha}\ \frac{Z(1 - e^{-\alpha\pi})}{(1 - Z)(1 - e^{-\alpha\pi}Z)}\right] \tag{60}$$

$$\underline{C} \times \underline{K}(Z) = \frac{(w_o + w_1\ e^{-\alpha T_1})\ Z(1 - e^{-\alpha\pi})}{(1 - Z)(1 - e^{-\alpha\pi}Z)} \tag{61}$$

$$(\underline{h}_{-1} \times \underline{W}) = W_1(1 - e^{-\alpha T_1}) \tag{62}$$

∴ From Eq. (50), the impulse response for this system is

$$P(s,Z) = \left[w_o Z^m + w_1\right] \frac{Z \dfrac{\alpha}{s(s + \alpha)}}{1 + w_1(1 - e^{-\alpha T_1})\dfrac{Z}{1-Z} + \dfrac{Z^2(1 - e^{-\alpha\pi})(w_o + w_1 e^{-\alpha T_1})}{(1-Z)(1-e^{-\alpha\pi}Z)}} \tag{64}$$

Consider that a step input is applied.

$$\left[F(s)\right]^{*} = \left[\frac{1}{s}\right]^{*} = \frac{1}{1-Z} = F(Z) \qquad (65)$$

$$z(Z,x_o) = \left[z(s,x_o)\right]^{*} \qquad (66)$$

Then

$$z(Z,x_o) = \frac{\frac{1}{1-Z}\left[w_o Z^m + w_1\right] Z^2 \frac{(1-e^{-\alpha\pi})}{(1-Z)(1-e^{-\alpha\pi}Z)}}{1 + \frac{w_1(1-e^{-\alpha T_1})Z}{1-Z} + \frac{Z^2(1-e^{-\alpha\pi})(w_o + w_1 e^{-\alpha T_1})}{(1-Z)(1-e^{-\alpha\pi}Z)}} \qquad (67)$$

Using the final value theorem for Z transforms

$$\lim_{t \to \infty} z(t,x_o) = \lim_{Z \to 1}\left[(1-Z) \quad z(Z, x_o)\right] \qquad (68)$$

$$\lim_{t \to \infty} z(t,x_o) = \frac{\left[w_o + w_1\right]}{w_1(1-e^{-\alpha T_1}) + w_o + w_1 e^{-\alpha T_1}} = \frac{w_o + w_1}{w_o + w_1} = 1 \qquad (69)$$

Therefore, the system if stable tracks a step perfectly eventually for any arbitrary error coefficients.

Equation (50) shows that the model as stated is completely linear and time invariant, whether or not it is stable depends on the vehicle dynamics and error weighting coefficients . The combination of Z^s in the numerator of Eq. (50) indicates that even though at $t = 0$ the input is T_k seconds away from x_o, still the output responds before this and the amount depends on the w_i. If for instance:

$$\left.\begin{array}{l} w_o = 1 \\ w_1 = 0 \end{array}\right\} \quad 1 \neq 0 \qquad (70)$$

$$p(z,s) = Z^{mk} \frac{Z}{D(Z)} \frac{H_v(s)}{s} = Z^{mk+1}\left[\frac{H_v(s)}{sD(Z)}\right] \quad (71)$$

Since

$$Z^{mk+1} = e^{-s(mk\tau+\tau)} = e^{-s(T_k+\tau)} = e^{-sT_k} \cdot e^{-s\tau} \quad (72)$$

the output response is "late" by τ seconds. If on the other hand:

$$\left.\begin{array}{l} w_k = 1 \\ w_i = 0 \end{array}\right\} i \neq k \quad (73)$$

$$P(Z,s) = \frac{Z}{D(Z)} \frac{H_v(s)}{s} = Z\left[\frac{H_v(s)}{sD(Z)}\right] \quad (74)$$

and the output starts responding after τ seconds to the input which is T_k seconds away from x_o at $t = 0$.

For arbitrary w_i the response is between these extremes.

Summary

In this outline a model of a human operator controlling a vehicle was analyzed. The model attempts to account for the fact that in many situations (i.e., driving) the operator has an input which is not a single point in time but an input which has spatial as well as time features. That is, he can look at the road ahead.

The sampling theorem in spatial coordinates was invoked in order to treat the time-space input as k discrete inputs to the operator simultaneously available. The model then states that the operator runs some sort of thought experiment in which he extrapolates his position and computes future error if he maintains the same control signal. This computed and weighted predicted error forms the basis for his control action.

His thought experiment requires τ seconds and during this time the control signal remains constant.

The model performs the thought experiment by computing from vehicle initial conditions and command signal, what the errors will be at the previewed points. These individual errors are simultaneously computed and weighted in a length of time requiring τ seconds.

The transformed impulse response for this model was derived and seen to be composed of discrete and continuous elements.

This impulse response was specifically evaluated for a first-order vehicle and two input points. It was seen that if stable, it eventually reaches a reference step height input.

ERRATA SHEET FOR R-[illegible]

Analysis of Predictor Model

NsG-107
M.I.T.

Page 12 line 5 from bottom

$\tau = \tau/k$ should read $\tau^* = \tau/k$

Page 13 Input into box $h_v^{(0)}(t) = h_v(t)$

$a_1 z_c + \dots + a_n z_c^{(n-1)} \delta^1(t)$ should read $[a_1 z_c + \dots + a_n z_c^{(n-1)}] \delta^1(t)$

Page 13 T_N should read T_k

Page 13 $g(0)\,\delta'(t)$ should read $g(0)\delta^1(t)$

Page 14 top line

$f(t)$ and $f(0)\delta'(t)$ should read $g(t)$ and $g(0)\delta^1(t)$

Page 14 omit "Breakline."

all $\delta'(t)$ should read $\delta^1(t)$

Page 16 T_N should read T_k

Page 22 first and second line from bottom

Second line $h_n^0(T_1)$ should read $h_v^{(0)}(T_1)$

first line $h_n^{(n-1)}(T_1)$ should read $h_v^{(n-1)}(T_1)$

first line $F(s)$ should read $G(s)$

Page 25 third line from top

$$+ \left\{ H(s) [a_n s^{-1} \sum_{i=0}^{k} w_i h_v^{(n-1)}(T_i) \right\}^{*}]$$ should read

$$+ \left\{ H(s) \left[a_n s^{-1} \right] \sum_{i=0}^{k} w_i h_v^{(n-1)}(T_i) \right\}^{*}]$$

Page 26

Eq. (28) $(1 \times (k + 1)$ should read $(1 \times (k + 1))$

Page 28 Line 2 from top Eq. (39)

$(\underline{h} - 1) \times \underline{W})$ should read $(\underline{h}_{-1} \times \underline{W})$

Page 30 Eq. 49

$\underline{h} - 1$ should read $\underline{h}_{-1}$

Page 32 Eq. 64

$(1 - e^{a\pi}Z)$ should read $(1 - e^{-a\pi}Z)$

(64) should read (63)

Insert the following after Eq. (63) corrected

"As a simple check on this system, the response to a step can be evaluated quite simply if the output is considered at sampled points and a large amount of time is allowed to pass.

From Eqs. (49), (50), and (63)

$$z(s, x_o) = \frac{F(Z)\left[w_o Z^m + w_1\right] Z \frac{a}{s(s+a)}}{1 + \frac{w_1(1 - e^{-aT_1})Z}{1 - Z} + \frac{Z^2(1-e^{-a\pi})(w_o + w_1 e^{-aT_1})}{(1-Z)(1-e^{-a\pi}Z)}} \qquad (64)$$