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Cylindrical Shock Model
of the
Plasma Pinch

Report No. 742

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February 1966

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Abstract

A model of the plasma pinch is formulated which represents the imploding current sheet as an impermeable piston that drives a gasdynamic shock wave ahead of it toward the axis of the discharge. This cylindrical piston-shock problem is solved without further reference to electromagnetic effects. First the Lagrangian equations are solved for a parabolic shock trajectory in the r - t plane yielding a first and second approximation for the piston trajectory. To determine the accuracy of the approximation, the same problem is solved for a straight shock in the r - t plane by the method of characteristics in using the Eulerian formulation. It is found that the solutions given by the two methods compare exactly where a solution to the problem using the method of characteristics exists. The results are in qualitative agreement with relevant experimental observations.

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LIST OF SYMBOLS

A	Initial nondimensional velocity of the shock
B	Nondimensional acceleration of the shock
c	Local sound speed of the medium
c_{∞}	Sound speed of undisturbed gas
$H(\psi, t)$	Trajectory of the particle ψ in the r-t plane
L	Initial radius of the piston
M	Mach number of the shock
p	Local pressure
p_1	Pressure immediately after the shock
p_{∞}	Pressure of the undisturbed gas
$P(t)$	Piston trajectory in the r-t plane
R	Gas constant
$R_s(t)$	Shock trajectory in the r-t plane
r	Radial coordinate
t	Time coordinate
t'	Time when the shock crosses a particle ψ
T_0	Nondimensionalizing time
U	Velocity of the shock
u	Local flow velocity
x	Nondimensional position coordinate
X_s	Nondimensional shock trajectory
x_{first}	First approximation to the trajectory of a particle ψ
γ	Ratio of specific heats
ϵ	$(\gamma - 1)/(\gamma + 1)$
ρ	Local density
ρ_1	Density immediately after the shock
ρ_{∞}	Density of the undisturbed gas
τ	Nondimensional time
ϕ	Nondimensional stream function identifying a particle
ψ	Stream function: mass between the piston and the particle
ψ_s	Value of the stream function at the shock

I. INTRODUCTION

The object of the research presented here is to formulate a theoretical model of the plasma pinch on the basis of gas dynamics in order to determine what effects observed experimentally might be due only to the gas dynamics of the problem.

The plasma pinch under consideration has been described by Jahn and von Jaskowsky.^{1,2,3,4} Briefly, the physical apparatus consists of two plane circular electrodes of four, five, or eight inches diameter, separated by a two inch gap of test gas. Typically the test gas is Argon, although several other gases have been used. The electrodes are connected to two different types of external circuits--simple lumped capacitor arrays, or pulse-forming networks^{3,4}--which are discharged through the gap via a gas-triggered switch.⁵ It is observed that the discharge across the electrodes begins as a cylindrical sheet at the perimeter of the electrodes and propagates radially inward, driven by the electromagnetic interaction of the discharge current with its own magnetic field. The radial transit time for the current sheet is on the order of a few microseconds. Such pinches have been studied quite extensively by a variety of photographic and internal probe techniques.^{1,2,6,7}

In the study which follows, this pinch process is idealized to the problem of a radially driven cylindrical piston, representing the current sheet, generating a gas dynamic strong shock which propagates ahead of the piston into the undisturbed gas in the center of the cylinder.

II. THEORY

A. DEVELOPMENT OF MASTER EQUATION

Consider a cylindrical piston of given height but variable radius. Suppose that suddenly the radius of this piston decreases so rapidly that a strong shock of very high Mach number is driven in front of the wall toward the axis of the cylinder. The question may be posed in two different manners. If we know the path of the piston in the r - t plane, what will the shock path look like? Conversely, if we know the shock path, what piston motion was necessary to drive it? From the formulation of the problem, it is found to be much easier to solve the latter case.

The first attempt to solve this problem was made from the Lagrangian viewpoint of following each particle. The claim is made that if we are considering two particles that are initially (before any shock has touched them) located at different radii in the cylinder, their respective radii will always be such that the particle initially closer to the axis will always be closer to the axis. That is, in the r - t plane, the particle paths will never intersect. This immediately leads to a law for conservation of mass and a method for labeling each particle. Consider Fig. 1 which shows the path of the piston $P(t)$, the path of the shock $R_s(t)$, and the path of any particle $H(\psi, t)$. ψ will be defined such that every particle beginning at the same radius will have the same ψ , but particles beginning at different radii will have different ψ . Once a particle is labeled by a ψ , it will retain this value throughout its history. Physically, the stream function ψ corresponds to the mass between the piston and the particle under question. On the figure, t' refers to the time when the shock passes over the particle ψ . Thus, if the shock trajectory is known, then the stream function may be said to be a function of t' . Conversely, t' may be regarded

as a function of ψ .^{8,9,10,11} Specifically, we define

$$\psi = \pi \rho_{\infty} \left[L^2 - R_s(t')^2 \right] \quad (1)$$

L is the initial radius of the chamber. $R_s(t')$ is the radial position of the particle ψ before the shock hits it. ρ_{∞} is the density of the undisturbed gas. At any later time, ψ may be found by taking the integral

$$\psi = - \int_{P(t)}^{H(\psi,t)} 2 \pi \rho(r,t) r dr \quad (2)$$

$\rho(r,t)$ is the density at any point on the r - t plane. Since there are only two independent variables, we may consider $r = r(\psi, t)$ or we may say that $\psi = \psi(r, t)$. The Lagrangian point of view makes ψ and t the independent variables.

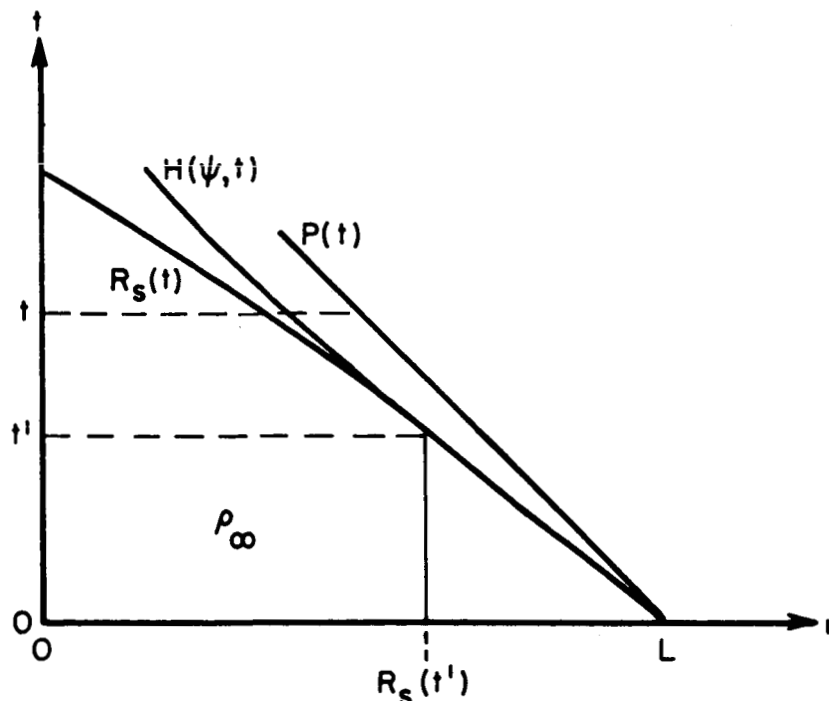


Figure 1

The momentum equation is seen to be

$$\rho \frac{\partial^2 r}{\partial t^2} = - \frac{\partial p}{\partial r} \quad (3)$$

where p is the pressure acting on the particle to accelerate it.

The process is assumed to be entirely isentropic except for a jump in entropy as the shock crosses the particle's path. Therefore, the ratio p/ρ^γ is a constant for each particle as it travels from the shock toward the axis. This constant is given by the conditions immediately after the shock. The pressure and density of a particle immediately after the shock has crossed it will be denoted by the subscript 1. γ is the ratio of specific heats, assumed constant.

$$\frac{p}{\rho^\gamma} = \frac{p_1}{\rho_1^\gamma} = \frac{p_1(\psi)}{\rho_1^\gamma(\psi)}$$

Note that p/ρ^γ is a function only of the streamline. This function is known if the shock trajectory is known. That is, p_1 and ρ_1 can be found by using the strong shock relations in conjunction with the perfect gas law. Specifically

$$\rho_1 = \frac{\gamma+1}{\gamma-1} \rho_\infty = \frac{\rho_\infty}{\epsilon} \quad (4)$$

where $\epsilon \equiv \frac{\gamma-1}{\gamma+1}$

$$\text{and } p_1 = p_\infty \frac{2\gamma}{\gamma+1} M^2 = p_\infty \gamma (1-\epsilon) \frac{U^2}{c_\infty^2} = p_\infty \gamma (1-\epsilon) \frac{\dot{R}_s^2(t')}{\gamma \frac{p_\infty}{\rho_\infty}}$$

or

$$p_1 = \rho_\infty (1 - \epsilon) \dot{R}_s^2(t') \quad (5)$$

Thus the isentropic condition is

$$\frac{p}{\rho^\gamma} = \rho_\infty^{1-\gamma} \epsilon^\gamma (1 - \epsilon) \dot{R}_s^2(t') \quad (6)$$

In these relations

- p_∞ is the pressure of the undisturbed gas
- M is the Mach number of the shock
- U is the velocity of the shock
- c_∞ is the speed of sound in the undisturbed gas
- $\dot{R}_s(t')$ is the velocity of the shock as it crosses the particle $\Psi = \Psi(t')$. That is, \dot{R}_s^2 is the square of the derivative of the shock trajectory with respect to time evaluated at the time t' .

These conservation relations will now be combined into one relationship.* First, by differentiation, equation (2) becomes

$$d\Psi = - 2\pi\rho r dr = -\pi\rho d(r^2) \quad (7)$$

The minus sign is a result of the stream function increasing as r is decreasing. After dividing through by ρ , this relation may now be integrated from the shock to an arbitrary stream line

Ψ to yield

$$R_s^2(t) - H^2(\Psi, t) = \int_{\Psi_s}^{\Psi} \frac{d\Psi}{\pi \rho} = \frac{1}{\pi} \int_{\Psi_s}^{\Psi} \frac{1}{\rho} \left(\frac{p}{\rho^\gamma} \right)^{\frac{1}{\gamma}} d\Psi$$

where Ψ_s is the value of Ψ at the shock at any time t . That is, $\Psi_s = \Psi_s(t)$. Note that this integration is done at the time t and not at the time t' . Substituting in the isentropic relation, we find

$$H^2(\Psi, t) = R_s^2(t) + \frac{1}{\pi} \int_{\Psi}^{\Psi_s} \left[\rho_\infty^{1-\gamma} \varepsilon^\gamma (1 - \varepsilon) \dot{R}_s^2(t') \right]^{\frac{1}{\gamma}} \frac{d\Psi}{\rho^{\frac{1}{\gamma}}} \quad (8)$$

If $R_s = R_s(t)$ is given, then we may find a function $t - t(R_s)$. For example, if $R_s = L - At - Bt^2$, then

$$t(R_s) = t = \frac{-A + \sqrt{A^2 - 4B(R_s - L)}}{2B} \quad (9)$$

From the definition of Ψ we note that

$$\Psi_s(t) = \pi \rho_\infty L^2 - \pi \rho_\infty R_s^2(t)$$

where Ψ_s is the value of Ψ at the shock at any time t . Hence

$$R_s(t) = \sqrt{L^2 - \frac{\Psi_s}{\pi \rho_\infty}}$$

Therefore

$$t(R_s) = t \left(\sqrt{L^2 - \frac{\Psi_s}{\pi \rho_\infty}} \right)$$

giving

$$\dot{R}_s(t') = t \left(\sqrt{L^2 - \frac{\psi}{\pi \rho_\infty}} \right)$$

which is a function only of ψ . Recall that t' is defined as the time when the particle ψ was crossed by the shock. In the example given above, equation (9) becomes

$$t' = t \left(\sqrt{L^2 - \frac{\psi}{\pi \rho_\infty}} \right) = \frac{-A + \sqrt{A^2 - 4B \left[\sqrt{L^2 - \frac{\psi}{\pi \rho_\infty}} - L \right]}}{2B}$$

yielding

$$\dot{R}_s(t') = -A - 2Bt' = - \sqrt{A^2 - 4B \left(\sqrt{L^2 - \frac{\psi}{\pi \rho_\infty}} - L \right)} \quad (10)$$

This relation will be used later. Let us now determine an expression for p . From the momentum equation, we find that

$$\frac{\partial p}{\partial r} dr = -\rho \frac{\partial^2 r}{\partial t^2} dr = \rho \frac{\partial^2 r}{\partial t^2} \frac{d\psi}{2\pi \rho r}$$

having made use of equation (7). Upon integration from the shock to any particle ψ , this becomes

$$p(\psi, t) = p_1(t) + \frac{1}{2\pi} \int_{\psi_s(t)}^{\psi} \frac{\partial^2 r(\psi, t)}{\partial t^2} \frac{d\psi}{r(\psi, t)}$$

Note that this integration is done at the time t , any arbitrary time after the shock has crossed the particle ψ . Substituting

equation (5) into this expression, the pressure is seen to be

$$P(\Psi, t) = \rho_{\infty} (1-\epsilon) \dot{R}_S^2(t) + \frac{1}{2\pi} \int_{\Psi_S(t)}^{\Psi} \frac{\partial^2 r(\Psi, t)}{\partial t^2} \frac{d\Psi}{r(\Psi, t)} \quad (11)$$

Putting this information back into equation (8), we find the master equation

$$H^2(\Psi, t) = R_S^2(t) - \frac{\epsilon}{\pi \rho_{\infty}} \int_{\Psi_S(t)}^{\Psi} \left\{ \frac{\rho_{\infty} (1-\epsilon) \dot{R}_S^2 \left(t' = t \left(\sqrt{L^2 - \frac{\Psi}{\pi \rho_{\infty}}} \right) \right)}{\rho_{\infty} (1-\epsilon) \dot{R}_S^2(t) + \frac{1}{2\pi} \int_{\Psi_S(t)}^{\Psi} \frac{\partial^2 r(\Psi, t)}{\partial t^2} \frac{d\Psi}{r(\Psi, t)}} \right\}^{\frac{1}{2}} d\Psi \quad (12)$$

Let us nondimensionalize the quantities appearing in the equation as follows

$$x = \frac{r}{L} ; \quad X_S = \frac{R_S}{L} ; \quad \vartheta = \frac{\Psi}{\pi L^2} ; \quad \tau = \frac{t}{T_0}$$

Note that $0 \leq x, X_S,$ and $\vartheta \leq 1$. T_0 is chosen so that τ is in microseconds, corresponding to the observation that all discharges have a pinch time of a few microseconds. The factor nondimensionalizing the stream function is the total mass contained in the chamber.

It should be mentioned that $H(\Psi, t)$ is exactly the same quantity as $r(\Psi, t)$. With these additions, the master equation

becomes

$$x^2(\vartheta, \tau) = x_s^2(\tau) + \varepsilon \int_{\vartheta}^{\vartheta_s = 1 - x_s^2(\tau)} \left\{ \frac{\left[\frac{dx_s}{d\tau} (\tau' = \tau(\sqrt{1 - \vartheta})) \right]^2}{\left[\frac{dx_s}{d\tau}(\tau) \right]^2 + \frac{1}{2(1-\varepsilon)} \int_{\vartheta_s = 1 - x_s^2(\tau)}^{\vartheta} \frac{\partial^2 x(\vartheta, \tau)}{\partial \tau^2} \frac{d\vartheta}{x(\vartheta, \tau)}} \right\} d\vartheta \quad (13)$$

This equation implies that a particle ϑ is, at a time τ , at a position, away from the shock by a distance equal to ε multiplied by an integral, the integrand of which consists of a numerator representing the isentropic condition and a denominator which is a constant fraction of the pressure. ε varies from 0 for $\gamma = 1$ to .25 for $\gamma = 1.667$. Recall that the entire integrand is $\frac{1}{\rho}$. The piston trajectory is $x(0, t)$; that is, the piston is always at the particle $\vartheta = 0$.

Since the right hand side of the equation involves a second derivative of the desired solution, a method of iteration must be used. Due to the difficulty in taking derivatives numerically, we should search for an analytic solution that may be placed back into the equation. From this analytic solution, a second approximation may be found, numerically if necessary, and compared to the first solution. If the comparison is good, then the solution may be assumed good. From streak and Kerr-Cell photographs, it seems that the luminous front in the plasma pinch always propagates toward the center in a relatively smooth manner. Hence, most interesting cases may be covered by assuming a parabolic shock trajectory:

$$R_s = L - \frac{A L t}{T_0} - \frac{B L t^2}{T_0^2}$$

or:

$$X_s = 1 - A\tau - B\tau^2$$

A is the initial nondimensional velocity of the shock and B is the shock's constant acceleration or deceleration toward the axis, depending upon whether B is positive or negative, respectively. Substituting this shock trajectory and equation (10), modified for the nondimensional variables and shock trajectory, into the nondimensional master equation, we find

$$x^2(\vartheta, \tau) = X_s^2(\tau) + \epsilon \int_{\vartheta}^{1 - X_s^2(\tau)} \left(\frac{A^2 + 4B(1 - \sqrt{1 - \vartheta})}{(A + 2B\tau)^2 - \frac{1}{2(1-\epsilon)} \int_{\vartheta}^{1 - X_s^2(\tau)} \frac{\partial^2 x(\vartheta, \tau)}{\partial \tau^2} \frac{d\vartheta}{x(\vartheta, \tau)}} \right)^{\frac{1}{2}} d\vartheta \quad (14)$$

As a first approximation, it may be assumed that the pressure does not change much between the shock and the piston. In effect, we assume

$$\frac{1}{2(1-\epsilon)} \int_{\vartheta}^{1 - X_s^2} \frac{\partial^2 x}{\partial \tau^2} \frac{d\vartheta}{x} \ll (A + 2B\tau)^2$$

This assumption is good when two conditions are met. First, we must not be too close to the axis. Otherwise, x will be small, making 1/x large. Secondly, the second derivative of x with respect to τ should be small and preferably negative. If $\frac{\partial^2 x}{\partial \tau^2} > 0$, then the denominator of the integrand of the large integral may approach zero. These conditions are met at least in the beginning stages of the piston's propagation. With this

assumption, equation (12) becomes

$$x^2(\vartheta, \tau) = X_S^2(\tau) + \frac{\varepsilon}{(A+2B\tau)^{2/\gamma}} \int_{\vartheta}^{1 - X_S^2} \left(A^2 + 4B - 4B \sqrt{1 - \vartheta} \right)^{1/\gamma} d\vartheta$$

which readily integrates to a first approximation for x

$$x_{\text{first}}^2(\vartheta, \tau) = X_S^2(\tau) + Q G(\tau) F(\tau) - Q G(\tau) H(\vartheta) \quad (15)$$

where

$$Q = \frac{2 \varepsilon \gamma (A^2 + 4B)^{\frac{1}{\gamma}}}{P^2 (\gamma + 1)} \quad \text{with} \quad P = \frac{4B}{A^2 + 4B}$$

$$G(\tau) = (A + 2B\tau)^{-\frac{2}{\gamma}} \quad (16)$$

$$F(\tau) = PX_S (1 - PX_S)^{\frac{\gamma+1}{\gamma}} + \frac{\gamma}{2\gamma+1} (1 - PX_S)^{\frac{2\gamma+1}{\gamma}}$$

$$H(\vartheta) = P \sqrt{1 - \vartheta} (1 - P \sqrt{1 - \vartheta})^{\frac{\gamma+1}{\gamma}} + \frac{\gamma}{2\gamma+1} (1 - P \sqrt{1 - \vartheta})^{\frac{2\gamma+1}{\gamma}}$$

In order to find the second approximation, we must substitute the second derivative of $x_{\text{first}}(\vartheta, \tau)$ with respect to τ into

the subintegral of equation (14). Denoting all first derivatives with respect to τ as primes and second derivatives as double primes, the second derivation may be written as

$$\frac{1}{x} \frac{\partial^2 x}{\partial \tau^2} = \frac{x''}{x} = \frac{2(A+2B\tau)^2 - 4BX_s + Q(GF'' + 2G'F + G''F - G''H)}{2x^2} \quad (17)$$

$$- \frac{[2X_s(-A-2B\tau) + QGF' + QG'F - QG'H]^2}{4x^4}$$

Letting $s = \frac{1}{\gamma}$ we find for the derivatives used above that

$$G' = -4sB(A + 2B\tau)^{-2s-1} \quad \text{and} \quad G'' = 8sB^2(2s+1)(A+2B\tau)^{-2s-2}$$

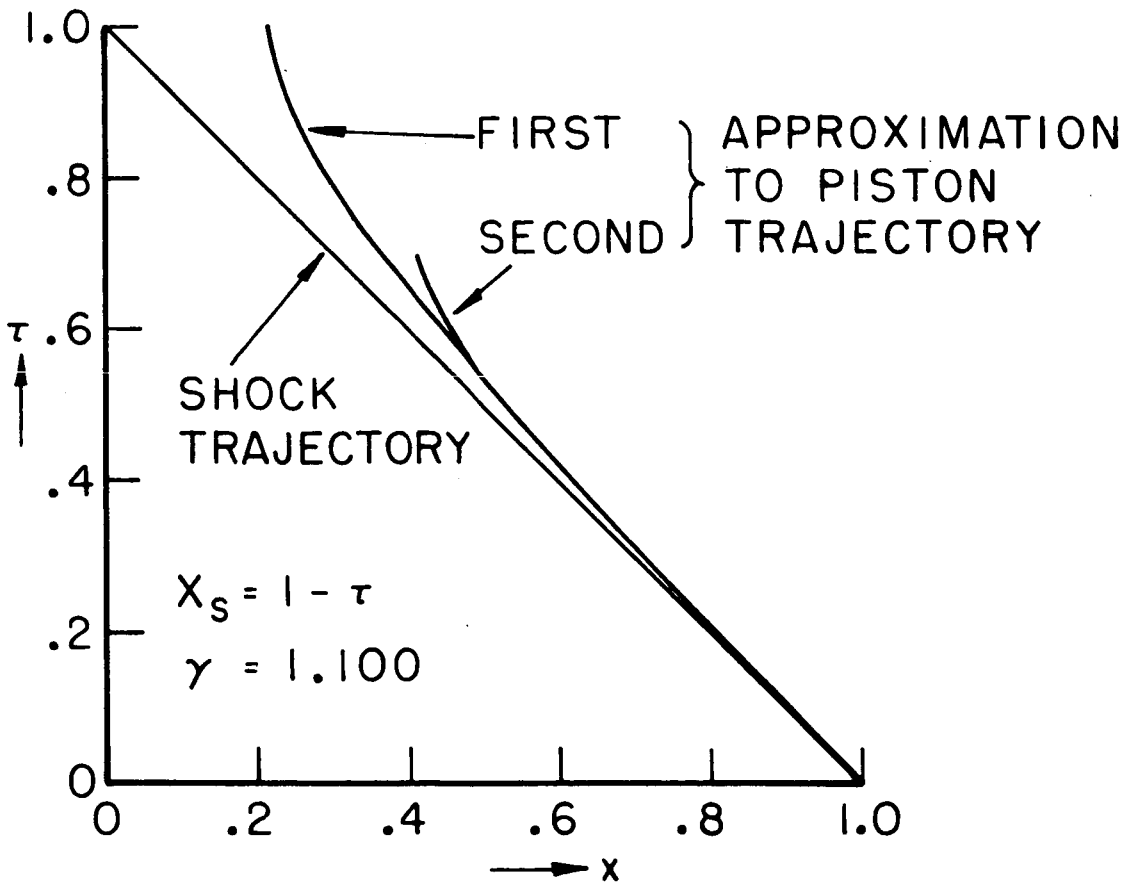
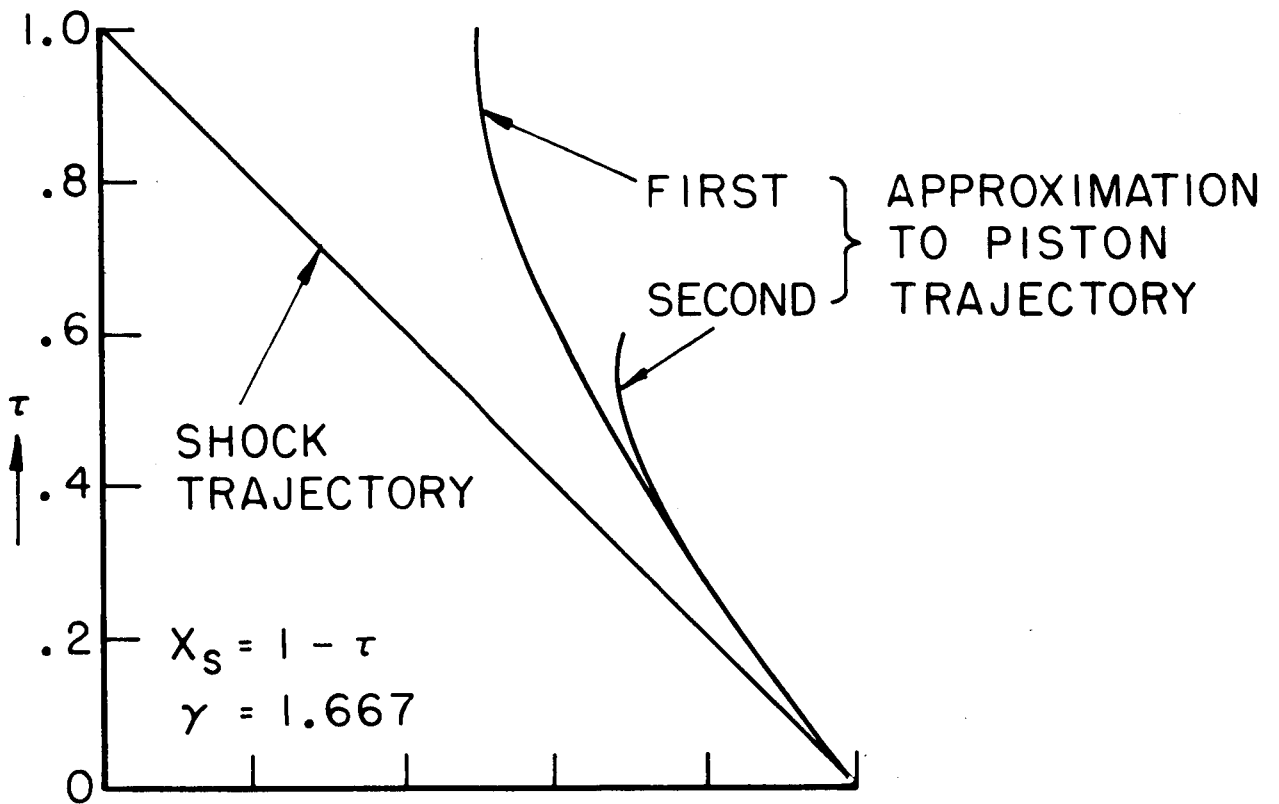
$$F' = P^2(s+1)X_s(1 - PX_s)^s(A + 2B\tau)$$

$$F'' = -P^2(s+1)(1 - PX_s)^s \left\{ (A+2B\tau)^2 - \frac{PsX_s(A+2B\tau)^2}{1 - PX_s} - 2BX_s \right\}$$

From this point, we invoke a computer program to perform a numerical integration to determine a second approximation. This program is shown and discussed in Appendix A.

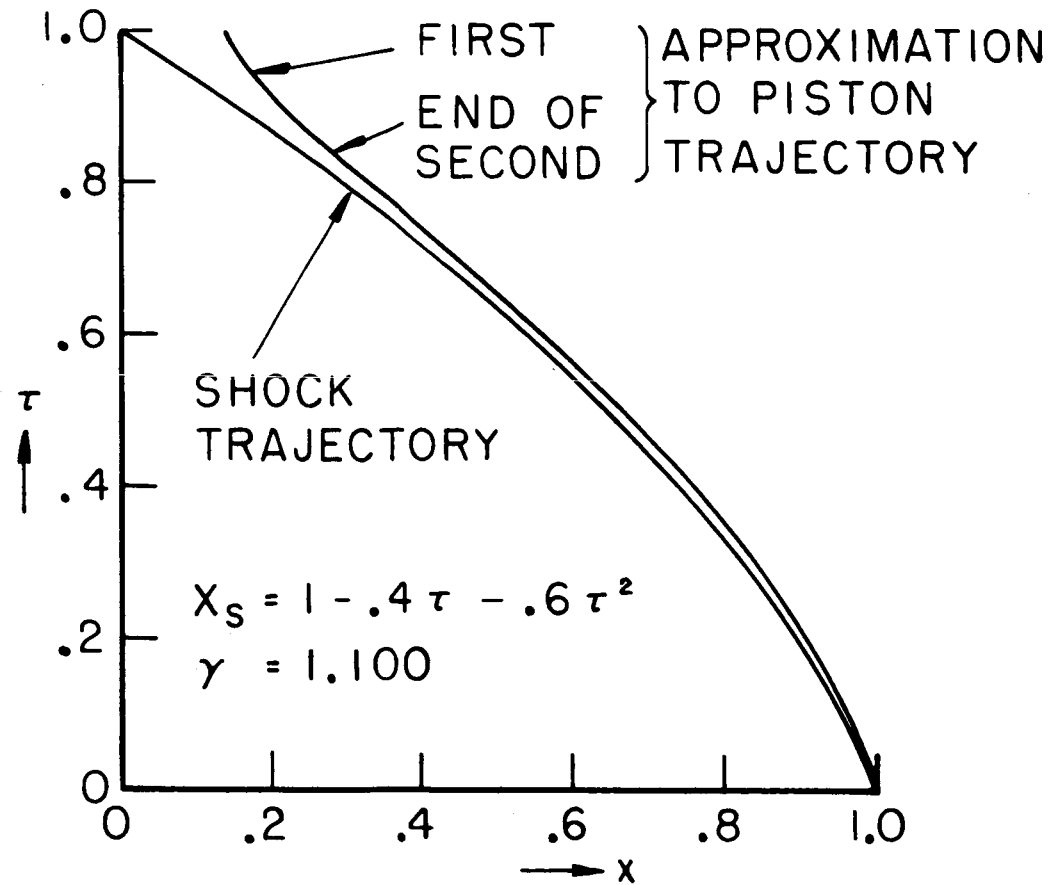
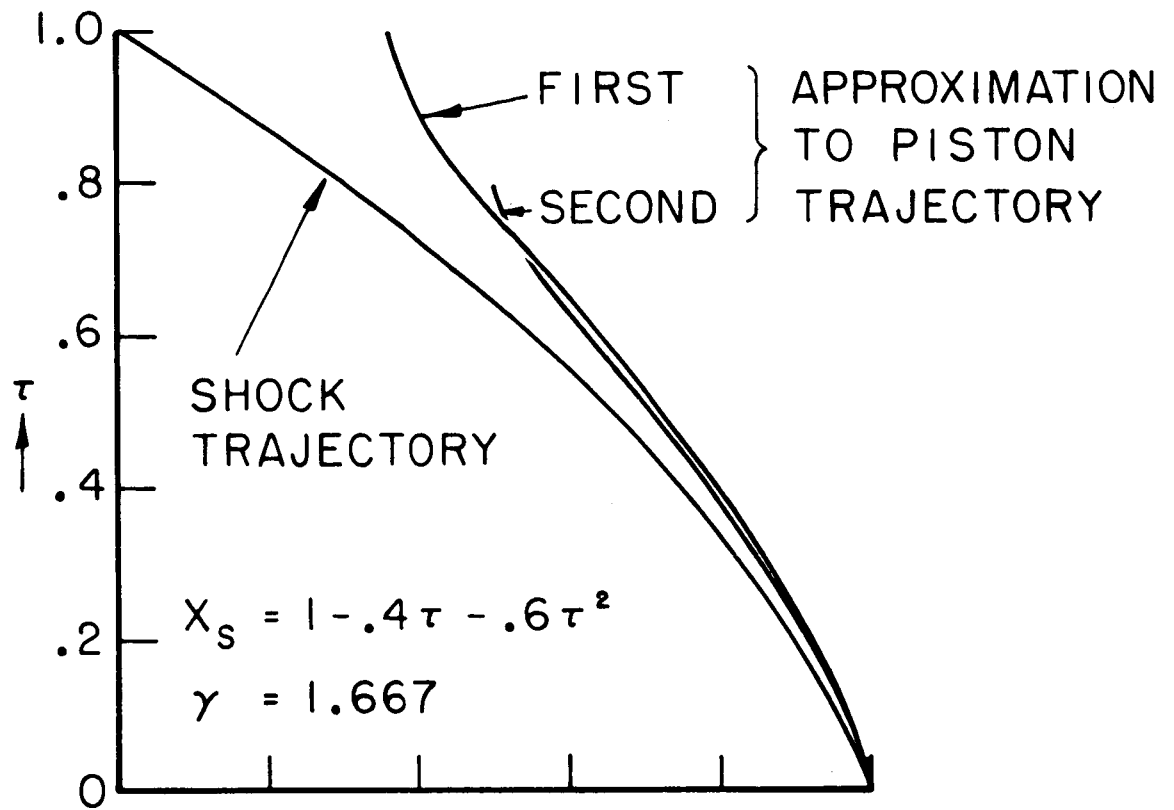
B. RESULTS FROM MASTER EQUATION

Graphs showing the piston trajectories for a straight shock, an accelerating shock, and a decelerating shock are shown in Figs. 2, 3, and 4. Two values of γ are used: 1.1 and 5/3. The latter value is the value for Argon at standard temperatures and pressures. Though this will not be close to the real value of γ after the Argon has experienced ionization, this value



PISTON TRAJECTORIES FOR CONSTANT VELOCITY SHOCK
 FIGURE 2

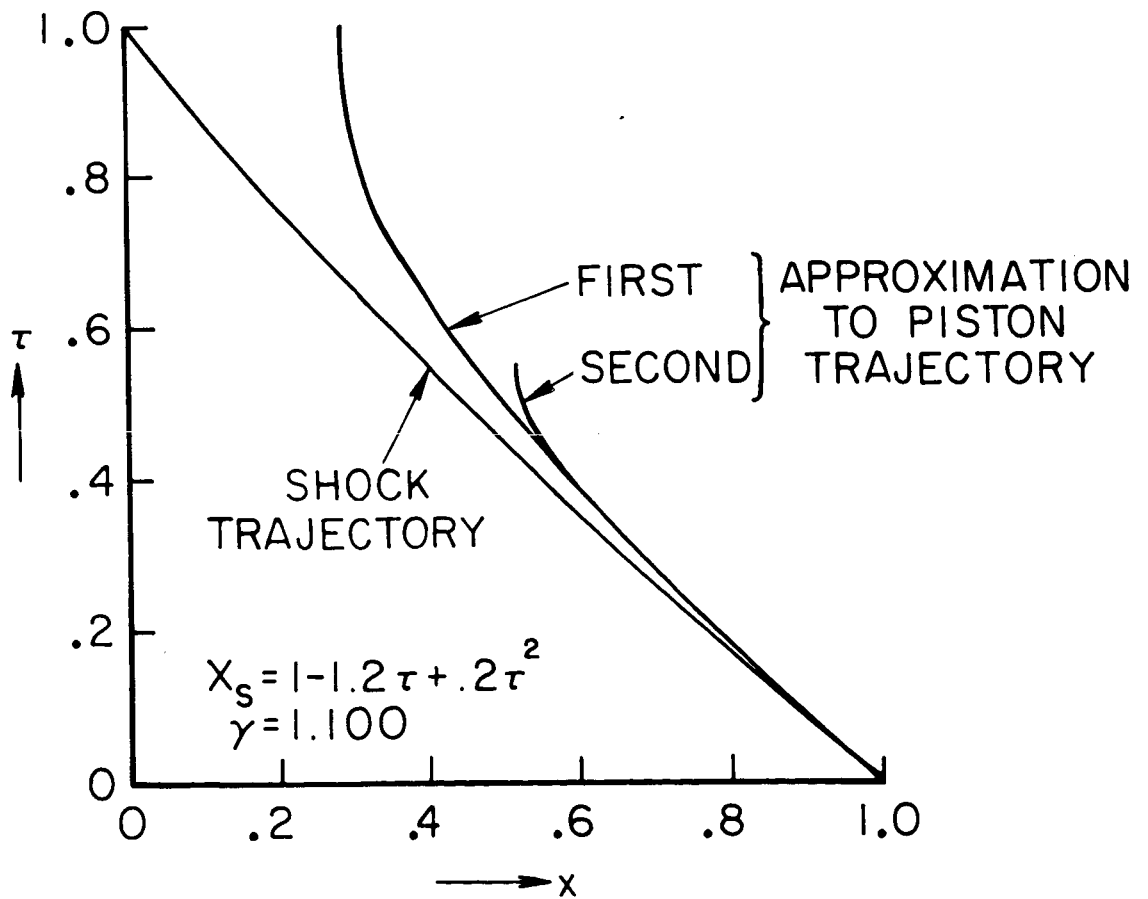
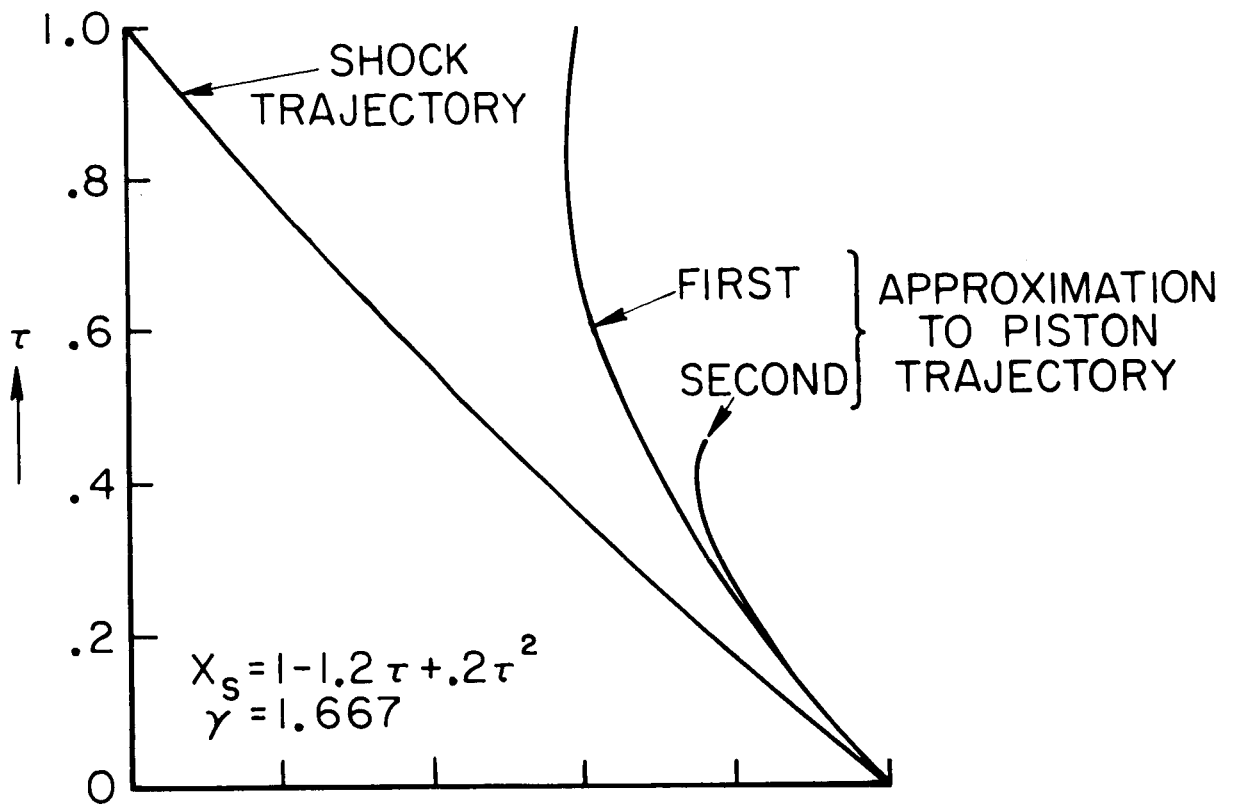
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PISTON TRAJECTORIES FOR ACCELERATING SHOCK

FIGURE 3

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PISTON TRAJECTORIES FOR DECELERATING SHOCK

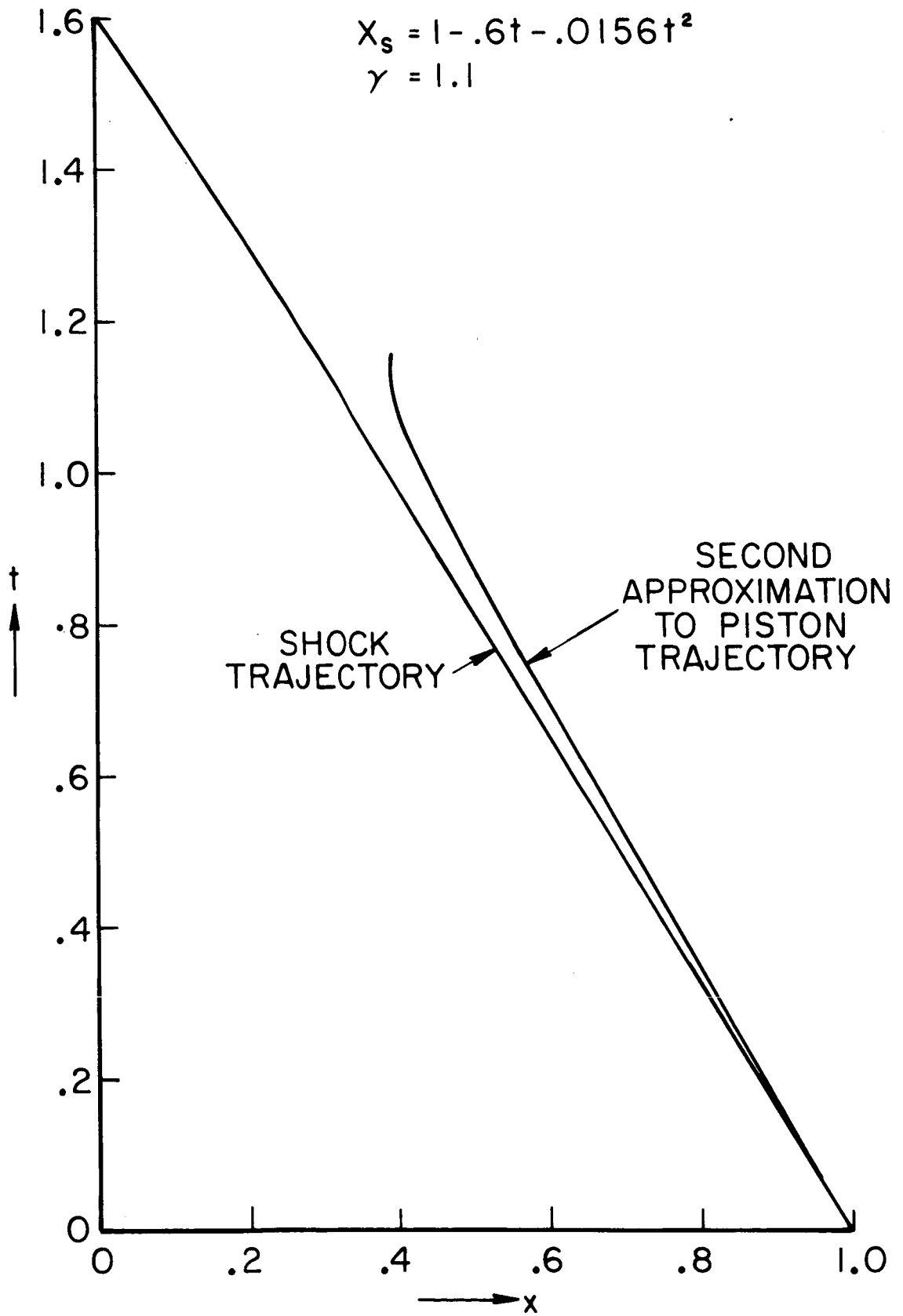
FIGURE 4

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will show the worst qualities of the master equation. As

$\epsilon = \frac{\gamma-1}{\gamma+1}$ becomes larger, due to a larger γ , the first approximation becomes worse in that the entire integral is multiplied by a larger number. Further, the subintegral, which is neglected in the first approximation, is made more significant by the factor $\frac{1}{1-\epsilon}$. In the pinch process, as observed in the laboratory, the Argon experiences first and second ionization, possibly even third; therefore, much of the energy available is absorbed in the ionization process. Hence, the real value of γ that may be expected would be closer to 1.1 than to 5/3.

Figures 2, 3, and 4 are based on a pinch time for the shock of 1.0 microseconds. This gives a convenient basis for comparison. There is no qualitative difference apparent if the time scale is expanded as may be seen from Fig. 5 which shows a pinch time of 1.6 microseconds. Figure 2 shows a constant velocity shock ($B = 0$) with the piston path computed for $\gamma = 1.1$ and 1.667. For $\gamma = 1.667$ it is seen that the first and second approximations for the piston trajectory are very close together until the piston reaches a radial position of 3/4. At this point, the second approximation diverges from the first approximation and ultimately turns back toward its initial position. This occurs as the denominator of the integrand of the master equation approaches zero, beyond which the second approximation is no longer calculated. Physically, a decrease in the denominator corresponds to a decrease in pressure at the piston. It is logical that the pressure decrease from the shock to the piston at a given time because in this region of the flow, there is, effectively, quasi-steady supersonic flow into a converging channel, which implies a decrease in velocity and a corresponding adverse pressure gradient. Since the conditions behind the shock are fixed, the pressure at the piston must be steadily decreasing as the gap between the shock and piston widens.



SECOND APPROXIMATION TO NEARLY LINEAR SHOCK

FIGURE 5

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When the pressure at the piston face reaches zero, the model develops a singularity. Note that for $\gamma = 1.1$ the piston travels slightly more than half way to the axis of the cylinder before turning around.

Figure 3 shows the results for an accelerating shock. For $\gamma = 1.1$ there is no difference large enough to be seen between the first and second approximations until the second approximation reaches the zero pressure limit. This occurs at $x(0, \tau) = .27$. Note that this radius is much smaller than the final radius of the piston for the constant velocity shock. This fact implies that the piston pushing an accelerating shock has control over the shock for a longer time than the piston pushing a constant velocity shock. For $\gamma = 1.667$, the accelerating shock has a piston path given by the second approximation that is closer to the center than the first approximation. This is due to the effect of the subintegral in the master equation. That is, in order to accelerate the flow, the pressure at the piston must be greater than the pressure at the shock. For the first approximation, this pressure difference is neglected. In the second approximation it is included. This effect is also present for the $\gamma = 1.1$ case; however, it is so small that it cannot be seen on the scale of Fig. 3.

Figure 4 shows the case of a decelerating shock. There is little new on this graph except that the piston turns back even sooner than it does for the constant velocity shock.

C. METHOD OF CHARACTERISTICS

It is of interest to determine what the real piston trajectory looks like after the time when the first and second approximations diverge. A third approximation was not attempted because of the inherent limitations on numerical calculations of the needed derivatives. Instead, it was decided to leave the master equation and to attempt to find a solution by the method of characteristics.

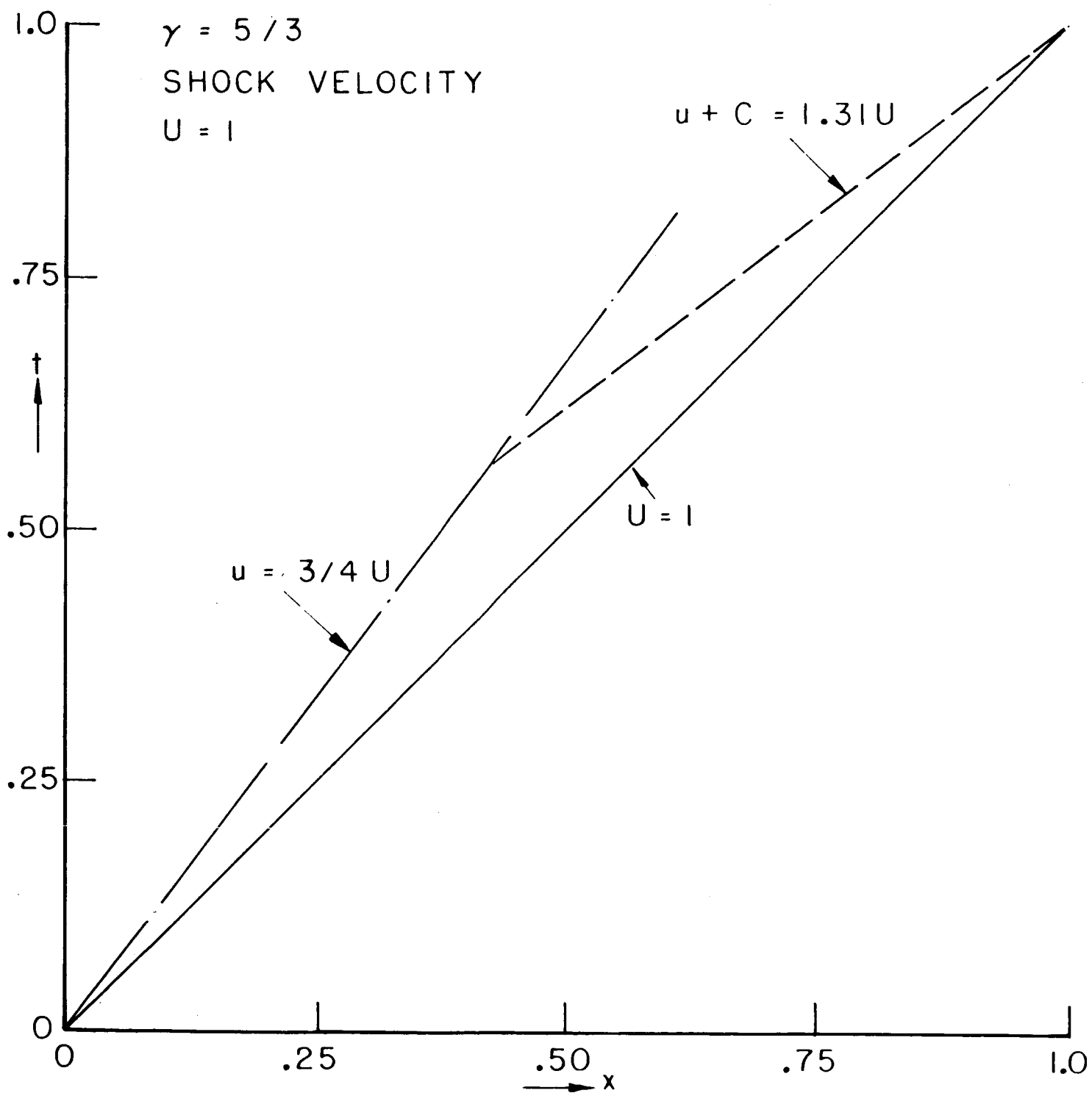
If we make the assumption of isentropic flow behind the shock, we may use the equations derived in other references.^{8,9} To have isentropic flow behind the shock, we must limit ourselves to a linear shock in the r-t plane. This will be adequate to examine the nature of the divergence of the previously found first and second approximations.

It is first instructive to consider the problem of one-dimensional, unsteady, isentropic flow in rectangular coordinates with no area change. For this problem, it is known that the characteristic directions in the t-x plane are $u + c$ and $u - c$ where u is the local flow velocity and c is the local sound speed. If a piston is pushed down a tube in such a manner as to set up a strong shock propagating ahead of it, u and c will remain constant throughout the flow field as long as the piston velocity is constant. From the strong shock relations, u and c are calculated to be

$$u = \frac{2}{\gamma + 1} U \quad c = \frac{\sqrt{2\gamma(\gamma - 1)}}{\gamma + 1} U$$

where U is the velocity of the shock. In this problem, the piston velocity is equal to the flow velocity. For $\gamma = 1.667$, Fig. 6 shows a piston traveling at a speed of $u = 3/4$ pushing a strong shock with speed $U = 1$. As seen from the figure, the last signal that may be sent from the piston and received by the shock before the shock reaches $x = 1$ must leave the piston before the piston is at $x = 1/2$. Thus the piston loses contact with the shock after the piston has propagated half way into the medium. The similar problem posed in cylindrical coordinates becomes slightly more difficult. The equations governing one-dimensional, unsteady, isentropic flow in cylindrical coordinates from the Eulerian viewpoint are written:

$$\begin{array}{l} \text{Mass conservation} \\ \text{or continuity} \end{array} \quad \frac{\partial(\rho u)}{\partial r} + \frac{\partial \rho}{\partial t} + \frac{\rho u}{r} = 0$$



PISTON AND SHOCK TRAJECTORY

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momentum or
Newton's Law $\frac{1}{\rho} \frac{\partial p}{\partial r} + u \frac{\partial u}{\partial r} + \frac{\partial u}{\partial t} = 0$

isentropic
relation $\frac{P}{\rho^\gamma} = \text{constant}$

perfect gas
relation $P = \rho R T$

and sound
speed $c^2 = \frac{dP}{d\rho}$

These equations may be put in a more useful form as shown by Courant and Friedrichs.¹²

Continuity $\rho_t + u \rho_r + \rho u_r + \frac{\rho u}{r} = 0$

Newton's Law $c^2 \rho_r + \rho u u_r + \rho u_t = 0$

(19)

sound speed $c^2 = \text{constant } \rho^{\gamma-1}$

where the subscripts here denote partial differentiation with respect to the subscript. The shock conditions apply across the shock trajectory $R_s = L - At$:

$$u_1 = -\frac{2A}{\gamma+1} \quad \rho_1 = \frac{\rho_\infty}{\epsilon} \quad M_1 = \frac{2}{\sqrt{2\gamma(\gamma-1)}}$$

$$P_1 = \frac{2\rho_\infty A^2}{\gamma+1} \quad c_1 = A \sqrt{\frac{2\gamma\epsilon}{\gamma+1}}$$

giving

$$c^2 = \frac{2A^2 \gamma \epsilon^\gamma}{(\gamma + 1) \rho_\infty^{\gamma-1}} \rho^{\gamma-1}$$

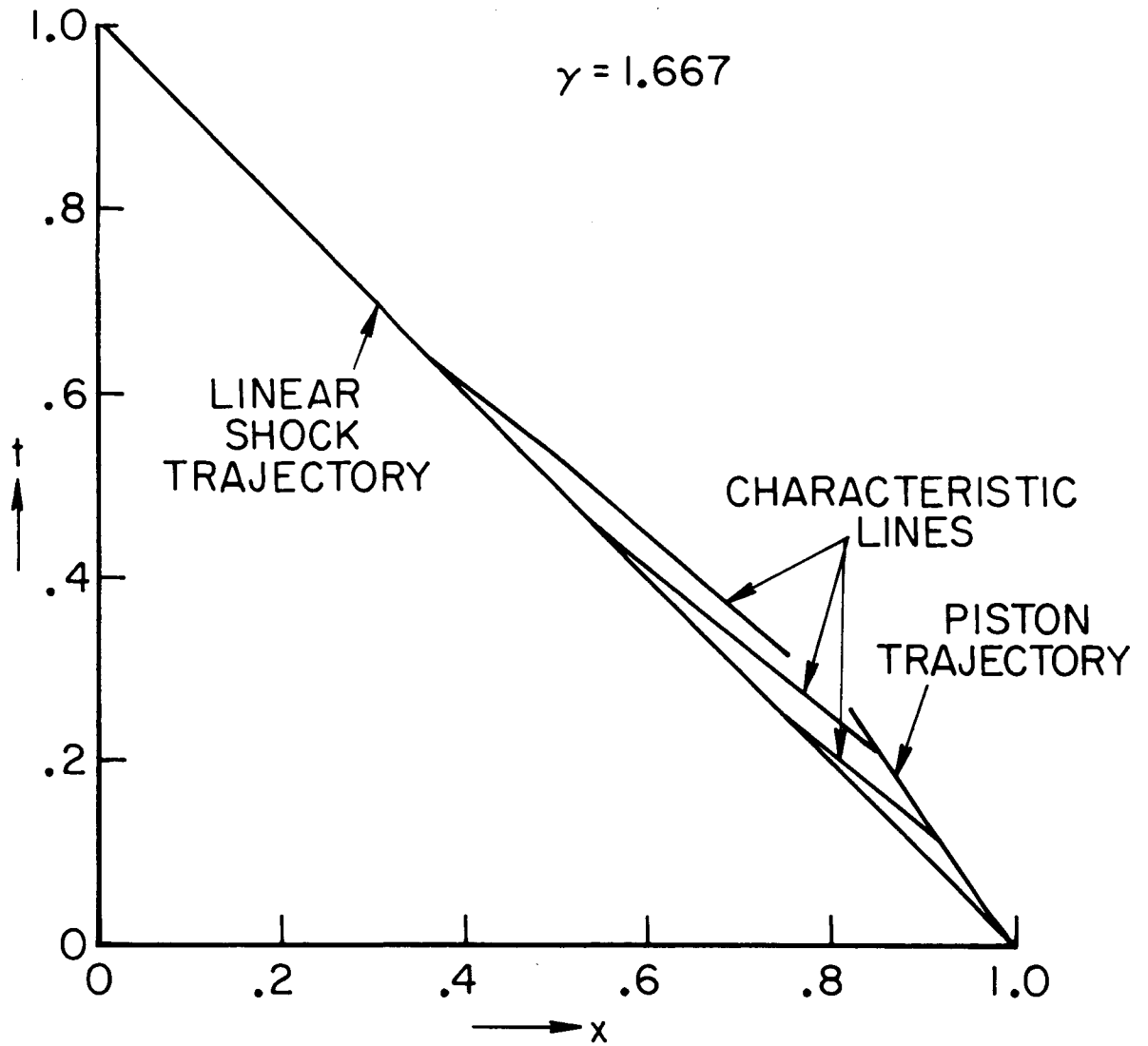
By the normal recipe outlined in Appendix B, we find the characteristic equations to be

$$\begin{aligned} \text{I: } dr &= (u + c) dt & \text{II: } dr &= (u - c) dt \\ \text{I: } du + \frac{2}{\gamma-1} dc + \frac{uc}{r} dt &= 0 & \text{II: } du - \frac{2}{\gamma-1} dc - \frac{uc}{r} dt &= 0 \end{aligned} \quad (20)$$

These equations are exactly the same as those for one-dimensional, isentropic, unsteady flow with constant area, except for the term $\frac{uc}{r} dt$ which adds the only complication to the problem. Once again we must resort to a computer program to find the solution. The program used may be found in Appendix C along with some comments on its structure.

D. RESULTS FROM CHARACTERISTICS

In Fig. 7 are shown the prescribed shock path, the piston trajectory as found from the method of characteristics, and three characteristic lines. The graph is drawn for $\gamma = 1.667$. If we consider the characteristics to run backwards from the shock to the piston, we note that they bend in such a manner as to intersect the piston trajectory much earlier than they would have if they had remained straight. Turning this around so that the characteristics run forward in time, we find that the piston loses control of the shock much sooner in the imploding cylindrical piston problem than it does in the constant-area piston-shock



PISTON TRAJECTORY AND CHARACTERISTIC LINES

FIGURE 7

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problem. Specifically, the piston loses control of the shock by the time the piston is only about $\frac{1}{4}$ of the way toward the center of the cylinder. Recall that this is the same distance that the first and second approximations to the master equation for a linear shock coincided. When the results from the approximate method are compared to the results obtained by the method of characteristics, it is seen that the piston trajectories for the given shock compare exactly. The longest characteristic line shown in the figure intersects the shock only at $x = .35$, because the program did not produce any characteristic lines that extended further. These lines were not produced due to convergence problems for the portion of the cylinder close to the axis. Even the longest characteristic line is cut off before it reaches the shock; this is also due to a fault of the program. However, by estimating the paths of future characteristic lines, it appears as though the fate of the shock is determined by that portion of the piston's trajectory that is very close to the beginning. After this very early control of the shock by the piston, it matters not what path the piston takes. Any message that is sent out from the piston toward the shock after the piston has traveled more than $\frac{1}{4}$ of the way toward the center, will not be received by the shock before the shock reaches the center of the cylinder. This discussion has been for $\gamma = 1.667$. For γ any smaller, the piston would control the shock for a longer period of time, but would behave qualitatively the same.

E. INTERPRETATION

We now know that the diverging portion of the first and second approximations to the master equation solution for the straight shock does not have physical relevance. However, the portion of the two approximations that do coincide is the true solution for the piston driving the prescribed straight shock.

Any answer from this time on must not be considered accurate unless some further boundary condition is prescribed.

When this argument is extended to the accelerating and decelerating shock trajectories, it would seem to imply that where the first and second approximations are very close together, the solution is good for the given boundary condition. Referring to Fig. 2, 3, and 4 it is seen that for a given pinch time and for a given γ , an accelerating shock is controlled more by its piston than is a decelerating shock.

III. SUMMARY

The two major conclusions from this analysis are given some qualified endorsement by experimental observations. First, the small separation between the piston and shock for the low γ cases evident in Figs. 2-5 is not inconsistent with the experimental inability to distinguish a shock front from the intense luminosity of the current sheet on streak and Kerr-Cell photographs. The theory suggests that if such separation is to be observed, it will be most evident in cases of rapidly decelerating fronts. These cases have not been studied intensively in the past because of their low dynamic efficiencies for propulsion purposes.

Second, the prediction that the shock trajectory is determined by the very early portion of the piston path concurs with the observed insensitivity of the pinch processes to the nature of the inner portions of the electrodes. In many experiments, all but the outer inch of electrode has been replaced by glass, or removed entirely, with minimal effect on the development of the pinch pattern.¹

Clearly, the definitive experiments must involve a positive identification of the shock fronts in relation to the current sheets, presumably by sensitive pressure gauges. Such experiments are currently in progress, and their results will soon be compared with this theory.

APPENDIX A

The following discussion concerns the numerical solution of the nondimensional master equation, equation (14), given a parabolic shock trajectory. If the shock has a constant velocity, $B = 0$, then $P = 0$, making $Q = \infty$ in equations (16). Therefore, the equations must be recast for the degenerate case of $B = 0$. Equation (14) becomes

$$x^2(\vartheta, \tau) = x_s^2(\tau) + \varepsilon \int_{\vartheta}^{1 - x_s^2} \left\{ \frac{A^2}{1 - x_s^2} - \frac{1}{2(1-\varepsilon)} \int_{\vartheta}^{\vartheta} \frac{\partial^2 x(\vartheta, \tau)}{\partial \tau^2} \frac{d\vartheta}{x(\vartheta, \tau)} \right\}^{1/2} d\vartheta \quad (14')$$

For the first approximation, neglecting the subintegral, eq. (14') integrates to

$$x_{\text{first}}^2 = x_s^2 + \varepsilon (1 - x_s^2) - \varepsilon \vartheta \quad (15')$$

When x_{first} is substituted back into (14'), x_{second} is found to be

$$x_{\text{second}}^2 = x_s^2 + \varepsilon \int_{\vartheta}^{1 - x_s^2} \left\{ \frac{2\varepsilon}{Q - \ln(F - \vartheta) - \frac{R}{(F - \vartheta)}} \right\}^{1/2} d\vartheta \quad (16')$$

where

$$Q = 1 + \epsilon + \ln \frac{x_s^2}{\epsilon} ; \quad R = \frac{(1-\epsilon)x_s^2}{\epsilon} \quad \text{and} \quad F = R + 1 \quad (17')$$

The actual program used appears at the end of this discussion. The card numbers shown are found at the end of each card printout. Cards numbered from 90 to 930 are for the equations (14), (15), (16), (17), and (18). Cards numbered from 940 to 1440 are for equations (14'), (15'), (16') and (17'). The program was run for $A = .999$ and $B = .001$ and was also run for $A = 1$ and $B = 0$. The answers were the same to three significant figures. The general philosophy of the program is first to determine what time steps to use, and then when the time steps are known, to calculate all quantities that are dependent only upon time for the first $\Delta \tau$. Then, at the time under consideration, the quantities that depend on ϕ are calculated. Specifically, the value of the integrand for $\phi = \phi_s$ is first calculated (the subintegral is zero at this point). Then the values of ϕ for which we want the particle positions spelled out are determined. The program does this by taking ϕ_s , rounding it off to the next lowest .05, and then using $\Delta \phi$ in steps of .05, the steps for which the two numerical integrations are performed. The subintegral is found for the interval from ϕ_s to the next lower ϕ . With this value of the subintegral, the total integral may be found giving x for the rounded off ϕ and the time $\Delta \tau$. Using the next lower value of ϕ , the next portion of the subintegral is added to the value obtained above. Similarly, the next portion of the entire integral is added on to the part already found. This procedure is carried on until we reach the value $\phi = 0$, which is the piston. At this time, we proceed to the

next time and repeat the entire process. Below are some comments on some cards that may not be obvious.

<u>Card Number</u>	<u>Comments</u>
85	If the shock is linear, $B = 0$, control is transferred to statement 200, card #940, to avoid dividing by zero as mentioned above.
130	TMAX is the pinch time of the shock.
140	R is the coefficient of the subintegral.
180	DELTAU is the time step used.
260	$G1 = G'$, $G2 = G''$, etc.
310,320	E and W are the parts of x''/x that depend only upon time.
330	XT is the part of x_{first}^2 that depends only upon time.
340	PHIS = \emptyset_s the value of \emptyset at the shock at the time TAU.
370	HS is the value $H(\emptyset_s)$.
380	GR = x''/x at the shock.
390	GAR is the value of the integrand at the shock.
410	If the difference between \emptyset_s and 0 is small, then the whole integral will be performed in one step.
420	HO is the value $H(0)$.
440	PF is the value of x''/x at $\emptyset = 0$.
460	PAM is the value of the integrand at $\emptyset = 0$.
530	If the difference between \emptyset_s and 0 is large, then the integral will be computed in steps.
540	First the integral is evaluated from \emptyset_s to SIGMA.
650, 660	GR and GAR are replaced by PF and PAM respectively for the next part of the integration.
715	If the denominator of the integrand is zero or negative, control is switched to card #820.

Card
Number

Comments

750

If SIGMA is equal to zero, then the piston position has been calculated for this time, in which case there is a special print out, TAU is increased by DELTAU and the entire integration is performed again. Otherwise, the value XFIRST, XSECOND, and $\emptyset = \text{SIGMA}$ are printed out followed by the next step of the integration.

820

Since the pressure, according to the second approximation, is now zero or negative, card #715, only the first approximation is calculated and printed by card #920.

940-1440

The same procedure is followed except that it is simplified by having a linear shock and no subintegral. The equations for this portion are shown in the initial part of this Appendix.

```

C      GLEN A. ROWELL--APPROXIMATE SOLUTION TO THE MASTER EQUATION
10  FORMAT (3F20.4)
20  FORMAT (3H A=F7.4,4H B=F7.4,8H GAMMA=F6.3,10H EPSILON=F6.4,
1     7H TMAX=F9.6//)
30  FORMAT(53H1  FIRST AND SECOND APPROXIMATION TO THE PISTON PATH//)
40  FORMAT (5H TAU=F7.4, 13H          XS=F6.4, 15H          PHIS=F6.4)
50  FORMAT (3F20.6)
60  FORMAT (2X,7HXFIRST=F8.6,13H          XSECOND=F8.6,10H          TAU=F7.4,
1     6H PHI=F4.2//)
61  FORMAT (1H0///)
65  FORMAT(4H XS=F7.4, 13H          XFIRST=F7.4, 10H          TAU=F8.4/)
18  READ 10,A, B, GAMMA
    PRINT 30
    IF(B) 12, 200, 12
12  S = 1./GAMMA
    P = 4.*B/(A*A+4.*B)
    EPSLON =(1.-S)/(1.+S)
    Q = 2.*EPSLON*GAMMA*(A*A+4.*B)**S/(P*P*(GAMMA+1.))
8   TMAX = (-A+SQRTE(A*A+4.*B))/(2.*B)
    R = 1./(2.*(1.-EPSLON))
    PRINT 20,A, B, GAMMA, EPSLON, TMAX
    NMAX = TMAX*20.
    TAUMAX = NMAX
    DELTAU = TAUMAX*.0025
    TAU = DELTAU
3   XS = 1.-A*TAU-B*TAU**2
    U = A+2.*B*TAU
    V = U*U
    Y = 1. - P*XS
    Z = Y**S
    G = U**(-2.*S)
    G1 = -4.*S*B*G/U
    G2 = 16.*S*B*B*(S+.5)*G/V
    F = P*XS*Z*Y+Z*Y*Y/(S+2.)
    F1 = P*P*(S+1.)*XS*Z*U
    F2 = -P*P*(S+1.)*Z*(V-P*S*XS*V/Y-2.*B*XS)
    E = 2.*V-4.*B*XS+Q*(G*F2 + 2.*G1*F1 +G2*F)
    W = Q*(G*F1+G1*F)-2.*XS*U
    XT = XS**2 + Q*G*F
    PHIS = 1.-XS**2
    PRINT 40, TAU, XS, PHIS
    RS = P*SQRTE(1.-PHIS)
    HS = RS*(1.-RS)**(S+1.)+(1.-RS)**(S+2.)/(S+2.)
    GR = (E-Q*G2*HS)/(2.*XS**2)-(W-Q*G1*HS)**2/(4.*XS**4)
    GAR = ((A*A+4.*B*(1.-XS))/V)**S
    L = 20.*PHIS
    IF(L-1) 70, 70, 80
70  HO = P*(1.-P)**(S+1.)+(1.-P)**(S+2.)/(S+2.)
    XFIRST = SQRTE(XT-Q*G*HO)
    PF = (E - Q*G2*HO)/(2.*XFIRST**2) - (W -Q*G1*HO)**2/(4.*XFIRST**4)
    SUBINT = (PF + GR)/2.*PHIS
    PAM = (A*A/(V-SUBINT*R))**S
    BIGINT = (PAM + GAR)/2.*PHIS
    X = SQRTE(XS*XS + EPSLON*BIGINT)
    PHI = 0.0
    PRINT 60, XFIRST, X, TAU, PHI
    TAU = TAU + DELTAU
    GO TO 3
80  C = L
    SIGMA = C*.05
    RSIGMA = P*SQRTE(1.-SIGMA)

```

```

HSIGMA = RSIGMA*(1.-RSIGMA)**(S+1.)+(1.-RSIGMA)**(S+2.)/(S+2.) 560
XFIRST = SQRTF(XT-Q*G*HSIGMA) 570
PF = (E-Q*G2*HSIGMA)/(2.*XFIRST**2)-(W-Q*G1*HSIGMA)**2 580
1 / (4.*XFIRST**4) 581
SUBINT = (PF + GR)/2.*(PHIS - SIGMA) 590
PAM = ((A*A + 4.*B*(1. - SQRTF(1. - SIGMA)))/(V - R*SUBINT))**S 600
BIGINT = (PAM + GAR)*.5*(PHIS - SIGMA) 610
X = SQRTF(XS*XS + EPSLON*BIGINT) 620
PRINT 50, XFIRST, X, SIGMA 630
SIGMA = SIGMA - .05 640
85 GR = PF 650
GAR = PAM 660
RSIGMA = P*SQRTF(1.-SIGMA) 670
HSIGMA = RSIGMA*(1.-RSIGMA)**(S+1.)+(1.-RSIGMA)**(S+2.)/(S+2.) 680
XFIRST = SQRTF(XT-Q*G*HSIGMA) 690
PF = (E-Q*G2*HSIGMA)/(2.*XFIRST**2)-(W-Q*G1*HSIGMA)**2 700
1 / (4.*XFIRST**4) 701
SUBINT = SUBINT + (PF + GR)/40. 710
IF ( V - R*SUBINT ) 120,120, 87 715
87 PAM = ((A*A + 4.*B*(1. - SQRTF(1. - SIGMA)))/(V - R*SUBINT))**S 720
BIGINT = BIGINT + (PAM + GAR)*.025 730
X = SQRTF(XS*XS + EPSLON*BIGINT) 740
IF (SIGMA) 100, 90, 100 750
100 PRINT 50, XFIRST, X, SIGMA 760
SIGMA = SIGMA - .05 770
GO TO 85 780
90 PRINT 60, XFIRST, X, TAU, SIGMA 790
110 TAU = TAU + DELTAU 800
IF (TAU - TMAX) 3, 18, 18 810
120 PRINT 61 820
H = P*(1.-P)**(S+1.)+(1.-P)**(S+2.)/(S+2.) 830
XFIRST = SQRTF(XS*XS + Q*G*F - Q*G*H) 840
PRINT 65, XS, XFIRST, TAU 850
130 TAU = TAU + DELTAU 860
G = (A + 2.*B*TAU)**(-2.*S) 870
XS = 1. - A*TAU - B*TAU**2 880
F = P*XS*(1. - P*XS)**(1.+S) + (1.-P*XS)**(2.+S)/(2.+S) 890
H = P*(1.-P)**(S+1.)+(1.-P)**(S+2.)/(S+2.) 900
XFIRST = SQRTF(XS*XS + Q*G*F - Q*G*H) 910
PRINT 65, XS, XFIRST, TAU 920
IF (TAU-TMAX) 130, 18, 18 930
200 S = 1./GAMMA 940
EPSLON = (1.-S)/(1.+S) 950
TMAX = 1./A 960
PRINT 20,A, B, GAMMA, EPSLON, TMAX 970
NMAX = 20.*TMAX 980
TAUMAX = NMAX 990
DELTAU = TAUMAX*.0025 1000
TAU = DELTAU 1010
4 XS = 1. - A*TAU 1020
XFIRST = SQRTF ((1. - EPSLON) * XS**2 + EPSLON ) 1030
Q = 1. + EPSLON + LOGF(XS*XS/EPSLON) 1040
R = (1. - EPSLON)*XS*XS/EPSLON 1050
F = R + 1. 1060
PHIS = 1.-XS**2 1070
PRINT 40, TAU, XS, PHIS 1080
GR = (2.*EPSLON/(Q - LOGF(F-PHIS) - R/(F-PHIS)))*S 1090
L = 20.*PHIS 1100
IF(L-1)170,170,180 1110
170 RO = (2.*EPSLON/(Q-LOGF(F) - R/F))*S 1120
BIGINT = (RO+GR)*PHIS/2. 1130

```

X = SQRTF(XS*XS + EPSLON*BIGINT)	1140
PHI = 0.0	1150
PRINT 60, XFIRST, X, TAU, PHI	1160
TAU = TAU + DELTAU	1170
GO TO 4	1180
180 C = L	1190
PHI = C*.05	1200
PF = (2.*EPSLON/(Q - LOGF(F-PHI) - R/(F-PHI)))*S	1210
BIGINT = (PF+GR)*(PHI-PHI)/2.	1220
X = SQRTF(XS*XS + EPSLON*BIGINT)	1230
X1 = SQRTF((1.-EPSLON)*XS**2 + EPSLON*(1.-PHI))	1240
PRINT 50, X1, X, PHI	1250
185 GR = PF	1260
PHI = PHI - .05	1270
Z = Q - LOGF(F-PHI)-R/(F-PHI)	1280
IF(Z) 300, 300, 182	1290
182 PF = (2.*EPSLON/Z)*S	1300
BIGINT = BIGINT + (PF+GR)*.025	1310
X = SQRTF(XS*XS + EPSLON*BIGINT)	1320
IF(PHI) 190,190,1100	1330
1100 X1 = SQRTF((1.-EPSLON)*XS**2 + EPSLON*(1.-PHI))	1340
PRINT 50, X1, X, PHI	1350
GO TO 185	1360
190 PRINT 60, XFIRST, X, TAU, PHI	1370
1110 TAU = TAU + DELTAU	1380
IF (TAU - TMAX) 4, 18, 18	1390
300 X = SQRTF((1. - EPSLON)*XS**2 + EPSLON)	1400
PRINT 65, XS, X, TAU	1410
TAU = TAU + DELTAU	1420
XS = 1. - A*TAU	1430
IF (TAU-TMAX) 300, 18, 18	1440
END	1450

APPENDIX B

The equations for continuity, Newton's law, and sound speed are

$$\rho_t + u \rho_r + \rho u_r + \frac{\rho u}{r} = 0$$

$$c^2 \rho_r + \rho u u_r + \rho u_t = 0$$

$$c^2 = \text{constant } \rho^{\gamma-1}$$

The equation for the speed of sound may be differentiated to yield

$$\frac{2}{\gamma-1} \frac{dc}{c} = \frac{d\rho}{\rho}$$

Substituting this into the two partial differential equations, we find

$$u_t + u u_r + \frac{2c}{\gamma-1} c_r = 0$$

$$c u_r + \frac{2}{\gamma-1} c_t + \frac{2u}{\gamma-1} c_r + \frac{uc}{r} = 0$$

In order to find the characteristic directions for this pair of equations, we form an arbitrary linear combination of them by multiplying the first equation by λ and adding them.^{12, 13}

$$L = \lambda u_t + (\lambda u + c) u_r + \frac{2}{\gamma-1} c_t + \left(\frac{2\lambda c}{\gamma-1} + \frac{2u}{\gamma-1} \right) c_r + \frac{uc}{r} = 0$$

We ask that this linear combination produce directional derivatives of u and c in the same directions. These directions, which depend upon r and t as well as the values of u and c at the point r, t , are the characteristic directions. For example, the derivative of u is taken in the directions

$$\frac{dt}{dr} = \frac{\lambda}{\lambda u + c}$$

If the derivative of c is taken in the same direction, it means that

$$\frac{\lambda}{\lambda u + c} = \frac{\frac{2}{\gamma - 1}}{\frac{2\lambda c}{\gamma - 1} + \frac{2u}{\gamma - 1}} = \frac{dt}{dr}$$

Solving the first equality for λ , we find that

$$\lambda = \pm 1$$

Hence there are two characteristic directions

$$dr = (u + c) dt \quad \text{and} \quad dr = (u - c) dt$$

Substituting $\lambda = 1$ and $u + c = \frac{dr}{dt}$ into L , we find

$$L = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial r} \frac{dr}{dt} + \frac{2}{\gamma - 1} \frac{\partial c}{\partial t} + \frac{2}{\gamma - 1} \frac{\partial c}{\partial r} \frac{dr}{dt} + \frac{uc}{r} = 0$$

or

$$L = \frac{du}{dt} + \frac{2}{\gamma - 1} \frac{dc}{dt} + \frac{uc}{r} = 0$$

This is the characteristic equation that corresponds to the characteristic direction given by $\lambda = 1$. Similarly, for $\lambda = -1$, we find the characteristic equation to be

$$du - \frac{2}{\gamma - 1} dc - \frac{uc}{r} dt = 0.$$

APPENDIX C

The general philosophy of this program is to calculate from the characteristic equations, equations (20), the characteristic directions and the values of the velocity and sound speed on a grid in the r - t plane. One set of grid lines is parallel to the shock trajectory at intervals of DT . The other set is perpendicular to the r axis and is spaced at intervals of DR . The piston position is calculated from the flow velocity of the particle that began at $x = 1$. Note that the flow velocities are negative since the flow direction is in the $-r$ direction. The notation used is shown in the figure below

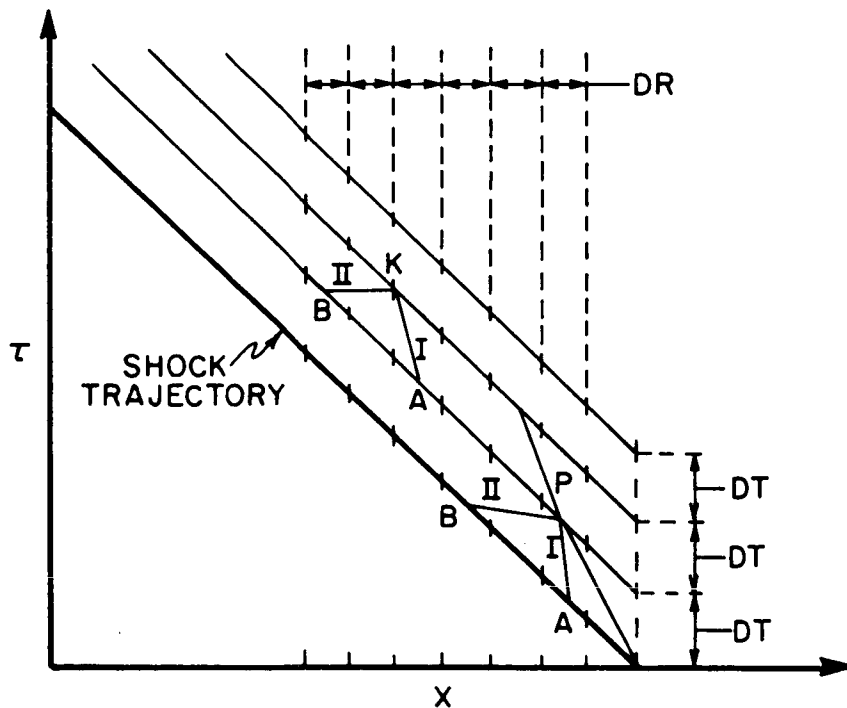


Figure 8

UI refers to the average flow velocity along the characteristic I. That is $UI = \frac{1}{2}(UA + UP)$, where UA refers to the flow velocity at the point A and UP at the point P. Similarly UII is the average flow velocity along a II characteristic. CI, CII, RI,

and RII are the average sound speeds and positions. The quantities RA, UA, CA, TA, RP, ..., RB, ..., RK, ... are the values of the position, flow velocity, sound speed, and time at the points A, P, B, and K. Initially, the piston position is found by assuming $UP = U$, the velocity behind the shock. Having this position, I and II characteristics are found assuming $UI = U - UII$ and $CI = C = CII$ where C is the speed of sound immediately following the shock. With the initial assumption that $DELUI$, ΔC along the I characteristic, is zero, $DELUI$, the change in flow velocity along the I characteristic may be found from the characteristic equations. With this information, $DELUII$ may be found. Then $DELUII$ may be calculated. With all this information, a new UP and CP may be found. Finally, a new average flow velocity of the piston may be found giving a new piston position. Meanwhile, the new values of UI are compared to the old values to determine whether the procedure should be done again. Further, the new values of RP and TP are also compared. When this has been done, the first grid point on the line parallel to the shock trajectory that is closer to the center of the cylinder is determined. For this point, RA, TA, RB, TB, UA, CA, UB, CB, RK, TK, UK, CK, UI, UII, CI, and CII are found and recorded. When this first line parallel to the shock is found, the conditions along the second line are found in much the same manner. However, the conditions at A and B are not so easily found because these points may fall between the previously recorded grid points. Therefore, a simple interpolation is performed.

L is defined in the program and printed out so that we know what leg of the program is being computed. The quantity A is read into the program from the data card, determines the shock pinch time as $1/A$. G represents Gamma.

10	FORMAT(51H1 STRAIGHT SHOCK BY CHARACTERISTICS, GLEN A. ROWELL///)	10
15	FORMAT(5F12.6)	20
17	FORMAT(6HGAMMA=F6.3, 11H DELTA R=F5.3, 11H DELTA T=F6.4,	30
1	5H A=F5.2,12H 1 PART IN F6.0///)	31
20	FORMAT (20H O=F13.6, 14H LEG NO.I2)	40
30	FORMAT (2F20.6)	50
40	FORMAT (9HORPISTON=F7.4, 12H TPISTON=F7.4 , 16H PISTON MACH=	60
1	F 7.4 /)	61
50	FORMAT(3H R(I2,2H)=F6.4,4H T(I2,2H)=F7.4,4H M(I2,2H)=F7.4)	70
60	FORMAT (2F9.4,12H UI+CI=F7.4, 2F9.4,14H UII-CII=F7.4)	80
70	FORMAT (4H RA=F6.4,6H TA=F7.4,10H RB=F6.4,6H TB=F7.4///)	90
80	FORMAT(1H1)	100
5	READ 15, G, DR, DT, A, PCENT	110
	PRINT 10	120
	PRINT 17, G, DR, DT, A, PCENT	130
	EPSLON = (G-1.)/(G + 1.)	140
	U = -2./(G+1.)	150
	C = SQRTF(-U*G*EPSLON)	160
	N = 1./DR	170
	DIMENSION R1(100), U1(200), T1(100), C1(200)	180
	R1(1) = 1. - DR	190
	T1(1) = DR	200
	DO 100 I = 2,N	210
	T1(I)=T1(I-1) + DR	220
100	R1(I)=R1(I-1) -DR	230
	V = U	240
	UI = U	250
	UII = U	260
	CI = C	270
	CII = C	280
	DELICI = 0.	290
	DELICII = 0.	300
	TP = DT/(1. + V)	310
	RP = 1. + V*TP	320
120	DELTI = DT/(1. + UI + CI)	330
	RA = RP - (UI + CI)* DELTI	340
	RI = (RA + RP) * .5	350
130	DELUI = -2. * DELCI/(G-1.) - UI*CI*DELTI/RI	360
	UP = U + DELUI	370
	UIA = (U +UP)*.5	380
	UII = UIA	390
	DELUII = DELUI	400
	DELTII = DT/(1. + UII - CII)	410
	RB = RP - (UII - CII)* DELTII	420
	RII = (RB + RP) * .5	430
	DELICII = (DELUII - UII * CII *DELTII/RII)*(G-1.)/2.	440
	CP = C + DELCII	450
	CII = (CP + C)*.5	460
	CI = (CII+CI)*.5	470
	O = ABSF((UIA - UI)/UIA*PCENT)	480
	L = 1	490
	PRINT 20, O, L	500
	DELICI =(DELICII+DELICI)*.5	510
	UI = (UIA + UI)*.5	520
	IF(O-1.) 150, 120, 120	530
150	V = (UP + U)*.5	540
	TG = DT/(1. + V)	550
	RG = 1. + V*TG	560
	P = ABSF((TG - TP)/TG * PCENT)	570
	Q = ABSF((RG - RP)/RG * PCENT)	580
	PRINT 30, P, Q	590

	RP = RG	600
	TP = TG	610
	IF (P+Q-1.) 160, 160, 120	620
160	TPISTN = TP*A	630
	GMPSTN =-UP/CP	640
	W = UI/A	650
	X = CI/A	660
	Y = UII/A	670
	Z = CII/A	680
	E = DELTI*A	690
	F = DELTII*A	700
	B = W + X	710
	D = Y - Z	720
	TA = TPISTN - E	730
	TB = TPISTN - F	740
	PRINT 40, RP, TPISTN, GMPSTN	750
	PRINT 60, W, X, B, Y, Z, D	760
	PRINT 70, RA, TA, RB, TB	770
	K = (1. - RP)/DR +1.	780
	R =R1(K)	790
	T1(K)=T1(K) + DT	800
	T =T1(K)	810
140	DELTI = DT/(1. + UI + CI)	820
	RA = R - (UI + CI)* DELTI	830
	RI = (RA + R) * .5	840
170	DELUI = -2. * DELCI/(G-1.) - UI*CI*DELTI/RI	850
	UK = U + DELUI	860
	UII = (U + UK)*.5	870
	DELTII = DT/(1. + UII - CII)	880
	RB = R - (UII - CII)* DELTII	890
	RII = (RB + R) * .5	900
	DELUII = DELUI	910
	DELSCII = (DELUII - UII * CII *DELTII/RII)*(G-1.)/2.	920
	DELCCI =(DELSCII+DELCCI)*.5	930
	CK = C + DELSCII	940
	CII = (CK + C) *.5	950
	CI = (CI+CII)*.5	960
	UIA = (U +UK)*.5	970
	O = ABSF((UIA - UI)/UIA*PCENT)	980
	L = 2	990
	PRINT 20, O, L	1000
	UI =(UIA+UI)*.5	1010
	IF (O - 100.) 175, 190, 190	1020
175	IF(O-1.) 180, 140, 140	1030
180	U1(K)= UK	1040
	C1(K)= CK	1050
	GMACH =-UK/CK	1060
	TR = T*A	1070
	W = UI/A	1080
	X = CI/A	1090
	Y = UII/A	1100
	Z = CII/A	1110
	E = DELTI*A	1120
	F = DELTII*A	1130
	B = W + X	1140
	D = Y - Z	1150
	TA = TR - E	1160
	TB = TR - F	1170
	PRINT 50, K, R, K, TR, K, GMACH	1180
	PRINT 60, W, X, B, Y, Z, D	1190
	PRINT 70, RA, TA, RB, TB	1200

	K = K+1	1210
	R = R1(K)	1220
	T1(K)=T1(K) + DT	1230
	T = T1(K)	1240
	IF (T-1.) 140, 190, 190	1250
190	TPAM = TP	1260
	RPAM = RP	1270
	UPAM = UP	1280
	CPAM = CP	1290
	PRINT 80	1300
	KMAX = K - 2	1310
	V = UP	1320
	TP = TP + DT/(1.+V)	1330
	IF (1. - TP) 5, 5, 195	1340
195	RP= RP+V* DT/(1.+V)	1350
	UI = UP	1360
	UII = UP	1370
	CI = CP	1380
	CII = CP	1390
	DELICI = 0.	1400
200	DELTI = DT/(1. + UI + CI)	1410
	RA = RP - (UI + CI)* DELTI	1420
	RI = (RA + RP) * .5	1430
	IA = (1.-RA)/DR	1440
	CA = (RA -R1(IA))*(C1(IA)-C1(IA + 1))/DR +C1(IA)	1450
	UA = (RA -R1(IA))*(U1(IA)-U1(IA + 1))/DR +U1(IA)	1460
	UI = (UA + UP)*.5	1470
	CI = (CA+CP)*.5	1480
	DELUI = -2. * DELCI/(G-1.) - UI*CI*DELTI/RI	1490
	UP = UA + DELUI	1500
	DELTII = DT/(1. + UII - CII)	1510
	RB = RP - (UII - CII)* DELTII	1520
	RII = (RB + RP) * .5	1530
	IB = (1.-RB)/DR	1540
	IF (KMAX-IB) 5, 205, 205	1550
205	CB = (RB -R1(IB))*(C1(IB)-C1(IB + 1))/DR +C1(IB)	1560
	UB = (RB -R1(IB))*(U1(IB)-U1(IB + 1))/DR +U1(IB)	1570
	UII = (UB + UP)*.5	1580
	CII= (CB+CP)*.5	1590
	DELUII= UP - UB	1600
	DELICII = (DELUII - UII * CII *DELTII/RII)*(G-1.)/2.	1610
	CP = CB + DELCII	1620
	CII = (CP + CB)*.5	1630
	DELICI = (CP-CA+DELICI)*.5	1640
	CI = ((CP + CA) * .5+CI)*.5	1650
	UIA = (UA+UP)*.5	1660
	O = ABSF((UIA - UI)/UIA*PCENT)	1670
	L = 3	1680
	PRINT 20, O, L	1690
	UI = (UIA+UI)*.5	1700
	IF(O-1.) 210, 200, 200	1710
210	V = (UPAM+UP)*.5	1720
	TG = DT/(1. + V) + TPAM	1730
	RG = RPAM + V*DT/(1.+V)	1740
	P = ABSF((TG - TP)/TG * PCENT)	1750
	Q = ABSF((RG - RP)/RG * PCENT)	1760
	PRINT 30, P, Q	1770
	RP = RG	1780
	TP = TG	1790
	IF (P+Q-1.) 220, 220, 200	1800
220	TPISTN = TP*A	1810

	GMPSTN = -UP/CP	1820
	W = UI/A	1830
	X = CI/A	1840
	Y = UII/A	1850
	Z = CII/A	1860
	E = DELTI*A	1870
	F = DELTII*A	1880
	B = W + X	1890
	D = Y - Z	1900
	TA = TPISTN - E	1910
	TB = TPISTN - F	1920
	PRINT 40, RP, TPISTN, GMPSTN	1930
	PRINT 60, W, X, B, Y, Z, D	1940
	PRINT 70, RA, TA, RB, TB	1950
	K = (1. - RP)/DR + 1.	1960
	KKEEP = K	1970
	R = R1(K)	1980
	T1(K) = T1(K) + DT	1990
	T = T1(K)	2000
	UK = UP	2010
	CK = CP	2020
230	DELTI = DT/(1. + UI + CI)	2030
	RA = R - (UI + CI) * DELTI	2040
	RI = (RA + R) * .5	2050
	IA = (1. - RA)/DR	2060
	CA = (RA - R1(IA)) * (C1(IA) - C1(IA + 1))/DR + C1(IA)	2070
	UA = (RA - R1(IA)) * (U1(IA) - U1(IA + 1))/DR + U1(IA)	2080
	UI = (UA + UK) * .5	2090
	CI = (CA + CK) * .5	2100
	DELUI = -2. * DELCI/(G-1.) - UI * CI * DELTI/RI	2110
	UK = UA + DELUI	2120
	DELTII = DT/(1. + UII - CII)	2130
	RB = R - (UII - CII) * DELTII	2140
	RII = (RB + R) * .5	2150
	IB = (1. - RB)/DR	2160
	IF (IB - KMAX) 233, 233, 250	2170
233	CB = (RB - R1(IB)) * (C1(IB) - C1(IB + 1))/DR + C1(IB)	2180
	UB = (RB - R1(IB)) * (U1(IB) - U1(IB + 1))/DR + U1(IB)	2190
	UII = (UB + UK) * .5	2200
	CII = (CB + CK) * .5	2210
	DELUII = UK - UB	2220
	DELSCII = (DELUII - UII * CII * DELTII/RII) * (G-1.)/2.	2230
	CK = CB + DELSCII	2240
	DELCI = (CK - CA + DELCI) * .5	2250
	CI = ((CK + CA) * .5 + CI) * .5	2260
	UIA = (UA + UK) * .5	2270
	O = ABSF((UIA - UI)/UIA * PCENT)	2280
	L = 4	2290
	PRINT 20, O, L	2300
	UI = (UIA + UI) * .5	2310
	IF (O - 100.) 235, 250, 250	2320
235	IF (O - 1.) 240, 230, 230	2330
240	J = N + K	2340
	U1(J) = UK	2350
	GMACH = -UK/CK	2360
	TR = T * A	2370
	W = UI/A	2380
	X = CI/A	2390
	Y = UII/A	2400
	Z = CII/A	2410
	E = DELTI * A	2420

	F = DELTII*A	2430
	B = W + X	2440
	D = Y - Z	2450
	TA = TR - E	2460
	TB = TR - F	2470
	PRINT 50, K, R, K, TR, K, GMACH	2480
	PRINT 60, W, X, B, Y, Z, D	2490
	PRINT 70, RA, TA, RB, TB	2500
	C1(J)= CK	2510
	K = K+1	2520
	R = R1(K)	2530
	T1(K)=T1(K) + DT	2540
	T =T1(K)	2550
	IF (T-1.) 230, 250, 250	2560
250	K = K - 1	2570
	KMAX = K	2580
	DO 260 I = KKEEP, K	2590
	II = I + N	2600
	U1(I)=U1(II)	2610
260	C1(I)=C1(II)	2620
	GO TO 190	2630
	END	2640

REFERENCES

1. Jahn, R. G. and von Jaskowsky, W.: "Structure of a Large-Radius Pinch Discharge," AIAA J. 1, 1809 (August 1963).
2. Jahn, R. G. and von Jaskowsky, W.: "Current Distributions in Large-Radius Pinch Discharges," AIAA J. 2, 1749 (October 1964).
3. Fourth Semi-Annual Progress Report for the period 1 January 1964 to 30 June 1964, NASA Research Grant NsG-306-63, Department of Aerospace and Mechanical Sciences Report No. 634c, Princeton University, Princeton, New Jersey.
4. Fifth Semi-Annual Progress Report for the period 1 July 1964 to 31 December 1964, NASA Research Grant NsG-306-63, Department of Aerospace and Mechanical Sciences Report No. 634d, Princeton University, Princeton, New Jersey.
5. Jahn, R. G., von Jaskowsky, W., and Casini, A. L.: "Gas-Triggered Inverse Pinch Switch," Rev. Sci. Instr. 34, 1439 (December 1963).
6. Corr, J. M.: Double Probe Studies in an 8" Pinch Discharge," M.S.E. Thesis, Department of Aerospace and Mechanical Sciences, Princeton University, Princeton, New Jersey, September 1964.
7. Wright, E. S.: "The Design and Development of Rogowski Coil Probes for Measurement of Current Density Distribution in a Plasma Pinch," M.S.E. Thesis, Department of Aerospace and Mechanical Sciences, Princeton University, Princeton, New Jersey, May 1965.
8. Chernyi, G. G.: "Gas Flow Around a Body at Hypersonic Speeds," Dokl. Akad. Nauk. SSSR 107, 221-224 (1956). M. D. Friedman Transl. No. C-106.
9. Chernyi, G. G.: "Adiabatic Motion of a Perfect Gas With Very Strong Shock Waves," Trudy III Vsesoyuznovo Matem. Sédza (Proc. of Third All-Union Mathematical Congress), Vol. 1, Izdat. Akad. Nauk USSR, Moscow, 1956.

REFERENCES-contd.

10. Chernyi, G. G.: "One-dimensional Unsteady Motion of a Perfect Gas With a Strong Shock Wave," Dokl. Akad. Nauk. SSSR 107, 657-660 (1956), M. D. Friedman Transl. No. C-107.
11. Freeman, N. C.: "On the Theory of Hypersonic Flow Past Plane and Axially Symmetric Bluff Bodies," J. Fluid Mech. 1, 366-386 (1956).
12. Courant and Friedrichs: Supersonic Flow and Shock Waves, Interscience Publishers, New York, 1948.
13. Lister, Mary: "The Numerical Solutions of Hyperbolic Partial Differential Equations by the Method of Characteristics," Chap. 15 of Mathematical Methods for Digital Computers, Edited by Ralston and Wilf, John Wiley & Sons, Inc., 1960.