NASA CONTRACTOR REPORT

NASA CR-443

TABLES OF NATURAL FREQUENCIES AND NODES FOR TRANSVERSE VIBRATION OF TAPERED BEAMS

by Han-chung Wang and Will J. Worley

Prepared under Grant No. NsG-434 *by*
UNIVERSITY OF ILLINOIS
Urbana, Ill.
for

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • APRIL 1966

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SUMMARY

The natural frequencies, nodal points and mode functions for transverse vibration of tapered beams are presented in this report.

The beams considered have the cross-sectional area bounded by the curve

$$\left|\frac{y}{h}\right|^{\beta} + \left|\frac{z}{b}\right|^{\gamma} = 1$$

with the thickness h and width b varying along the beam according to the relations

$$h = h_o\left(\frac{X}{\ell}\right)^{\phi} \quad \text{and} \quad b = b_o\left(\frac{X}{\ell}\right)^{\psi}$$

where β, γ, ϕ and ψ are positive constants not necessarily integers.

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INTRODUCTION

1. Statement of the Problem

The geometrical properties of solid bodies generated by revolving the line defined as

$$\left|\frac{X}{\ell}\right|^{\alpha} + \left|\frac{y}{h}\right|^{\beta} = 1 \qquad (4.1)^{*}$$

and of bodies bounded by the surface defined as

$$\left|\frac{X}{\ell}\right|^{\alpha} + \left|\frac{y}{h}\right|^{\beta} + \left|\frac{z}{b}\right|^{\gamma} = 1 \qquad (4.2)$$

have been published in the first and second reports [1, 2]** under this grant. The current report treats the dynamic response of a general class of tapered beams. The shape of beams includes the bodies generated by Eqs. (4.1) and (4.2) with the parameter $\alpha = 1$. Special cases of bodies of the above shapes have been tested widely in NACA and NASA reports as well as in the technical journals, [3, 4, 5, 6, 7]. Beams of special shapes also have been applied as designs for high speed machine guns as reported in a recent paper by Elder [8].

To achieve more nearly complete information on the applications of this class of bodies, the results for the mode functions and for the natural frequencies of vibration of tapered beams are presented in this report. This report includes much of the existing data as special cases. The results appear in tables and in graphical form. The complete and detailed derivations are reported by Wang [9].

2. Profiles of the Beams

The beams whose flexural rigidity and mass per unit length vary according to two arbitrary powers of the longitudinal coordinate are considered in this report. The relationship of the variations may be written as

$$EI = E_o I_o \left(\frac{X}{\ell}\right)^m$$
$$\rho A = \rho_o A_o \left(\frac{X}{\ell}\right)^n \qquad (4.3)$$

* The notation (4.1) is adopted to aid in cross-referencing equations from the first three reports under this grant.

** Number in brackets refer to the References.

where E_oI_o is the bending rigidity and ρ_oA_o is the mass per unit length at the larger end of the beam where $X = \ell$.

The relations of Eq. (4.3) can be considered as a homogeneous beam with the moment of inertia and cross-sectional area varying with powers m and n respectively. The relations can be applied for a general class of cross-sections with varying thickness and varying width. An important group of beam shapes can be considered as shown in Fig. 1. The cross-section of the beam is symmetrical and its first quadrant is bounded by the curve of the equation

$$\left(\frac{y}{h}\right)^{\beta} + \left(\frac{z}{b}\right)^{\gamma} = 1 \qquad (4.4)$$

where b represents half of the width and h represents half of the thickness of the beam. These parameters vary according to the relations

$$b = b_o\left(\frac{X}{\ell}\right)^{\psi}, \qquad h = h_o\left(\frac{X}{\ell}\right)^{\phi} \qquad (4.5)$$

The constants ψ and ϕ are positive but not necessarily integers.

The selection of different values for the parameters γ and β in Eq. (4.4) permits the cross-section of the beam to be varied from the diamond shape, $\gamma = \beta = 1$, through the elliptical shape, $\gamma = \beta = 2$, to the rectangular shape, γ and $\beta >> 1$. The moment of inertia and the area for this group of cross-section may be expressed in terms of γ and β [10], which gives

$$I = \frac{4}{3} b_o h_o^3 \left[\frac{\Gamma(\frac{1}{\gamma}+1)\ \Gamma(\frac{3}{\beta}+1)}{\Gamma(\frac{1}{\gamma}+\frac{3}{\beta}+1)}\right]\left(\frac{X}{\ell}\right)^{\psi+3\phi}$$

$$A = 4b_oh_o \left[\frac{\Gamma(\frac{1}{\gamma}+1)\ \Gamma(\frac{1}{\beta}+1)}{\Gamma(\frac{1}{\gamma}+\frac{1}{\beta}+1)}\right]\left(\frac{X}{\ell}\right)^{\psi+\phi} \qquad (4.6)$$

Comparison of Eqs. (4.6) with Eqs. (4.3) yields the relationships

$$m = \psi + 3\phi, \qquad n = \psi + \phi$$

The beam described by Eqs. (4.3) can also be considered as a nonhomogeneous beam with uniform cross-section, provided the modulus of elasticity and the density vary as powers m and n. The longitudinal vibration of beams of this type has been treated by Lindholm and Doshi [11] .

3. Symbols

X	horizontal coordinate along the length of the beam
y	vertical coordinate perpendicular to X
z	horizontal coordinate perpendicular to X
ℓ	reference length of the beam
L	actual beam length
b	half the width of the beam
h	half the thickness of the beam
b_o	the width of the beam at the larger end
h_o	the thickness of the beam at the larger end
x	dimensionless abscissa, X/ℓ
α	exponent of (X/ℓ)
β	exponent of (y/h)
γ	exponent of (z/b)
ψ	exponent of x for width variation
ϕ	exponent of x for thickness variation
m	exponent of x for area moment of inertia variation
n	exponent of x for cross-sectional area variation
EI	bending rigidity of the beam (elastic modulus times moment of inertia of the cross-section)
ρA	mass per unit length of the beam
θ	$= 4 - m + n$
u	x^{θ}
δ_x	$x \frac{d}{dx}$
δ_u	$u \frac{d}{du}$
ω	circular frequency
p	eigenvalue , $\rho A_o \ell^4 \omega^2 / (EI_o)$
s	indicial root
${}_0F_3$	generalized hypergeometric function
$y(x)$	mode function

$U(x, \xi)$	influence function for beam deflections
$K(x, \xi)$	kernel of homogeneous integral equations
c	truncation of the beam, dimensionless; see Fig. 1
a_r	coefficients of the series
$P_u Q_v$	$\delta^u P \ \delta^v Q - \delta^v P \ \delta^u Q$
r_g	radius of gyration

4. Acknowledgement

This project was sponsored by the National Aeronautics and Space Administration, Office of Advanced Research and Technology, Applied Mathematics Branch, of which Dr. Raymond H. Wilson is Chief.

The investigation was part of the work of the Engineering Experiment Station of which Professor Ross J. Martin is Director and was conducted in the Department of Theoretical and Applied Mechanics of which Professor Thomas J. Dolan is Head. The authors wish to acknowledge the assistance of Mr. Tom E. Breuer, an undergraduate student, for contributions to various phases of the project.

Both the ILLIAC II and the IBM 7094 computer facilities were used. The ILLIAC II was constructed in the Digital Computer Laboratory, now known as the Department of Computer Science, University of Illinois with support from the Atomic Energy Commission, grant USAEC AT(11-1)-415, and from the Office of Naval Research, grant NONOR-1832 (15). The IBM 7094 computer facility is partially supported by the National Science Foundation under NSF GP700.

SOLUTIONS OF MODE FUNCTIONS

Neglecting rotatory inertia, the differential equation for the mode functions of vibrating beams is given as

$$\frac{d^2}{dX^2}\, EI(X)\, \frac{d^2y}{dX^2} - \rho\omega^2 A(X)\, y = 0 \qquad (4.7)$$

For a homogeneous beam, having the profile described by the relations

$$I = I_o \left(\frac{X}{\ell}\right)^m \quad \text{and} \quad A = A_o\left(\frac{X}{\ell}\right)^n \qquad (4.8)$$

Eq. (4.7) can be expanded as

$$x^4\frac{d^4y}{dx^4} + 2mx^3\frac{d^3y}{dx^3} + m(m-1)x^2\frac{d^2y}{dx^2} - px^\theta y = 0 \qquad (4.9)$$

where $x = (X/\ell)$, $p = \rho A_o \ell^4\omega^2/(EI_o)$ and $\theta = 4 - m + n$.

Upon introducing the differential operator δ_x to represent $x\frac{d}{dx}$, Eq. (4.9) may be written as

$$\delta_x(\delta_x - 1)(\delta_x + m - 2)(\delta_x + m - 3)\, y - px^\theta y = 0 \qquad (4.10)$$

Let $u = x^\theta$ and let $\delta_u = u\frac{d}{du}$, thus $\delta_x^r y = \theta^r \delta_u^r y$

and Eq. (4.10) yields

$$\delta_u(\delta_u - \frac{1}{\theta})(\delta_u + \frac{m-2}{\theta})(\delta_u + \frac{m-3}{\theta})y - \frac{p}{\theta^4}uy = 0 \qquad (4.11)$$

Equation (4.11) is a type of generalized hypergeometric equation [12] , which possesses a general solution with linear combinations of four generalized hypergeometric functions. It can be written as

$$y = A_1 y_o + A_2 y_1 + A_3 y_{2-m} + A_4 y_{3-m} \qquad (4.12)$$

where each series, in hypergeometric series notation, is defined as

$$
\begin{aligned}
y_o &= {}_0F_3\left(-; 1-\frac{1}{\theta}, 1+\frac{m-2}{\theta}, 1+\frac{m-3}{\theta}; \frac{p}{\theta^4}u\right) \\
y_1 &= u^{\frac{1}{\theta}}\ {}_0F_3\left(-; 1+\frac{1}{\theta}, 1+\frac{m-1}{\theta}, 1+\frac{m-2}{\theta}; \frac{p}{\theta^4}u\right) \\
y_{2-m} &= u^{\frac{2-m}{\theta}}\ {}_0F_3\left(-; 1-\frac{1}{\theta}, 1-\frac{m-1}{\theta}, 1-\frac{m-2}{\theta}; \frac{p}{\theta^4}u\right) \\
y_{3-m} &= u^{\frac{3-m}{\theta}}\ {}_0F_3\left(-; 1+\frac{1}{\theta}, 1-\frac{m-2}{\theta}, 1-\frac{m-3}{\theta}; \frac{p}{\theta^4}u\right)
\end{aligned} \tag{4.13}
$$

provided that m is not an integer. When m is an integer, then logarithmic terms appear in the general solution.

The solutions of the differential equation (4.10) also can be obtained by the standard series method, the method of Frobenius. The series in Eq. (4.12) will have the form

$$y_s = x^s \sum_{r=o}^{\infty} a_r x^{r\theta} \tag{4.14}$$

with the recurrence formular of the coefficients being

$$a_r = \frac{p}{(s+r\theta)(s+r\theta-1)(s+r\theta+m-2)(s+r\theta+m-3)} a_{r-1} \tag{4.15}$$

where s = 0, 1, 2-m and 3-m are the roots of the indicial equation.

When the parameter m is an integer, then two equal roots of s may exist for the indicial equation or the denominator of the recurrence formula of Eq. (4.15) becomes zero. When equal roots occur, the series solutions of Eq. (4.13) are dependent on each other. When the denominator of the recurrence formula is zero, the series solutions are meaningless. In these cases, by applying the theorems due to Frobenius [13], the independent series solution for the repeated root, $s = s_o$, is

$$
\begin{aligned}
y_{s_o} &= \log x \cdot (y)_{s=s_o} \\
&- x^{s_o} \sum_{r=1}^{\infty} \left[\sum_{\eta=1}^{r} \left(\frac{1}{s_o+\eta\theta} + \frac{1}{s_o+\eta\theta-1} + \frac{1}{s_o+\eta\theta+m-2} + \frac{1}{s_o+\eta\theta+m-3} \right) \right] a_r x^{r\theta}
\end{aligned} \tag{4.16}
$$

The series solution for the root, $s = s_1$, which makes the recurrence formula of the coefficients undefined, is given by

$$y_{s_1} = \log x \cdot (y)_{s=s_1}$$

$$+x^{s_1}\left\{1+\sum_{r=1}^{\infty}\left[\frac{1}{s-s_1} - \sum_{\eta=1}^{r}\left(\frac{1}{s+\eta\theta}+\frac{1}{s+\eta\theta-1}+\frac{1}{s+\eta\theta+m-2}+\frac{1}{s+\eta\theta+m-3}\right)\right]_{s=s_1} a_r x^{r\theta}\right\} \quad (4.17)$$

Again, when $\theta = 0$, the series of Eq. (4.14), does not apply. For this case, the recurrence formula of the coefficients is again undefined; however, Eq. (4.10) is then of the Euler-Cauchy type for which a general solution always exists of the form

$$y = A_1 x^{s_1} + A_2 x^{s_2} + A_3 x^{s_3} + A_4 x^{s_4} \quad (4.18)$$

where s_1, s_2, s_3, and s_4 are the roots of the auxiliary equation

$$s(s-1)(s+m-2)(s+m-3) - p = 0 \quad (4.19)$$

Vibration can occur only when Eq. (4.19) has two or four non-real roots. This condition introduces the trigonometric function of $\log x$ into the solution for y. Suppose s_1, s_2 are two real roots and s_3, s_4 are two complex roots represented by $\alpha_1 \pm i\alpha_2$, then the solution for the normal function has the form

$$y = A_1 x^{s_1} + A_2 x^{s_2} + x^{\alpha_1}\left[A_3 \cos(\alpha_2 \log x) + A_4 \sin(\alpha_2 \log x)\right] \quad (4.20)$$

FREQUENCY EQUATIONS AND NODAL POINTS

To establish the frequency equations for tapered beams with different end conditions, two separate geometrical catagories of beams are treated. First, the complete tapered beam, which is gradually narrowed toward a point, is considered. The end coordinates of the beam are o and ℓ. Next, the truncated beam, with end coordinates $c \cdot \ell$ and ℓ, as indicated in Fig. 1, is considered. The origin, for the truncated beam lies beyond the end of the beam and serves as a reference point for the variation of the area moment of inertia and for the cross-sectional area of the beam.

1. Complete Tapered Beams

Since one end of the beam is at the origin, the arbitrary constants A_3 and A_4 in the general solution, Eq. (4.12), must vanish if finite values exist for the deflection, moment and shear at the end x = o. Hence, the general solution of the mode function for beams with any combination of parameters m and n can be written as

$$y = A_1 P + A_2 Q \qquad (4.21)$$

where P and Q are the series of y_o and y_1 as defined in Eq. (4.13). The frequency equation is then obtained using the boundary conditions at $X = \ell$, that is, at x = 1. Three different cases are considered below.

A. Cantilever Beams

The base of the beam is fixed and the tip is free. Both the deflection and slope vanish at x = 1. Application of the boundary conditions to Eq. (4.21) and elimination of the constants A_1 and A_2 yields the frequency equation

$$P\frac{dQ}{dx} - Q\frac{dP}{dx} = 0 \;, \qquad \text{at } x = 1 \qquad (4.22)$$

Representation of $\delta^u P \cdot \delta^v Q - \delta^v P \cdot \delta^u Q$ in symbolic form as $P_u Q_v$ yields the relations $P_u Q_v = -P_v Q_u$ and $\delta(P_u Q_{u+1}) = P_u Q_{u+2}$. Using this notation, the frequency equation yields

$$\frac{1}{x} P Q_1 = 0 \;, \qquad \text{at } x = 1 \qquad (4.23)$$

Since P and Q are in series form, as in Eq. (4.14), their derivatives and products are in series form as well.

Let the series be defined as $U = PQ_1$. Then U is a solution of a fifth order differential equation

$$(\delta-1)(\delta+m-2)(\delta+m-3)(\delta+m-4)(\delta+2m-5)U+2(2\delta+\theta+2m-6)px^{\theta}U=0 \quad (4.24)$$

Solving Eq. (4.24) and comparing the corresponding terms of U with Eq. (4.23) the series which satisfies both Eq. (4.24) and (4.23) is established as

$$U = \sum_{r=0}^{\infty} a_r p^r x^{1+r\theta} \quad (4.25)$$

Substitution of U into Eq. (4.23) yields the frequency equation as a polynomial of p

$$\sum_{r=o}^{\infty} a_r p^r = 0 \quad (4.26)$$

with the recurrence formula of coefficients

$$a_r = \frac{-2(2r\theta-\theta+2m-4)}{r\theta(r\theta+m-1)(r\theta+m-2)(r\theta+m-3)(r\theta+2m-4)} a_{r-1} \quad (4.27)$$

B. Antisymmetrical Mode Vibration of Free-Free Beams

Tapered beams with both ends free are considered to consist of two equal halves fitted together at their large ends as shown in Fig. 2. Each half is a section of a solid which has the profile shown in Fig. 1. When the beam vibrates in the antisymmetrical mode, the deflection and the bending moment are zero at the middle section where the two halves are joined together, that is

$$y = 0 \qquad \text{and } EI(x)\frac{d^2y}{dx^2} = 0 \qquad \text{at } x = 1$$

Substitution of these conditions into Eq. (4.21) and applying the differential operator P_uQ_v, the frequency equation becomes

$$\frac{1}{x^2}U = 0 \qquad \text{at } x = 1 \quad (4.28)$$

where $U = PQ_2 - PQ_1$. The function U is a solution of a fifth order differential equation

$$(\delta-\theta-1)(\delta+m-4)(\delta+m-3)(\delta+2m-5)(\delta+m-2)U+2(2\delta+\theta+2m-6)px^{\theta}U = 0 \tag{4.29}$$

Hence, the frequency equation may again be represented by a polynomial of p, of the form of Eq. (4.26), with the coefficients

$$a_r = \frac{-2(2r\theta+\theta+2m-4)}{r\theta(r\theta+\theta+m-1)(r\theta+\theta+m-2)(r\theta+\theta+m-3)(r\theta+\theta+2m-4)} a_{r-1} \tag{4.30}$$

The locations of the nodes for the normal mode vibration, can be obtained by substituting the corresponding natural frequency, p_i, into Eq. (4.21) and solving for x with y equal to zero. Since P and Q are generalized hypergeometric functions, the equation, which gives the nodal points for cantilever beams and free-free beams of antisymmetrical mode, can be represented by

$$\begin{aligned} &{}_0F_3\left(-; 1+\frac{1}{\theta}, 1+\frac{m-1}{\theta}, 1+\frac{m-2}{\theta}; \frac{p_i}{\theta^4}\right) {}_0F_3\left(-; 1-\frac{1}{\theta}, 1+\frac{m-2}{\theta}, 1+\frac{m-3}{\theta}; p_i\frac{x^{\theta}}{\theta^4}\right) \\ &-x\,{}_0F_3\left(-; 1-\frac{1}{\theta}, 1+\frac{m-2}{\theta}, 1+\frac{m-3}{\theta}; \frac{p_i}{\theta^4}\right) {}_0F_3\left(-; 1+\frac{1}{\theta}, 1+\frac{m-1}{\theta}, 1+\frac{m-2}{\theta}; p_i\frac{x^{\theta}}{\theta^4}\right) = 0 \end{aligned} \tag{4.31}$$

C. Symmetrical Mode Vibration of Free-Free Beams

For the symmetrical mode vibration of the free-free beam, Fig. 2, the slope and shear are zero at the middle section of the beam, hence

$$\frac{dy}{dx} = 0 \qquad \text{and } \frac{d}{dx} EI(x) \frac{d^2y}{dx^2} = 0 \qquad \text{at } x = 1$$

Upon substituting the end conditions into Eq. (4.21) and using the notation P_uQ_v, the frequency equation may be written as

$$\frac{d}{dx}(x^{m-3}U) = 0 \tag{4.32}$$

at x = 1, where

$$U = PQ_3 + (m-3)PQ_2 - (m-2)PQ_1$$

As discussed in the previous sections, U may be represented by a series which is a solution of a sixth order differential equation

$$(\delta - \theta - 1)(\delta - \theta + m - 2)(\delta - \theta + m - 3)(\delta + m - 4)(\delta + m - 3)(\delta + 2m - 5)\, U$$

$$+ 2(2\delta + \theta + 2m - 6)(\delta + m - 3)\, p x^{\theta}\, U = 0 \qquad (4.33)$$

Substituting the solution U into Eq. (4.32) and evaluating at $x = 1$, gives the equation for the eigenvalues p as

$$\sum_{r=0}^{\infty} (r\theta + \theta + m - 2)\, a_r p^r = 0 \qquad (4.34)$$

with the recurrence formulas for the coefficients

$$a_r = \frac{-2(2r\theta + \theta + 2m - 4)}{r\theta(r\theta + m - 1)(r\theta + \theta + m - 2)(r\theta + \theta + m - 3)(r\theta + \theta + 2m - 4)}\, a_{r-1} \qquad (4.35)$$

The locations of the nodes for the normal mode vibration in this case can be solved from the equation

$$\left[\frac{p_i}{(\theta + 1)(\theta + m - 1)(\theta + m - 2)}\, {}_0F_3\left(-;\, 2 + \frac{1}{\theta},\, 2 + \frac{m-1}{\theta},\, 2 + \frac{m-2}{\theta};\, \frac{p_i}{\theta^4}\right) \right.$$

$$\left. + {}_0F_3\left(-;\, 1 + \frac{1}{\theta},\, 1 + \frac{m-1}{\theta},\, 1 + \frac{m-2}{\theta};\, \frac{p_i}{\theta^4}\right) \right] {}_0F_3\left(-;\, 1 - \frac{1}{\theta},\, 1 + \frac{m-2}{\theta},\, 1 + \frac{m-3}{\theta};\, p_i \frac{x^{\theta}}{\theta^4}\right)$$

$$- \left[\frac{p_i x}{(\theta - 1)(\theta + m - 2)(\theta + m - 3)}\, {}_0F_3\left(-;\, 1 - \frac{1}{\theta},\, 1 + \frac{m-2}{\theta},\, 1 + \frac{m-3}{\theta};\, \frac{p_i}{\theta^4}\right) \right.$$

$$\left. {}_0F_3\left(-;\, 1 + \frac{1}{\theta},\, 1 + \frac{m-1}{\theta},\, 1 + \frac{m-2}{\theta};\, p_i \frac{x^{\theta}}{\theta^4}\right) \right] = 0 \qquad (4.36)$$

2. Truncated Tapered Beams

A. Exact Solutions

A beam which is gradually reduced to a small cross-section instead of to a point, may be considered as a beam truncated at the location $x = c$ as shown in Fig. 1. The total length of the beam is $(1 - c) \cdot \ell$, where ℓ is a reference length for the tapering. Since the beam does not start from the origin, the general solution of the mode function must be written in the form of Eq. (4.12) with four series. By substituting the end conditions into the mode function, a fourth order determinant equation of the eigenvalues p may be obtained.

Consider the case for a cantilever beam; the end conditions are

$$y = y' = 0 \qquad \text{at } x = 1$$

and $$EIy'' = (EIy'')' = 0 \qquad \text{at } x = c$$

which gives the frequency equation

$$\begin{vmatrix} \left[y_o\right]_{x=1} & \left[y_1\right]_{x=1} & \left[y_{2-m}\right]_{x=1} & \left[y_{3-m}\right]_{x=1} \\ \left[y_o'\right]_{x=1} & \left[y_1'\right]_{x=1} & \left[y_{2-m}'\right]_{x=1} & \left[y_{3-m}'\right]_{x=1} \\ \left[y_o''\right]_{x=c} & \left[y_1''\right]_{x=c} & \left[y_{2-m}''\right]_{x=c} & \left[y_{3-m}''\right]_{x=c} \\ \left[(x^m y_o'')'\right]_{x=c} & \left[(x^m y_1'')'\right]_{x=c} & \left[(x^m y_{2-m}'')'\right]_{x=c} & \left[(x^m y_{3-m}'')'\right]_{x=c} \end{vmatrix} = 0 \qquad (4.37)$$

where y_o, y_1, y_{2-m} and y_{3-m} are defined by Eqs. (4.13), (4.16), or (4.17).

B. Approximate Solutions

The calculation of the natural frequencies from the characteristic equation, Eq. (4.37), involves tedious numerical computations since each element in the determinant is a combination of generalized hypergeometric series. For practical purposes and in order to supply results to check the exact solutions, two approximate methods are introduced for calculating the upper bound and the lower bound of the approximate fundamental frequencies.

The Ritz method is one of the approximate methods which has been widely used to determine the upper bound of the natural frequencies for elastic systems. The frequency of the fundamental mode is calculated by minimizing the expression for the energies, which, for beams having the prescribed profile is

$$p = \int_c^1 x^m \left(\frac{d^2y}{dx^2}\right)^2 dx \Big/ \int_c^1 x^n y^2 dx \tag{4.38}$$

As an example, consider a cantilever beam. In order to satisfy the boundary conditions at free end, a one term approximation is assumed for the second derivative of the beam deflection, as

$$y'' = 12a\,(x - c)^2 \tag{4.39}$$

Integration of Eq. (4.39) gives the deflection curve

$$y = a\left[(x - c)^4 + C_1 x + C_2\right]$$

where $C_1 = -4\,(1 - c)^3$ and $C_2 = 4(1-c)^3 - (1-c)^4$ in order to satisfy the boundary conditions $y = y' = 0$ at $x = 1$.

The lower bound approximate frequency of the fundamental mode is obtained from the expression

$$p = 1 \Big/ \int_c^1 K(\xi, \xi)\, d\xi \tag{4.41}$$

where

$$K(x, \xi) = \frac{EI_0}{\ell^3} \xi^n U(x, \xi) \tag{4.42}$$

is the kernel of a homogeneous integral equation for the beam profile of current interest. The influence function for beam deflections, $U(x, \xi)$, is the static deflection of the beam at x with a unit load applied at a distance ξ measured from the origin. For a cantilever beam, $U(x, \xi)$ can be expressed as

$$U(x, \xi) = \frac{\ell^3}{EI_0} \left[\frac{1}{(2-m)(3-m)} x^{3-m} - \frac{\xi}{(1-m)(2-m)} x^{2-m} + \frac{(2-m)\xi - (1-m)}{(1-m)(2-m)} x - \frac{(3-m)\xi - (2-m)}{(2-m)(3-m)} \right] \tag{4.43}$$

for $m \neq 1$, 2 and 3, and

$$\begin{aligned} U(x, \xi) &= \frac{\ell^3}{EI_0} \left[\frac{x^2}{2} - \xi x(\log x - 1) - x - \xi + \frac{1}{2} \right], && \text{for } m = 1 \\ U(x, \xi) &= \frac{\ell^3}{EI_0} \left[x(\log x - 1) + \xi(\log x - x) + \xi + 1 \right], && \text{for } m = 2 \\ U(x, \xi) &= \frac{\ell^3}{EI_0} \left[x\left(1 - \frac{\xi}{2}\right) - \log x - \frac{\xi}{2x} + \xi - 1 \right], && \text{for } m = 3 \end{aligned} \tag{4.44}$$

NUMERICAL RESULTS AND DISCUSSION OF TABLES

1. Natural Frequencies and Nodal Points for Complete Tapered Beams

The natural frequencies for different vibrational modes of complete tapered beams can be calculated by solving for the roots of polynominals of Eqs. (4.26) and (4.34). The coefficients of the polynominals, for cantilever beams, for free-free beams executing antisymmetrical vibrational modes and for free-free beams executing symmetrical vibrational modes, can be generated from Eqs. (4.27), (4.30) and (4.35) respectively. The recurrence formulas indicate that the coefficients reduce rapidly as the number of terms increases. Therefore, in the numerical calculations, the first five frequencies are computed with the first sixteen terms of the series. Computer programs are written to generate the coefficients as well as to solve for the roots.

The exponents, for a uniform beam, are $\psi = \phi = 0$ or $m = 0$, $\theta = 4$, and the frequency equation for cantilever beams, Eq. (4.26), becomes

$$\sum_{r=0}^{\infty} \frac{(-1)^r}{(4\,r)\,!} (4\,p)^r = 0 \tag{4.45}$$

where $p = \omega^2 \left(\frac{\rho A_o}{E I_o}\right) \ell^4$ as defined earlier. Let the beam length be L, which is equal to ℓ^o for the complete tapered beam, and let the frequency constant K be $\sqrt{p}$. Then the natural frequency can be represented as

$$\omega = K \sqrt{\frac{E I_o}{\rho A_o}} \bigg/ L^2 \tag{4.46}$$

The first five frequency constants, K, for uniform cantilever beams are obtained from Eq. (4.45) using the first 16 terms of the series. These frequency constants are

3.51601, 22.0345, 61.6972, 120.911, 199.860

The substitution of each frequency into Eq. (4.31) establishes the locations the nodes for the corresponding natural frequency, which are given in the following table.

Uniform Cantilever Beam

ψ	ϕ	mode	frequency constant, K	locations of nodes, (X/L)				
.00	.00	1st	3.51601	1.00000				
		2nd	22.03449	.21656	1.00000			
		3rd	61.69721	.13232	.49645	1.00000		
		4th	120.90191	.09444	.35591	.64166	1.00000	
		5th	199.85953	.07345	.27678	.50009	.72125	1.00000

In the table, the first two columns display the combinations of the exponents ψ and ϕ, the third and fourth columns indicate corresponding modes and their natural frequencies. The remaining columns show the locations of the nodes for different modes.

For cantilever beams with other combinations of ψ and ϕ, the results for frequencies and nodes for the first five modes are listed in Table 1. Page one of Table 1 lists data for the combinations of ψ and ϕ corresponding to beams of constant thickness with width varying as x^{ψ}. Page two of the table lists corresponding data for beams of constant width with thickness varying as x^{ϕ}. The frequency data of these two cases are also plotted in Figs. 3 and 4 as the ratio of frequencies of tapered beams to those of uniform beams. Fig. 5 indicates the variation of the ratio of frequencies for the beams with taper both in width and thickness according to x^{ϕ}. The ratio of frequencies for the first three modes, for 81 combinations of ψ and ϕ, are plotted in Fig. 6 in three dimensional form. The figures reveal the variation of the natural frequencies of different modes as the taper of the beam varies. It is of interest to note that the frequencies of the fundamental mode increase when the beams taper either in width or in thickness. The frequencies of the higher modes increase as the taper of the width increases, that is, as ψ decreases. The higher mode frequencies decrease as the taper on the thickness increases, that is, as ϕ increases.

The shapes of the first four normal modes for the vibration of tapered cantilever beams are plotted in Figs. 7, 8 and 9. Fig. 7 shows the change of mode shapes and the shifting of the nodal points for constant thickness beams as the exponent ψ increases. Fig. 8 shows those for beams with constant width and varying thickness. Fig. 9 displays those for beams with both width and thickness varying as a same power of x, that is $\psi = \phi$. The amplitudes of the deflections are normalized to the deflection at the free end.

For free-free beams of antisymmetrical and symmetrical mode vibration, the frequency constant K and the nodal points are also computed. For a uniform free-free beam, vibrating in antisymmetrical modes, the frequencies can be obtained from Eq. (4.26) with coefficients given by Eq. (4.30) and with m = 0, θ = 4, that is

$$\sum_{r=0}^{\infty} \frac{(-1)^r}{(4r+3)!} (4p)^r = 0 \qquad (4.47)$$

The substitution of the first five roots of Eq. (4.47) into Eq. (4.31) gives the locations of the nodes for the corresponding mode. The results are listed as follows:

Uniform Free-Free Beam of Antisymmetrical Mode

ψ	ϕ	mode	frequency constant, K	locations of nodes (X/L)				
.00	.00	1st	5.59332	.44832				
		2nd	30.22585	.18889	.71161			
		3rd	74.63888	.12020	.45291	.81825		
		4th	138.79131	.08814	.33213	.60005	.86666	
		5th	222.68295	.06959	.26221	.47372	.68421	.89474

The results for other combinations of ψ and ϕ are listed in Table 2 in the same order as in Table 1. The variation of the ratio of the frequencies of tapered beams to the frequencies of uniform beams are plotted in Fig. 10. The nodal points listed here are for one half of the beam length. The nodes of the other half of the beam are symmetrically located. The notation L represents half the total length of the free-free beam.

For free-free beams of symmetrical mode vibration, the frequencies and nodal points are evaluated from Eqs. (4.34) and (4.36) respectively. The results are listed in Table 3. For a uniform beam, the frequency equation is given as

$$\sum_{r=0}^{\infty} \frac{(-1)^r}{(4r+1)!} (4p)^r = 0 \qquad (4.48)$$

which gives the first five roots and the corresponding nodes as follows:

Uniform Free-Free Beam of Symmetrical Mode

ψ	ϕ	mode	frequency constant, K	locations of nodes (X/L)					
.00	.00	1st	15.41821	.2642	1.0000				
		2nd	49.96486	.1469	.5536	1.0000			
		3rd	104.24769	.1017	.3832	.6924	1.0000		
		4th	178.26972	.0778	.2931	.5295	.7647	1.0000	
		5th	272.03098	.0630	.2372	.4286	.6190	.8095	1.0000

The above calculated results for frequencies and nodes for general tapered beams are checked with existing results of some special cases which appear in References [14, 15, 16, 17, 18, 19, 20].

2. Natural Frequencies of Truncated Beams

For truncated cantilever beams the frequency constants, K in Eq. (4.46), are the roots of Eq. (4.37). The numerical results are obtained by using the method of regula falsi. The series involved in each of the elements of Eq. (4.37) are calculated using 16 terms. The frequency constants K for the first two vibrational modes are presented in Table 4 for beams truncated at $0.2\ \ell$ and at $0.4\ \ell$. The combinations of the exponents ψ and ϕ again include the beams with constant thickness, with constant width and with both thickness and width varying as the same power.

The upper bound and the lower bound approximations for the fundamental natural frequencies are calculated for six different degrees of truncation as well as for complete tapered beams. The values for the lower bound approximation are evaluated from Eq. (4.41) with the kernel defined in Eq. (4.42). The values of the upper bound approximation are evaluated from Eq. (4.38) with the deflection curve defined as Eq. (4.40). The approximate values of K are listed in Table 5 with 84 different combinations of ψ and ϕ.

Comparisons of the correct frequencies of truncated beams with the approximate frequencies appear in Figs. 12, 13 and 14. The results for constant thickness beams with varying width appear in Fig. 12, while Fig. 13 displays results for constant width beams with varying thickness and Fig. 14 gives the results for beams for which both width and thickness vary. The frequencies of the upper bound approximation for constant thickness beams are closer to the correct results than those for constant width beams. This implies that the assumed deflection curve is more nearly correct for the constant thickness beams. The lower bound approximations also yield more nearly correct results for beams with constant thickness than for beams with constant width. This is true because the frequencies of the higher modes for constant thickness beams are larger than those for constant width beams.

3. Radii of Gyration of Cross-Section

The evaluation of circular frequencies from the frequency constants K, as defined in Eq. (4.46), involves the calculation of the radius of gyration, $(I_o/A_o)^{1/2}$. For the beams with cross-sections bounded by Eq. (4.4), the area moments of inertia and the cross-sectional areas with different values of γ and β were listed in the first report [1]. The radius of gyration of the cross-section at the large end of the beam, calculated from Eq. (4.6), is

see errata

$$\frac{r_g}{h_o} = \frac{1}{3} \frac{\left(\frac{3}{\beta} + 1\right) \left(\frac{1}{\gamma} + \frac{1}{\beta} + 1\right)}{\left(\frac{1}{\beta} + 1\right) \left(\frac{1}{\gamma} + \frac{3}{\beta} + 1\right)} \qquad (4.49)$$

The results of Eq. (4.49) are listed in Table 6 with the same combinations of α and β as those used in the first report.

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TABLE 1. NATURAL FREQUENCIES AND NODES FOR TAPERED CANTILEVER BEAMS

Ψ	Φ	MODE	FREQUENCY CONSTANT, K	LOCATIONS OF NODES (X/L)				
.10	.00	1ST	3.84759	1.00000				
		2ND	22.91518	.23054	1.00000			
		3RD	63.04715	.14189	.50266	1.00000		
		4TH	122.74233	.10160	.36153	.64453	1.00000	
		5TH	202.19116	.07916	.28164	.50319	.72293	1.00000
.20	.00	1ST	4.18617	1.00000				
		2ND	23.79954	.24375	1.00000			
		3RD	64.40342	.15105	.50861	1.00000		
		4TH	124.58941	.10851	.36696	.64732	1.00000	
		5TH	204.52963	.08469	.28636	.50624	.72459	1.00000
.30	.00	1ST	4.53188	1.00000				
		2ND	24.68792	.25626	1.00000			
		3RD	65.76606	.15985	.51432	1.00000		
		4TH	126.44310	.11519	.37221	.65005	1.00000	
		5TH	206.87497	.09006	.29095	.50922	.72621	1.00000
.40	.00	1ST	4.88482	1.00000				
		2ND	25.58063	.26812	1.00000			
		3RD	67.13513	.16831	.51982	1.00000		
		4TH	128.30356	.12166	.37731	.65272	1.00000	
		5TH	209.22717	.09527	.29542	.51215	.72781	1.00000
.50	.00	1ST	5.24506	1.00000				
		2ND	26.47796	.27941	1.00000			
		3RD	68.51067	.17647	.52511	1.00000		
		4TH	130.17072	.12793	.38225	.65533	1.00000	
		5TH	211.58626	.10035	.29977	.51502	.72938	1.00000
.60	.00	1ST	5.61263	1.00000				
		2ND	27.38015	.29017	1.00000			
		3RD	69.89271	.18434	.53022	1.00000		
		4TH	132.04458	.13402	.38705	.65788	1.00000	
		5TH	213.95227	.10529	.30402	.51783	.73092	1.00000
.80	.00	1ST	6.36984	1.00000				
		2ND	29.19990	.31028	1.00000			
		3RD	72.67645	.19930	.53992	1.00000		
		4TH	135.81270	.14570	.39625	.66281	1.00000	
		5TH	218.70500	.11482	.31221	.52331	.73394	1.00000
1.00	.00	1ST	7.15646	1.00000				
		2ND	31.04131	.32874	1.00000			
		3RD	75.48660	.21333	.54900	1.00000		
		4TH	139.60798	.15678	.40498	.66755	1.00000	
		5TH	223.48545	.12392	.32004	.52860	.73685	1.00000

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Ψ	Φ	MODE	FREQUENCY CONSTANT, K	LOCATIONS OF NODES (X/L)				
.00	.10	1ST	3.75540	1.00000				
		2ND	21.53823	.22510	1.00000			
		3RD	58.22532	.13605	.49123	1.00000		
		4TH	112.59111	.09607	.34845	.63288	1.00000	
		5TH	184.78438	.07402	.26843	.48864	.71254	1.00000
.00	.20	1ST	3.98423	1.00000				
		2ND	20.99322	.23211	1.00000			
		3RD	54.81987	.13863	.48536	1.00000		
		4TH	104.54557	.09677	.34037	.62355	1.00000	
		5TH	170.27172	.07380	.25953	.47657	.70323	1.00000
.00	.30	1ST	4.20172	1.00000				
		2ND	20.40273	.23769	1.00000			
		3RD	51.48127	.14012	.47883	1.00000		
		4TH	96.76554	.09660	.33162	.61360	1.00000	
		5TH	156.32171	.07285	.25005	.46380	.69324	1.00000
.00	.40	1ST	4.40698	1.00000				
		2ND	19.76973	.24191	1.00000			
		3RD	48.20998	.14058	.47157	1.00000		
		4TH	89.25128	.09560	.32216	.60296	1.00000	
		5TH	142.93450	.07123	.23998	.45027	.68250	1.00000
.00	.50	1ST	4.59896	1.00000				
		2ND	19.09684	.24483	1.00000			
		3RD	45.00642	.14003	.46353	1.00000		
		4TH	82.00310	.09381	.31196	.59156	1.00000	
		5TH	130.11026	.06896	.22928	.43590	.67089	1.00000
.00	.60	1ST	4.77655	1.00000				
		2ND	18.38646	.24647	1.00000			
		3RD	41.87105	.13849	.45463	1.00000		
		4TH	75.02126	.09126	.30096	.57928	1.00000	
		5TH	117.84916	.06610	.21793	.42060	.65832	1.00000
.00	.80	1ST	5.08332	1.00000				
		2ND	16.86125	.24588	1.00000			
		3RD	35.80663	.13245	.43390	1.00000		
		4TH	61.85787	.08395	.27624	.55160	1.00000	
		5TH	95.01719	.05871	.19314	.38675	.62966	1.00000
.00	1.00	1ST	5.31510	1.00000				
		2ND	15.20717	.23980	1.00000			
		3RD	30.01981	.12230	.40833	1.00000		
		4TH	49.76335	.07376	.24730	.51858	1.00000	
		5TH	74.44003	.04931	.16529	.34763	.59497	1.00000

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Ψ	Φ	MODE	FREQUENCY CONSTANT, K	LOCATIONS OF NODES (X/L)				
.10	.10	1ST	4.08637	1.00000				
		2ND	22.38253	.23812	1.00000			
		3RD	59.52071	.14497	.49731	1.00000		
		4TH	114.35302	.10272	.35396	.63583	1.00000	
		5TH	187.01330	.07930	.27318	.49183	.71431	1.00000
.10	.20	1ST	4.31441	1.00000				
		2ND	21.80267	.24428	1.00000			
		3RD	56.06091	.14698	.49135	1.00000		
		4TH	106.22908	.10297	.34578	.62659	1.00000	
		5TH	172.39803	.07869	.26418	.47984	.70509	1.00000
.10	.30	1ST	4.53085	1.00000				
		2ND	21.17865	.24911	1.00000			
		3RD	52.66817	.14797	.48473	1.00000		
		4TH	98.37077	.10239	.33694	.61674	1.00000	
		5TH	158.34546	.07739	.25461	.46715	.69520	1.00000
.10	.40	1ST	4.73475	1.00000				
		2ND	20.51322	.25268	1.00000			
		3RD	49.34290	.14797	.47741	1.00000		
		4TH	90.77835	.10102	.32741	.60621	1.00000	
		5TH	144.85576	.07544	.24444	.45371	.68456	1.00000
.10	.50	1ST	4.92505	1.00000				
		2ND	19.80887	.25501	1.00000			
		3RD	46.08556	.14700	.46932	1.00000		
		4TH	83.45211	.09889	.31714	.59491	1.00000	
		5TH	131.92909	.07287	.23366	.43943	.67307	1.00000
.10	.60	1ST	5.10058	1.00000				
		2ND	19.06779	.25613	1.00000			
		3RD	42.89657	.14508	.46039	1.00000		
		4TH	76.39234	.09602	.30607	.58276	1.00000	
		5TH	119.56565	.06973	.22222	.42421	.66062	1.00000
.10	.80	1ST	5.40200	1.00000				
		2ND	17.48294	.25464	1.00000			
		3RD	36.72535	.13835	.43962	1.00000		
		4TH	63.07337	.08812	.28123	.55534	1.00000	
		5TH	96.52915	.06181	.19723	.39055	.63226	1.00000
.10	1.00	1ST	5.62660	1.00000				
		2ND	15.77062	.24780	1.00000			
		3RD	30.83211	.12757	.41406	1.00000		
		4TH	50.82352	.07737	.25215	.52262	1.00000	
		5TH	75.74766	.05191	.16916	.35161	.59792	1.00000

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Ψ	Φ	MODE	FREQUENCY CONSTANT, K	LOCATIONS OF NODES (X/L)				
.20	.10	1ST	4.42440	1.00000				
		2ND	23.23123	.25045	1.00000			
		3RD	60.82253	.15355	.50316	1.00000		
		4TH	116.12162	.10917	.35930	.63871	1.00000	
		5TH	189.24912	.08443	.27781	.49495	.71605	1.00000
.20	.20	1ST	4.65164	1.00000				
		2ND	22.61712	.25585	1.00000			
		3RD	57.30847	.15504	.49712	1.00000		
		4TH	107.91934	.10900	.35103	.62956	1.00000	
		5TH	174.53124	.08346	.26871	.48304	.70691	1.00000
.20	.30	1ST	4.86701	1.00000				
		2ND	21.96005	.26001	1.00000			
		3RD	53.86166	.15556	.49043	1.00000		
		4TH	99.98279	.10804	.34212	.61980	1.00000	
		5TH	160.37614	.08183	.25905	.47043	.69712	1.00000
.20	.40	1ST	5.06950	1.00000				
		2ND	21.26259	.26298	1.00000			
		3RD	50.48250	.15514	.48305	1.00000		
		4TH	92.31226	.10633	.33252	.60938	1.00000	
		5TH	146.78397	.07957	.24880	.45707	.68658	1.00000
.20	.50	1ST	5.25804	1.00000				
		2ND	20.52706	.26479	1.00000			
		3RD	47.17144	.15379	.47492	1.00000		
		4TH	84.90802	.10388	.32219	.59819	1.00000	
		5TH	133.75492	.07673	.23794	.44288	.67521	1.00000
.20	.60	1ST	5.43143	1.00000				
		2ND	19.75552	.26542	1.00000			
		3RD	43.92890	.15151	.46596	1.00000		
		4TH	77.77034	.10071	.31106	.58615	1.00000	
		5TH	121.28914	.07332	.22641	.42775	.66289	1.00000
.20	.80	1ST	5.72734	1.00000				
		2ND	18.11131	.26311	1.00000			
		3RD	37.65095	.14413	.44517	1.00000		
		4TH	64.29584	.09224	.28610	.55898	1.00000	
		5TH	98.04816	.06490	.20125	.39427	.63480	1.00000
.20	1.00	1ST	5.94455	1.00000				
		2ND	16.34084	.25557	1.00000			
		3RD	31.65133	.13275	.41962	1.00000		
		4TH	51.89071	.08095	.25691	.52655	1.00000	
		5TH	77.06235	.05451	.17297	.35551	.60081	1.00000

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Ψ	Φ	MODE	FREQUENCY CONSTANT, K	LOCATIONS OF NODES (X/L)				
.30	.10	1ST	4.76957	1.00000				
		2ND	24.08461	.26218	1.00000			
		3RD	62.13082	.16182	.50879	1.00000		
		4TH	117.89695	.11542	.36447	.64153	1.00000	
		5TH	191.49183	.08942	.28231	.49801	.71775	1.00000
.30	.20	1ST	4.99599	1.00000				
		2ND	23.43679	.26689	1.00000			
		3RD	58.56259	.16283	.50267	1.00000		
		4TH	109.61637	.11487	.35613	.63246	1.00000	
		5TH	176.67138	.08812	.27313	.48617	.70871	1.00000
.30	.30	1ST	5.21022	1.00000				
		2ND	22.74709	.27044	1.00000			
		3RD	55.06178	.16292	.49593	1.00000		
		4TH	101.60165	.11356	.34715	.62280	1.00000	
		5TH	162.41377	.08618	.26339	.47365	.69901	1.00000
.30	.40	1ST	5.41124	1.00000				
		2ND	22.01795	.27287	1.00000			
		3RD	51.62881	.16211	.48850	1.00000		
		4TH	93.85304	.11152	.33749	.61247	1.00000	
		5TH	148.71918	.08364	.25307	.46038	.68857	1.00000
.30	.50	1ST	5.59794	1.00000				
		2ND	21.25152	.27418	1.00000			
		3RD	48.26409	.16040	.48034	1.00000		
		4TH	86.37083	.10877	.32711	.60139	1.00000	
		5TH	135.58774	.08053	.24213	.44627	.67731	1.00000
.30	.60	1ST	5.76911	1.00000				
		2ND	20.44970	.27438	1.00000			
		3RD	44.96803	.15779	.47136	1.00000		
		4TH	79.15528	.10531	.31592	.58946	1.00000	
		5TH	123.01965	.07686	.23053	.43123	.66510	1.00000
.30	.80	1ST	6.05931	1.00000				
		2ND	18.74637	.27131	1.00000			
		3RD	38.58345	.14980	.45056	1.00000		
		4TH	65.52530	.09632	.29088	.56254	1.00000	
		5TH	99.57422	.06796	.20521	.39792	.63730	1.00000
.30	1.00	1ST	6.26895	1.00000				
		2ND	16.91781	.26312	1.00000			
		3RD	32.47746	.13786	.42504	1.00000		
		4TH	52.96490	.08451	.26158	.53040	1.00000	
		5TH	78.38411	.05711	.17672	.35934	.60365	1.00000

TABLE 1. NATURAL FREQUENCIES AND NODES FOR TAPERED CANTILEVER BEAMS

Ψ	Φ	MODE	FREQUENCY CONSTANT, K	LOCATIONS OF NODES (X/L)				
.40	.10	1ST	5.12194	1.00000				
		2ND	24.94292	.27335	1.00000			
		3RD	63.44562	.16980	.51421	1.00000		
		4TH	119.67903	.12150	.36949	.64427	1.00000	
		5TH	193.74142	.09429	.28670	.50101	.71943	1.00000
.40	.20	1ST	5.34749	1.00000				
		2ND	24.26185	.27743	1.00000			
		3RD	59.82330	.17037	.50803	1.00000		
		4TH	111.32020	.12059	.36108	.63529	1.00000	
		5TH	178.81845	.09267	.27745	.48925	.71047	1.00000
.40	.30	1ST	5.56052	1.00000				
		2ND	23.53993	.28043	1.00000			
		3RD	56.26857	.17007	.50124	1.00000		
		4TH	103.22733	.11895	.35205	.62572	1.00000	
		5TH	164.45835	.09045	.26764	.47681	.70086	1.00000
.40	.40	1ST	5.75999	1.00000				
		2ND	22.77940	.28236	1.00000			
		3RD	52.78184	.16889	.49378	1.00000		
		4TH	95.40070	.11661	.34234	.61549	1.00000	
		5TH	150.66135	.08764	.25725	.46361	.69052	1.00000
.40	.50	1ST	5.94477	1.00000				
		2ND	21.98229	.28323	1.00000			
		3RD	49.36352	.16685	.48559	1.00000		
		4TH	87.84056	.11357	.33190	.60451	1.00000	
		5TH	137.42756	.08427	.24624	.44959	.67937	1.00000
.40	.60	1ST	6.11362	1.00000				
		2ND	21.15039	.28303	1.00000			
		3RD	46.01401	.16393	.47659	1.00000		
		4TH	80.54718	.10985	.32068	.59269	1.00000	
		5TH	124.75719	.08037	.23457	.43464	.66728	1.00000
.40	.80	1ST	6.39792	1.00000				
		2ND	19.38815	.27925	1.00000			
		3RD	39.52286	.15537	.45579	1.00000		
		4TH	66.76175	.10034	.29555	.56602	1.00000	
		5TH	101.10733	.07100	.20910	.40151	.63975	1.00000
.40	1.00	1ST	6.59979	1.00000				
		2ND	17.50155	.27046	1.00000			
		3RD	33.31050	.14289	.43032	1.00000		
		4TH	54.04608	.08805	.26615	.53416	1.00000	
		5TH	79.71292	.05970	.18043	.36310	.60642	1.00000

TABLE 1. NATURAL FREQUENCIES AND NODES FOR TAPERED CANTILEVER BEAMS

Ψ	Φ	MODE	FREQUENCY CONSTANT, K	LOCATIONS OF NODES (X/L)				
.50	.10	1ST	5.48156	1.00000				
		2ND	25.80637	.28400	1.00000			
		3RD	64.76696	.17751	.51944	1.00000		
		4TH	121.46787	.12741	.37437	.64696	1.00000	
		5TH	195.99795	.09905	.29098	.50395	.72107	1.00000
.50	.20	1ST	5.70617	1.00000				
		2ND	25.09250	.28752	1.00000			
		3RD	61.09063	.17768	.51321	1.00000		
		4TH	113.03085	.12616	.36590	.63806	1.00000	
		5TH	180.97245	.09713	.28166	.49227	.71220	1.00000
.50	.30	1ST	5.91792	1.00000				
		2ND	24.33867	.29001	1.00000			
		3RD	57.48205	.17701	.50638	1.00000		
		4TH	104.85989	.12422	.35682	.62858	1.00000	
		5TH	166.50990	.09464	.27179	.47990	.70268	1.00000
.50	.40	1ST	6.11574	1.00000				
		2ND	23.54703	.29149	1.00000			
		3RD	53.94163	.17549	.49889	1.00000		
		4TH	96.95525	.12159	.34707	.61844	1.00000	
		5TH	152.61052	.09157	.26134	.46679	.69244	1.00000
.50	.50	1ST	6.29851	1.00000				
		2ND	22.71946	.29195	1.00000			
		3RD	50.46977	.17314	.49068	1.00000		
		4TH	89.31721	.11829	.33659	.60756	1.00000	
		5TH	139.27439	.08797	.25027	.45285	.68139	1.00000
.50	.60	1ST	6.46495	1.00000				
		2ND	21.85763	.29137	1.00000			
		3RD	47.06685	.16993	.48168	1.00000		
		4TH	81.94603	.11431	.32533	.59585	1.00000	
		5TH	126.50177	.08383	.23854	.43798	.66942	1.00000
.50	.80	1ST	6.74316	1.00000				
		2ND	20.03664	.28694	1.00000			
		3RD	40.46916	.16082	.46089	1.00000		
		4TH	68.00520	.10432	.30012	.56942	1.00000	
		5TH	102.64750	.07402	.21293	.40503	.64215	1.00000
.50	1.00	1ST	6.93705	1.00000				
		2ND	18.09205	.27760	1.00000			
		3RD	34.15046	.14784	.43545	1.00000		
		4TH	55.13427	.09155	.27065	.53784	1.00000	
		5TH	81.04880	.06228	.18408	.36679	.60915	1.00000

TABLE 1. NATURAL FREQUENCIES AND NODES FOR TAPERED CANTILEVER BEAMS

Ψ	Φ	MODE	FREQUENCY CONSTANT, K	LOCATIONS OF NODES (X/L)				
.60	.10	1ST	5.84844	1.00000				
		2ND	26.67514	.29418	1.00000			
		3RD	66.09489	.18496	.52450	1.00000		
		4TH	123.26350	.13316	.37911	.64959	1.00000	
		5TH	198.26140	.10369	.29516	.50684	.72269	1.00000
.60	.20	1ST	6.07203	1.00000				
		2ND	25.92884	.29719	1.00000			
		3RD	62.36461	.18477	.51822	1.00000		
		4TH	114.74832	.13160	.37060	.64077	1.00000	
		5TH	183.13340	.10150	.28579	.49523	.71390	1.00000
.60	.30	1ST	6.28241	1.00000				
		2ND	25.14344	.29922	1.00000			
		3RD	58.70225	.18375	.51135	1.00000		
		4TH	106.49930	.12937	.36147	.63138	1.00000	
		5TH	168.56846	.09875	.27586	.48294	.70447	1.00000
.60	.40	1ST	6.47850	1.00000				
		2ND	24.32092	.30028	1.00000			
		3RD	55.10819	.18192	.50384	1.00000		
		4TH	98.51672	.12647	.35168	.62133	1.00000	
		5TH	154.56667	.09545	.26535	.46991	.69432	1.00000
.60	.50	1ST	6.65916	1.00000				
		2ND	23.46308	.30036	1.00000			
		3RD	51.58284	.17928	.49562	1.00000		
		4TH	90.80081	.12292	.34117	.61055	1.00000	
		5TH	141.12825	.09161	.25422	.45605	.68338	1.00000
.60	.60	1ST	6.82309	1.00000				
		2ND	22.57145	.29944	1.00000			
		3RD	48.12654	.17579	.48661	1.00000		
		4TH	83.35184	.11870	.32987	.59894	1.00000	
		5TH	128.25335	.08725	.24244	.44126	.67152	1.00000
.60	.80	1ST	7.09501	1.00000				
		2ND	20.69188	.29441	1.00000			
		3RD	41.42238	.16618	.46584	1.00000		
		4TH	69.25565	.10825	.30461	.57274	1.00000	
		5TH	104.19474	.07702	.21669	.40848	.64451	1.00000
.60	1.00	1ST	7.28073	1.00000				
		2ND	18.68930	.28455	1.00000			
		3RD	34.99732	.15271	.44046	1.00000		
		4TH	56.22945	.09503	.27506	.54143	1.00000	
		5TH	82.39172	.06486	.18769	.37042	.61182	1.00000

TABLE 1. NATURAL FREQUENCIES AND NODES FOR TAPERED CANTILEVER BEAMS

Ψ	Φ	MODE	FREQUENCY CONSTANT, K	LOCATIONS OF NODES (X/L)				
.80	.10	1ST	6.60401	1.00000				
		2ND	28.42930	.31329	1.00000			
		3RD	68.77063	.19919	.53411	1.00000		
		4TH	126.87519	.14423	.38822	.65467	1.00000	
		5TH	202.80911	.11268	.30325	.51246	.72585	1.00000
.80	.20	1ST	6.82529	1.00000				
		2ND	27.61914	.31539	1.00000			
		3RD	64.93265	.19833	.52777	1.00000		
		4TH	118.20383	.14211	.37963	.64601	1.00000	
		5TH	187.47620	.10999	.29377	.50099	.71721	1.00000
.80	.30	1ST	7.03265	1.00000				
		2ND	26.77137	.31660	1.00000			
		3RD	61.16289	.19670	.52086	1.00000		
		4TH	109.79881	.13936	.37043	.63679	1.00000	
		5TH	172.70647	.10676	.28375	.48885	.70795	1.00000
.80	.40	1ST	7.22499	1.00000				
		2ND	25.88775	.31693	1.00000			
		3RD	57.46172	.19430	.51331	1.00000		
		4TH	101.66040	.13597	.36058	.62692	1.00000	
		5TH	158.50002	.10302	.27314	.47598	.69799	1.00000
.80	.50	1ST	7.40113	1.00000				
		2ND	24.96983	.31634	1.00000			
		3RD	53.82950	.19113	.50508	1.00000		
		4TH	93.78885	.13195	.35002	.61633	1.00000	
		5TH	144.85704	.09875	.26192	.46227	.68724	1.00000
.80	.60	1ST	7.55975	1.00000				
		2ND	24.01897	.31481	1.00000			
		3RD	50.26657	.18714	.49608	1.00000		
		4TH	86.18438	.12729	.33867	.60492	1.00000	
		5TH	131.77768	.09399	.25003	.44766	.67560	1.00000
.80	.80	1ST	7.81849	1.00000				
		2ND	22.02259	.30869	1.00000			
		3RD	43.34957	.17660	.47537	1.00000		
		4TH	71.77754	.11598	.31331	.57918	1.00000	
		5TH	107.31037	.08296	.22406	.41522	.64909	1.00000
.80	1.00	1ST	7.98725	1.00000				
		2ND	19.90401	.29790	1.00000			
		3RD	36.71176	.16225	.45011	1.00000		
		4TH	58.44082	.10190	.28364	.54839	1.00000	
		5TH	85.09876	.06998	.19476	.37750	.61700	1.00000

TABLE 1. NATURAL FREQUENCIES AND NODES FOR TAPERED CANTILEVER BEAMS

Ψ	Φ	MODE	FREQUENCY CONSTANT, K	LOCATIONS OF NODES (X/L)				
1.00	.10	1ST	7.38863	1.00000				
		2ND	30.20649	.33090	1.00000			
		3RD	71.47303	.21260	.54313	1.00000		
		4TH	130.51427	.15477	.39688	.65954	1.00000	
		5TH	207.38463	.12130	.31099	.51788	.72891	1.00000
1.00	.20	1ST	7.60720	1.00000				
		2ND	29.33359	.33224	1.00000			
		3RD	67.52760	.21115	.53675	1.00000		
		4TH	121.68684	.15216	.38823	.65104	1.00000	
		5TH	191.84694	.11816	.30143	.50655	.72042	1.00000
1.00	.30	1ST	7.81115	1.00000				
		2ND	28.42435	.33276	1.00000			
		3RD	63.65066	.20899	.52981	1.00000		
		4TH	113.12596	.14895	.37898	.64197	1.00000	
		5TH	176.87248	.11451	.29133	.49456	.71132	1.00000
1.00	.40	1ST	7.99934	1.00000				
		2ND	27.48032	.33246	1.00000			
		3RD	59.84256	.20610	.52226	1.00000		
		4TH	104.83184	.14512	.36909	.63227	1.00000	
		5TH	162.46145	.11036	.28064	.48183	.70154	1.00000
1.00	.50	1ST	8.17056	1.00000				
		2ND	26.50287	.33129	1.00000			
		3RD	56.10363	.20245	.51403	1.00000		
		4TH	96.80474	.14068	.35849	.62186	1.00000	
		5TH	148.61396	.10572	.26934	.46828	.69098	1.00000
1.00	.60	1ST	8.32347	1.00000				
		2ND	25.49314	.32923	1.00000			
		3RD	52.43416	.19803	.50505	1.00000		
		4TH	89.04486	.13562	.34711	.61065	1.00000	
		5TH	135.33020	.10057	.25737	.45383	.67955	1.00000
1.00	.80	1ST	8.56825	1.00000				
		2ND	23.38030	.32219	1.00000			
		3RD	45.30443	.18666	.48442	1.00000		
		4TH	74.32746	.12353	.32168	.58534	1.00000	
		5TH	110.45425	.08880	.23121	.42172	.65351	1.00000
1.00	1.00	1ST	8.71926	1.00000				
		2ND	21.14566	.31058	1.00000			
		3RD	38.45377	.17150	.45930	1.00000		
		4TH	60.68014	.10866	.29193	.55506	1.00000	
		5TH	87.83399	.07507	.20166	.38434	.62200	1.00000

TABLE 2. NATURAL FREQUENCIES AND NODES FOR TAPERED FREE-FREE BEAMS OF ANTISYMMETRICAL MODE

Ψ	Φ	MODE	FREQUENCY CONSTANT, K	LOCATIONS OF NODES (X/L)					
.10	.00	1ST	16.04270	.2810	1.0000				
		2ND	51.07784	.1575	.5604	1.0000			
		3RD	105.85465	.1094	.3892	.6954	1.0000		
		4TH	180.37029	.0838	.2982	.5327	.7665	1.0000	
		5TH	274.62509	.0679	.2417	.4317	.6211	.8107	1.0000
.20	.00	1ST	16.67084	.2969	1.0000				
		2ND	52.19662	.1676	.5669	1.0000			
		3RD	107.46787	.1168	.3950	.6984	1.0000		
		4TH	182.47738	.0897	.3032	.5359	.7682	1.0000	
		5TH	277.22591	.0727	.2460	.4348	.6232	.8118	1.0000
.30	.00	1ST	17.30292	.3118	1.0000				
		2ND	53.32124	.1774	.5732	1.0000			
		3RD	109.08740	.1240	.4007	.7013	1.0000		
		4TH	184.59102	.0953	.3080	.5390	.7699	1.0000	
		5TH	279.83343	.0774	.2502	.4378	.6252	.8129	1.0000
.40	.00	1ST	17.93916	.3260	1.0000				
		2ND	54.45176	.1867	.5793	1.0000			
		3RD	110.71325	.1310	.4061	.7041	1.0000		
		4TH	186.71124	.1009	.3127	.5421	.7715	1.0000	
		5TH	282.44766	.0820	.2543	.4408	.6272	.8140	1.0000
.50	.00	1ST	18.57977	.3395	1.0000				
		2ND	55.58823	.1958	.5851	1.0000			
		3RD	112.34544	.1377	.4114	.7068	1.0000		
		4TH	188.83803	.1062	.3173	.5451	.7731	1.0000	
		5TH	285.06863	.0865	.2583	.4437	.6292	.8150	1.0000
.60	.00	1ST	19.22493	.3523	1.0000				
		2ND	56.73065	.2045	.5907	1.0000			
		3RD	113.98401	.1443	.4165	.7095	1.0000		
		4TH	190.97142	.1114	.3218	.5481	.7747	1.0000	
		5TH	287.69633	.0908	.2622	.4465	.6311	.8161	1.0000
.80	.00	1ST	20.52951	.3762	1.0000				
		2ND	59.03360	.2210	.6013	1.0000			
		3RD	117.28031	.1568	.4263	.7147	1.0000		
		4TH	195.25801	.1215	.3304	.5538	.7778	1.0000	
		5TH	292.97194	.0992	.2698	.4521	.6349	.8182	1.0000
1.00	.00	1ST	21.85396	.3980	1.0000				
		2ND	61.36083	.2365	.6112	1.0000			
		3RD	120.60229	.1687	.4357	.7197	1.0000		
		4TH	199.57111	.1311	.3387	.5593	.7809	1.0000	
		5TH	298.27457	.1073	.2770	.4575	.6386	.8202	1.0000

TABLE 2. NATURAL FREQUENCIES AND NODES FOR TAPERED FREE - FREE BEAMS OF ANTISYMMETRICAL MODE

Ψ	Φ	MODE	FREQUENCY CONSTANT, K	LOCATIONS OF NODES (X/L)					
.00	.10	1ST	15.08138	.2770	1.0000				
		2ND	47.18989	.1518	.5507	1.0000			
		3RD	97.11908	.1038	.3766	.6855	1.0000		
		4TH	164.86205	.0786	.2850	.5188	.7577	1.0000	
		5TH	250.41947	.0631	.2287	.4164	.6081	.8030	1.0000
.00	.20	1ST	14.71346	.2884	1.0000				
		2ND	44.46738	.1556	.5474	1.0000			
		3RD	90.21738	.1050	.3693	.6783	1.0000		
		4TH	151.95371	.0786	.2765	.5076	.7503	1.0000	
		5TH	229.67805	.0625	.2198	.4035	.5964	.7960	1.0000
.00	.30	1ST	14.31692	.2984	1.0000				
		2ND	41.79778	.1583	.5436	1.0000			
		3RD	83.54284	.1053	.3615	.6706	1.0000		
		4TH	139.54486	.0779	.2673	.4958	.7423	1.0000	
		5TH	209.80682	.0613	.2103	.3900	.5839	.7885	1.0000
.00	.40	1ST	13.89400	.3074	1.0000				
		2ND	39.18153	.1599	.5393	1.0000			
		3RD	77.09574	.1048	.3530	.6624	1.0000		
		4TH	127.63570	.0764	.2576	.4832	.7338	1.0000	
		5TH	190.80585	.0595	.2003	.3759	.5707	.7804	1.0000
.00	.50	1ST	13.44673	.3152	1.0000				
		2ND	36.61912	.1606	.5345	1.0000			
		3RD	70.87632	.1034	.3437	.6538	1.0000		
		4TH	116.22631	.0744	.2472	.4699	.7247	1.0000	
		5TH	172.67526	.0571	.1899	.3609	.5564	.7716	1.0000
.00	.60	1ST	12.97695	.3221	1.0000				
		2ND	34.11102	.1602	.5292	1.0000			
		3RD	64.88489	.1012	.3338	.6445	1.0000		
		4TH	105.31692	.0716	.2362	.4557	.7148	1.0000	
		5TH	155.41518	.0542	.1789	.3451	.5412	.7621	1.0000
.00	.80	1ST	11.97619	.3332	1.0000				
		2ND	29.25974	.1566	.5167	1.0000			
		3RD	53.58713	.0946	.3113	.6239	1.0000		
		4TH	84.99889	.0644	.2119	.4243	.6926	1.0000	
		5TH	123.50709	.0472	.1552	.3108	.5070	.7403	1.0000
.00	1.00	1ST	10.90236	.3410	1.0000				
		2ND	24.63138	.1490	.5014	1.0000			
		3RD	43.20475	.0850	.2848	.6000	1.0000		
		4TH	66.68312	.0550	.1845	.3880	.6662	1.0000	
		5TH	95.08264	.0386	.1294	.2721	.4667	.7138	1.0000

TABLE 2. NATURAL FREQUENCIES AND NODES FOR TAPERED FREE-FREE BEAMS OF ANTISYMMETRICAL MODE

Ψ	Φ	MODE	FREQUENCY CONSTANT, K	LOCATIONS OF NODES (X/L)					
.10	.10	1ST	15.68218	.2928	1.0000				
		2ND	48.25902	.1618	.5574	1.0000			
		3RD	98.65841	.1110	.3825	.6886	1.0000		
		4TH	166.87082	.0842	.2901	.5222	.7596	1.0000	
		5TH	252.89746	.0677	.2331	.4196	.6102	.8042	1.0000
.10	.20	1ST	15.29172	.3032	1.0000				
		2ND	45.49287	.1649	.5540	1.0000			
		3RD	91.68921	.1118	.3752	.6815	1.0000		
		4TH	153.87080	.0838	.2814	.5111	.7522	1.0000	
		5TH	232.03995	.0667	.2240	.4068	.5987	.7973	1.0000
.10	.30	1ST	14.87364	.3125	1.0000				
		2ND	42.77983	.1671	.5502	1.0000			
		3RD	84.94729	.1116	.3672	.6740	1.0000		
		4TH	141.37030	.0827	.2722	.4993	.7444	1.0000	
		5TH	212.05267	.0652	.2144	.3934	.5863	.7899	1.0000
.10	.40	1ST	14.43004	.3207	1.0000				
		2ND	40.12034	.1683	.5459	1.0000			
		3RD	78.43290	.1107	.3587	.6659	1.0000		
		4TH	129.36955	.0810	.2623	.4869	.7360	1.0000	
		5TH	192.93569	.0631	.2044	.3793	.5732	.7819	1.0000
.10	.50	1ST	13.96281	.3280	1.0000				
		2ND	37.51488	.1685	.5410	1.0000			
		3RD	72.14631	.1090	.3494	.6574	1.0000		
		4TH	117.86867	.0786	.2519	.4737	.7270	1.0000	
		5TH	174.68916	.0604	.1938	.3644	.5591	.7732	1.0000
.10	.60	1ST	13.47367	.3343	1.0000				
		2ND	34.95393	.1678	.5357	1.0000			
		3RD	66.08781	.1065	.3394	.6483	1.0000		
		4TH	106.86784	.0756	.2408	.4596	.7173	1.0000	
		5TH	157.31317	.0573	.1827	.3487	.5440	.7638	1.0000
.10	.80	1ST	12.43560	.3446	1.0000				
		2ND	30.02742	.1635	.5233	1.0000			
		3RD	54.65624	.0993	.3168	.6280	1.0000		
		4TH	86.36714	.0678	.2164	.4284	.6954	1.0000	
		5TH	125.17338	.0498	.1588	.3145	.5101	.7422	1.0000
.10	1.00	1ST	11.32564	.3517	1.0000				
		2ND	25.31434	.1553	.5081	1.0000			
		3RD	44.14037	.0891	.2903	.6044	1.0000		
		4TH	67.86891	.0579	.1888	.3924	.6694	1.0000	
		5TH	96.51744	.0407	.1328	.2759	.4702	.7161	1.0000

TABLE 2. NATURAL FREQUENCIES AND NODES FOR TAPERED FREE-FREE BEAMS OF ANTISYMMETRICAL MODE

Ψ	Φ	MODE	FREQUENCY CONSTANT, K	LOCATIONS OF NODES (X/L)					
.20	.10	1ST	16.28719	.3077	1.0000				
		2ND	49.33404	.1713	.5639	1.0000			
		3RD	100.20407	.1180	.3882	.6917	1.0000		
		4TH	168.88618	.0896	.2950	.5255	.7614	1.0000	
		5TH	255.38217	.0721	.2373	.4227	.6124	.8054	1.0000
.20	.20	1ST	15.87465	.3173	1.0000				
		2ND	46.52435	.1739	.5604	1.0000			
		3RD	93.16742	.1133	.3808	.6847	1.0000		
		4TH	155.79447	.0889	.2862	.5144	.7541	1.0000	
		5TH	234.40858	.0708	.2281	.4100	.6009	.7985	1.0000
.20	.30	1ST	15.43540	.3259	1.0000				
		2ND	43.76795	.1756	.5565	1.0000			
		3RD	86.35815	.1178	.3728	.6772	1.0000		
		4TH	143.20236	.0875	.2769	.5028	.7464	1.0000	
		5TH	214.30526	.0690	.2184	.3966	.5887	.7912	1.0000
.20	.40	1ST	14.97140	.3335	1.0000				
		2ND	41.06531	.1764	.5522	1.0000			
		3RD	79.77653	.1165	.3642	.6693	1.0000		
		4TH	131.11007	.0854	.2670	.4904	.7381	1.0000	
		5TH	195.07231	.0666	.2083	.3826	.5757	.7833	1.0000
.20	.50	1ST	14.48444	.3402	1.0000				
		2ND	38.41688	.1762	.5473	1.0000			
		3RD	73.42282	.1144	.3549	.6609	1.0000		
		4TH	119.51771	.0827	.2565	.4773	.7292	1.0000	
		5TH	176.70983	.0637	.1976	.3678	.5617	.7748	1.0000
.20	.60	1ST	13.97613	.3461	1.0000				
		2ND	35.82312	.1752	.5420	1.0000			
		3RD	67.29729	.1117	.3448	.6519	1.0000		
		4TH	108.42548	.0794	.2453	.4634	.7197	1.0000	
		5TH	159.21796	.0604	.1864	.3522	.5467	.7655	1.0000
.20	.80	1ST	12.90096	.3555	1.0000				
		2ND	30.80149	.1703	.5296	1.0000			
		3RD	55.73196	.1039	.3222	.6320	1.0000		
		4TH	87.74213	.0712	.2208	.4324	.6981	1.0000	
		5TH	126.84654	.0524	.1624	.3181	.5131	.7442	1.0000
.20	1.00	1ST	11.75495	.3620	1.0000				
		2ND	26.00374	.1615	.5146	1.0000			
		3RD	45.08263	.0932	.2956	.6088	1.0000		
		4TH	69.06147	.0608	.1930	.3966	.6725	1.0000	
		5TH	97.95907	.0429	.1361	.2796	.4736	.7184	1.0000

TABLE 2. NATURAL FREQUENCIES AND NODES FOR TAPERED FREE - FREE BEAMS OF ANTISYMMETRICAL MODE

Ψ	Φ	MODE	FREQUENCY CONSTANT, K	LOCATIONS OF NODES (X/L)					
.30	.10	1ST	16.89661	.3218	1.0000				
		2ND	50.41502	.1805	.5701	1.0000			
		3RD	101.75608	.1247	.3938	.6947	1.0000		
		4TH	170.90811	.0949	.2997	.5287	.7631	1.0000	
		5TH	257.87359	.0765	.2414	.4258	.6145	.8066	1.0000
.30	.20	1ST	16.46240	.3307	1.0000				
		2ND	47.56186	.1827	.5666	1.0000			
		3RD	94.65202	.1246	.3863	.6878	1.0000		
		4TH	157.72475	.0938	.2909	.5177	.7560	1.0000	
		5TH	236.78396	.0749	.2321	.4131	.6031	.7998	1.0000
.30	.30	1ST	16.00231	.3386	1.0000				
		2ND	44.75219	.1839	.5626	1.0000			
		3RD	87.77546	.1238	.3783	.6804	1.0000		
		4TH	145.04106	.0921	.2815	.5062	.7484	1.0000	
		5TH	216.56462	.0728	.2224	.3999	.5910	.7926	1.0000
.30	.40	1ST	15.51818	.3457	1.0000				
		2ND	42.01646	.1843	.5583	1.0000			
		3RD	81.12665	.1222	.3696	.6726	1.0000		
		4TH	132.85724	.0897	.2716	.4939	.7402	1.0000	
		5TH	197.21571	.0701	.2121	.3859	.5781	.7847	1.0000
.30	.50	1ST	15.01169	.3519	1.0000				
		2ND	39.32513	.1838	.5534	1.0000			
		3RD	74.70584	.1198	.3603	.6643	1.0000		
		4TH	121.17343	.0868	.2610	.4809	.7314	1.0000	
		5TH	178.73730	.0670	.2014	.3712	.5643	.7763	1.0000
.30	.60	1ST	14.48436	.3573	1.0000				
		2ND	36.68863	.1824	.5481	1.0000			
		3RD	68.51333	.1168	.3502	.6555	1.0000		
		4TH	109.98981	.0833	.2497	.4671	.7220	1.0000	
		5TH	161.12957	.0634	.1901	.3556	.5494	.7671	1.0000
.30	.80	1ST	13.37229	.3660	1.0000				
		2ND	31.58196	.1769	.5358	1.0000			
		3RD	56.81429	.1085	.3275	.6359	1.0000		
		4TH	89.12387	.0745	.2251	.4364	.7008	1.0000	
		5TH	128.52653	.0549	.1659	.3216	.5161	.7461	1.0000
.30	1.00	1ST	12.19031	.3719	1.0000				
		2ND	26.69958	.1676	.5209	1.0000			
		3RD	46.03153	.0972	.3009	.6130	1.0000		
		4TH	70.26080	.0637	.1972	.4008	.6755	1.0000	
		5TH	99.40757	.0450	.1393	.2833	.4769	.7207	1.0000

TABLE 2. NATURAL FREQUENCIES AND NODES FOR TAPERED FREE - FREE BEAMS OF ANTISYMMETRICAL MODE

Ψ	Φ	MODE	FREQUENCY CONSTANT, K	LOCATIONS OF NODES (X/L)					
.40	.10	1ST	17.51065	.3352	1.0000				
		2ND	51.50197	.1894	.5761	1.0000			
		3RD	103.31447	.1313	.3992	.6976	1.0000		
		4TH	172.93664	.1001	.3044	.5318	.7649	1.0000	
		5TH	260.37175	.0807	.2454	.4288	.6166	.8077	1.0000
.40	.20	1ST	17.05512	.3434	1.0000				
		2ND	48.60544	.1911	.5725	1.0000			
		3RD	96.14306	.1308	.3916	.6908	1.0000		
		4TH	159.66165	.0987	.2955	.5210	.7578	1.0000	
		5TH	239.16609	.0788	.2361	.4162	.6053	.8011	1.0000
.40	.30	1ST	16.57447	.3508	1.0000				
		2ND	45.76257	.1919	.5685	1.0000			
		3RD	89.19922	.1296	.3836	.6835	1.0000		
		4TH	146.88642	.0967	.2860	.5095	.7503	1.0000	
		5TH	218.83077	.0765	.2262	.4030	.5933	.7939	1.0000
.40	.40	1ST	16.07045	.3573	1.0000				
		2ND	42.97383	.1919	.5641	1.0000			
		3RD	82.48326	.1277	.3749	.6759	1.0000		
		4TH	134.61109	.0940	.2760	.4974	.7422	1.0000	
		5TH	199.36589	.0736	.2159	.3891	.5805	.7861	1.0000
.40	.50	1ST	15.54461	.3631	1.0000				
		2ND	40.23965	.1911	.5593	1.0000			
		3RD	75.99541	.1251	.3655	.6677	1.0000		
		4TH	122.83585	.0908	.2654	.4845	.7336	1.0000	
		5TH	180.77160	.0702	.2051	.3745	.5668	.7778	1.0000
.40	.60	1ST	14.99840	.3681	1.0000				
		2ND	37.56047	.1894	.5540	1.0000			
		3RD	69.73595	.1218	.3554	.6590	1.0000		
		4TH	111.56086	.0870	.2541	.4707	.7243	1.0000	
		5TH	163.04798	.0664	.1938	.3590	.5521	.7687	1.0000
.40	.80	1ST	13.84960	.3762	1.0000				
		2ND	32.36883	.1834	.5418	1.0000			
		3RD	57.90325	.1130	.3327	.6397	1.0000		
		4TH	90.51236	.0779	.2293	.4402	.7033	1.0000	
		5TH	130.21336	.0575	.1694	.3252	.5191	.7480	1.0000
.40	1.00	1ST	12.63170	.3816	1.0000				
		2ND	27.40185	.1736	.5271	1.0000			
		3RD	46.98707	.1013	.3061	.6172	1.0000		
		4TH	71.46690	.0666	.2012	.4049	.6785	1.0000	
		5TH	100.86293	.0472	.1426	.2869	.4802	.7229	1.0000

TABLE 2. NATURAL FREQUENCIES AND NODES FOR TAPERED FREE - FREE BEAMS OF ANTISYMMETRICAL MODE

Ψ	Φ	MODE	FREQUENCY CONSTANT, K	LOCATIONS OF NODES (X/L)					
.50	.10	1ST	18.12946	.3480	1.0000				
		2ND	52.59495	.1979	.5818	1.0000			
		3RD	104.87926	.1376	.4044	.7004	1.0000		
		4TH	174.97178	.1051	.3089	.5349	.7666	1.0000	
		5TH	262.87666	.0849	.2493	.4318	.6187	.8089	1.0000
.50	.20	1ST	17.65293	.3556	1.0000				
		2ND	49.65511	.1992	.5782	1.0000			
		3RD	97.64051	.1369	.3968	.6937	1.0000		
		4TH	161.60520	.1034	.2999	.5242	.7596	1.0000	
		5TH	241.55497	.0827	.2399	.4193	.6075	.8023	1.0000
.50	.30	1ST	17.15199	.3625	1.0000				
		2ND	46.76913	.1997	.5742	1.0000			
		3RD	90.62948	.1353	.3887	.6866	1.0000		
		4TH	148.73843	.1011	.2905	.5128	.7522	1.0000	
		5TH	221.10368	.0801	.2300	.4061	.5956	.7952	1.0000
.50	.40	1ST	16.62827	.3685	1.0000				
		2ND	43.93743	.1994	.5698	1.0000			
		3RD	83.84638	.1331	.3800	.6790	1.0000		
		4TH	136.37164	.0982	.2804	.5007	.7442	1.0000	
		5TH	201.52288	.0770	.2197	.3923	.5829	.7875	1.0000
.50	.50	1ST	16.08326	.3739	1.0000				
		2ND	41.16046	.1982	.5650	1.0000			
		3RD	77.29152	.1303	.3706	.6710	1.0000		
		4TH	124.50496	.0948	.2697	.4879	.7357	1.0000	
		5TH	182.81269	.0734	.2088	.3777	.5693	.7793	1.0000
.50	.60	1ST	15.51829	.3786	1.0000				
		2ND	38.43865	.1963	.5597	1.0000			
		3RD	70.96514	.1267	.3605	.6624	1.0000		
		4TH	113.13864	.0908	.2583	.4743	.7265	1.0000	
		5TH	164.97322	.0693	.1973	.3623	.5547	.7703	1.0000
.50	.80	1ST	14.33292	.3860	1.0000				
		2ND	33.16211	.1898	.5476	1.0000			
		3RD	58.99883	.1174	.3378	.6434	1.0000		
		4TH	91.90761	.0812	.2335	.4440	.7059	1.0000	
		5TH	131.90701	.0601	.1728	.3286	.5219	.7499	1.0000
.50	1.00	1ST	13.07914	.3909	1.0000				
		2ND	28.11056	.1795	.5330	1.0000			
		3RD	47.94925	.1053	.3111	.6212	1.0000		
		4TH	72.67976	.0695	.2053	.4090	.6814	1.0000	
		5TH	102.32514	.0493	.1458	.2905	.4834	.7251	1.0000

TABLE 2. NATURAL FREQUENCIES AND NODES FOR TAPERED FREE-FREE BEAMS OF ANTISYMMETRICAL MODE

Ψ	Φ	MODE	FREQUENCY CONSTANT, K	LOCATIONS OF NODES (X/L)					
.60	.10	1ST	18.75316	.3602	1.0000				
		2ND	53.69398	.2062	.5874	1.0000			
		3RD	106.45044	.1438	.4095	.7032	1.0000		
		4TH	177.01355	.1101	.3133	.5379	.7683	1.0000	
		5TH	265.38832	.0889	.2532	.4347	.6207	.8100	1.0000
.60	.20	1ST	18.25593	.3673	1.0000				
		2ND	50.71092	.2071	.5838	1.0000			
		3RD	99.14442	.1427	.4019	.6966	1.0000		
		4TH	163.55538	.1081	.3043	.5273	.7614	1.0000	
		5TH	243.95063	.0866	.2437	.4223	.6096	.8035	1.0000
.60	.30	1ST	17.73493	.3736	1.0000				
		2ND	47.78188	.2073	.5797	1.0000			
		3RD	92.06621	.1409	.3937	.6896	1.0000		
		4TH	150.59710	.1055	.2948	.5160	.7540	1.0000	
		5TH	223.38338	.0837	.2338	.4092	.5978	.7964	1.0000
.60	.40	1ST	17.19172	.3793	1.0000				
		2ND	44.90730	.2066	.5753	1.0000			
		3RD	85.21605	.1385	.3850	.6821	1.0000		
		4TH	138.13886	.1024	.2847	.5040	.7462	1.0000	
		5TH	203.68665	.0803	.2233	.3954	.5852	.7889	1.0000
.60	.50	1ST	16.62768	.3843	1.0000				
		2ND	42.08759	.2052	.5705	1.0000			
		3RD	78.59420	.1353	.3756	.6742	1.0000		
		4TH	126.18083	.0987	.2739	.4913	.7378	1.0000	
		5TH	184.86059	.0765	.2124	.3809	.5717	.7807	1.0000
.60	.60	1ST	16.04405	.3886	1.0000				
		2ND	39.32319	.2030	.5652	1.0000			
		3RD	72.20091	.1315	.3654	.6657	1.0000		
		4TH	114.72316	.0945	.2625	.4778	.7287	1.0000	
		5TH	166.90530	.0723	.2009	.3655	.5572	.7719	1.0000
.60	.80	1ST	14.82223	.3955	1.0000				
		2ND	33.95182	.1960	.5532	1.0000			
		3RD	60.10104	.1218	.3427	.6470	1.0000		
		4TH	93.30962	.0844	.2376	.4478	.7084	1.0000	
		5TH	133.60754	.0626	.1761	.3320	.5248	.7517	1.0000
.60	1.00	1ST	13.53261	.4000	1.0000				
		2ND	28.82571	.1853	.5388	1.0000			
		3RD	48.91807	.1092	.3161	.6252	1.0000		
		4TH	73.89939	.0723	.2093	.4129	.6843	1.0000	
		5TH	103.79419	.0515	.1490	.2940	.4865	.7273	1.0000

TABLE 2. NATURAL FREQUENCIES AND NODES FOR TAPERED FREE - FREE BEAMS OF ANTISYMMETRICAL MODE

Ψ	Φ	MODE	FREQUENCY CONSTANT, K	LOCATIONS OF NODES (X/L)					
.80	.10	1ST	20.01576	.3830	1.0000				
		2ND	55.91034	.2220	.5979	1.0000			
		3RD	109.61211	.1558	.4192	.7086	1.0000		
		4TH	181.11694	.1196	.3219	.5438	.7715	1.0000	
		5TH	270.43190	.0968	.2606	.4404	.6247	.8122	1.0000
.80	.20	1ST	19.47788	.3892	1.0000				
		2ND	52.84103	.2222	.5943	1.0000			
		3RD	102.17166	.1541	.4116	.7021	1.0000		
		4TH	167.47572	.1171	.3128	.5333	.7648	1.0000	
		5TH	248.76226	.0940	.2511	.4281	.6138	.8058	1.0000
.80	.30	1ST	18.91740	.3947	1.0000				
		2ND	49.82607	.2218	.5902	1.0000			
		3RD	94.95921	.1518	.4034	.6953	1.0000		
		4TH	154.33448	.1141	.3032	.5222	.7577	1.0000	
		5TH	227.96313	.0907	.2410	.4152	.6021	.7989	1.0000
.80	.40	1ST	18.33567	.3996	1.0000				
		2ND	46.86588	.2206	.5859	1.0000			
		3RD	87.97500	.1488	.3946	.6881	1.0000		
		4TH	141.69343	.1105	.2930	.5105	.7500	1.0000	
		5TH	208.03464	.0869	.2304	.4015	.5898	.7915	1.0000
.80	.50	1ST	17.73395	.4039	1.0000				
		2ND	43.96086	.2186	.5811	1.0000			
		3RD	81.21926	.1452	.3852	.6803	1.0000		
		4TH	129.55267	.1064	.2822	.4979	.7418	1.0000	
		5TH	188.97685	.0827	.2194	.3872	.5765	.7836	1.0000
.80	.60	1ST	17.11332	.4076	1.0000				
		2ND	41.11139	.2159	.5759	1.0000			
		3RD	74.69226	.1410	.3750	.6721	1.0000		
		4TH	117.91239	.1018	.2707	.4846	.7330	1.0000	
		5TH	170.78995	.0781	.2078	.3719	.5623	.7749	1.0000
.80	.80	1ST	15.81892	.4135	1.0000				
		2ND	35.58051	.2081	.5640	1.0000			
		3RD	62.32536	.1305	.3524	.6539	1.0000		
		4TH	96.13392	.0909	.2456	.4550	.7132	1.0000	
		5TH	137.02913	.0677	.1828	.3387	.5303	.7552	1.0000
.80	1.00	1ST	14.45767	.4172	1.0000				
		2ND	30.27530	.1967	.5499	1.0000			
		3RD	50.87562	.1171	.3257	.6328	1.0000		
		4TH	76.35894	.0780	.2171	.4206	.6898	1.0000	
		5TH	106.75287	.0558	.1553	.3009	.4927	.7315	1.0000

TABLE 2. NATURAL FREQUENCIES AND NODES FOR TAPERED FREE-FREE BEAMS OF ANTISYMMETRICAL MODE

Ψ	Φ	MODE	FREQUENCY CONSTANT, K	LOCATIONS OF NODES (X/L)					
1.00	.10	1ST	21.29928	.4040	1.0000				
		2ND	58.15127	.2368	.6078	1.0000			
		3RD	112.79963	.1671	.4285	.7137	1.0000		
		4TH	185.24694	.1287	.3300	.5495	.7747	1.0000	
		5TH	275.50256	.1045	.2678	.4459	.6285	.8143	1.0000
1.00	.20	1ST	20.72162	.4093	1.0000				
		2ND	54.99597	.2365	.6041	1.0000			
		3RD	105.22490	.1650	.4208	.7075	1.0000		
		4TH	171.42275	.1258	.3209	.5392	.7681	1.0000	
		5TH	253.60104	.1012	.2581	.4338	.6178	.8081	1.0000
1.00	.30	1ST	20.12233	.4141	1.0000				
		2ND	51.89532	.2355	.6001	1.0000			
		3RD	97.87833	.1622	.4126	.7008	1.0000		
		4TH	158.09865	.1223	.3112	.5282	.7612	1.0000	
		5TH	232.57011	.0975	.2480	.4210	.6064	.8013	1.0000
1.00	.40	1ST	19.50265	.4184	1.0000				
		2ND	48.84971	.2338	.5958	1.0000			
		3RD	90.76018	.1588	.4038	.6937	1.0000		
		4TH	145.27482	.1184	.3010	.5166	.7537	1.0000	
		5TH	212.40985	.0933	.2374	.4075	.5942	.7941	1.0000
1.00	.50	1ST	18.86367	.4221	1.0000				
		2ND	45.85953	.2315	.5910	1.0000			
		3RD	83.87066	.1548	.3944	.6863	1.0000		
		4TH	132.95142	.1139	.2901	.5043	.7457	1.0000	
		5TH	193.12044	.0888	.2262	.3932	.5811	.7863	1.0000
1.00	.60	1ST	18.20635	.4254	1.0000				
		2ND	42.92513	.2283	.5859	1.0000			
		3RD	77.21004	.1502	.3843	.6783	1.0000		
		4TH	121.12860	.1089	.2786	.4912	.7371	1.0000	
		5TH	174.70194	.0838	.2145	.3781	.5671	.7779	1.0000
1.00	.80	1ST	16.83970	.4304	1.0000				
		2ND	37.22491	.2198	.5742	1.0000			
		3RD	64.57622	.1389	.3616	.6606	1.0000		
		4TH	98.98528	.0973	.2533	.4620	.7179	1.0000	
		5TH	140.47814	.0727	.1892	.3451	.5357	.7587	1.0000
1.00	1.00	1ST	15.40683	.4334	1.0000				
		2ND	31.75062	.2076	.5604	1.0000			
		3RD	52.85972	.1247	.3350	.6400	1.0000		
		4TH	78.84556	.0836	.2246	.4281	.6951	1.0000	
		5TH	109.73896	.0601	.1614	.3076	.4987	.7355	1.0000

TABLE 3. NATURAL FREQUENCIES AND NODES FOR TAPERED FREE-FREE BEAMS OF SYMMETRICAL MODE

Ψ	Φ	MODE	FREQUENCY CONSTANT, K	LOCATIONS OF NODES (X/L)				
.10	.00	1ST	5.96531	.46975				
		2ND	31.09310	.20187	.71811			
		3RD	75.99965	.12912	.45938	.82068		
		4TH	140.64556	.09492	.33769	.60327	.86795	
		5TH	225.03065	.07504	.26696	.47693	.68618	.89554
.20	.00	1ST	6.34515	.48935				
		2ND	31.96779	.21423	.72426			
		3RD	77.35779	.13770	.46560	.82303		
		4TH	142.50713	.10146	.34306	.60642	.86922	
		5TH	227.38572	.08032	.27159	.48008	.68812	.89532
.30	.00	1ST	6.73274	.50735				
		2ND	32.84992	.22601	.73009			
		3RD	78.74329	.14597	.47159	.82531		
		4TH	144.37609	.10780	.34828	.60950	.87045	
		5TH	229.74813	.08546	.27609	.48317	.69003	.89709
.40	.00	1ST	7.12804	.52396				
		2ND	33.73947	.23726	.73562			
		3RD	80.12618	.15395	.47738	.82753		
		4TH	146.25239	.11395	.35335	.61251	.87166	
		5TH	232.11789	.09045	.28048	.48620	.69190	.89785
.50	.00	1ST	7.53098	.53933				
		2ND	34.63642	.24802	.74089			
		3RD	81.51643	.16167	.48297	.82968		
		4TH	148.13605	.11993	.35827	.61546	.87284	
		5TH	234.49499	.09532	.28476	.48917	.69374	.89859
.60	.00	1ST	7.94150	.55362				
		2ND	35.54077	.25834	.74591			
		3RD	82.91404	.16914	.48837	.83177		
		4TH	150.02705	.12574	.36306	.61834	.87400	
		5TH	236.87944	.10007	.28893	.49209	.69555	.89933
.80	.00	1ST	8.78507	.57938				
		2ND	37.37162	.27776	.75529			
		3RD	85.73135	.18339	.49867	.83578		
		4TH	153.83114	.13691	.37227	.62392	.87624	
		5TH	241.67039	.10923	.29701	.49779	.69909	.90076
1.00	.00	1ST	9.65836	.60201				
		2ND	39.23190	.29576	.76387			
		3RD	88.57805	.19683	.50835	.83957		
		4TH	157.66458	.14753	.38103	.62928	.87840	
		5TH	246.49069	.11799	.30474	.50328	.70252	.90214

TABLE 3. NATURAL FREQUENCIES AND NODES FOR TAPERED FREE-FREE BEAMS OF SYMMETRICAL MODE

Ψ	Φ	MODE	FREQUENCY CONSTANT, K	LOCATIONS OF NODES (X/L)				
.00	.10	1ST	5.78131	.46626				
		2ND	28.95721	.19634	.71191			
		3RD	69.97776	.12340	.44749	.81453		
		4TH	128.81296	.08950	.32457	.59079	.86277	
		5TH	205.46280	.07000	.25386	.46207	.67480	.89113
.00	.20	1ST	5.95253	.48209				
		2ND	27.69303	.20243	.71182			
		3RD	65.44420	.12554	.44146	.81058		
		4TH	119.18417	.08998	.31642	.58097	.85862	
		5TH	188.91291	.06966	.24498	.44980	.66476	.88726
.00	.30	1ST	6.10602	.49608				
		2ND	26.43320	.20725	.71133			
		3RD	61.03814	.12668	.43479	.80639		
		4TH	109.90489	.08963	.30764	.57055	.85417	
		5TH	173.03329	.06863	.23555	.43686	.65403	.88309
.00	.40	1ST	6.24080	.50845				
		2ND	25.17752	.21085	.71042			
		3RD	56.75954	.12686	.42743	.80191		
		4TH	100.97510	.08851	.29820	.55945	.84940	
		5TH	157.82389	.06695	.22557	.42319	.64252	.87858
.00	.50	1ST	6.35582	.51939				
		2ND	23.92577	.21326	.70908			
		3RD	52.60829	.12612	.41932	.79712		
		4TH	92.39474	.08664	.28806	.54759	.84427	
		5TH	143.28472	.06467	.21501	.40871	.63015	.87369
.00	.60	1ST	6.45002	.52904				
		2ND	22.67767	.21450	.70728			
		3RD	48.58426	.12447	.41040	.79198		
		4TH	84.16375	.08407	.27718	.53488	.83871	
		5TH	129.41568	.06182	.20384	.39335	.61679	.86836
.00	.80	1ST	6.57120	.54491				
		2ND	20.19084	.21341	.70218			
		3RD	40.91702	.11847	.38975	.78038		
		4TH	68.74930	.07688	.25292	.50640	.82610	
		5TH	103.68770	.05459	.17959	.35957	.58656	.85609
.00	1.00	1ST	6.59365	.55667				
		2ND	17.71250	.20723	.69469			
		3RD	33.75518	.10874	.36453	.76654		
		4TH	54.73005	.06707	.22482	.47277	.81089	
		5TH	80.63878	.04552	.15259	.32087	.55036	.84104

TABLE 3. NATURAL FREQUENCIES AND NODES FOR TAPERED FREE-FREE BEAMS OF SYMMETRICAL MODE

Ψ	Φ	MODE	FREQUENCY CONSTANT, K	LOCATIONS OF NODES (X/L)				
.10	.10	1ST	6.15072	.48586				
		2ND	29.79616	.20850	.71824			
		3RD	71.28543	.13174	.45385	.81703		
		4TH	130.58940	.09579	.33002	.59410	.86413	
		5TH	207.70799	.07503	.25850	.46536	.67686	.89198
.10	.20	1ST	6.31897	.50013				
		2ND	28.50364	.21387	.71800			
		3RD	66.69875	.13336	.44773	.81316		
		4TH	120.88277	.09584	.32177	.58439	.86005	
		5TH	191.05561	.07432	.24952	.45317	.66693	.88817
.10	.30	1ST	6.46911	.51279				
		2ND	27.21538	.21805	.71738			
		3RD	62.23955	.13404	.44099	.80905		
		4TH	111.52566	.09511	.31291	.57407	.85568	
		5TH	175.07348	.07295	.24000	.44031	.65631	.88406
.10	.40	1ST	6.60016	.52403				
		2ND	25.93119	.22107	.71637			
		3RD	57.90776	.13380	.43356	.80467		
		4TH	102.51802	.09363	.30340	.56308	.85100	
		5TH	159.76156	.07096	.22993	.42673	.64492	.87962
.10	.50	1ST	6.71110	.53398				
		2ND	24.65083	.22297	.71495			
		3RD	53.70329	.13268	.42541	.79998		
		4TH	93.85979	.09144	.29319	.55133	.84596	
		5TH	145.11982	.06839	.21927	.41234	.63267	.87480
.10	.60	1ST	6.80083	.54279				
		2ND	23.37400	.22375	.71310			
		3RD	49.62598	.13068	.41645	.79493		
		4TH	85.55088	.08857	.28224	.53874	.84051	
		5TH	131.14822	.06527	.20802	.39706	.61946	.86956
.10	.80	1ST	6.91196	.55728				
		2ND	20.82924	.22185	.70794			
		3RD	41.85190	.12403	.39576	.78359		
		4TH	69.98044	.08081	.25786	.51054	.82814	
		5TH	105.21501	.05753	.18357	.36345	.58954	.85750
.10	1.00	1ST	6.92273	.56801				
		2ND	18.29207	.21497	.70049			
		3RD	34.58267	.11370	.37051	.77005		
		4TH	55.80481	.07046	.22961	.47721	.81324	
		5TH	81.96057	.04798	.15634	.32492	.55371	.84272

TABLE 3. NATURAL FREQUENCIES AND NODES FOR TAPERED FREE-FREE BEAMS OF SYMMETRICAL MODE

Ψ	Φ	MODE	FREQUENCY CONSTANT, K	LOCATIONS OF NODES (X/L)				
.20	.10	1ST	6.52781	.50388				
		2ND	30.64254	.22011	.72424			
		3RD	72.60046	.13979	.45998	.81945		
		4TH	132.37318	.10190	.33531	.59734	.86545	
		5TH	209.96057	.07994	.26303	.46858	.67889	.89282
.20	.20	1ST	6.69293	.51680				
		2ND	29.32163	.22482	.72387			
		3RD	67.96065	.14094	.45379	.81566		
		4TH	122.58873	.10155	.32698	.58773	.86144	
		5TH	193.20567	.07887	.25396	.45647	.66906	.88906
.20	.30	1ST	6.83959	.52831				
		2ND	28.00492	.22841	.72314			
		3RD	63.44830	.14119	.44697	.81163		
		4TH	113.15379	.10046	.31805	.57751	.85715	
		5TH	177.12101	.07718	.24435	.44370	.65854	.88501
.20	.40	1ST	6.96679	.53854				
		2ND	26.69218	.23091	.72205			
		3RD	59.06333	.14056	.43950	.80734		
		4TH	104.06828	.09866	.30847	.56662	.85255	
		5TH	161.70657	.07490	.23420	.43019	.64727	.88064
.20	.50	1ST	7.07350	.54763				
		2ND	25.38318	.23234	.72056			
		3RD	54.80561	.13908	.43130	.80274		
		4TH	95.33216	.09616	.29820	.55499	.84760	
		5TH	146.96230	.07206	.22346	.41589	.63515	.87589
.20	.60	1ST	7.15866	.55568				
		2ND	24.07757	.23269	.71865			
		3RD	50.67498	.13675	.42232	.79780		
		4TH	86.94532	.09300	.28719	.54252	.84225	
		5TH	132.88810	.06869	.21212	.40070	.62207	.87073
.20	.80	1ST	7.25951	.56895				
		2ND	21.47475	.23005	.71345			
		3RD	42.79401	.12950	.40159	.78669		
		4TH	71.21884	.08471	.26269	.51458	.83012	
		5TH	106.74961	.06046	.18749	.36726	.59246	.85887
.20	1.00	1ST	7.25837	.57875				
		2ND	18.87860	.22252	.70605			
		3RD	35.41729	.11861	.37635	.77344		
		4TH	56.88678	.07385	.23431	.48154	.81552	
		5TH	83.28960	.05044	.16004	.32889	.55700	.84435

TABLE 3. NATURAL FREQUENCIES AND NODES FOR TAPERED FREE-FREE BEAMS OF SYMMETRICAL MODE

Ψ	Φ	MODE	FREQUENCY CONSTANT, K	LOCATIONS OF NODES (X/L)				
.30	.10	1ST	6.91251	.52052				
		2ND	31.49632	.23122	.72993			
		3RD	73.92286	.14757	.46590	.82181		
		4TH	134.16430	.10784	.34045	.60051	.86675	
		5TH	212.22046	.08471	.26744	.47174	.68089	.89364
.30	.20	1ST	7.07437	.53226				
		2ND	30.14701	.23533	.72945			
		3RD	69.22991	.14828	.45964	.81809		
		4TH	124.30202	.10712	.33205	.59099	.86281	
		5TH	195.36306	.08333	.25829	.45971	.67115	.88993
.30	.30	1ST	7.21740	.54275				
		2ND	28.80179	.23839	.72863			
		3RD	64.66439	.14814	.45277	.81413		
		4TH	114.78924	.10570	.32305	.58087	.85859	
		5TH	179.17588	.08134	.24861	.44702	.66074	.88594
.30	.40	1ST	7.34061	.55210				
		2ND	27.46048	.24041	.72746			
		3RD	60.22621	.14716	.44525	.80992		
		4TH	105.62587	.10358	.31341	.57009	.85407	
		5TH	163.65890	.07878	.23838	.43360	.64958	.88163
.30	.50	1ST	7.44300	.56042				
		2ND	26.12277	.24140	.72591			
		3RD	55.91522	.14535	.43703	.80542		
		4TH	96.81186	.10080	.30309	.55857	.84921	
		5TH	148.81207	.07568	.22756	.41938	.63758	.87696
.30	.60	1ST	7.52346	.56780				
		2ND	24.78834	.24136	.72397			
		3RD	51.73126	.14271	.42802	.80058		
		4TH	88.34706	.09737	.29203	.54622	.84396	
		5TH	134.63529	.07207	.21614	.40427	.62464	.87188
.30	.80	1ST	7.61380	.57997				
		2ND	22.12734	.23801	.71874			
		3RD	43.74332	.13488	.40727	.78969		
		4TH	72.46450	.08857	.26743	.51853	.83205	
		5TH	108.29149	.06337	.19135	.37100	.59533	.86022
.30	1.00	1ST	7.60053	.58894				
		2ND	19.47208	.22988	.71139			
		3RD	36.25901	.12345	.38204	.77672		
		4TH	57.97592	.07721	.23893	.48577	.81773	
		5TH	84.62586	.05290	.16369	.33280	.56022	.84595

TABLE 3. NATURAL FREQUENCIES AND NODES FOR TAPERED FREE-FREE BEAMS OF SYMMETRICAL MODE

Ψ	Φ	MODE	FREQUENCY CONSTANT, K	LOCATIONS OF NODES (X/L)				
.40	.10	1ST	7.30476	.53593				
		2ND	32.35749	.24186	.73535			
		3RD	75.25262	.15511	.47161	.82409		
		4TH	135.96279	.11361	.34544	.60361	.86802	
		5TH	214.48772	.08938	.27175	.47485	.68285	.89444
.40	.20	1ST	7.46322	.54664				
		2ND	30.97974	.24542	.73477			
		3RD	70.50652	.15542	.46530	.82044		
		4TH	126.02268	.11256	.33698	.59418	.86414	
		5TH	197.52780	.08769	.26253	.46290	.67321	.89079
.40	.30	1ST	7.60249	.55623				
		2ND	29.60600	.24799	.73387			
		3RD	65.88782	.15491	.45839	.81656		
		4TH	116.43204	.11082	.32793	.58416	.86000	
		5TH	181.23810	.08542	.25278	.45028	.66290	.88685
.40	.40	1ST	7.72161	.56480				
		2ND	28.23606	.24957	.73263			
		3RD	61.39641	.15359	.45083	.81243		
		4TH	107.19079	.10842	.31824	.57348	.85555	
		5TH	165.61856	.08260	.24247	.43694	.65185	.88261
.40	.50	1ST	7.81954	.57244				
		2ND	26.86962	.25016	.73103			
		3RD	57.03213	.15147	.44258	.80801		
		4TH	98.29884	.10537	.30788	.56207	.85078	
		5TH	150.66916	.07926	.23159	.42280	.63997	.87801
.40	.60	1ST	7.89520	.57922				
		2ND	25.50630	.24975	.72905			
		3RD	52.79481	.14854	.43356	.80327		
		4TH	89.75611	.10168	.29677	.54983	.84562	
		5TH	136.38979	.07541	.22010	.40778	.62715	.87301
.40	.80	1ST	7.97481	.59040				
		2ND	22.78702	.24577	.72381			
		3RD	44.69982	.14018	.41281	.79260		
		4TH	73.71740	.09239	.27208	.52239	.83394	
		5TH	109.84064	.06627	.19515	.37468	.59814	.86153
.40	1.00	1ST	7.94921	.59862				
		2ND	20.07249	.23707	.71653			
		3RD	37.10783	.12824	.38759	.77990		
		4TH	59.07225	.08056	.24347	.48992	.81990	
		5TH	85.96933	.05535	.16730	.33664	.56338	.84752

TABLE 3. NATURAL FREQUENCIES AND NODES FOR TAPERED FREE-FREE BEAMS OF SYMMETRICAL MODE

Ψ	Φ	MODE	FREQUENCY CONSTANT, K	LOCATIONS OF NODES (X/L)				
.50	.10	1ST	7.70451	.55026				
		2ND	33.22605	.25206	.74052			
		3RD	76.58974	.16241	.47714	.82630		
		4TH	137.76862	.11924	.35031	.60664	.86926	
		5TH	216.76231	.09393	.27596	.47790	.68478	.89523
.50	.20	1ST	7.85944	.56006				
		2ND	31.81981	.25513	.73985			
		3RD	71.79048	.16235	.47078	.82272		
		4TH	127.75066	.11787	.34179	.59730	.86545	
		5TH	199.69988	.09196	.26667	.46602	.67523	.89163
.50	.30	1ST	7.99483	.56885				
		2ND	30.41752	.25725	.73888			
		3RD	67.11858	.16150	.46383	.81892		
		4TH	118.08217	.11584	.33269	.58737	.86137	
		5TH	183.30764	.08943	.25686	.45348	.66502	.88775
.50	.40	1ST	8.10973	.57673				
		2ND	29.01891	.25842	.73758			
		3RD	62.57391	.15987	.45625	.81486		
		4TH	108.76303	.11316	.32296	.57679	.85700	
		5TH	167.58556	.08637	.24649	.44022	.65408	.88357
.50	.50	1ST	8.20310	.58375				
		2ND	27.62368	.25864	.73594			
		3RD	58.15633	.15746	.44798	.81053		
		4TH	99.79315	.10986	.31255	.56549	.85231	
		5TH	152.53357	.08279	.23555	.42617	.64232	.87904
.50	.60	1ST	8.27384	.59000				
		2ND	26.23144	.25790	.73394			
		3RD	53.86561	.15426	.43894	.80588		
		4TH	91.17242	.10592	.30141	.55336	.84724	
		5TH	138.15160	.07872	.22399	.41123	.62962	.87411
.50	.80	1ST	8.34252	.60029				
		2ND	23.45375	.25331	.72869			
		3RD	45.66350	.14539	.41819	.79542		
		4TH	74.97754	.09617	.27664	.52616	.83577	
		5TH	111.39706	.06915	.19890	.37830	.60090	.86282
.50	1.00	1ST	8.30437	.60783				
		2ND	20.67981	.24409	.72147			
		3RD	37.95372	.13296	.39300	.78298		
		4TH	60.17573	.08388	.24794	.49397	.82200	
		5TH	87.32002	.05781	.17086	.34041	.56647	.84904

TABLE 3. NATURAL FREQUENCIES AND NODES FOR TAPERED FREE-FREE BEAMS OF SYMMETRICAL MODE

Ψ	Φ	MODE	FREQUENCY CONSTANT, K	LOCATIONS OF NODES (X/L)				
.60	.10	1ST	8.11171	.56362				
		2ND	34.10197	.26187	.74545			
		3RD	77.93421	.16950	.48250	.82845		
		4TH	139.58179	.12473	.35504	.60961	.87048	
		5TH	219.04425	.09839	.28008	.48089	.68668	.89601
.60	.20	1ST	8.26298	.57262				
		2ND	32.66723	.26448	.74471			
		3RD	73.08178	.16909	.47609	.82494		
		4TH	129.48601	.12306	.34648	.60035	.86672	
		5TH	201.87928	.09615	.27073	.46909	.67722	.89246
.60	.30	1ST	8.39438	.58070				
		2ND	31.23633	.26619	.74367			
		3RD	68.35665	.16793	.46911	.82121		
		4TH	119.73962	.12076	.33734	.59052	.86271	
		5TH	185.38451	.09338	.26086	.45663	.66711	.88863
.60	.40	1ST	8.50494	.58795				
		2ND	29.80902	.26698	.74232			
		3RD	63.75872	.16600	.46151	.81723		
		4TH	110.34258	.11783	.32757	.58004	.85842	
		5TH	169.55988	.09008	.25044	.44345	.65627	.88451
.60	.50	1ST	8.59363	.59443				
		2ND	28.38496	.26686	.74065			
		3RD	59.28779	.16332	.45323	.81298		
		4TH	101.29476	.11428	.31713	.56883	.85381	
		5TH	154.40532	.08628	.23943	.42947	.64462	.88005
.60	.60	1ST	8.65935	.60019				
		2ND	26.96374	.26580	.73862			
		3RD	54.94365	.15986	.44419	.80841		
		4TH	92.59603	.11011	.30595	.55681	.84883	
		5TH	139.92068	.08199	.22782	.41462	.63205	.87520
.60	.80	1ST	8.71689	.60968				
		2ND	24.12752	.26065	.73338			
		3RD	46.63435	.15051	.42345	.79816		
		4TH	76.24491	.09992	.28111	.52986	.83756	
		5TH	112.96074	.07201	.20259	.38185	.60360	.86409
.60	1.00	1ST	8.66598	.61661				
		2ND	21.29402	.25094	.72623			
		3RD	38.82667	.13762	.39829	.78597		
		4TH	61.28636	.08719	.25233	.49794	.82405	
		5TH	88.67790	.06026	.17439	.34413	.56951	.85054

TABLE 3. NATURAL FREQUENCIES AND NODES FOR TAPERED FREE-FREE BEAMS OF SYMMETRICAL MODE

Ψ	Φ	MODE	FREQUENCY CONSTANT, K	LOCATIONS OF NODES (X/L)				
.80	.10	1ST	8.94826	.58784				
		2ND	35.87586	.28040	.75467			
		3RD	80.64518	.18308	.49272	.83258		
		4TH	143.23018	.13531	.36417	.61535	.87283	
		5TH	223.63016	.10703	.28805	.48672	.69038	.89753
.80	.20	1ST	9.09185	.59547				
		2ND	34.38400	.28220	.75380			
		3RD	75.68635	.18205	.48625	.82919		
		4TH	132.97867	.13311	.35553	.60627	.86919	
		5TH	206.26014	.10431	.27860	.47507	.68109	.89407
.80	.30	1ST	9.21490	.60236				
		2ND	32.89580	.28317	.75266			
		3RD	70.85474	.18032	.47923	.82559		
		4TH	123.07650	.13031	.34632	.59661	.86531	
		5TH	189.56025	.10107	.26863	.46276	.67117	.89035
.80	.40	1ST	9.31647	.60854				
		2ND	31.41095	.28330	.75124			
		3RD	66.15020	.17786	.47160	.82176		
		4TH	113.52360	.12691	.33650	.58632	.86116	
		5TH	173.53047	.09734	.25811	.44973	.66054	.88634
.80	.50	1ST	9.39549	.61408				
		2ND	29.92909	.28256	.74951			
		3RD	61.57252	.17469	.46331	.81766		
		4TH	104.31985	.12291	.32600	.57532	.85670	
		5TH	158.17068	.09313	.24701	.43591	.64911	.88201
.80	.60	1ST	9.45086	.61901				
		2ND	28.44975	.28093	.74746			
		3RD	57.12144	.17076	.45427	.81326		
		4TH	95.46505	.11832	.31477	.56351	.85189	
		5TH	143.48078	.08844	.23529	.42122	.63678	.87730
.80	.80	1ST	9.48552	.62712				
		2ND	25.49611	.27478	.74223			
		3RD	48.59751	.16052	.43357	.80340		
		4TH	78.80130	.10730	.28982	.53702	.84102	
		5TH	116.10986	.07768	.20982	.38878	.60887	.86654
.80	1.00	1ST	9.40851	.63297				
		2ND	22.54303	.26417	.73523			
		3RD	40.57372	.14678	.40850	.79168		
		4TH	63.52904	.09374	.26089	.50562	.82800	
		5TH	91.41526	.06515	.18131	.35138	.57542	.85343

TABLE 3. NATURAL FREQUENCIES AND NODES FOR TAPERED FREE-FREE BEAMS OF SYMMETRICAL MODE

Ψ	Φ	MODE	FREQUENCY CONSTANT, K	LOCATIONS OF NODES (X/L)				
1.00	.10	1ST	9.81405	.60923				
		2ND	37.67907	.29763	.76314			
		3RD	83.38550	.19593	.50235	.83648		
		4TH	146.90791	.14543	.37287	.62086	.87509	
		5TH	228.24542	.11533	.29569	.49236	.69396	.89900
1.00	.20	1ST	9.94950	.61577				
		2ND	36.12995	.29874	.76217			
		3RD	78.32020	.19437	.49584	.83321		
		4TH	136.50064	.14276	.36418	.61194	.87156	
		5TH	210.67030	.11217	.28616	.48084	.68484	.89563
1.00	.30	1ST	10.06378	.62168				
		2ND	34.58428	.29908	.76095			
		3RD	73.38202	.19214	.48880	.82974		
		4TH	126.44264	.13951	.35492	.60246	.86781	
		5TH	193.76527	.10853	.27611	.46868	.67510	.89201
1.00	.40	1ST	10.15593	.62700				
		2ND	33.04173	.29863	.75947			
		3RD	68.57077	.18922	.48116	.82604		
		4TH	116.73384	.13569	.34505	.59235	.86379	
		5TH	177.53030	.10441	.26551	.45580	.66467	.88811
1.00	.50	1ST	10.22488	.63177				
		2ND	31.50187	.29736	.75771			
		3RD	63.88623	.18560	.47287	.82209		
		4TH	107.37406	.13129	.33451	.58154	.85947	
		5TH	161.96526	.09982	.25433	.44214	.65345	.88390
1.00	.60	1ST	10.26950	.63601				
		2ND	29.95420	.29524	.75564			
		3RD	59.32808	.18125	.46384	.81785		
		4TH	98.35312	.12631	.32325	.56993	.85482	
		5TH	147.07000	.09477	.24252	.42761	.64134	.87933
1.00	.80	1ST	10.28050	.64296				
		2ND	26.89264	.28820	.75047			
		3RD	50.58920	.17022	.44321	.80834		
		4TH	81.38650	.11453	.29822	.54389	.84432	
		5TH	119.28795	.08328	.21685	.39549	.61395	.86889
1.00	1.00	1ST	10.17662	.64792				
		2ND	23.81939	.27682	.74362			
		3RD	42.34886	.15570	.41825	.79707		
		4TH	65.80021	.10021	.26919	.51299	.83176	
		5TH	94.18135	.07001	.18807	.35841	.58111	.85621

TABLE 4. NATURAL FREQUENCIES FOR TRUNCATED BEAMS

		C = 0.2		C = 0.4	
Ψ	Φ	1ST MODE	2ND MODE	1ST MODE	2ND MODE
.1	.0	3.67847	22.36876	3.61365	22.22533
.2	.0	3.84786	22.70905	3.71272	22.41816
.3	.0	4.02157	23.05519	3.81430	22.61243
.4	.0	4.20091	23.40771	3.91801	22.80770
.5	.0	4.38517	23.76588	4.02307	23.00573
.6	.0	4.57712	24.13146	4.13134	23.20523
.7	.0	4.77262	24.50256	4.24163	23.40584
.8	.0	4.97587	24.88047	4.35377	23.60921
.9	.0	5.18301	25.26468	4.46853	23.81437
1.0	.0	5.39691	25.65562	4.58573	24.02130
.0	.1	3.61104	21.40556	3.56582	21.56926
.0	.2	3.70551	20.77436	3.61455	21.10498
.0	.3	3.79561	20.14155	3.66175	20.64266
.0	.4	3.88164	19.50751	3.70660	20.18308
.0	.5	3.96387	18.87360	3.75005	19.72495
.0	.6	4.04063	18.24194	3.79130	19.27075
.0	.7	4.11409	17.61204	3.83125	18.82074
.0	.8	4.18075	16.98365	3.86828	18.36944
.0	.9	4.23912	16.36086	3.90247	17.97842
.0	1.0	4.29265	15.74243	3.93432	17.59643
.1	.1	3.77736	21.73113	3.66352	21.75592
.2	.2	4.04253	21.41178	3.81260	21.47170
.3	.3	4.31454	21.07969	3.96323	21.18252
.4	.4	4.58885	20.73267	4.11464	20.88849
.5	.5	4.86505	20.37156	4.26638	20.59206
.6	.6	5.14136	19.99895	4.41749	20.29139
.7	.7	5.41396	19.61430	4.56803	19.98790
.8	.8	5.68160	19.21557	4.71720	19.68703
.9	.9	5.94408	18.80675	4.86446	19.40364
1.0	1.0	6.19721	18.38376	5.00722	19.13450

TABLE 5. APPROXIMATE FUNDAMENTAL FREQUENCIES

Ψ	Φ	C	UPPER BOUND	LOWER BOUND	Ψ	Φ	C	UPPER BOUND	LOWER BOUND
.00	.00	.00	3.46410	3.53009	.00	.60	.00	4.57891	5.13806
		.10	3.46410	3.53009			.10	4.13003	4.58113
		.20	3.46410	3.53009			.20	3.92379	4.27828
		.30	3.46410	3.53009			.30	3.79515	4.07590
		.40	3.46410	3.53009			.40	3.70554	3.92952
		.50	3.46410	3.53009			.50	3.63910	3.81882
		.75	3.46410	3.53009			.75	3.52972	3.63569
.00	.10	.00	3.68862	3.79204	.00	.80	.00	4.79400	5.69898
		.10	3.59159	3.68841			.10	4.27443	4.98299
		.20	3.55178	3.64280			.20	4.02933	4.56564
		.30	3.52681	3.61297			.30	3.87526	4.28366
		.40	3.50935	3.59139			.40	3.76697	4.07954
		.50	3.49645	3.57492			.50	3.68584	3.92557
		.75	3.47570	3.54703			.75	3.54938	3.67299
.00	.20	.00	3.89954	4.05588	.00	1.00	.00	4.89898	6.27495
		.10	3.71413	3.85364			.10	4.36092	5.40881
		.20	3.63587	3.76033			.20	4.10184	4.87170
		.30	3.58693	3.69910			.30	3.93573	4.50434
		.40	3.55281	3.65480			.40	3.81669	4.23801
		.50	3.52763	3.62107			.50	3.72577	4.03760
		.75	3.48703	3.56424			.75	3.56785	3.71133
.00	.30	.00	4.09560	4.32206	.10	.00	.00	3.78444	3.86568
		.10	3.83053	4.02565			.10	3.67314	3.75169
		.20	3.71576	3.88267			.20	3.62115	3.69739
		.30	3.64415	3.78846			.30	3.58526	3.65957
		.40	3.59431	3.72032			.40	3.55780	3.63046
		.50	3.55755	3.66854			.50	3.53561	3.60684
		.75	3.49811	3.58171			.75	3.49402	3.56231
.00	.40	.00	4.27532	4.59096	.10	.10	.00	4.00639	4.13031
		.10	3.93954	4.20432			.10	3.80288	3.91666
		.20	3.79081	4.00978			.20	3.71002	3.81446
		.30	3.69814	3.88105			.30	3.64854	3.74524
		.40	3.63371	3.78795			.40	3.60329	3.69349
		.50	3.58616	3.71732			.50	3.56805	3.65270
		.75	3.50892	3.59944			.75	3.50561	3.57944
.00	.50	.00	4.43706	4.86287	.10	.20	.00	4.21433	4.39723
		.10	4.03983	4.38953			.10	3.92704	4.08853
		.20	3.86039	4.14166			.20	3.79498	3.93642
		.30	3.74859	3.97686			.30	3.70906	3.83419
		.40	3.67084	3.85768			.40	3.64691	3.75866
		.50	3.61336	3.76741			.50	3.59927	3.69991
		.75	3.51946	3.61744			.75	3.51693	3.59684

TABLE 5. APPROXIMATE FUNDAMENTAL FREQUENCIES

Ψ	Φ	C	UPPER BOUND	LOWER BOUND	Ψ	Φ	C	UPPER BOUND	LOWER BOUND
.10	.30	.00	4.40690	4.66685	.20	.00	.00	4.11047	4.20899
		.10	4.04439	4.26717			.10	3.88993	3.98278
		.20	3.87539	4.06324			.20	3.78311	3.87085
		.30	3.76650	3.92643			.30	3.70949	3.79299
		.40	3.68849	3.82597			.40	3.65338	3.73330
		.50	3.62920	3.74844			.50	3.60824	3.68508
		.75	3.52800	3.61450			.75	3.52414	3.59481
.10	.40	.00	4.58258	4.93946	.20	.10	.00	4.32982	4.47647
		.10	4.15365	4.45242			.10	4.02164	4.15437
		.20	3.95063	4.19490			.20	3.87305	3.99235
		.30	3.82055	4.02195			.30	3.77328	3.88149
		.40	3.72787	3.89542			.40	3.69909	3.79810
		.50	3.65777	3.79830			.50	3.64075	3.73200
		.75	3.53879	3.63242			.75	3.53571	3.61213
.10	.50	.00	4.73962	5.21532	.20	.20	.00	4.53467	4.74663
		.10	4.25347	4.64414			.10	4.14712	4.33284
		.20	4.02000	4.33135			.20	3.95874	4.11879
		.30	3.87085	4.12073			.30	3.83414	3.97333
		.40	3.76488	3.96700			.40	3.74284	3.86506
		.50	3.68489	3.84949			.50	3.67201	3.78026
		.75	3.54930	3.65061			.75	3.54703	3.62971
.10	.60	.00	4.87608	5.49465	.20	.30	.00	4.72361	5.01978
		.10	4.34245	4.84219			.10	4.26510	4.51802
		.20	4.08282	4.47258			.20	4.03955	4.25014
		.30	3.91706	4.22278			.30	3.89174	4.06849
		.40	3.79936	4.04072			.40	3.78446	3.93419
		.50	3.71049	3.90201			.50	3.70193	3.82988
		.75	3.55954	3.66906			.75	3.55808	3.64757
.10	.80	.00	5.07807	6.06430	.20	.40	.00	4.89506	5.29619
		.10	4.48208	5.25671			.10	4.37428	4.70976
		.20	4.18599	4.76927			.20	4.11480	4.38636
		.30	3.99587	4.43667			.30	3.94575	4.16697
		.40	3.86007	4.19457			.40	3.82378	4.00548
		.50	3.75683	4.01104			.50	3.73043	3.88084
		.75	3.57913	3.70675			.75	3.56885	3.66568
.10	1.00	.00	5.16633	6.64945	.20	.50	.00	5.04723	5.57605
		.10	4.56063	5.69483			.10	4.47328	4.90791
		.20	4.25446	5.08473			.20	4.18382	4.52742
		.30	4.05419	4.66363			.30	3.99583	4.26876
		.40	3.90864	4.35697			.40	3.86064	4.07894
		.50	3.79617	4.12539			.50	3.75745	3.93314
		.75	3.59752	3.74550			.75	3.57934	3.68407

TABLE 5. APPROXIMATE FUNDAMENTAL FREQUENCIES

Ψ	Φ	C	UPPER BOUND	LOWER BOUND	Ψ	Φ	C	UPPER BOUND	LOWER BOUND
.20	.60	.00	5.17811	5.85955	.30	.30	.00	5.04571	5.38093
		.10	4.56070	5.11232			.10	4.49254	4.77819
		.20	4.24589	4.67330			.20	4.20822	4.44343
		.30	4.04162	4.37386			.30	4.01987	4.21469
		.40	3.89487	4.15456			.40	3.88223	4.04501
		.50	3.78290	3.98679			.50	3.77573	3.91288
		.75	3.58955	3.70272			.75	3.58835	3.68092
.20	.80	.00	5.36656	6.43796	.30	.40	.00	5.21275	5.66119
		.10	4.69487	5.53927			.10	4.60130	4.97633
		.20	4.34633	4.97934			.20	4.28334	4.58425
		.30	4.11895	4.59396			.30	4.07378	4.31618
		.40	3.95475	4.31229			.40	3.92146	4.11817
		.50	3.82880	4.09811			.50	3.80416	3.96495
		.75	3.60907	3.74081			.75	3.59910	3.69923
.20	1.00	.00	5.43765	7.03230	.30	.50	.00	5.35985	5.94509
		.10	4.76478	5.98943			.10	4.69915	5.18080
		.20	4.41039	5.30421			.20	4.35183	4.72995
		.30	4.17494	4.82725			.30	4.12355	4.42101
		.40	4.00209	4.47868			.40	3.95812	4.19353
		.50	3.86750	4.21482			.50	3.83106	4.01838
		.75	3.62738	3.77995			.75	3.60957	3.71781
.30	.00	.00	4.44230	4.56015	.30	.60	.00	5.48494	6.23279
		.10	4.11441	4.22339			.10	4.78465	5.39144
		.20	3.95000	4.05059			.20	4.41298	4.88048
		.30	3.83681	3.93044			.30	4.16883	4.52919
		.40	3.75086	3.83867			.40	3.99206	4.27108
		.50	3.68199	3.76484			.50	3.85634	4.07317
		.75	3.55446	3.62758			.75	3.61975	3.73666
.30	.10	.00	4.65893	4.83064	.30	.80	.00	5.65943	6.81998
		.10	4.24779	4.40155			.10	4.91267	5.83058
		.20	4.04088	4.17655			.20	4.51033	5.19589
		.30	3.90105	4.02181			.30	4.24449	4.75556
		.40	3.79676	3.90525			.40	4.05104	4.43274
		.50	3.71456	3.81282			.50	3.90175	4.18682
		.75	3.56602	3.64509			.75	3.63921	3.77515
.30	.20	.00	4.86056	5.10415	.30	1.00	.00	5.71288	7.42349
		.10	4.37426	4.58654			.10	4.97328	6.29253
		.20	4.12718	4.30751			.20	4.56961	5.53018
		.30	3.96218	4.11657			.30	4.29796	4.99526
		.40	3.84061	3.97403			.40	4.09704	4.60316
		.50	3.74583	3.86217			.50	3.93977	4.30591
		.75	3.57732	3.66287			.75	3.65743	3.81470

TABLE 5. APPROXIMATE FUNDAMENTAL FREQUENCIES

Ψ	Φ	C	UPPER BOUND	LOWER BOUND
.40	.00	.00	4.77996	4.91929
		.10	4.34650	4.47354
		.20	4.12184	4.23667
		.30	3.96724	4.07198
		.40	3.85026	3.94659
		.50	3.75687	3.84613
		.75	3.58497	3.66064
.40	.10	.00	4.99375	5.19293
		.10	4.48122	4.65818
		.20	4.21353	4.36715
		.30	4.03187	4.16627
		.40	3.89632	4.01499
		.50	3.78949	3.89519
		.75	3.59653	3.67834
.40	.20	.00	5.19199	5.46988
		.10	4.60836	4.84961
		.20	4.30027	4.50267
		.30	4.09320	4.26398
		.40	3.94022	4.08561
		.50	3.82076	3.94564
		.75	3.60782	3.69631
.40	.30	.00	5.37318	5.75035
		.10	4.72660	5.04762
		.20	4.38141	4.64317
		.30	4.15091	4.36511
		.40	3.98181	4.15846
		.50	3.85061	3.99746
		.75	3.61883	3.71456
.40	.40	.00	5.53563	6.03450
		.10	4.83460	5.25206
		.20	4.45623	4.78862
		.30	4.20463	4.46963
		.40	4.02090	4.23353
		.50	3.87895	4.05066
		.75	3.62955	3.73307
.40	.50	.00	5.67746	6.32247
		.10	4.93095	5.46274
		.20	4.52403	4.93897
		.30	4.25400	4.57755
		.40	4.05734	4.31081
		.50	3.90571	4.10523
		.75	3.63999	3.75184
.40	.60	.00	5.79655	6.61438
		.10	5.01420	5.67949
		.20	4.58408	5.09417
		.30	4.29869	4.68884
		.40	4.09093	4.39031
		.50	3.93080	4.16117
		.75	3.65014	3.77089
.40	.80	.00	5.95664	7.21036
		.10	5.13539	6.13056
		.20	4.67798	5.41897
		.30	4.37251	4.92155
		.40	4.14892	4.55595
		.50	3.97569	4.27718
		.75	3.66953	3.80978
.40	1.00	.00	5.99199	7.82303
		.10	5.18602	6.60404
		.20	4.73208	5.76267
		.30	4.42325	5.16770
		.40	4.19350	4.73045
		.50	4.01298	4.39868
		.75	3.68766	3.84975
.50	.00	.00	5.12348	5.28650
		.10	4.58609	4.73319
		.20	4.29863	4.42920
		.30	4.10081	4.21766
		.40	3.95158	4.05711
		.50	3.83290	3.92898
		.75	3.61569	3.69398
.50	.10	.00	5.33427	5.56339
		.10	4.72183	4.92424
		.20	4.39098	4.56422
		.30	4.16575	4.31491
		.40	3.99776	4.12735
		.50	3.86554	3.97914
		.75	3.62724	3.71187
.50	.20	.00	5.52896	5.84385
		.10	4.84930	5.12200
		.20	4.47803	4.70432
		.30	4.22722	4.41562
		.40	4.04169	4.19985
		.50	3.89680	4.03069
		.75	3.63851	3.73004

TABLE 5. APPROXIMATE FUNDAMENTAL FREQUENCIES

Ψ	Φ	C	UPPER BOUND	LOWER BOUND	Ψ	Φ	C	UPPER BOUND	LOWER BOUND
.50	.30	.00	5.70600	6.12806	.60	.00	.00	5.47284	5.66186
		.10	4.96716	5.32627			.10	4.83306	5.00233
		.20	4.55910	4.84944			.20	4.48037	4.62823
		.30	4.28485	4.51979			.30	4.23752	4.36756
		.40	4.08321	4.27460			.40	4.05484	4.17027
		.50	3.92658	4.08364			.50	3.91006	4.01341
		.75	3.64950	3.74848			.75	3.64662	3.72760
.50	.40	.00	5.86365	6.41614	.60	.10	.00	5.68047	5.94208
		.10	5.07406	5.53688			.10	4.96949	5.19968
		.20	4.63346	4.99953			.20	4.57324	4.76782
		.30	4.33830	4.62739			.30	4.30271	4.46781
		.40	4.12213	4.35159			.40	4.10111	4.24238
		.50	3.95481	4.13798			.50	3.94272	4.06468
		.75	3.66020	3.76719			.75	3.65815	3.74569
.50	.50	.00	6.00000	6.70820	.60	.20	.00	5.87142	6.22612
		.10	5.16857	5.75364			.10	5.09695	5.40365
		.20	4.70039	5.15454			.20	4.66044	4.91253
		.30	4.38721	4.73842			.30	4.36423	4.57156
		.40	4.15829	4.43082			.40	4.14503	4.31677
		.50	3.98141	4.19370			.50	3.97394	4.11736
		.75	3.67061	3.78617			.75	3.66940	3.76406
.50	.60	.00	6.11289	7.00434	.60	.30	.00	6.04411	6.51410
		.10	5.24922	5.97637			.10	5.21410	5.61406
		.20	4.75916	5.31443			.20	4.74127	5.06228
		.30	4.43120	4.85286			.30	4.42172	4.67879
		.40	4.19150	4.51230			.40	4.18644	4.39344
		.50	4.00629	4.25082			.50	4.00364	4.17144
		.75	3.68073	3.80541			.75	3.68036	3.78269
.50	.80	.00	6.25815	7.60910	.60	.40	.00	6.19677	6.80614
		.10	5.36292	6.43911			.10	5.31955	5.83071
		.20	4.84923	5.64861			.20	4.81500	5.21703
		.30	4.50299	5.09196			.30	4.47482	4.78951
		.40	4.24840	4.68197			.40	4.22515	4.47239
		.50	4.05061	4.36921			.50	4.03174	4.22694
		.75	3.70004	3.84471			.75	3.69104	3.80160
.50	1.00	.00	6.27495	8.23093	.60	.50	.00	6.32744	7.10230
		.10	5.40292	6.92384			.10	5.41188	6.05342
		.20	4.89778	6.00168			.20	4.88089	5.37671
		.30	4.55081	5.34461			.30	4.52316	4.90368
		.40	4.29146	4.86060			.40	4.26098	4.55360
		.50	4.08714	4.49316			.50	4.05816	4.28383
		.75	3.71808	3.88509			.75	3.70143	3.82078

TABLE 5. APPROXIMATE FUNDAMENTAL FREQUENCIES

Ψ	Φ	C	UPPER BOUND	LOWER BOUND
.60	.60	.00	6.43391	7.40267
		.10	5.48960	6.28200
		.20	4.93820	5.54127
		.30	4.56637	5.02130
		.40	4.29377	4.63708
		.50	4.08282	4.34213
		.75	3.71151	3.84023
.60	.80	.00	6.56390	8.01622
		.10	5.59514	6.75613
		.20	5.02406	5.88482
		.30	4.63594	5.26684
		.40	4.34950	4.81082
		.50	4.12653	4.46294
		.75	3.73074	3.87994
.60	1.00	.00	6.56171	8.64718
		.10	5.62389	7.25185
		.20	5.06668	6.24724
		.30	4.68063	5.52604
		.40	4.39093	4.99362
		.50	4.16224	4.58935
		.75	3.74869	3.92073
.80	.00	.00	6.18902	6.43723
		.10	5.34864	5.56883
		.20	4.85869	5.04607
		.30	4.52044	4.68026
		.40	4.26723	4.40465
		.50	4.06786	4.18708
		.75	3.70907	3.79571
.80	.10	.00	6.38977	6.72429
		.10	5.48542	5.77840
		.20	4.95214	5.19487
		.30	4.58590	4.78660
		.40	4.31356	4.48058
		.50	4.10049	4.24061
		.75	3.72057	3.81420
.80	.20	.00	6.57267	7.01561
		.10	5.61184	5.99444
		.20	5.03915	5.34883
		.30	4.64728	4.89652
		.40	4.35734	4.55884
		.50	4.13159	4.29558
		.75	3.73178	3.83296
.80	.30	.00	6.73607	7.31124
		.10	5.72654	6.21672
		.20	5.11899	5.50787
		.30	4.70423	5.01000
		.40	4.39840	4.63944
		.50	4.16107	4.35199
		.75	3.74270	3.85200
.80	.40	.00	6.87814	7.61123
		.10	5.82809	6.44505
		.20	5.19093	5.67193
		.30	4.75636	5.12703
		.40	4.43656	4.72235
		.50	4.18885	4.40984
		.75	3.75332	3.87131
.80	.50	.00	6.99682	7.91563
		.10	5.91505	6.67923
		.20	5.25420	5.84094
		.30	4.80330	5.24757
		.40	4.47163	4.80759
		.50	4.21485	4.46912
		.75	3.76364	3.89089
.80	.60	.00	7.08982	8.22447
		.10	5.98592	6.91909
		.20	5.30803	6.01484
		.30	4.84467	5.37163
		.40	4.50342	4.89514
		.50	4.23899	4.52983
		.75	3.77365	3.91075
.80	.80	.00	7.18799	8.85557
		.10	6.07323	7.41518
		.20	5.38434	6.37703
		.30	4.90919	5.63019
		.40	4.55650	5.07717
		.50	4.28135	4.65554
		.75	3.79271	3.95129
.80	1.00	.00	7.14647	9.50473
		.10	6.07763	7.93210
		.20	5.41391	6.75801
		.30	4.94702	5.90259
		.40	4.59437	5.26844
		.50	4.31528	4.78698
		.75	3.81046	3.99292

TABLE 5. APPROXIMATE FUNDAMENTAL FREQUENCIES

Ψ	Φ	C	UPPER BOUND	LOWER BOUND
1.00	.00	.00	6.92820	7.24569
		.10	5.89209	6.17242
		.20	5.25670	5.49065
		.30	4.81608	5.01052
		.40	4.48749	4.65004
		.50	4.23031	4.36731
		.75	3.77234	3.86499
1.00	.10	.00	7.12130	7.53977
		.10	6.02787	6.39369
		.20	5.35007	5.64868
		.30	4.88149	5.12308
		.40	4.53373	4.72988
		.50	4.26284	4.42316
		.75	3.78380	3.88388
1.00	.20	.00	7.29537	7.83849
		.10	6.15190	6.62122
		.20	5.43621	5.81190
		.30	4.94241	5.23930
		.40	4.57721	4.81212
		.50	4.29374	4.48049
		.75	3.79496	3.90304
1.00	.30	.00	7.44868	8.14184
		.10	6.26280	6.85480
		.20	5.51437	5.98022
		.30	4.99846	5.35914
		.40	4.61776	4.89673
		.50	4.32292	4.53928
		.75	3.80583	3.92248
1.00	.40	.00	7.57934	8.44984
		.10	6.35914	7.09423
		.20	5.58379	6.15356
		.30	5.04926	5.48258
		.40	4.65518	4.98371
		.50	4.35030	4.59953
		.75	3.81638	3.94220
1.00	.50	.00	7.68521	8.76249
		.10	6.43944	7.33933
		.20	5.64369	6.33185
		.30	5.09442	5.60960
		.40	4.68929	5.07306
		.50	4.37580	4.66125
		.75	3.82663	3.96220
1.00	.60	.00	7.76392	9.07980
		.10	6.50219	7.58991
		.20	5.69329	6.51501
		.30	5.13355	5.74018
		.40	4.71991	5.16478
		.50	4.39933	4.72442
		.75	3.83656	3.98247
1.00	.80	.00	7.82856	9.72840
		.10	6.56883	8.10688
		.20	5.75852	6.89566
		.30	5.19222	6.01191
		.40	4.76994	5.35527
		.50	4.44016	4.85515
		.75	3.85545	4.02384
1.00	1.00	.00	7.74597	10.39568
		.10	6.54654	8.64398
		.20	5.77350	7.29493
		.30	5.22233	6.29764
		.40	4.80384	5.55516
		.50	4.47214	4.99172
		.75	3.87298	4.06632
.25	.25	.00	4.79257	5.06144
		.10	4.31996	4.55190
		.20	4.08352	4.27855
		.30	3.92706	4.09233
		.40	3.81259	3.95397
		.50	3.72391	3.84593
		.75	3.56771	3.65520
.75	.75	.00	7.01561	8.48425
		.10	5.93616	7.12264
		.20	5.27732	6.16033
		.30	4.82584	5.47252
		.40	4.49205	4.96365
		.50	4.23235	4.57509
		.75	3.77249	3.92315
.90	.90	.00	7.49528	9.61462
		.10	6.32607	8.01807
		.20	5.58736	6.82582
		.30	5.07015	5.95653
		.40	4.68243	5.31135
		.50	4.37786	4.82073
		.75	3.83297	4.00830

TABLE 6. RADII OF GYRATION FOR DIFFERENT CROSS-SECTIONS

γ	β	RADIUS/h	γ	β	RADIUS/h	γ	β	RADIUS/h
1.00	1.00	.40825	2.00	1.00	.47809	4.00	1.00	.52298
	1.25	.42008		1.25	.48467		1.25	.52641
	1.50	.43033		1.50	.49042		1.50	.52943
	1.75	.43930		1.75	.49549		1.75	.53212
	2.00	.44721		2.00	.50000		2.00	.53452
	2.50	.46057		2.50	.50767		2.50	.53864
	3.00	.47140		3.00	.51394		3.00	.54203
	3.50	.48038		3.50	.51918		3.50	.54488
	4.00	.48795		4.00	.52361		4.00	.54730
	5.00	.50000		5.00	.53071		5.00	.55120
	7.00	.51640		7.00	.54045		7.00	.55658
	10.00	.53108		10.00	.54924		10.00	.56148
1.25	1.00	.43355	2.50	1.00	.49507	5.00	1.00	.53300
	1.25	.44345		1.25	.50043		1.25	.53576
	1.50	.45204		1.50	.50513		1.50	.53821
	1.75	.45958		1.75	.50929		1.75	.54038
	2.00	.46625		2.00	.51299		2.00	.54233
	2.50	.47753		2.50	.51930		2.50	.54566
	3.00	.48671		3.00	.52448		3.00	.54842
	3.50	.49433		3.50	.52881		3.50	.55074
	4.00	.50076		4.00	.53248		4.00	.55271
	5.00	.51102		5.00	.53837		5.00	.55589
	7.00	.52502		7.00	.54647		7.00	.56030
	10.00	.53759		10.00	.55380		10.00	.56431
1.50	1.00	.45227	3.00	1.00	.50709	7.00	1.00	.54495
	1.25	.46075		1.25	.51161		1.25	.54694
	1.50	.46813		1.50	.51558		1.50	.54870
	1.75	.47463		1.75	.51909		1.75	.55027
	2.00	.48038		2.00	.52223		2.00	.55168
	2.50	.49014		2.50	.52759		2.50	.55410
	3.00	.49809		3.00	.53200		3.00	.55611
	3.50	.50471		3.50	.53569		3.50	.55779
	4.00	.51030		4.00	.53882		4.00	.55923
	5.00	.51924		5.00	.54385		5.00	.56156
	7.00	.53145		7.00	.55079		7.00	.56478
	10.00	.54244		10.00	.55708		10.00	.56773
1.75	1.00	.46667	3.50	1.00	.51605	10.00	1.00	.55427
	1.25	.47408		1.25	.51995		1.25	.55567
	1.50	.48055		1.50	.52338		1.50	.55691
	1.75	.48625		1.75	.52643		1.75	.55802
	2.00	.49130		2.00	.52915		2.00	.55902
	2.50	.49989		2.50	.53381		2.50	.56073
	3.00	.50691		3.00	.53764		3.00	.56216
	3.50	.51275		3.50	.54085		3.50	.56335
	4.00	.51770		4.00	.54358		4.00	.56438
	5.00	.52561		5.00	.54798		5.00	.56604
	7.00	.53645		7.00	.55404		7.00	.56834
	10.00	.54622		10.00	.55955		10.00	.57045

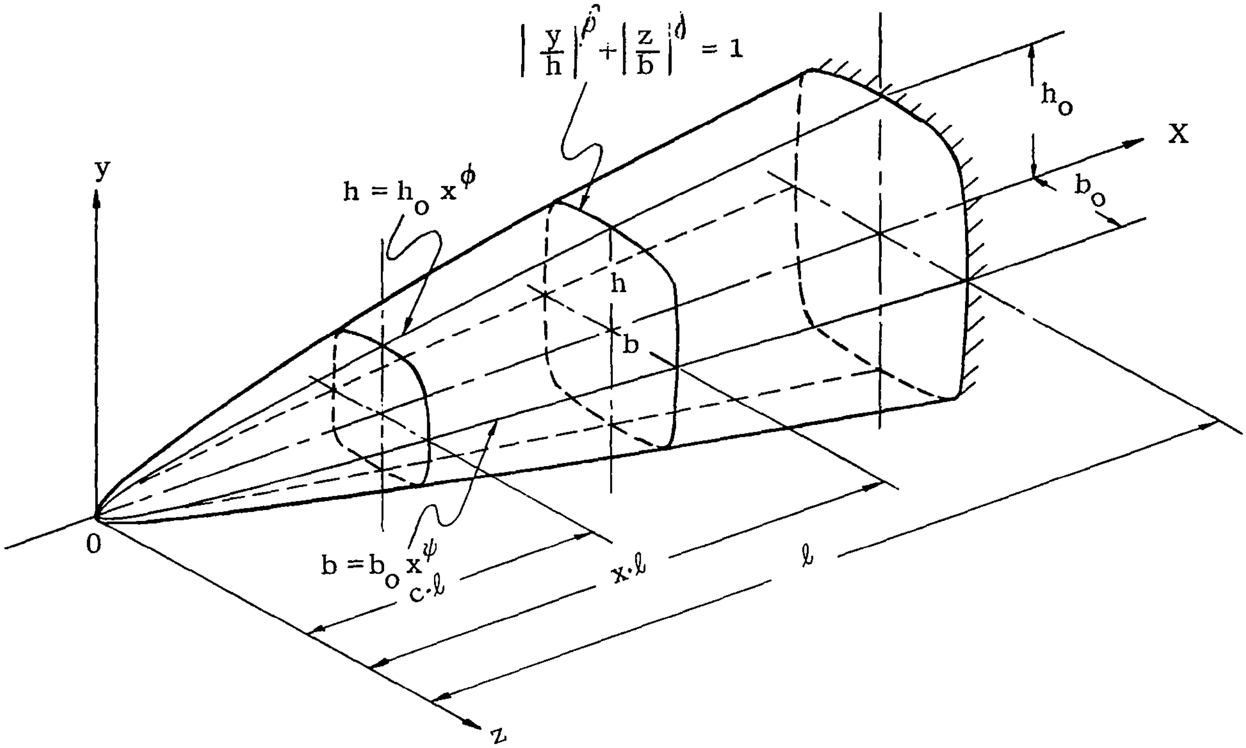

Fig. 1 Cantilever Beam

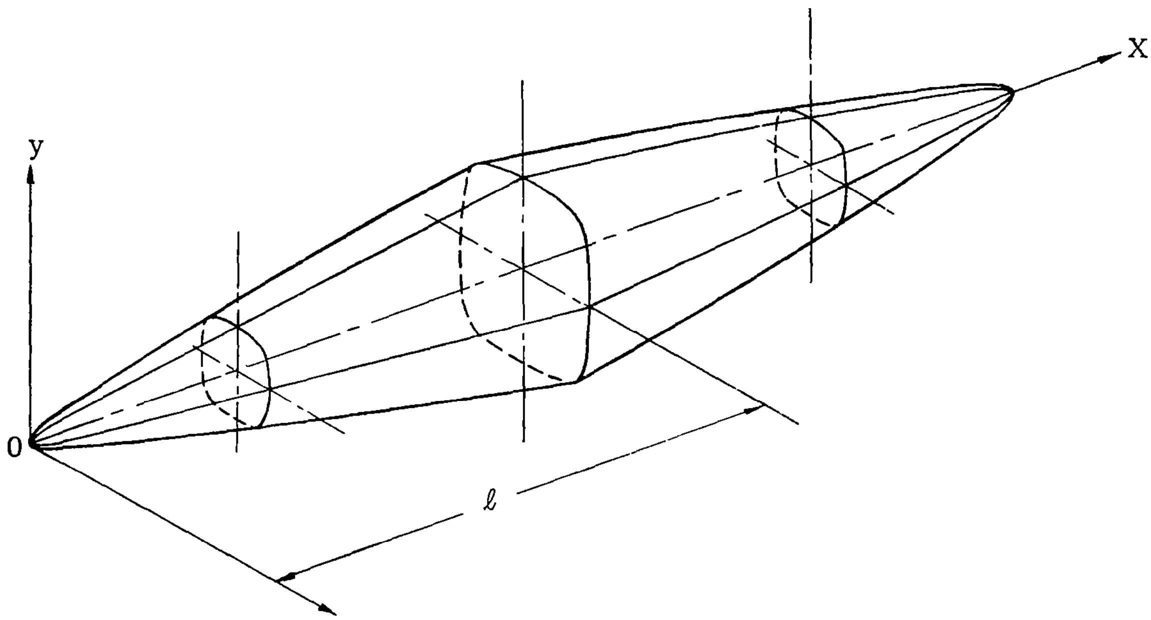

Fig. 2 Free-Free Beam

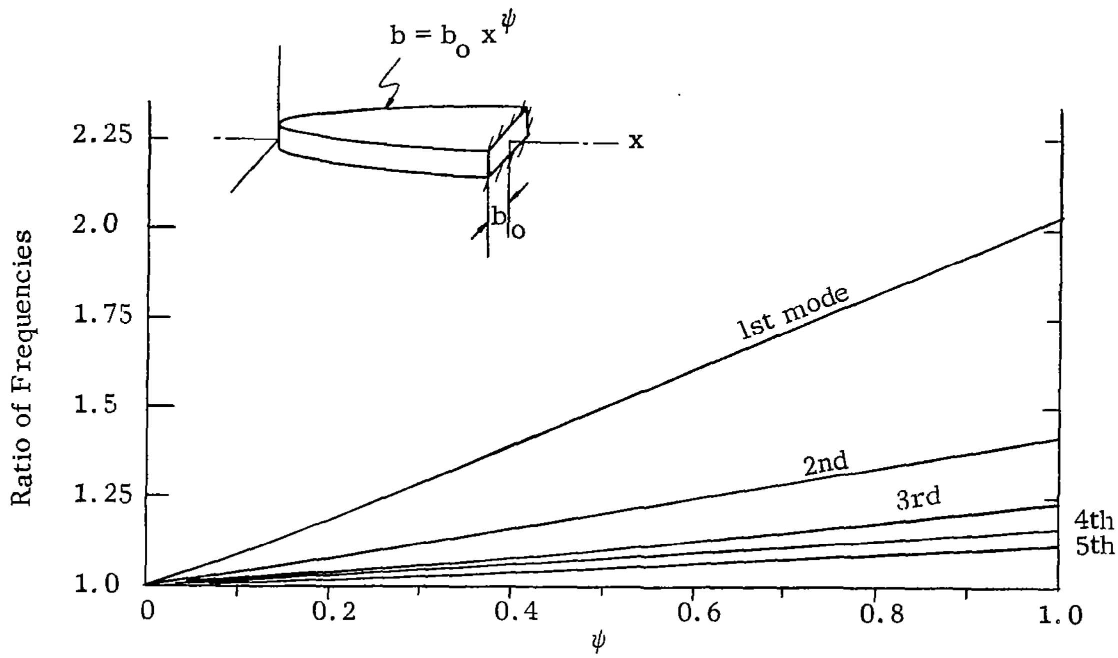

Fig. 3 Ratio of Frequencies of Cantilever Beams with Cross-Section Varying in Width to Those of Uniform Cross-Section

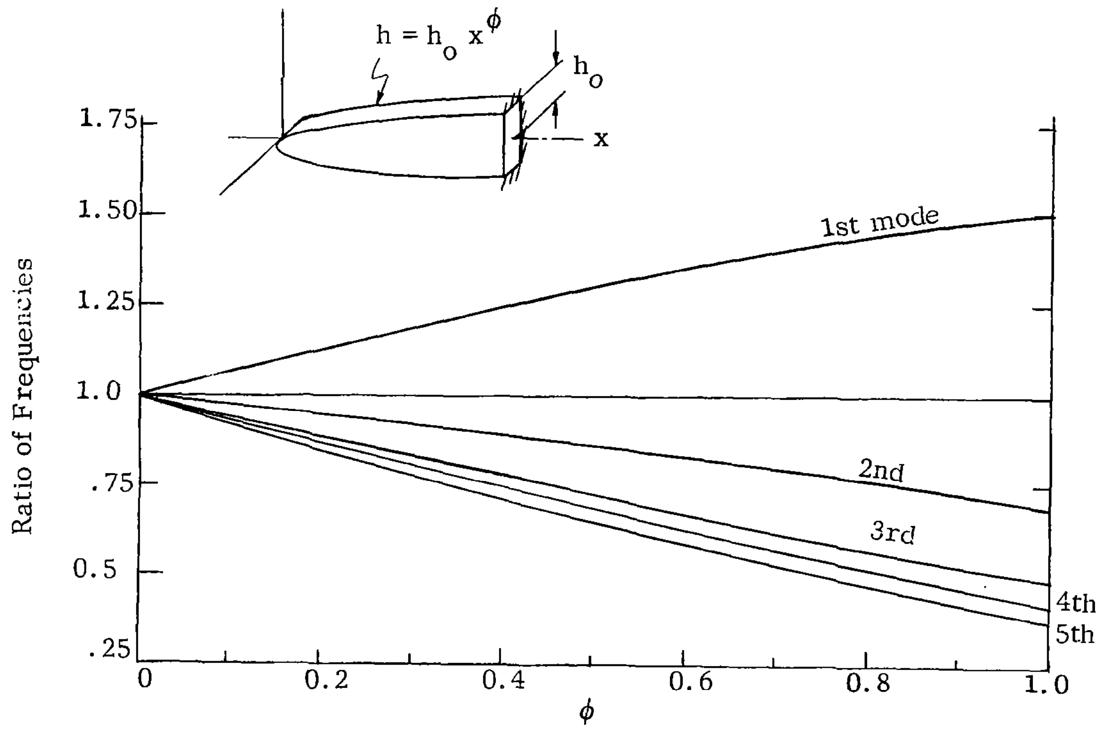

Fig. 4 Ratio of Frequencies of Cantilever Beams with Cross-Section Varying in Thickness to Those of Uniform Cross-Section

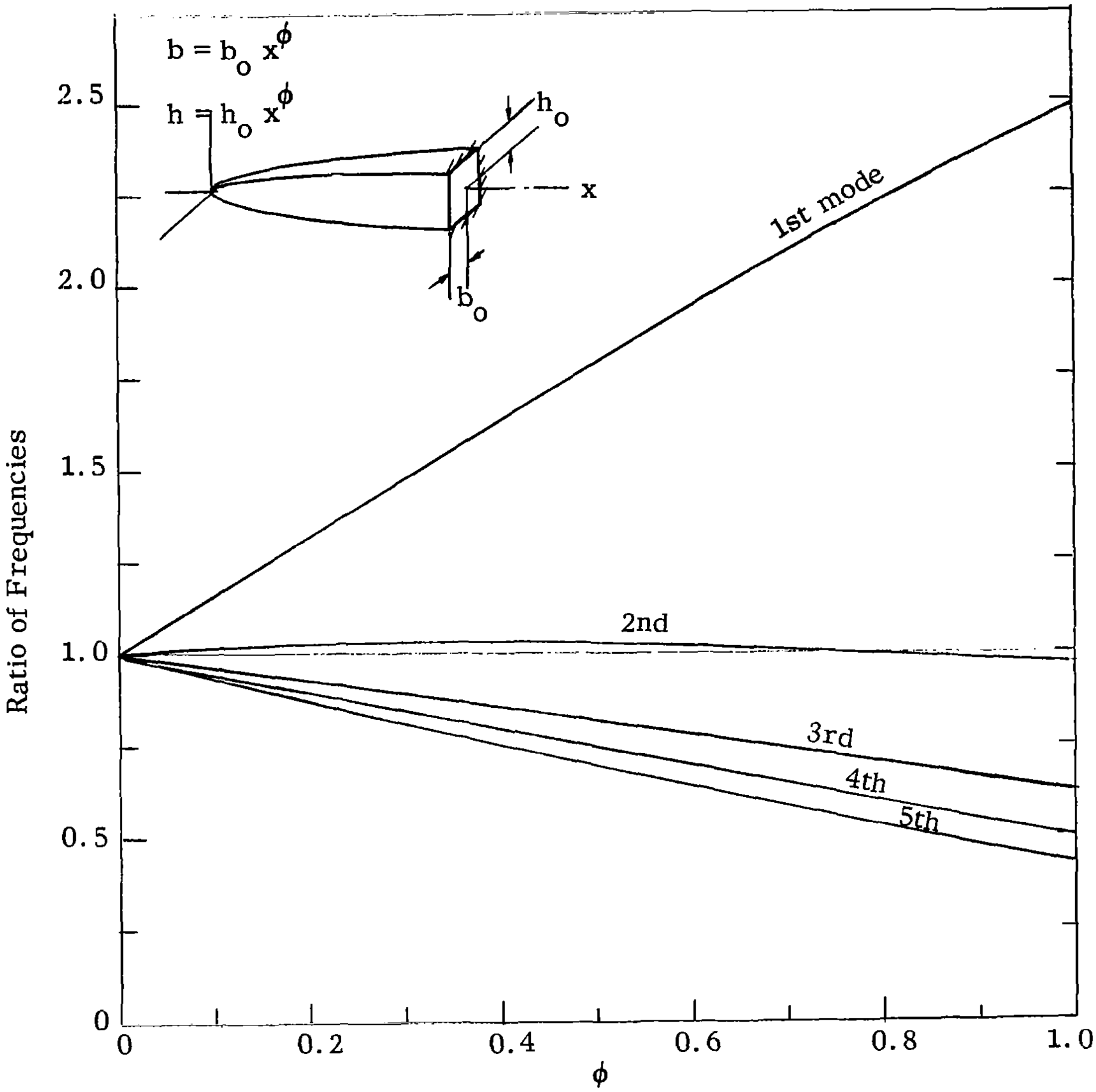

Fig. 5 Ratio of Frequencies of Cantilever Beams with Cross-Section Varying Both in Width and Thickness to Those of Uniform Cross-Section

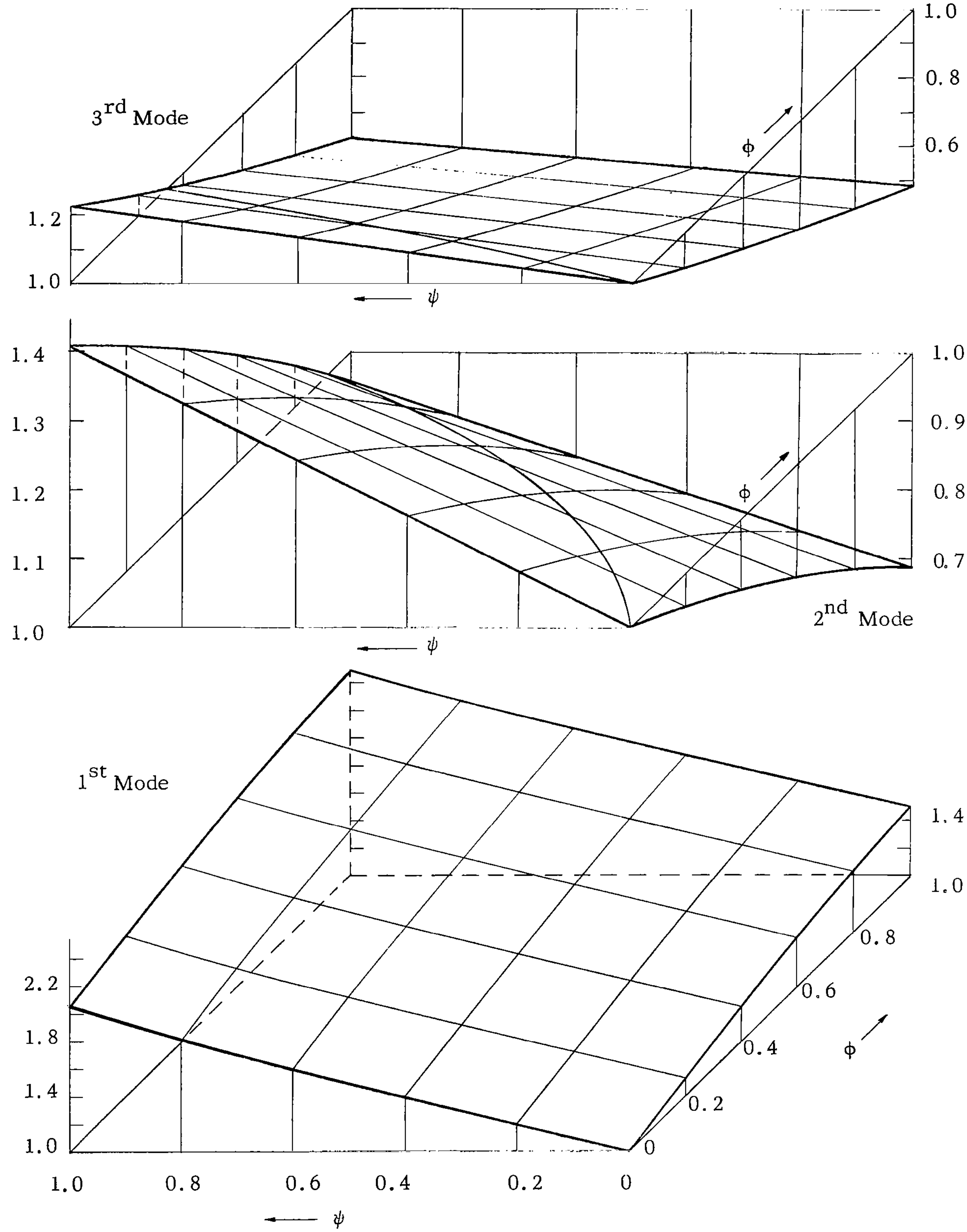

Fig. 6 Ratio of Frequencies of Tapered Cantilever Beams to Frequencies of a Uniform Beam

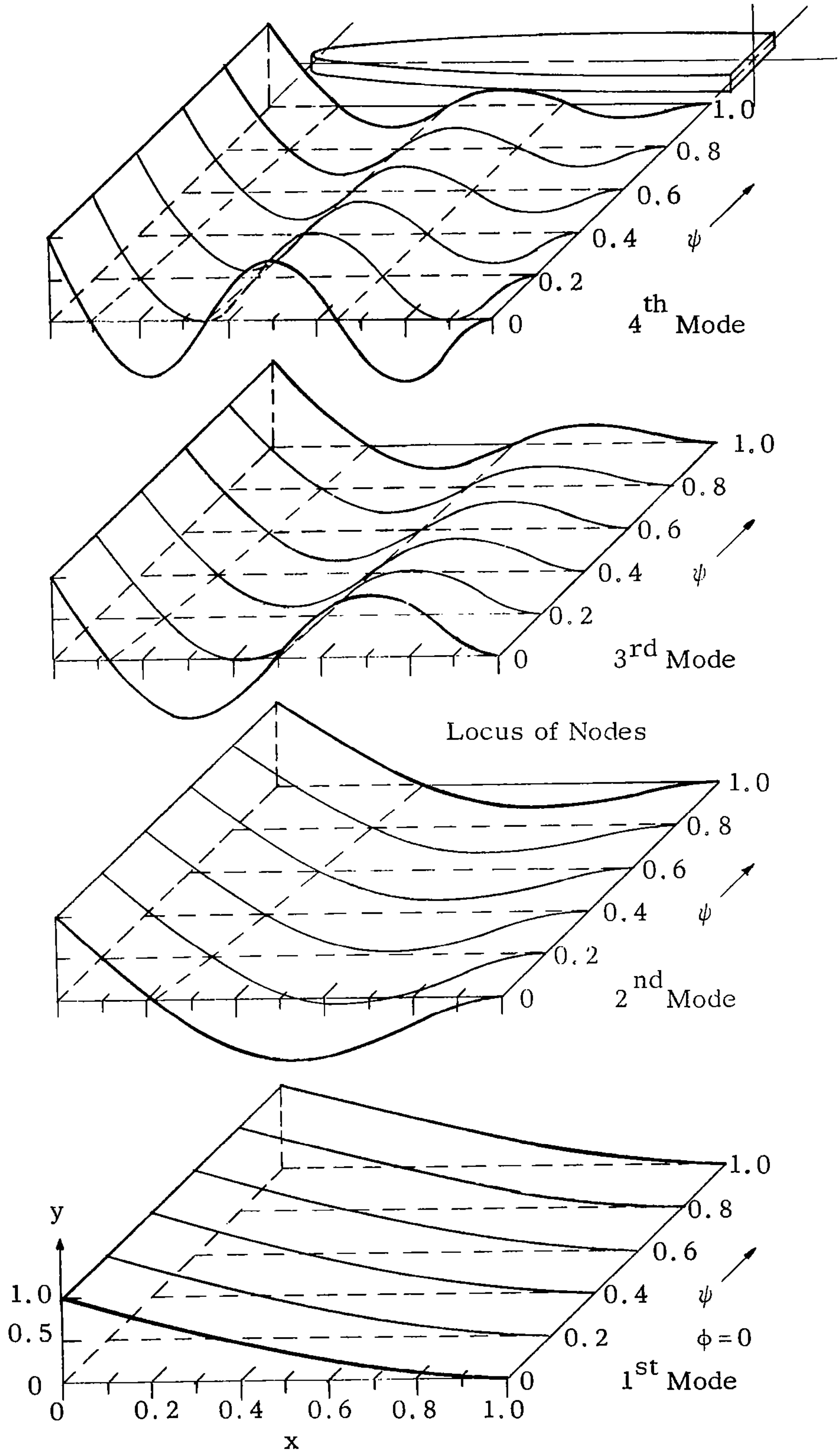

Fig. 7 Mode Shapes for Cantilever Beams with Varying Width and Constant Thickness

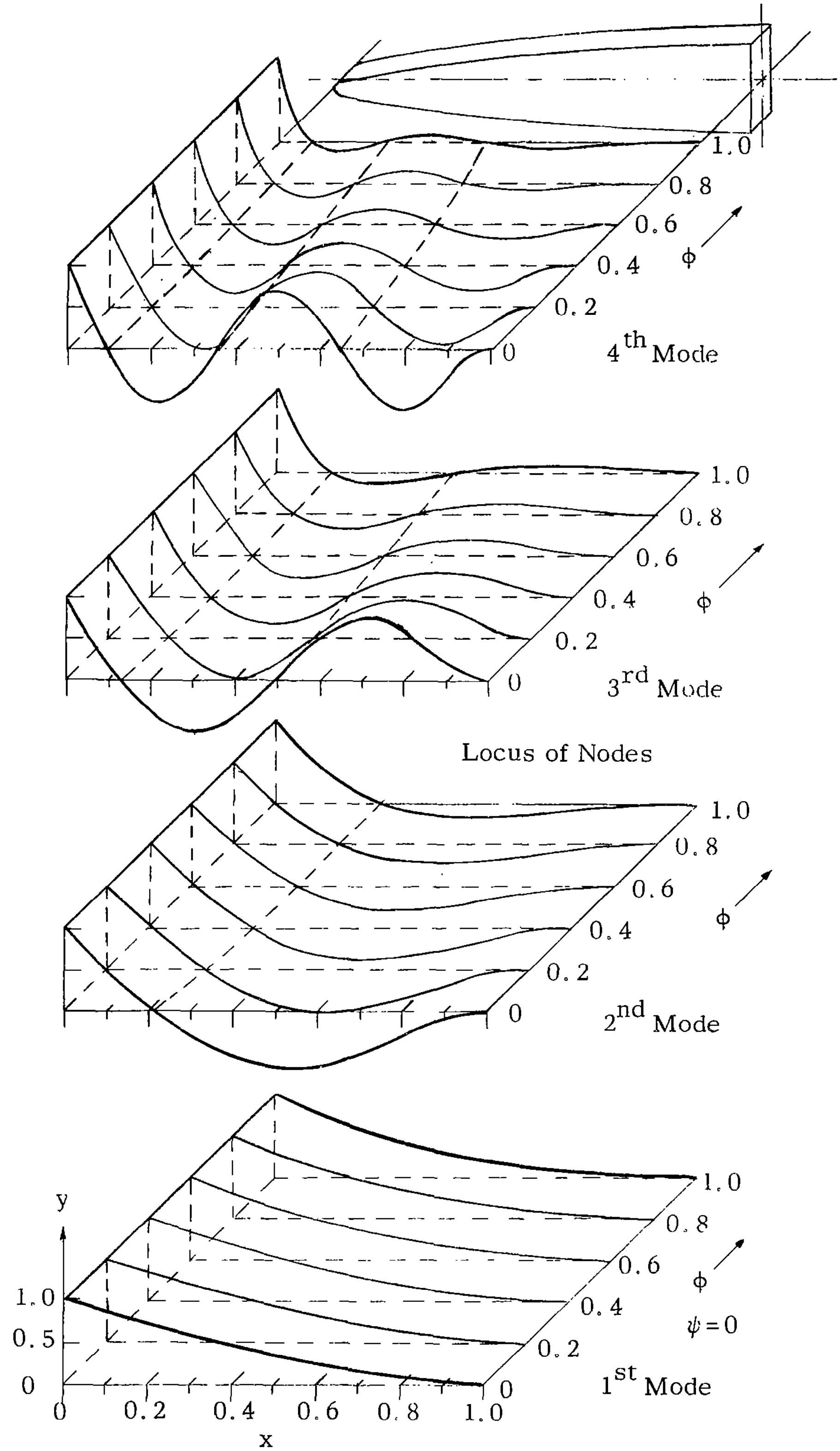

Fig. 8 Mode Shapes for Cantilever Beams with Varying Thickness and Constant Width

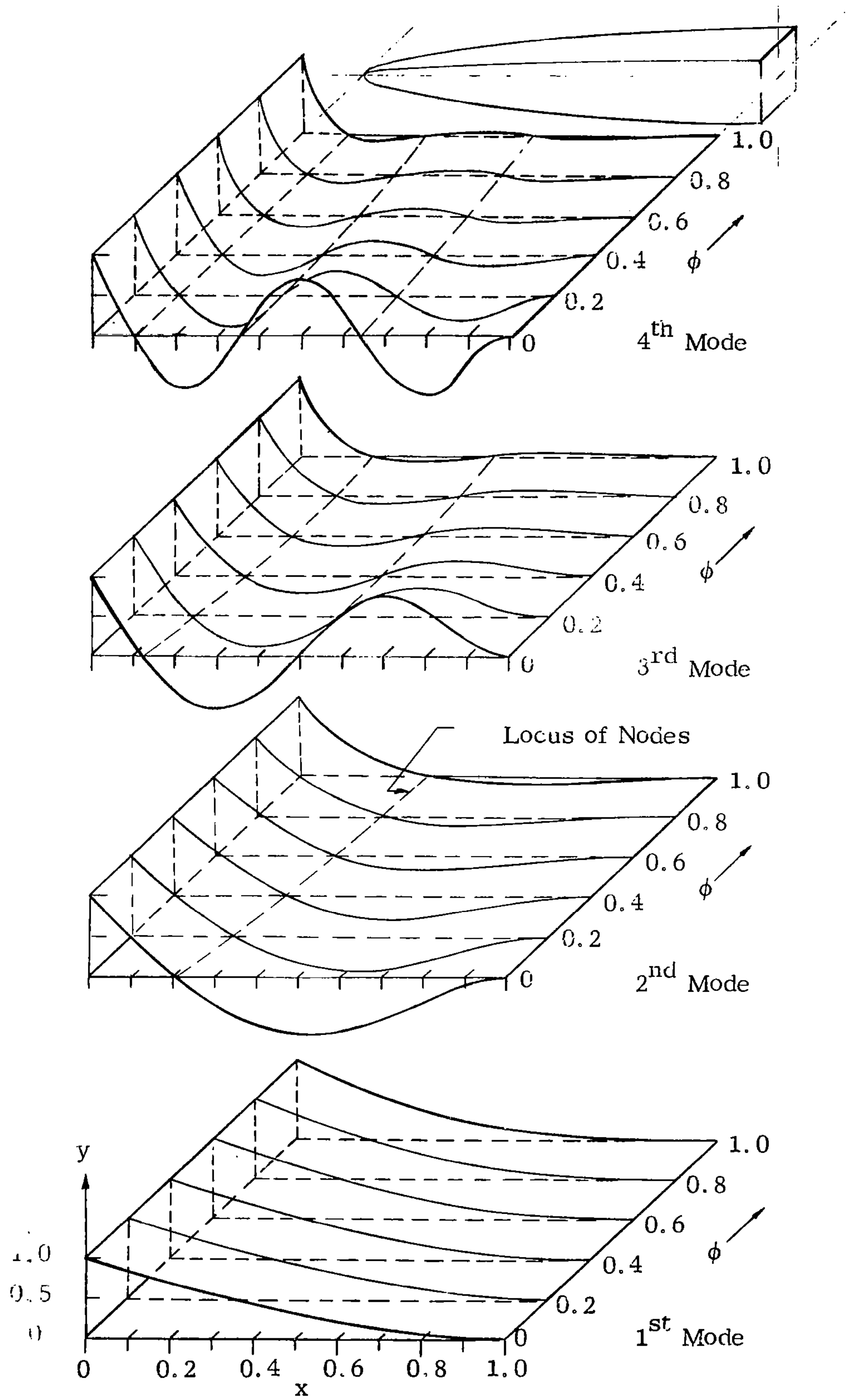

Fig. 9 Mode Shapes for Cantilever Beams with Varying Width and Thickness

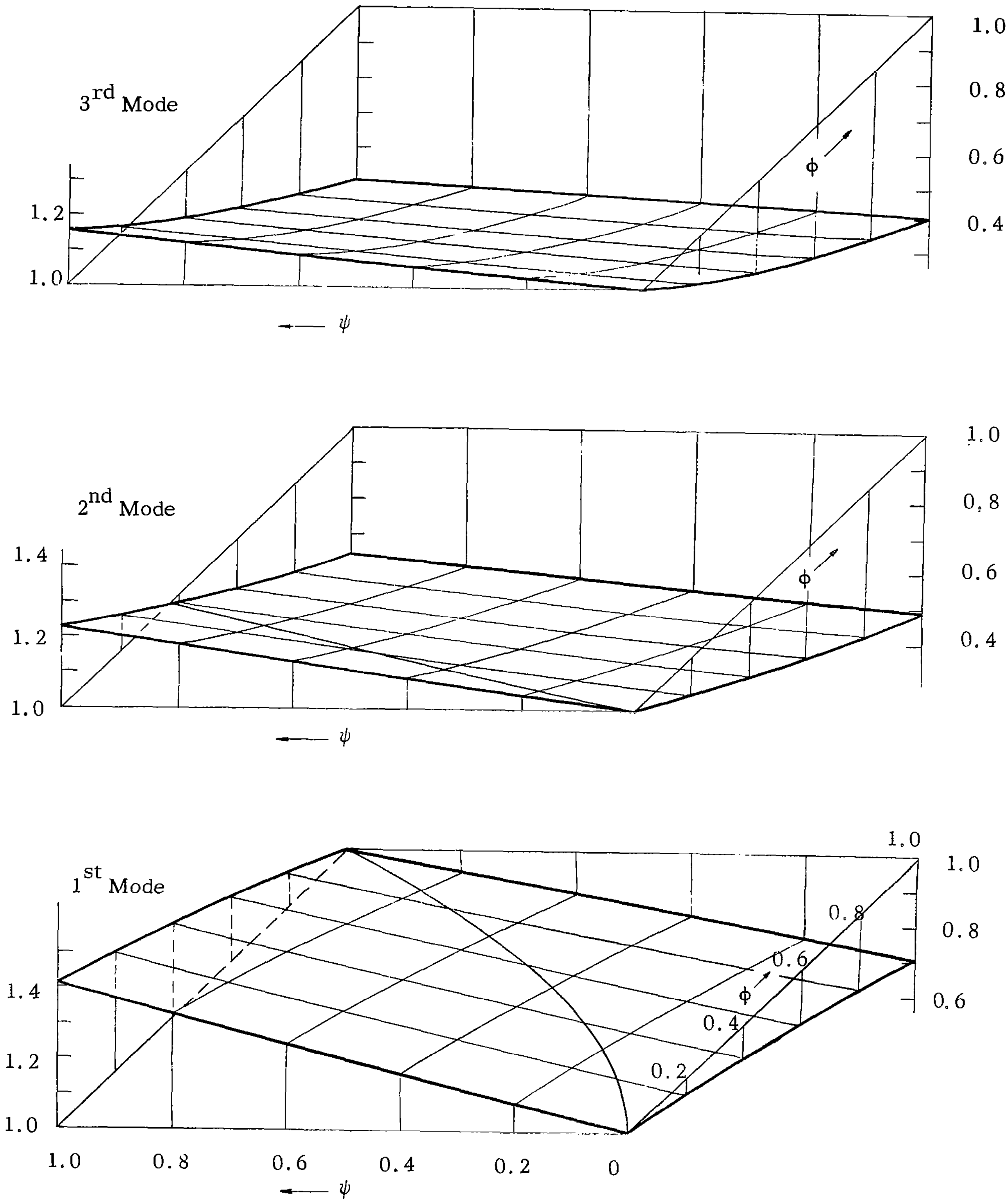

Fig.10 Ratio of Frequencies of Antisymmetrical Mode of Tapered Free-Free Beams to Frequencies of a Uniform Beam

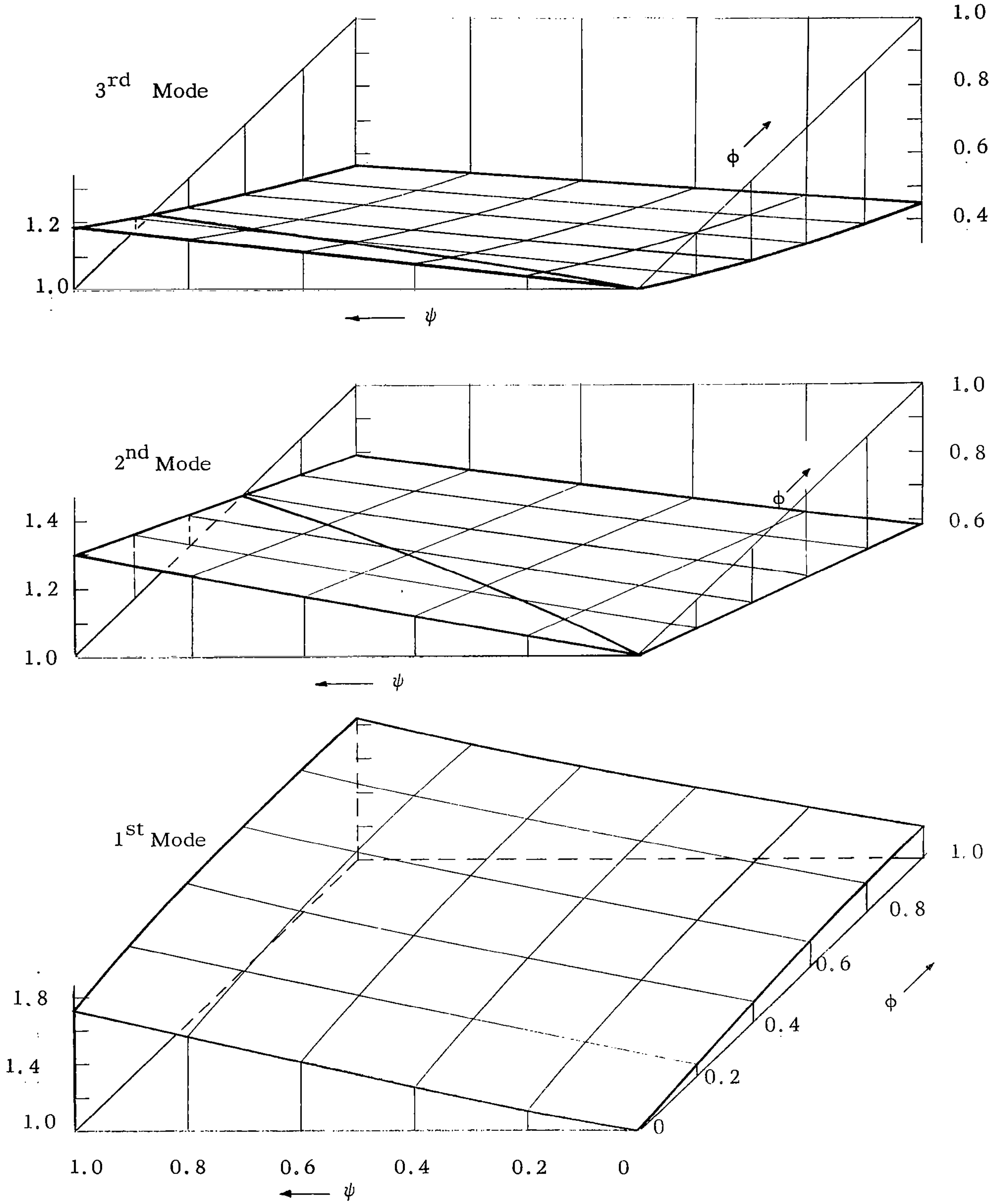

Fig. 11 Ratio of Frequencies of Symmetrical Mode of Tapered Free-Free Beams to Frequencies of a Uniform Beam

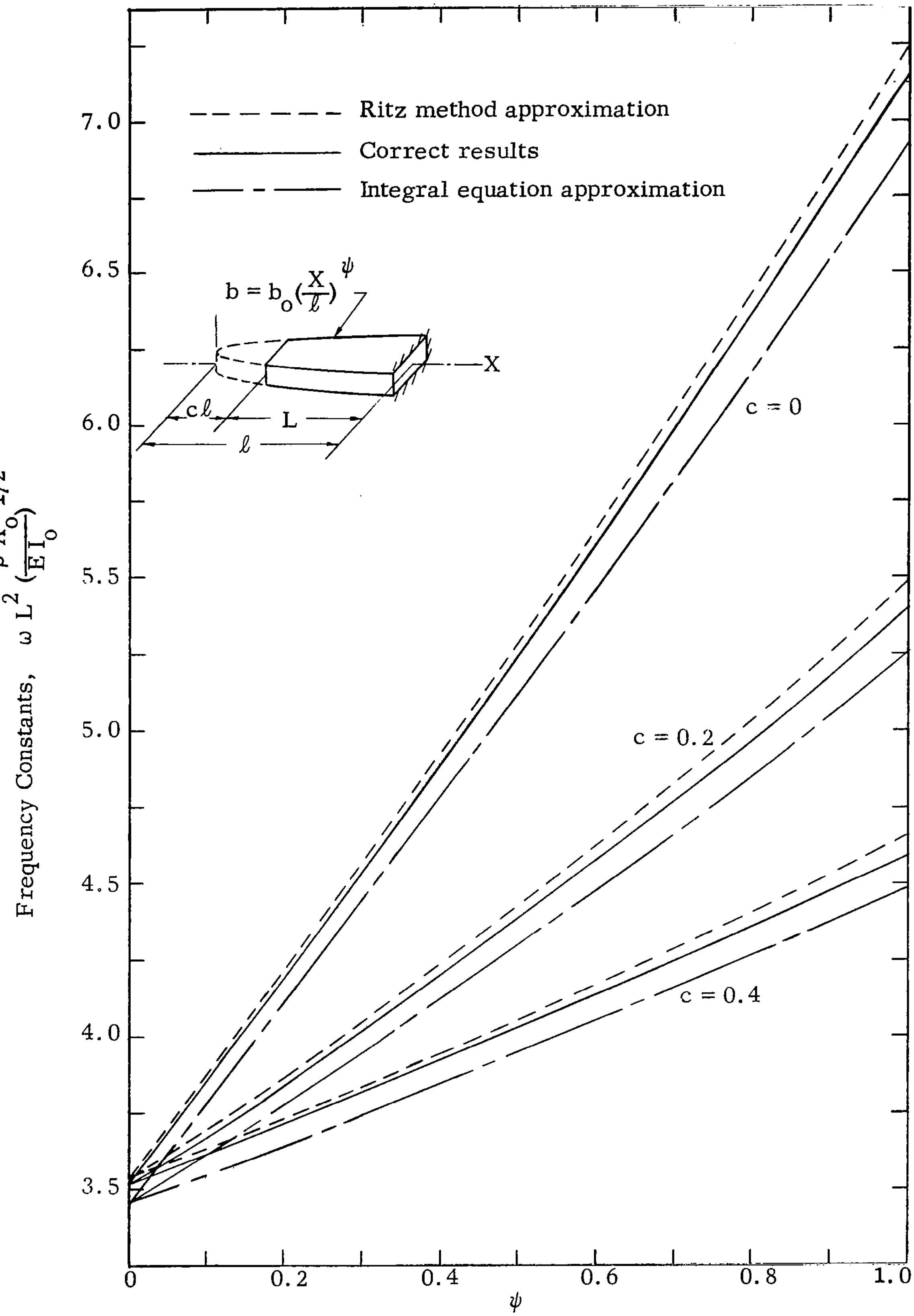

Fig.12 Fundamental Frequencies of Truncated Cantilever Beams with Varying Width

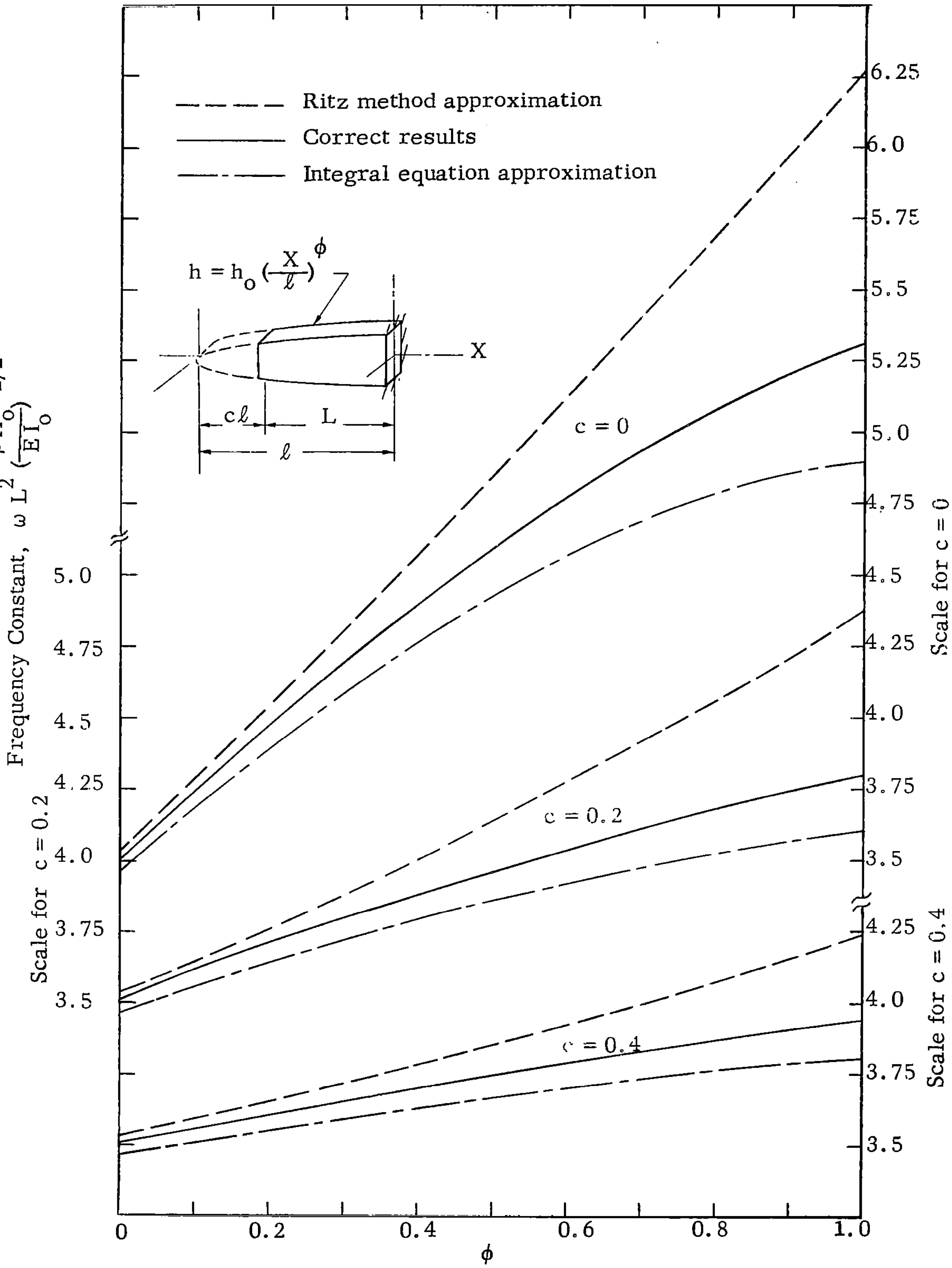

Fig. 13 Fundamental Frequencies of Truncated Cantilever Beams with Varying Thickness

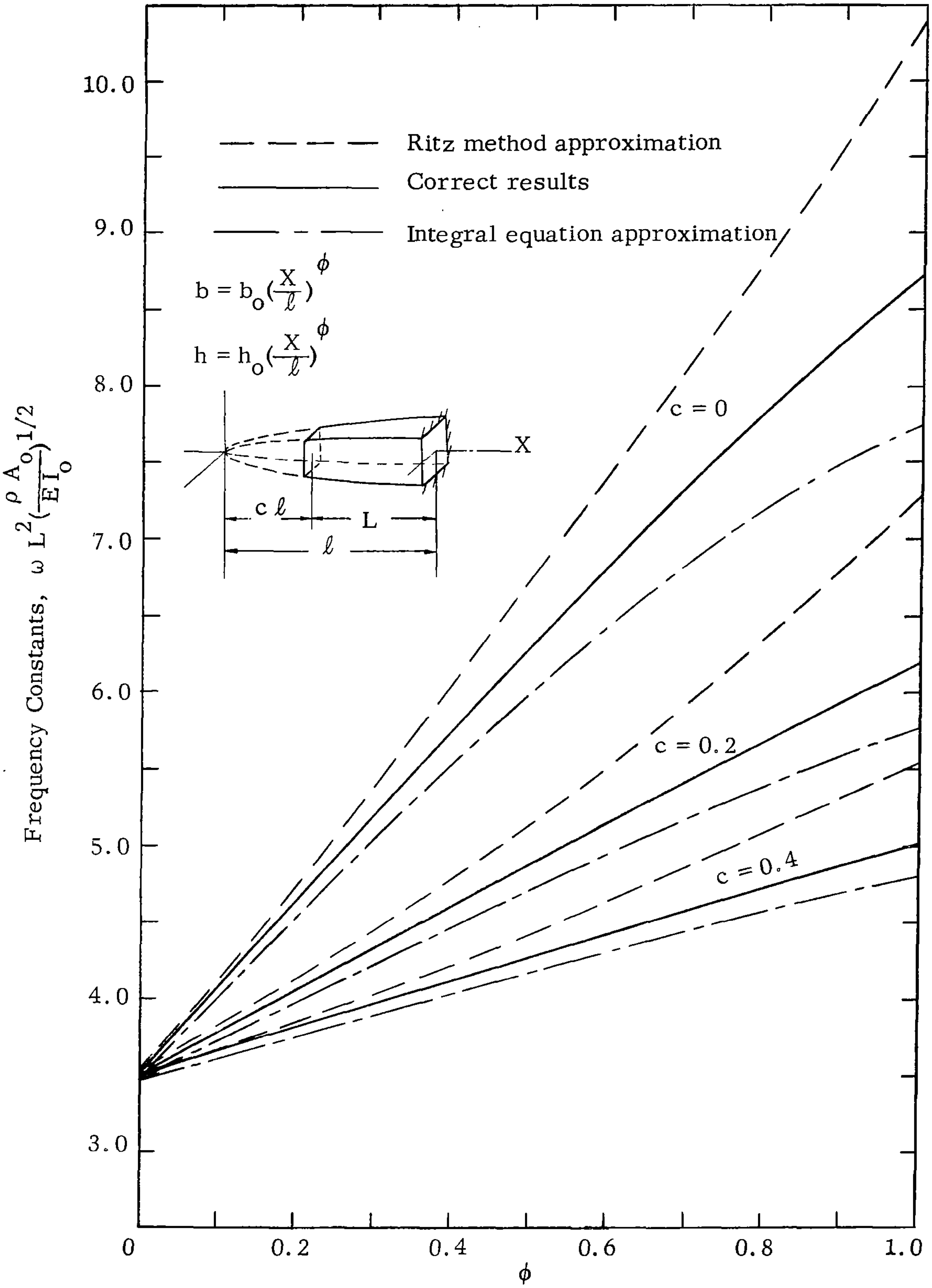

Fig. 14 Fundamental Frequencies of Truncated Cantilever Beams with Varying Width and Thickness

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