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ON THE ASYMPTOTIC STABILITY OF FEEDBACK CONTROL SYSTEMS CONTAINING A SINGLE TIME-VARYING ELEMENT

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ABSTRACT

Rowland, James Richard, Ph.D., Purdue University, June, 1966. On the Asymptotic Stability of Feedback Control Systems Containing a Single Time-Varying Element. Major Professor: Zenonas V. Rekasius.

Three contributions are presented in this thesis. The first contribution is showing that the Popov Criterion, a powerful stability result developed for feedback systems having a single time-invariant nonlinearity, does not apply without modification when the nonlinear characteristic varies with time. The damped Mathieu equation provides a counter-example for the desired purpose.

The second contribution is closely related to the first. A frequency domain criterion is developed to guarantee global asymptotic stability for systems containing a time-varying nonlinear element in the loop. The criterion, known as the Improved Criterion, takes advantage of additional information related to the rate at which the nonlinear characteristic varies with time and represents a considerable improvement over previous criteria.

The third contribution is the Sinusoidal Criterion, which guarantees asymptotic stability for linear feedback systems containing a single sinusoidal gain. The class of systems to which the Sinusoidal Criterion applies is not as large as for the Improved Criterion. However, when both criteria are applied to systems having a single sinusoidal gain, the Sinusoidal Criterion yields a much wider stability

sector.

Both criteria developed in this thesis are independent of the order of the system. The width of the stability sector for each criterion depends upon the transfer function of the linear plant and additional information about the time rate at which the separate element varies. Examples are provided to illustrate the particular effectiveness of each criterion.

Evaluation of these results are discussed from the point of view of their importance to the field of stability theory and the implications for further research.

CHAPTER 1

INTRODUCTION

1.1 Motivation

The analysis and design of automatic control systems frequently requires an investigation of system stability. When the system dynamics are described by a set of linear differential equations with constant coefficients, then the familiar techniques of linear servomechanisms theory, such as root locus and the Nyquist Criterion, may be successfully applied to determine system stability. Even when the equations are nonlinear, some criteria which guarantee stability, such as the Popov Criterion, may often provide useful information. However, when the system contains a time-varying element, the problem becomes considerably more difficult and the above techniques are not directly applicable.

There are numerous physical systems which contain nonstationary elements. These systems may have time-varying parameters because of the action of a process outside the system itself [1,2]. Wide ranges of air density around a rocket or space vehicle may cause time variations in certain parameters of the systems. The resistance of a carbon microphone and the capacitance of a condenser microphone are time-varying parameters. In certain mechanical systems the effective mass or stiffness of a component may vary with time. A pendulum whose pivot point is caused to oscillate in a vertical position is described by the

Mathieu equation [3], which has a time-varying gain. The design of optimal control systems sometimes requires a controller which possesses time varying parameters.

The increased use of more complicated systems in space and missile applications has accelerated the demand for new and improved stability criteria. However, only meager results for time-varying systems are presently available. The purpose of this thesis is to develop new stability criteria for feedback control systems containing a single time-varying element.

1.2 Notation

Vector-matrix notation is used consistently throughout the thesis. Some small English letters, such as b, c, and x, are used to represent n-dimensional column vectors. The capital English letter A designates an n by n constant matrix, and the letter K is reserved to represent a scalar gain constant. The letters V and W denote scalar functions. Other scalar constants are represented by small Greek letters, such as α, β, and γ. The letters f, g, and h are used as real-valued continuous functions. The transpose of a vector or a matrix is shown by a capital T superscript, such as x^T or A^T. The matrix inequality $P \geq 0$ implies that the associated quadratic form x^TPx is non-negative for all x. The statement that the pair (A,b) is completely controllable means the vectors b, Ab, ..., $A^{n-1}b$ are linearly independent. That the pair (A, c^T) is completely observable means the vectors c, A^Tc, ..., $(A^{n-1})^Tc$ are linearly independent. Special notation may be introduced at times to conform with popular usage in the literature, but new symbols are carefully defined at that particular point in the thesis.

1.3 Definitions of Stability

It is important to define the precise meaning of the stability to which later criteria refer. Consider the system described by the vector differential equation

$$\dot{x} = h(x, t) \tag{1.1}$$

where x is an n-vector representing the state of the system and h(x, t) is a real-valued vector function which is continuous in both x and t.

Assume that the equilibrium state x_e being investigated is located at the origin in the state space and that

$$h(0, t) = 0 \quad \text{for all } t \geq t_o. \tag{1.2}$$

Moreover, let the norm of x be represented by $||x||$.

Definition 1:

If for any given $\epsilon > 0$ there exists another positive real number $\delta(\epsilon, t_o)$ such that for every initial state satisfying the inequality

$$||x(t_o)|| < \delta$$

the trajectory satisfies the inequality

$$||x(t)|| < \epsilon$$

for all $t \geq t_o$, then the equilibrium state $x_e = 0$ is said to be stable in the sense of Liapunov.

Definition 2:

If the origin of the state space is stable and, in addition, every trajectory starting sufficiently close to the equilibrium state $x_e = 0$ converges to x_e as time $\to \infty$, then the system (1.1)-(1.2) is said to be asymptotically stable.

If the region of asymptotic stability includes the entire state space, then the system is globally asymptotically stable. If the value of δ in Definitions 1 and 2 is independent of t_o, then the equilibrium state of the system (1.1)-(1.2) is, respectively, uniformly stable or uniformly asymptotically stable.

An inherent deficiency of the above definitions is that stability in the Liapunov sense is a local concept. This means that one may be able to insure stability for small initial conditions, while large initial conditions result in an unstable behavior. However, the criteria obtained in later chapters guarantee global asymptotic stability.

1.4 Liapunov's Stability Theorems

In 1892 Liapunov [4] developed his theory of the stability of dynamic systems. He investigated the stability problem by two distinct methods. The first method, which has since met with very little success, required an explicit solution of the differential equations describing the system behavior. The second method was based on the physical reasoning that a dissipative system perturbed from its equilibrium state will always return to it. To facilitate this theory, Liapunov introduced an energy-like function which, together with its time derivative, must satisfy certain requirements to predict either system stability or instability. This technique, which does not require the explicit solution of the system equations, has become known as the "Direct" or "Second Method" of Liapunov.

The following theorems provide a basis for the development of the stability criteria in later chapters.

Theorem 1:

If there exists a real-valued continuous function $V(x, t)$ with the following properties:

a). $V(x, t)$ has continuous first partial derivatives

b). $V(x, t)$ is positive definite, i.e. $V(x, t) \geq W_1(x) > 0$ for all $x > 0$ and all $t \geq 0$, and $V(0, t) = 0$.*

c). $\lim_{|x_i| \to \infty} V(x, t) = \infty$ for all x_i.

where x_i for $i = 1, \ldots, n$ represents the components of the n-vector x.

d). there exists some region including the origin in the state space in which $\dot{V}(x, t) \leq 0$,

then the equilibrium state $x_e = 0$ of the system (1.1)-(1.2) is stable in the sense of Liapunov.

The theorem for asymptotic stability is somewhat more restrictive.

Theorem 2:

If there exists a real-valued continuous scalar function $V(x, t)$ which satisfies conditions (a), (b), and (c) of Theorem 1, and in some region including the equilibrium state the condition $\dot{V}(x, t) < 0$ is satisfied, where $0 < W_1(x) \leq V(x, t) \leq W_2(x)$, and $W_1(x)$ and $W_2(x)$ are positive definite, then the equilibrium state $x_e = 0$ of the system (1.1)-(1.2) is asymptotically stable.

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* In this context, $W_1(x)$ is a positive definite function dominated by $V(x, t)$.

If $h(x, t)$ in (1.1) is not explicitly a function of time, then the condition $\dot{V} < 0$ may be replaced by $\dot{V} \leq 0$, where the curve $\dot{V} = 0$ is not a trajectory of the system (1.1).

The theorems of Liapunov result in sufficient, rather than necessary, conditions for the stability of systems. For this reason, if a stability theorem is not satisfied, one cannot conclude on this basis that the system is unstable. Instability can, however, be proven if certain other conditions specified in some of Liapunov's theorems for instability are satisfied. Generally, the result obtained by Liapunov theory is quite restrictive.

Liapunov theory has become more useful in recent years as researchers have discovered improved techniques for selecting tentative Liapunov functions and for constraining their time derivatives. It will be demonstrated in later chapters of this thesis that these improved techniques give a result which is less restrictive than before.

1.5 Applications of Liapunov Theory

Although Liapunov's theorems were available even before the turn of the century, it has only been within the last ten years that interest has become accelerated. Earlier works include books and papers by Lur'e [5], Malkin [6], Letov [7], and Zubov [8]. In particular, Lur'e suggested using a Liapunov function composed of a quadratic term plus an integral term. A summary of the early results is given in a book by Hahn [9]. Kalman and Bertram [10] and LaSalle and Lefschetz [11] have also produced works on Liapunov theory. In general, the results pertain to systems of low order, which presents a serious handicap in view of modern work in the area.

The big impetus for current research using the Second Method came as a result of a stability criterion developed by an entirely different technique. This renewed interest in stability stems directly from a breakthrough in the early part of this decade by the Rumanian scientist, V.M. Popov. Using functional analysis, Popov [12, 13] developed a frequency domain criterion to guarantee asymptotic stability for feedback systems containing a single time-invariant nonlinearity in a finite sector (0, K). The Popov investigation reveals an entirely new insight into the stability problem; the width of the sector containing the nonlinearity depends only upon the transfer function of the linear plant. Moreover, the Popov result is independent of the order of the system, which represents a remarkable advantage.

The relevance of the Popov Criterion to Liapunov theorists became apparent shortly thereafter in a paper by Kalman [14]. Essentially, the Popov Criterion pointed the direction in which stability work should proceed. Kalman solved the indirect control problem, and then Rekasius [15] followed by showing that Popov's result holds for the direct control problem. More recently, Brockett [16, 17] has utilized Liapunov theory to obtain even stronger stability results for the single time-invariant nonlinearity system.

1.6 Derivation of the Popov Inequality Using Liapunov Theory

The Popov Criterion applies to a system described by

$$\dot{x} = Ax + bf(\sigma) \tag{1.3}$$

$$\sigma = c^T x \tag{1.4}$$

where

$$0 < \sigma\ f(\sigma) < K\sigma^2 \quad \text{for } \sigma \neq 0$$
$$f(0) = 0 \tag{1.5}$$

Choose as a tentative Liapunov function the form first proposed by Lur'e [5].

$$V(x, \sigma) = x^T P x + \beta \int_0^\sigma f(z)\, dz \tag{1.6}$$

Evaluating its time derivative along the trajectories of the system (1.3)-(1.5), one obtains

$$\dot{V}(x, \sigma) = x^T[A^T P + PA]x + (2b^T P + \beta c^T A)\ x\ f(\sigma)$$
$$+ \beta c^T b \quad f^2(\sigma) \tag{1.7}$$

Constrain $\dot{V}(x, \sigma)$ to be of the form

$$\dot{V}(x, \sigma) = -\ [q^T x - \sqrt{\gamma}\ \ f(\sigma)]^2 - f(\sigma)\ [\sigma - \frac{1}{K} f(\sigma)]. \tag{1.8}$$

The resulting equations are

$$A^T P + PA = -\ qq^T \tag{1.9}$$

$$2b^T P + \beta c^T A = 2\sqrt{\gamma}\ q^T - c^T \tag{1.10}$$

$$\gamma = \frac{1}{K} - \beta c^T b \tag{1.11}$$

In (1.9)-(1.11), q is a real n-vector. The existence of such a vector satisfying (1.9)-(1.11) is guaranteed under certain conditions specified in the following lemma due to Kalman [14].

<u>Lemma</u>:

If there exists a real non-negative number γ, two real n-vectors b and m, where $m = \beta A^T c + c$, and an asymptotically stable matrix A such that (A, b) is completely controllable, then a real n-vector q satisfying (1.9)-(1.11) exists if and only if the inequality

$$\gamma - \text{Re}[m^T(j\omega I - A)^{-1}\ b] \geq 0 \tag{1.12}$$

holds for all real values of ω .

The proof of another lemma which is analogous to Kalman's lemma is given in Appendix I.

The existence of a real n-vector q means that the first term of $\dot{V}(x, \sigma)$ in (1.8) is negative semi-definite. Because of (1.5) the second term is non-positive. If the condition of complete observability is imposed, then $V(x, \sigma)$ in (1.6) is positive definite. Moreover, $\dot{V}(x, \sigma)$ is not a trajectory of the system (1.3)-(1.5). Therefore, the system is globally asymptotically stable if (1.12) holds for all real ω.

Using the Laplace transformation, one can eliminate the vector x from (1.3) and (1.4) to obtain

$$\left.\frac{L\,[\sigma(t)]}{L\,[f(\sigma,t)]}\right|_{x(0)\,=\,0} = c^T\,(sI - A)^{-1}\,b \tag{1.13}$$

Defining the open-loop transfer function of the linear part of the system as the negative of the left hand side of (1.13), one may write

$$G(s) = -\left.\frac{L\,[\sigma(t)]}{L\,[f(\sigma,\,t)]}\right|_{x(0)\,=\,0} = -\,c^T(sI - A)^{-1}\,b \tag{1.14}$$

Taking the derivative of (1.4) and then eliminating the vector x as before, one has

$$sG(s) = -\left.\frac{L\,[\dot{\sigma}(t)]}{L\,[f(\sigma,\,t)]}\right|_{x(0)\,=\,0} = -c^TA\,(sI - A)^{-1}\,b - c^Tb \tag{1.15}$$

Equations (1.14)-(1.15) with $s = j\omega$ may be used in the inequality (1.12) to yield the Popov Criterion

$$\frac{1}{K} + \mathrm{Re}[(1 + j\omega\beta)\,G(j\omega)] \geq 0 \tag{1.16}$$

which must hold for all real ω and some real scalar constant β .

If one defines

$$Y(j\omega) = \omega \operatorname{Im} G(j\omega) \tag{1.17}$$

$$X(j\omega) = \operatorname{Re} G(j\omega) \tag{1.18}$$

then (1.16) becomes

$$\frac{1}{K} + X(j\omega) - \beta \ Y(j\omega) \geq 0 \tag{1.19}$$

Using the foregoing substitutions, Popov gave a geometrical interpretation for his criterion. He defined a modified frequency plane, shown for a particular third order system in Fig. 1.1, with $Y(j\omega)$ or $\omega \operatorname{Im} G(j\omega)$ as ordinate and $X(j\omega)$ or $\operatorname{Re} G(j\omega)$ as the abscissa. Thus, a straight line with slope $1/\beta$ could be drawn tangent to and completely above the frequency plot. The intersection of this line with the negative $\operatorname{Re} G(j\omega)$ axis yielded a permissible value of $-1/K$. By moving this straight line to various positions, one could obtain the largest value of K. Aizerman and Gantmacher [18] described the Popov treatment and its implications in a recent monograph.

Kalman also developed an "effective" procedure for calculating the elements of the q vector and the P matrix once an acceptable value of K and its corresponding β have been determined from the frequency domain inequality. This procedure will be utilized in Chapter 3.

The significance of the Popov Criterion has been well expressed by Lefschetz [19], who noted that Popov reduced the problem of searching for the individual elements of an n by n matrix P to the much simpler problem of searching for a single scalar constant β.

1.7 Organization of the Thesis

Following this introductory material, a counter-example is presented in Chapter 2 to show that Popov's Criterion, which applies to a certain

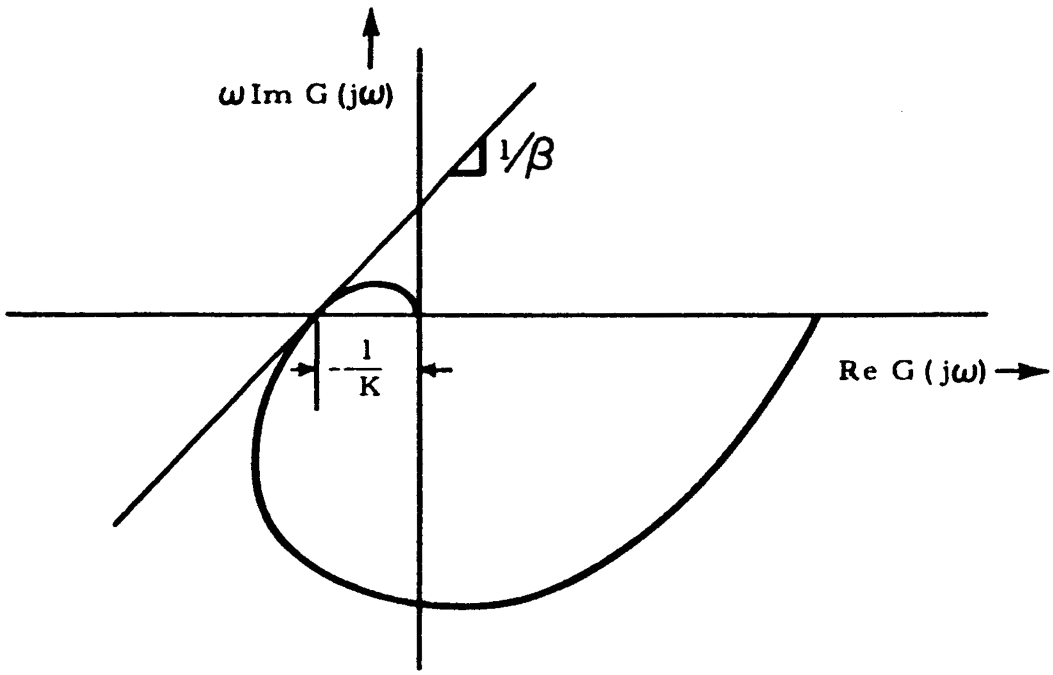

Fig. 1.1 Geometrical Interpretation of the Popov Criterion (1.16)

class of stationary systems, needs modification for systems in which the separate element varies with time.

Chapter 3 contains a frequency domain criterion which guarantees global asymptotic stability for feedback systems containing a single time-varying nonlinear element confined in a finite sector.

Another stability criterion is developed in Chapter 4 for feedback systems in which the single time-varying element is linear and varies sinusoidally with time.

Chapter 5 presents an evaluation of the results of the thesis with recommendations for further study.

CHAPTER 2

THE NONSTATIONARY PROBLEM AND POPOV'S CRITERION

2.1 Introduction

The aim of this chapter is to gain more insight into the stability of feedback systems having a time-varying element. It is shown that the Popov Criterion must be modified for the nonstationary case. One method of modification is then developed in Chapter 3.

Before the development of Popov's Criterion, Aizerman [20] conjectured that if the feedback system obtained by replacing the nonlinearity by a linear gain K_1, where $0 < K_1 < K$, were asymptotically stable for any K_1, then the corresponding nonlinear system should also be asymptotically stable. In other words, Aizerman contended that the Routh-Hurwitz Criterion for linear time-invariant systems should be applicable as well to nonlinear systems. That this conjecture was untrue in general was demonstrated through several counter-examples by Krasovskii [21], Pliss [22], Dewey and Jury [23], and others. The Popov Criterion later provided new insight into the single nonlinearity problem.

An interesting parallel exists between the foregoing and the potential application of the Popov Criterion to certain nonstationary systems. Suppose one considers the case in which the nonlinearity in the Popov problem is allowed to vary with time. Upon first observation, one might be tempted to use Popov's result to predict system stability. In fact, at least one such attempt has actually been made [24]. The statement of the nonstationary problem is given in the next section and then a counter-example is presented in the following section to disprove the above conjecture.

2.2 Statement of the Problem

Consider the following equations which describe a feedback system with a single time-varying nonlinear element in the loop (Fig. 2.1).

$$\dot{x} = Ax + b\, f(\sigma, t) \tag{2.1}$$

$$\sigma = c^T x \tag{2.2}$$

where x is an n-vector which represents the state of the system, A is an asymptotically stable n by n constant matrix, b and c are n-vectors, and σ and $f(\sigma, t)$ are the input and output, respectively, of the time-varying nonlinear element. The output $f(\sigma, t)$ is a real-valued continuous scalar function of σ and t.

Furthermore, let $f(\sigma, t)$ be confined to a sector (Fig. 2.2) in the following manner.

$$\begin{aligned} 0 < \sigma\, f(\sigma, t) < K\sigma^2 \quad &\text{for} \quad \sigma \neq 0 \\ f(0, t) = 0 \quad &\text{for all } t \geq 0. \end{aligned} \tag{2.3}$$

If the lower limit on $f(\sigma, t)$ in (2.3) had been some value other than zero, then a "pole-shifting" technique [15] could be used to rearrange (2.1)-(2.2) such that (2.3) holds.

The problem is to determine sufficient conditions which must be satisfied by the linear plant in order for the system (2.1)-(2.3) to be globally asymptotically stable.

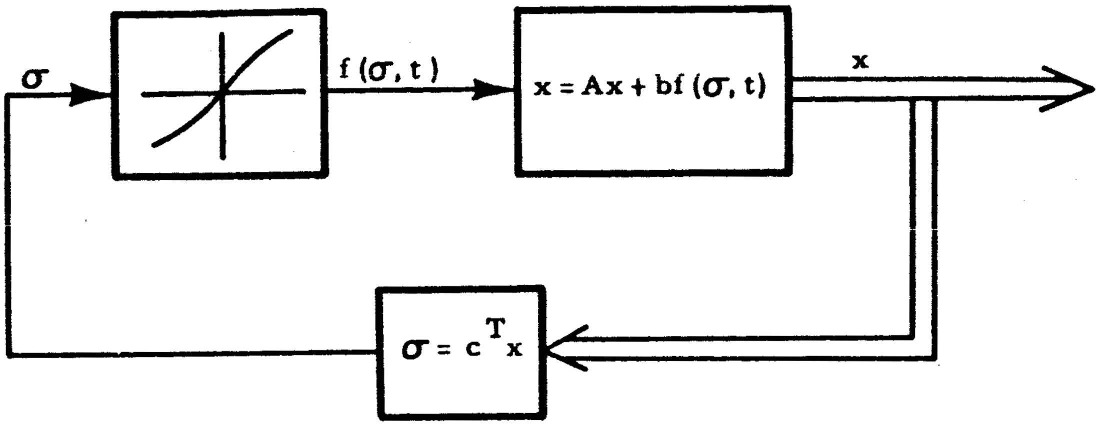

Fig. 2.1 Schematic Diagram of the System (2.1) Through (2.3)

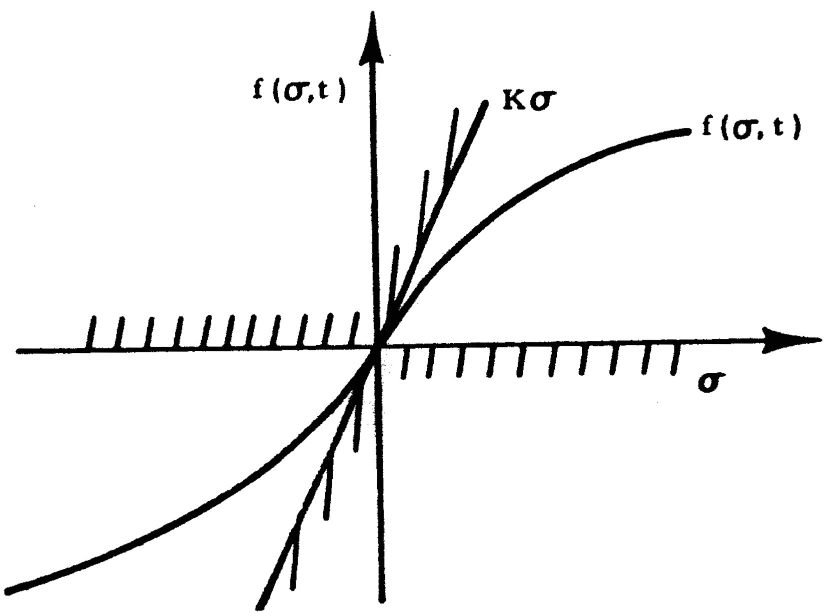

Fig. 2.2 Input-Output Characteristic of the Time-Varying Nonlinear Element (2.3)

2.3 A Counter-Example

The purpose of this section is to prove by means of a counter-example [25] that the Popov result cannot be applied without modifications to systems having a time-varying element in the loop. Specifically, a particular class of systems described by the Mathieu equation with a small damping term will be investigated. Referring to the literature, one can easily confirm that the members of a certain subclass of these systems are indeed unstable. An actual analog computer simulation of a particular member of this subclass will provide a specific example of instability.

(A) An arbitrary application of the Popov Criterion:

Consider the Mathieu equation with a small damping term.

$$\ddot{x} + 2\rho\dot{x} + (\mu - 2\xi \cos 2t)x = 0 \tag{2.4}$$

where $\xi > 0$, $\mu > 0$, and $\gamma = \mu - 2\xi - \epsilon > 0$. Both ϵ and ρ are small and positive.

One can make the following identifications in applying the Popov Criterion.

$$\sigma = -x \tag{2.5}$$

$$f(\sigma, t) = \frac{K}{2}\left(1 - \frac{2\xi}{\mu-\gamma} \cos 2t\right)\sigma \tag{2.6}$$

$$K = 2(\mu-\gamma) \tag{2.7}$$

$$G(s) = \frac{1}{s^2 + 2\rho s + \gamma} \tag{2.8}$$

Since $\mu - 2\xi - \gamma = \epsilon > 0$, the sector requirement (2.3) on $f(\sigma, t)$ is satisfied. Moreover, since ρ and γ in (2.8) are both positive, the condition that A be asymptotically stable is also satisfied.

Thus

$$\frac{1}{K} + \mathrm{Re}[(1 + j\omega\beta)G(j\omega)] = \frac{1}{K} + \frac{(\gamma - \omega^2) + 2\beta\rho\omega^2}{(\gamma - \omega^2)^2 + 4\rho^2\omega^2} \qquad (2.9)$$

If one selects $\beta = \frac{1}{2\rho}$, the right hand side of (2.9) becomes

$$\frac{1}{K} + \frac{\gamma}{(\gamma - \omega^2)^2 + 4\rho^2\omega^2} \geq \frac{1}{K} > 0 \qquad (2.10)$$

Therefore, an arbitrary application of the Popov Criterion to (2.4) apparently (although incorrectly as shown below) guarantees global asymptotic stability for any value of K such that $0 < K < \infty$.

(B) Proof of instability:

The damping term in (2.4) can be eliminated by the substitution

$$x = e^{-\rho t}\, y\,. \qquad (2.11)$$

The resulting equation is

$$\ddot{y} + (\zeta - 2\xi \cos 2t)\, y = 0 \qquad (2.12)$$

where

$$\zeta = \mu - \rho^2 \qquad (2.13)$$

The stability of equation (2.12) is discussed in detail by McLachlan [3, 26]. Cunningham [1] also has a pertinent discussion. In particular, one may refer to Figure 9.11, page 273, in [1] to see that (2.12) is unstable for certain values of ζ and ξ for which $(\zeta - 2\xi) > 0$.

The instability of (2.12) does not guarantee the instability of (2.4) for all positive values of ρ. However, for sufficiently small values of ρ, one would expect the boundaries between stable and unstable solutions of (2.4) to be very near those exhibited by (2.12). Hence, if a point (ξ, ζ) is chosen in the interior of an unstable region of (2.12) and far away from the boundary, then for small values of ρ the solution of (2.4) is also unstable.

McLachlan [3] and Hayashi [27, 28] present the stability boundaries which substantiate the foregoing conclusions (Fig. 2.3). In particular, consider the system described by (2.4)-(2.8) in which $\rho = 0.1$, $\gamma = 0.17$, and ϵ is arbitrarily small ($\epsilon > 0$). Combining (2.6) and (2.13), one has

$$\zeta = \frac{K}{2} - \rho^2 + \gamma = \frac{K}{2} + 0.16 \tag{2.14}$$

The fact that $\gamma = \mu - 2\xi - \epsilon$, together with (2.13), implies that

$$\zeta = 2\xi + \gamma - \rho^2 + \epsilon = 2\xi + 0.16 \tag{2.15}$$

if ϵ is arbitrarily small. This means that in Fig. 2.3 the straight line (2.15) may be drawn to determine system stability. Since ζ is a known function of K by (2.14), points on the straight line correspond to various values of K. Therefore, stability ranges may be determined as

$0 < K < 1.24$	stable
$1.24 < K < 2.70$	unstable
$2.70 < K < 7.30$	stable
$7.30 < K < 11.85$	unstable

As K increases to higher values, the system stability continues to alternate between stable and unstable behavior.

Consider the specific case in which $K = 1.68$. Therefore, $\zeta = 1.0$ and $\xi = 0.42$, i.e.

$$\ddot{x} + 0.2\dot{x} + (1.01 - 0.84 \cos 2\, t)x = 0, \tag{2.16}$$

and the system (2.16) is unstable according to Fig. 2.3. The system was actually simulated on an analog computer, and the unstable solution (Fig. 2.4) was verified. The choice of $\beta = \frac{1}{2\rho} = 5.0$ in (2.9) shows that the Popov Criterion is satisfied by the above system. Therefore, the Popov Criterion in its present form cannot be applied to systems containing a single time-varying element.

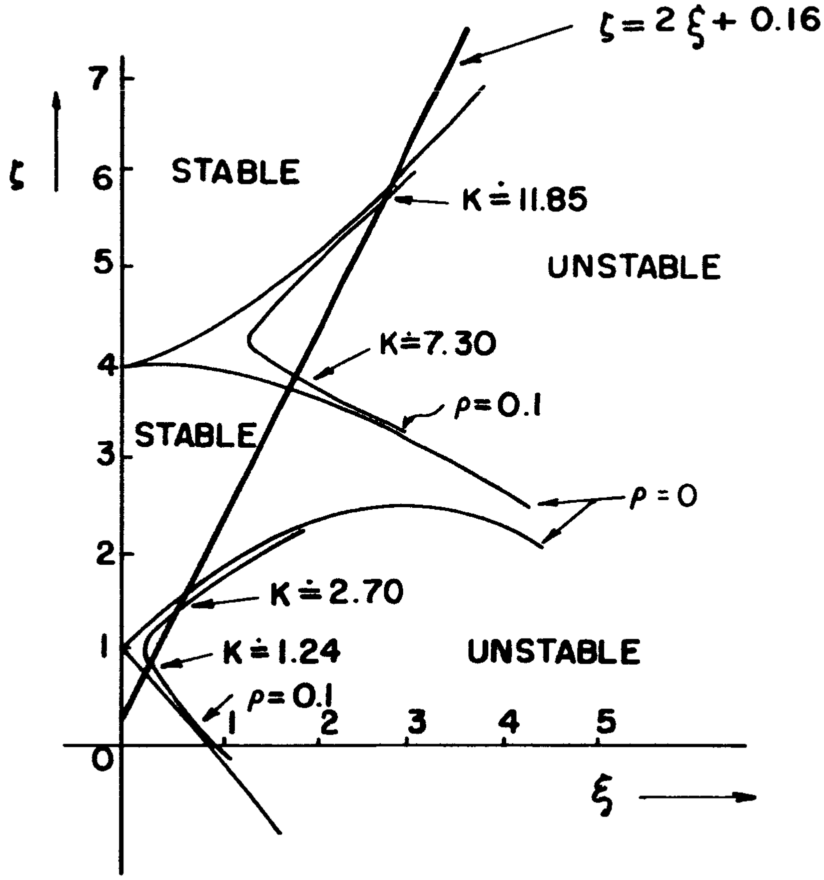

Fig. 2.3 Determination of Stable Ranges for K in the System (2.4)-(2.8)

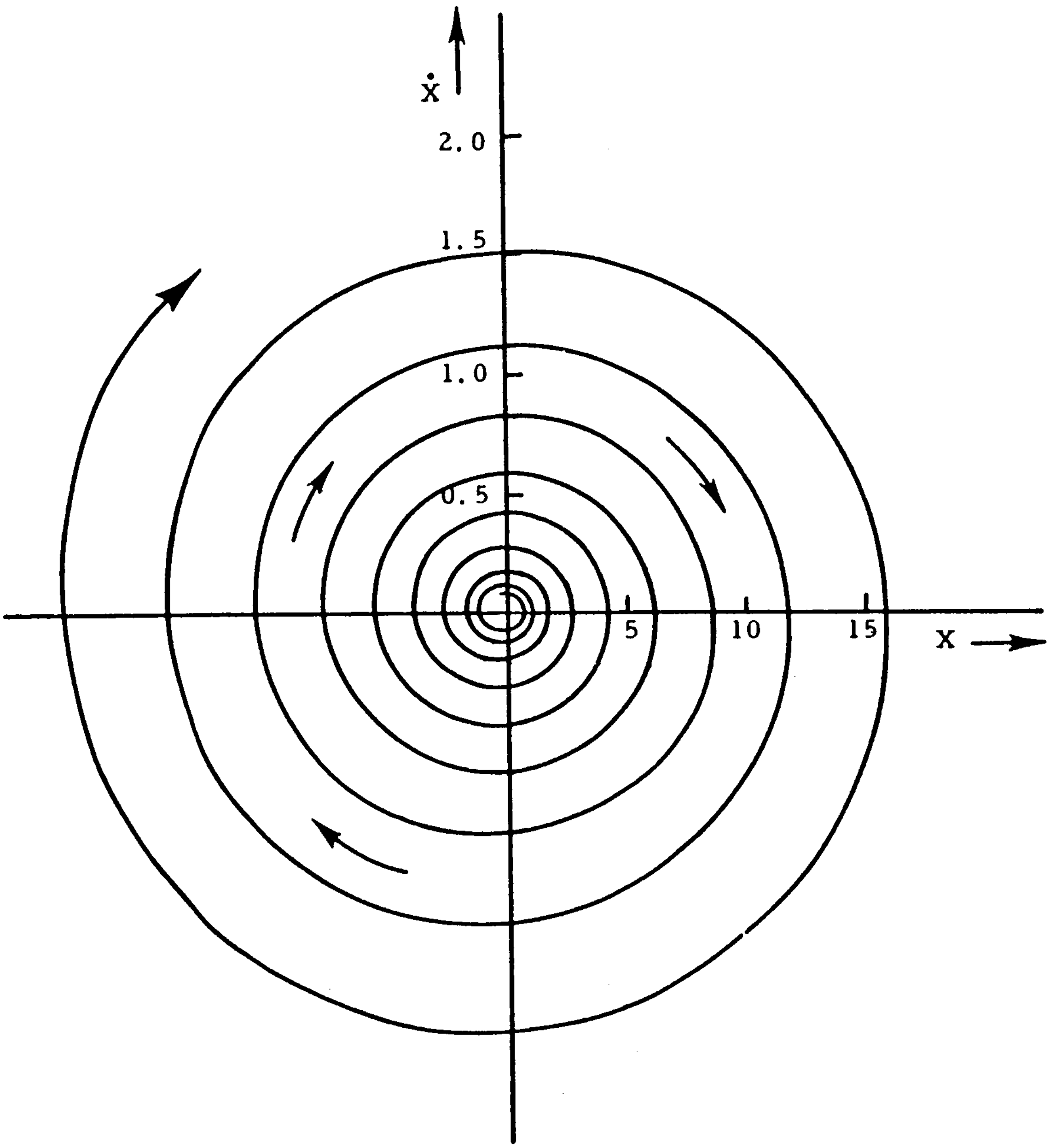

Fig. 2.4 A Specific Example of Instability Obtained Via Analog Simulation for the Damped Mathieu Equation in (2.16)

2.4 The Rozenvasser Criterion

Although Popov's result for stationary systems does not apply in general to the time-varying case, Rozenvasser [29] has observed that the inequality (1.16) is valid for the nonstationary system (2.1)-(2.3) if the scalar β is set equal to zero. This corollary to Popov's work will be referred to as the Rozenvasser Criterion

$$\frac{1}{K} + \mathrm{Re}\ G(j\omega) \geq 0 \tag{2.17}$$

which must hold for all real ω. The Rozenvasser Criterion is important because at the outset of this investigation it was the only generally applicable frequency domain criterion for feedback systems having a single time-varying element. Thus, all criteria developed in the following chapters will be compared with the Rozenvasser Criterion.

The inequality (2.17) has a simple geometrical interpretation in the $G(j\omega)$ plane (Fig. 2.5). A vertical straight line tangent to the curve at its left extremity intersects the negative real axis at $-1/K$. This yields the largest value of K.

Example:

Consider the third order system described by

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= -6x_1 - 11x_2 - 6x_3 + f(\sigma, t) \\ \sigma &= -x_1 \end{aligned} \tag{2.18}$$

where $f(\sigma, t)$ satisfies (2.3). In the notation of (2.1)-(2.2), the values of A, b, and c may be written as

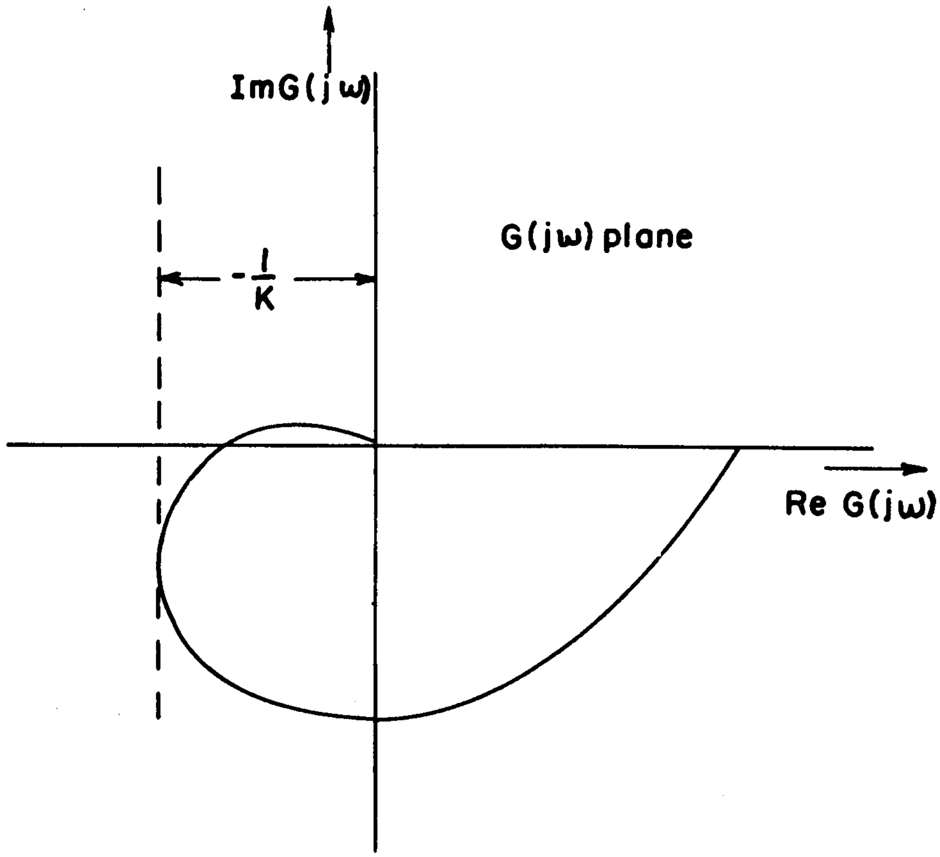

Fig. 2.5 Geometrical Interpretation of the Rozenvasser Criterion (2.17)

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \; ; \quad b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \; ; \quad c = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \tag{2.19}$$

Using (2.19) in (1.14), one can easily obtain the open-loop transfer function of the linear plant as

$$G(s) = \frac{1}{(s + 1)(s + 2)(s + 3)} \tag{2.20}$$

Using the geometrical interpretation of (2.17) in Fig. 2.5, one finds that the Rozenvasser Criterion yields a maximum K of 27.95. The result obtained by applying the Popov Criterion to a stationary nonlinear system having the same linear plant is 60.0, which is the same as the Routh-Hurwitz sector for the corresponding linear time-invariant system. The Popov inequality (1.16) required a corresponding β of 6/11 to obtain the widest sector.

2.5 Summary and State of the Art

Rozenvasser appears to have been the first to have considered the problem of a single nonstationary element in an otherwise linear time-invariant system. Although his result is quite restrictive, it does form the foundation upon which the results of the following chapters are built. Recently, Sandberg [30] used functional analysis to obtain results similar to (2.17).

Rozenvasser's work applies to feedback systems having a time-varying nonlinearity. However, a number of stability criteria have been developed recently for linear systems containing a time-varying gain. Among these are results by Bongiorno [31] obtained for systems with a periodically varying gain and results by Narendra and Goldwyn [32] and

Brockett and Forys [33] obtained by using Liapunov theory for systems having a general time-varying gain.

The next chapter deals with the problem of including the scalar constant β in the stability inequality. An improved criterion is developed by placing an additional restriction upon the rate at which the nonlinearity may vary with time.

CHAPTER 3

AN IMPROVED CRITERION

3.1 Introduction

Sufficient conditions for global asymptotic stability will be obtained in this chapter for feedback systems containing a single time-varying nonlinear element whose input-output characteristic lies within a finite sector (Fig. 2.2). An improved frequency domain criterion which retains the scalar constant β (as in the Popov Criterion) will be developed. This new criterion utilizes information related to the rate at which the nonlinear characteristic varies with time. Finally, it will be shown that this new result represents a considerable improvement over the Rozenvasser Criterion.

The problem has been stated fully in Section 2.2, but certain pertinent equations are rewritten here for convenience.

$$\dot{x} = Ax + b\ f(\sigma, t) \tag{2.1}$$

$$\sigma = c^T x \tag{2.2}$$

The sector requirement may be formally expressed as

$$\begin{gathered} 0 < \sigma\ f(\sigma, t) < K\sigma^2 \text{ for } \sigma \neq 0 \\ f(0, t) = 0 \end{gathered} \tag{2.3}$$

for all $t \geq 0$. Moreover, the linearized system obtained by replacing $f(\sigma, t)$ by $K_1\sigma$ in (2.1) is assumed to be asymptotically stable, where $0 < K_1 < K$.

The problem is to determine sufficient conditions which must be satisfied by the linear plant in order for the system (2.1)-(2.3) to be globally asymptotically stable.

3.2 Development of an Improved Criterion for the Nonstationary Problem

Consider as a tentative Liapunov function

$$V(x, \sigma, t) = x^T P x + \beta \int_0^\sigma f(z, t)\, dz \qquad (3.1)$$

where $P = P^T > 0$, i.e., $x^T P x > 0$ for all $x \neq 0$ and β is a scalar constant. Evaluating its time derivative along the trajectories of the system (2.1)-(2.3), one obtains

$$\dot{V}(x, \sigma, t) = x^T[A^T P + PA]x + (2b^T P + \beta c^T A)x\, f(\sigma, t)$$
$$+ \beta c^T b\, f^2(\sigma, t) + \beta \int_0^\sigma \frac{\partial f(z, t)}{\partial t}\, dz \qquad (3.2)$$

One may constrain $\dot{V}(x, \sigma, t)$ to be of the form

$$\dot{V}(x, \sigma, t) = - [q^T x - \sqrt{\gamma}\, f(\sigma, t)]^2 - f(\sigma, t)\, [\sigma - \frac{1}{K} f(\sigma, t)]$$
$$- \beta\, [\alpha_1 \sigma^2 + \alpha_2 \sigma\, f(\sigma, t) + \alpha_3\, f^2(\sigma, t) -$$
$$\int_0^\sigma \frac{\partial f(z, t)}{\partial t}\, dz] \qquad (3.3)$$

by equating the coefficients of corresponding terms in (3.2) and (3.3).

$$A^T P + PA = - qq^T - \beta\alpha_1 cc^T \qquad (3.4)$$
$$2b^T P + \beta c^T A = 2\sqrt{\gamma}\, q^T - (1 + \beta\alpha_2)\, c^T \qquad (3.5)$$
$$\gamma = \frac{1}{K} - \beta\alpha_3 - \beta c^T b \qquad (3.6)$$

In (3.4)-(3.5) q is a real n-dimensional column vector. The necessary and sufficient condition for the existence of this real n-vector is given by the following lemma which is analogous to Kalman's Lemma [14] discussed in Section 1.6.

Lemma:

If there exist a real non-negative number γ, a real number ℓ, three real n-vectors b, m, and r, and a stable matrix A such that (A, b) is completely controllable, then a real n-vector q satisfying the equations

$$A^T P + PA = -qq^T - \ell rr^T \tag{3.7}$$

$$2Pb + m = 2\sqrt{\gamma}\, q \tag{3.8}$$

exists if and only if the inequality

$$\gamma - \mathrm{Re}[m^T(j\omega I - A)^{-1} b] - \ell \,|\, r^T(j\omega I - A)^{-1} b \,|^2 \geq 0 \tag{3.9}$$

holds for all real ω.

The proof of the above lemma is given in Appendix I.

If one sets $m^T = \beta c^T A + (1 + \beta\alpha_2)c^T$, $\ell = \beta\alpha_1$, and $r = c$, then the lemma implies the existence of a real n-vector q such that (3.4)-(3.6) are satisfied, provided (3.9) holds for all real ω.

Let the inequality

$$\beta \int_0^\sigma \frac{\partial f(z, t)}{\partial t}\, dz \leq \beta\, [\alpha_1\sigma^2 + \alpha_2\sigma\, f(\sigma, t) + \alpha_3\, f^2(\sigma, t)] \tag{3.10}$$

hold for all σ and all $t \geq 0$, where α_1, α_2, and α_3 are scalar constants.

Utilizing the lemma, (2.3), and (3.10), one may therefore prove that $\dot{V}(x, \sigma, t)$ is negative semidefinite. Furthermore, one may observe that

$$W_1(x) \leq V(x, \sigma, t) \leq W_2(x) \tag{3.11}$$

where

$$W_1(x) = x^T P x \tag{3.12}$$

$$W_2(x) = x^T [P + \beta K\ cc^T]\ x \tag{3.13}$$

Therefore, the asymptotic stability of A, the complete observability of (A, c^T), equations (3.11)-(3.13), and the assumption* that every linearized system

$$\dot{x} = Ax + bK_1\ c^T x \tag{3.14}$$

where

$$0 < K_1 < K \tag{3.15}$$

is asymptotically stable implies that $V(x, \sigma, t)$ is positive definite. The conditions for Liapunov's theorem for stability (Section 1.4) have been satisfied.

Letting $s = j\omega$, one may use (1.14)-(1.15) in the stability inequality (3.9) to yield the following frequency domain relationship for stability.

$$\frac{1}{K} - \beta\alpha_3 + \mathrm{Re}\ [(1 + \beta\alpha_2 + j\omega\beta)G(j\omega)] - \beta\alpha_1\ |G(j\omega)|^2 \geq 0 \tag{3.16}$$

which must hold for all real ω. The schematic diagram given in Fig. 2.1 for the system (2.1)-(2.3) utilized vector-matrix notation. An equivalent representation in terms of the plant transfer function is shown in Figure 3.1.

Although the result in (3.16) is a sufficient condition for the stability of (2.1)-(2.3), one would like to be able to guarantee global asymptotic stability. If time did not appear explicitly in $V(x, \sigma, t)$ and $\dot{V}(x, \sigma, t)$, then one could use the following argument to prove global asymptotic stability, where $f(\sigma)$ replaces $f(\sigma, t)$. The asymptotic stability of A guarantees that $\dot{V} \not\equiv 0$ on any trajectory of the

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*This assumption is needed to allow the possibility of negative values of β in (3.1) and in the resulting theorem.

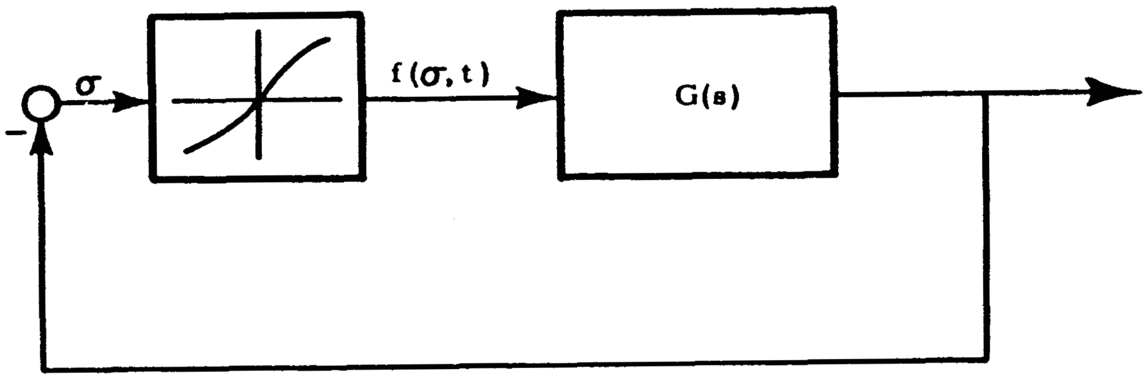

Fig. 3.1 An Equivalent Transfer Function Representation of the System Shown in Vector-Matrix Form in Fig. 2.1

system since $\dot{V} = 0$ implies that $\sigma = \frac{1}{K} f(\sigma)$, which violates (2.3) unless $\sigma = 0$. If one sets $\sigma = f(\sigma) = 0$ in (2.1), then the fact that A is asymptotically stable implies that the limit, as $t \to \infty$, of $x(t)$ is zero. Thus, for the stationary case, the fact that V is positive definite and $\dot{V}$ is negative semi-definite and not a trajectory of the system is sufficient to prove global asymptotic stability by Liapunov's theorems.

However, this is not a valid proof for time-varying systems. LaSalle [30] has proposed the following system to illustrate that a negative semi-definite $\dot{V}$ is not sufficient to guarantee asymptotic stability for time-varying systems.

$$\dot{x}_1 = x_2 \tag{3.17}$$

$$\dot{x}_2 = -x_1 - (2 + e^t)x_2 \tag{3.18}$$

Choosing $V = \frac{1}{2}(x_1^2 + x_2^2)$, one has $\dot{V} = -(2 + e^t)x_2 \leq 0$, which equals zero only when x_2 is zero. The only solution on $x_2 = 0$ is $x_1 = x_2 = 0$. However, $x_1(t) = 1 + e^{-t}$ is a solution and the system is not asymptotically stable even though $\dot{V}$ is negative semi-definite. Therefore, $\dot{V}$ must be negative definite to prove asymptotic stability.

A slight modification must be made to guarantee asymptotic stability in the present problem. A negative definite $\dot{V}$ may be obtained by constraining $\dot{V}$ in a different way. One may achieve the desired result by constraining $\dot{V}$ as

$$\begin{aligned}\dot{V}(x, \sigma, t) = &- [q^T x - \sqrt{\gamma}\, f(\sigma, t)]^2 - f(\sigma, t)\,[\sigma - \tfrac{1}{K} f(\sigma, t)] \\ &- \beta\,[\alpha_1\sigma^2 + \alpha_2\sigma\, f(\sigma, t) + \alpha_3\, f^2(\sigma, t) - \\ &\int_0^\sigma \frac{\partial f(z, t)}{\partial t}\, dz\,] - \sum_{i=1}^{n} \rho_i\, (d_i^T x)^2\end{aligned} \tag{3.19}$$

where $\rho_1, \rho_2, \ldots, \rho_n$ are arbitrarily small positive constants and $d_1, d_2, \ldots, d_n$ are arbitrary linearly independent n-vectors. The resulting frequency domain inequality is

$$\frac{1}{K} - \beta\alpha_3 + \text{Re}\,[(1 + \beta\alpha_2 + j\omega\beta)G(j\omega)] - \beta\alpha_1 \,|\, G(j\omega) \,|^2$$
$$- \sum_{i=1}^{n} \rho_i \,|\, d_i^T\,(j\omega I - A)^{-1}\, b \,|^2 \geq 0 \qquad (3.20)$$

which must hold for all real ω. The new term becomes negligible in a practical application because $\rho_1, \rho_2, \ldots, \rho_n$ may each be chosen arbitrarily small. It is shown in Appendix II that an equivalent statement is to replace K in (3.16) by K_{max}, where $K = K_{max}$ appears in the criterion but global asymptotic stability is guaranteed in the sector (2.3) dependent upon K. The above results may be expressed in the form of a theorem [34].

<u>Theorem</u>:

If there exist real numbers β, K_{max}, α_1, α_2, and α_3 such that

a) the inequality (3.16)

$$\frac{1}{K_{max}} - \beta\alpha_3 + \text{Re}[(1 + \beta\alpha_2 + j\omega\beta)G(j\omega)] - \beta\alpha_1 \,|\, G(j\omega) \,|^2 \geq 0$$

holds for all real ω,

b) the inequality (3.10)

$$\beta \int_0^\sigma \frac{\partial f(z,\, t)}{\partial t}\, dz \leq \beta\, [\alpha_1\sigma^2 + \alpha_2\sigma\, f(\sigma,\, t) + \alpha_3\, f^2(\sigma,\, t)]$$

holds for all σ and all $t \geq 0$,

c) the linearized system (3.14) is asymptotically stable for all K_1 in the interval $(0, K)$,

then the system (2.1)-(2.3) is globally asymptotically stable for all $K = K_{max} - \epsilon$, where ϵ is an arbitrarily small positive constant.

Let condition (a) of the theorem be designated as the Improved Criterion and condition (b) as the Integral Constraint. Consider the following special subcases of the Improved Criterion.

$$\text{Subcase I:} \quad \alpha_1 \neq 0 \;, \; \alpha_2 = \alpha_3 = 0. \tag{3.21}$$

$$\text{Subcase II:} \quad \alpha_2 \neq 0 \;, \; \alpha_1 = \alpha_3 = 0. \tag{3.22}$$

$$\text{Subcase III:} \quad \alpha_3 \neq 0 \;, \; \alpha_1 = \alpha_2 = 0. \tag{3.23}$$

In applying the theorem, the Integral Constraint is first used to determine the smallest values of α_1, α_2, or α_3 for the particular subcase. Then the Improved Criterion is used with β being varied to find the largest K.

One will observe that when $\beta = 0$, the Improved Criterion becomes identical to the Rozenvasser Criterion (2.17). When $\beta \neq 0$ gives the best sector for any one of the three subcases above, then there is an improvement over Rozenvasser's result, i.e., the largest value of K which can be obtained by using the Improved Criterion is greater than the largest value which can be obtained by using (2.17) alone.

3.3 Comparison with Previous Criteria

Let us consider the system discussed in Section 2.4 whose plant equations are given by

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= -6x_1 - 11x_2 - 6x_3 + f(\sigma, t) \\ \sigma &= -x_1 \end{aligned} \tag{2.18}$$

where the function $f(\sigma, t)$ satisfies (2.3). The open-loop transfer function of the plant may be obtained by using (1.14).

$$G(s) = \frac{1}{(s + 1)(s + 2)(s + 3)} \tag{2.20}$$

Using (2.20) in (3.16), one obtains the results shown in Figures 3.2, 3.3, and 3.4 for Subcases I, II, and III, respectively. In each case, $\beta = 0$ occurs for sufficiently large values of α_1, α_2, and α_3, and the Rozenvasser sector is obtained. When either α_1, α_2, or α_3 is equal to zero, then $K_{max} = 60.0$, which is the best result that can be obtained for this system with a stationary nonlinearity by using the Popov Criterion. The latter result also happens to be the Routh-Hurwitz sector for this particular system (2.20).

Now consider a particular form of the function $f(\sigma, t)$:

$$f(\sigma, t) = g(t)\, h(\sigma)\sigma \tag{3.24}$$

where $g(t)$ and $h(\sigma)$ satisfy the inequalities:

$$0 \leq g(t) \leq 1 \quad \text{for all } t \geq 0 \tag{3.25}$$

$$|h(\sigma)| < K \quad \text{for all } \sigma \tag{3.26}$$

Furthermore, for the purpose of this example, let:

$$g(t) = (at + b)e^{-\rho t} \quad (t \geq 0) \tag{3.27}$$

where a, b, and ρ are non-negative constants.

A curve of $g(t)$ versus time is shown in Figure 3.5. The time function $g(t)$ corresponds to the instantaneous voltage across the capacitor in the critical case of the natural response of a three-element RLC parallel circuit. ρ is determined by the element values, while a and b are determined by initial conditions in the circuit.

The conditions which must be imposed upon a, b, and ρ, according to (3.25) and (3.27) are

$$0 \leq b \leq 1 \tag{3.28}$$

$$t^* = \frac{a - b\rho}{a\rho} \geq 0 \tag{3.29}$$

$$g(t^*) = 1 \tag{3.30}$$

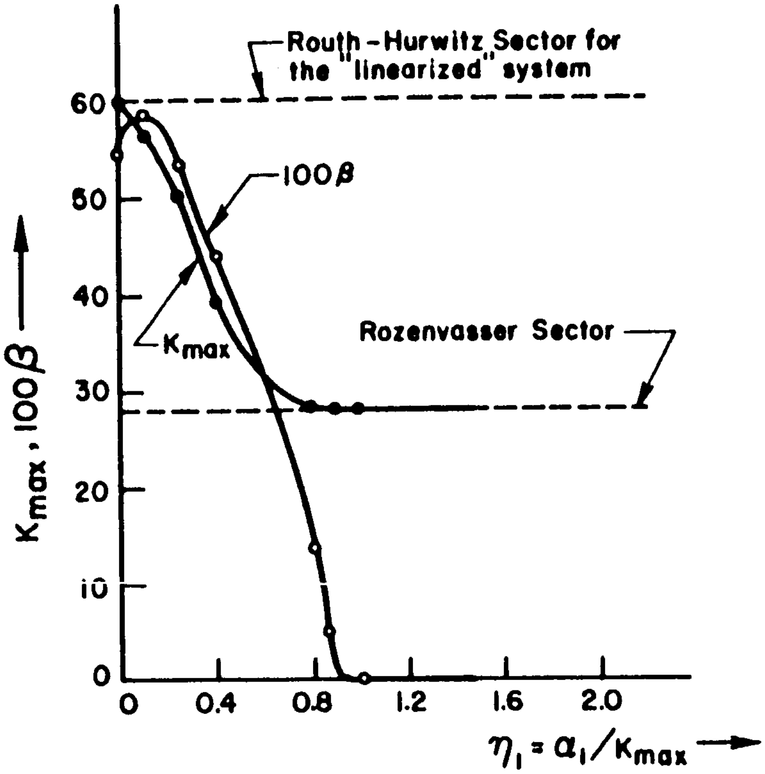

Fig. 3.2 Use of Subcase I on the System Described by (2.18)

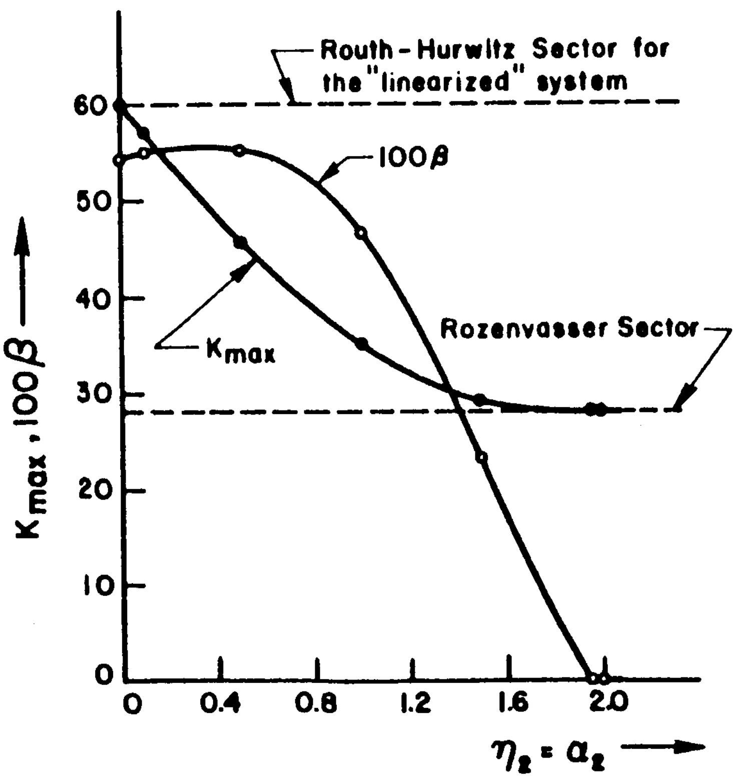

Fig. 3.3 Use of Subcase II on the System Described by (2.18)

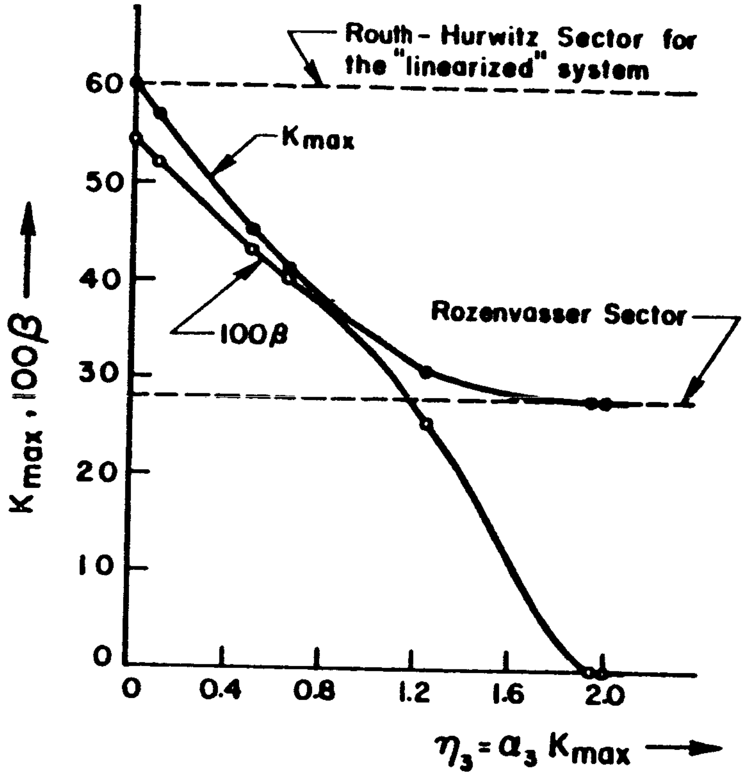

Fig. 3.4 Use of Subcase III on the System Described by (2.18)

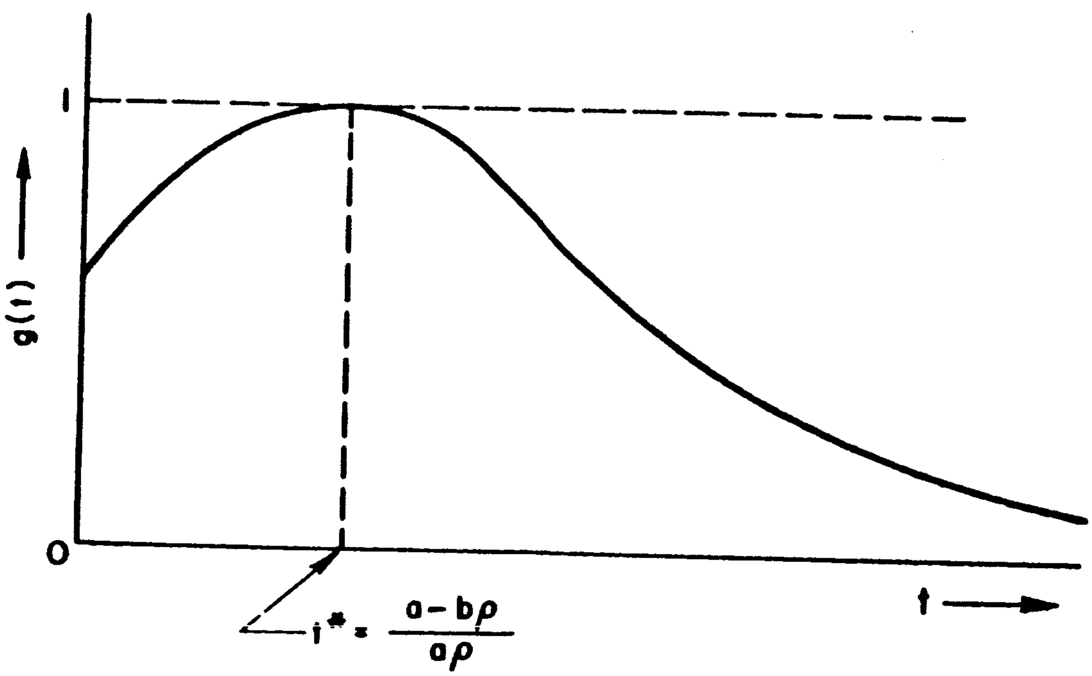

Fig. 3.5 A Typical Curve of the Function Given by (3.27)

For this particular time-varying nonlinearity, one may determine the integral which appears on the left side of the Integral Constraint for each of the three subcases.

$$\beta \int_0^\sigma \frac{\partial f(z, t)}{\partial t}\, dz = \beta \int_0^\sigma [(a - b\rho) - a\rho t]e^{-\rho t}\, h(\sigma)\, \sigma d\sigma$$

$$< \beta(a - b\rho - a\rho t)e^{-\rho t}\, \frac{K\sigma^2}{2} \tag{3.31}$$

Referring to Figure 3.5, one can see that the maximum positive slope of $g(t)$ occurs at $t = 0$. In the closed interval $[0, t^*]$, the slope of $g(t)$ is still positive, but $g(t)$ is greater than $g(0)$. In the interval $(t^*, +\infty)$, the slope of $g(t)$ is always negative and $g(t)$ is positive. The significance of these observations is that if the following inequalities, which correspond to the Integral Constraint for each subcase and utilize (3.31) hold for $t = 0$, then they are true for all $t > 0$.

$$\beta \int_0^\sigma \frac{\partial f(z, t)}{\partial t}\, dz < \beta(a - b\rho - a\rho t)e^{-\rho t}\, \frac{K\sigma^2}{2} \leqq \beta\alpha_1 \sigma^2 \tag{3.32}$$

$$\beta(a - b\rho - a\rho t)e^{-\rho t}\, \frac{K\sigma^2}{2} \leqq \beta\alpha_2\, \sigma f(\sigma, t) \tag{3.33}$$

$$\beta(a - b\rho - a\rho t)e^{-\rho t}\, \frac{K\sigma^2}{2} \leqq \beta\alpha_3\, f^2(\sigma, t) \tag{3.34}$$

Equations (3.32)-(3.34) may be simplified to give (at $t = 0$):

$$\alpha_1 = \frac{K(a - b\rho)}{2} \tag{3.35}$$

$$\alpha_2 = \frac{a - b\rho}{2b} \tag{3.36}$$

$$\alpha_3 = \frac{a - b\rho}{2Kb^2} \tag{3.37}$$

where α_1, α_2, and α_3 have been chosen as the smallest values which satisfy (3.32)-(3.34) when β is positive.

Define:

$$\eta_1 = \frac{\alpha_1}{K_{max}} \tag{3.38}$$

$$\eta_2 = \alpha_2 \tag{3.39}$$

$$\eta_3 = K_{max}\alpha_3 \tag{3.40}$$

There are definite choices of a, b, and ρ in the present example for which Subcase I is best, and other choices of a, b, and ρ for which II is best. Two of these cases are given below.

Case 1.

$a = 0.9701$
$b = 0.4000$
$\rho_* = 0.4253$
$t = 1.9390$

$\eta_1 = 0.400$
$\eta_2 = 1.000$
$\eta_3 = 2.500$

Subcase	K_{max}	Corresponding β
I	39.41	0.4382
II	35.00	0.4686
III	27.95	0.0000

Therefore, Subcase I gives the best results for Case 1.

Case 2.

$a = 1.5142$
$b = 0.8000$
$\rho_* = 0.8927$
$t = 0.5918$

$\eta_1 = 0.400$
$\eta_2 = 0.500$
$\eta_3 = 0.625$

Subcase	K_{max}	Corresponding β
I	39.41	0.4382
II	45.47	0.5549
III	42.45	0.4068

Therefore, Subcase II gives the best results for Case 2.

By taking $K = K_{max}$, one may show from (3.35)-(3.40) that:

$$b = \eta_1/\eta_2 \tag{3.41}$$

$$\eta_2^{\ 2} = \eta_1 \ \eta_3 \tag{3.42}$$

Equations (3.28) and (3.41) require that $\eta_1 \leq \eta_2$ for this example. This information, together with (3.42) and the fact that the curves of K_{max} versus η_2 and K_{max} versus η_3 in Figures 3.3 and 3.4, respectively, are identical for all of the points calculated numerically, means that for the particular plant (2.20) and the specific form of $f(\sigma, t)$ given by (3.24)-(3.27), Subcase III can never give a larger K_{max} than Subcase II. However, this does not rule out the possibility that III may indeed be better than II for other cases.

One may use the procedure described in Appendix I to calculate the n-vector q in (3.3) and the n by n matrix P in (3.1). As an example, the values of q and P for Subcase I in both cases given above are

$$q = \begin{pmatrix} 0.8803 \\ 1.4840 \\ 0.9036 \end{pmatrix} ; \qquad P = \begin{pmatrix} 8.4691 & 3.9093 & 0.6402 \\ 3.9093 & 2.3351 & 0.4555 \\ 0.6402 & 0.4555 & 0.1439 \end{pmatrix}$$

which means both $V(x, \sigma, t)$ and $\mathring{V}(x, \sigma, t)$ are fully determined for the particular values of K_{max} and β which were obtained.

3.4 Conclusions

Sufficient conditions for the stability of feedback systems with a single time-varying nonlinear element contained in a finite sector have been given by Rozenvasser. The results reported in this chapter take advantage of more information which might be available about the nonlinearity by placing an upper bound upon a certain integral, $\beta \int_0^\sigma \frac{\partial f(z, t)}{\partial t} \, dz$, which occurs in the expression for $\dot{V}(x, t)$. If this additional information is known, then a frequency domain criterion which often gives better results than the Rozenvasser Criterion may be developed.

If the only information available about the nonlinearity is that it lies in a finite sector, then the Improved Criterion developed in this chapter cannot give a better result than the Rozenvasser inequality. However, if in addition one knows that $f(\sigma, t)$ is governed by (3.24)-(3.27) and that $\frac{dg(t)}{dt} \leq \psi$ (where ψ is a constant) for all $t \geq 0$, then Subcase I may often be used to obtain a better sector than that obtainable by Rozenvasser. Moreover, if still more information about $g(t)$ and its slope in certain intervals are known, then the two remaining subcases may also be useful. In general, any of the three subcases may be used only if the particular α corresponding to that subcase can be found. This does not restrict $f(\sigma, t)$ in every case to the form given by (3.24)-(3.27).

An example was given in which Subcase I yielded a better sector than either Subcase II or III. A second example showed that Subcase II can sometimes give a better result for certain nonlinearities than either of the other two subcases. No example has yet been found in which Subcase III gives the best results of the three subcases, but it is conjectured that such an example may indeed exist.

Although the particular example in Section 3.3 illustrated only three special subcases of the Improved Criterion, this restriction is certainly not necessary. Allowing α_1, α_2, and α_3 to be nonzero simultaneously may yield for some systems an even wider sector than can be obtained by using any one of the three subcases.

CHAPTER 4

A STABILITY CRITERION FOR FEEDBACK SYSTEMS CONTAINING A SINGLE SINUSOIDAL GAIN

4.1 Introduction

As described in the preceding chapters, the recent trend in stability investigations has been toward developing frequency domain criteria which guarantee asymptotic stability for feedback systems having a single nonlinear and/or time-varying element in an otherwise linear system. The Popov Criterion, which applies to systems having a single stationary nonlinearity, and the Improved Criterion of Chapter 3 for time-varying systems represent important stability results of this type. These criteria are, in general, independent of the order of the system and require that the nonlinear characteristic remains in a finite sector (0, K) for all time. For stationary systems the value of K from the Popov Criterion depends only upon the open-loop transfer function of the linear plant, but for systems with a time-varying element the Improved Criterion shows that K also depends upon the rate at which the characteristic varies. In general, when more information is available about the nonlinear time-varying element, a wider stability sector should be obtained. The purpose of this chapter is to develop a stability criterion for systems which contain an element about which a substantial amount of information is known. Specifically, the element is linear and varies sinusoidally with time at a single constant frequency.

4.2 Statement of the Problem

Consider an unforced linear feedback system containing a single sinusoidal gain in the loop (Fig. 4.1).

$$\dot{x} = Ax + b\, g(t)\sigma \tag{4.1}$$

$$\sigma = c^T x \tag{4.2}$$

$$g(t) = \frac{K}{2}\,[1 + N \sin(\omega_o t + \phi)] \tag{4.3}$$

where x is an n-vector which represents the state of the system, A is an asymptotically stable n by n constant matrix, b and c are n-vectors, and σ is the input to the time-varying element whose variation is described by (4.3).

The scalar N in (4.3) is a non-negative constant which does not exceed unity. Bounds on g(t) may be obtained from (4.3) as

$$\frac{K}{2}\,(1-N) \leq g(t) \leq \frac{K}{2}\,(1+N) \tag{4.4}$$

for all $t \geq 0$, which indicates that g(t) lies within a finite range which has a lower limit of zero only for N = 1.

The problem is to determine sufficient conditions which must be satisfied by the linear plant in order for the system (4.1)-(4.3) to be asymptotically stable.

4.3 Derivation of the Sinusoidal Criterion

A Liapunov function consisting of a quadratic term plus an integral term has dominated the work in previous chapters. Attempting to be more general, one may consider the following as a tentative Liapunov function.

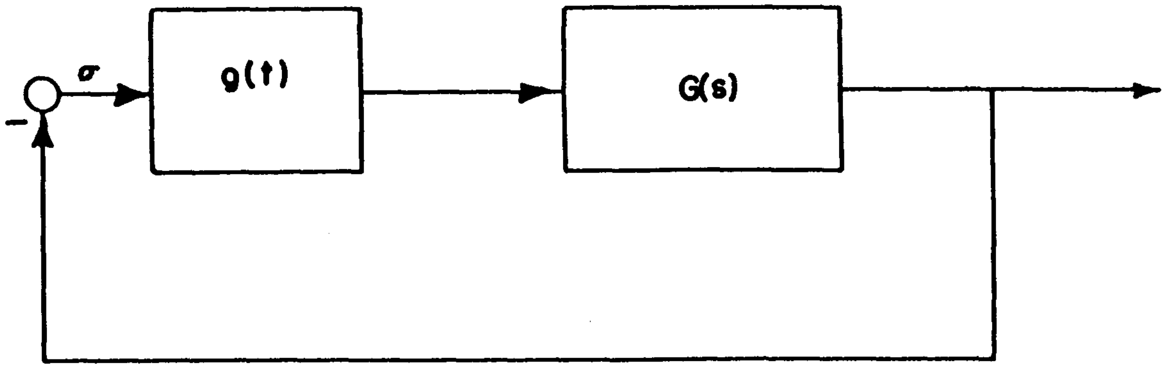

Fig. 4.1 Block Diagram of the Sinusoidally Varying Linear System (4.1)-(4.3)

$$V(x, t) = x^T P x + x^T P_s x \sin(\omega_o t + \phi) + x^T P_c x \cos(\omega_o t + \phi) \tag{4.5}$$

Evaluating its time derivative along the trajectories of the system (4.1)-(4.3), one obtains

$$\begin{aligned} \dot{V}(x, t) = {} & x^T[A^T P + PA + \frac{K}{2}(Pbc^T + cb^T P)]x \\ & + x^T[A^T P_s + P_s A - \omega_o P_c + \frac{KN}{2}(Pbc^T + cb^T P) \\ & \qquad + \frac{K}{2}(P_s bc^T + cb^T P_s)]x \sin(\omega_o t + \phi) \\ & + x^T[A^T P_c + P_c A + \omega_o P_s + \frac{K}{2}(P_c bc^T + cb^T P_c)]x \cos(\omega_o t + \phi) \\ & + x^T[\frac{KN}{2}(P_c bc^T + cb^T P_c)]x \sin(\omega_o t + \phi)\cos(\omega_o t + \phi) \\ & + x^T[\frac{KN}{2}(P_s bc^T + cb^T P_s)]x \sin^2(\omega_o t + \phi) \end{aligned} \tag{4.6}$$

Constrain $\dot{V}(x, t)$ to be of the form

$$\begin{aligned} \dot{V}(x, t) = {} & -[q^T x + q_s{}^T x \sin(\omega_o t + \phi) + q_c{}^T x \cos(\omega_o t + \phi)]^2 \\ & -(d^T x)^2 \, [1 - \delta \sin(\omega_o t + \phi) - \alpha^2 \sin^2(\omega_o t + \phi) \\ & \qquad - \beta\alpha\omega_o \cos(\omega_o t + \phi) \\ & \qquad - a_1 \sin(\omega_o t + \phi)\cos(\omega_o t + \phi) \\ & \qquad - a_2 \cos^2(\omega_o t + \phi)] \end{aligned} \tag{4.7}$$

where q, q_s, q_c, and d are real n-vectors and δ, α, β, a_1, and a_2 are scalar constants which add to the generality of (4.7). By equating the coefficients of corresponding terms in (4.6) and (4.7), one obtains

$$A^T P + PA + \frac{K}{2}(Pbc^T + cb^T P) = -qq^T - dd^T \tag{4.8}$$

$$\begin{aligned} & A^T P_s + P_s A - \omega_o P_c + \frac{KN}{2}(Pbc^T + cb^T P) + \frac{K}{2}(P_s bc^T + cb^T P_s) \\ & \qquad = -qq_s{}^T - q_s q^T + \delta\, dd^T \end{aligned} \tag{4.9}$$

$$A^T P_c + P_c A + \omega_o P_s + \frac{K}{2}(P_c bc^T + cb^T P_c) = -qq_c{}^T - q_c q^T + \beta\alpha\omega_o dd^T \tag{4.10}$$

$$\frac{KN}{2}(P_c bc^T + cb^T P_c) = -q_s q_c^T - q_c q_s^T + a_1 dd^T \tag{4.11}$$

$$\frac{KN}{2}(P_s bc^T + cb^T P_s) = -q_s q_s^T + \alpha^2 dd^T \tag{4.12}$$

$$0 = -q_c q_c^T + a_2 dd^T \tag{4.13}$$

Equations (4.8)-(4.13) will be utilized to obtain a single matrix equation and a single vector equation from which the frequency domain inequality follows directly. From (4.13), one finds

$$q_c = \sqrt{a_2}\; d \tag{4.14}$$

Post-multiplying and pre-multiplying (4.12) by an arbitrary n-vector y, one may write

$$y^T \left[\frac{KN}{2}(P_s bc^T + cb^T P_s)\right] y = -(q_s^T y)^2 + \alpha^2 (d^T y)^2 \tag{4.15}$$

or

$$KN\,(b^T P_s y)(c^T y) = -(q_s^T y)^2 + \alpha^2 (d^T y)^2 \tag{4.16}$$

If one sets $c^T y = 0$, then

$$\alpha^2 (d^T y)^2 - (q_s^T y)^2 = 0 \tag{4.17}$$

or

$$(\alpha d^T y - q_s^T y)(\alpha d^T y + q_s^T y) = 0 \tag{4.18}$$

Therefore, either c is proportional to $(\alpha d - q_s)$ or to $(\alpha d + q_s)$, i.e., either

$$\alpha d - q_s = k_a c \tag{4.19}$$

or

$$\alpha d + q_s = k_b c \tag{4.20}$$

where k_a and k_b are proportionality constants.

Performing the post-multiplication and pre-multiplication of (4.11) by y and regrouping terms, one may write

$$KN(b^T P_c y)(c^T y) = -2(q_s{}^T y)(q_c{}^T y) + a_1 (d^T y)^2 \tag{4.21}$$

Substituting (4.14) into (4.21) and setting $c^T y = 0$, one obtains

$$(d^T y)\ [a_1 d - 2\sqrt{a_2}\ q_s]^T y = 0 \tag{4.22}$$

Therefore, either the n-vectors c and d must be linearly dependent or c is proportional to $(a_1 d - 2\sqrt{a_2}\ q_s)$, i.e., either

$$d = k_c c \tag{4.23}$$

or

$$a_1 d - 2\sqrt{a_2}\ q_s = k_d c \tag{4.24}$$

where k_c and k_d are proportionality constants. When (4.23) is placed in (4.19), the result is

$$q_s = (\alpha k_c - k_a) c \tag{4.25}$$

Substituting (4.14), (4.23), and (4.25) in (4.11) and (4.12), one obtains

$$P_c b = [\frac{2}{KN}(k_a - \alpha k_c)\sqrt{a_2}\ k_c + \frac{a_1 k_c{}^2}{KN}]c \tag{4.26}$$

$$P_s b = \frac{k_a}{KN}(2\alpha\, k_c - k_a)c \tag{4.27}$$

In summarizing the foregoing, one should observe that the net effect of (4.11)-(4.13) is that the vectors d, q_s, q_c, $P_s b$, and $P_c b$ are each proportional to the n-vector c. Of the two proportionality constants, k_a and k_c, one may be selected arbitrarily. Let

$$k_c = \sqrt{\frac{K}{4\alpha}} \tag{4.28}$$

be the arbitrary choice. One will observe that the use of (4.20) instead of (4.19) and/or (4.24) instead of (4.23) would yield the same result as above, except one or both of the two different arbitrary constants would appear in a different position in the resulting equations.

The effect of (4.10) will be considered next. Because of (4.14) and (4.23), q must be proportional to some linear combination of c and $A^T c$ unless q_c is zero. One cannot obtain a frequency domain inequality when $q_c \neq 0$ because in such a case one would obtain from (4.8)-(4.9) a frequency domain equality which would not be useful in the stability investigation. Therefore, the very nature of (4.10) demands that q_c be zero. From (4.14), this means that a_2 must also be zero.

Since both P_s and P_c are symmetric, equations (4.26)-(4.27) imply

$$P_s = \alpha_{so}\, cc^T \qquad (4.29)$$

$$P_c = \alpha_{co}\, cc^T \qquad (4.30)$$

However, an inspection of (4.10) when q_c equals zero shows that

$$\alpha_{co} = 0 \qquad (4.31)$$

Therefore, from (4.26)

$$a_1 = 0 \qquad (4.32)$$

Using (4.23) and (4.29) in (4.10), one finds

$$\omega_o\, \alpha_{so} = \beta\, \omega_o\, \alpha\, k_c^2 \qquad (4.33)$$

where k_c is given by (4.28). Thus,

$$\alpha_{so} = \frac{\beta k}{4} \qquad (4.34)$$

and

$$P_s = \frac{\beta K}{4}\, cc^T \qquad (4.35)$$

Substituting (4.35) and (4.28) in (4.27), one finds

$$k_a = \sqrt{\frac{K\alpha}{4}} \pm \frac{K}{2}\sqrt{\frac{\alpha}{K} - \beta N\, c^T b} \qquad (4.36)$$

Therefore

$$q_s = \frac{K}{2} \sqrt{\frac{\alpha}{K} - \beta N\ c^T b}\ \ c \tag{4.37}$$

The result of placing the foregoing in (4.9) is

$$N\ Pb + P_s b + \frac{\beta}{2} A^T c = - \sqrt{\frac{\alpha}{K} - \beta N\ c^T b}\ \ q + \frac{\delta}{4\alpha} c \tag{4.38}$$

Using

$$\bar{P} = P - P_s/N \tag{4.39}$$

$$\bar{q} = q - q_s/N \tag{4.40}$$

one may write (4.38) in the form

$$\bar{P}b + (\frac{\alpha}{2N^2} - \frac{\delta}{4\alpha N})c + \frac{\beta}{2N} A^T c = - \frac{1}{N} \sqrt{\frac{\alpha}{K} - \beta N\ c^T b}\ \ \bar{q} \tag{4.41}$$

Equations (4.8), (4.9), and (4.12) yield

$$A^T\bar{P} + \bar{P}A = - \bar{q}\bar{q}^T - (1 + \frac{\delta}{N} - \frac{\alpha^2}{N^2}) \frac{K}{4\alpha} cc^T \tag{4.42}$$

The six formidable equations (4.8)-(4.13) have been reduced to (4.41)-(4.42) to which the lemma of Appendix I may be applied. The resulting frequency domain inequality is

$$\frac{1}{N^2} (\frac{\alpha}{K} - \beta N\ c^T b) + 2\ [\frac{\alpha}{2N^2} - \frac{\delta}{4\alpha N}]\ \mathrm{Re}\ G(j\omega)$$
$$+ \frac{\beta}{N} [\mathrm{Re}\ j\omega\ G(j\omega) + c^T b] - (1 + \frac{\delta}{N} - \frac{\alpha^2}{N^2}) \frac{K}{4\alpha}\ |G(j\omega)|^2 \geq 0 \tag{4.43}$$

Simplifying, one has

$$\frac{1}{K} + \mathrm{Re}\ [(1 - \frac{\delta N}{2\alpha^2} + j\omega \frac{\beta N}{\alpha})\ G(j\omega)]$$
$$- (1 + \frac{\delta}{N} - \frac{\alpha^2}{N^2}) \frac{KN^2}{4\alpha^2}\ |G(j\omega)|^2 \geq 0 \tag{4.44}$$

which must hold for all real ω, some real scalar constant β, some real positive scalar α, and some real non-negative scalar N, where $0 \leq N \leq 1$.

The expression for $\dot{V}(x, t)$ in (4.7) is negative semi-definite only for certain values of the scalars α, β, ω_o, and δ. The relationships which must exist between these scalars in order for $\dot{V}(x, t)$ to be negative semi-definite will now be determined. Using (4.32), the result $a_2 = 0$, and (4.37), one may write

$$\dot{V}(x, t) = - [q^T x + \frac{K}{2} \sqrt{\frac{\alpha}{K} - \beta N\ c^T b}\ \ \sigma \sin (\omega_o t + \phi)]^2 - m(t)\sigma^2 \tag{4.45}$$

where

$$m(t) = \frac{K}{4\alpha} (1 - \delta \sin (\omega_o t + \phi) - \alpha^2 \sin^2(\omega_o t + \phi) - \beta\alpha\omega_o \cos (\omega_o t + \phi)) \tag{4.46}$$

The function m(t) in (4.46) must be non-negative for $\dot{V}(x, t)$ to be negative semi-definite.

<u>Lemma</u>:

The function m(t) given by (4.46) is non-negative for all $t \geq 0$ if and only if there exists some real scalar constant Θ in the interval $(0, \pi/2)$ such that

$$\alpha \leq \sqrt{\frac{1 - \delta(\sin \Theta + \cos \Theta \cot \Theta)}{1 + \cos^2 \Theta}} \tag{4.47}$$

$$\beta\omega_o = \cos \Theta\ [2\alpha + \frac{\delta}{\alpha \sin \Theta}] \tag{4.48}$$

where δ is a real positive scalar constant in (4.47)-(4.48), α and ω_o are real positive scalar constants, and β is a real scalar constant. If the scalar δ is negative, then the scalar constant Θ must belong to the interval $(\frac{3\pi}{2}, 2\pi)$ in (4.47)-(4.48). If δ is zero, then the following relationships must hold.

$$\alpha^2 + \left(\frac{\beta\omega_o}{2}\right)^2 \leq 1 \qquad \text{for } \alpha \geq \sqrt{1/2} \tag{4.49}$$

$$\alpha\,\beta\omega_o \leq 1 \qquad \text{for } \alpha < \sqrt{\tfrac{1}{2}} \tag{4.50}$$

<u>Proof of Lemma:</u>

Observe that $m(t)$ is both periodic and continuous. Therefore, let t^* be some value of t at which

$$\left.\frac{d\,m(t)}{dt}\right|_{t\,=\,t^*} = 0 \tag{4.51}$$

Then the inequality

$$m(t^*) \geq 0 \tag{4.52}$$

is both necessary and sufficient for $m(t)$ to be non-negative for all $t \geq 0$.

Setting the first derivative of $m(t)$ at $t = t^*$ equal to zero, one has

$$\frac{d\,m(t^*)}{dt} = \frac{K\omega_o}{4\alpha}\left[-\delta\cos(\omega_o t^* + \phi) - 2\alpha^2 \sin(\omega_o t^* + \phi)\cos(\omega_o t^* + \phi) + \beta\omega_o\alpha\sin(\omega_o t^* + \phi)\right] = 0 \tag{4.53}$$

To simplify notation, let

$$\theta = \omega_o t^* + \phi \tag{4.54}$$

Using (4.54) and solving (4.52) and (4.53) simultaneously yields (4.47)-(4.48) when δ is non-zero. For the case in which δ equals zero, (4.53) becomes

$$(\sin\theta)(\beta\omega_o\alpha - 2\alpha^2\cos\theta) = 0 \tag{4.55}$$

Therefore, either

$$\sin\theta = 0 \tag{4.56}$$

or

$$\cos \Theta = \frac{\beta\omega_o}{2\alpha} \tag{4.57}$$

If (4.57) is true, then the application of (4.52) yields (4.49) for $\alpha^2 \geq \frac{1}{2}$. However, if (4.56) is true, then (4.50) is obtained from (4.52). This result is valid for $\alpha^2 < \frac{1}{2}$. A family of curves showing permissible values of α and $\beta\omega_o$ for various values of δ is shown in Fig. 4.2.

All of the conditions developed in this section can now be brought together in the form of a theorem for asymptotic stability.

Theorem:

The system (4.1)-(4.3) is asymptotically stable* if there exist numbers $\alpha > 0$, β, δ, N (where $0 \leq N \leq 1$), and $K > 0$ such that

a) $$K_{max} = K + \epsilon \tag{4.58}$$

where ϵ is an arbitrarily small real positive scalar constant,

b) the inequality (4.44) with K_{max} replacing K

$$\frac{1}{K_{max}} + \mathrm{Re}\ [(1 - \frac{\delta N}{2\alpha^2} + j\omega \frac{\beta N}{\alpha})\ G(j\omega)] - (1 + \frac{\delta}{N} - \frac{\alpha^2}{N^2}) \frac{K_{max} N^2}{4\alpha^2} |\ G(j\omega)\ |^2 \geq 0$$

holds for all real ω, some real scalar constant β, some real positive scalar α, and some real non-negative scalar N, where $0 \leq N \leq 1$,

c) there exists some real scalar constant Θ in the interval $(0, \pi/2)$ such that (4.47)-(4.48)

$$\alpha \leq \sqrt{\frac{1 - \delta(\sin \Theta + \cos \Theta \cot \Theta}{1 + \cos^2 \Theta}}$$

*The technique of Section 3.2 and Appendix II has been utilized to make $\dot{V}(x, t)$ negative definite.

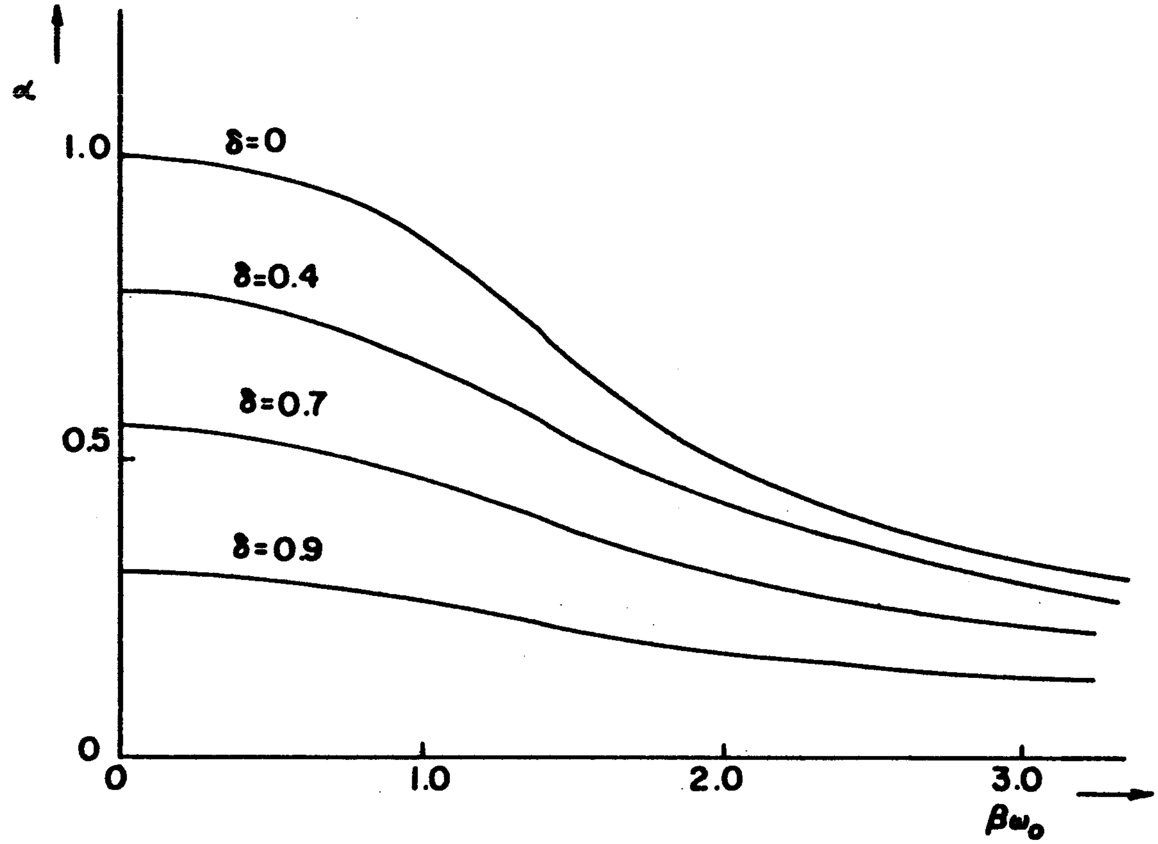

Fig. 4.2 A Family of Curves of α Versus $\beta\omega_o$ for Various Values of δ such that $m(t)$ in (4.46) is Non-Negative

$$\beta\omega_o = \cos\theta\,[2\alpha + \frac{\delta}{\alpha\sin\theta}]$$

hold for $\delta > 0$ and the equations (4.49)-(4.50)

$$\alpha^2 + (\frac{\beta\omega_o}{2})^2 \leq 1 \qquad \text{for } \alpha \geq \sqrt{1/2}$$

$$\alpha\beta\omega_o \leq 1 \qquad \text{for } \alpha < \sqrt{1/2}$$

hold when $\delta = 0$, and

d) the stationary system (3.14) is asymptotically stable for all K_1 in the interval $(0, K)$.

The stability criterion resulting from the above theorem is quite general. The result is valid for any frequency of sinusoidal variation. As the frequency approaches zero, α and N approach unity and the Popov sector is obtained. Moreover, as the frequency approaches infinity, β in (4.49) must be chosen to be zero. Figure 4.3 shows a diagram of the sector obtained from the criterion, which shall subsequently be referred to as the Sinusoidal Criterion.

4.4 Relationship with Previous Criteria

The Rozenvasser Criterion is a special case of (4.44) for the class of systems under consideration and may be obtained by setting $\alpha = 1$, $\delta = 0$, $\beta = 0$, and $N = 1$.

By choosing $V(x, t)$ as in (3.1) and letting $f(\sigma, t) = g(t)\sigma$, where $g(t)$ is a time varying gain, one may constrain $\dot{V}(x, t)$ in the form

$$\dot{V}(x, t) = -\,[q^T x - \sqrt{\frac{1}{K} - \beta c^T b}\;\; g(t)\sigma]^2 - g(t)\sigma^2\,[1 - \frac{g(t)}{K}] + \frac{\beta}{2}\frac{d\,g(t)}{dt}\sigma^2 \qquad (4.59)$$

to guarantee asymptotic stability for systems containing a single time-

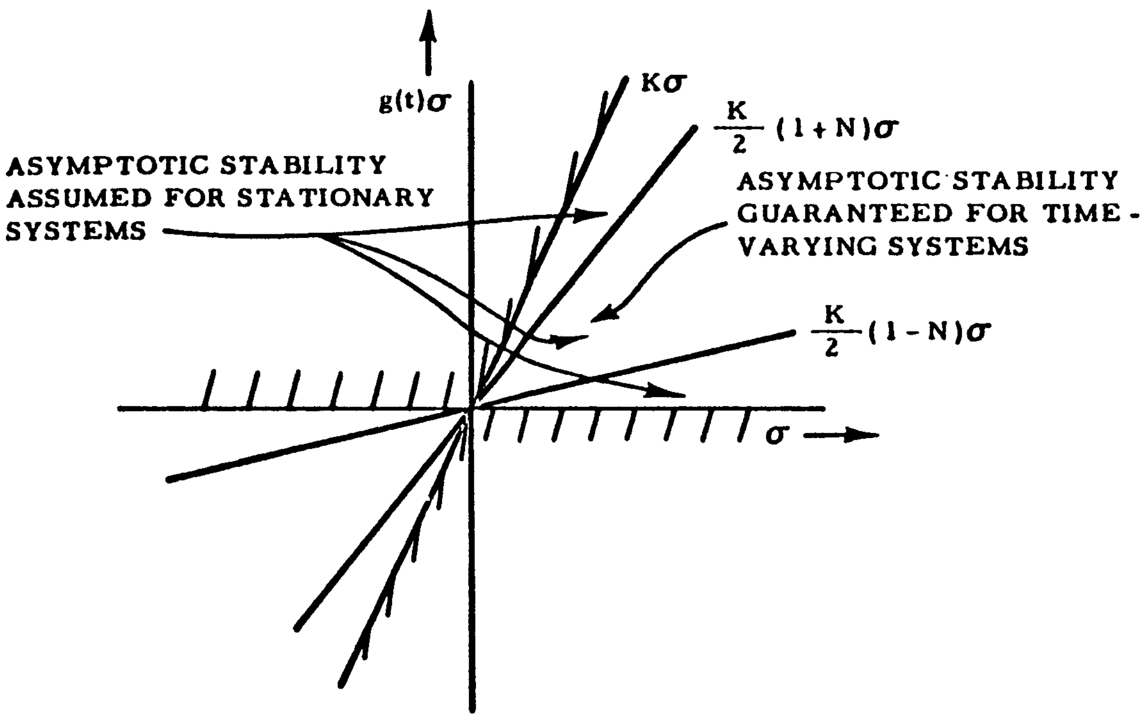

Fig. 4.3 Stability Sector Obtained by the Sinusoidal Criterion

varying gain g(t) if both

$$\frac{\beta}{2} \frac{d\,g(t)}{dt} - g(t)\left[1 - \frac{g(t)}{K}\right] \qquad (4.60)$$

holds for all $t \geq 0$ and (1.16) holds for all real ω. This result, which was established very early during this thesis investigation, is equivalent to a more recent result by Brockett and Forys [32]. For $\delta = 0$, the Sinusoidal Criterion of this chapter yields the same result as (1.16) and (4.60) when applied to the system (4.1)-(4.3). However, when $\delta \neq 0$ gives the best sector in (4.44), then there is an improvement over the above result.

The Sinusoidal Criterion may be applied directly by choosing N = 1 and then varying α and δ or β and δ to find the largest K. However, one may also utilize other techniques to obtain a graphical interpretation. One will observe that the Popov inequality (1.16) is obtained by choosing $\delta = 0$ and $\alpha = N$. The value of N will be in general less than unity. Thus, a graphical interpretation in the form of a modified frequency response (Fig. 1.1) may be used to find the widest sector in Fig. 4.3. The new criterion gives a sector whose lower limit is non-zero when $N \neq 1$. To compare this result with criteria which guarantee stability in a sector (0, K), one may use a "pole-shifting" technique to rearrange (4.1)-(4.3) such that the lower limit for a new gain h(t) defined below is zero [15]. This shifting is only for the purpose of comparison and does not affect the result of the theorem.

Define

$$h(t) = \frac{KN}{2}\left[1 + \sin(\omega_o t + \phi)\right] \qquad (4.61)$$

Therefore,

$$h(t) = g(t) - \frac{K}{2}(1-N) \tag{4.62}$$

The new system equations become

$$\dot{x} = A_1 x + b\, h(t)\sigma \tag{4.63}$$

$$\sigma = c^T x \tag{4.64}$$

where

$$A_1 = A + \frac{K}{2}(1-N)\, bc^T \tag{4.65}$$

With this new formulation h(t) belongs to the range (0, KN), and asymptotic stability is guaranteed if the peak-to-peak sinusoidal variation does not exceed KN.

An example will be presented to illustrate the new criterion when $\delta = 0$ and $\alpha = N$.

Example

Consider the system described by the following set of differential equations.

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= -2.8345\, x_1 - 11x_2 - 6x_3 + g(t)\sigma \end{aligned} \tag{4.66}$$

$$\sigma = -x_1 \tag{4.67}$$

$$g(t) = \frac{K}{2}[1 + N \sin(\omega_o t + \phi)] \tag{4.68}$$

The open-loop transfer function of the linear plant is

$$G(s) = \frac{1}{s^3 + 6s^2 + 11s + 2.8345} \tag{4.69}$$

Using the modified frequency response described in Fig. 1.1, one finds that the maximum value of K which satisfies (1.16) is 63.16 with a corresponding β of 0.545.

Let the value of ω_o in (4.68) be 1.60 for this example. One may then use (4.49), where N replaces α, to find that N equals 0.8998. Therefore, the value of KN obtained by using the largest possible K in (1.16) is 56.83.

Using this value of KN, one may write the system equations in the form of (4.72)-(4.65). The resulting plant transfer function is

$$G(s) = \frac{1}{s^3 + 6s^2 + 11s + 6} = \frac{1}{(s + 1)(s + 2)(s + 3)} \qquad (2.20)$$

and the new time-varying gain h(t) becomes

$$h(t) = \frac{56.83}{2} [1 + \sin (1.6t + \phi)] \qquad (4.70)$$

The reason for selecting the somewhat awkward values in (4.69) was to arrive eventually at the transfer function in (2.20), which was the basis of an example in Section 3.3. This example was examined by using the Improved Criterion, which had been developed for feedback systems containing a single time-varying nonlinear element. The only additional information available was an upper bound on an integral involving the time rate of change of the nonlinear characteristic. The maximum gain obtained by using Subcase I of the Improved Criterion was only 39.41, compared to 56.83 by using the new criterion. Thus the new criterion based on a linear system having a single sinusoidal gain has yielded a much better result for this particular case than a previous criterion developed for feedback systems containing a single time-varying nonlinear element.

Returning to the original problem described by (4.66)-(4.69), one may verify that the maximum value of KN does not occur at that value of β for which K is maximum (Fig. 4.4). If the Popov line in Fig. 1.1 is tangent to the curve at a slightly lower frequency $\omega = 3.04$, compared to $\omega = 3.31$ for maximum K, then the corresponding lower value of β yields a larger value of 0.921 for N (compared with 0.8998 before). The net result is that the maximum value of KN is 57.62 with a corresponding β of 0.484.

The next curve (Fig. 4.5) shows the maximum gain as a function of the frequency ω_o of the sinusoidal variation. This curve indicates that larger gains are allowed for lower frequencies according to the Sinusoidal Criterion.

Choosing $\alpha = N$ and $\delta = 0$ enables one to utilize the graphical interpretation in Fig. 1.1. However, a pole-shifting technique must later be applied to obtain a sector whose lower limit is zero. This indirect procedure is unnecessary. If N is chosen to be unity, then one may vary α to obtain the same result as before. This eliminates the pole-shifting and the need for finding a new G(s).

Suppose one has a system composed of the plant in (2.20) and a sinusoidal gain h(t) described by (4.61) where $N = 1$. Then by choosing $\alpha = 0.921$, $\beta = 0.484$, and $\delta = 0$, one may use (4.44) directly to guarantee asymptotic stability in the range (0, 57.62).

Let $G_1(j\omega)$ be the plant transfer function in the formulation in which (4.60) applies and a wedge sector is obtained. Let $G_2(j\omega)$ be the plant transfer function when the lower limit of the sinusoidal gain is zero, i.e., as in (4.61). Then the relationship between $G_1(j\omega)$

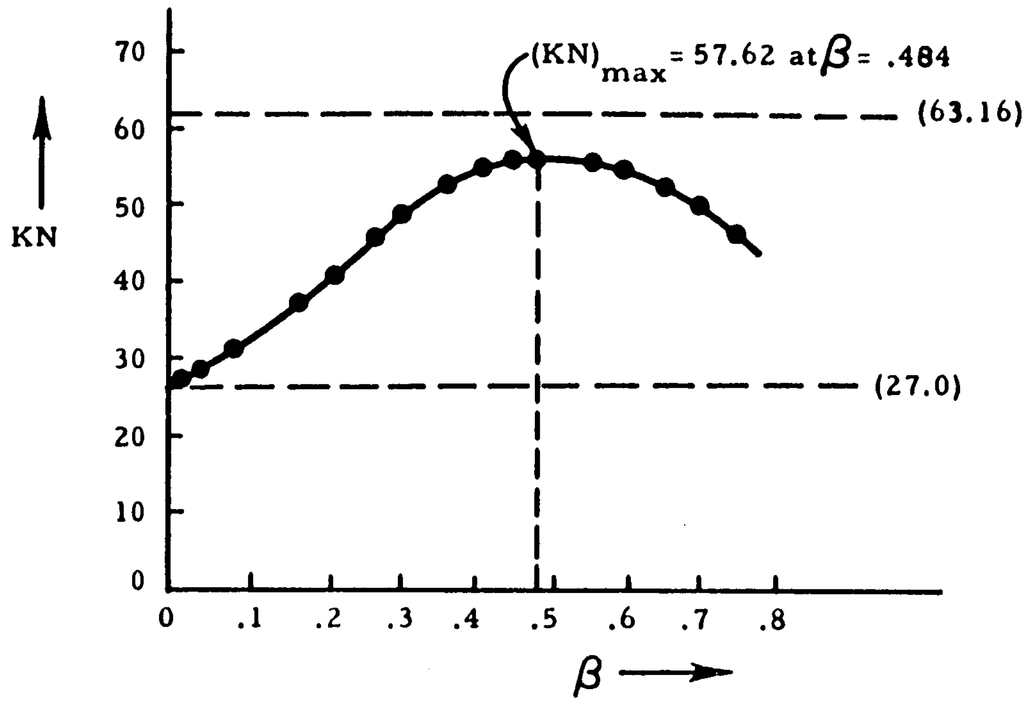

Fig. 4.4 A Plot of KN Versus β for the Example

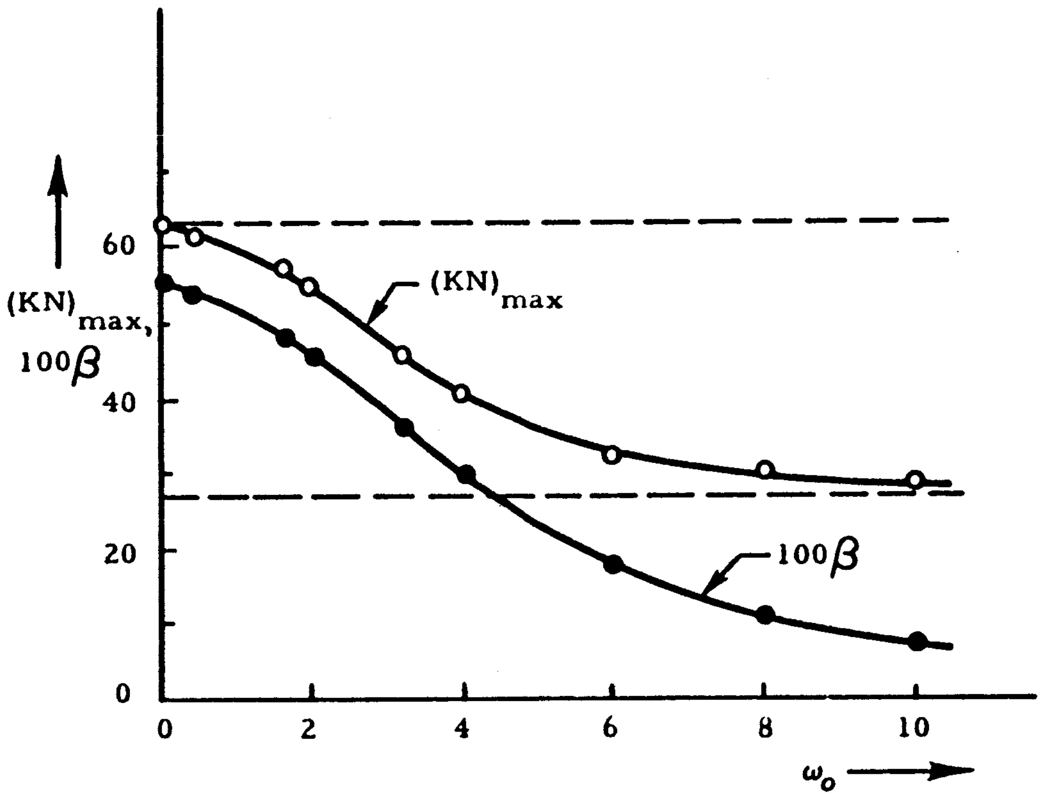

Fig. 4.5 Plots of KN Versus ω_o and β Versus ω_o for the Example

and $G_2(j\omega)$ is

$$G_2(j\omega) = \frac{G_1(j\omega)}{1 + \frac{K}{2\alpha}(1 - \alpha)\, G_1(j\omega)} \tag{4.71}$$

and the two formulations of the problem lead to the same result.

4.5 Application of Criteria to the Damped Mathieu Equation

A counter-example using the damped Mathieu equation was presented in Section 2.3 to show that Popov's Criterion needs modification for time-varying systems. The particular equation has a single sinusoidal gain and thus falls into the class of problems to which the criterion developed in this chapter applies. This problem is particularly interesting because the exact stability boundaries (Fig. 2.3) are recorded in the literature [3, 27, 28]. The Rozenvasser Criterion will first be applied and then the improvement offered by both the Improved Criterion and the Sinusoidal Criterion will be demonstrated. The differential equation is

$$\ddot{x} + 2\rho\dot{x} + [\mu + 2\xi \sin(2t + \frac{3\pi}{2})]x = 0 \tag{4.72}$$

which may be rearranged as

$$G(s) = \frac{1}{s^2 + 2\rho s + \gamma} \tag{2.8}$$

$$g(t) = \frac{K}{2}[1 + \sin(2t + \frac{3\pi}{2})] \tag{4.73}$$

The relationships between K, γ, ρ, μ, ξ, and ζ are

$$K = 2(\mu - \gamma) \tag{2.7}$$

$$\mu = 2\xi + \gamma \tag{4.74}$$

$$\zeta = \mu - \rho^2 \tag{2.13}$$

Therefore,

$$\xi = \frac{K}{4} \tag{4.75}$$

$$\zeta = \frac{K}{2} + \gamma - \rho^2 \tag{4.76}$$

The problem is to determine curves in the (ξ, ζ) plane as γ is varied. The results obtained from the Rozenvasser Criterion, the Improved Criterion, and the Sinusoidal Criterion will be compared.

In applying the Rozenvasser Criterion, one may first calculate

$$\text{Re } G(j\omega) = \frac{\gamma - \omega^2}{(\gamma - \omega^2)^2 + (2\rho\omega)^2} \tag{4.77}$$

Setting the derivative of Re $G(j\omega)$ equal to zero, one finds the Rozenvasser gain K_R as a function of ρ and γ.

$$K_R = 4\rho \; (\sqrt{\gamma} + \rho) \tag{4.78}$$

Combining (4.78) with (4.75)-(4.76), one obtains

$$\zeta = \frac{\xi^2}{\rho^2} \tag{4.79}$$

Consequently, if $\rho = 0.1$,

$$\zeta = 100 \; \xi^2 \tag{4.80}$$

which expresses ζ as a function of ξ for the Rozenvasser Criterion.

Moreover, for $\rho = 0.1$, the largest values of K that could be obtained by the Improved Criterion (Subcase I) and Sinusoidal Criterion are tabulated in Table 4.1 and displayed in the form of curves shown in Fig. 4.6. The region between the particular curve and the ζ axis is the region of asymptotic stability for that criterion. Thus, the Sinusoidal Criterion gives a larger region of asymptotic stability than either the Rozenvasser Criterion or the Improved Criterion for the

Table 4.1 Application of the Rozenvasser Criterion, the Improved Criterion, and the Sinusoidal Criterion to the System Described by (2.8) and (4.78) as γ is Varied and ρ is Constant at 0.1.

Rozenvasser Criterion:

γ	K_R	ζ_R	ξ_R
0.17	.204	0.26	.051
0.50	.323	0.66	.081
1.00	.440	1.21	.110
2.00	.606	2.28	.151
3.00	.733	3.35	.183

Improved Criterion:

γ	K_I	β_I	ω^*	ζ_I	ξ_I
0.17	0.204	0.00	0.50	0.26	.051
0.50	0.323	0.00	0.80	0.66	.081
1.00	0.458	0.97	1.08	1.22	.114
2.00	0.846	3.40	1.45	2.31	.211
3.00	1.240	4.00	1.76	3.61	.310

Sinusoidal Criterion:

γ	K_S	β_S	α_S	δ	ω^*	ζ_S	ξ_S
0.17	0.235	.40	0.916	0.0	0.51	0.28	.059
0.50	0.430	.54	0.842	0.0	0.82	0.71	.107
1.00	0.680	.60	0.800	0.0	1.14	1.33	.170
2.00	1.199	.64	0.768	0.0	1.58	2.59	.300
3.00	1.68	.66	0.752	0.0	1.92	3.84	.420

ω^* is that value of ω in the search for the largest K_I or K_S such that the left hand side of (3.16) or (4.44), respectively, attains its smallest value.

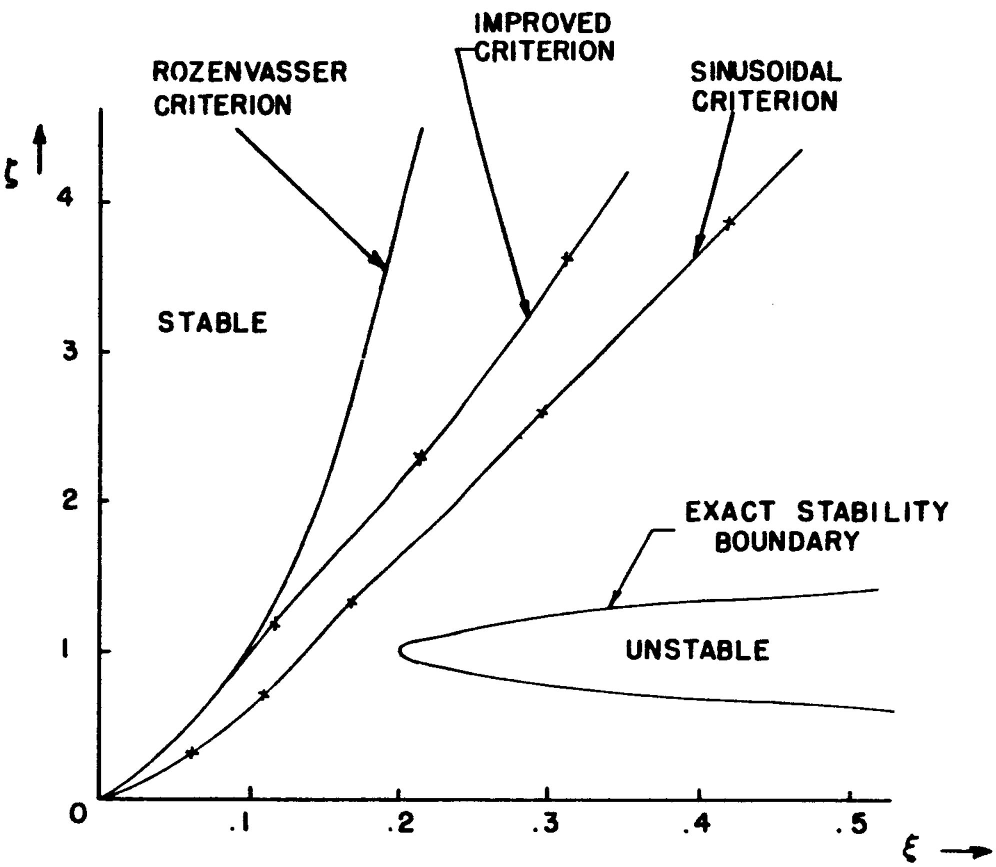

Fig. 4.6 A Comparison between the Rozenvasser Criterion and the Sinusoidal Criterion for the Damped Mathieu Equation

system described by (4.72), (4.73), and (2.8), when γ is varied.

4.6 Summary

Sufficient conditions were obtained in this chapter to guarantee global asymptotic stability for linear feedback systems containing a single sinusoidal gain. The new result, known as the Sinusoidal Criterion, represents a considerable improvement over both the Rozenvasser Criterion and the Improved Criterion, which were developed for a class of time-varying nonlinear systems.

The Rozenvasser Criterion, the Improved Criterion, and the Sinusoidal Criterion were each applied to the damped Mathieu equation. When several values of γ were considered with ρ held constant at 0.1, one could obtain curves in the (ξ, ζ) plane for each criterion. These curves illustrated that the Sinusoidal Criterion yielded a larger range of asymptotic stability than either of the other two criteria. In addition, Subcase I of the Improved Criterion yielded a better result than the Rozenvasser Criterion.

CHAPTER 5

CONCLUSIONS

5.1 Evaluations of Results

The recent trend in stability theory has been toward developing criteria which apply to a wide class of systems rather than being concerned about methods which apply only to a particular system. For many years the Second Method of Liapunov served as a tool for investigating the stability of particular equations. Usually these systems were of low order, and the techniques developed almost always depended greatly upon this fact.

The work of Popov [12, 13] in the early years of this decade is of special interest in the study of the problems of stability theory. Popov used functional analysis to develop a stability criterion for feedback systems containing a single time-invariant nonlinear element in a finite sector of its input-output plane. His stability criterion was significant, not only because it was independent of the order of the system, but also because the permissible value of K depended only upon the transfer function of the linear plant.

The Popov result was soon derived through the use of Liapunov theory by Kalman [14]. As a part of his work, Kalman presented an effective procedure by which one could construct the Liapunov function to guarantee global asymptotic stability. Rozenvasser [29] observed that a special case of the Popov Criterion ($\beta = 0$) was applicable to systems in which the nonlinearity varied with time.

Upon the works of these investigators are based the results of this thesis. It was first shown via a counter-example that Popov's Criterion must be modified (for $\beta \neq 0$) to apply to the case of a time-varying nonlinearity. Once this need was established, the Improved Criterion was developed in Chapter 3 to guarantee global asymptotic stability for feedback systems having a single time-varying nonlinear element confined to a finite sector of its input-output plane. This criterion represents a considerable improvement over the Rozenvasser Criterion. However, although the Rozenvasser Criterion requires only that the continuous nonlinearity remains in the finite sector for all time, the Improved Criterion also requires additional information related to the rate at which the nonlinear characteristic varies with time.

The utilization of more knowledge about the separate element is fundamental to this thesis. There is developed in Chapter 4 a stability criterion for feedback systems containing a single element about which a substantial amount of information is known. Specifically, the element is linear and varies sinusoidally with time. This new result, known as the Sinusoidal Criterion, yields a stability sector which is much larger than could be obtained by either the Improved Criterion or the Roznevasser Criterion when applied to this special case.

The stability criteria of this thesis represent a considerable improvement over previous results. In every case the criteria are independent of the order of the system and yield a sector which depends upon the transfer function of the linear plant and additional information about the separate element. The criteria guarantee global asymptotic stability for feedback systems containing a single time-varying element.

5.2 Proposed Extension of the Sinusoidal Criterion

The ideas of this and the following sections are presented as suggestions for further investigation. One will recall that the Rozenvasser Criterion was developed by using a Liapunov function consisting of only a quadratic term, i.e.,

$$V(x) = x^T P x \tag{5.1}$$

The Sinusoidal Criterion of Chapter 4 was obtained by choosing

$$V(x, t) = x^T[P + P_s \sin(\omega_o t + \phi) + P_c \cos(\omega_o t + \phi)]x \tag{4.5}$$

and constraining $\dot{V}(x, t)$ to be of the form

$$\begin{aligned}\dot{V}(x, t) = - &[q^T x + q_s^T x \sin(\omega_o t + \phi) + q_c^T x \cos(\omega_o t + \phi)]^2 \\ &- (d^T x)\,[1 - \delta \sin(\omega_o t + \phi) - \alpha^2 \sin^2(\omega_o t + \phi) \\ &\quad - \beta\alpha\omega_o \cos(\omega_o t + \phi) \\ &\quad - a_1 \sin(\omega_o t + \phi) \cos(\omega_o t + \phi) \\ &\quad - a_2 \cos^2(\omega_o t + \phi)]\end{aligned} \tag{4.7}$$

The criterion resulting from (4.5)-(4.7) yielded a much better result than the Rozenvasser Criterion for feedback systems having a single sinusoidal gain (4.1)-(4.3).

The proposed extension is to consider a more general Liapunov function of the form

$$V(x, t) = x^T[P + \sum_{k=1}^{N} P_{sk} \sin(k\omega_o t + \phi) + \sum_{n=1}^{N} P_{cn} \cos(n\omega_o t + \phi)]x \tag{5.2}$$

and to constrain $\dot{V}(x, t)$ as

$$\dot{V}(x, t) = -[q^T x + \sum_{k=1}^{N} q_{sk}^T x \sin (k\omega_o t + \phi) + \sum_{n=1}^{N} q_{cn}^T x \cos (n\omega_o t + \phi)]^2$$

$$- (d^T x)^2 [1 - \sum_{k=0}^{N} \sum_{n=0}^{N} a_{nk} \sin (k\omega_o t + \phi) \cos (n\omega_o t + \phi)] \tag{5.3}$$

While the Rozenvasser Criterion and the Sinusoidal Criterion used a V-function from a truncated Fourier Series, the proposed extension permits one to include terms in both V and $\dot{V}$ which are higher than the fundamental frequency. A more general stability criterion should be the result.

5.3 Extension to Systems Containing Several Time-Varying Nonlinearities

The second area for further investigation is based on earlier results by Ibrahim and Rekasius [38], who developed criteria for feedback systems with more than one time-invariant nonlinear element. In particular, one should be able to use the techniques of Chapter 3 to develop a new criterion which applies when the nonlinear elements vary with time. The suggested Liapunov function is

$$V(x, t) = x^T P x + \sum_{i=1}^{m} \beta_i \int_0^{\sigma_i} f_i(z_i, t)\, dz_i \tag{5.4}$$

which is analogous to a Liapunov function used in [38] to obtain a criterion to insure global asymptotic stability for single loop systems with several nonlinear elements. For the proposed investigation, the nonlinear characteristics should each be confined to a finite sector.

The rates at which these nonlinear characteristics vary with time will undoubtedly play an important role.

The above two ideas appear to be fruitful areas for further investigation in stability theory for feedback systems having time-varying elements.

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APPENDIX I

PROOF OF THE MODIFIED KALMAN LEMMA

A proof for the lemma in Section 3.2 is provided in this appendix. As a part of the sufficiency proof, a procedure due to Kalman [14] is outlined for the construction of a Liapunov function.

Necessity:

Adding the quantity $(-j\omega P + j\omega P)$ to the left hand side of (3.7) and rearranging terms, one obtains

$$(-j\omega I - A^T)P + P(j\omega I - A) = qq^T + \ell rr^T \tag{I.1}$$

Pre-multiplying (I.1) by $b^T(-j\omega I - A^T)^{-1}$ and post-multiplying by $(j\omega I - A)^{-1}b$, one has

$$b^T P(j\omega I - A)^{-1}b + b^T(-j\omega I - A^T)^{-1}Pb = b^T(-j\omega I - A^T)^{-1} qq^T(j\omega I - A)^{-1}b$$
$$+ b^T(-j\omega I - A^T)^{-1} \ell rr^T (j\omega I - A)^{-1}b \tag{I.2}$$

From (3.8), one has

$$Pb = \sqrt{\gamma}\, q - \frac{1}{2} m \tag{I.3}$$

$$b^T P = \sqrt{\gamma}\, q^T - \frac{1}{2} m^T \tag{I.4}$$

Using (I.3) and (I.4), one may write (I.2) as

$$\sqrt{\gamma}\, q^T (j\omega I - A)^{-1}b - \frac{1}{2} m^T(j\omega I - A)^{-1}b + b^T(-j\omega I - A^T)^{-1} \sqrt{\gamma}\, q$$
$$- \frac{1}{2} b^T(-j\omega I - A^T)^{-1}m = b^T(-j\omega I - A^T)^{-1} qq^T (j\omega I - A)^{-1}b$$
$$+ b^T(-j\omega I - A^T)^{-1} \ell rr^T (j\omega I - A)^{-1}b \tag{I.5}$$

Adding γ to both sides of (I.5), then rearranging terms and factoring, one obtains

$$[b^T(-j\omega I - A^T)^{-1}q - \sqrt{\gamma}\,][q^T(j\omega I - A)^{-1}b - \sqrt{\gamma}\,] =$$
$$-\ell\,[b^T(-j\omega I - A^T)^{-1}r]\,[r^T(j\omega I - A)^{-1}b]$$
$$-\frac{1}{2}m^T(j\omega I - A)^{-1}b - \frac{1}{2}b^T(-j\omega I - A^T)^{-1}m + \gamma \qquad (I.6)$$

Since the quantity on the left hand side of (I.6) is real and non-negative, then it follows that inequality (3.9)

$$\gamma - \mathrm{Re}\,[m^T(j\omega I - A)^{-1}b] - \ell\,|\,r^T(j\omega I - A)^{-1}b\,|^2 \geq 0$$

must hold for all real ω.

Thus the necessity proof is completed. The sufficiency proof is obtained in a manner analogous to the proof by Kalman in [14].

Sufficiency:

The sufficiency proof consists of showing that (3.9) implies that (3.7)-(3.8) must hold where A, b, and c define the system (2.1)-(2.2). Let the system be described by its phase variables, i.e., let

$$A = \begin{bmatrix} 0 & 1 & 0 & & \\ 0 & 0 & 1 & & \\ \vdots & & & & \\ 0 & & & 0 & 1 \\ -a_0 & & & -a_{n-2} & -a_{n-1} \end{bmatrix};\; b = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 1 \end{bmatrix};\; c = \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-2} \\ c_{n-1} \end{bmatrix}$$

The open-loop transfer function of the linear plant is given by (1.14). Inequality (3.9) may be written

$$\gamma - \mathrm{Re}\,m^T(j\omega I - A)^{-1}b - \ell\,|\,r^T(j\omega I - A)^{-1}b\,|^2 = \left|\frac{M(j\omega)}{N(j\omega)}\right|^2 \geq 0 \qquad (I.7)$$

Factorization of (I.7) yields

$$\frac{|M(j\omega)|^2}{|N(j\omega)|^2} = \frac{M(j\omega)}{N(j\omega)} \cdot \frac{M(-j\omega)}{N(-j\omega)} \tag{I.8}$$

where the poles and zeros of $\frac{M(j\omega)}{N(j\omega)}$ are in the left half plane. From (I.8), one forms

$$\frac{H(j\omega)}{N(j\omega)} = \sqrt{\gamma} - \frac{M(j\omega)}{N(j\omega)} \tag{I.9}$$

If the coefficients of $H(j\omega)$, arranged in the order of ascending powers, are identified with the n-vector q, then

$$\frac{H(j\omega)}{N(j\omega)} = q^T(j\omega I - A)^{-1}b \tag{I.10}$$

The vector q so defined satisfies (3.7) and (3.8). Therefore, (3.9) implies the equations (3.7)-(3.8), which completes the sufficiency proof. Moreover, the n-vector q can be effectively computed by using the above procedure. Note that the asymptotic stability of A in (3.7) implies that P is non-negative. For P to be positive definite, the pair (A, c^T) should be completely observable.

APPENDIX II

AN EQUIVALENCE PROOF FOR STABILITY INEQUALITIES

The purpose of this appendix is to show that the inequality (3.16)

$$\frac{1}{K_{max}} - \beta\alpha_3 + \mathrm{Re}\,[(1 + \beta\alpha_2 + j\omega\beta)G(j\omega)] - \beta\alpha_1\,|\,G(j\omega)\,|^2 \geq 0$$

is equivalent to the inequality (3.20)

$$\frac{1}{K} - \beta\alpha_3 + \mathrm{Re}\,[(1 + \beta\alpha_2 + j\omega\beta)\,G(j\omega)]$$

$$- \beta\alpha_1\,|\,G(j\omega)\,|^2 - \sum_{i=1}^{n} \rho_i\,|\,d_i^T(j\omega I - A)^{-1}b\,|^2 \geq 0$$

where

$$K = K_{max} - \epsilon \tag{II.1}$$

To prove this equivalence, one must show that (3.16) implies (3.20) and, in addition, that (3.20) implies (3.16). Let

$$\epsilon_1 = \max_{0 \leq \omega} \left[\sum_{i=1}^{n} \rho_i\,|\,d_i^T(j\omega I - A)^{-1}b\,|^2\right] \tag{II.2}$$

where ρ_1, ρ_2, ..., ρ_n are arbitrarily small positive quantities. Therefore, (3.20) implies (3.16) if

$$\frac{1}{K} - \epsilon_1 = \frac{1}{K_{max}} \tag{II.3}$$

or, simplifying,

$$K = K_{max} - \epsilon_1\,K\,K_{max} \tag{II.4}$$

One may choose ϵ in (II.1) such that

$$\epsilon = \epsilon_1 \, K \, K_{max} \tag{II.5}$$

and, therefore, (3.20) implies (3.16).

Next, one must show that, given ϵ, the values of ρ_i and d_i may be properly chosen. This means that (3.20) must follow from (3.16). The value of ϵ_1 is immediately known from (II.5) when ϵ has been specified. Since the scalar constants $\rho_1, \rho_2, \ldots, \rho_n$ and the linearly independent n-vectors $d_1, d_2, \ldots, d_n$ are all arbitrary, one may choose

$$\rho = \rho_1 = \rho_2 = \ldots = \rho_n \tag{II.6}$$

Therefore, one may select ρ and $d_1, d_2, \ldots, d_n$ such that

$$n \, \rho \max_{0 \leq \omega} \left[\sum_{i=1}^{n} \left| d_i^{\,T} (j\omega I - A)^{-1} b \right|^2 \right] = \epsilon_1 = \frac{\epsilon}{K \, K_{max}} \tag{II.7}$$

which shows that (3.16) implies (3.20).

Errata Sheet for Technical Report TR-EE66-2, Purdue University, "On the Asymptotic Stability of Feedback Control Systems Containing a Single Time-Varying Element" by Z. V. Rekasins and J. R. Rowland, January, 1966.

Page Number	Now Reads	Should Read
6, line 2	..the curve $\dot{V} = 0$..	..the curve $\dot{V} \equiv 0$ ..
9, line 8	Moreover, $\dot{V}(x,\sigma)$ is not...	Moreover, the curve $\dot{V} \equiv 0$ is not...
13, line 7	$0 < K_1 < K$	$0 \leq K_1 \leq K$
25, last line	$0 < K_1 < K$	$0 \leq K_1 \leq K$
28, eqn. (3.15)	$0 < K_1 < K$	$0 \leq K_1 \leq K$
28, last line	$\dot{V} \neq 0$	$\dot{V} \not\equiv 0$
31, 4th line after (3.20)	where $K = K_{max}$	where $K_{max} = K + \epsilon$
31, condition (c) of theorem	in the interval $(0,K)$,	in the closed interval $[0, K_{max}]$,
36, 7th line after (3.31)	...utilize (3.31) hold..	...utilize (3.31), hold...
41, 5th line after (4.3)	The scalar N in (4.3) is a non-negative constant	The scalar N in (4.3) is a real positive constant
50, 2nd line of theorem and last line of condition (b)	...where $0 \leq N \leq 1$..	...where $0 < N \leq 1$ ...
50, next to last line of condition (b)	...some real non-negative scalar N,...	...some real positive scalar N, ...
52, condition (d)	...in the interval $(0,K)$	...in the closed interval $[0,K_{max}]$.
54, eqn. (4.60)	$-$	$\leq$

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