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FREE VIBRATIONS OF THIN

ISOTROPIC OBLATE SPHEROIDAL SHELLS

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FOREWORD

This report presents the solution to the problem of free vibration of a thin isotropic oblate spheroidal shell, which is the idealized case for the empty Centaur liquid oxygen tank. It is then a first step in the direction of deriving more exact vibration modes for use in the Centaur thrust buildup problem. This study was conducted, in part, under the provisions of contract NAS3-3232.

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SUMMARY

As a preliminary step toward formulating and solving the sloshing problem in the Centaur Liquid Oxygen (LO_2) tank, this report presents a theoretical study of the free-vibration characteristics of such an elastic tank. Since the LO_2 tank has essentially the shape of an oblate spheroidal shell, this study develops equations and obtains numerical values for the frequencies and mode shapes arising in a shell of this shape.

The vibration problem is solved by Galerkin's method. In deriving the differential equations of motion, membrane theory and harmonic axisymmetric motion are assumed. The equations of motion lead to two ordinary differential equations with variable coefficients. These two equations can be reduced to one ordinary second-order differential equation with variable coefficients. This eigenvalue problem is solved by Galerkin's method. It is shown that, as the eccentricity of the oblate spheroid goes to zero, Galerkin's solution for the oblate spheroid yields the exact solution for the sphere. It is also shown that two sets of frequencies exist for the oblate spheroidal shell. After the frequencies are determined, the tangential displacements are obtained as solutions of a homogeneous system of equations.

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SECTION I

INTRODUCTION

1.1 GENERAL

The Centaur Liquid Oxygen (LO_2) tank presents a sloshing problem which is difficult to analyze. An introductory approach to the problem involves consideration of free vibrations in a thin oblate spheroidal shell approximating the shape of the LO_2 tank. As the first step toward resolving the sloshing problem for an elastic tank, the solution of this theoretical problem has application not only for the Centaur LO_2 tank, but also for Saturn upper-stage tanks and as a basis for further studies of thin orthotropic tank shells which will, in turn, have application in analyzing vehicle tanks having stiffeners.

1.2 PREVIOUS LITERATURE

While the literature on free and forced vibrations of spherical shells is almost endless, only two papers exist (according to the author's knowledge) dealing with the free vibration of ellipsoidal shells; and numerical results are lacking. Due to the complexity of the problem, it appears that a closed-form analytical solution does not exist.

One of these papers was written by Chintsun Hwang (Reference 1-1), who proposed that the problem be solved by a numerical integration of the differential equations. Hwang applied membrane theory to solve the problem of the free vibration of a thin hemiellipsoidal shell, assuming a free-edge condition. However, the validity of his analysis is subject to question, for, as W.H. Hoppmann, II, has noted (Reference 1-2), membrane theory may not be adequate to describe the behavior of spherical segments. The second numerical method, proposed by N. Shiraishi and F. L. Di Maggio (Reference 1-3), developed a perturbation method to solve the problem of axisymmetric vibration of prolate spheroidal shells.

There is a paper by Love (Reference 1-4, 1888) on the free vibrations of isotropic elastic shells. In particular, for a spherical shell, he deduced that the characteristic shapes are expressible as associated Legendre functions; and he discussed the frequency equation. W. H. Hoppmann, II (Reference 1-5, 1961), has discussed both the free and forced vibrations of a thin elastic orthotropic spherical shell - the general case of Love's spherical shell problem.

W. E. Baker (Reference 1-6) has conducted experimental studies of actual spherical shells, by explosive loading and use of a vibration exciter. His experimental results demonstrated the physical existence of the two sets of frequencies, first predicted by H. Lamb (Reference 1-7).

SECTION II

EQUATIONS OF MOTION

2.1 DERIVATION OF DIFFERENTIAL EQUATIONS OF MOTION

2.1.1 BASIC ASSUMPTIONS. Let us assume that a thin isotropic oblate spheroidal shell is subjected to membrane-type stresses, i.e., that its deformation is momentless. Let us also assume that its free vibrations are symmetric and extensional. If this is the case, the torsional displacement can be neglected, i.e., $u \equiv 0$. Figure 2-1 illustrates an element of such a shell showing the normal displacement, w, and the tangential displacement, v.

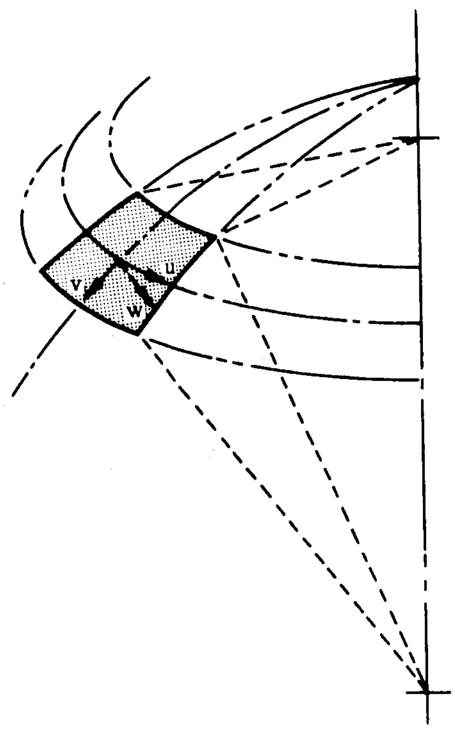

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Figure 2-1. Displacements of the Oblate Spheroidal Shell

The present study is conducted in terms of a curvilinear coordinate system. Within this system, the principal curvatures of the shell surface are defined by the radii, R_1, and R_2, which are functions of only one variable: the inclination, φ, of the normal vector in space (see Figure 2-2). A derivation of these two parameters is given in Appendix A (see Subsection A.4), where they are shown to be the result of a transformation from the oblate spheroidal coordinate system. Figure 2-2 also shows the horizontal angle increment, $d\theta$. The orthogonal vectors, X, Y, and Z, represent the components of force; and the vectors, N_φ and N_θ, are the forces per unit length of the section, corresponding to the stresses, σ_φ and σ_θ.

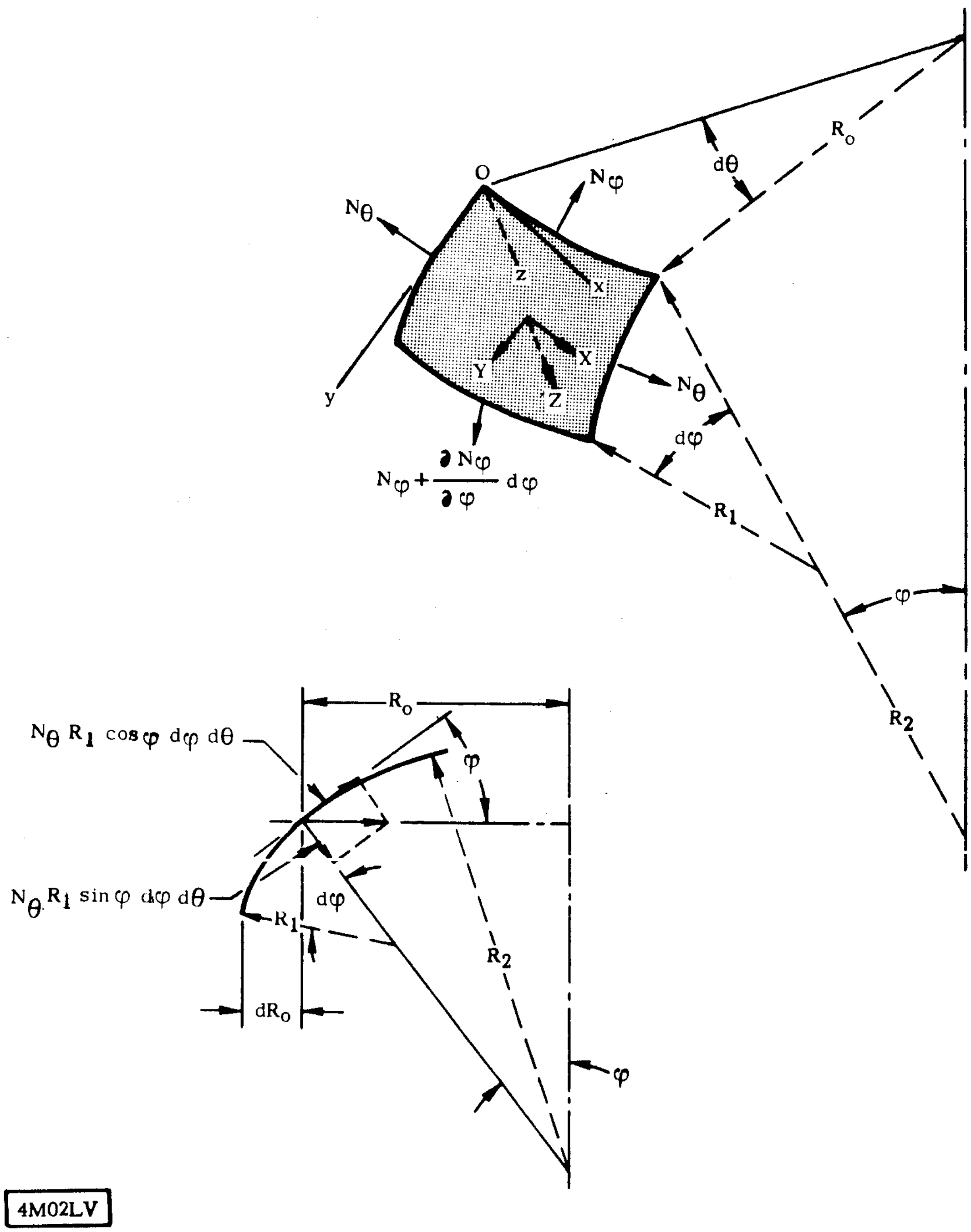

Figure 2-2. Elements of the Oblate Spheroidal Surface

2.1.2 DIFFERENTIAL EQUATIONS OF MOTION. Using Timoshenko's assumptions, the two differential equations of motion are then

$$R_o \frac{\partial N_\varphi}{\partial \varphi} + N_\varphi \frac{\partial R_o}{\partial \varphi} - N_\theta R_1 \cos\varphi = \rho R_o R_1 h \frac{\partial^2 v}{\partial t^2} \tag{2.1}$$

and

$$N_\varphi R_o + N_\theta R_1 \sin\varphi = \rho R_o R_1 h \frac{\partial^2 w}{\partial t^2} \tag{2.2}$$

(see Reference 2-1, pages 434 and 435),

where

ρ = the specific mass of the shell

h = the thickness of the shell

v = tangential displacement, positive toward south pole

w = normal displacement, positive toward center

and

$R_o = R_2 \sin\varphi$

The strain, expressed in terms of displacements, can be written

$$\epsilon_\varphi = \frac{1}{R_1}\left(\frac{\partial v}{\partial \varphi} - w\right) \tag{2.3}$$

$$\epsilon_\theta = \frac{1}{R_2}(v \cot\varphi - w) \tag{2.4}$$

(see Reference 2-1, page 446).

If E is the shell's modulus of elasticity and ν is Poisson's ratio, the forces per unit length of the section are then

$$N_\varphi = \frac{Eh}{(1-\nu^2)}(\epsilon_\varphi + \nu\epsilon_\theta) \tag{2.5}$$

$$N_\theta = \frac{Eh}{(1-\nu^2)}(\epsilon_\theta + \nu\epsilon_\varphi) \tag{2.6}$$

Substituting Equations A.19 (from Appendix A), 2.3, and 2.4 into Equations 2.5 and 2.6 yields

$$N_\varphi = \frac{Eh}{(1-\nu^2)} \cdot \frac{\sqrt{1 - e^2\cos^2\varphi}}{a}\left\{\frac{(1 - e^2\cos^2\varphi)}{(1-e^2)} \cdot \frac{\partial v}{\partial \varphi} + \nu(\cot\varphi)v - \left[\frac{(1 - e^2\cos^2\varphi)}{(1-e^2)} + \nu\right] w\right\} \tag{2.7}$$

and similarly

$$N_\theta = \frac{Eh}{(1-\nu^2)} \cdot \frac{\sqrt{1-e^2\cos^2\varphi}}{a} \left\{ \nu \frac{(1-e^2\cos^2\varphi)}{(1-e^2)} \frac{\partial v}{\partial \varphi} + (\cot\varphi) v - \left[1+\nu \frac{(1-e^2\cos^2\varphi)}{(1-e^2)} \right] w \right\} \tag{2.8}$$

where the radius, a, is as shown in Figure A-1.

Assuming separation of variables, the displacements, v and w, can be expressed as

$$v = v_n(\varphi) \cdot T_n(t) \tag{2.9}$$

and

$$w = w_n(\varphi) \cdot T_n(t) \tag{2.10}$$

where the subscripts designate the frequency mode. Assuming harmonic motion,

$$\frac{d^2T_n}{dt^2} + p_n^2 T_n = 0 \tag{2.11}$$

where p_n is the angular frequency of the shell.

Substituting Equations 2.7 through 2.11 into Equation 2.1 and 2.2 and substituting

$$x = \cos\varphi \tag{2.12}$$

yields the differential equations of motion in the following form:

$$(1-x^2)^2 (1-e^2x^2)^3 \frac{d^2v_n}{dx^2} - 2x (1-x^2)(1-e^2x^2)^2 \left[1+(1-2x^2)e^2\right] \frac{dv_n}{dx}$$

$$- (1-e^2)(1-e^2x^2) \left\{\nu + (1-\nu)x^2 - \left[(1+\nu)x^2 - \nu x^4\right] e^2\right\} v_n$$

$$+ (1-x^2)^{3/2} (1-e^2x^2)^2 \left[(1+\nu) - (\nu+x^2)e^2\right] \frac{dw_n}{dx}$$

$$- x (1-x^2)^{1/2} (1-e^2x^2) \left[4(1-x^2) - (2x^2 - 3x^4 + 1)e^2\right] e^2 w_n$$

$$= -A (1-e^2)^2 (1-x^2) p_n^2 v_n \qquad (2.13)^*$$

*See footnote page 2-5.

and

$$- (1-e^2x^2)^2\left[\nu(1-e^2) + (1-e^2x^2)\right](1-x^2)\frac{dv_n}{dx}$$

$$+ (1-e^2x^2)\ (1-e^2)x\left[\nu(1-e^2x^2) + (1-e^2)\right]v_n$$

$$- (1-e^2x^2)\ (1-x^2)^{1/2}\left[2\nu\,(1-e^2x^2)\ (1-e^2) + (1-e^2x^2)^2 + (1-e^2)^2\right]w_n$$

$$= -A\,(1-e^2)^2\,(1-x^2)^{1/2}\,p_n^2\,w_n \qquad (2.14)^*$$

where

$$A = \rho\,\frac{(1-\nu^2)a^2}{E} = (1+\nu)\,B \qquad (2.15)$$

2.2 REDUCTION TO ONE SECOND-ORDER DIFFERENTIAL EQUATION

The last two ordinary differential equations (Equations 2.13 and 2.14) can be reduced to a single second-order differential equation with variable coefficients in the following manner.

2.2.1 NORMAL AND TANGENTIAL DISPLACEMENTS. Let us first obtain, from Equation 2.14, the normal displacement, w_n, expressed as

$$w_n = \frac{-\alpha(x)\dfrac{dv_n}{dx} + \beta(x)\,v_n}{\gamma(x)} \qquad (2.16)$$

where

$$\alpha(x) = (1-e^2x^2)^2\left[\nu(1-e^2) + (1-e^2x^2)\right](1-x^2) \qquad (2.17)$$

$$\beta(x) = (1-e^2x^2)\ (1-e^2)x\left[\nu\,(1-e^2x^2) + (1-e^2)\right] \qquad (2.18)$$

$$\gamma(x) = (1-x^2)^{1/2}\left\{(1-e^2x^2)\left[2\nu(1-e^2x^2)\ (1-e^2) + (1-e^2x^2)^2\right.\right.$$

$$\left.\left. + (1-e^2)^2\right] - A\,(1-e^2)^2p_n^{\,2}\right\} \qquad (2.19)$$

---

*Lengthy algebraic manipulations are necessary to derive Equations 2.13 and 2.14. These algebraic calculations were checked on an IBM 7094 digital computer using FORMAC (Formula Manipulation Compiler).

The differential equation of the tangential displacement, v_n, can now be derived by substitution of Equation 2.16 into Equation 2.13:

$$
\begin{aligned}
&(1-x^2)^2(1-e^2x^2)^3\frac{d^2v_n}{dx^2} - 2x(1-x^2)(1-e^2x^2)^2\left[1+(1-2x^2)e^2\right]\frac{dv_n}{dx} \\
&- (1-e^2)(1-e^2x^2)\left\{\nu+(1-\nu)x^2 - \left[(1+\nu)x^2-\nu x^4\right]e^2\right\}v_n \\
&+ (1-x^2)^{3/2}(1-e^2x^2)^2\left[1+\nu-(\nu+x^2)e^2\right]\frac{1}{\gamma^2}\Bigg\{-\alpha\gamma\frac{d^2v_n}{dx^2} \\
&+ \left[\gamma(\beta-\alpha')+\alpha\gamma'\right]\frac{dv_n}{dx} + (\beta'\gamma-\beta\gamma')v_n\Bigg\} \\
&- e^2x(1-x^2)^{1/2}(1-e^2x^2)\left[4(1-x^2)-(2x^2-3x^4+1)e^2\right]\cdot\left[\frac{-\alpha\frac{dv_n}{dx}+\beta v_n}{\gamma}\right] \\
&= -A(1-e^2)^2(1-x^2)p_n^2v_n \qquad (2.20)
\end{aligned}
$$

where α', β', and γ' are the first derivatives of the respective functions (see Equations 2.22, 2.23, and 2.24).

2.2.2 THE SECOND-ORDER DIFFERENTIAL EQUATION IN FINAL FORM. Equation 2.20 can be expressed in the following form:

$$
\begin{aligned}
&\left\{(1-x^2)^2(1-e^2x^2)^3\gamma^2 - (1-x^2)^{3/2}(1-e^2x^2)^2\left[(1+\nu)-(\nu+x^2)e^2\right]\alpha\gamma\right\}\frac{d^2v_n}{dx^2} \\
&- \Bigg\{2x(1-x^2)(1-e^2x^2)^2\left[1+(1-2x^2)e^2\right]\gamma^2 \\
&- (1-x^2)^{3/2}(1-e^2x^2)^2\left[(1+\nu)-(\nu+x^2)e^2\right]\left[\gamma(\beta-\alpha')+\alpha\gamma'\right] \\
&- x(1-x^2)^{1/2}(1-e^2x^2)\left[4(1-x^2)-(2x^2-3x^4+1)e^2\right]e^2\alpha\gamma\Bigg\}\frac{dv_n}{dx} \\
&+ \Bigg\{A(1-e^2)^2p_n^2(1-x^2)\gamma^2 + (1-x^2)^{3/2}(1-e^2x^2)^2\left[(1+\nu)-(\nu+x^2)e^2\right](\beta'\gamma-\beta\gamma') \\
&- (1-e^2)(1-e^2x^2)\left[\nu+(1-\nu)x^2-\{(1+\nu)x^2-\nu x^4\}e^2\right]\gamma^2 \\
&- e^2x(1-x^2)^{1/2}(1-e^2x^2)\left[4(1-x^2)-(2x^2-3x^4+1)e^2\right]\beta\gamma\Bigg\}v_n = 0 \qquad (2.21)
\end{aligned}
$$

where α, β, and γ are specified by Equations 2.17, 2.18, and 2.19 and their first derivatives by

$$\alpha' = \frac{d\alpha}{dx} = -2x\,(1-e^2x^2)\Big\{\Big[2e^2(1-x^2)+(1-e^2x^2)\Big]\Big[\nu\,(1-e^2)+(1-e^2x^2)\Big] + e^2(1-x^2)\,(1-e^2x^2)\Big\} \tag{2.22}$$

$$\beta' = \frac{d\beta}{dx} = \Big[(1-3e^2x^2)\,(1-e^2)\Big]\Big[\nu\,(1-e^2x^2)+(1-e^2)\Big] - 2\nu e^2x^2(1-e^2)\,(1-e^2x^2) \tag{2.23}$$

and

$$\begin{aligned}\gamma' = \frac{d\gamma}{dx} = &-\frac{x}{\sqrt{1-x^2}}\Big\{(1-e^2x^2)\Big[2\,\nu(1-e^2x^2)\,(1-e^2)+(1-e^2x^2)^2 \\ &+(1-e^2)^2\Big]-A\,(1-e^2)^2p_n^2\Big\} \\ &+\sqrt{1-x^2}\,\Big\{-2e^2x\Big[2\,\nu\,(1-e^2x^2)\,(1-e^2)+(1-e^2x^2)^2+(1-e^2)^2\Big] \\ &-4e^2x\,(1-e^2x^2)\Big[\nu(1-e^2)+(1-e^2x^2)\Big]\Big\}\end{aligned} \tag{2.24}$$

Equation 2.21 is a second-order differential equation with variable coefficients. Due to the complexity of the problem a closed-form analytical solution for the oblate spheriod does not seem to exist.

Galerkin's method was applied to solve Equation 2.21, since the authors regard it as one of the most feasible methods for this type of problem. Since we have used in Galerkin's method the mode shapes of a vibrating thin spherical shell, this problem must be discussed first.

2.3 CLOSED-FORM SOLUTION FOR THE SPHERE

The differential equation of the sphere can be derived by the substitution of $e = 0$ into Equation 2.21.

$$(1-x^2)\frac{d^2v_n}{dx^2} - 2x\frac{dv_n}{dx} + \left[\frac{(2-Bp_n^2)\,(Ap_n^2+1-\nu)}{(1-\nu-Bp_n^2)} - \frac{1}{1-x^2}\right]v_n = 0 \tag{2.25}$$

where the constants A and B are as defined in Equation 2.15.

In solving Equations 2.21 and 2.25 for a complete shell, it should be noted that there are no physical boundary conditions. The conditions to be imposed are that the displacements should be single-valued and bounded at every point of the shell, including the north and south poles. This condition yields the expressions,

$$v_n = P_n^1(x) \tag{2.26}$$

and

$$\frac{(2-Bp_n^2)\,(Ap_n^2 + 1-\nu)}{(1-\nu - Bp_n^2)} = n\,(n+1) \qquad n = 1, 2, 3, \ldots \tag{2.27}$$

where the index numbers, n, refer to the frequency mode, and $P_n^1(x)$ represents the associated Legendre functions of degree n and order 1.

Equation 2.27 represents the frequency equation, which yields two sets of frequencies: one set of which is bounded and the other unbounded (see References 1-4 and 1-5). The upper limit of the bounded set is:

$$\lim_{n \to \infty} p''_n = \sqrt{\frac{(1-\nu)}{B}} \tag{2.28}$$

where p''_n represents the frequencies in the bounded set.

The normal displacement, w_n, can be derived from Equation 2.16 in the following form:

$$w_n(x) = \frac{n(n+1)\,P_n(x)}{(2-Bp_n^2)} \qquad n = 1, 2, 3 \ldots \tag{2.29}$$

where the $P_n(x)$ are the Legendre ploynominals.

SECTION III

SOLUTION BY GALERKIN'S METHOD

3.1 EXPRESSION IN TERMS OF LEGENDRE FUNCTIONS

3.1.1 USE OF DIFFERENTIAL OPERATOR, L. Equation 2.21 can be expressed by a linear differential operator and the unknown tangential displacement, v_n.

$$L(v_n) = 0 \tag{3.1}$$

The linear differential operator can be written in the form

$$L(\;) = E_1 \frac{d^2(\;)}{dx^2} + E_2 \frac{d(\;)}{dx} + E_3(\;) \tag{3.2}$$

3.1.2 COEFFICIENTS, E_1, E_2, E_3. The coefficients of the differential operator are then

$$E_1 = \left\{ (1-x^2)^2 (1-e^2x^2)^3 \gamma^2 - (1-x^2)^{3/2} (1-e^2x^2)^2 [(1+\nu) - (\nu + x^2) e^2] \alpha\gamma \right\} \tag{3.3}$$

$$\begin{aligned} E_2 = -\Big\{ & 2x(1-x^2)(1-e^2x^2)^2 [1 + (1-2x^2) e^2] \gamma^2 \\ & - (1-x^2)^{3/2} (1-e^2x^2)^2 [(1+\nu) - (\nu + x^2) e^2] [\gamma(\beta - \alpha') + \alpha\gamma'] \\ & - x(1-x^2)^{1/2} (1-e^2x^2) [4(1-x^2) - (2x^2 - 3x^4 + 1) e^2] e^2 \alpha\gamma \Big\} \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} E_3 = \Big\{ & A(1-e^2)^2 p_n^2 (1-x^2)\gamma^2 + (1-x^2)^{3/2}(1-e^2x^2)^2 [(1+\nu) - (\nu + x^2) e^2] (\beta'\gamma - \beta\gamma') \\ & - (1-e^2)(1-e^2x^2) [\nu + (1-\nu)x^2 - \{(1+\nu)x^2 - \nu x^4\} e^2] \gamma^2 \\ & - e^2 x (1-x^2)^{1/2} (1-e^2x^2) [4(1-x^2) - (2x^2 - 3x^4 + 1) e^2] \beta\gamma \Big\} \end{aligned} \tag{3.5}$$

3.1.3 DERIVATION OF FREQUENCY EQUATION DETERMINANT. The solution of Equation 3.1 can now be obtained in the following manner:

Let us express an approximate solution of Equation 3.1 in the form

$$\bar{v}_n(x) = \sum_{i=1} a_i \cdot \varphi_i(x) \tag{3.6}$$

where the a_i are undetermined coefficients and where $\varphi_i(x)$ is a certain system of functions, satisfying the boundary conditions and being the first N functions of a system which is complete in the interval, $-1 \leq x \leq +1$. The condition that $L(\bar{v})$ be identically equal to zero is equivalent to the requirement that the expression $L(\bar{v})$ be orthogonal to all the functions of the system $\varphi_i(x)$, $i = 0, 1, 2, 3, \ldots$ We choose as system $\varphi_i(x)$ the associated Legendre functions of order 1 (P_i^1), since these functions represent the exact analytical solution for the sphere.

N orthogonality conditions can be satisfied for the undetermined a_i coefficients. Therefore

$$\int_{-1}^{1} L(\bar{v}_n)\ \varphi_j(x)\, dx = 0 \tag{3.7}$$

Substituting Equation 3.6 into Equation 3.7,

$$\sum_{i=1}^{N} \left\{ \int_{-1}^{1} L\left[\varphi_i(x)\right] \cdot \varphi_j(x)\, dx \right\} a_i = 0 \qquad j = 1, 2, 3, \ldots, N \tag{3.8}$$

The non-trivial solution of Equation 3.8 represents the frequency equation of the problem, which can be written in determinant form as follows:

$$\begin{vmatrix} \int_{-1}^{1} L\left[\varphi_1(x)\right] \varphi_1 dx, & \int_{-1}^{1} L\left[\varphi_2(x)\right] \varphi_1 dx, & \ldots, & \int_{-1}^{1} L\left[\varphi_n(x)\right] \varphi_1 dx \\ \int_{-1}^{1} L\left[\varphi_1(x)\right] \varphi_2\, dx, & \int_{-1}^{1} L\left[\varphi_2(x)\right] \varphi_2\, dx, & \ldots, & \int_{-1}^{1} L\left[\varphi_n(x)\right] \varphi_2\, dx \\ \vdots & \vdots & & \vdots \\ \int_{-1}^{1} L\left[\varphi_1(x)\right] \varphi_n\, dx, & \int_{-1}^{1} L\left[\varphi_2(x)\right] \varphi_n\, dx, & \ldots, & \int_{-1}^{1} L\left[\varphi_n(x)\right] \varphi_n\, dx \end{vmatrix} = 0 \tag{3.9}$$

Equation 3.8 can also be written in another form, namely:

$$\sum_{i=1}^{N} A_{ij} \cdot a_i = 0 \qquad j = 1, 2, 3, \ldots, N \tag{3.10}$$

where

$$A_{ij} = \int_{-1}^{1} L\left[\varphi_i(x)\right] \cdot \varphi_j(x)\, dx \tag{3.11}$$

The determinant of Equation 3.9 can be written by the use of Equation 3.11 as

$$\det\left[A_{ij}\right] = 0 \tag{3.12}$$

3.1.4 USE OF MODAL FUNCTIONS OF THE SPHERE. Thus far, the functions $\varphi_i(x)$ have remained undetermined; it is now necessary to select some modal functions which fit the problem. All the necessary conditions can be satisfied if it is assumed that the oblate spheroid vibrates in a manner similar to a sphere. The modal functions of the sphere are known; we therefore chose as set $\varphi_i(x)$ the associated Legendre function

$$\varphi_i(x) = P_i^1(x) \tag{3.13}$$

The approximate tangential displacement of the oblate spheroidal shell can then be expressed as

$$\bar{v}_n(x) = \sum_{i=1}^{N} a_i \cdot P_i^1(x) \tag{3.14}$$

3.1.5 RESULTING EQUATION. The frequency equation can now be obtained by the substitution of Equation 3.13 into Equations 3.11 and 3.12. Thus

$$A_{ij} = \int_{-1}^{1} L\left[P_i^1(x)\right] \cdot P_j^1(x)\, dx \tag{3.15}$$

or, in determinant form,

$$\det\left[A_{ij}\right] = \begin{vmatrix} \int_{-1}^{1} L\left[P_1^1(x)\right] P_1^1(x)\, dx, & \ldots, & \int_{-1}^{1} L\left[P_N^1(x)\right] P_1^1(x)\, dx \\ \vdots & & \vdots \\ \int_{-1}^{1} L\left[P_1^1(x)\right] P_N^1(x)\, dx, & \ldots, & \int_{-1}^{1} L\left[P_N^1(x)\right] P_N^1(x)\, dx \end{vmatrix} = 0 \tag{3.16}$$

3.2 PROPERTIES OF THE FREQUENCY EQUATION, det $[A_{ij}] = 0$, FOR A SPHERICAL SHELL

3.2.1 DIFFERENTIAL OPERATOR FOR THE SPHERE. The differential operator of Equation 3.2 becomes, in the case of e = 0, the following:

$$L(\ ) = (1-x^2)\frac{d^2(\ )}{dx^2} - 2x\frac{d(\ )}{dx} + \left[\frac{(2-Bp_n^2)(Ap_n^2+1-\nu)}{(1-\nu-Bp_n^2)} - \frac{1}{1-x^2}\right]\cdot(\ ) \qquad (3.17)$$

Let

$$\lambda_n = \frac{(2-Bp_n^2)(Ap_n^2+1-\nu)}{(1-\nu-Bp_n^2)} \qquad (3.18)$$

where p_n is the angular frequency. Equation 3.17 can then be written as

$$L(\ ) = (1-x^2)\frac{d^2(\ )}{dx^2} - 2x\frac{d(\ )}{dx} + \left[\lambda_n - \frac{1}{1-x^2}\right](\ ) \qquad (3.19)$$

It should be noted that the quantity, P_i^1, in the equation,

$$A_{ij} = \int_{-1}^{1} L[P_i^1(x)]\,P_j^1(x)\,dx = \int_{-1}^{1}\left\{(1-x^2)\frac{d^2P_i^1}{dx^2} - 2x\frac{dP_i^1}{dx} + \left[\lambda_n - \frac{1}{1-x^2}\right]P_i^1\right\}P_j^1\,dx \qquad (3.20)$$

satisfies Legendre's differential equation, i.e.,

$$(1-x^2)\frac{d^2P_i^1}{dx^2} - 2x\frac{dP_i^1}{dx} - \frac{P_i^1}{1-x^2} = -n(n+1)P_i^i \qquad n = 1, 2, 3, \ldots, N \qquad (3.21)$$

Substituting Equation 3.21 into Equation 3.20, one has

$$A_{ij} = \int_{-1}^{1}[\lambda_n - n(n+1)]\,P_i^1P_j^1dx = [\lambda_n - n(n+1)]\int_{-i}^{1}P_i^1P_j^1dx \qquad (3.22)$$

Since

$$\int_{-1}^{1}P_i^1(x)\,P_j^1(x)\,dx = 0 \qquad \text{if } i \neq j \qquad (3.23)$$

$$\int_{-1}^{1} P_i^1(x) P_j^1(x)\, dx = \frac{2n(n+1)}{(2n+1)} \qquad \text{if } i = j = n \tag{3.24}$$

The substitution of Equations 3.23, 3.24, and 3.22 into the determinant of Equation 3.16 yields

$$\det[A_{ij}] = \begin{vmatrix} \frac{4}{3}(\lambda_n - 2) & 0 & 0 & \cdots \\ 0 & \frac{12}{5}(\lambda_n - 6) & 0 & \cdots \\ 0 & 0 & \frac{24}{7}(\lambda_n - 12) & \cdots \\ \vdots & \vdots & \vdots & \end{vmatrix} = 0 \tag{3.25}$$

3.2.2 ROOTS OF THE FREQUENCY EQUATION. Equation 3.25 can be expressed as

$$(\lambda_n - 2)(\lambda_n - 6)(\lambda_n - 12)(\lambda_n - 20) \ldots = 0 \tag{3.26}$$

The roots of Equation 3.26 represent the frequencies. It can be seen that these roots satisfy the homogeneous equation system of Equation 3.10. One has then

$$\left.\begin{aligned} (\lambda_n - 2)\, a_1 &= 0 \\ (\lambda_n - 6)\, a_2 &= 0 \\ (\lambda_n - 12)\, a_3 &= 0 \\ \vdots\qquad & \quad \vdots \end{aligned}\right\} \tag{3.27}$$

For the first mode shape, $\lambda_1 = 2$; thus a_1 is arbitrary. It also follows that

$$a_2 = a_3 = a_4 = \ldots = 0 \tag{3.28}$$

Thus

$$v_1(x) = a_1 P_1^1(x), \text{ etc.} \tag{3.29}$$

3.3 PROPERTIES OF THE FREQUENCY EQUATION, det $[A_{ij}] = 0$, FOR AN OBLATE SPHEROIDAL SHELL

3.3.1 THE DETERMINANT. In the case of an oblate spheriodal shell, the determinant of the homogeneous equation system (Equation 3.10) can be written in the following form:

$$\det [A_{ij}] = \begin{vmatrix} A_{11} & 0 & A_{31} & 0 & A_{51} & \cdots \\ 0 & A_{22} & 0 & A_{42} & 0 & \cdots \\ A_{13} & 0 & A_{33} & 0 & A_{53} & \cdots \\ 0 & A_{24} & 0 & A_{44} & 0 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \end{vmatrix} = 0 \tag{3.30}$$

3.3.2 FORM OF THE DETERMINANT. The determinant of Equation 3.30 can now be examined separately for odd and even indices.

3.3.2.1 Legendre Functions and their Derivatives. The associated Legendre functions can be expressed in the following form:

$$P_i^1(x) = \sqrt{1-x^2}\,(\beta_o + \beta_2 x^2 + \ .\ .\ .\ + \beta_{i-1} x^{i-1}) \tag{3.31}$$

where

$i = 1, 3, 5, \ldots$ = any odd integer;

and

$$P_k^1(x) = \sqrt{1-x^2}\,(\gamma_1 x + \gamma_3 x^3 + \ .\ .\ .\ + \gamma_{k-1} x^{k-1}) \tag{3.32}$$

where

$k = 2, 4, 6, \ldots$ = any even integer;

and

$\beta_o \ldots \beta_{i-1}$ and $\gamma_1 \ldots \gamma_{k-1}$ are constants.

The first and second derivatives of the $P_i^1(x)$, for odd indices, are

$$\frac{d\,P_i^1(x)}{dx} = \frac{1}{\sqrt{1-x^2}}\left[\beta_o' x + \beta_1' x^3 + \ .\ .\ .\ + \beta_i' x^i\right] \tag{3.33}$$

where

i = any odd integer, and

k = any even integer.

Equation 3.40 can also be expressed in the form

$$A_{ik} = \int_{-1}^{1} \Big\{ (E_{1,0}\beta_o''\gamma_1 x + \ldots + E_{1,22}\beta_{i+1}\gamma_{k-1}x^{22+i+k})$$

$$+ (E_{2,1}\beta_o'\gamma_1 x^3 + \ldots + E_{2,23}\beta_i'\gamma_{k-1}x^{22+i+k})$$

$$+ (E_{3,0}\beta_o\gamma_1 x + \ldots + E_{3,20}\beta_{i-1}\gamma_{k-1}x^{20+i+k}) \Big\} dx = 0 \qquad (3.41)$$

since i+k = any odd integer. Similarly, we can write

$$A_{ki} =$$

$$\int_{-1}^{1} \Big\{ (E_{1,0} + \ldots + E_{1,22}x^{22})(\gamma_1 x + \ldots + \gamma_{k+1}x^{k+1})(\beta_o + \ldots + \beta_{i-1}x^{i-1})$$

$$+ (E_{2,1}x + \ldots + E_{2,23}x^{23})(\gamma_1 + \ldots + \gamma_k x^k)(\beta_o + \ldots + \beta_{i-1}x^{i-1})$$

$$+ (E_{2,1}x + \ldots + E_{2,23}x^{20})(\gamma_1 x + \ldots + \gamma_{k-1}x^{k-1})$$

$$(\beta_o + \ldots + \beta_{i-1}x^{i-1})(1 - x^2) \Big\} dx$$

$$= \int_{-1}^{1} \Big\{ E_{1,0}\gamma_1\beta_o x + \ldots + E_{2,22}\gamma_{k+1}\beta_{i-1}x^{22+k+i} \Big\} dx = 0 \qquad (3.42)$$

where

k+i = any odd integer.

It has thus been proven by the integrals of Equations 3.41 and 3.42 that

$$A_{ik} = A_{ki} = 0 \qquad (3.43)$$

where

i+k = any odd integer.

SECTION IV

CALCULATION OF MODE SHAPES

The mode shapes of the thin oblate spheroidal shell can be calculated from Equations 3.14, 3.15, and 3.10.

4.1 HOMOGENEOUS EQUATION SYSTEMS FOR EVEN AND ODD MODES

It can be readily seen from the determinant of Equation 3.30 that the even numbered modes have been separated from the odd numbered modes. For odd modes, the homogeneous equation system is

$$\left.\begin{array}{l} A_{1\,1}a_1 + A_{3\,1}a_3 + A_{5\,1}a_5 + \cdots = 0 \\ A_{1\,3}a_1 + A_{3\,3}a_3 + A_{5\,3}a_5 + \cdots = 0 \\ A_{1\,5}a_1 + A_{3\,5}a_3 + A_{5\,5}a_5 + \cdots = 0 \\ \quad\vdots \qquad\quad \vdots \qquad\quad \vdots \qquad\qquad \vdots \end{array}\right\} \tag{4.1}$$

For even modes, the homogeneous equation system is

$$\left.\begin{array}{l} A_{2\,2}a_2 + A_{4\,2}a_4 + \cdots = 0 \\ A_{2\,4}a_2 + A_{4\,4}a_4 + \cdots = 0 \\ \quad\vdots \qquad\quad \vdots \qquad\qquad \vdots \end{array}\right\} \tag{4.2}$$

The coefficients A_{ij} of the unknown quantity, a_i, can then be calculated for each corresponding frequency.

4.2 EVALUATION OF UNDETERMINED CONSTANT, a_n

The solutions of Equations 4.1 and 4.2 leave one undetermined constant. This problem can be solved in either of two ways:

a. Set the arbitrary constant $a_n = 1$. In this case, the homogeneous equation system is reduced to an inhomogeneous equation system and the equation system becomes determined.

b. Determine the unknown constant, a_n, from the following equation:

$$\int_{-1}^{1} v_n^2 \, dx = \int_{-1}^{1} \left[\sum_{i=1}^{n} a_i P_i^1(x) \right]^2 dx = 1 \tag{4.3}$$

The eigenfrequencies, p_n, satisfying Equation 3.30, were found digitally on an IBM 7094 computer by first bracketing solutions and then using the regula falsi to locate the exact values of p_n for the zeros of the determinant.

For any particular solution, p_n, the undetermined coefficients, a_i, can be determined up to a multiplicative factor. By arbitrarily setting $a_n = 1$, the a_i ($i = 1,2, \ldots n-1, n+1, \ldots N$) are uniquely determined.

4.3 CALCULATION OF NORMAL MODES

A straightforward numerical calculation of w_n is not possible from Equation 2.16 because of the vanishing denominator and a singularity in $\frac{dv_n}{dx}$ occurring at the poles (where $x = \pm 1$). However, a modification of Equation 2.16 can be effected. As derived previously, Equation 2.16 was written

$$w_n = \frac{-\alpha(x) \frac{dv_n}{dx} + \beta(x) v_n}{\gamma(x)} \tag{4.4}$$

Equation 3.14 can be expressed as

$$v_n = \sum_{i=1}^{N} a_i P_i^1(x) = \sqrt{1-x^2} \sum_{i=1}^{N} a_i \frac{d}{dx} \left[P_i(x) \right] \tag{4.5}$$

Differentiating Equation 4.5, we then obtain

$$\frac{dv_n}{dx} = -\frac{1}{\sqrt{1-x^2}} \cdot \sum_{i=1}^{N} a_i \left\{ x \frac{d}{dx} \left[P_i(x) \right] - (1-x^2) \frac{d^2}{dx^2} \left[P_i(x) \right] \right\} \tag{4.6}$$

Equation 2.17 can be written

$$\alpha(x) = (1-x^2) [\ldots] \tag{4.7}$$

and Equation 2.19 can be written

$$\gamma(x) = \sqrt{1-x^2} \{\ldots\} \tag{4.8}$$

Substituting the last four equations into Equation 4.4, one has

$$w_n = \frac{(1-x^2)\,[.\;\;.\;\;.]\,\dfrac{1}{\sqrt{1-x^2}}\displaystyle\sum_{i=1}^{N} a_i\,\{\text{POLYNOMIALS}\} + \beta(x)\,\sqrt{1-x^2}\displaystyle\sum_{i=1}^{N} a_i \frac{d}{dx}\left[P_i(x)\right]}{\sqrt{1-x^2}\,\{.\;\;.\;\;.\}} \tag{4.9}$$

4.4 EQUATION FOR COMPUTATION OF w_n

The singularities can now be removed, and Equation 4.9 is then ready for computation in the form

$$w_n = \frac{[.\;\;.\;\;.]\displaystyle\sum_{i=1}^{N} a_i\,\{\text{POLYNOMIALS}\} + \beta(x)\displaystyle\sum_{i=1}^{N} a_i \frac{d}{dx}\left[P_i(x)\right]}{\{.\;\;.\;\;.\}} \tag{4.10}$$

SECTION V

FREE VIBRATION OF IDEALIZED CENTAUR LIQUID OXYGEN TANK

5.1 INTRODUCTORY DATA AND METHODS

The theoretical developments of the previous sections can now be applied to the problem of free vibrations in the idealized Centaur Liquid Oxygen (LO_2) tank, which has the shape of an oblate spheroid.

5.1.1 TANK PARAMETERS. The physical data characterizing the Centaur LO_2 tank are as given in Table 5-1.

TABLE 5-1. PHYSICAL CHARACTERISTICS OF THE CENTAUR LIQUID OXYGEN TANK

Parameter	Symbol	Value	Units
Major axis	a	60	in.
Minor axis	b	43.5	in.
Eccentricity	e	0.68874887	ND
Poisson ratio	ν	0.28	ND
Modulus of elasticity	E	2.8×10^7	lb/in.^{2}
Density	ρ	7.34×10^{-4}	$\frac{\text{lb-sec}^2}{\text{in.}^4}$

5.1.2 NUMERICAL CALCULATIONS FOR SPHERE. The natural angular frequencies of the corresponding sphere with radius a = 60 inches, as found by the use of Equations 2.27 and 2.28, are as given in Table 5-2.

5.1.3 METHODS OF COMPUTATION FOR OBLATE SPHEROID. The frequencies of the oblate spheroid were calculated from the determinant of Equation 3.30.

5.1.3.1 Gaussian Integration. The elements of the determinant were evaluated by Gaussian integration (see Reference 5-1). It was shown, that the integrand for the determination of an element, A_{ij}, is an exact polynomial of degree $(21 + i + j)$.

These two frequencies are located between the bounded and unbounded sets and therefore do not cause any difficulties.

5.1.3.2 Legendre Functions and Derivatives for Computer Input. The Legendre function and its first and second derivatives have been calculated for computer use by the following recursion formulas:

$$P_n^1(x) = \frac{(2n-1)}{(n-1)} \cdot x\, P_{n-1}^1(x) - \frac{n}{(n-1)} P_{n-2}^1 \tag{5.3}$$

$$\frac{d}{dx}\left[P_n^1(x)\right] = -\frac{x}{1-x^2}\, n\, P_n^1(x) + \frac{1}{1-x^2}(n+1)\, P_{n-1}^1(x) \tag{5.4}$$

and

$$\frac{d^2}{dx^2}\left[P_n^1(x)\right] = \frac{2x}{(1-x^2)} \frac{d}{dx}\left[P_n^1(x)\right] - \frac{1}{(1-x^2)}\left[n(n+1) - \frac{1}{1-x^2}\right] P_n^1(x) \tag{5.5}$$

(see Reference 5-2).

5.1.3.3 Computational Methods for Frequencies and Mode Shapes. The frequencies were found as the roots of the determinant of Equation 3.30 by a modified regula falsi method, which was proposed by Dr. Kahan of the University of Toronto.

The numerical computation of the tangential displacement, v_n, is based upon Equations 3.10, 3.14, and 3.15. The computation of the normal displacement, w_n, is based upon Equations 2.16, 4.5, and 4.10.

5.2 NUMERICAL CALCULATIONS FOR CENTAUR LO_2 TANK

The two sets of frequencies and mode shapes were found for the case of a thin oblate spheroidal shell in a manner similar to their derivation in the case of the thin spherical shell.

5.2.1 UNBOUNDED SET OF FREQUENCIES. One group of resulting frequencies forms an unbounded set.

5.2.1.1 First Mode, n=1. Table 5-4 illustrates the convergence of frequency values obtained by Galerkin's method. Table 5-5 presents the calculated mode shapes.

5.2.1.2 Second Mode, n=2. Tables 5-6 and 5-7 show the calculated frequencies and coefficients, a_i, for the second mode.

TABLE 5-4. CALCULATION OF FREQUENCIES FOR UNBOUNDED SET, FIRST MODE

Frequency		
Summation Limit N	Angular Frequency p_1' (^{1}/sec)	
1	-	
3	6,821.4	Angular Frequency
5	7,266.4	
7	7,511.5	$p_1' = 7,869.1\ ^1/\text{sec}$
9	7,680.0	
11	7,767.0	
13	7,846.1	
15	7,869.1	

TABLE 5-5. CALCULATION OF MODE SHAPES FOR UNBOUNDED SET, FIRST MODE

Mode Shape, n = 1		
Index Number i	Coefficient a_i	
1	1.00	
3	0.115	
5	0.00972	Tangential Mode Shape
7	-0.00131	
9	-0.00157	$v_1(x) = \sum_{i=1}^{15} a_i P_i^1(x)$
11	-0.000902	
13	-0.000511	
15	-0.000074	

TABLE 5-6. CALCULATION OF FREQUENCIES FOR UNBOUNDED SET, SECOND MODE

Frequency		
Summation Limit N	Angular Frequency $\dot{p}_2$ (^{1}/sec)	
2	14,120	Angular Frequency
4	11,770	$\dot{p}_2 = 10{,}774.0\ ^1/\text{sec}$
6	11,070	
8	10,840	
10	10,780	
12	10,774.0	

TABLE 5-7. CALCULATION OF MODE SHAPES FOR UNBOUNDED SET, SECOND MODE

Mode Shape, n = 2		
Index Number i	Coefficient a_i	
2	1.00	Tangential Mode Shape
4	0.224	$v_2(x) = \sum_{i=2}^{12} a_i P_i^1(x)$
6	0.03738	
8	0.004500	
10	-0.000471	
12	-0.000515	

5.2.2 BOUNDED SET OF FREQUENCIES. The other group of resulting frequencies formed a bounded set. Within this set, the first mode represents the rigid-body motion in the case of the vibrating thin spherical shell. Galerkin's method, however, does not present the rigid-body motion in the case of a vibrating thin oblate spheroidal shell with i = 15 terms.

The lowest bounded frequencies (n = 2) and the mode shape were calculated; and these, together with the coefficients, a_i, are presented in Tables 5-10 and 5-11.

5.3 MODE-SHAPE AND FREQUENCY PLOTS

Figure 5-1 shows the tangential and normal mode shapes of the unbounded set of frequencies for mode numbers n = 1, 2, 3.

Figure 5-2 shows the tangential and normal mode shapes of the bounded set of frequencies for mode number n = 2

Figure 5-3 shows the angular frequencies as a function of eccentricity for the second mode of the unbounded set.

TABLE 5-10. CALCULATION OF FREQUENCIES FOR BOUNDED SET, SECOND MODE

Frequency		
Summation Limit N	Angular Frequency p_2'' (^{1}/sec)	
2	394.5	
4	698.6	Angular Frequency
6	703.4	p_2'' = 734.85 ^{1}/sec
8	717.0	
10	725.68	
12	731.2	
14	734.85	

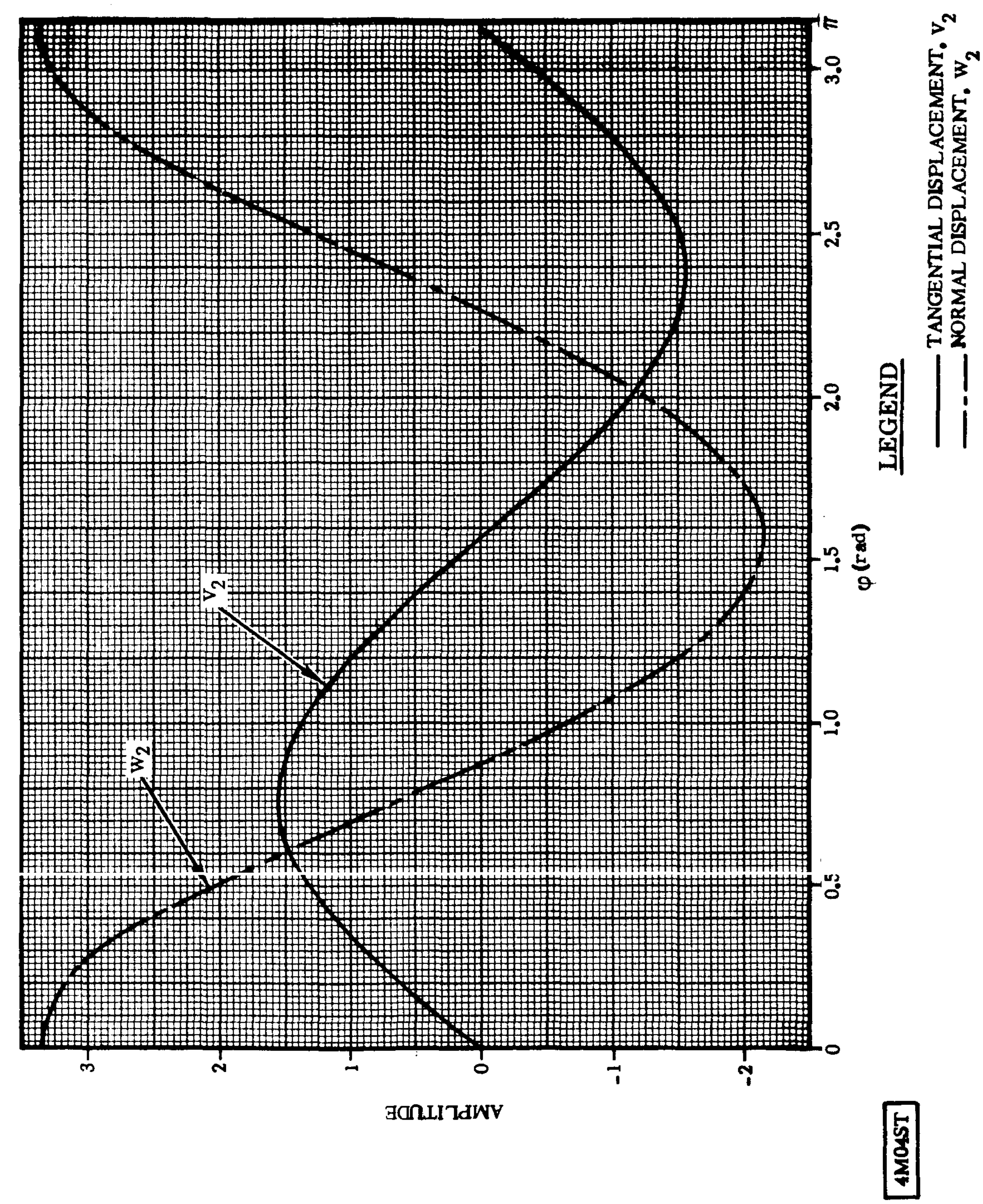

Figure 5-2. Mode Shapes of Bounded Set of Frequencies, Second Mode

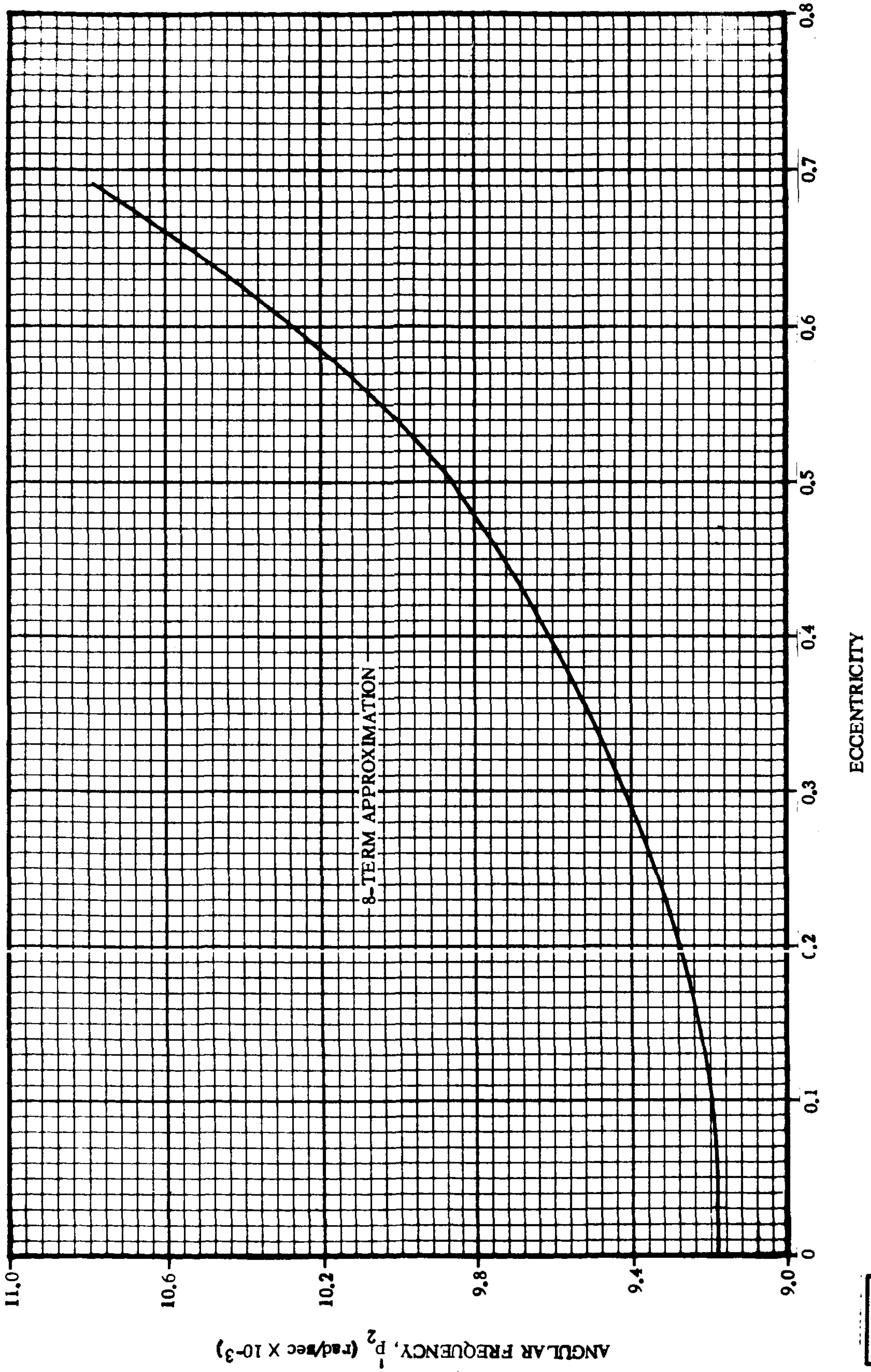

Figure 5-3. Angular Frequency versus Eccentricity for Unbounded Set, Second Mode

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SECTION VI

CONCLUSIONS AND RECOMMENDATIONS

In this report, the authors have presented the solution of the problem of free vibration of a thin isotropic oblate spheroidal shell. The solution of this problem is an extension of Love's old problem (1888) of free symmetrical vibration of thin spherical shells (isotropic), presented in Reference 1-4 (pages 535-537). It is believed that the present work is original.

As shown in this report, the series resulting from the use of Galerkin's method was found to converge rapidly for small eccentricities; and the convergence of the series for larger eccentricities was also satisfactory ($e = 0.688$, 11 terms). The existence of two sets of frequencies for the complete shell (bounded and unbounded) was also established in this report (see References 1-2, 1-4, and 1-6).

As the sizes of advanced upper-stage vehicles increase, propellant tanks with stiffeners will have more technical advantages than pressurized tanks. Vehicle tanks with stiffeners can be represented by <u>orthotropic</u> shell theory. The present theory for isotropic shells can easily be extended, however, to include this type of tank; it should therefore have many technical applications such as:

a. A better physical representation of the Centaur liquid oxygen (LO_2) tank

b. Representation of the LO_2 tank of the Saturn V (upper-stage)

c. For future advanced upper-stage vehicles.

SECTION VII

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APPENDIX A

A.1 INTRODUCTION

This study has been conducted in terms of a curvilinear coordinate system based upon the radii, R_1 and R_2, which express the principal curvatures of the surface as functions of φ alone. Appendix A is provided for the purpose of deriving the R_1, R_2 system, $f(\varphi)$, by transformation from the oblate spheroidal coordinate system, $g(\alpha, \beta, \theta)$.

A.2 OBLATE SPHEROIDAL COORDINATES

Consider the oblate spheroidal coordinate system,

$$\left.\begin{aligned} x &= p\cosh\alpha\cos\beta\cos\theta \\ y &= p\cosh\alpha\cos\beta\sin\theta \\ z &= p\sinh\alpha\sin\beta \end{aligned}\right\} \qquad \text{(A.1)}$$

and

$$p = ae \qquad \text{(A.2)}$$

where a is the radius of the equivalent sphere and e is the eccentricity. Let us consider the effect on the equation if α, β, and θ are each held constant in turn.

A.2.1 θ = CONSTANT. This produces planes, since $\frac{y}{x} = \tan\theta$. Also, if ξ is the intersection of the plane with the x-y plane (see Figure A-1),

$$\left.\begin{aligned} \xi^2 &= x^2 + y^2 \\ \xi^2 &= p^2\cosh^2\alpha\cos^2\beta \\ \text{or} \quad \xi &= p\cosh\alpha\cos\beta \end{aligned}\right\} \qquad \text{(A.3)}$$

A.2.2 β = CONSTANT. This produces an hyperboloid, since

$$\frac{\xi^2}{p^2\cos^2\beta} - \frac{z^2}{p^2\sin^2\beta} = 1 \qquad \text{(A.4)}$$

A.2.3 α = CONSTANT. This produces an ellipsoid, since

$$\frac{\xi^2}{p^2 \cosh^2\alpha} + \frac{z^2}{p^2 \sinh^2\alpha} = 1 \tag{A.5}$$

For the equation of the oblate spheroid, however, α must be a constant. We can therefore write

$$\left.\begin{aligned} \cosh\alpha &= C \\ \text{or} \quad e &= \frac{1}{\cosh\alpha} = \frac{1}{C} \end{aligned}\right\} \tag{A.6}$$

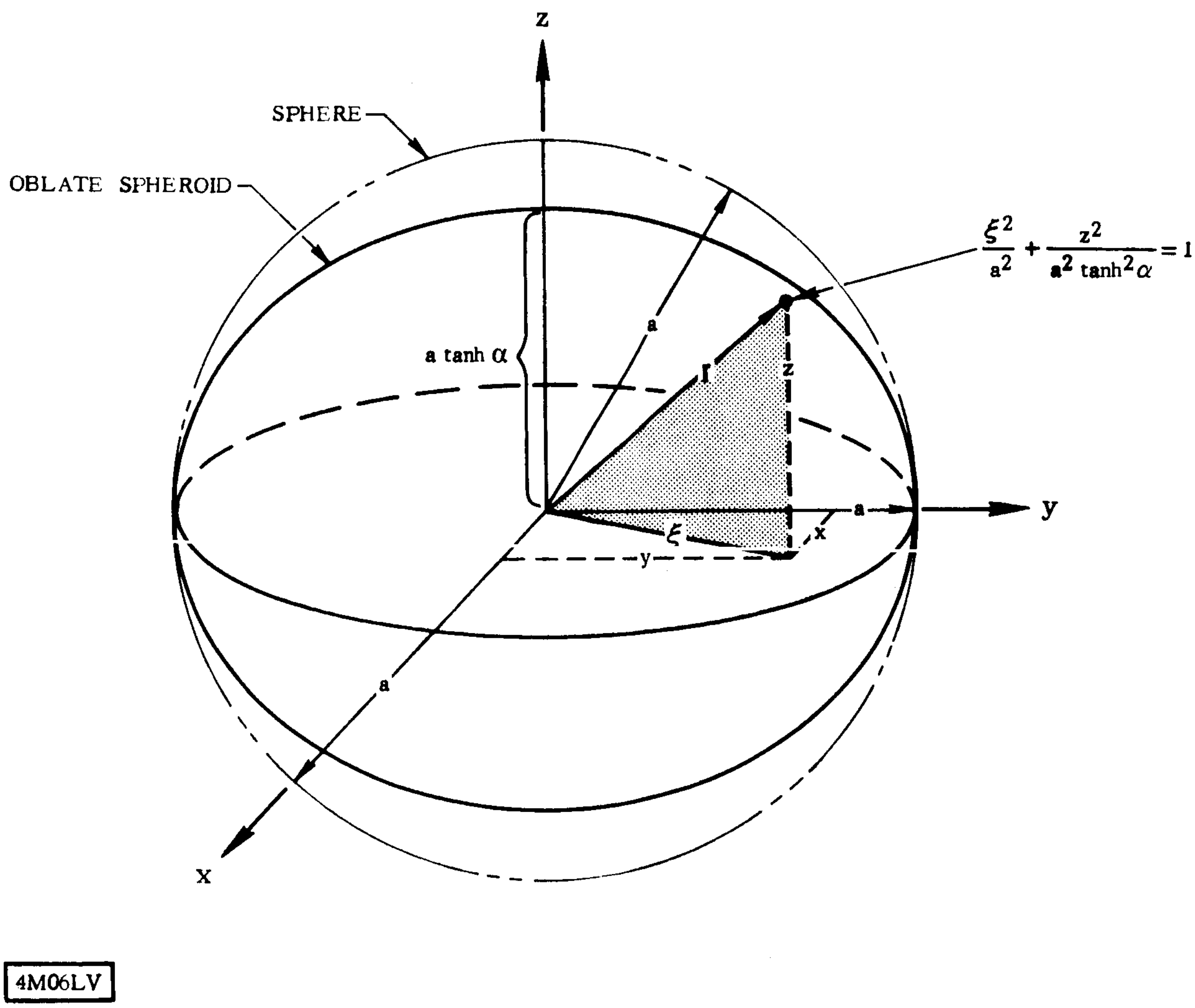

Figure A-1. Vectorial Representation of the Oblate Spheroidal Surface

A.2.4 RESULTING SPECIAL COORDINATES. In view of these considerations, the equations of the oblate spheroidal shell can be written in the following form:

$$\left.\begin{aligned} x &= pC\cos\beta\cos\theta = a\cos\beta\cos\theta \\ y &= pC\cos\beta\sin\theta = a\cos\beta\sin\theta \\ z &= p\sqrt{C^2-1}\,\sin\beta = a\sqrt{1-e^2}\,\sin\beta \end{aligned}\right\} \tag{A.7}$$

A.3 VECTORIAL REPRESENTATION OF THE SURFACE

Let the position vector of the ellipsoid be represented in the following form:

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = a\cos\beta\cos\theta\mathbf{i} + a\cos\beta\sin\theta\mathbf{j} + a\sqrt{1-e^2}\sin\beta\mathbf{k} \tag{A.8}$$

Equation A.5 represents a rotational ellipsoid (see Figure A-1). Equation A.5 can now be written by the use of Equations A.2 and A.6 as

$$\frac{\xi^2}{a^2} + \frac{z^2}{a^2\tanh^2\alpha} = 1 \tag{A.9}$$

This equation represents an oblate spheroidal surface.

A.4 PROPERTIES OF THE SURFACE

A.4.1 FIRST FUNDAMENTAL FORM. An element of the arc of the ellipsoid can be expressed as

$$ds^2 = E^2 d\beta^2 + 2EF\,d\beta\,d\theta + G\,d\theta^2 \tag{A.10}$$

(see Reference 1-4), where

$$\left.\begin{aligned} E &= \mathbf{r}_\beta \cdot \mathbf{r}_\beta = a^2(1-e^2\cos^2\beta) \\ F &= \mathbf{r}_\beta \cdot \mathbf{r}_\theta = 0 \\ G &= \mathbf{r}_\theta \cdot \mathbf{r}_\theta = a^2\cos^2\beta \end{aligned}\right\} \tag{A.11}$$

The discriminant is

$$\sqrt{EG-F^2} = a^2\cos\beta\sqrt{1-e^2\cos^2\beta} \tag{A.12}$$

A.4.2 NORMAL UNIT VECTOR. The normal unit vector of the oblate spheroid is

$$\mathbf{N} = \frac{(\mathbf{r}_\theta \times \mathbf{r}_\beta)}{\sqrt{EG-F^2}}$$

$$= \frac{1}{\sqrt{1-e^2\cos^2\beta}}\left(\sqrt{1-e^2}\cos\beta\cos\theta\,\mathbf{i} + \sqrt{1-e^2}\cos\beta\sin\theta\,\mathbf{j} + \sin\beta\,\mathbf{k}\right) \quad (A.13)$$

or

$$\mathbf{N} = N_1\mathbf{i} + N_2\mathbf{j} + N_3\mathbf{k} \quad (A.14)$$

where $\mathbf{i}, \mathbf{j}$, and $\mathbf{k}$ are the unit vectors of the coordinate system.

From Equation A.13, it can be easily calculated that

$$\cos\varphi = \frac{N_3}{|\mathbf{N}|} = \frac{\sin\beta}{\sqrt{1-e^2\cos^2\beta}} \quad (A.15)$$

(see Figure A-2), where φ is the angle in space between the z axis and the normal vector.

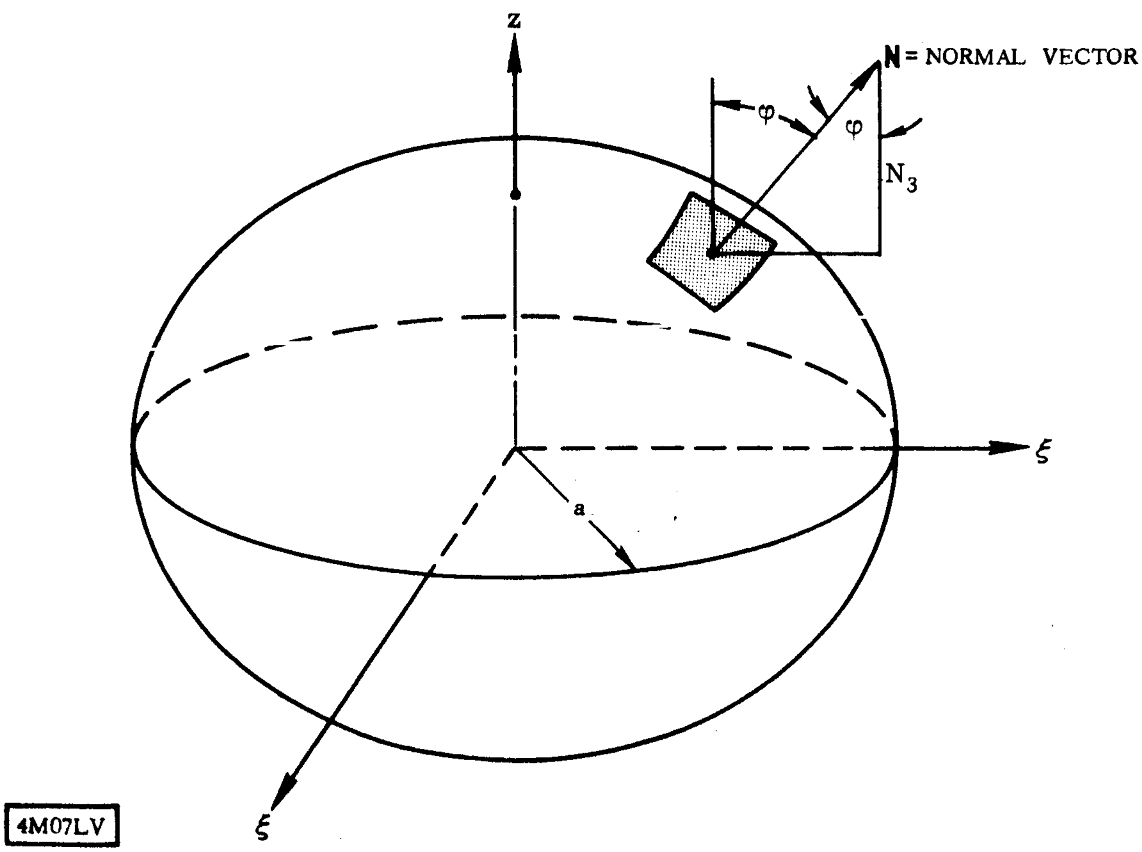

Figure A-2. Normal Unit Vector, $\mathbf{N}$

A.4.3 SECOND FUNDAMENTAL FORM. The terms of the second fundamental form of the surface are

$$\left.\begin{aligned} e^* &= \frac{(r_{\beta\beta} r_\beta r_\theta)}{\sqrt{EG-F^2}} = \sqrt{\frac{1-e^2}{1-e^2\cos^2\beta}} \\ f &= (r_{\beta\theta}\, r_\beta r_\theta) = 0 \qquad \text{Since} \qquad r_\beta \cdot r_\theta = 0 \\ g &= \frac{(r_{\theta\theta}\, r_\beta r_\theta)}{\sqrt{EG-F^2}} = \frac{a\sqrt{1-e^2}\cdot \cos^2\beta}{\sqrt{1-e^2\cos^2\beta}} \end{aligned}\right\} \tag{A.16}$$

A.4.4 PRINCIPAL CURVATURES. The normal curvatures of the surface, $\varkappa_1$ and $\varkappa_2$, in the two principle directions are

$$\left.\begin{aligned} \varkappa_1 &= \frac{1}{R_1} = \frac{e^*}{E} = \frac{\sqrt{1-e^2}}{a(1-e^2\cos^2\beta)^{3/2}} \\ \varkappa_2 &= \frac{1}{R_2} = \frac{g}{G} = \frac{\sqrt{1-e^2}}{a\sqrt{1-e^2\cos^2\beta}} \end{aligned}\right\} \tag{A.17}$$

The two principal radii of the surface can then be written as

$$\left.\begin{aligned} R_1 &= \frac{a(1-e^2\cos^2\beta)^{3/2}}{\sqrt{1-e^2}} = \frac{(1-e^2)R_2^3}{a^2} \\ R_2 &= \frac{a\sqrt{1-e^2\cos^2\beta}}{\sqrt{1-e^2}} \end{aligned}\right\} \tag{A.18}$$

A.5 COORDINATES EMPLOYED IN THIS STUDY

Equation A. 18 can now be expressed as a function of the variable φ alone (see Equation A. 15) in the following form:

$$\left.\begin{aligned} R_1 &= \frac{(1-e^2)\,a}{(1-e^2\cos^2\varphi)^{3/2}} \\[2em] R_2 &= \frac{a}{(1-e^2\cos^2\varphi)^{1/2}} \end{aligned}\right\} \qquad \text{(A. 19)}$$

The principle radii, R_1 and R_2, are illustrated in Figure 2-2. Having now derived these quantities from those of the oblate spheroidal coordinate system (Equations A.1 and A.2), their introduction into this study in Equations 2.1 and 2.2 is explained. The simplification effected by this transformation of coordinates will readily be appreciated.

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