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THEORETICAL AND EXPERIMENTAL RESEARCH ON PARAMETER TRACKING SYSTEMS

by Lee Gregor Hofmann, Paul M. Lion, and John J. Best

Prepared under Contract No. NAS 1-4221 *by*
SYSTEMS TECHNOLOGY, INC.
Hawthorne, Calif.
for Langley Research Center

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ABSTRACT

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Parameter tracking systems based upon a generalization of the equation error approach are synthesized for the case wherein the unknown plant is linear, time-invariant and uncorrupted by noise. Only the plant input and output are assumed to be measurable. The order of the unknown plant to which this method is applicable and the number of model parameters to be adjusted are arbitrary.

The report proves that these parameter tracking systems are completely stable and shows that multiple differentiations of the input and output are not required in order to obtain a set of state variables for this class of plant. It is further shown that, for a large class of system inputs, the rate of parameter convergence can always be increased by increasing the gain in the parameter adjustment loops provided that a number of independent generalized equation errors equal in number to the parameters being adjusted are defined in the parameter tracking system.

Weighting filters for a large class of time-varying, linear and nonlinear response error parameter tracking systems are derived, and the connection between response error and equation error systems is shown.

Experimental results illustrating these analytical findings are also included in the report. Author

FOREWORD

This research was sponsored by the National Aeronautics and Space Administration, Langley Research Center under Contract No. NAS1-4221. Mr. Paul Rempfer was the technical monitor for the Research Center.

The studies reported here were conducted at the Princeton, N. J. branch of Systems Technology, Inc. They began 15 July 1964 and were concluded in August 1965. Mr. Dunstan Graham was the Technical Director of the Company under whose general supervision this project was carried out. Project Engineer for the Company was Lee Gregor Hofmann.

The authors gratefully acknowledge the contributions to this effort of Mr. Daniel B. McElwain and Mr. Paul P. Chen of Systems Technology, Inc., and the cooperation of personnel at Electronic Associates, Inc. Princeton Computation Center where the experimental phase of the project was pursued. The typescript was painstakingly composed by Miss Helen Perna.

This report is the final report and concludes the work on Contract No. NAS1-4221.

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SECTION I

INTRODUCTION AND SUMMARY

The quest for a simple, economical means for determining a mathematical model of an unknown plant has led to an identification scheme commonly called parameter tracking. A parameter tracking system is an adaptive servomechanism which operates to null a measure of an error in the system by continuously adjusting the parameters of a model of the unknown plant. If the system performs as intended, nulling of the error will indicate that the parameters of the model match those of the unknown plant.

Two basic types of parameter tracking systems have been previously identified in the literature. These are the equation error based system (Vide References 1-6) and the response error based system (Vide References 7-14). Equation error is traditionally obtained by summing the input and output of the plant and appropriate time derivatives of these variables weighted by estimates of the plant parameters. If (in the absence of noise) these estimates are correct, the plant differential equation will be satisfied and the sum will be zero. If the estimates are incorrect, the sum will be the nonzero quantity called equation error.

Response error is the difference between a particular output of the unknown plant and the corresponding output of a model dynamically similar in structure but with parameters which are estimates of those of the unknown plant. When the estimated parameter values equal those of the plant (in the absence of noise), the difference, the response error, is zero.

Considerable attention has been given these systems in the literature as may be appreciated from our list of selected references. But beyond formal statements of the basic concepts of parameter tracking, analytical advances have been painfully slow. Perhaps the greatest deficiencies among the tools for analyzing parameter tracking systems prior to this program were lack of techniques for determining system stability, and lack of means to improve parameter convergence rates beyond what appeared to be a certain maximum. (Vide References 3 and 9.) This compensation problem appeared to be an increasingly severe one with the number of parameters being adjusted For example, Reference 9 states, "On theoretical grounds the (parameter settling) time would tend to increase with powers of 2^n where n is the number of parameters, considering the geometry of angular sectors in hyper-space." This must be considered as a limitation only in the context of systems using a scalar error quantity, but nonetheless it reflects the state-of-the-art at the time.

The remarks above are not to be construed to say that the stability and compensation problems have been treated without some success, but rather that the practicality of results was limited. For example, stability in the small was investigated in References 7 and 8 for response error systems. The unknown plants were first and second order forced by a step or a sinusoidal input. In Reference 3, Miller analyzed equation error system stability. His results proved that the system was stable but not necessarily asymptotically stable. An attempt to prove asymptotic stability came near success, but in the end was forced to resort to a plausibility argument. References 7 and 8 also addressed the compensation problem. There it was found that under constant coefficient assumptions the parameter adjustment loop dynamics in a response error system consisted of weighting filter dynamics plus an integration in a closed-loop. This analysis pertained only to the step input, one adjustable parameter case where the model could identically match the plant. With this knowledge in hand, it was shown how parameter adjustment loop dynamics could be improved by inserting lead to compensate for weighting filter lag. Hagen, in Reference 12, came to the same conclusion concerning the parameter adjustment loop dynamics while using a less restrictive analysis based upon Zadeh's time-varying system function and the partial-system expansion method. Hagen's approach to compensation was to configure the system in such a way that the weighting filters had approximately unity transmittances. This could be interpreted as including the use of lead compensation as in References 7 and 8, or, alternatively, as the use of different plant variables or model parameters.

OBJECTIVES

In view of the apparent need for more practical analytical tools, the research objectives of this program were to advance the state-of-the-art in parameter tracking theory with respect to:

- Mathematical formulations used for analysis
- Determining system stability
- Compensating system performance

In addition, a fairly extensive experimental program was outlined which would further the emphasis on practicality. This served to:

- Confirm analytical results
- Suggest new analytical approaches
- Explore the practical aspects of the theoretical synthesis.

SUMMARY OF THEORETICAL RESULTS

Initial research was concerned with deriving the weighting filter equations (References 9 and 11) for a broad class of time-varying linear and non-linear models. These weighting filters are necessary elements in the implementation of response error systems. While weighting filters for time-varying models had been developed heuristically in Reference 9, and simultaneously, for linear systems only, by a rather involved method in Reference 12, the new derivation is both more general and complete. It is accomplished by performing a variational expansion of the model equations and decomposing the resulting linear system of equations to reflect the influence of the individual parametric variations.

By employing a general vector form of the response error in the above derivation, the conceptual connection between response error and equation error systems was then demonstrated. Significant in this is the fact that a way to obtain weighting filters with transmittances of unity as suggested by Hagen becomes apparent. This involves choosing the unspecified coefficients in the general form of the response error appropriately. Not too surprisingly, this choice of coefficients results in an equation error system. This demonstrates the connection between response error and equation error systems. The equation error system is a special case of the general response error system. It is not merely understanding the basis for distinctions between these systems which lends significance to this discovery, but rather it is appreciation of the fact that analytical techniques pertaining to equation error systems can be applied in an approximate sense for the analysis of response error systems. Used in this way, the analysis becomes increasingly exact as the weighting filter transmittances approach unity.

Careful sifting of the body of literature pertaining to equation error systems exposed an important concept of generating sets of state variables for constant coefficient linear plants with noise-free signals. Reference 5 shows that it is in fact possible to obtain a complete set of plant state variables from measured plant input and output signals. (Also see Reference 4.) A set of state variables can be obtained, for example, by passing the input and output signals through separate series of constant coefficient filters (such as lags). These filters are called state variable filters. The output of each stage of the filter series is a state variable. Any one set of state variables may then be summed, weighted by the estimates of the plant parameters. This sum is the generalized equation error. When the generalized equation error is zero over time, it can be shown that the estimates of the plant parameters are linearly related to the coefficients of the plant.

It is, however, a well known fact that the state variables describing any one plant are not unique. This particular feature allows linearly independent generalized equation errors to be generated. Considered collectively, the independent generalized equation errors comprise a vector.

The idea of generalized equation error is key to the research results in two ways. First, the disadvantage heretofore associated with equation error systems, i.e. the necessity for determining a complete of state variables by either direct measurement or by successively differentiating the directly measurable signals, is removed, at least for constant coefficient plants with noise-free signals. A set of state variables is seldom directly measurable in practice, and differentiations cannot actually be realized, so that the concept of a generalized equation error which can be generated from the plant input and output using such "nice" operators as lags, assumes considerable practical importance. Secondly, the fact that a generalized equation error vector can be defined permits us to configure a parameter tracking system wherein the convergence rate may be increased without bound by increasing the parameter adjustment loop gain. The fact that generalized equation error is an algebraic function of the model parameters enables a non-negative performance criterion to be found which is also an algebraic function of the model parameters. It is then possible to have the parameters adjust along a steepest descent path in parameter space on the criterion surface. Using Liapunov's second method, it is proven that the generalized equation error system is asymptotically stable in the large about the matching point in parameter space for certain plant inputs. This is true when the input contains a number of sinusoids of different frequency at least equal to one half the number of model parameters. Since a random signal over any finite interval can be regarded as periodic with an infinite number of separate frequencies, this result will also hold for almost all random inputs over an interval which may be made arbitrarily large.

Miller, in Reference 3, showed experimentally the existence of an optimum value of parameter adjustment loop gain in an equation error system. For any higher or lower parameter adjustment loop gain the settling time for the model parameters increased. This result was, however, a direct consequence of the fact that the criterion surface was not positive definite, having closed contours. (In these systems equation error was a scalar quantity.)

Now, the convergence properties of parameter trackers are primarily determined by the shape of the criterion surface in parameter space. This is a concept which has not received sufficient consideration in the literature. Ideally, we would want the criterion surface, at each instant, to be positive definite with closed contours surrounding the matching point, i.e. the point in parameter space where the error is identically zero. To synthesize a system which can be made to converge at an arbitrarily rapid rate simply by turning up the parameter adjustment loop gain, we prove that it is sufficient to be able to generate such a criterion surface by defining a number of linearly independent generalized equation errors equal to the number of parameters to be adjusted. Here, then, equation error is a vector quantity. Furthermore, we show that the parameters must approach their asymptotic values monotonically.

The final analytical result achieved is a proof that a steep descent in one set of parameter coordinates is equivalent to a steepest descent in the same coordinate axes but with different parameter scaling.

ORGANIZATION OF THE REPORT

Analytical aspects of the research program including the connection between equation error and response error systems, the generalization of equation error, proof of asymptotic stability for generalized equation error systems, the improvement of parameter convergence rates, and the transformation of steep descent into steepest descent coordinates are treated in detail in Section II.

In Section III, the generalized equation error system performance is compared with response error system performance, and the analytical results of Section II are confirmed and illustrated with experimental results. Some of the performance limitations of generalized equation error systems are also explored with respect to practical parameter adjustment rates, rates at which time-varying plant parameters may be tracked, and the existence of parameter biases because of noise in the measured plant signals.

The generalized equation error technique was applied to track three parameters of a simulated human pilot in a compensatory tracking task. (Simulated remnant effects were included.) This is an example wherein the signal to noise ratios may tend to be rather small (on the order three in comparison to infinity for which the theoretical results are strictly applicable). Performance of this system was not impressive when a random plant input was used, but a way to surmount the small signal to noise ratio problem using a quasi-random plant input consisting of sums of non-harmonically related sine waves was suggested by the outcome of this experiment.

Potentially promising directions for research in the immediate future are discussed in Section IV along with certain supporting qualitative observations made in the course of this program.

Appendix A summarizes the derivation of weighting filters for a large class of time-varying linear and nonlinear models in response error systems, and provides the mathematical framework for showing the conceptual connection between response error and equation error systems. This has been included as an appendix in order to preserve continuity in the central presentation with respect to basic assumptions such as "constant-coefficient linear plant".

The relationship of equation error to response error is shown in Appendix B. There, it is also shown that the weighting filters for equation error systems have transmittances of unity.

Definitions for the different types of stability of concern in the body of the report and theorem proofs are given in Appendix C.

SECTION II

ANALYSIS

In the first portion of this Section, we develop the connection between response and equation error systems under the assumption that the plant is linear with constant coefficients. This type of plant is the subject for analysis in the following subsections which present a generalization of the equation error type of parameter tracking system, sufficient conditions for its complete stability, and a method for reducing the time required for identification of the plant.

RELATION OF EQUATION ERROR TO RESPONSE ERROR

A comprehensive formulation of response error parameter tracking systems is given in Appendix A. A key factor in rendering this formulation comprehensive is the definition of response error as a vector quantity. Appendix B shows that equation error is a special case of response error. Furthermore, the weighting filter transmittances are unity for this case.

Understanding the connection between response error and equation error is of importance to the analysis of parameter tracking systems because it enables us to apply the analytical techniques that we shall presently develop for equation error systems in the approximate analysis of response error systems. And, it turns out that the accuracy of such an approximation is indicated by the degree to which the weighting filter transmittances of a response error system approach unity.

The development of these points is carried out in considerable detail in Appendices A and B. Such detail, however, is not necessary for the main course of our presentation. Therefore, the relationships between equation error and response error have been specialized here to preserve continuity.

Consider a linear model having a single input and output described by the following differential equation

$$\left(p^n + \sum_{k=0}^{n-1} \alpha_k p^k\right) z(t) = \left(\sum_{j=0}^{m} \beta_j p^j\right) x(t) \qquad (2\text{-}1)$$

where $p \equiv \frac{d}{dt}$ and $m < n$. The model output variable is $z(t)$ and the model and plant input is $x(t)$. The plant output variable will be denoted by $y(t)$. Equation 2-1 can be rewritten in the form of Eqs A-4 and B-2 of the Appendices as

$$\dot{\underline{z}} = \left[\begin{array}{c|ccc} 0 & \diagdown & & \\ & & 1 & \diagdown \\ 0 & & & \\ \hline -\alpha_0 & -\alpha_1 & \cdots & -\alpha_{n-1} \end{array}\right] \underline{z} + \left[\begin{array}{ccc} & 0 & \\ \hline \beta_0 & \cdots & \beta_m \end{array}\right] \underline{x} \qquad (2\text{-}2)$$

where $\underline{z} = \text{col}(z, pz, \ldots p^{n-1} z)$

and $\underline{x} = \text{col}(x, px, \ldots p^m x)$

Response error is defined by Eq A-5 as

$$\underline{e}_r = C(\underline{z} - \underline{y}) + D(\dot{\underline{z}} - \dot{\underline{y}}) \qquad (2\text{-}3)$$

where $\underline{y}$ and $\dot{\underline{y}}$ are plant state variables which correspond to $\underline{z}$ and $\dot{\underline{z}}$. Upon choosing the matrices, C and D, as in Eq B-3

$$C = -\left[\begin{array}{c|ccc} 0 & \diagdown & & \\ \vdots & & 1 & \diagdown \\ 0 & & & \\ \hline -\alpha_0 & -\alpha_1 & \cdots & -\alpha_{n-1} \end{array}\right] \qquad D = \left[\diagdown 1 \diagdown\right] \qquad (2\text{-}4)$$

and using Eq 2-2 to eliminate $\underline{z}$ and $\underline{\dot{z}}$ from Eq 2-3, the response error vector components become:

$$e_{r_i} = y_{i+1} - \dot{y}_i \qquad i = 1, 2, \ldots n-1 \tag{2-5}$$

$$e_{r_n} = \sum_{j=0}^{m} \beta_j x_j - \sum_{k=0}^{n-1} \alpha_k y_k - \dot{y}_n \tag{2-6}$$

Now, by the definition of $\underline{y}$, the e_{r_i} in Eq 2-5 are identically zero and, by the definition of $\underline{x}$ following Eq 2-2, Eq 2-6 may be rewritten:

$$e_{r_n} = \left(\sum_{j=0}^{m} \beta_j p^j\right) x(t) - \left(p^n + \sum_{k=0}^{n-1} \alpha_k p^k\right) y(t) \tag{2-7}$$

Upon comparison of Eq 2-7 with Eq 2-1, it is evident that Eq 2-7 is the expression for the equation error.

MATHEMATICAL FORMULATION OF THE EQUATION ERROR PARAMETER TRACKING PROBLEM

Consider now a plant described by a linear constant coefficient differential equation. In operator form, the relationship between input and output can be written in a form analogous to the equation for the model:

$$\left(p^n + \sum_{k=0}^{n-1} a_k p^k\right) y(t) = \left(\sum_{j=0}^{m} b_j p^j\right) x(t) \qquad (m<n) \tag{2-8}$$

where $p \equiv \frac{d}{dt}$.

Our problem may then be formulated as follows: given only the input to the system, x(t), and the output, y(t), determine the parameter vectors,

$$\begin{aligned} \underline{a} &= \mathrm{col}(a_0, \ldots, a_{n-1}) \\ \underline{b} &= \mathrm{col}(b_0, \ldots, b_m) \end{aligned} \tag{2-9}$$

as rapidly as possible. ($\underline{a}$ and $\underline{b}$ are n and m+1 vectors respectively. Hence we have p = n+m+1 parameters to identify).

GENERALIZED EQUATION ERROR

As a measure of error, the following "generalized equation error" is defined. (It is a generalization of an idea introduced by Rucker in Reference 5.)

$$e_0 = \sum_{k=0}^{n} \alpha_k \, y_k + \sum_{j=0}^{n} \beta_j \, x_j \qquad (2\text{-}10)$$

where $y_k = M_k y$ and $x_j = M_j x$. (All the coefficients in Eq 2-10 will not be needed for a particular case.)

M_k in this equation symbolizes a linear, constant coefficient filter. These filters can have a fairly general form. For example, we may use

$$M_k(s) = \frac{(s + c)^{k+r}}{H(s)} \qquad (2\text{-}11)$$

where r is some fixed positive or negative integer. H(s) is an arbitrary filter chosen by the system designer. Actually, the filters may have a still more general form; e.g., the "free" poles and zeros (those excluding H(s)) may differ for each value of k, or we may use a combination of "free" poles and zeros. It is necessary only that the equation, $e_0 \equiv 0$, imply a linear relationship between the first n derivatives of y and the first m derivatives of x. The values of

$$\underline{\alpha} = \mathrm{col}(\alpha_0, \ldots, \alpha_n)$$

and

$$\underline{\beta} = \mathrm{col}(\beta_0, \ldots, \beta_m) \qquad (2\text{-}12)$$

which make $e_0 \equiv 0$ are denoted by $\underline{\alpha}^*$ and $\underline{\beta}^*$, and will be referred to as the "matching point" in parameter space. The components of $\underline{\alpha}^*$ and $\underline{\beta}^*$ will then be linear combinations of the plant parameters, $\underline{a}$ and $\underline{b}$.

As an example, consider the second order system

$$\ddot{y} + a_1\dot{y} + a_0 y = b_1\dot{x} + b_0 x \tag{2-13}$$

and the operator $M_k(s) = \left(\frac{1}{s+1}\right)^k$. The generalized equation error is then defined as

$$e_0 = \alpha_2\ y_2(t) + \alpha_1\ y_1(t) + \alpha_0 y(t) + \beta_2\ x_2(t) + \beta_1\ x_1(t) \tag{2-14}$$

where we have taken $\beta_0 = 0$. Setting $e_0 \equiv 0$ and $\underline{\alpha} = \underline{\alpha}^*$, $\underline{\beta} = \underline{\beta}^*$, and manipulating Eq 2-14 yields:

$$\left(\alpha_2^* + \alpha_1^*(s+1) + \alpha_0^*(s+1)^2\right)\left(\frac{1}{s+1}\right)^2 Y(s) = -\left(\beta_2^* + \beta_1^*(s+1)\right)\left(\frac{1}{s+1}\right)^2 X(s) \tag{2-15}$$

Comparing Eqs 2-13 and 2-15, we have

$$\alpha_0^* = 1 \qquad\qquad \beta_1^* = -b_1$$

$$\alpha_1^* + 2\alpha_0^* = a_1 \qquad\qquad \beta_1^* + \beta_2^* = -b_0$$

$$\alpha_2^* + \alpha_1^* + \alpha_0^* = a_0$$

Therefore, if we can find a way to drive e_0 to zero, and keep it there, the unknown parameters of the plant ($\underline{a}$, $\underline{b}$) can be calculated from the known values of the model parameters ($\underline{\alpha}^*$, $\underline{\beta}^*$).

Equation 2-15 shows why this method is restricted to constant coefficient systems since the $\left(\frac{1}{s+1}\right)^2$ on each side can be cancelled only when α^* and β^* are constants.

The most convenient form of the M_k from the viewpoint of data reduction is:

$$M_k(s) = \frac{s^k}{H(s)} \tag{2-16}$$

In this case, $\alpha_k^* = a_k$ and $\beta_k^* = b_k$. If one chooses $H(s) = 1$ in Eq 2-16, then the traditional equation error system results. If $H(s) = \frac{1}{s^n}$ in Eq 2-16 the system suggested by Puri and Weygandt in Reference 4 results. In what follows, filters of type described by Eq 2-16 will be used except where noted otherwise; indices will be adjusted accordingly (i.e. we take $\alpha_n = 1$).

PARAMETER ADJUSTMENT LAW

As a criterion surface, consider $F = e_0^2/2$. To follow the gradient of F, the parameter adjustment laws are

$$\begin{aligned} \dot{\alpha}_j &= -k \frac{\partial F}{\partial \alpha_j} = -k\, e_0\, y_j \qquad j = 0, \ldots, n-1 \\ \dot{\beta}_i &= -k \frac{\partial F}{\partial \beta_i} = -k\, e_0\, x_i \qquad i = 0, \ldots, m \end{aligned} \tag{2-17}$$

where k is a constant, the parameter adjustment loop gain.

The values $\underline{\alpha}^*$ and $\underline{\beta}^*$ are defined by

$$0 \equiv \sum_{k=0}^{n} \alpha_k^* y_k + \sum_{k=0}^{m} \beta_k^* x_k \tag{2-18}$$

Subtracting Eq 2-18 from Eq 2-10 requiring that $\alpha_n = \alpha_n^* = 1$, we have

$$e_0 = \sum_{j=0}^{n-1} \Delta\alpha_j\, y_j + \sum_{i=0}^{m} \Delta\beta_i\, x_i \tag{2-19}$$

where

$$\begin{aligned} \Delta\alpha_j &= \alpha_j - \alpha_j^* \\ \Delta\beta_i &= \beta_i - \beta_i^* \end{aligned} \tag{2-20}$$

define the parameter differences. In more compact notation, Eq 2-19 can be written

$$e_0 = \underline{w}_0' \underline{\gamma} \tag{2-21}$$

where $\underline{w}_0 = \text{col}(y_0, \ldots, y_{n-1}, x_0, \ldots, x_m)$
and $\underline{\gamma} = \text{col}(\Delta\alpha_0, \ldots \Delta\alpha_{n-1}, \Delta\beta_0, \ldots, \Delta\beta_m)$ are both p-vectors.

If we substitute Eq 2-19 into Eq 2-17, there results a linear, homogeneous set of p differential equations in terms of parameter differences:

$$\begin{aligned} \dot{\Delta\alpha}_j &= -k\left(\sum_{i=0}^{n-1} y_j\, y_i\, \Delta\alpha_i + \sum_{i=0}^{m} y_j\, x_i\, \Delta\beta_i\right) \\ \dot{\Delta\beta}_j &= -k\left(\sum_{i=0}^{n-1} x_j\, y_i\, \Delta\alpha_i + \sum_{i=0}^{m} x_j\, x_i\, \Delta\beta_i\right) \end{aligned} \tag{2-22}$$

or, in matrix notation

$$\dot{\underline{\gamma}} = -kA(t)\, \underline{\gamma} \tag{2-23}$$

where $A(t) = \underline{w}_0\underline{w}_0'$, is a (p x p) time-varying matrix. These equations show the means for adjusting the parameters.

PROOF OF STABILITY

The most critical question to be answered concerning the system given by Eq 2-23 is whether the estimated parameter values ($\underline{\alpha}$, $\underline{\beta}$) converge to their proper values ($\underline{\alpha}^*$, $\underline{\beta}^*$). This property can be established by proving the asymptotic stability of Eq 2-23 about the origin of the parameter space ($\underline{\gamma}$-space).

Rigorous definitions of the various types of stability plus theorems are summarized in Appendix C. Two points should be noted:

1) To prove Eq 2-23 is asymptotically stable, it must first be shown that it is stable.

2) Since the system is linear, any stability proofs will hold globally.

Stability can be shown by finding a Liapunov function $V(\gamma(t))$ which is positive definite and whose total time derivative is negative semi-definite. A satisfactory Liapunov function is

$$V = \frac{1}{2}\left(\sum_{j=0}^{n-1} \Delta\alpha_j^2 + \sum_{j=0}^{m} \Delta\beta_j^2\right) = \frac{1}{2}\,(\underline{\gamma}'\ \underline{\gamma}) \tag{2-24}$$

The time derivative of Eq 2-24 is

$$\dot{V} = \underline{\gamma}'\ \dot{\underline{\gamma}} = -\,k\ \underline{\gamma}'\ A\ \underline{\gamma} \tag{2-25}$$

using Eq 2-23. Since $A = \underline{w}_0\underline{w}_0'$, we have

$$\dot{V} = -\,k\ (\underline{\gamma}'\ \underline{w}_0)\ (\underline{w}_0'\ \underline{\gamma}) = -\,k\ e_0^2 \tag{2-26}$$

where Eq 2-21 has been used.

Thus the derivative $\dot{V}$ is clearly negative semi-definite, proving the stability of Eq 2-23. It is not, however, negative definite, since it is possible to have $e_0 = 0$ when $\underline{\gamma} \neq 0$. From Eq 2-19 it can be seen that, at any instant of time, the set of points in parameter space ($\underline{\gamma}$-space) for which $e_0 = 0$ is an (n+m) dimensional hyperplane. However, it is possible to prove asymptotic stability for certain types of system input, x, since this hyperplane is not fixed in parameter space but varies with time. This is a result of the fact that, for these inputs, $e_0(t)$ is not zero over an interval unless $\underline{\gamma} = 0$.

Asymptotic stability can be proven using the following theorem:

Theorem 1 Given: A linear differential system whose coefficients are continuous functions of t. If one can find a Liapunov function, $V(\underline{\gamma})$, differentiable with respect to $\underline{\gamma}$, such that:

a. $V(\underline{\gamma}) > \xi\ \|\underline{\gamma}\|$

b. $V(0) = 0$

c. $\dot{V} \leq 0$ for all $\underline{\gamma}$, t

d. the origin is the only solution along which $\lim_{t\to\infty} \dot{V} = 0$ then the origin is completely stable.

This theorem is based on an extension of a theorem proven by LaSalle for autonomous systems (Reference 15). Further details are given in Appendix C. It is important to note that V is not explicitly dependent upon t and, hence, its contours are time invariant. The point of the theorem can be made clear by noting that V is a non-increasing function of time which is bounded from below and therefore must approach a limit. The origin is the only set on which $\dot{V}$ approaches zero. Hence V must approach the origin as a limit.

The Liapunov function given by Eq 2-24 satisfies all conditions given by Theorem 1 except possibly d. If $\dot{V} \longrightarrow 0$ as $t \longrightarrow \infty$, then, by Eq 2-26, we also have $e_0 \longrightarrow 0$. For periodic inputs, this can result only if there exist solutions such that $e_0 \equiv 0$ over some interval. Consider Eq 2-21. This is a single algebraic equation with p unknowns $(\Delta\alpha_k, \Delta\beta_k)$. If the "coefficients" (y_k, x_k) are linear combinations of more than p linearly independent functions, then the only solution which makes $e_0 \equiv 0$ is the trivial solution, $\underline{\gamma} = 0$. This will be the case if x contains, say more than p/2 sinusoids of different frequencies (none of which have phase shifts of $k\pi$ radians through the plant). For the case where the input contains exactly p/2 sinusoids (none of which have phase shifts of $k\pi$ radians through the plant), it also turns out that $e_0 \equiv 0$ implies $\underline{\gamma} = 0$.

Hence the following result has been proved.

> If the input is a real, periodic signal with p/2 or more separate frequencies, none of which have phase shifts of $k\pi$ radians through the plant, then the parameter tracking system described by Eqs 2-17 and 2-19 is completely stable.

That is, the system will identify the correct parameters regardless of the accuracy of the initial guess.

Consider now the case of a random input, x. Since a random signal over any finite interval can be regarded as a periodic function with an infinite number of separate frequencies, we may argue that the result will also hold for almost all random inputs over that interval (which can be made arbitrarily large).

A more rigorous result is possible, however, using the concept of stability in the mean. Again, precise definitions are presented in Appendix C.

When x is a random function, the elements of A(t) in Eq 2-23 are continuous random functions of t. Simply put, stability in the mean implies that the expected value of a positive definite Liapunov function approaches zero. This ensures that we will have convergence for all random inputs, x, except for a set of measure zero.

It is then possible to prove the stability in this case also, using an analog of the Theorem 1 for the case of systems with random coefficients. The proof itself is outlined in Appendix C.

Theorem 2 Given a Liapunov function, differentiable with respect to γ, such that

a. $V(\underline{\gamma}) > 0$ for all $\underline{\gamma} \neq 0$

b. $V(0) = 0$

c. $E(\dot{V}) \leq 0$

d. the origin is the only set such that $\lim E(\dot{V}) \longrightarrow 0$ as $t \longrightarrow \infty$, then the origin is completely stable in the mean.

In Theorem 2, E( ) indicates the expectation operator, where the average is taken over the ensemble of inputs at any time t.

Again, using the Liapunov function $V = \underline{\gamma}' \, \underline{\gamma}$, we can show that $\dot{V} \leq - e_0^2$. Clearly then, $E(\dot{V}) \leq 0$. Also, for almost all inputs, $\dot{V} \longrightarrow 0$ as $t \longrightarrow \infty$ only if $\| \gamma \| \longrightarrow 0$. Therefore the final condition of the Theorem holds. We may conclude that, for random inputs, the parameter tracking system defined by Eqs 2-17 and 2-19 is completely stable in the mean.

SYNTHESIS OF SYSTEMS WITH RAPID CONVERGENCE RATE

One property of the system described above is that one cannot increase the rate of convergence simply by turning up the gain, k. In Reference 3, Miller showed experimentally the existence of an optimum value of k for equation error systems. For any higher or lower gain, the time required for the parameter values to converge increased. This is a result of the fact that the surface generated by $F = e_0^2/2$ is not positive definite with closed constant F contours. It is, in fact, positive semi-definite and the contours are (n+m) dimensional hyperplanes. It is only because the gain, k, cannot be made infinite and, hence, the parameter vector cannot actually keep up with the $e_0 \equiv 0$ hyperplane that the system ultimately converges.

It would be desirable, then, to be able to synthesize a system which can be made to converge at an arbitrarily rapid rate, simply by turning up the gain, k. To synthesize such a system where p parameters are to be adjusted, it is sufficient to be able to generate p generalized equation errors which are linearly independent at each instant of time; that is, for the error vector defined by $\underline{e} = \text{col}(e_0, \ldots, e_{n+m})$, the following condition must hold:

Condition 1: At each instant, t, e = 0 if and only if $\underline{\gamma} = 0$.
In other words, all components of the error vector vanish only when the parameter vector is at the matching point.

Previously, the generalized equation error was defined by Eq 2-10:

$$e_0 = y_n + \sum_{k=0}^{n-1} \alpha_k y_k + \sum_{k=0}^{m} \beta_k x_k \tag{2-27}$$

This was also written as:

$$e_0 = \underline{w}_0' \underline{\gamma} \tag{2-28}$$

where $\underline{w}_0$ and $\underline{\gamma}$ were defined following Eq 2-21. Additional equation errors are defined in the following manner:

$$e_i = y_{n+i} + \sum_{k=0}^{n-1} \alpha_k y_{k+i} + \sum_{k=0}^{m} \beta_k x_{k+i} \tag{2-29}$$

In similar manner as before, this can be written as

$$e_i = \underline{w}_i' \underline{\gamma} \tag{2-30}$$

where $\underline{w}_i = \text{col}(y_i, \ldots, y_{n-1+i}, x_i, \ldots, x_{m+i})$. (2-31)

The relation between the error vector and the parameter difference vector can be written

$$\underline{e} = W(t)\ \underline{\gamma} \tag{2-32}$$

where W is a (p x p) matrix whose i^{th} row is $\underline{w}_{i-1}$.

Condition 1 will hold if and only if det $W(t) \neq 0$ for all t. This will be true only for certain inputs, x. For instance, if the input contains exactly p/2 frequencies, none of which have phase shifts of kπ radians through the plant, it can be shown that Condition 1 holds for all t. For periodic inputs with more than p/2 frequencies, Condition 1 will hold except at isolated instants. As will be seen, this does not effect the subsequent conclusions significantly.

Let us assume now that Condition 1 holds except at isolated instants. Then the surface, $F = \underline{e}'\underline{e}/2$, is positive definite with closed contours, except at those instants.

The parameter adjustment law for steepest descent on F is

$$\begin{aligned} \dot{\underline{\gamma}} &= -\ k\ \text{grad}\ F \\ &= -\ k\ \frac{\partial \underline{e}'}{\partial \underline{\gamma}}\ \underline{e} \end{aligned} \tag{2-33}$$

From Eq 2-32, $\dfrac{\partial \underline{e}'}{\partial \underline{\gamma}} = W'$, and therefore Eq 2-33 can be written

$$\dot{\underline{\gamma}} = -\ k\ A\ \underline{\gamma} \tag{2-34}$$

where

$$A = W'W. \tag{2-35}$$

We now show that Eq 2-34 is asymptotically stable. Use the Liapunov function

$$V = \frac{1}{2}\,\underline{\gamma}'\ \underline{\gamma} \tag{2-36}$$

Taking the derivative, we have

$$\dot{V} = - k \underline{\gamma}' A \underline{\gamma} = - k \underline{\gamma}' W' W \underline{\gamma} \tag{2-37}$$

$$\dot{V} = - k \underline{e}'\underline{e} \tag{2-38}$$

According to our assumption of Condition 1, $\dot{V}$ will then be negative except for isolated instants when it vanishes. Hence, by Theorem 1, we conclude the complete stability of Eq 2-34.

More than this can be said, however. Note that since A has the form W'W, a sufficient condition for A to be positive definite is that det $W \neq 0$. This, of course, is true if and only if Condition 1 holds.

Assume now that the input is such that Condition 1 is satisfied for all t. In this case, if $\lambda(t)$ is the smallest eigenvalue of $A(t)$, then $\lambda(t)$ is real and greater than zero for all t. For inputs such that Condition 1 fails to hold at certain instants, then $\lambda(t)$ vanishes at these points and is greater than zero everywhere else. $\lambda(t)$ is, in effect, a measure of the linear independence of the generalized equation errors.

To determine the convergence properties, the following theorem of Wintner will be useful. (Vide Reference 16).

Theorem 3 Let $L(t)$ be a square matrix depending continuously on t for all t. Let $H(t)$ be the Hermitian matrix whose elements are

$$h_{ij}(t) = \frac{1}{2}(l_{ij}(t) + \bar{l}_{ji}(t))$$

Then for every solution $\underline{\gamma}(t)$ of the differential equation, $\dot{\underline{\gamma}} = L \underline{\gamma}$

$$\| \underline{\gamma}(0) \| \exp \int_0^t m(\tau)\, d\tau \leq \| \underline{\gamma}(t) \| \leq \| \underline{\gamma}(0) \| \exp \int_0^t M(\tau)\, d\tau$$

where $m(t)$ is the minimum eigenvalue of $H(t)$ and $M(t)$ is the maximum eigenvalue of $H(t)$.

In the case under consideration here, in Eq 2-34, $L(t) = - k A(t)$. Since $- k A(t)$ is real and symmetric, it is equal to $H(t)$. If $\lambda(t)$ is the minimum eigenvalue of $A(t)$, then $- k \lambda(t)$ is the maximum eigenvalue of $- k A(t)$. Therefore, the norm of any solution of Eq 2-34 is bounded by:

$$\| \underline{\gamma}(t) \| \leq \| \underline{\gamma}(0) \| \, e^{-k \int_0^t \lambda(\tau) \, d\tau} \qquad (2\text{-}39)$$

Since $\lambda(t)$ is never negative, the function

$$q(t) = \int_0^t \lambda(\tau) \, d\tau \qquad (2\text{-}40)$$

is non-decreasing. Since q is non-decreasing, isolated points at which $\lambda = 0$ are of no significance. What is necessary is that $\lambda(t)$ be $\neq 0$ except for isolated instants, and that it not decay "too rapidly," so that $q \longrightarrow \infty$ as $t \longrightarrow \infty$. Then, the rate of convergence of the system given by Eq 2-34 can be increased by increasing the gain k.

This result cannot be shown to be the case for the system with only one equation error (Eq 2-23). With only one equation error the maximum eigenvalue is zero for all t. The maximum eigenvalue for any system with p adjustable parameters and less than p independent equation errors is also zero. In such cases, the determinant of the matrix, is identically zero and, therefore, $\lambda(t)$ will be identically zero.

Note that $\lambda(t)$ is a function of the input x and the plant parameters ($\underline{a}$, $\underline{b}$). If x is a periodic function with sufficiently many frequencies (p/2 or more) then we are assured that $q(t)$ increases without bound as $t \longrightarrow \infty$. This will also be true if x is a stationary random function.

STEEP DESCENT, STEEPEST DESCENT AND CRITERION FUNCTIONS

Because the power present in various elements of the matrix, A, in Eq 2-34 may be greatly different, it often turns out to be advantageous to make the parameter adjustment loop gain different for each component of $\dot{\underline{\gamma}}$. This is in order to obtain parameter responses which are more satisfactory for particular sets of initial parameter values. Use of different parameter adjustment loop gains for the components of $\dot{\underline{\gamma}}$ raises the question, "With respect to what criterion function is a steepest descent parameter adjustment being made?". Heretofore, we have adjusted $\underline{\gamma}$ so that it would move along a steepest descent path on the criterion surface F; i.e. the parameter adjustment law was given by

$$\dot{\underline{\gamma}} = - k \nabla_{\gamma} F \tag{2-41}$$

where $\nabla_{\gamma} = \left(\frac{\partial}{\partial \gamma_1}, \ldots, \frac{\partial}{\partial \gamma_p}\right)$. Using a different gain for each component of $\underline{\gamma}$ gives a parameter adjustment law

$$\dot{\underline{\gamma}} = - K \nabla_{\gamma} F \tag{2-42}$$

where K = diag (k_0, ..., k_p) is a (p x p) matrix with positive elements.

The control law, Eq 2-42, clearly implies that in $\underline{\gamma}$-space, the adjustment is one of "steep descent," in distinction to "steepest descent."

Consider now, however, a transformation from γ coordinates to ζ coordinates; i.e.

$$\underline{\zeta} = R \, \underline{\gamma} \tag{2-43}$$

where R is a (p x p) non-singular, <u>diagonal</u> matrix.

The gradient operator (a covariant vector) transforms according to the law (Reference 17):

$$\nabla_{\zeta} F = (R')^{-1} \nabla_{\gamma} F = R^{-1} \nabla_{\gamma} F \tag{2-44}$$

Substituting Eqs 2-43 and 2-44 into Eq 2-42 yields

$$\dot{\underline{\zeta}} = - (RKR) \nabla_{\zeta} F \tag{2-45}$$

Then, if we choose $R = K^{-1/2}$ (i.e. $r_{ii} = \sqrt{\frac{1}{k_i}}$, $i = 0, \ldots, p - 1$), we have

$$\dot{\underline{\zeta}} = - \nabla_{\zeta} F \tag{2-46}$$

Therefore, a steep descent adjustment in γ coordinates according to Eq 2-42 corresponds to a steepest descent adjustment in ζ coordinates where $\underline{\zeta}$ is related to $\underline{\gamma}$ by the transformation

$$\underline{\zeta} = K^{-1/2} \underline{\gamma} \tag{2-47}$$

Since Eq 2-46 has the same form as Eq 2-41, it is clear that the previously proved conclusions as to stability and rate of convergence still apply using control law given by Eq 2-46.

The discussion, up to this point, has been based upon the criterion function, $F = \underline{e}'\underline{e}/2$. It is possible to prove all previous results using a slightly more general criterion

$$F = \frac{1}{2}(\underline{e}' \; Q \; \underline{e}) \tag{2-48}$$

where $Q = \text{diag}\,(q_0^2, \ldots, q_p^2)$. Since Q is symmetric and positive definite, the arguments used previously are not affected. The error criterion defined by Eq 2-48 weighs each generalized equation error differently.

SECTION III

EMPIRICAL STUDY OF PARAMETER TRACKING SYSTEM PERFORMANCE

An experimental program was undertaken to provide a better understanding of, and a greater appreciation for parameter tracking system performance. The objectives and results of that program are presented in this Section.

Objectives of the experimental program were

- To compare alternative designs for parameter tracking systems.
- To unify the analytical results obtained for certain systems in Section II.
- To investigate the effect of measurement noise on parameter tracking performance.
- To study the case where plant parameters are time-varying.
- To apply the knowledge gained to the design of a simulated pilot parameter tracking system.

For present purposes, it is convenient to consider each objective in turn in separate subsections. The particular procedures used to achieve each objective are described in the appropriate subsection.

COMPARISON OF RESPONSE ERROR AND EQUATION ERROR SYSTEMS

The most promising approaches to parameter tracking have been called the response error method and the equation error method. These descriptions actually refer to the method for defining error between desired signals and actual signals. Parameter tracking systems are also further characterized by the criterion function and parameter adjustment laws used. The criterion function and parameter adjustment laws are elements common to equation error and response error parameter tracking systems.

The two definitions of error are compared in Figure 3-1. There, it is important to note that Y_m represents the model in operational notation with coefficients which are the elements of the matrices, N_m and D_m. Y_m

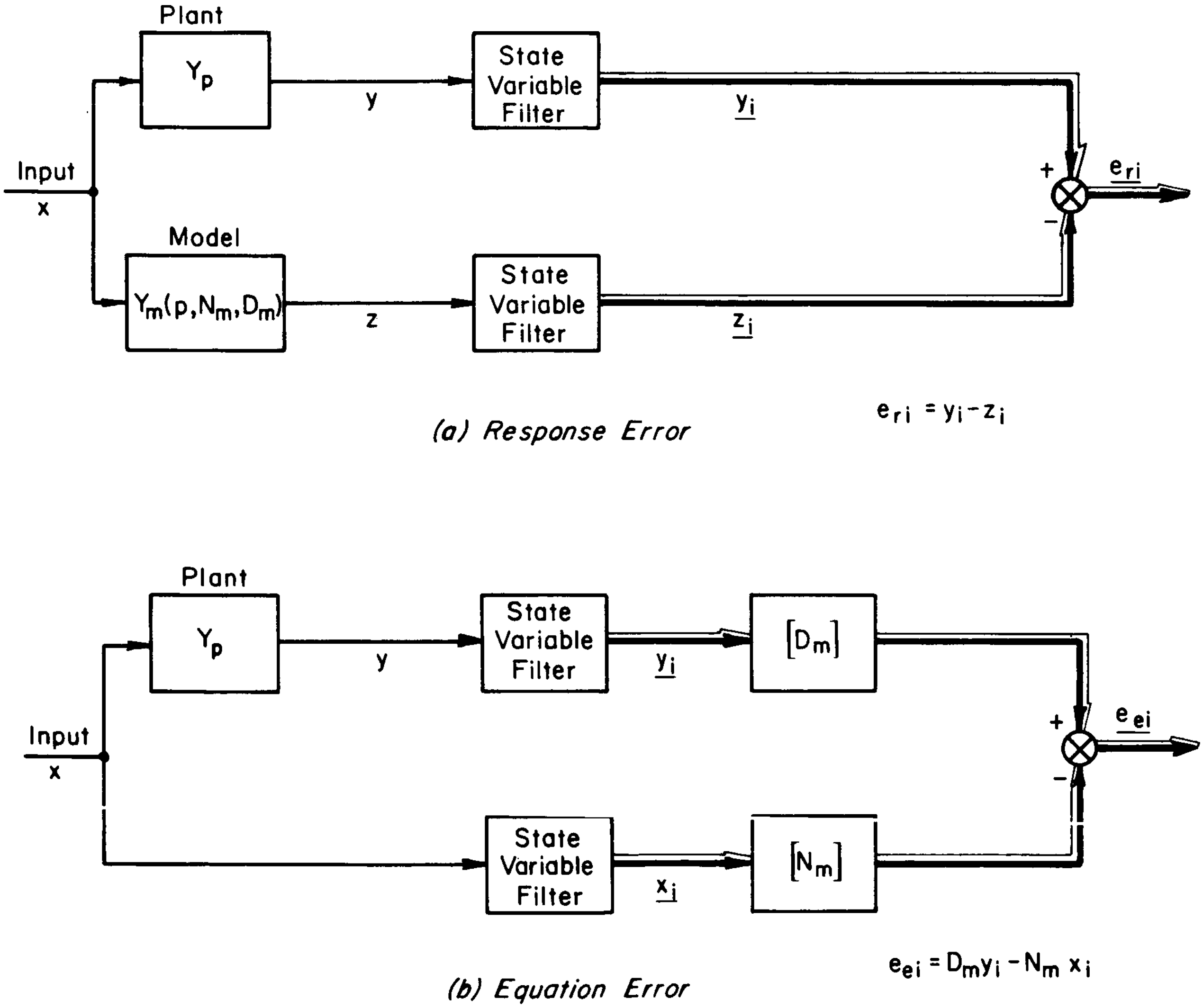


Figure 3-1. Formulation of Error Signals

is not a transfer function. The similarity between the response error and equation error schemes has been deliberately emphasized in Figure 3-1.

Through the use of state variable filters a single signal is transformed into a group of signals (vector). This provides a set of independent errors (error vector) in both formulations for a broad class of inputs. Rucker (Reference 5) first conceived of using state variable filters to overcome the problems associated with successive differentiation.

Two quadratic criterion functions are considered. They are

$$F_1 = \frac{1}{2}\left[e(\ )_0 + \lambda e(\ )_1\right]^2 \qquad (3\text{-}1)$$

and

$$F_2 = \frac{1}{2}\left[e(\ )_0^2 + \lambda e(\ )_1^2\right] \qquad (3\text{-}2)$$

where the $e(\ )_0$ and $e(\ )_1$ are the components of the error vector. F_1 includes the criterion functions used in References 7-9. Figure 3-2 shows the parameter adjustment laws which result in a steep descent on the criterion surfaces, F_1 and F_2, when these functions are plotted as functions of the model parameters. Notice that the required number of multipliers and summing amplifiers is different in each case. The table below summarizes equipment requirements.

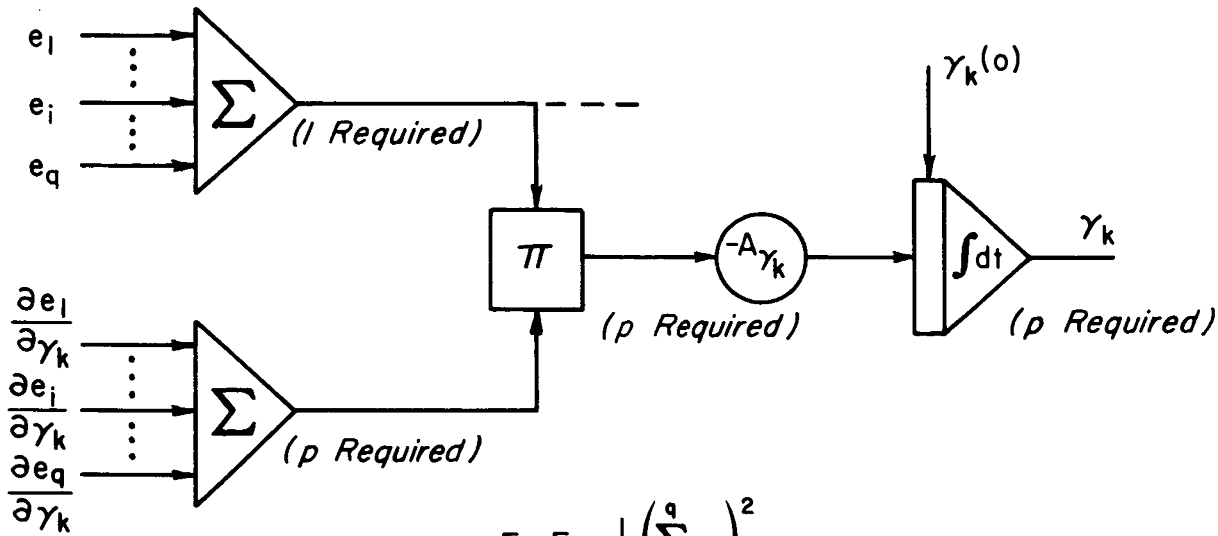
TABLE I

EQUIPMENT REQUIREMENTS FOR ADJUSTMENT OF p PARAMETERS WITH q COMPONENTS OF ERROR

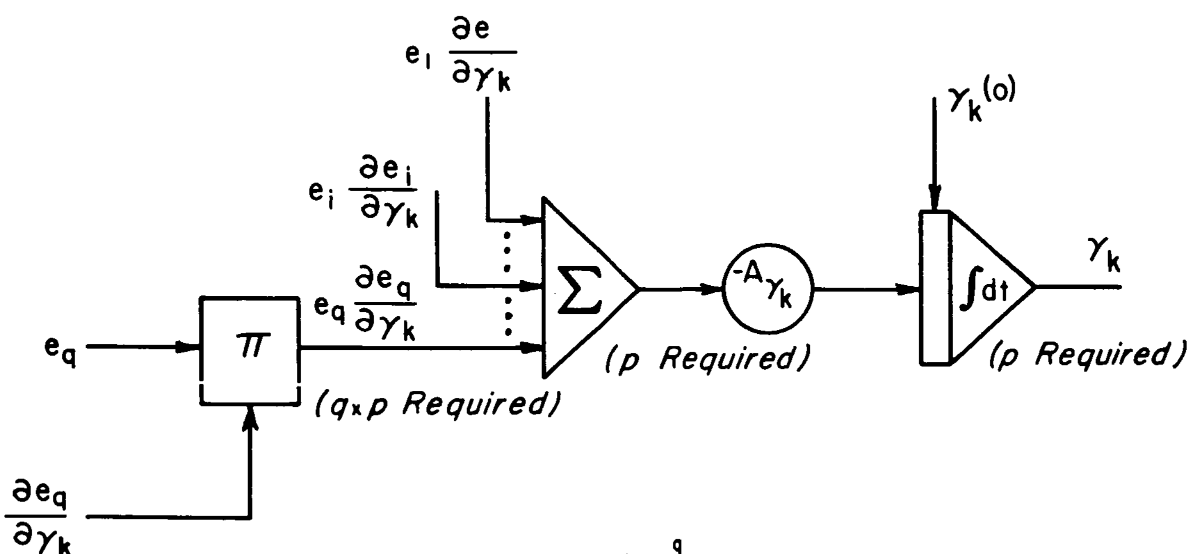
CRITERION	MULTIPLIERS	SUMMERS	INTEGRATORS
F_1	p	p+1	p
F_2	pxq	p	p

Equation defining path of steep descent:

$$\dot{\gamma}_k = -A_{\gamma_k} \frac{\partial F}{\partial \gamma_k}$$



$$F = F_1 = \frac{1}{2}\left(\sum_i^q e_i\right)^2$$



$$F = F_2 = \frac{1}{2} \sum_i^q (e_i)^2$$

Figure 3-2. Parameter Adjustment Law for the Parameter γ_k Using the Criteria F_1 and F_2

Here q is the order of the error vector and p is the number of model parameters being adjusted. The above table shows that q times as many multipliers are required to mechanize the F_2 criterion function than to mechanize the F_1 criterion function. These equipment requirements hold independently of whether response error or equation error is used. In the comparison experiments, p = 2 and q = 1,2.

The general experimental set-up used to perform the comparison experiments is shown in Figure 3-3. The plant considered is second order with a single zero,

$$Y_p = \frac{Y(s)}{X(s)} = \frac{b_1 s + b_0}{s^2 + a_1 s + a_0} = \frac{s+1}{s^2+2(0.5)2s+(2)^2} \qquad (3\text{-}3)$$

two parameters, b_1 and a_0, are treated as unknown but the order of the system is assumed known. Thus the differential equation in operational form representing the model is:

$$(p^2 + \alpha_1 p + \alpha_0)\ z = (\beta_1 p + \beta_0)\ x \qquad (3\text{-}4)$$

where

$$p \equiv \frac{d}{dt}$$

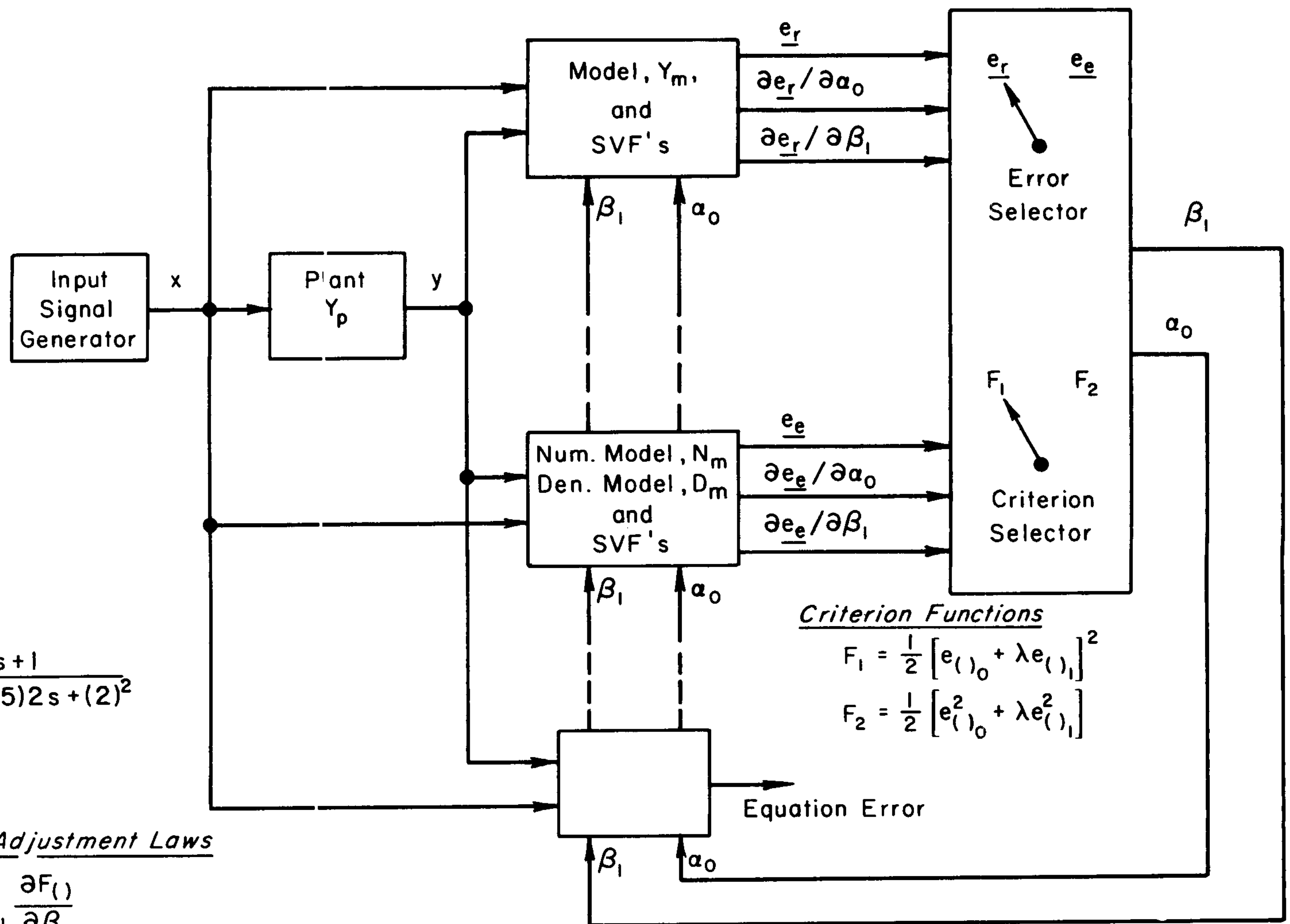
$\beta_0 = b_0 = 1.0$

β_1 is "unknown"

α_0 is "unknown"

$\alpha_1 = a_1 = 2.0$

Provision is made to use either the response error vector, $\underline{e}_r$, or the equation error vector, $\underline{e}_e$, in either the F_1 or F_2 criterion function. The possible combinations available enable us to investigate four different, but comparable, mechanizations of a parameter tracking system. All com-



$$Y_p = \frac{s+1}{s^2 + 2(0.5)2s + (2)^2}$$

Parameter Adjustment Laws

$$\dot{\beta}_1 = -A_{\beta_1} \frac{\partial F_{(\,)}}{\partial \beta_1}$$

$$\dot{\alpha}_0 = -A_{\alpha_0} \frac{\partial F_{(\,)}}{\partial \alpha_0}$$

SVF State Variable Filters

Figure 3-3. Experimental Setup for Comparison Between Designs binations are of interest to us here. The analysis of Section II pertains to the $\underline{e}_e$ - F_2 combination and is an approximation for the $\underline{e}_r$ - F_2 combination. The $\underline{e}_r$ - F_1 combination was investigated in References 7-9 and is of continuing interest for the sake of comparison. Further the relationship between $\underline{e}_r$ and $\underline{e}_e$ gives rise to our interest in the $\underline{e}_e$ - F_1 combination.

The state variable filters have transfer functions

$$F_n(s) = \frac{s^n}{H(s)} = \frac{s^n}{1+\frac{2(1.0)}{10}s+\frac{s^2}{(10)^2}} \qquad n = 0,1,2,3 \tag{3-5}$$

The state variable filter, $F_n(s)$, is used to obtain the n^{th} approximate time derivative of its input signal. Components of the response error vector are obtained by filtering the (y - z) signal through the $F_0(s)$ and $F_1(s)$ state variable filters.

$$\underline{e}_r = \begin{Bmatrix} e_{r_0} \\ e_{r_1} \end{Bmatrix} = \mathcal{L}^{-1}\left(\begin{Bmatrix} F_0(s) \\ F_1(s) \end{Bmatrix} \left[Y(s) - Z(s)\right]\right) \tag{3-6}$$

Components of the equation error vector are obtained by filtering the y and x signals through $F_0(s)$, $F_1(s)$, $F_2(s)$ and $F_3(s)$. If Eq 3-4 specifies the model then one generalized equation error is:

$$e_{e_0} = \mathcal{L}^{-1}\left(\left[F_2 + 2F_1 + \alpha_0 F_0\right] Y(s) - \left[\beta_1 F_1 + F_0\right] X(s)\right) \tag{3-7}$$

<u>Vide</u> Eq 2-10 <u>et</u> <u>seq</u>. Continuing in the manner of Section II, the second generalized equation error is defined by Eq 2-29 with i = 1.

$$e_{e_1} = \mathcal{L}^{-1}\left(\left[F_3 + 2F_2 + \alpha_0 F_1\right] Y(s) - \left[\beta_1 F_2 + F_1\right] X(s)\right) \tag{3-8}$$

The equation error vector is:

$$\underline{e}_e = \begin{Bmatrix} e_{e_0} \\ e_{e_1} \end{Bmatrix} \tag{3-9}$$

In Eqs 3-7 and 8, α_0 and β_1 are treated as constants in the inverse Laplace transformation.

In general, both response error and equation error provide sufficient information to identify the plant. Performance of the two systems can, however, be markedly different. Figure 3-4 shows time responses and x - y plots for response and equation error systems. The input in each case is a sinusoid and the criterion is F_2. The oscillatory nature of the performance of the response error system, although not always present, can be expected to occur for sufficiently high adjustment loop gains. In fact, response error systems can be made to appear unstable. Figure 3-5 shows an apparent instability occurring in the response error system. The conditions are identical to those for Figure 3-4 except that the β_1 adjustment loop gain, A_{β_1}, was increased by a factor of 10. A similar effect can be induced by increasing the amplitude of the input signal. This effectively increases the gain in every parameter adjustment loop by the square of the factor by which the input amplitude was increased. Thus response error systems will be vulnerable to instability caused by increases in input amplitude. Another apparently unstable case is shown in Figure 3-6. One frequency predominates in this latter case.

It is not difficult to develop a plausibility argument to account for the unstable behavior of the response error system. Refer to Figure A-2. There it may be appreciated that the open-loop dynamics of each parameter adjustment loop consist of an integrator and a weighting filter. For simplicity, consider all parameters except one to be constant. Assume that the model is a constant coefficient linear system. Then the weighting filter may be represented by a transfer function. The open loop also contains one constant and two time-varying gain factors. Assuming that all these gain factors are constant, and that there is an excess of two poles over zeros in the weighting filter transfer function, instability of the closed-loop parameter adjustment system is unavoidable as the open-loop gain is increased. These conditions prevail for the α_0 parameter adjustment loop in the response error configurations of this experiment.

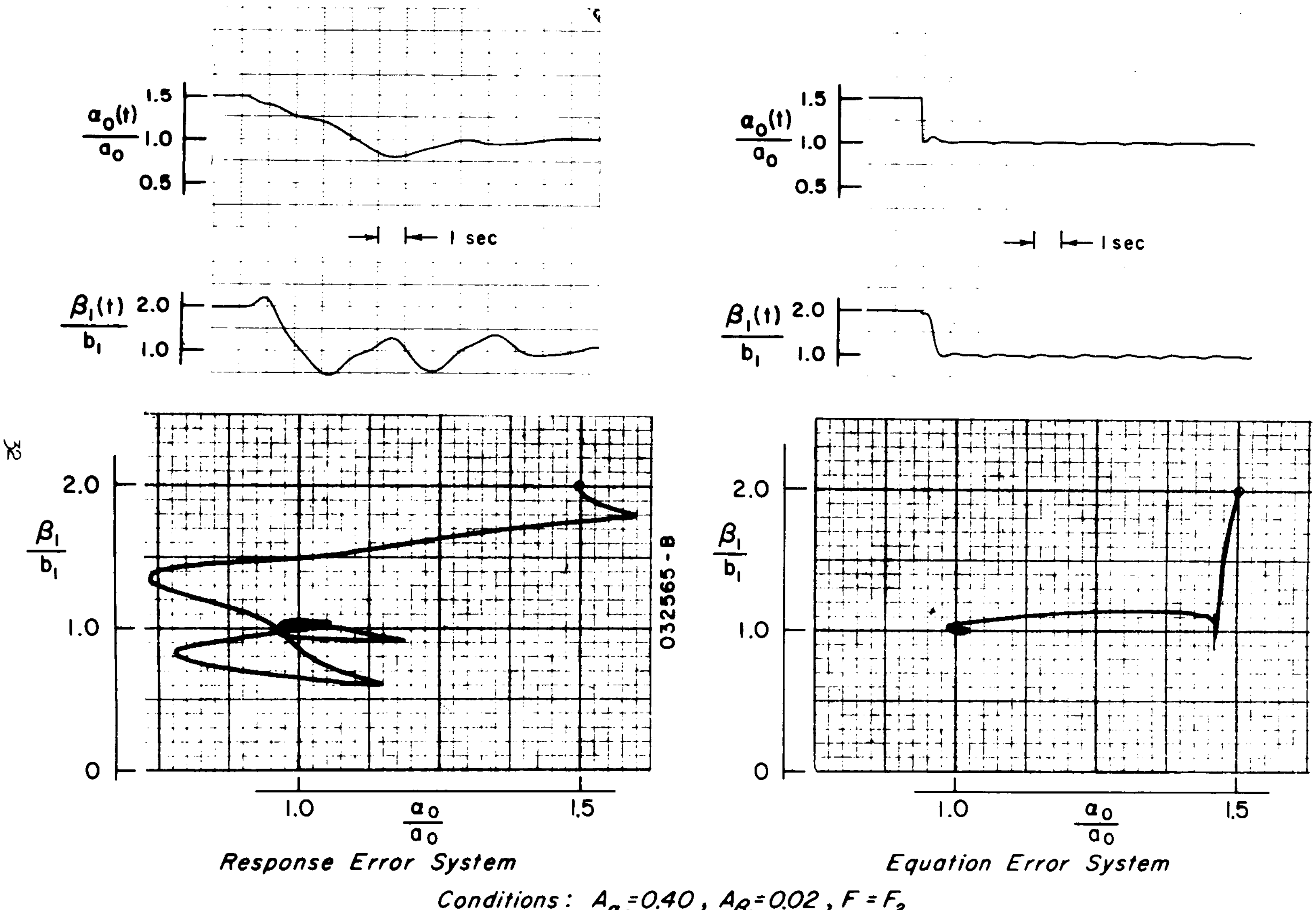
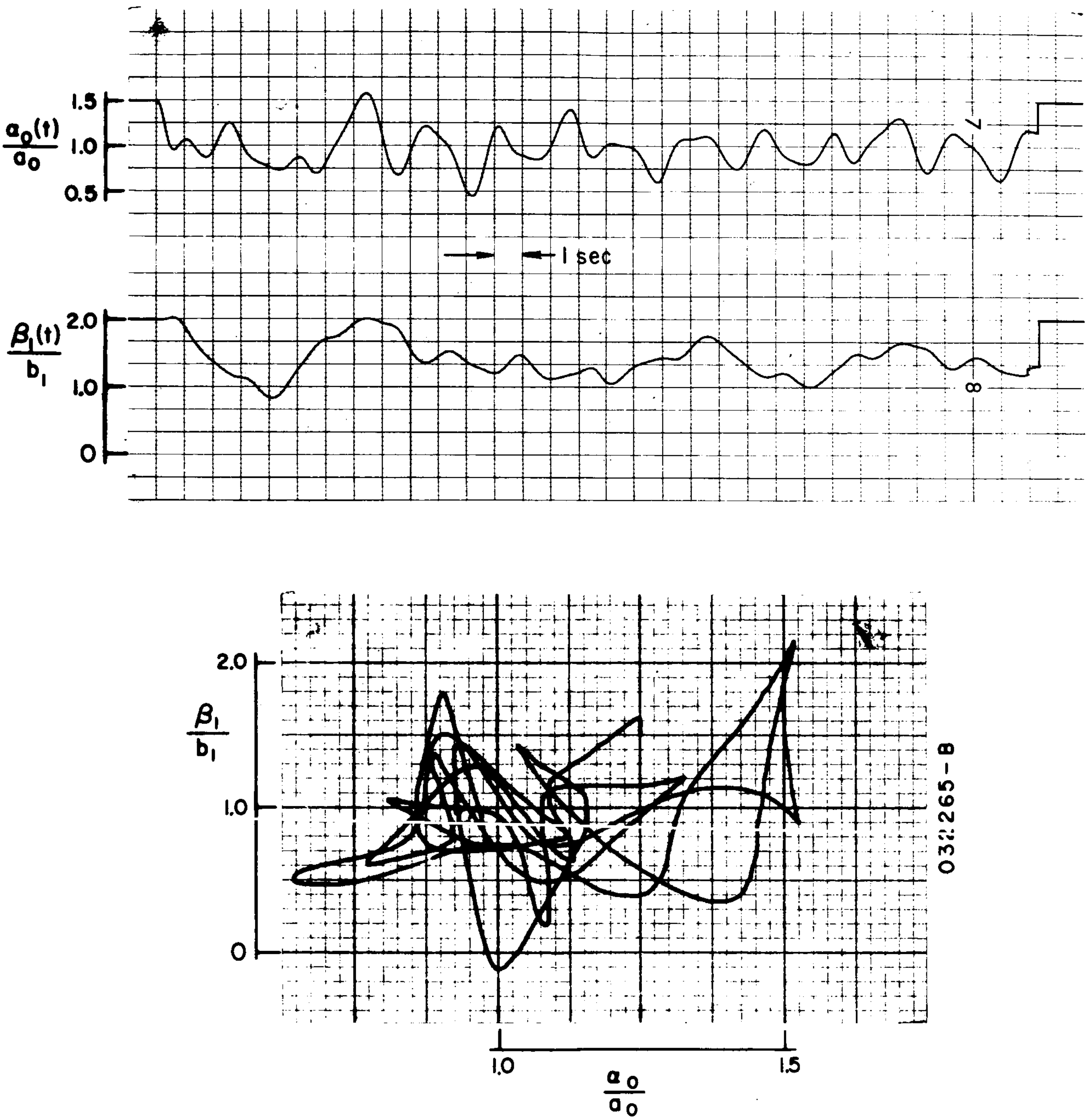
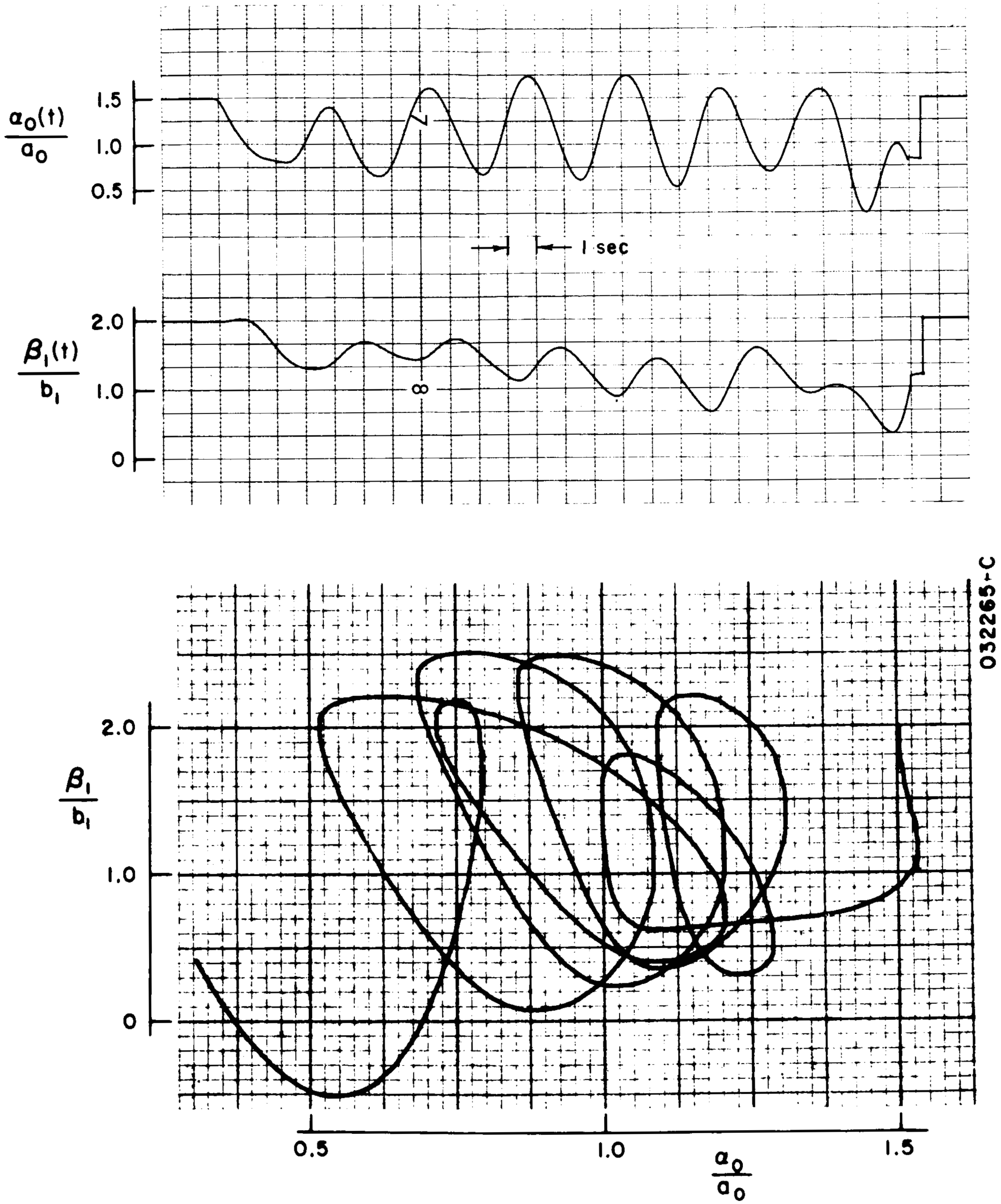


Figure 3-4. Comparison of Response and Equation Error Systems



Conditions: $A_{a_0} = 0.40$, $A_{\beta_1} = 0.20$, $F = F_2$

Figure 3-5. Apparent Instability in the Response Error System, Example 1



Conditions: $A_{a_0} = 0.40$, $A_{\beta_1} = 0.20$, $F = F_2$

Figure 3-6. Apparent Instability in the Response Error System, Example 2

An interesting phenomenon was observed when the input frequency was varied. Performance of the response error system is sharply degraded as input frequency is increased. Figure 3-7 shows a comparison between systems for gains optimized at $\omega_i = 1.0$ rad/sec when ω_i is increased to 1.6 rad/sec. This is because of the fact that the system amplitude ratio increases ($\omega_n = 2.0$ rad/sec) as the input frequency goes from 1.0 to 1.6. This increase effectively increases the parameter adjustment loop gain and leads to the oscillatory behavior described earlier (Figures 3-5 and 6).

The sensitivity of response error parameter tracking system performance to input amplitude and frequency is a notable limitation. There are techniques available by which we might conceivably circumvent this limitation, e.g. by the use of limiters in the parameter adjustment loops, but the introduction of these techniques would raise other subtle questions regarding system performance. The important point is that equation error systems, being stable, do not exhibit this performance sensitivity to input amplitude and frequency. Once the parameter adjustment loop gains are made high, the system tolerates large changes in operating point conditions. (Compare the equation error and response error system responses in Figure 3-7.) The equation error system behaves to a much greater degree as a well designed feedback system should behave.

The conclusion reached is that equation error systems have two distinct and noteworthy advantages over response error systems. These are:

- Equation error systems are always stable feedback control systems.
- Such systems can be operated with high loop gains and thus they may be made relatively insensitive to changes in the operating conditions.

In the above discussion it was assumed that the systems being compared were properly designed. The criterion used was $F_2 = \frac{1}{2}\left(e(\ )_0^2 + \lambda e(\ )_1^2\right)$, $\lambda = 1$ which defines a positive definite surface with closed contours of constant F_2 for the inputs used. The effect of alternative criterion functions on the performance of equation error systems will be presented next.

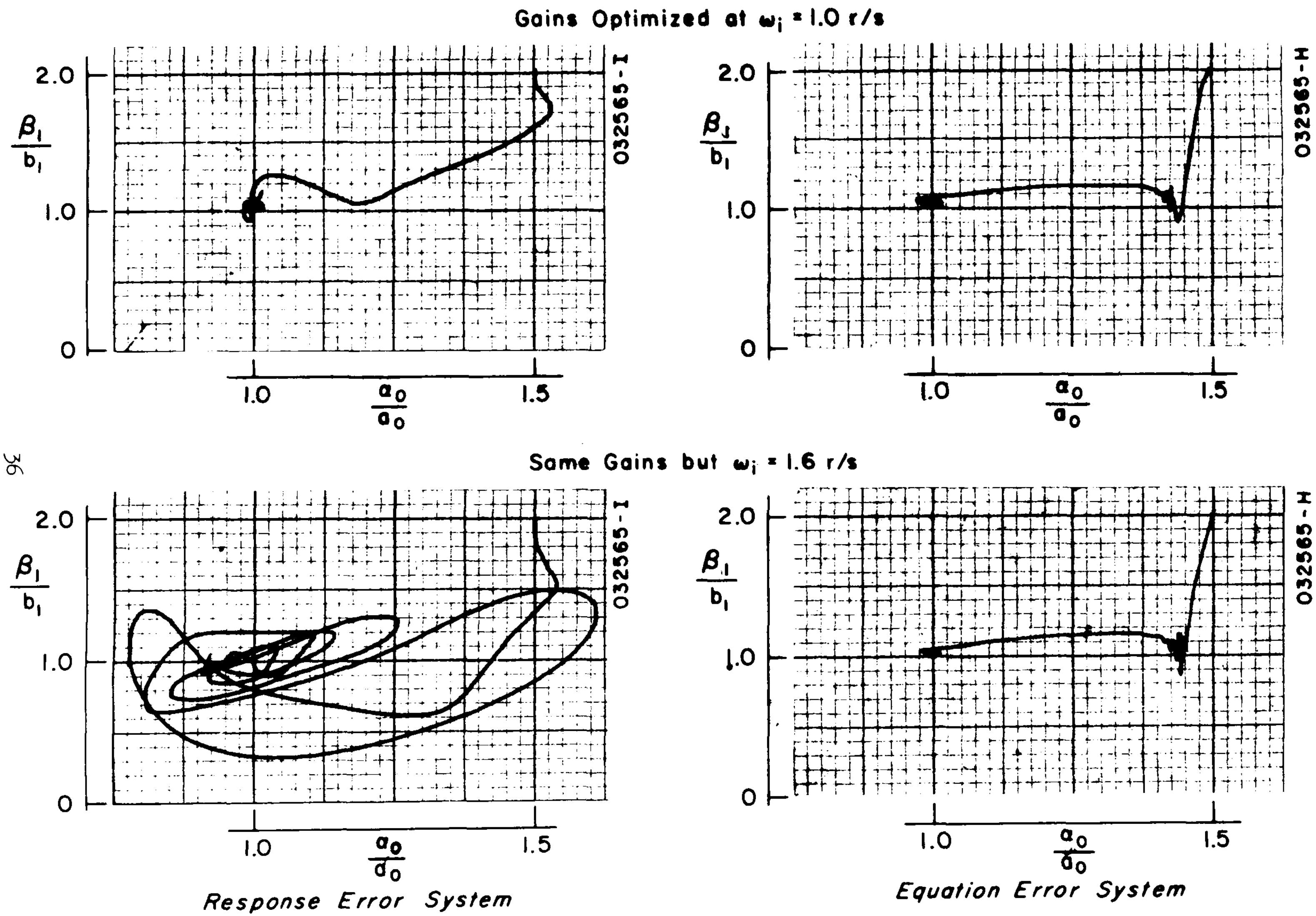


Figure 3-7. Sensitivity of Systems to Input Frequency

First consider the case where λ is zero, that is consider $F_2 = \frac{1}{2}e_{e_0}^2$. This criterion function is positive semi-definite; the contours of constant F_2 are not closed. Figure 3-8 shows that there is an optimum adjustment of loop gain in this situation. This has previously been demonstrated experimentally by Miller. (Vide Reference 3.) Of course, stability is still assured, but increasing gain does not necessarily decrease the convergence time. Also, we cannot now expect to be able to achieve a desired tolerance by increasing gain as we could when $F = \frac{1}{2}\left(e_{e_0}^2 + e_{e_1}^2\right)$.

Finally consider the criterion $F = F_1 = \frac{1}{2}\left(e_{e_0} + e_{e_1}\right)^2$. Figure 3-9 shows the effect of increasing the adaptive loop gains by a factor of five. The convergence time increases as it did when $F_2 = \frac{1}{2}e_{e_0}^2$ was used. Clearly the desired effect of adding a second error component is not being achieved. The inevitable conclusion is that the criterion function, F_2, is superior to F_1 even though it requires more equipment to mechanize.

PERFORMANCE VERIFICATION FOR A THREE PARAMETER EQUATION ERROR SYSTEM

In Section II, sufficiency conditions were formulated under which equation error parameter tracking systems could always be made completely stable. Furthermore, additional sufficiency conditions were obtained for determining when the adjusting model parameters would converge to those values present in the plant, and when decaying exponential bounds could be placed upon the responses of the converging model parameters. For the cases in which the plant and model are identical in form and order, and the input signal consists of a sum of randomly phased sinusoids with different frequencies, none of which have phase shifts if kπ radians through the plant; the convergence properties of the system can be defined in terms of regions on a very simple diagram. Figure 3-10 is this diagram. (When the forms or orders of the model and plant are not alike or when some model parameters are fixed the situation is not so simply described.) The three regions defined on the diagram account for all possibilities of present interest. All solutions are stable, of course, by virtue of Eq 2-24 through 26. The three regions on the diagram correspond to the following performances:

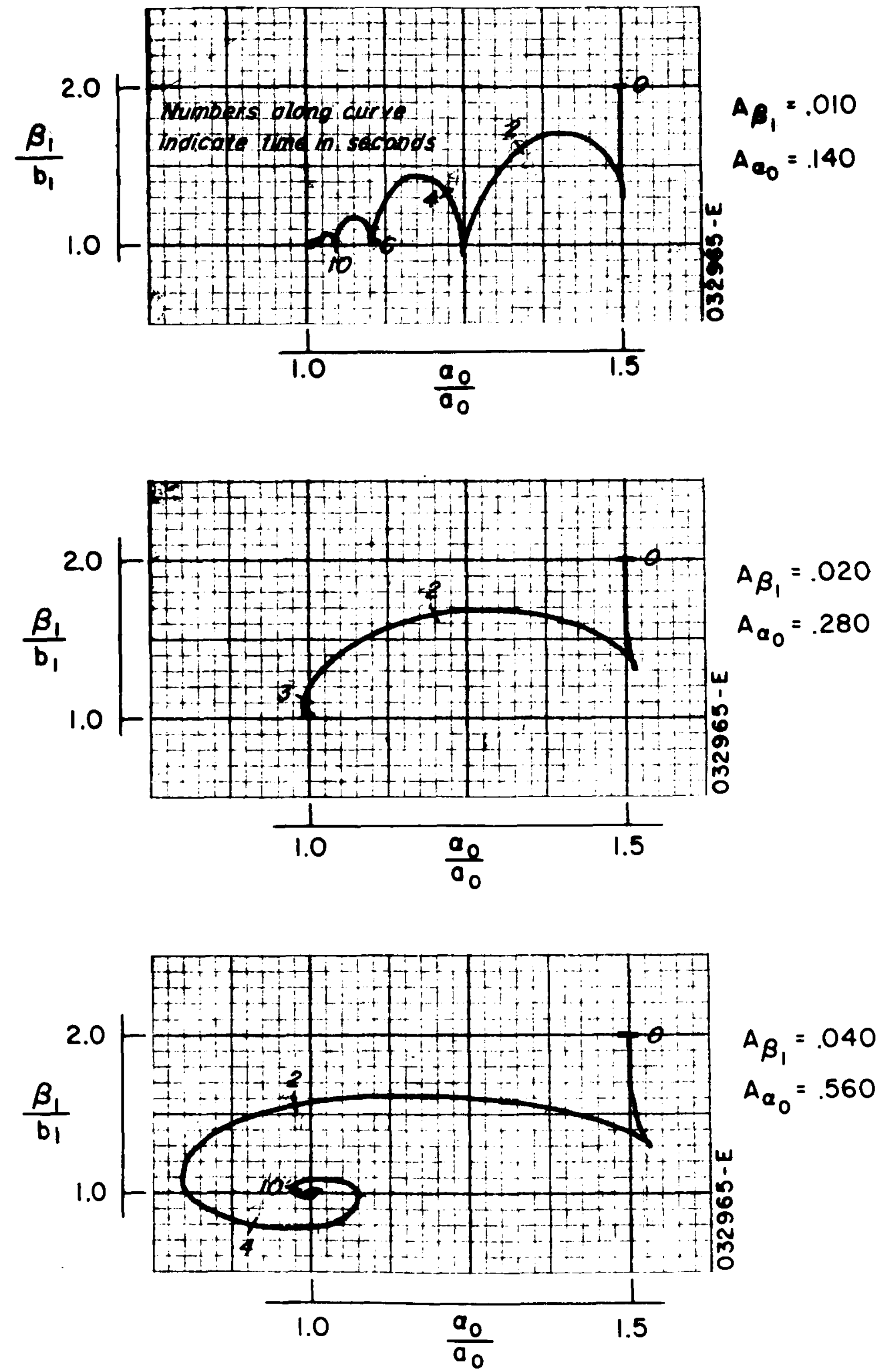
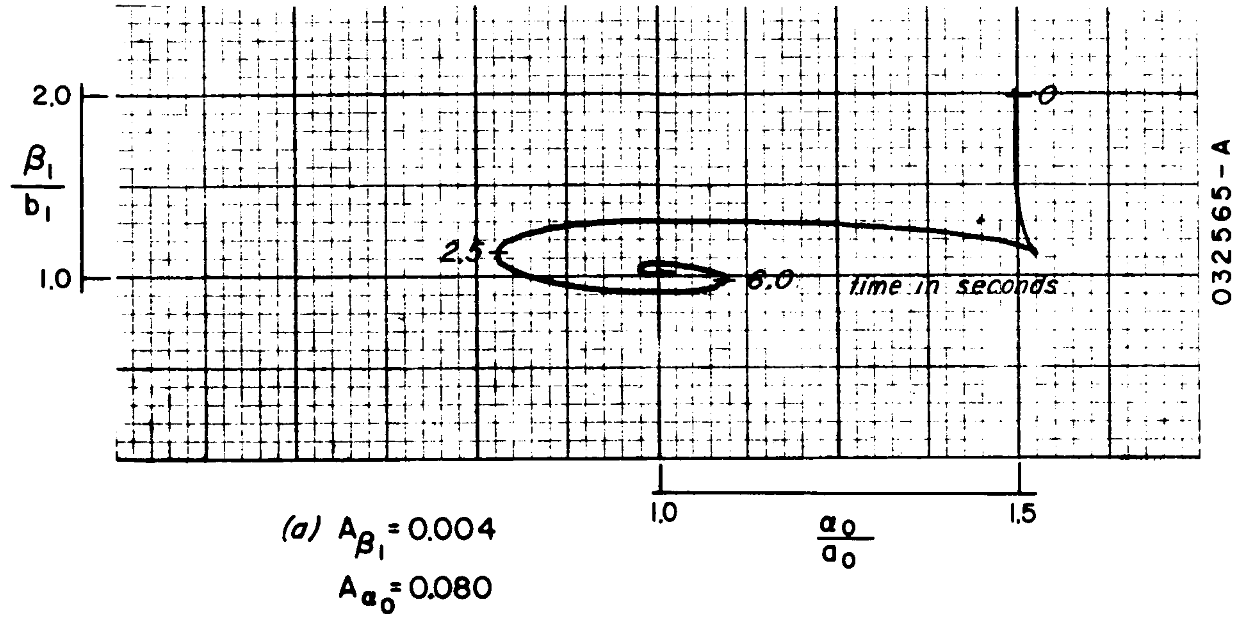


Figure 3-8. Gain Optimization for Equation Error System When λ = 0.0



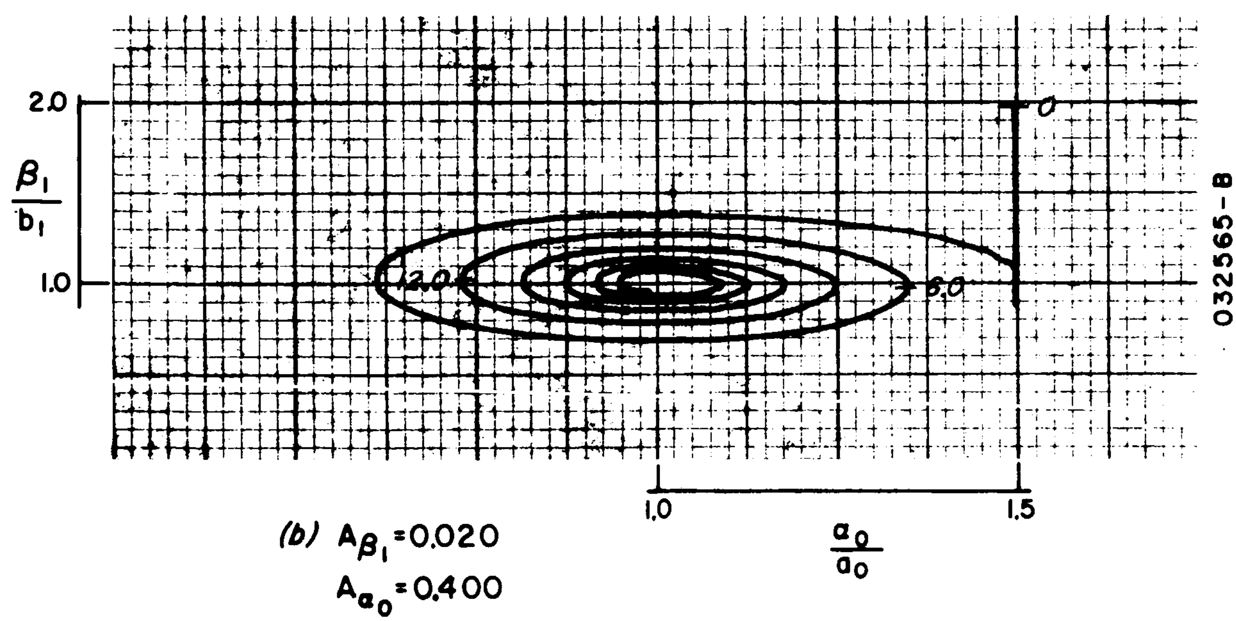
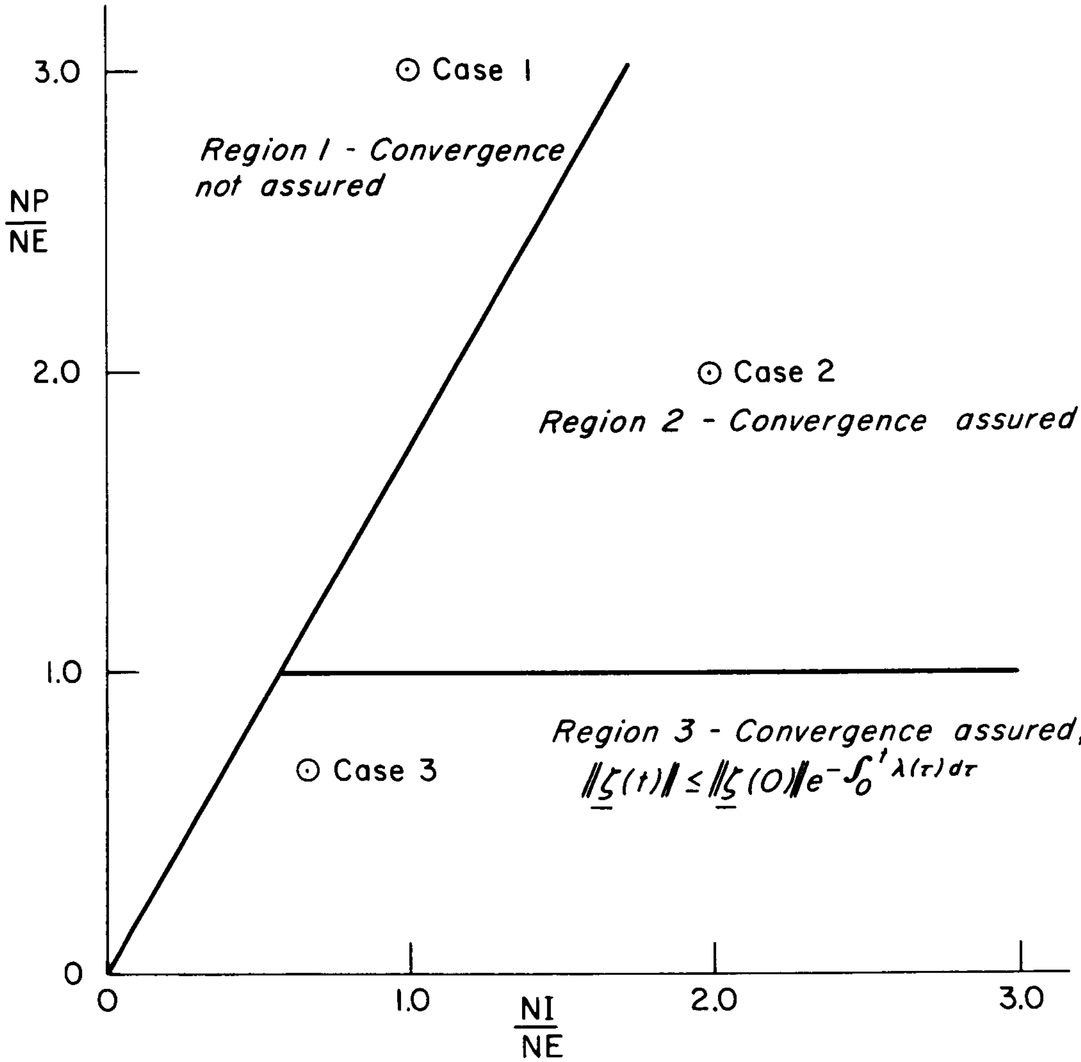


Figure 3-9. Effect of Gain on Equation Error System When $F = F_1 = \frac{1}{2}\left(e_{e_0} + e_{e_1}\right)^2$



NP = Number of adjustable parameters

NI = Number of sinusoidal input components

NE = Number of independent error components

Figure 3-10. Convergence Properties of Parameter Tracking Systems for Inputs Which Are Sums of Sine Waves

1. Convergence to the plant parameter values is not assured. Only in certain coincidental situations will the correct answer be obtained.

2. Convergence to the plant parameter values is assured. The correct answer can always be obtained.

3. Convergence to the plant parameter values is assured, and the norm of the transformed parameter vector, $\underline{\zeta}$, is bounded from above by:

$$\| \underline{\zeta}(t) \| \leq \| \underline{\zeta}(0) \| \, e^{-\int_0^t \lambda(\tau)\, d\tau}$$

A lower bound on $\| \underline{\zeta}(t) \|$ is obviously zero. The above inequality also requires that $\| \underline{\zeta}(t) \|$ decrease monotonically in time.

Also shown in Figure 3-10, as the circled points, are the conditions for three experimental cases which are representative of the data taken to verify the theoretical predictions.

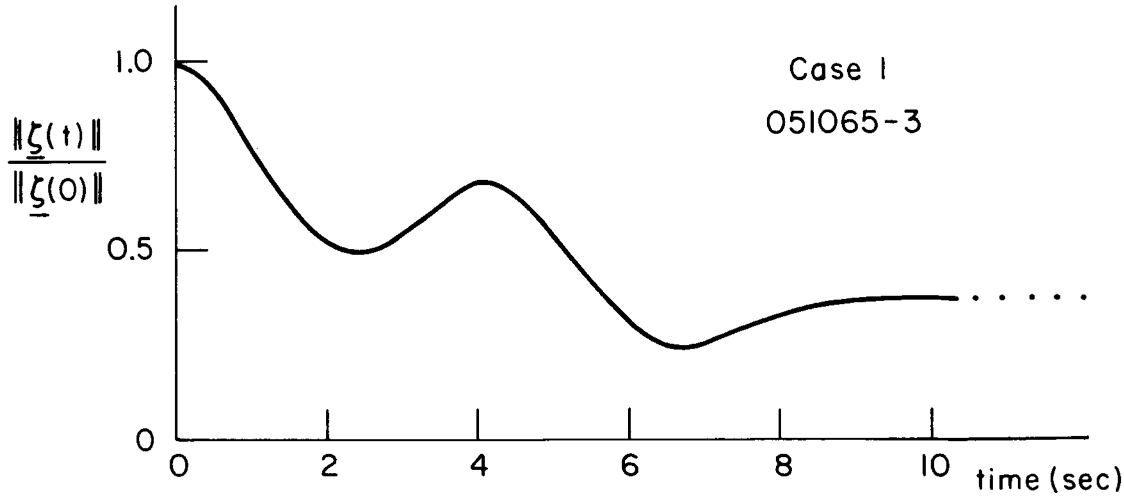
Figure 3-11 shows the possible convergence situations for the three cases. In Case 1 there are three parameters, β_1, α_0 and α_1, to adjust, NP = 3; and using a scalar equation error, NE = 1. The system is excited by a single sine wave, NI = 1. The norm of the transformed parameter* vector for this case is plotted in part a. of Figure 3-11. It may be seen in the Figure that the system does not converge to the correct solution, and the norm does not approach zero. The actual final value of the parameter vector depends on the initial values of the adjustable parameters, the input amplitude and phase and all the plant parameters.

Increasing the number of error components will not produce a convergent situation. This can be appreciated in connection with Figure 3-10 by noting that, according to the definitions of the axes, the number of

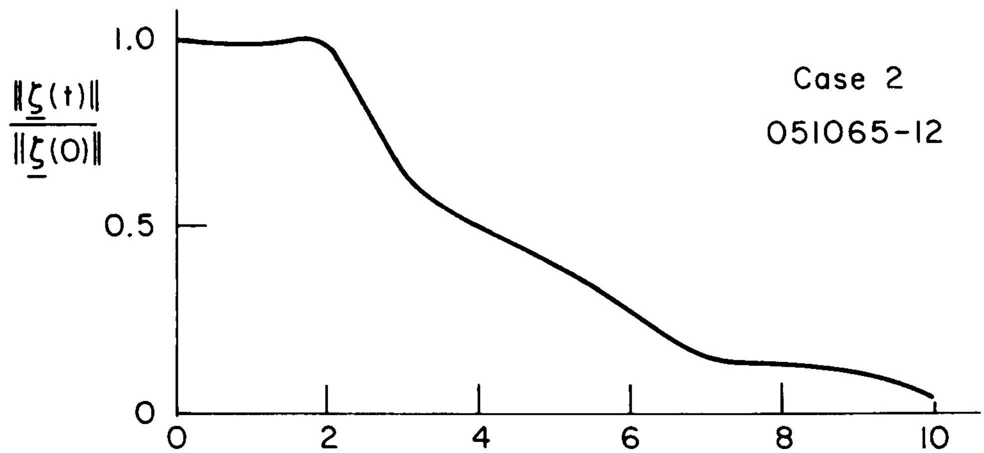
---

*Using Eq 2-47, the transformed parameter vector for these cases is given by:

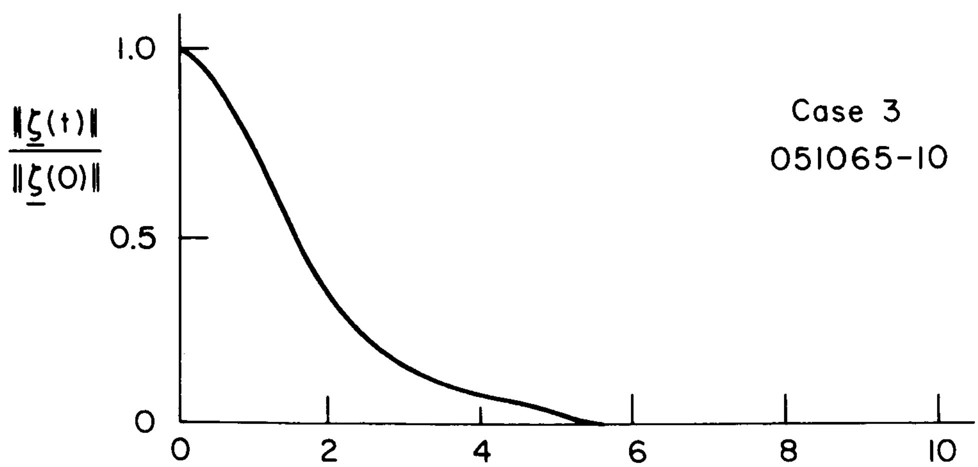
$$\underline{\zeta}(t) = \begin{Bmatrix} (\beta_1(t) - b_1)/\sqrt{A_{\beta_1}} \\ (\alpha_0(t) - a_0)/\sqrt{A_{\alpha_0}} \\ (\alpha_1(t) - a_1)/\sqrt{A_{\alpha_1}} \end{Bmatrix}$$



(a) A Nonconvergent Case



(b) Convergent Case With Unspecified Bound



(c) Convergent Case With a Monotonically Decreasing Upper Bound on the Parameter Vector Norm

Figure 3-11. Convergence Property Experiments error components can only move the location of particular case points radically on this graph. Therefore, Case 1 cannot be moved into another region on Figure 3-10 by increasing the number of error components. The only way to induce convergence is to increase the number of inputs or decrease the number of parameters or both. This was actually done to produce Case 2 and the transient shown in part b. of Figure 3-11. Here, NP = 2, NI = 2 and NE = 1.

In Figure 3-11 b. the parameters converge to the proper values. The norm is reduced to 10 percent of its initial value in 10.0 sec.

In Case 3, shown in Figure 3-11 c., the number of error components was increased from one to three, NE = 3. The parameter adjustment loop gains and the number of sinusoidal components in the input signal, however, are here the same as for Case 2. Convergence is now much more rapid; the norm is reduced to 10 percent of its initial value in only 2.5 sec.

Case 3 typifies the situation for which a monotonically decreasing upper bound on the transformed parameter vector norm can be demonstrated. Assuming ideal components the bound can be made to converge as rapidly as desired by increasing parameter adjustment loop gains. In the experiments described here the gains were set at moderate levels so that the limits of data recording equipment would not be exceeded.

In actual practice, the factors which might limit the parameter adjustment loop gain are:

- Non-ideal effects in the computing elements of the parameter adjustment loops (e.g. saturation, high frequency dynamics)
- The presence of disturbance inputs to the plant which may be regarded as measurement noise entering the system at the sensors.

The first factor might be of considerable practical importance if very high gain systems were considered. Investigation of its effects, however, is beyond the scope of the present research. The second factor is now to be considered both in connection with the prototype second order system under discussion and, later, in connection with simulated pilot parameter tracking experiments.

MEASUREMENT NOISE EFFECT IN EQUATION ERROR SYSTEMS

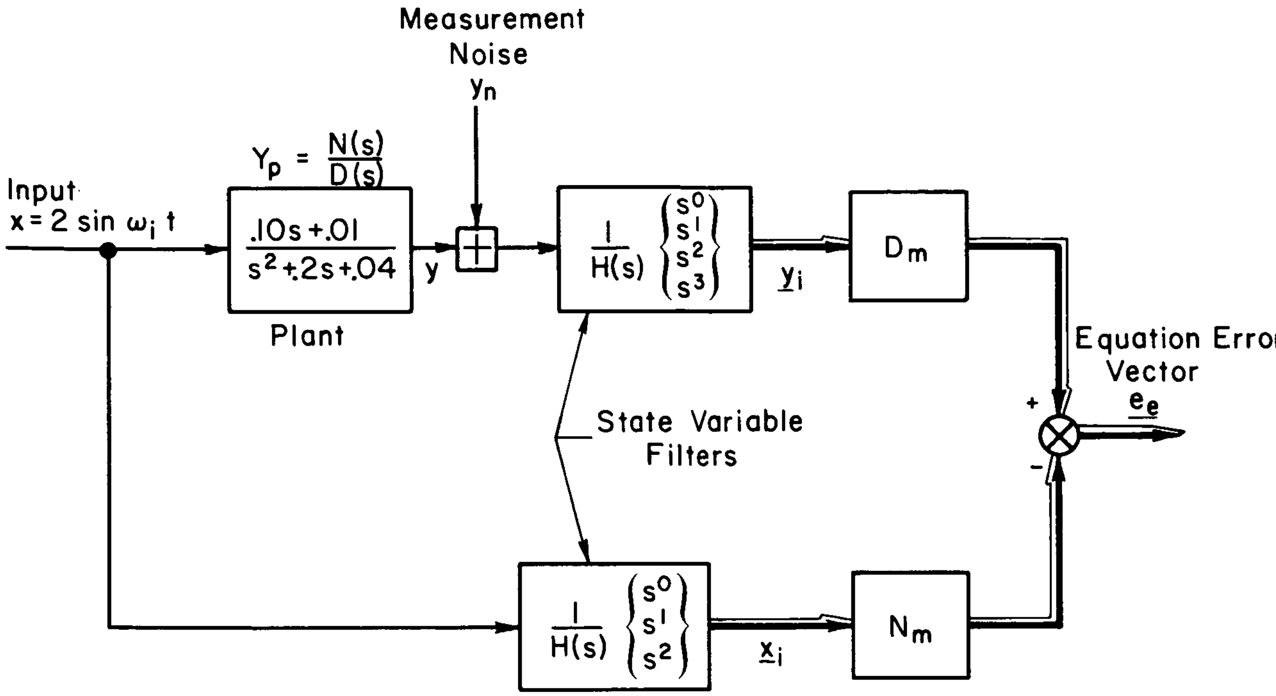
The sensitivity of parameter tracking system performance to measurement noise is an important practical consideration. Unfortunately, at present, the theory falls short of providing an analytical basis for treating this problem. Future research is clearly needed in this area.

In the present program, an empirical approach to minimizing the effects of noise is taken. The structure of the equation error system and the characteristics of the plant input spectrum indicate an obvious path to follow. Results of the experiments further point to two possible ways of attacking the problem analytically. These approaches for the future will be described in the concluding remarks on the experiments.

The set-up for the measurement noise experiments is shown in Figure 3-12. The plant is second order with a single zero. Noise enters additively at the plant output. The model differential equation has the same form as that for the plant. Adjustable model parameters are β_1 and α_0. The equations describing the simulated system are summarized in Figure 3-12.

The state variable filters contain a second order cut-off filter, $1/H(s)$, with a fixed damping ratio of unity and a variable natural frequency, ω_f. With ω_f at its nominal value, 1.0 rad/sec, the state variable filter bandwidth exceeds that of the plant by nearly a decade, so that in this case the state variables x_i and y_i are approximately $d^i x/dt^i$ and $d^i y/dt^i$. The exact equation error is therefore closely approximated by e_{e_0}.

The input used throughout these experiments is a single sine wave. This is sufficient to assure that the model parameters converge, and that an upper bound which is a decaying exponential exists for the parameter responses. The two parameters may then be made to converge as rapidly as desired by increasing the parameter adjustment loop gains when no noise is present. The input frequency, ω_i, is varied in some runs with the nominal value being 0.2 rad/sec. Parameter adjustment loop gains are such that moderately rapid convergence is achieved under nominal conditions for this input when no noise is present.



Cut-Off Filter Incorporated In The State Variable Filters

$$\frac{1}{H(s)} = \frac{\omega_f^2}{s^2 + 2(1.0)\omega_f s + \omega_f^2} \qquad (\omega_f \text{ is varied})$$

$\underline{y}_i = \text{col}\,(y_0, y_1, y_2, y_3)$ $\qquad \mathcal{L}(y_i) = s^i \left(Y(s) - Y_n(s)\right)/H(s)$

$\underline{x}_i = \text{col}\,(x_0, x_1, x_2)$ $\qquad \mathcal{L}(x_i) = s^i X(s)/H(s)$

$\underline{e}_e = \text{col}\,(e_{e_0}, e_{e_1})$

$$D_m = \begin{bmatrix} \alpha_0 & .2 & 1 & 0 \\ 0 & \alpha_0 & .2 & 1 \end{bmatrix} \qquad N_m = \begin{bmatrix} .01 & \beta_1 & 0 \\ 0 & .01 & \beta_1 \end{bmatrix}$$

$$\underline{e}_e = D_m \underline{y}_i - N_m \underline{x}_i$$

Figure 3-12. Configuration for Measurement Noise Experiments

Measurement noise is obtained by shaping unit white noise with a filter having the transfer function:

$$\left[\frac{1.89\ s}{\frac{s^2}{(0.374)^2} + \frac{2(0.8)}{0.374}\ s + 1}\right]^2$$

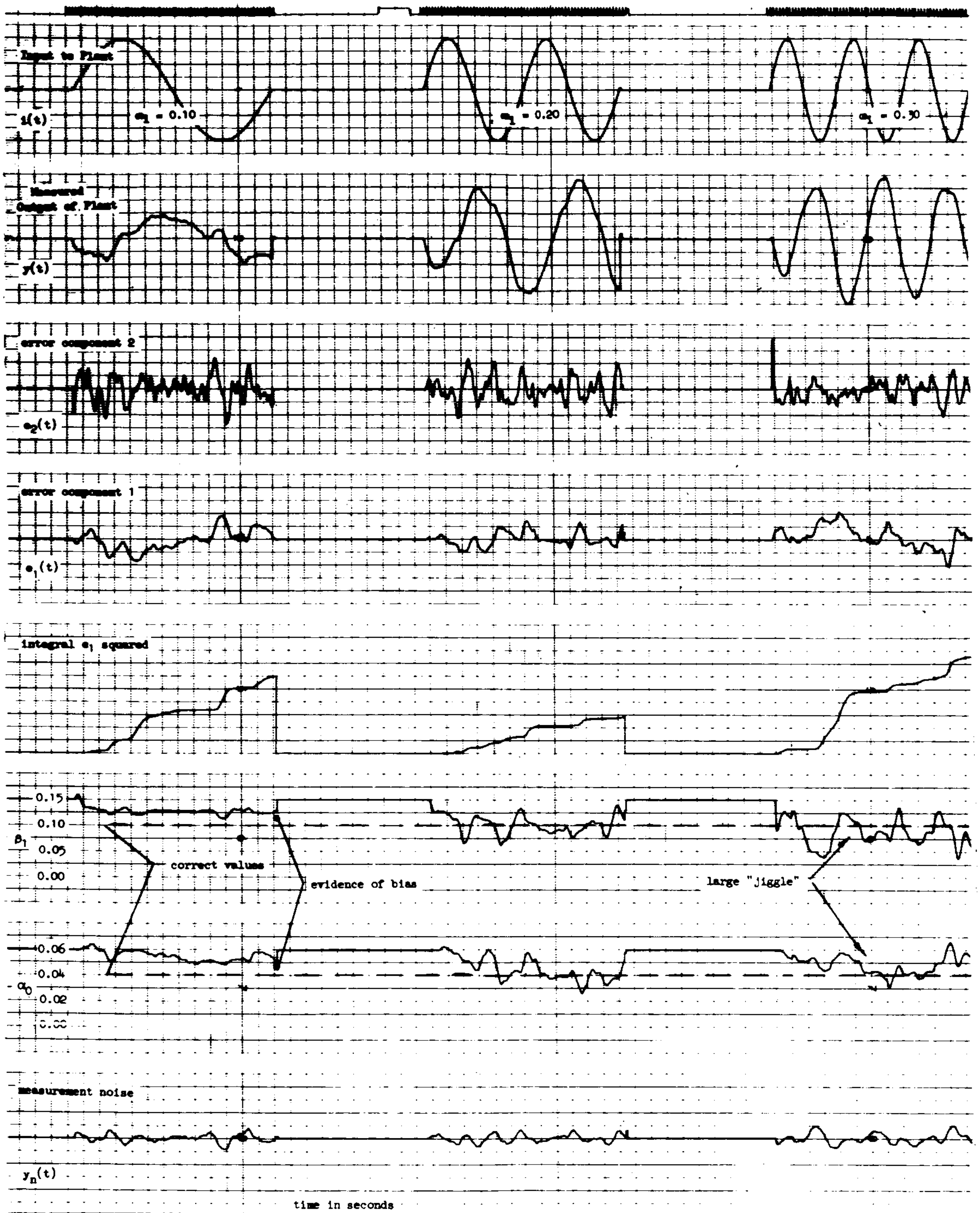
This concentrates the measurement noise power in a small, given frequency range. Moving the input frequency into that range then provides a crucial test of the parameter tracking system's ability to suppress noise effects. Note that the nominal input frequency, $\omega_i = 0.2$ rad/sec, occurs at a frequency which is less than the break frequencies of the shaping filter.

A very practical reason for using this shaping filter in the experiment also exists since, by cutting off the low frequencies, the low frequency variability, or, alternately, the required run length, is reduced.

Figure 3-13 shows some results from the measurement noise experiments. Scaling is omitted since for present purposes numerical detail would be an encumbrance. The input frequency is varied in these three runs.

The first, and perhaps most important characteristic to be noted is the presence of an offset, or bias, in the mean value of the adjusted parameters. In Run a the offset is substantial, in Run b it is very small, while in Run c it is moderately small. That the bias is a function of input frequency is not surprising since the signal to noise ratio seen by the parameter tracking mechanism depends on the gain of the plant and, to a greater extent, on the gain of the state variable filters at the input frequency. Integral-square-error, shown on channel four, reflects the bias in the adjusted parameters, i.e. it is smallest in Run b, where the bias is smallest.

A second aspect of this problem is parameter "jiggle." This term is used to characterize the variance of the parameter about its mean value. Parameter jiggle increases from Run a to Run c. The apparent reason for this is that as the frequency separation between two spectral components of the input and the measurement noise becomes less, the parameter tracking system interprets



Run a Run b Run c

Input Frequency Varied

Figure 3-13 Parameter Tracking in the Presence of Measurement Noise.

the noise as arising increasingly from low frequency variations of the plant parameters. The parameter tracking system is able to track low frequency variations of the plant parameters, as we shall show later, and therefore it responds to these apparent variations. To obtain an appreciation for this viewpoint, assume a sinusoidal signal in the plant, $\sin \alpha t$, is modulated cosinusoidally by a time-varying plant parameter, $\cos \beta t$. The modulation product is:

$$\sin \alpha t \cos \beta t = \frac{1}{2} \sin(\alpha + \beta)\ t + \frac{1}{2} \sin(\alpha - \beta)\ t \qquad (3\text{-}10)$$

Suppose the sum and difference frequency components appear in the measured plant output. The parameter tracking system operates using essentially DC information, such as that contained in the square of the measured plant output signal, to determine the rate of parameter adjustment.

$$(\text{measured plant output})^2 =$$

$$\sin^2 \alpha t \cos^2 \beta t = \frac{1}{4}\left[1+\cos 2\beta t - \cos 2\alpha t - \frac{1}{2} \cos(\alpha-\beta)t - \frac{1}{2} \cos(\alpha+\beta)\ t\right] \qquad (3\text{-}11)$$

Now consider a situation wherein the measured plant output has two components: $1/2 \cos \gamma t$ arising from the plant input signal and $1/2 \cos \delta t$, a spectral component of measurement noise. As in the case of parametric modulation there are two frequency components in the signal upon which the parameter tracking system must operate. If we take $\gamma = (\alpha - \beta)$ and $\delta = (\alpha + \beta)$, the two situations are identical from a frequency content standpoint. Note also that the frequency separation of the two components $(\delta - \gamma)$ is 2β. In the following table, the sources of jiggle are matched with the frequency components of jiggle they cause for three cases of interest.

In Table II, the sources of the various jiggle components are of general interest. We must keep in mind, however, that the "jiggle components" listed undergo integration in the parameter adjustment loops. This means that the low frequency components are more important than high frequency components because the amplitude ratio of the integrator

TABLE II

INTERPRETATION OF TERMS IN EQ 3-11 FOR THREE CASES

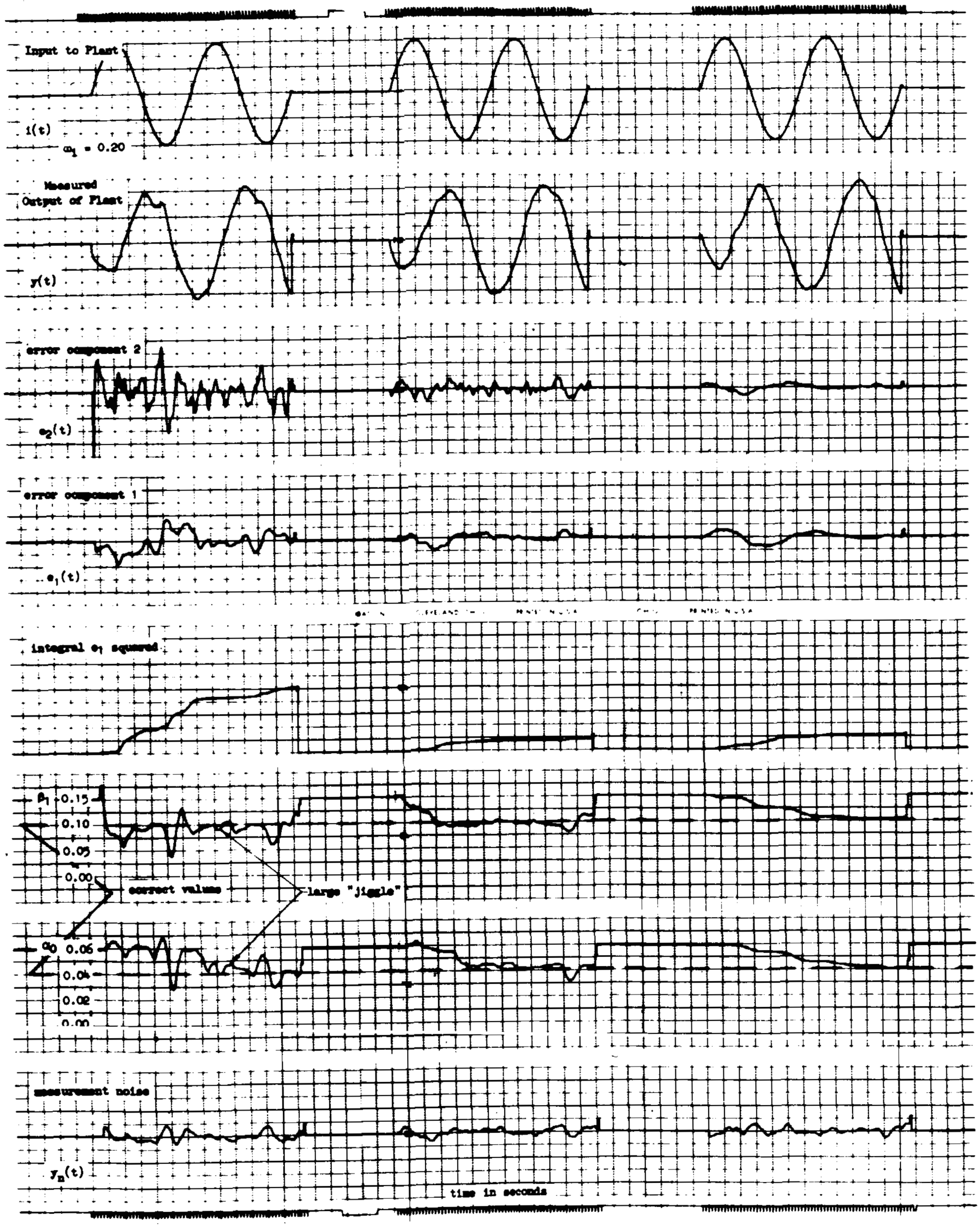
Term of Eq 3-11	CASES		
	Constant Plant Parameters, No Measurement Noise, ($\beta = 0$)	Time-Varying Plant Parameter, No Measurement Noise, ($\beta \neq 0$)	Constant Plant Parameters, With Measurement Noise ($\beta \neq 0$ or $\beta = 0$)
1st	DC Parameter Rate of Change Signal.		
2nd		Component of Jiggle is an Effect of Time-Varying Plant Parameter Frequency, β.	For $\beta \neq 0$, Component of Jiggle at Difference Frequency of Plant Output Frequency, ($\alpha - \beta$), and Measurement Noise Spectral Component Frequency, ($\alpha + \beta$). For $\beta = 0$, Bias on DC Parameter Rate of Change Signal.
3rd	Component of Jiggle is an Effect of Plant Input Frequency, α.		Component of Jiggle at Sum Frequency of Plant Output Frequency, ($\alpha - \beta$), and Measurement Noise Spectral Component Frequency, ($\alpha + \beta$).
4th		Component of Jiggle is a Difference Frequency Modulation Effect from $\sin \alpha t \cos \beta t$	Component of Jiggle is an Effect of Plant Output Frequency, ($\alpha - \beta$).
5th		Component of Jiggle is a Sum Frequency Modulation Effect from $\sin \alpha t \cos \beta t$	Component of Jiggle is an Effect of Measurement Noise Spectral Component Frequency, ($\alpha + \beta$).

weights the signals according to the inverse of the frequency. That this is an important consideration when measurement noise is present is especially true for the interpretation of the second term of Eq 3-11. This term has a frequency which decreases with decreasing separation in frequency between the plant output signal and the measurement noise spectral component. When this frequency is very low, this term effectively is a short term DC bias on the parameter rate of change signals which the integration in the closed parameter adjustment loop can offset. In terms of the effect on the parameter adjustments of the closed-loop system, this produces an oscillation at that low frequency. The superposition of these oscillations for all the measurement noise spectral components gives rise to a low frequency jiggle effect such as that observable in Figure 3-13.

A similar argument can be constructed to justify the existence of a DC parameter bias (of finite power), but there is nothing to be gained over the explanation advanced by Elkind in Reference 18.

The possibility of filtering the parameter signals to reduce jiggle exists. It appears from the frequency content of the signals shown in Figure 3-13 that a substantial decrease in the amount of jiggle would be achieved. However, filters represent additional complexity and their inclusion should be weighted against alternative schemes which, possibly, are more attractive. (Because, for example, such a filter would not be effective in reducing parameter biases.) Fortunately, a degree of freedom exists in the equation error system design that has not yet been exploited. This involves specification of the cut-off filter segment, $1/H(s)$, of the state variable filters.

Figure 3-14 shows time records for an experiment in which the cut-off filter undamped natural frequency, ω_f, is varied. It is obvious that the effects of ω_f on integral-squared-error (channel 5) and parameter adjustment (channels 6 and 7) can be used to advantage. While it is not clear what the effect on parameter bias is, it is clear that as ω_f is decreased, the parameter jiggle is markedly reduced, but the parameter convergence time is increased. This is the tradeoff we must make in selecting an optimum bandwidth for the cut-off filters. Mean square error and response



$\omega_f = 0.2$ $\omega_f = 0.6$ $\omega_f = 1.0$

Figure 3-14 State Variable Filter Optimization.

time are plotted with arbitrary weighting in Figure 3-15 which shows that an optimum ω_f exists, and that this might be approximately equal to 0.6 rad/sec.

It seems then, that a good index of parameter tracking performance, especially when measurement noise is present, might be the mean square of the parameter vector norm. Inasmuch as the analytical results presented in Section II are expressed in terms of the parameter vector norm, it might be feasible to extend the results to cases including measurement noise, and then to optimize the performance bound with respect to the cut-off filter parameters.

A much more pedestrian approach was attempted during this program. This approach assumed that the transient response of the parameter tracking system to initial parameter offsets had died out; that is, that the mean square equation error had been minimized by the parameter tracking action. The steady model parameter values obtained under this condition are different from the known plant parameters because of the noise bias effect. Our interest was to minimize the mean square of the parameter vector norm with respect to the cut-off filter parameters; however, the problem was fraught with algebraic complexity even for the simplest meaningful case. Because of this, the approach was abandoned.

Another engineering approach to the problem would be to maximize the signal to noise ratio of the generalized equation errors by computing the optimum fixed form or free form cut-off filter. This would require that input and measurement noise spectra be substantially different, however, since the potential rewards from the optimum filter approach depend strongly upon this factor. When parameter tracking is performed using, as the plant input, quantities occurring in a natural environment, this requirement may not be met, and therefore, optimum filtering cannot constitute an effective approach. The human pilot performing a compensatory control task when the input to the closed-loop pilot controlled element system is shaped unit white noise is an example of such a situation. This particular problem is considered in detail in the last subsection along with a practical way for circumventing the problems introduced by the measurement noise or, more correctly for the case of the human pilot, the remnant.

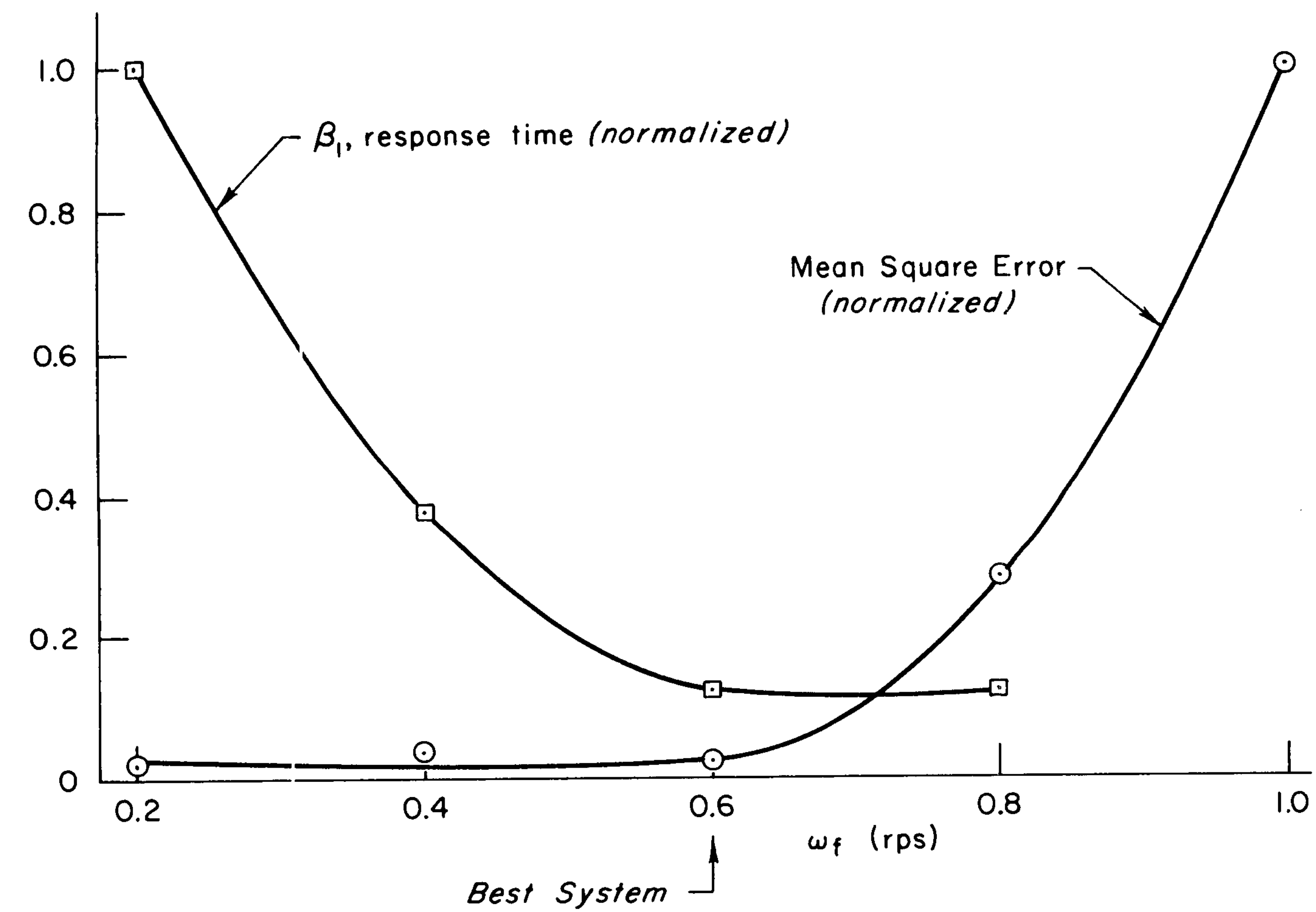


Figure 3-15. State Variable Filter Optimization

TRACKING TIME-VARYING PLANT PARAMETERS

A preliminary effort, was undertaken to determine the efficacy of the generalized equation error system in tracking time-varying parameters. In addition to analytical work, a system was set up on the analog computer which would track a single varying parameter. The general conclusion from this portion of the work is that, under certain conditions, time-varying parameters can indeed be tracked.

The analysis of such systems is, of necessity, approximate. Consider the plant shown in Figure 3-12 to be time-varying. The numerator and denominator are now denoted by N(s, t) and D(s, t) respectively.* In this case, it is no longer true that $e \equiv 0$ when the parameter values are on their correct (time-varying) values. This is because the dynamics of the filter, $1/H(s)$, can no longer be commuted with the dynamics of numerator and denominator, as was done in Eq 2-15. The exact equation error ($1/H(s) = 1$) is, of course, an exception to this last remark.

To show this more explicitly, consider the time-varying n^{th} order plant

$$\sum_{k=0}^{n} a_k(t)y^{(k)} = \sum_{j=0}^{m} b_j(t)x^{(j)} \tag{3-12}$$

or, in operator notation

$$D(s, t)\, Y(s) - N(s, t)\, X(s) = 0. \tag{3-13}$$

where

$$D(s, t) = \sum_{k=0}^{n} a_k(t)s^k \tag{3-14}$$

*From this point, we will use a purely formal notation in which s may denote the operator d/dt or the Laplace transform variable whichever is appropriate.

$$N(s,\ t) = \sum_{j=0}^{m} b_j(t)s^j \tag{3-15}$$

and X(s) and Y(s) are the Laplace transforms of x(t) and y(t) respectively.

The generalized equation error shown in Figure 3-12 can be written in operator form

$$E(s,\ t) = \widetilde{D}(s,\ t)\ Y_1(s) - \widetilde{N}(s,\ t)\ X_1(s) \tag{3-16}$$

where

$$\widetilde{D}(s,\ t) = \sum_{k=0}^{n} \alpha_k(t)s^k$$

$$\widetilde{N}(s,\ t) = \sum_{j=0}^{m} \beta_j(t)s^k \tag{3-17}$$

and

$$Y_1(s) = Y(s)/H(s) \qquad X_1(s) = X(s)/H(s) \tag{3-18}$$

Let us now multiply Eq 3-13 by 1/H(s), which yields:

$$(1/H(s))\ D(s,\ t)\ Y(s) - (1/H(s))\ N(s,\ t)\ X(s) = 0 \tag{3-19}$$

If the plant were time-invariant ($N(s,\ t) = N(s)$ and $D(s,\ t) = D(s)$), then the operator (1/H(s)) could be commuted with D(s) and N(s). If this result is subtracted from Eq 3-16 there results an expression for the generalized equation error which is linear and homogeneous in the parameter differences:

$$E(s) = \left(\sum_{k=0}^{n} \Delta\alpha_k s^k\right) Y_1(s) - \left(\sum_{k=0}^{m} \Delta\beta_k s^k\right) X_1(s) \tag{3-20}$$

It is then possible to show conditions on the input such that $e \equiv 0$ if and only if

$$\widetilde{N}(s, t) = N(s) \text{ and } \widetilde{D}(s, t) = D(s)$$

In other words, $e \equiv 0$ if and only if the parameter values have converged. Asymptotic stability of the system is then assured if the parameters are adjusted along the gradient of $F = e^2/2$.

In the time-varying case, $(1/H(s))$ does not commute with $N(s, t)$ and $D(s, t)$. These results cannot, then, be rigorously proved. However, if the bandwidth of $(1/H(s))$ is sufficiently large, we have

$$Y_1(s) \doteq Y(s) \text{ and } X_1(s) \doteq X(s)$$

over the frequency range of interest. In that case, Eq 3-16 can be written:

$$E(s) \doteq \widetilde{D}(s, t)\, Y(s) - \widetilde{N}(s, t)\, X(s) \tag{3-21}$$

Subtracting Eq 3-13 from this leads to an expression equivalent to Eq 3-20, i.e.

$$E(s) \doteq \left(\sum_{k=1}^{n} \Delta\alpha_k s^k \right) Y(s) - \left(\sum_{k=0}^{m} \Delta\beta_k s^k \right) X(s) \tag{3-22}$$

(For the exact equation error, $1/H(s) = 1$, this equation is exact).

If the bandwidth of $1/H(s)$ is sufficient, then we may expect the results for the time-invariant case to hold approximately.

Using Eq 3-22, we can develop differential equations for the parameter differences in exactly the same manner as before. These equations have the form

$$\dot{\underline{\gamma}} = - kA(t)\, \underline{\gamma} + \dot{\underline{a}} \tag{3-23}$$

where again

$$\underline{\gamma} = \text{col}(\Delta\alpha_1, \ldots, \Delta\alpha_{n-1}; \Delta\beta_1, \ldots, \Delta\beta_n)$$

and

$$\underline{a} = \text{col}(a_1, \ldots, a_n; b_1, \ldots, b_n)$$

Thus, in the time-varying case, the equations are inhomogeneous and the forcing functions are the plant parameter rates of change. We might expect from Eq 3-23 that high plant parameter rates of change would lead to large values of $\underline{\gamma}$ and, therefore, to inaccurate tracking. Secondly, we might expect a lag in the tracking since the future behavior of $\dot{\underline{a}}$ is not known. A modest computer effort was undertaken to verify these predictions.

A functional block diagram of the set-up for the experiments is shown in Figure 3-16. The plant is a first order system with a single time-varying parameter, a(t). For this experiment a(t) is usually varied sinusoidally, although there are some runs where a(t) is a ramp. For the most part the input, as well, is a sinusoid although a few runs are with a ramp input.

The state variable filters are (arbitrarily) chosen to be:

$$F_k(s) = \frac{s^k}{H(s)} = \frac{s^k}{s+1} \tag{3-24}$$

These filters are used throughout the experiment. No attempt is made to "optimize" the tracking by changing the bandwidth or form of the cut-off filter, 1/H(s). In light of the foregoing analysis, it seems clear that the performance could be improved by using this degree of freedom to advantage.

The parameter adjustment law in this case is

$$\dot{\alpha} = - k\ e\ y. \tag{3-25}$$

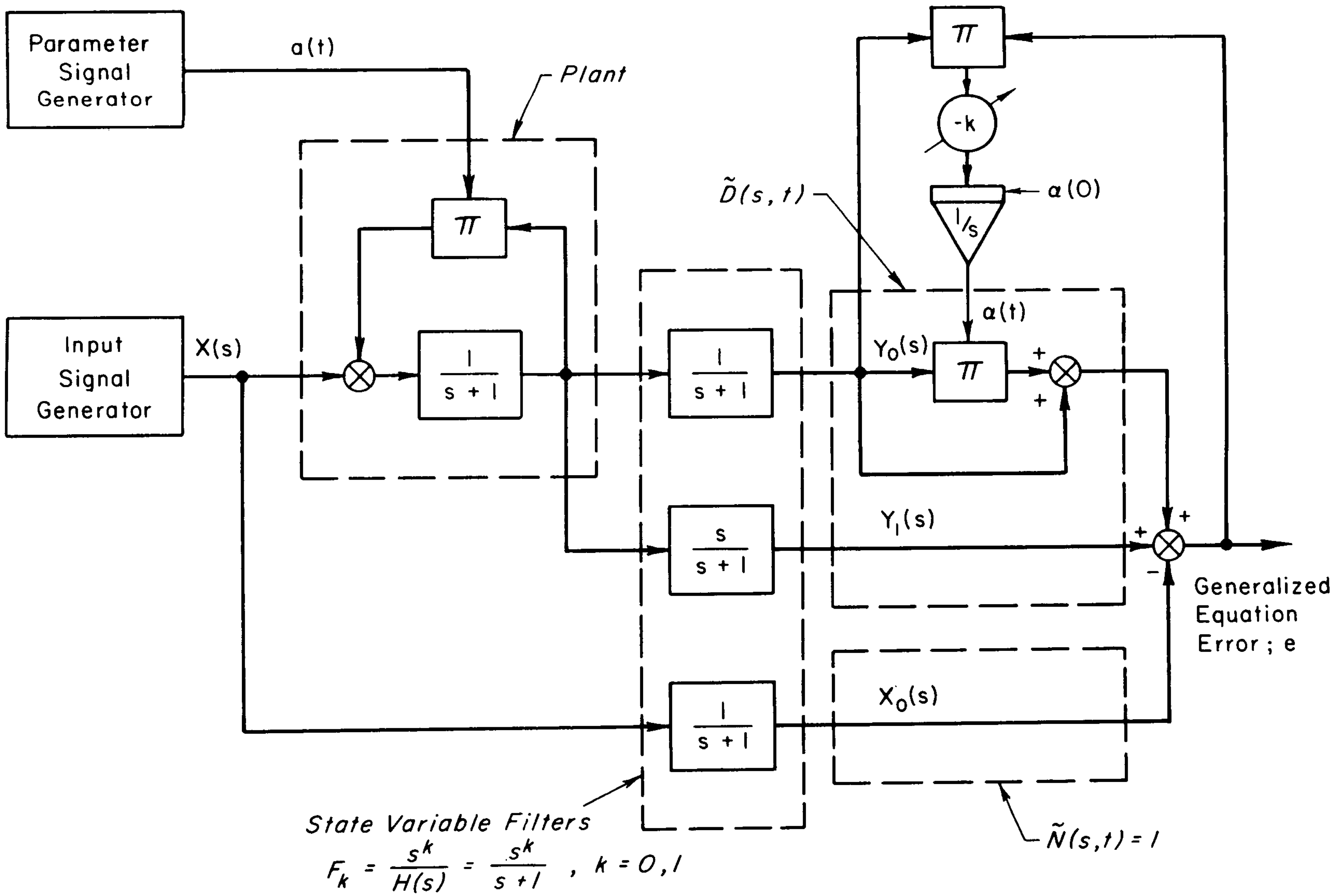


Figure 3-16. Functional Block Diagram for Tracking the Time-Varying Parameter of a Simple Plant

which leads to the following differential equation for the parameter difference, $\Delta\alpha = \alpha - a$:

$$\Delta\dot{\alpha} = - k\, y_1^2\, \Delta\alpha - \dot{a} \qquad (3\text{-}26)$$

(This equation is, of course, approximate as are Eq 3-22 and 3-23).

The objective of the parameter tracking system is to make $\Delta\alpha$ zero so that $\alpha(t)$ traces the same path as the unknown parameter, $a(t)$. This is impossible to do exactly since Eq 3-26 is inhomogeneous, and the forcing function, $\dot{a}$, is unknown in advance. However, it is possible to approach this ideal quite closely, as will be shown.

The major portion of the study uses a sinusoidally varying parameter and a sinusoidal input. The tracking performance is studied as a function of three things: input frequency, ω_i; parameter variation frequency, ω_a; and parameter adjustment loop gain, k. The frequencies range over the following values:

$$\omega_i = 0.7,\ 1,\ 1.4,\ 2.24,\ 3.17 \text{ rad/sec}$$

$$\omega_a = 0.1,\ 0.224,\ 0.317,\ 0.448 \text{ rad/sec}$$

Note that the input frequencies extend past the break frequency of the state variable filter.

Each of these combinations of frequencies is run at three different levels of parameter adjustment loop gain, which will be referred to as high, intermediate, and low.

Three different effects, or types of distortion, are observed in the parameter responses. These are bumpiness, amplitude reduction, and lag. The effect of changing the experimental variables on these types of distortion is summarized in Table III. A check mark indicates a strong dependency. The most serious of these distortions is probably amplitude reduction because it would be impossible to detect in an actual situation, and it would significantly affect any subsequent analyses based upon the distorted parameter estimates. Note that the second row is stated in terms of amplitude reduction. Therefore, it is desirable that each of the qualities in the table be decreased.

TABLE III

EFFECTS OF EXPERIMENTAL VARIABLES UPON DISTORTION OF THE PARAMETER RESPONSES

	EFFECT OF INCREASING		
	ω_i	ω_a	k
Bumpiness	decreases ✓	increases ✓	increases
Amplitude reduction	increases ✓	increases	decreases ✓
Lag	increases ✓	increases	decreases

One of the interesting points to note from Table III is that the most important parameter is ω_i, the input frequency, since it effects all three qualities strongly. To a large extent, however, the adverse effects of a high input frequency (on amplitude reduction and lag) can be compensated for by increasing the parameter adjustment loop gain. In addition, the one adverse effect of increasing gain (on bumpiness) is relatively small.

The effects of ω_i, ω_a and k can be explained, in a qualitative sense, by considering Eq 3-26. Increased lag and amplitude reduction both mean, in general, a larger value of $\Delta\alpha$. The homogeneous part of this equation is asymptotically stable and hence tends to reduce $\Delta\alpha$. The inhomogeneous part tends to increase $\Delta\alpha$. Increasing k increases the first term relative to the second, and hence we would expect the lag and amplitude reduction to decrease. Increasing ω_a and ω_i would have the opposite effect; i.e. the second term would increase relative to the first. We would expect lag and amplitude reduction to increase.

Bumpiness is an effect which results from high values of $\dot{\alpha}$. Equation 3-26 can be rewritten:

$$\dot{\alpha} + k\, y_1\, \alpha = - k\, y_1(\dot{y}_1 - x_1) \qquad (3\text{-}27)$$

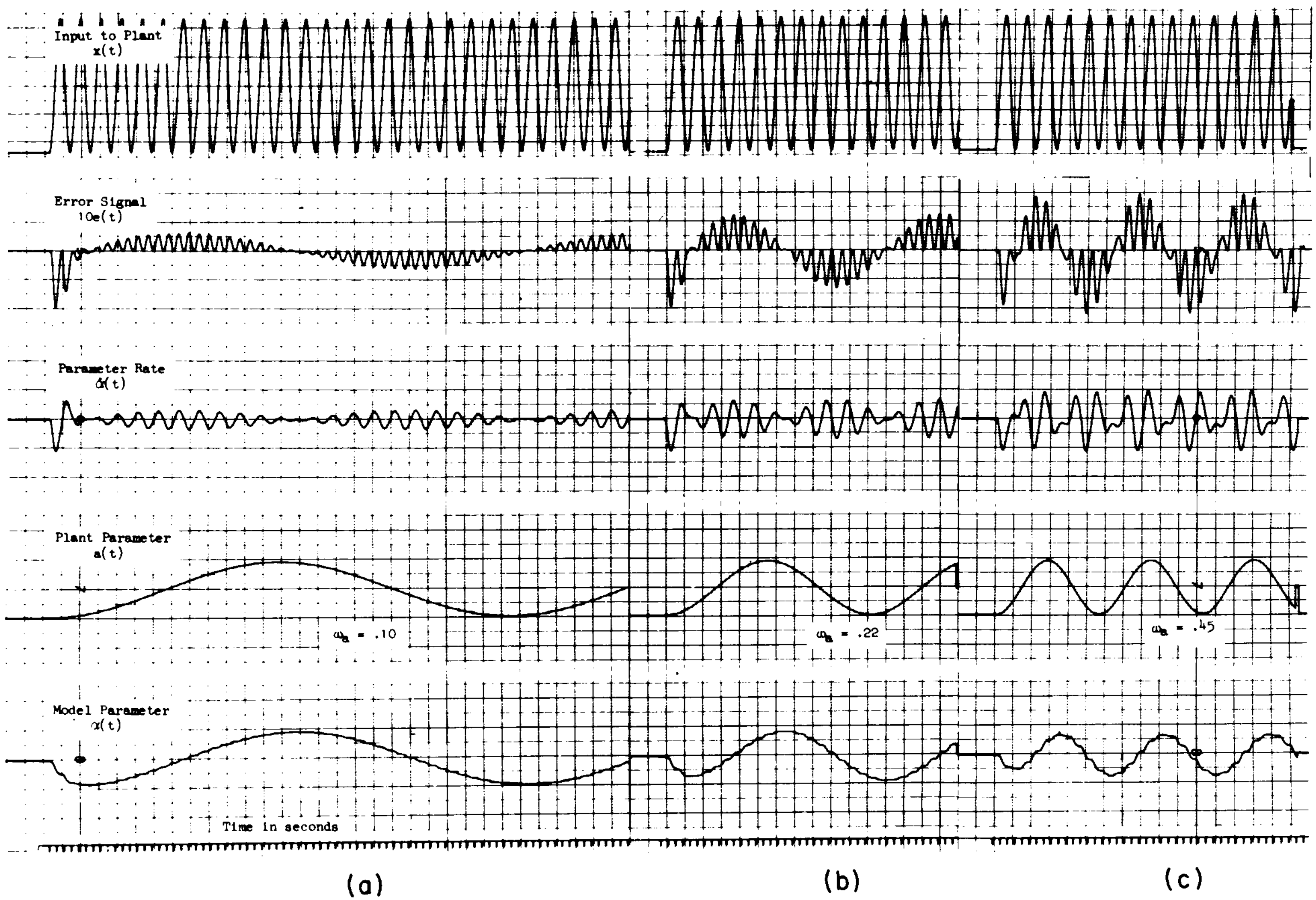
From this equation, it is easy to appreciate why an increase in k increases bumpiness. Also, since increasing ω_i reduces the amplitudes of y_1, $\dot{y}_1$, and x_1 because of the frequency response characteristic of $1/H(s)$, we might expect this to reduce bumpiness as indeed it does. Finally, the effect of increasing ω_a can be seen by examining Eq 3-26. An increase in ω_a increases $\dot{a}$, which leads to an increase in $\dot{\alpha}$.

Examples of the effects of increasing k, ω_i, and ω_a are shown in Figures 3-17 through 3-19, respectively. These runs are all for a sine wave input and sinusoidally varying parameter, a. Note that α was not initially at the correct value, but needed some time to "catch up."

Results involving ramp inputs and/or ramp variation of the parameter are shown in Figure 3-20. In Figure 3-20 a, the parameter variation is sinusoidal, but the input used is a ramp. Again the initial estimate $\alpha(0)$ is unequal to the correct value. The trace of $\alpha(t)$ shows that the system corrected this error rapidly and then tracked a(t) almost perfectly with only a slight lag.

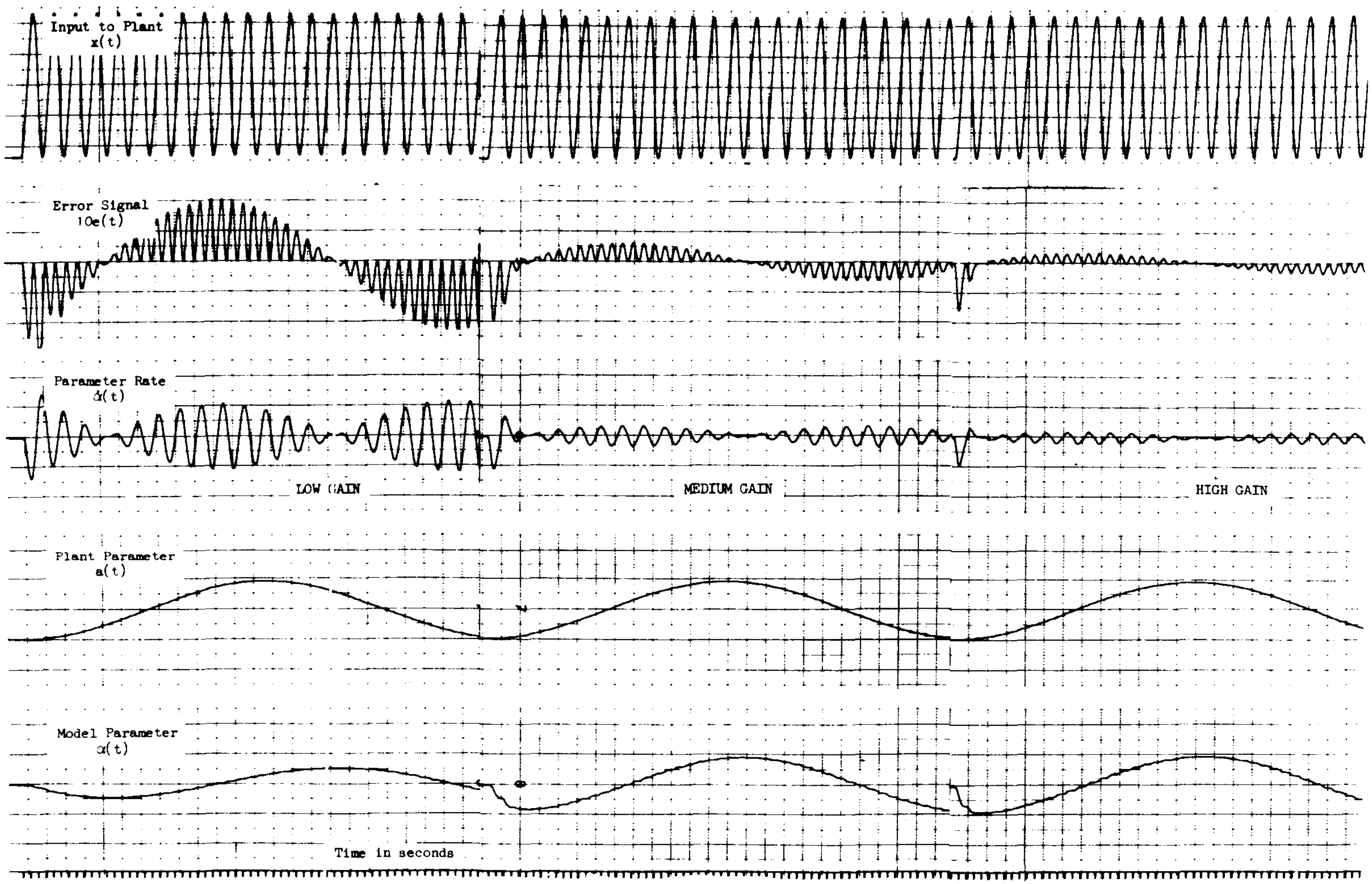
Figure 3-20 shows the case of a ramp variation in the parameter, $a(t) = c\, t$, and a sinusoidal input. The tracking system correctly reproduced the slope of the variation, c, with a very small lag and with little bumpiness. Again, the bumpiness decreases as input frequency increases.

Finally, the most intriguing run of all is the one with a ramp variation of the parameter and a ramp input. As can be seen in Figure 3-20 c, the run can be separated into two segments. After first lagging behind, $\alpha(t)$ catches up using a larger slope than the correct one. Once it has caught up, it changes slope abruptly (at point A) and is "on" for the remainder of the run.



ω_i = 2.2; Medium gain

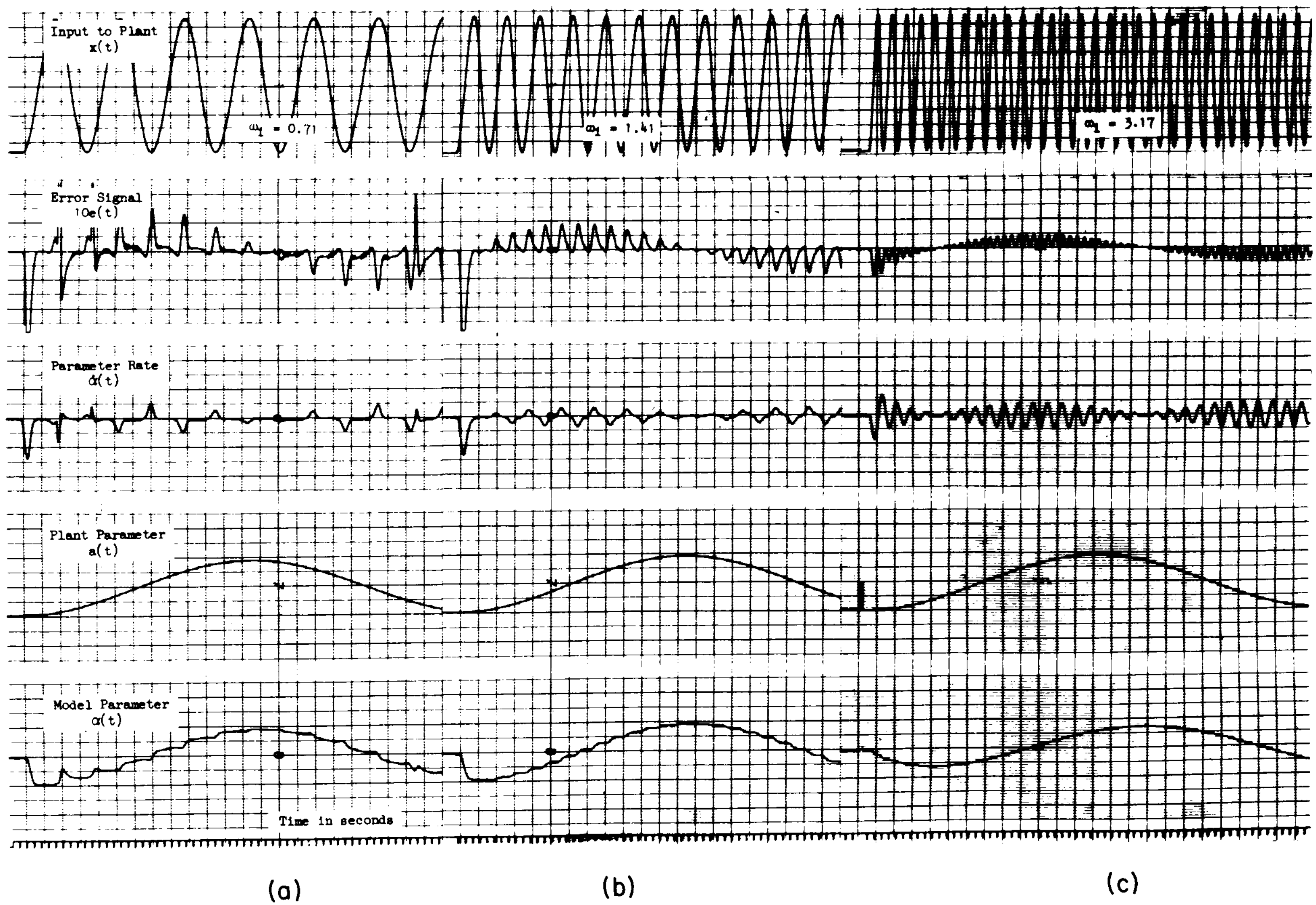
Figure 3-17 Effect of Increasing Parameter Frequency.



(a) (b) (c)

$\omega_i = 2.2$; $\omega_a = 0.1$

Figure 3-18 Effect of Increasing Gain.



Medium gain; $\omega_a = .1$

Figure 3-19 Effect of Increasing Input Frequency.

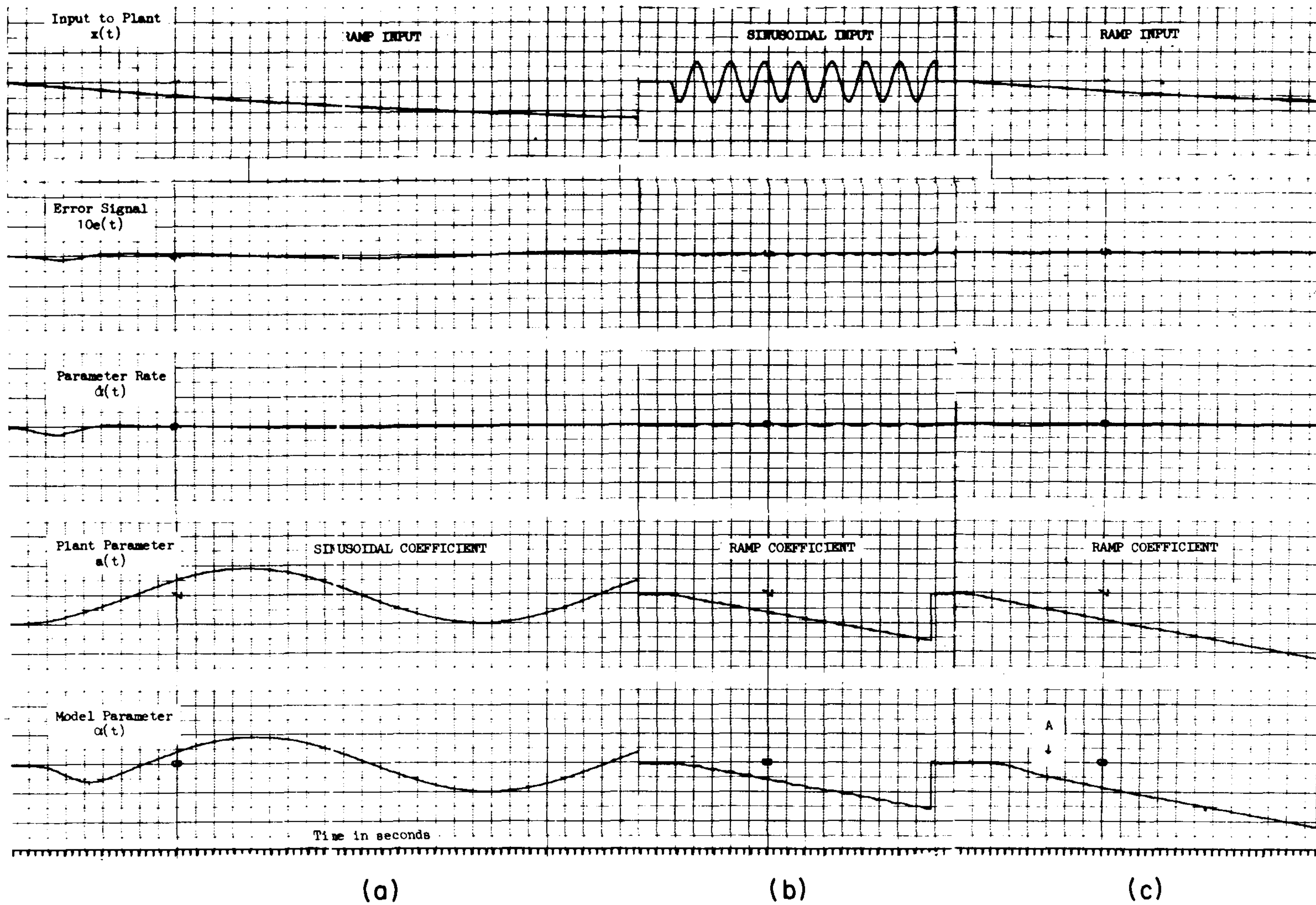


Figure 3-20 Ramp Input and/or Ramp Coefficient.

One conclusion which can be drawn from these last few runs, is that the data at the beginning of any run should be ignored to give the tracking parameters a chance to "catch up." The overall conclusion to be drawn from this study is highly encouraging: time-varying parameters can indeed be successfully tracked with only small errors provided the variation is not "too fast." Further research is indicated to add some quantitative meaning to the phrase "too fast" and to determine other things, such as the effect of two parameters varying, the effects of changing the filter $1/H(s)$, etc. Based on these preliminary data, this approach to the identification of time-varying systems seems profitable.

APPLICATION OF THE GENERALIZED EQUATION ERROR METHOD TO PILOT PARAMETER TRACKING

A simulated pilot parameter tracking experiment, chosen to exemplify practical application of the generalized equation error method, is discussed in this subsection. The object in this experiment is to identify the describing function parameters of a known, quasi-linear representation of a human pilot performing a compensatory control task. By comparing parameter values identified with the values known to exist in the simulation, the validity of the approach may be established.

The unknown plant in this example has been viewed by previous investigators as either the simulated pilot describing function or the entire closed-loop simulated pilot- vehicle system. Elements of the closed-loop other than the simulated human pilot are assumed known in either case. References 18, 19 and 20 point out that a proper quasi-linear model of a human pilot must include a remnant* signal as well as a linear transmission path. In the presence of the remnant, the only strictly correct viewpoint is to treat the entire closed-loop system as the unknown plant (Vide References 18 and 19.) although this might be unnecessary in certain cases. Exploration of this latter question is a secondary objective of this application since the validity of results obtained using pilot param-

*By definition the remnant is the signal, uncorrelated with the input to the closed-loop system, which must be included in the quasi-linear representation of an element in order that the output of the quasi-linear element model exactly equal that of the actual element for a given input.

eter tracking techniques, e.g. References 9, 13 and 21, has not yet been demonstrated when a remnant signal is included.

The functional block diagram of the simulated pilot-vehicle tracking system plus a generalized equation error parameter tracking system is shown in Figure 3-21. Switches indicate the capability for tracking parameters from the outside or inside of the loop closed around the simulated pilot, and for including a simulated remnant signal or not.

The linear constant coefficient describing function for the simulated pilot is:

$$Y_p = \frac{M(s)}{E(s)} = \frac{K(Ts+1)}{(\tau s+1)} \tag{3-28}$$

where $1/(\tau s+1)$ is a Padé approximation to the reaction time delay, $e^{-\tau s}$. The three parameters to be "tracked" are K, KT and τ. Although not shown in Figure 3-21, provision is made to introduce step and sinusoidal variations, δK_p($p \equiv$ plant or simulated pilot), in K from a constant value, K_0, in the simulated pilot describing function. Provision is also made to introduce step variations, δKT_p and $\delta\tau_p$ in KT and τ from constant values, KT_0 and τ_0, respectively in the simulated pilot describing function. Step changes; δK_m, δKT_m and $\delta\tau_m$($m \equiv$ model); can also be introduced into the model parameters. The object of the parameter tracking function is to null the quantities, ΔK, ΔKT and $\Delta\tau$, which are the differences between the respective model and describing function parameters.

A remnant signal model for the simulated pilot is constructed from the data in Reference 22. There, evidence is given supporting the proposition that the proper remnant power spectrum for injection at the error point in the pilot-vehicle loop is proportional to the input power spectrum to the closed-loop. The proportionality factor (remnant power/input power) is here taken to be 0.112.

The input spectrum to the closed-loop is white noise shaped by two first order lags with break frequencies at 1.0 rad/sec as in Reference 14. The filters for shaping the input and remnant spectra from unit white

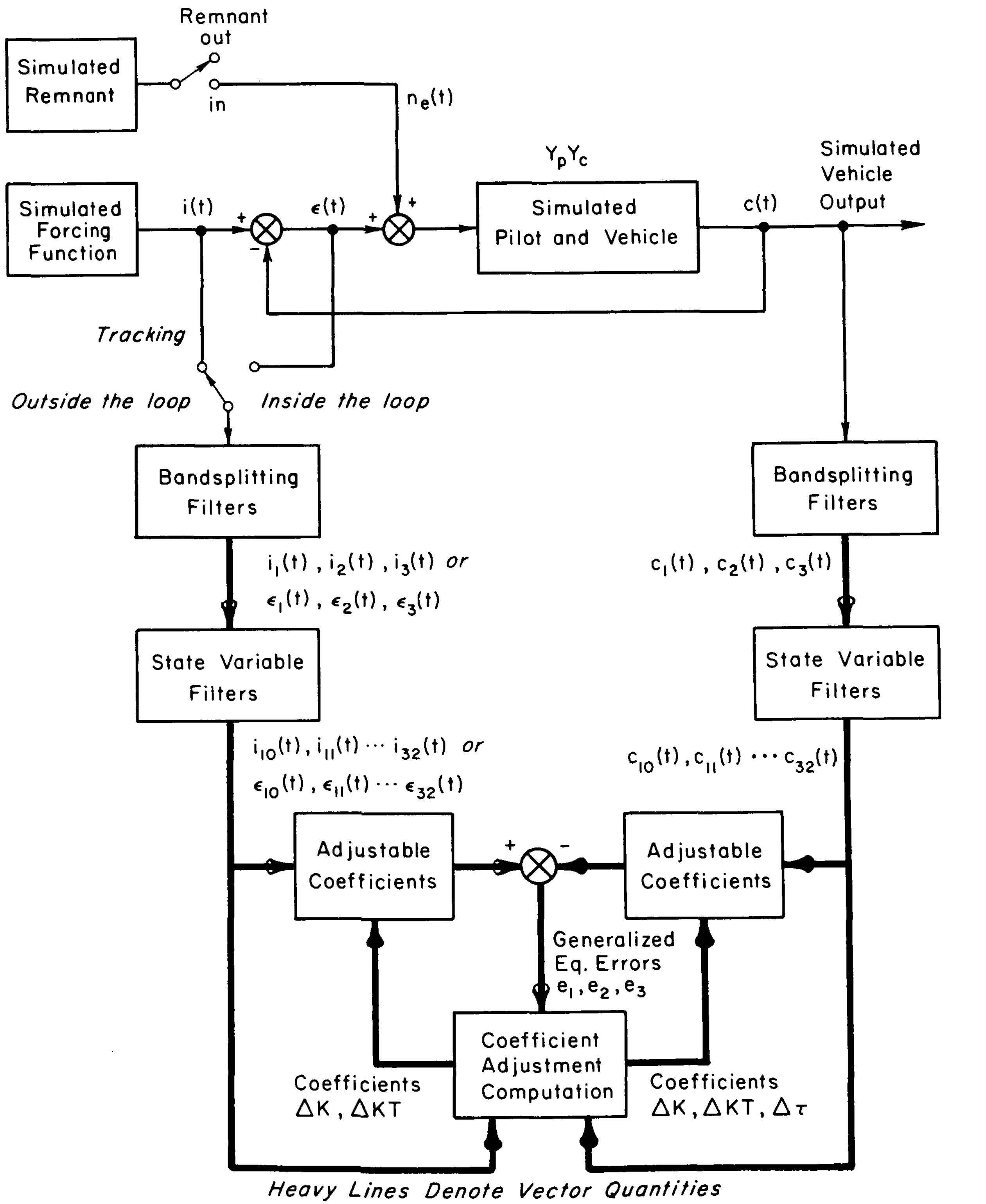


Figure 3.21. Functional Block Diagram of the Simulated Pilot Parameter Tracking Experiment

noise are respectively:

$$\frac{a}{(1+s)^2} \qquad \frac{0.334a}{(1+s)^2}$$

The constant, a, is adjusted to obtain the proper input signal level to the simulated closed-loop pilot-vehicle system.

$$\overline{i^2(t)} = (6.12)^2 \text{ (volts)}^2$$

The vehicle transfer function is:

$$Y_c = \frac{C(s)}{M(s)} = \frac{1}{s} \tag{3-29}$$

The transfer functions for the simulated closed-loop pilot-vehicle system are:

$$\frac{C(s)}{I(s)} = \frac{C(s)}{N_e(s)} = \frac{K(Ts+1)}{s(\tau s+1) + K(Ts+1)} \tag{3-30}$$

Handling qualities theory, e.g. Reference 22 can be used to show a proper pilot describing function for this choice for Y_c should be approximately:

$$Y_p = \frac{1.0(0.0s+1)}{(0.2s+1)} \tag{3-31}$$

These values are used in the simulated human pilot describing function. Despite the fact that the lead time constant is zero, we have made provision for its adjustment in the model. This presents an opportunity to adjust three parameters. This seems to have been the point at which previous investigators have bogged down using the equation error parameter tracking system approach.

Independent generalized equation errors in this parameter tracking system are generated by passing each measured signal from the simulated pilot-vehicle system through a bandsplitting filter. Each output of the bandsplitting filter is passed through a state variable filter, each section of which produces a low pass filtered derivative of its input signal. The remaining details of the parameter tracking system are executed in the customary way. (Vide Section II and Figure 3-21). This approach has several advantages:

1. It should tend to maximize the independence of the generalized equation errors, i.e. it should tend to maximize $\lambda(t)$ (Vide p. 20.)
2. It is the simplest configuration to mechanize in terms of the number of generalized computing elements required.
3. It provides low passed equation error as an output. This is valuable for estimating the system performance quality.
4. Only a modest number of approximate, successive differentiations is required using this approach. The number required is equal to the order of the model being used; two, in this case.

The bandsplitting filter consists of low pass, bandpass and high pass sections designed such that the sum of the section outputs equals the bandsplitting filter input. The bandpass section covers the frequency decade in the cross-over frequency region for $Y_p \, Y_c$. The low pass, bandpass and high pass sections, respectively, have transfer functions:

$$B_1(s) = \frac{0.316}{s+0.316} \qquad (3\text{-}32)$$

$$B_2(s) = \frac{(3.16-0.316)\ s}{(s+0.316)\ (s+3.16)} \qquad (3\text{-}33)$$

$$B_3(s) = \frac{s}{s+3.16} \qquad (3\text{-}34)$$

The state variable filter sections have transfer functions

$$F_n(s) = \frac{s^n}{H(s)} = \frac{s^n}{1+2\,\frac{(0.5)}{1.0}s + \frac{s^2}{(1.0)^2}} \qquad n = 0,1,2 \qquad (3\text{-}35)$$

where n denotes the section used to obtain a particular state variable (or approximate time derivative in this case).

The measured signals from the simulated pilot-vehicle system are $i(t)$ (or $\epsilon(t)$ when tracking inside the loop) and $c(t)$. Each drives a separate bandsplitting filter. The bandsplitting filter section outputs for the $i(t)$ input are: $i_1(t)$, $i_2(t)$ and $i_3(t)$. For the $c(t)$ input they are: $c_1(t)$, $c_2(t)$ and $c_3(t)$. To see how these signals are processed further, let us consider one typical variable, $c_3(t)$. The processing for any other variable would be the same.

$c_3(t)$ is a typical input to a state variable filter. The outputs of the F_0, F_1 and F_2 sections of this filter are, respectively, $c_{30}(t)$, $c_{31}(t)$ and $c_{32}(t)$. These are, respectively, the zeroth, first and second time derivatives of $c_3(t)$ after being low pass filtered by $1/H(s)$. Again, let us select a typical output variable, $c_{31}(t)$.

$c_{31}(t)$ is the first time derivative of the plant output when regarded as a component of the third generalized equation error, e_3. Hence $c_{31}(t)$ is then multiplied by the appropriate model parameter and is summed with similar terms in the other variables; $i_{30}(t)$, $i_{31}(t)$, $c_{30}(t)$ and $c_{32}(t)$; to form the generalized equation error, e_3.

The criterion function used is:

$$F(\underline{e}) = (e_1^2 + e_2^2 + e_3^2)/2 \tag{3-36}$$

The generalized equation errors, e_1 and e_2, are similar to e_3, only these are generated using the outputs from the $B_1(s)$ and $B_2(s)$ sections, respectively, of the bandsplitting filter. The partial derivative of the criterion function with respect to each model parameter is formed, and the rate of adjustment of each model parameter is made proportional to the negative of the partial derivative with respect to that parameter. For example,

$$\Delta\dot{K} = - A_K \frac{\partial F}{\partial K} \tag{3-37}$$

where A_K is the proportionality constant referred to as the K parameter adjustment loop gain.

The parameter adjustment loop gain matrix, K, used in this system is

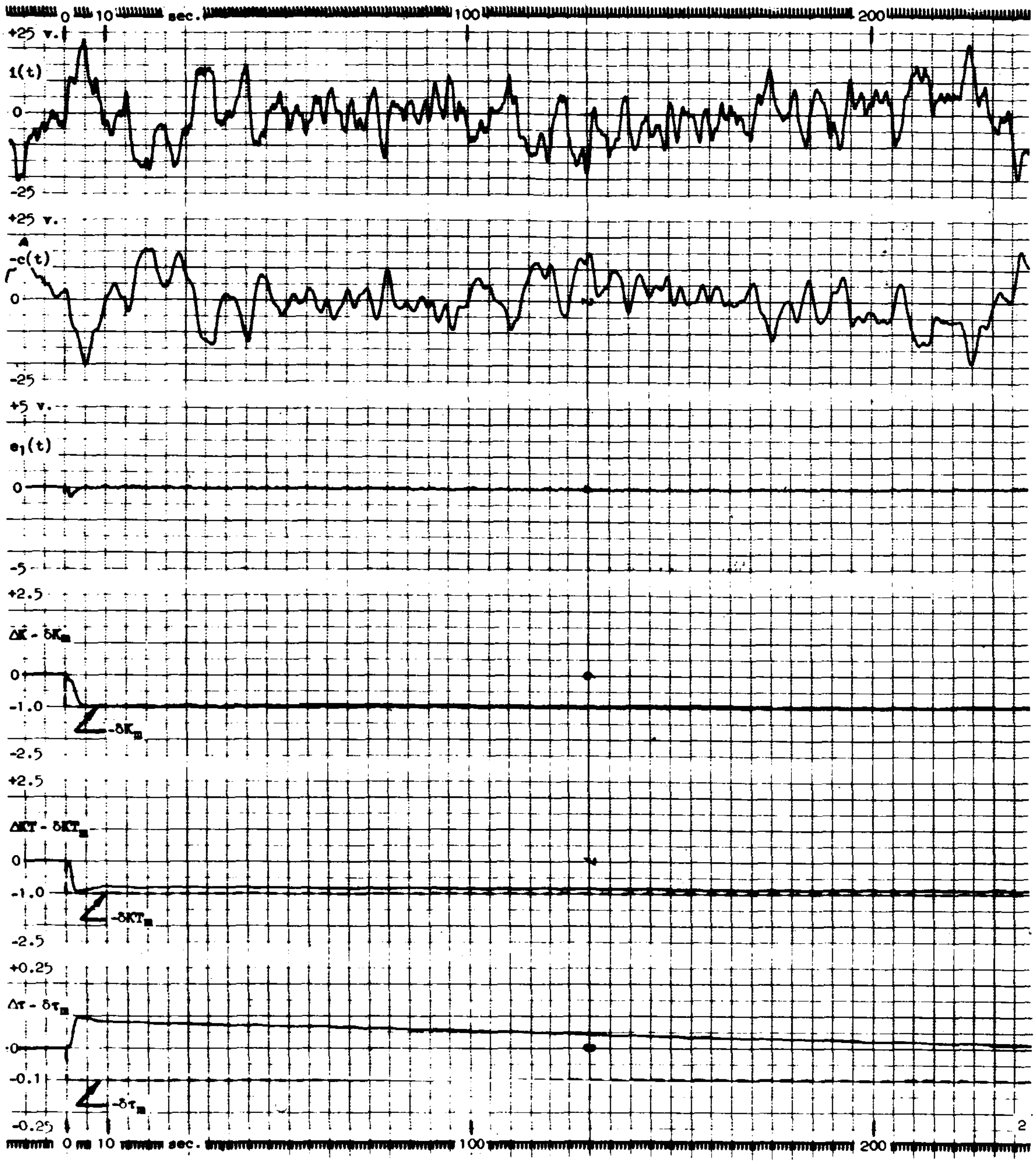
$$K = \begin{bmatrix} A_K & 0 & 0 \\ 0 & A_{KT} & 0 \\ 0 & 0 & A_\tau \end{bmatrix} = \begin{bmatrix} 0.01 & 0 & 0 \\ 0 & 0.01 & 0 \\ 0 & 0 & 0.001 \end{bmatrix} \qquad (3\text{-}38)$$

when three parameters are being adjusted. When two parameters are being adjusted, $A_\tau = 0$; and when only one parameter, K, is being adjusted, $A_\tau = A_{KT} = 0$.

First, we will show a typical result to demonstrate that the three-parameter tracking system operates as predicted in the absence of remnant. In Figure 3-22 responses of the three (Δ) parameters to step changes in the model parameters; δK_m, δKT_m and $\delta\tau_m$; are shown. The asymptotic values which the responses should approach are shown dashed. Here, tracking is performed from the outside of the loop. It makes little difference whether outside or inside the loop tracking is used as is to be expected when there is no remnant. In fact, even the shapes of the parameter responses are similar under these conditions.

To casual observation, the performance of the $\Delta\tau$ adjustment in Figure 3-22 might seem to be contrary to our theoretical expectation. However, it is well to remember that the theory requires only that the length of the parameter vector <u>in</u> ζ <u>coordinates</u> be monotonically decreasing, and that these time responses show differently scaled components of the parameter vector in γ coordinates.

For the parameter adjustment loop gains used in Figure 3-22 (given by Eq 3-38) the parameter coordinates in which a steepest descent on the surface, $F = \underline{e}'\underline{e}/2$, is being performed are:



Conditions: 3 adjustable parameters, outside the loop tracking, no remnant

Figure 3-22 Pilot Parameter Tracking System Responses to Model Parameter Perturbations.

$$\underline{\zeta} = K^{-1/2} \, \underline{\gamma} = \begin{Bmatrix} 10 \; \Delta K \\ 10 \; \Delta KT \\ 10 \sqrt{10} \; \Delta\tau \end{Bmatrix} \qquad (3\text{-}39)$$

To assure ourselves that $\| \underline{\zeta} \|$ is decreasing let us compute $\| \underline{\zeta} \|$ several times during the response. Taking the time in seconds at which the parameter perturbations are introduced as zero, it may be determined from Figure 3-22 that

$$\| \underline{\zeta}(0) \| = \left[10^2 + 10^2 + \left(\frac{10}{\sqrt{10}} \right)^2 \right]^{1/2} \doteq 14.5$$

$$\| \underline{\zeta}(2.5) \| \doteq 12.0$$

$$\| \underline{\zeta} = (10) \| \doteq 6.03$$

$$\| \underline{\zeta} = (100) \| \doteq 5.19$$

The norm of $\underline{\zeta}$ is decreasing monotonically as it should.

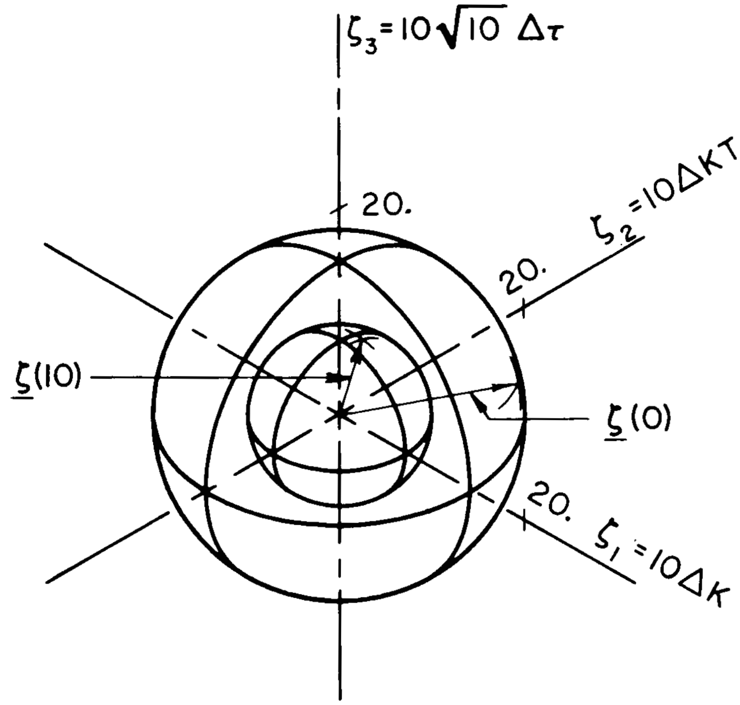
The next observation is that the $\Delta\tau$ response is quite slow after the first five seconds of the response, and furthermore, that $\Delta\tau$ is larger than it was at t = 0 although it is moving in the right direction. The fact that $\Delta\tau$ has increased might ordinarily lessen our confidence in the system's performance. Let us consider some implications of these points.

Notice that the (low pass filtered) equation error, e_1, is quite small compared to the plant input and output after the first five seconds of parameter adjustment. This indicates that, from the equation error viewpoint, the model resembles the plant quite closely after t = 5 sec despite the large deviation of $\Delta\tau$ from its asymptotic value. This is reasonable because the effects of τ are most apparent in the vicinity of 5 rad/sec, and beyond, while the bandwidth of the input extends only to 1.0 rad/sec, and the cut-off filter of the approximate differentiations has a break frequency of 1.0 rad/sec. A way to correct this problem is to weight the high frequency components of the error vector more heavily. When a bandsplitting filter is used to obtain the independent generalized equation errors which are the components of the error vector, it is a simple matter to weight the error vector components according to frequency because each generalized equation error emphasizes a different frequency region. Weighting is then a matter of choosing the positive elements for the diagonal matrix, Q, and using $F = \underline{e}'Q\underline{e}/2$ as the criterion function instead of $F = \underline{e}'\underline{e}/2$. Another way to correct this problem is to increase each parameter adjustment loop gain by the same factor.

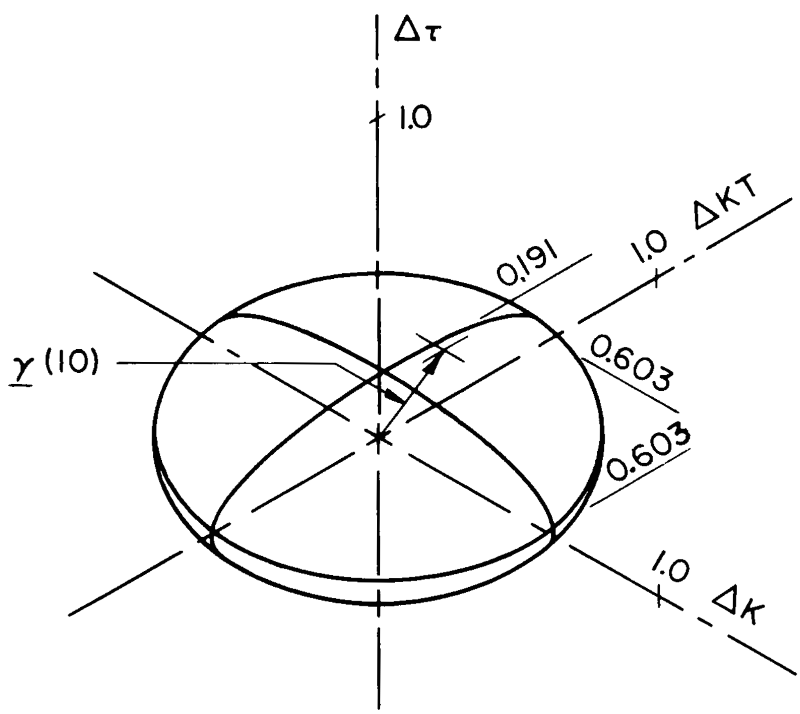
Next, let us look at the spheroids described by the norm of the parameter vector in ζ coordinates at two particular times in Figure 3-23 a. First, let us note that when the length of any component of the parameter vector approaches the length of the parameter vector norm, the length of the other components of the vector approach zero. This is apparent for the $\underline{\zeta}(10)$ vector in Figure 3-23 a wherein $10\sqrt{10}\,\Delta\tau$ is approaching the norm while $10\,\Delta K$ and $10\,\Delta KT$ are approaching zero. Thus we may conclude that when one or more components of $\underline{\zeta}$ increase from their initial values, the remaining components of $\underline{\zeta}$ approach origin of the ζ coordinates more closely at a given value of $\|\underline{\zeta}\|$ than if all components decreased simultaneously. Secondly, by transforming the spheroid of radius $\|\underline{\zeta}(10)\|$ into γ coordinates as shown in Figure 3-23 b, we can determine the maximum uncertainty in each parameter vector component in γ coordinates, presumably the coordinates of interest. More than this may be said, however. Note that the transformation from ζ to γ coordinates is

$$\underline{\gamma} = \underline{K}^{1/2}\,\underline{\zeta} \qquad (3\text{-}40)$$

which implies that the relative maximum uncertainty in each parameter vector in γ coordinates is controlled by the designer's choice of the positive elements in the diagonal matrix, K. Thus, by deciding the relative deviation from the origin that can be tolerated for each component of the parameter vector in γ coordinates, the relative values of the elements of the matrix, K may be chosen. In order to be assured of reducing the norm of the parameter vector in ζ coordinates to a certain fraction of the initial norm of the parameter vector in ζ coordinates, it is sufficient to set



(a) Spheroid



(b) Ellipsoid

Figure 3-23. Spheroid of Parameter Values in ζ Coordinates and a Corresponding Ellipsoid in γ Coordinates

$$e^{-k \int_0^T \lambda(\tau)\, d\tau} \quad *$$

equal to that fraction. Solution gives different combinations of k and T for a given K which will accomplish this. T is here the run length time required. In order that $\| \underline{\zeta}(T) \|$ be smaller than a given constant, c, i.e. the deviation from the origin for each component of the parameter vector in γ coordinates at time, T, must be smaller than certain constants for the respective components; it is sufficient that the initial norm of the parameter vector in ζ coordinates be smaller than:

$$c\, e^{k \int_0^T \lambda(\tau)\, d\tau}$$

This implies an ellipsoid in γ coordinates within which the initial parameter values must lie which, in turn, implies a region in which the initial parameters of the model must lie.

Because τ parameter deviations do not give rise to any significant portion of the equation errors for the configuration of parameter tracking system used, the remainder of the experiments is conducted with $\Delta\tau$ set equal to zero. While the $\Delta\tau$ parameter could have been retained, very long times would be required for the parameters to approach their asymptotic values because of the limited gain available in the simulation. On the other hand, changing the configuration of the system so that $F = \underline{e}'Q\, \underline{e}/2$ could be used would also require more gain than was available. Figure 3-24 shows the response to step perturbations in the model parameters, δK_m and δKT_m, of the two parameter system tracking from outside the loop in the absence of remnant. All possible combinations of algebraic sign are used for the perturbations. Responses from all combinations of ± 100% model parameter perturbations are well behaved and rapid. Figure 3-25 shows responses to the same perturbations for inside the loop tracking.

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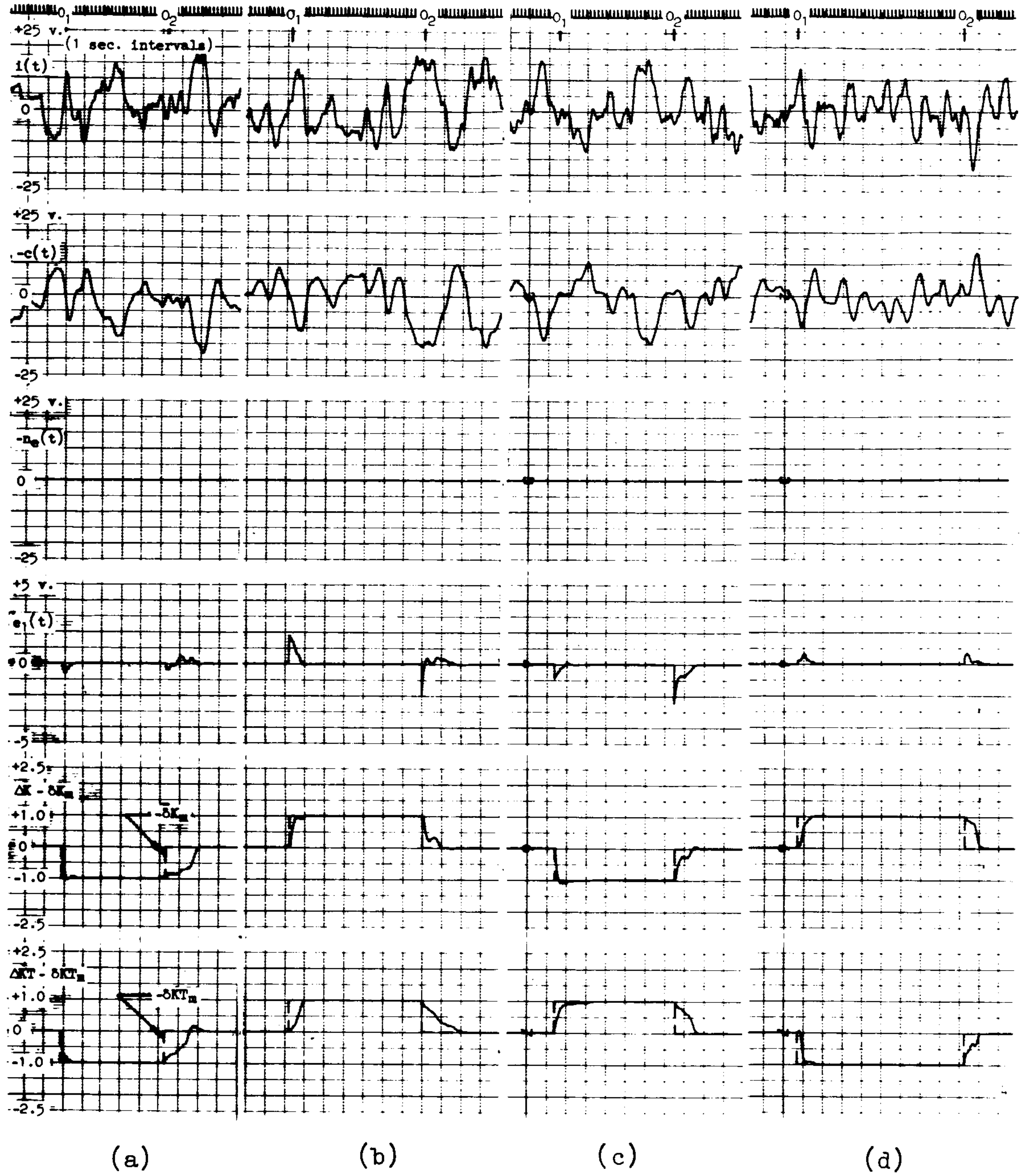
*Here the parameter adjustment law is assumed to be:

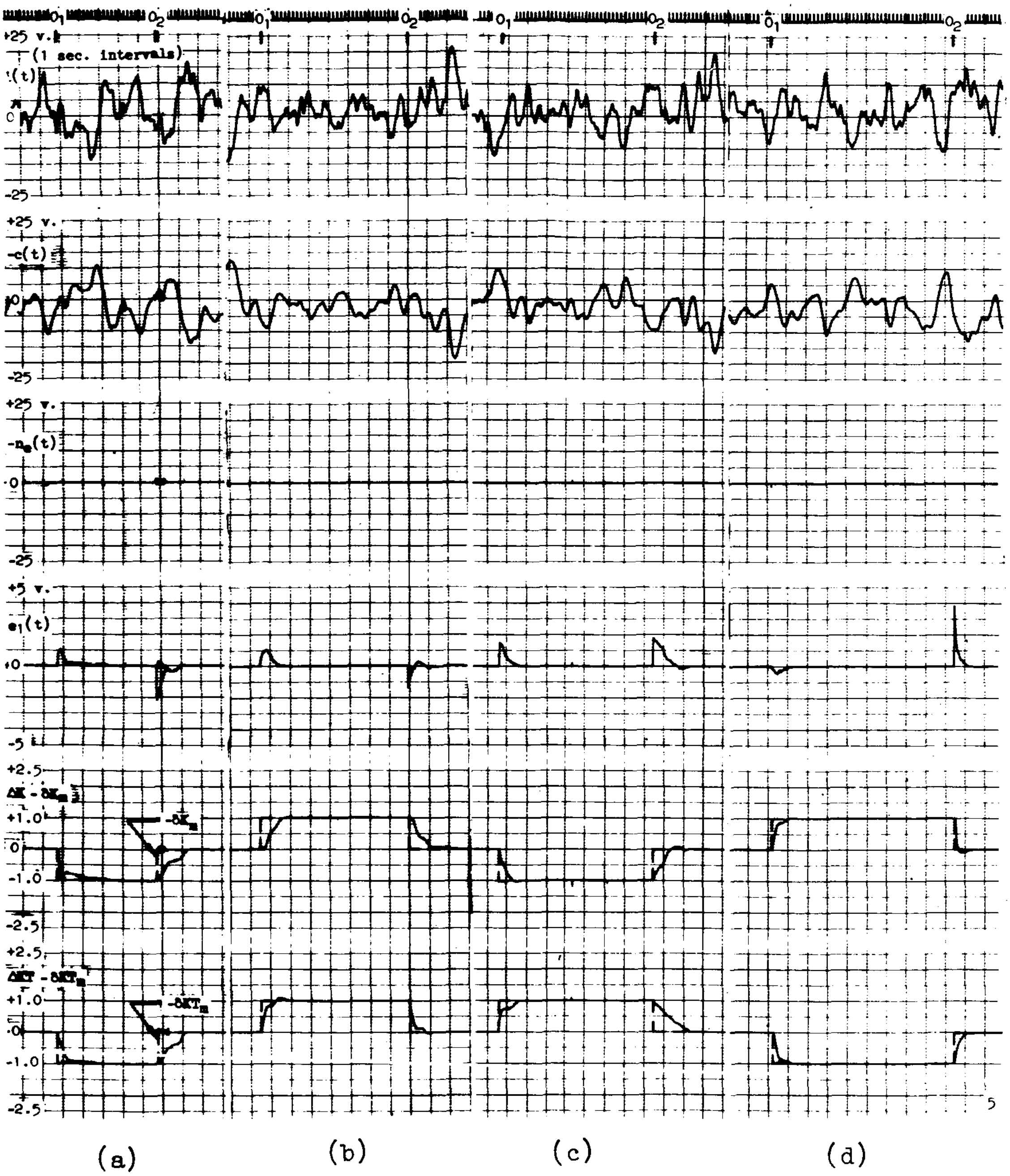
$$\dot{\underline{\gamma}} = - k\, K\, A\, \underline{\gamma} \qquad (3\text{-}41)$$

$\lambda(t)$ is the mimimum eigenvalue of $K^{1/2} A\, K^{1/2}$.



Conditions: 2 adjustable parameters, outside the loop tracking, no remnant

Figure 3-24 Pilot Parameter Tracking System Responses to Model Parameter Perturbations.



Conditions: 2 adjustable parameters, inside the loop tracking, no remnant

Figure 3-25 Pilot Parameter Tracking System Responses to Model Parameter Perturbations.

There is here no essential difference in the performance for inside and outside the loop tracking, when there is no remnant. This is, of course, the performance which is expected.

The same experiments are rerun including the remnant input. Figure 3-26 shows responses comparable to the traces of Figures 3-24 a and 3-25 a. Notice that the remnant gives rise to substantial variation of the adjusting parameters in comparison to their behavior when there is no remnant. In fact, it would be difficult to identify correct parameter values from these data.

The rapid variations in the parameters also mask the so-called parameter biases predicted for inside the loop tracking by Elkind (Reference 18) and Rucker (Reference 5). We hoped these would be clearly observable in Figure 3-26 b, but not in Figure 3-26 a.

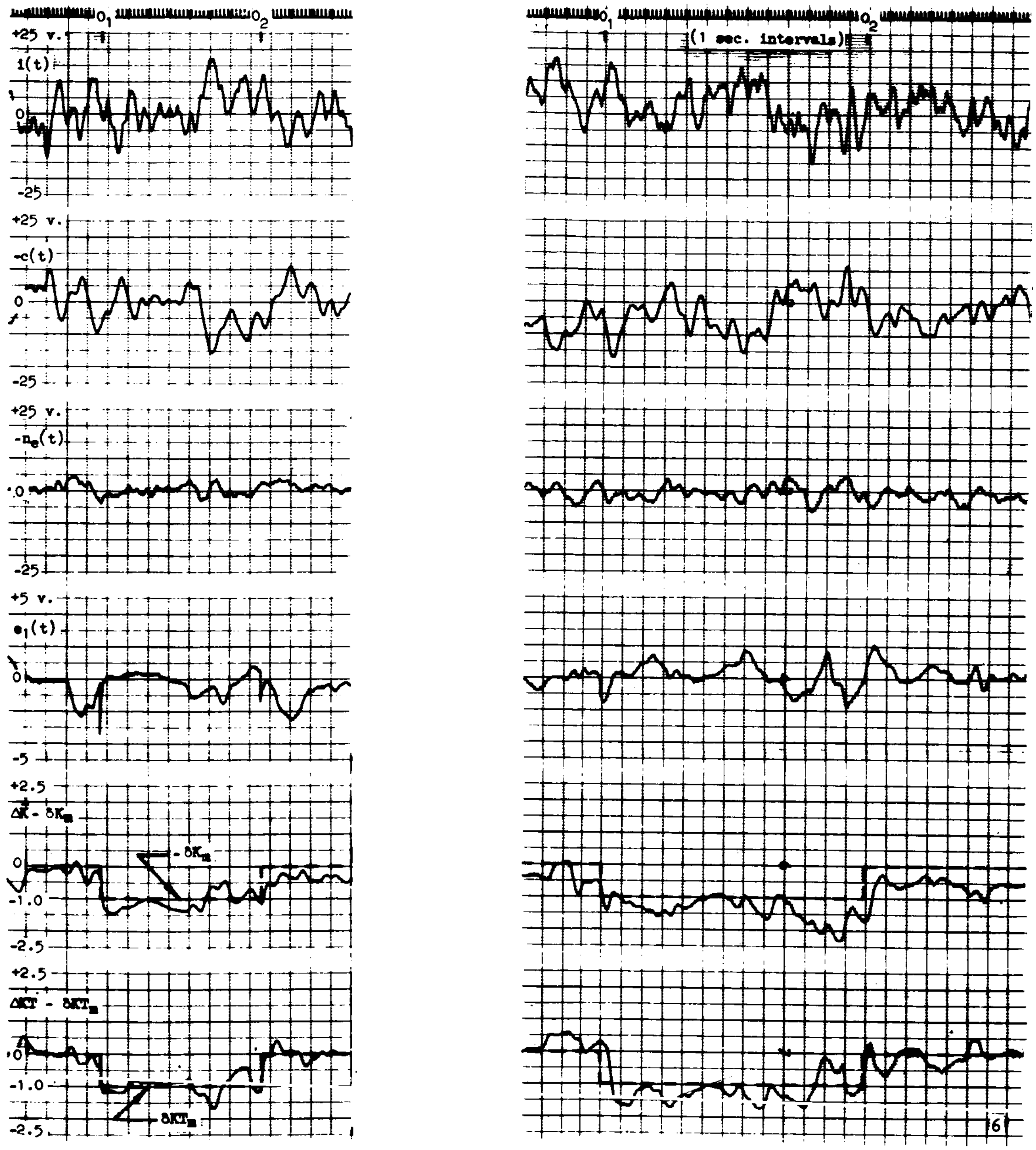
Figure 3-27 shows parameter responses comparable to the responses in Figure 3-24 a (outside the loop tracking, no remnant) and Figure 3-26 a (outside the loop tracking, with remnant) for parameter perturbations made in the plant. The first response (no remnant) shows that the parameter responses are similar to, although slower than, responses for perturbations in the model parameters. Responses are slower for two reasons:

1. Information containing the effect of the plant parameter perturbation is delayed by the state variable filter. That is, it is delayed by $1/H(s)$ which in this case is:

$$\frac{1}{H(s)} = \frac{1}{1+2\,\frac{(0.5)}{\sqrt{10}}\,s+\frac{s^2}{10}} \qquad (3\text{-}42)$$

2. Perturbations in the plant parameters produce a transient effect because of the temporary time-varying quality of the coefficients. In order to identify the constant model of the plant, this transient must die out so that the plant may again be considered constant coefficient in nature.

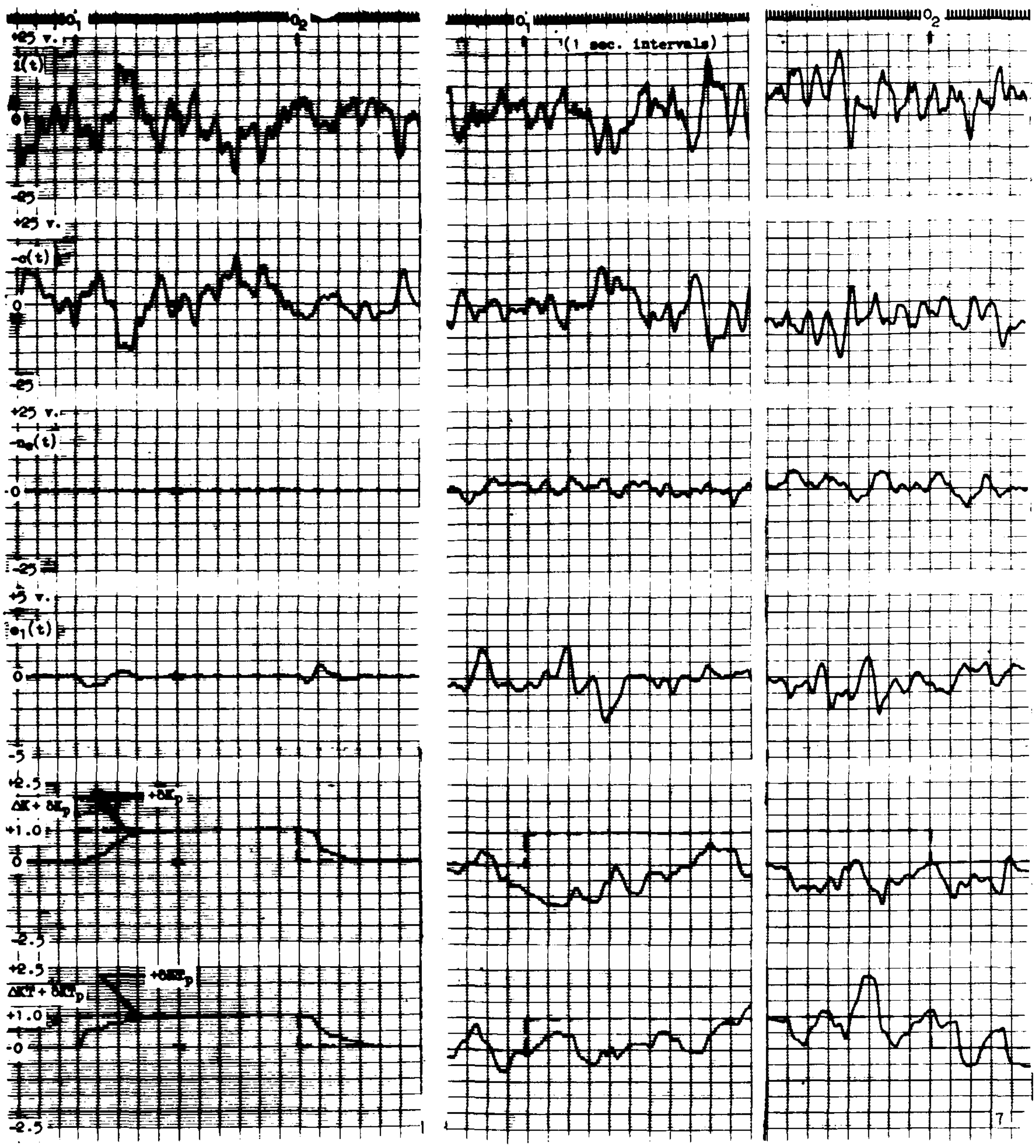
The second response, Figure 3-27 b is a repeat of the first with remnant included. This record is comparable to that of Figure 3-26 a. Here it is evident that parameter tracking system fails to cope with



(a) Outside the loop tracking. (b) Inside the loop tracking.

Conditions: 2 adjustable parameters, remnant included.

Figure 3-26 Pilot Parameter Tracking System Responses to Model Parameter Perturbations.



(a) No remnant

(b) Remnant included

Conditions: 2 adjustable parameters, outside the loop tracking.

Figure 3-27 Pilot Parameter Tracking System Responses to Plant Parameter Perturbations.

the remnant.

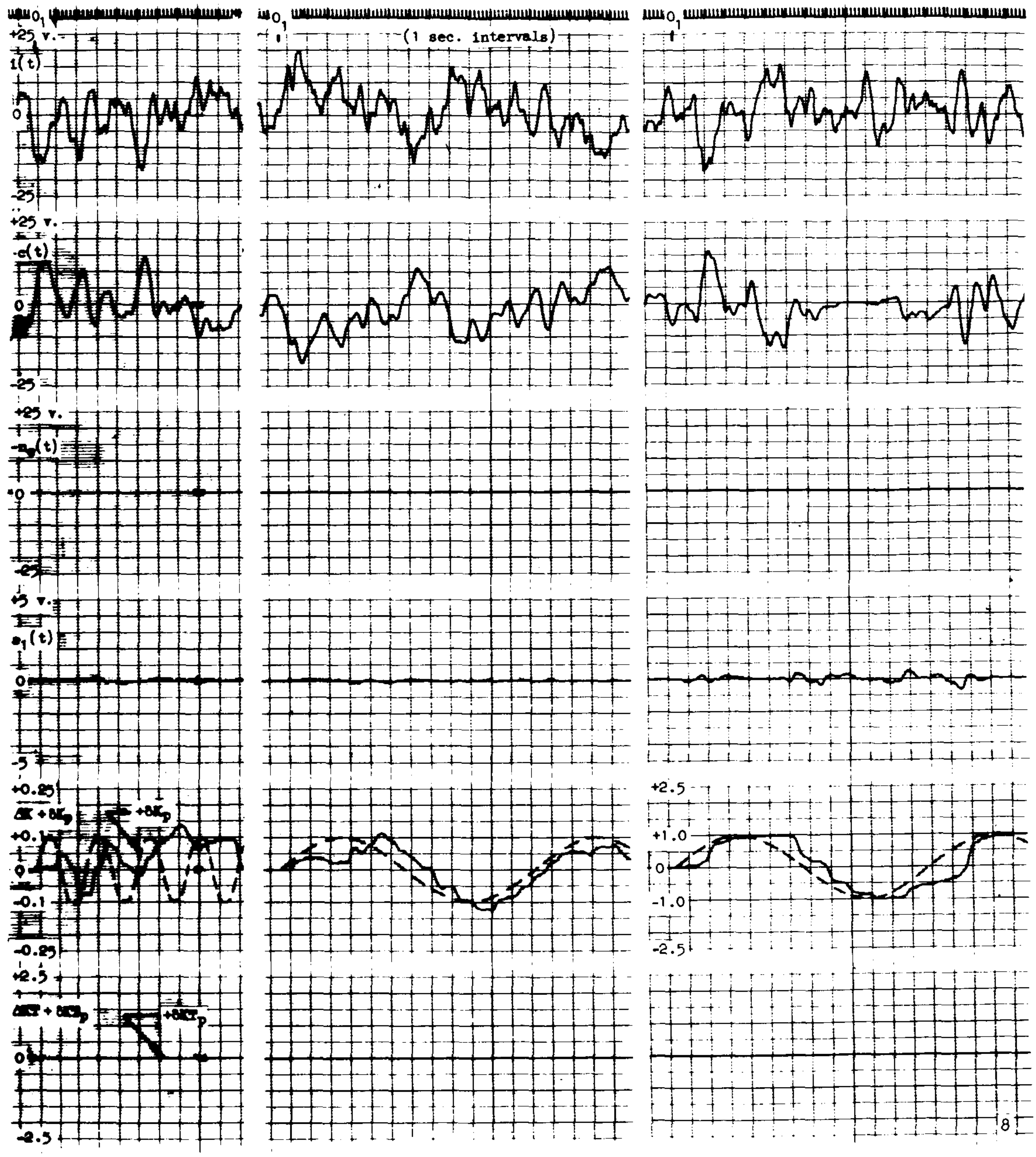
Tracking a sinusoidally varying K in the plant is shown in Figure 3-28. Here KT is fixed at its correct value in the model. The results indicate that the ability to track time-varying parameters increases with decreasing amplitude and/or frequency of the parameter time-variation. When the ΔKT parameter is free to track also, it is observed that its interaction with the ΔK parameter is slight whenever conditions are such that the low-passed equation error is held to small values by the parameter tracking action. Here again, it is found that the parameter tracking system could not operate successfully in the presence of remnant.

Conclusions drawn from this series of experiments are:

1. The pilot parameter tracking system works as predicted by the theory in the absence of remnant. However, because there is considerable evidence to support the existence of the remnant as part of a human pilot model, and the pilot parameter tracking system as simulated was not truly capable of coping with the remnant, we can only conclude that such an application needs more research before pilot parameter tracking constitutes a well established technique. Since we have previously demonstrated the qualitative equivalence of equation error and response error systems, this conclusion places data previously obtained using any pilot parameter tracking technique in a questionable position. Much of this "questionable" data, however, agrees well with results obtained by rigorously established techniques. We therefore have reason to believe that additional research specifically oriented toward the remnant problem would be successful in bringing pilot parameter tracking to a satisfactory status.

2. The so-called parameter biases because of remnant in equation error systems have not been observed in the simulated pilot parameter tracking experiments.

3. One way around the problems caused by remnant induced fluctuations (and biases) in the tracking parameters is to use a quasi-random input to the closed-loop system which is a sum of non-harmonically related sine waves with random phases. The research reported in Reference 22 contains measurements that tend to show that pilot remnant has a continuous power spectrum even when the input spectrum has power only at discrete frequencies. This fact may be used to advantage to discriminate against remnant effects. Signals



(a) $\delta K_p = 0.1 \sin 0.5t$

(b) $\delta K_p = 0.1 \sin 0.1t$

(c) $\delta K_p = 1.0 \sin 0.1t$

Conditions: 1 adjustable parameter, outside the loop tracking, no remnant.

Figure 3-28 Pilot Parameter Tracking System Responses to Sinusoidal Plant Parameter Variation.

containing a remnant component may be bandpass filtered at the sinusoidal input frequencies. Note that bandpass filters would perform a function similar to the band-splitting filters used in the simulated pilot parameter tracking experiments described above. That is, they would enable linearly independent generalized equation errors to be formed. However, the signal to noise ratio of the generalized equation errors generated using bandpass filters will be improved by a factor which is approximately the inverse of the filter pass-band. Parameter bias would also be appropriately reduced. Of course, it is obvious here that a special input signal to the closed-loop system is required.

4. The experiments (without remnant) indicate that it might be feasible to track pilot parameters which vary fairly slowly. An example of such a situation might be typified by the adaption of a pilot to a change in vehicle transfer function.

SECTION IV

RECOMMENDATIONS FOR FURTHER STUDY

Areas for research on parameter tracking which seem most likely to yield significant results for only modest expenditures of effort are summarized in this Section. These areas are:

- Analysis of simplified parameter tracking system mechanizations.
- Development of techniques for suppressing the effects of measurement noise.
- Engineering analyses and experiments to determine parameter tracking system capability to track time-varying plant parameters.
- Validation of alternative pilot parameter tracking techniques.

We shall consider each area in turn below.

Table I shows that a considerable number of analog multipliers (p x q) is required to mechanize the most desirable criterion function, F_2, when p parameters are to be adjusted. If a sufficient number of error vector components are included so that a non-increasing upper bound on the norm of the parameter vector can be found, then $q = p$. Then p^2 multipliers are required to obtain $\partial F_2/\partial \underline{\gamma}$ from $\underline{e}$. From a practical viewpoint, it is clear that if these multipliers could be replaced by simpler logic elements (relays), one major equipment requirement would be considerably lessened. This would make the parameter tracking technique available to a broader class of users having access to analog computers of modest capacity. On the other hand, such a development would also enable special purpose parameter tracking computers to be realized at lower cost. At the present time, however, additional theoretical development is necessary to establish that such simplified systems would perform satisfactorily with regard to stability and convergence rate. One configuration which might be investigated easily would use

$$\dot{\underline{\gamma}} = - k \frac{\partial \underline{e}'}{\partial \underline{\gamma}} \text{ sgn } \underline{e} \tag{4-1}$$

where

$$\text{sgn } \underline{e} = \text{col}(\text{sgn } e_1, \text{ sgn } e_2, \dots \text{ sgn } e_q) \tag{4-2}$$

for the parameter adjustment law. Another adjustment law for which the rate of parameter adjustment is independent of the plant input magnitude is also of interest. This is:

$$\dot{\underline{\gamma}} = -k \text{ sgn } \frac{\partial \underline{e}'}{\partial \underline{\gamma}} \text{ sgn } \underline{e} \tag{4-3}$$

In Section III, the so-called measurement noise is shown to impose a significant performance limitation on parameter tracking systems using the equation error approach. If we are to avoid using special plant inputs, the only remaining degree of freedom through which the system may be optimized is by choice of the cut-off filter, $1/H(s)$. That this approach can be successful is shown empirically in Section III. What is needed here is a sound engineering approach for determining analytically what the cut-off filter transfer function should be. One straightforward analytical approach, described on page 52 of Section III, was found to be intractable. It may be that a more sophisticated approach would prove fruitful. An alternative might be to merely minimize the noise to signal ratio of the equation error with respect to the parameters of the cut-off filter, or to determine the optimal cut-off filter for minimizing that quantity.

While the problem of identifying the parameters of a linear time invariant noise-free plant via parameter tracking has effectively been solved in Section II, we have considered only briefly, in Section III, a simple case where one plant parameter is time-varying. Extensions of the approach taken in Section III would be valuable, in establishing more firmly, suitable techniques for analysis. These analyses, however, must, of necessity, be approximate. They would serve to determine bounds upon parameter tracking performance, such as upon the tracking error as a function of the cut-off filter bandwidth, parameter adjustment loop gain and input spectral characteristics. The bounds should also be determined experimentally to check the regions over which the approximate analyses might be expected to be accurate.

The fact that some question can be raised as to the complete validity of the pilot parameter tracking technique in the presence of remnant, makes research on this point a pressing matter. For the reason explained on page 83, we have good cause to believe that complete validation is largely a matter of picking up a number of loose ends. Two approaches for accomplishing this are suggested below.

Research results reported here and the work reported by Weirwille in Reference 23 might be combined to develop an analytical synthesis of a response error pilot parameter tracking system. Briefly, an appropriate expansion of the model equations such as in Appendix A plus Weirwille's viewpoint on appropriate criteria for real-time determination of the best constant coefficient model may be used to show the proper form of the model equations for synthesis. It is suspected that the proper form will be similar to that reported by Meissinger in Reference 24. It is also expected that in the properly synthesized parameter tracking system, the response error will be an algebraic function of the adjustable parameters enabling a steepest descent solution to actually be realized. Thus, on-line solutions for the parameters would be achievable to any desired degree of accuracy in the absence of remnant provided certain sufficiency conditions on the input to the plant, similar to those developed in Section II, are satisfied. It appears that even more may be said about the performance of the system in the presence of remnant if the remnant meets certain sufficiency conditions. If this is the case, it would appear that the model parameters will be asymptotically stable about the values present in the plant. An approach similar to that used by Elkind (Reference 8) might be used to evaluate the effect of remnant on parameter values at any given time.

According to Elkind, parameter values determined by the equation error pilot parameter tracking method will be biased because of remnant effects. A pressing need, then, is to find a practical way around this point. The necessity for this is all the more apparent in the light of results from the simulated pilot parameter tracking experiment reported here. The large fluctuations observable in the tracking parameters are because of remnant at signal to noise ratios of three (a typical value for pilot-compensatory tracking systems). In the absence of remnant the parameter tracking system operated correctly and as predicted by the theory.

One way around both of these problems is to use a quasi-random input to the closed-loop system which is a sum of non-harmonically related sine waves with random phases. The research reported in Reference 22 contains measurements that tend to show that pilot remnant has a continuous power spectrum when the input spectrum has power only at discrete frequencies. This fact may be used to advantage to discriminate against remnant effects. Signals containing a remnant component may be bandpass filtered at the sinusoidal input frequencies. The signal to noise ratio of the output signal with respect to the original signal will then be improved by a factor which is approximately the inverse of the filter pass-band. Parameter bias would also be appropriately reduced. Of course, it is obvious that a special input is required. However, this input is of a sufficiently general form, being quasi-random, to be tailored for most laboratory applications.

APPENDIX A

DERIVATION OF TIME-VARYING WEIGHTING FILTERS FOR RESPONSE ERROR SYSTEMS

This effort is directed toward eliminating mathematical approximations in analysis of self-adjusting systems. Central in this commentary is emphasis on the time-varying quality of the dynamic systems considered. This implies that analysis must be performed in terms of functional relationships rather than value relationships. Functional relationships are the proper domain of variational calculus (in distinction to differential calculus).

VARIATIONAL CONCEPTS

A few words are sufficient to introduce the basic variational concepts we will need.

A variation, δz_i, defines a change in the functional relationship of z_i and t and must not be confused with a change, Δz_i, in the value of a given function, $z_i(t)$, due to a change Δt, in the independent vairable. (Vide Reference 25)

The variation of a function $f_i = f_i(\underline{z};\ \underline{x};\ \underline{\gamma};\ t)$ where z_i and γ_k are the dependent variables, and t and x_i are the independent variables, is given by:

$$\delta f_i = f_i(z_1+\delta z_1,\ldots,z_n+\delta z_n; x_1,\ldots,x_n\ ;\ \gamma_1+\delta\gamma_1,\ldots,\gamma_m+\delta\gamma_m\ ;\ t) - f_i(z_1,\ldots,z_n; x_1,\ldots,x_n\ ;\ \gamma_1,\ldots,\gamma_m\ ;\ t) \qquad \text{(A-1)}$$

If f_i is suitably differentiable, the above equation may be expanded in a

Taylor series as:

$$\delta f_i = \sum_{s=1}^{n} \frac{\partial f_i}{\partial z_s} \delta z_s + \sum_{k=1}^{m} \frac{\partial f_i}{\partial \gamma_k} \delta \gamma_k + \sum_{k=1}^{m} \sum_{s=1}^{n} \frac{\partial^2 f_i}{\partial \gamma_k \partial z_s} \delta \gamma_k \delta z_s$$

$$+ \frac{1}{2} \sum_{s=1}^{n} \sum_{t=1}^{m} \frac{\partial^2 f_i}{\partial z_s \partial z_t} \delta z_s \delta z_t + \frac{1}{2} \sum_{k=1}^{m} \sum_{r=1}^{m} \frac{\partial^2 f_i}{\partial \gamma_k \partial \gamma_r} \delta \gamma_k \delta \gamma_r + \dots \qquad \text{(A-2)}$$

The terms of order one, two, ... etc., in δz_s and/or $\delta \alpha_k$ respectively constitute the first order variation, $\delta^1 f_i$, of f_i; the second order variation, $\delta^2 f_i$, of f_i; etc.

The usefulness of the variational technique for the parameter tracking problem is embodied in the following property: (Reference 26)

> If t is an independent variable (and, accordingly $\delta t = 0$) the operators δ and d/dt are commutative; that is:

$$\frac{d}{dt} (\delta z_i) = \delta \left(\frac{dz_i}{dt} \right) \qquad \text{(A-3)}$$

This extremely important commutative property enables us to formulate a system of differential equations which relate dynamic changes in model coefficients to the attendant changes in the model output variables without requiring a constant coefficient restriction.

DEFINITION OF PARAMETER TRACKING SYSTEM ELEMENTS

Response error parameter tracking systems can be considered as composed of two functional blocks; the model, and the parameter adjustment laws, as shown in Figure A-1. The interaction of these functional blocks and their coupling with the plant to be modeled are also indicated in this figure.

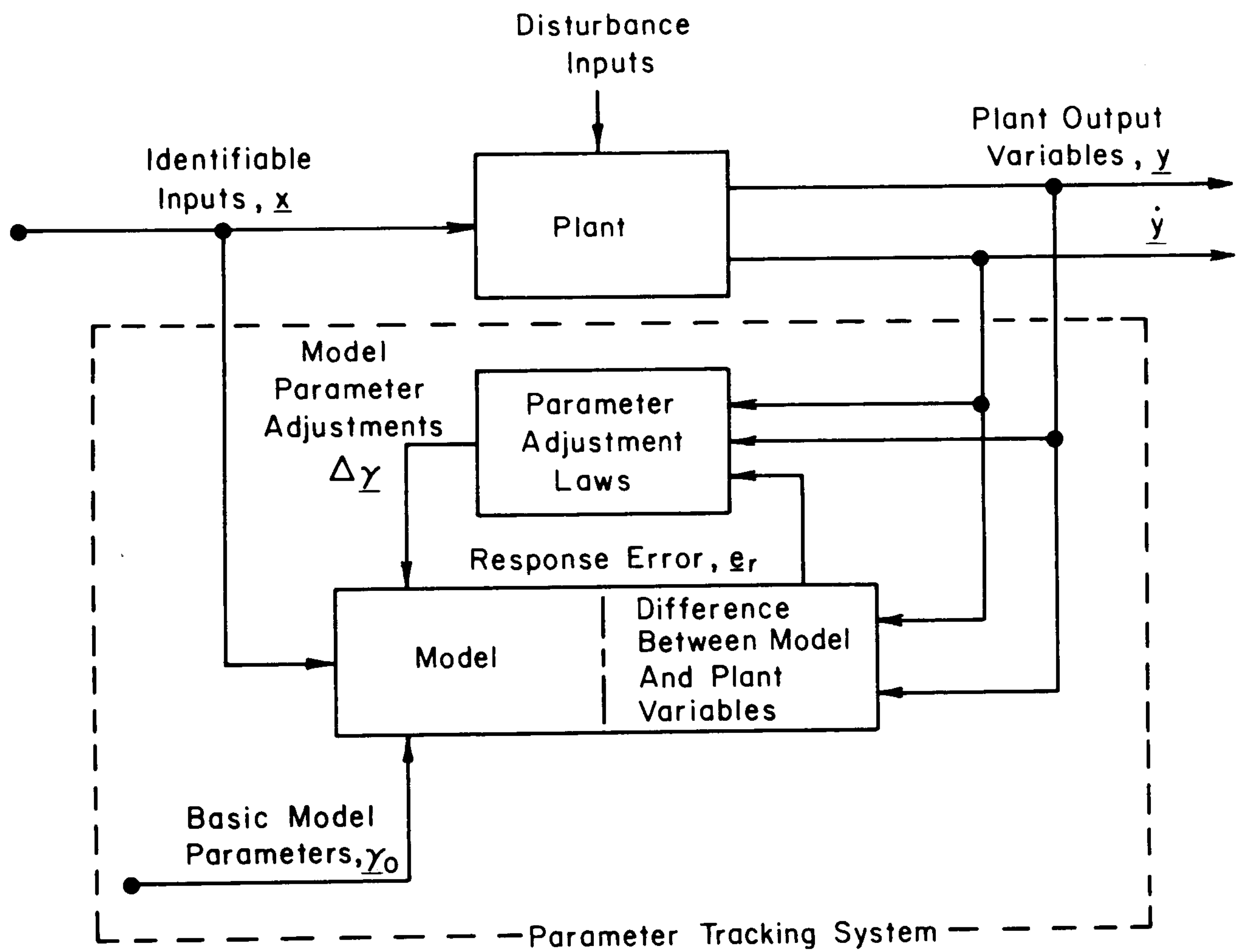


Figure A-1. Functional Vector Block Diagram for Response Error Parameter Tracking Systems

The form of the n^{th} order model is defined by

$$\underline{\dot{z}} = \underline{f}(\underline{z}; \underline{x}; \underline{\gamma}; t)$$
$$\underline{z}(0) = \underline{z}^0$$
$$\underline{z} = \mathrm{col}(z_1, z_2, \ldots z_n) \qquad \text{(A-4)}$$
$$\underline{x} = \mathrm{col}(x_1, x_2, \ldots x_l)$$
$$\underline{\gamma} = \mathrm{col}(\gamma_0, \gamma_1, \ldots \gamma_m)$$

where the z_i are the state variables of the model, t is the independent variable, time; the γ_k are the coefficients or parameters of the model and are treated as dependent variables. x_j are the inputs to the system. These signals are the identifiable inputs to the plant which are available as signals. The x_j are considered to be independent variables.

The response error vector, $\underline{e}_r$, is defined by m linear combinations of the differences of each of the n output variables of the plant, y_i, and the corresponding output variables of the model, z_i, and the time derivatives of these differences.

$$\underline{e}_r = C(\underline{z} - \underline{y}) + D(\underline{\dot{z}} - \underline{\dot{y}}) = G\underline{v} \qquad \text{(A-5)}$$

where $G = \left[C \;\vdots\; D\right]$, $\underline{v} = \begin{Bmatrix} \underline{z} - \underline{y} \\ \underline{\dot{z}} - \underline{\dot{y}} \end{Bmatrix}$. C and D are m x n matrices, G is a m x 2n matrix. The columns of C and D which correspond to the components of $\underline{y}$ and $\underline{\dot{y}}$ which are not measurable are zero. The remaining elements of C and D are specified by the designer.

Complicated as the above description might seem, it does have the virtue of encompassing a great many cases of practical interest. For a linear model with simple coefficients,* the Taylor series expansion of f_i

*Simple coefficients of a linear differential equation are those occurring linearly, such as:

$$\sum_{i=0}^{n} \alpha_i \binom{i}{z} = \sum_{i=0}^{l} \beta_i \binom{i}{x} \qquad \dot{z}_i = \sum_{j=1}^{n} \gamma_{ij} z_j + \sum_{j=n+1}^{n+l} \gamma_{ij} x_{j-n}$$

terminates after terms of second order, and for both simple and non-simple coefficients the expansion is linear in the δz_i. When nonlinear model equations are considered, the expansion can always be linearized in the δz_i by truncating the series after terms of first order in δz_i.

Linearity can be used to advantage in that the superposition principle can be employed to separate the solutions, δz_i, into components ascribable to the individual $\delta\gamma_k$. This yields the expression

$$\delta z_{ik} = \sum_{s=1}^{n} \left(\frac{\partial f_i}{\partial z_s} + \sum_{k=1}^{m} \frac{\partial^2 f_i}{\partial z_s \partial \gamma_k} \delta\gamma_k \right) \delta z_{sk} + \frac{\partial f_i}{\partial \gamma_k} \delta\gamma_k \qquad \text{(A-6)}$$

$\delta z_{ik}(0) = 0$, $k = 1, 2, \ldots m$ where $\delta z_i = \sum_{k=1}^{m} \delta z_{ik}$. This is exact for linear models with simple coefficients. (For non-simple coefficients, the problem can be reduced to this form plus auxiliary algebraic constraints among the parameters.) For nonlinear models with simple coefficients, Eq A-6 is a first order approximation in the δz_{ik} variables. The interpretation of δz_{ik} is as follows: δz_{ik} is the component of δz_i which is ascribable to the parameter variation $\delta\gamma_k$ in the presence of the structural change in the system of equations because of the other parameter variations in the system, $\delta\gamma_r$, $r \neq k$. Equations A-6, using δ to indicate infinitesimal variations, define a <u>weighting filter</u> for each parameter. For infinitesimal variations, Eq A-6 may be written in vector form as:

$$\delta \dot{\underline{z}}_k = \frac{\partial \underline{f}}{\partial \underline{z}} \delta \underline{z}_k + \frac{\partial \underline{f}}{\partial \gamma_k} \delta\gamma_k \qquad \text{(A-7)}$$

If Eq A-7 is considered to define a linear system in the δz_{ik}, then the $(\partial \underline{f}/\partial\gamma_k)\, \delta\gamma_k$ term must be the forcing applied to this system.

Then, using Δ to indicate variations of finite size, let us define

$$z_i = z_{0i} + \sum_{k=1}^{m} \Delta z_{ik}$$

$$\gamma_k = \gamma_{0k} + \Delta\gamma_k \qquad \text{(A-8)}$$

where γ_{0k} is the base component of the k^{th} model coefficient which is given a priori--not necessarily a constant--$\Delta\gamma_k$ is the manipulated component of the k^{th} model parameter. z_{0i} is the model response resulting when all $\Delta\gamma_k$ are constrained to zero. Δz_i gives the change in the model response due to the $\Delta\gamma_k$. A base portion of the model may be defined as:

$$\begin{aligned} \dot{\underline{z}}_0 &= f_0(\underline{z}_0;\ \underline{x};\ \underline{\gamma}_0;\ t) \\ \underline{z}_0(0) &= \underline{z}_0^0 \end{aligned} \qquad \text{(A-9)}$$

By applying the Taylor series expansion and the superposition principle to Eq A-9, and considering finite variations, manipulated portions of the model are defined by

$$\Delta\dot{z}_{ik} = \sum_{s=1}^{n} \left(\frac{\partial f_{0i}}{\partial z_s} + \sum_{k=1}^{m} \frac{\partial^2 f_{0i}}{\partial z_s \partial\gamma_k} \Delta\gamma_k \right) \Delta z_{sk} + \frac{\partial f_{0i}}{\partial\gamma_k} \Delta\gamma_k \qquad \text{(A-10)}$$

$$\Delta z_{ik}(0) = 0 \qquad k = 1,2, \dots m$$

for linear models with simple coefficients. Equation A-10 also defines the first order approximation in the Δz_{ik} variables for nonlinear models with simple coefficients.

Similarity of the definitions for the manipulated portions of the model and the weighting filters is apparent.

The fact that

$$\frac{\partial f_i}{\partial\gamma_k} = \frac{\partial f_{0i}}{\partial\gamma_k} + \sum_{s=1}^{n} \frac{\partial^2 f_{0i}}{\partial z_s \partial\gamma_k} \Delta z_s$$

can be used to obtain an alternate expression for Eq A-10 in vector form which will be useful at a later point.

$$\begin{aligned} \Delta\dot{\underline{z}}_k &= \frac{\partial \underline{f}_0}{\partial \underline{z}} \Delta\underline{z}_k + \frac{\partial \underline{f}}{\partial \underline{\gamma}} \Delta\gamma \\ \Delta\underline{z}_k(0) &= 0 \qquad k = 1,2, \dots m \end{aligned} \qquad \text{(A-11)}$$

By adding the base model equations and the finite variational equations according to Eq A-8, we may convince ourselves that these definitions are consistent with the original definition of the model, Eq A-4.

Next, we will examine the way these two sets of variational equations are used in parameter tracking systems. Consider a criterion function, $F(\underline{e}_r)$, which is a positive definite function of a response error vector. At a stationary point which is a minimum of this function with respect to the model variables (the z_i and $\dot{z}_j$), the rate of change of the criterion function in the direction of any model variable must be zero. This property of a stationary point means that the first variation of the criterion function, Eq A-12, must vanish everywhere in the infinitesimal region about the stationary point. Therefore, the coefficients of the component variations of the criterion function first variation must vanish independently. The first variation of the criterion function, Eq A-16, can be expressed as a function of the infinitesimal component variations, $\delta\gamma_k$, δz_{ik} and $\delta\dot{z}_{ik}$ (given by Eq A-7). The coefficients of the component variations are functions of the z_i, $\dot{z}_i$, y_i and $\dot{y}_i$. A successful parameter tracking system may be defined as one which servoes these coefficients in the criterion function first variation to zero.

Now, it is clear that the model parameters provide a convenient means for manipulating the z_i variables. This constitutes the use of the equations in finite variations, Eq A-10 or A-11. The equations show the means for servoing the coefficients of δz_{ik} and $\delta\dot{z}_{ik}$ to zero though the control of each manipulated parameter on the model variables is not of an uncoupled nature.

Consider the first variation of the criterion function:

$$\delta^1 F(\underline{e}_r) = \left[\frac{\partial F(\underline{e}_r)}{\partial \underline{e}_r}\right]' \delta\underline{e}_r \tag{A-12}$$

$$
\begin{aligned}
\delta \underline{e}_r &= C\delta\underline{z} + D\delta\underline{\dot{z}} + \delta C(\underline{z}-\underline{y}) + \delta D(\underline{\dot{z}}-\underline{\dot{y}}) \\
&= C\left\{\sum_{k=1}^{m} \delta z_{ik}\right\} + D\left\{\sum_{k=1}^{m} \Delta\delta z_{ik}\right\} + \left[\sum_{j=1}^{n} \frac{\partial C_{ij}}{\partial \gamma_k}(z_j - y_j)\right]\left\{\delta\gamma_k\right\} \\
&\quad + \left[\sum_{j=1}^{n} \frac{\partial D_{ij}}{\partial \gamma_k}(\dot{z}_j - \dot{y}_j)\right]\left\{\delta\gamma_k\right\} \qquad \text{(A-13)} \\
&= H\underline{w}
\end{aligned}
$$

Where the m x(2n + m) matrix, H, is

$$
H = \left[\, C \,\vdots\, D \,\vdots\, E \,\right] \qquad \text{(A-14)}
$$

and where

$$
E_{ik} = \sum_{j=1}^{n} \left[\frac{\partial C_{ij}}{\partial \gamma_k}(z_j - y_j) + \frac{\partial D_{ij}}{\partial \gamma_k}(\dot{z}_j - \dot{y}_j)\right]
$$

and the (2n + m) vector, $\underline{w}$ is:

$$
\underline{w} = \left\{\begin{array}{c} \delta\underline{z} \\ \hline \delta\underline{\dot{z}} \\ \hline \delta\underline{\gamma} \end{array}\right\} \qquad \text{(A-15)}
$$

Then:

$$
\delta^1 F(\underline{e}_r) = \left[\frac{\partial F(\underline{e}_r)}{\partial \underline{e}_r}\right]' H\underline{w} \qquad \text{(A-16)}
$$

The first variation of the criterion function is also used to approximate the first order sensitivity of the criterion function to changes in the model parameters. This is accomplished by defining the (2n + m) x m matrix, W:

$$W = \begin{bmatrix} \delta z_{ik}/\delta\gamma_k \\ ----- \\ \delta\dot{z}_{ik}/\delta\gamma_k \\ ----- \\ \ddots\, 1\, \ddots \end{bmatrix} \tag{A-17}$$

Note that $W\delta\underline{\gamma} = \underline{w}$. The control law employed to adjust (or manipulate) the parameters of the model is:

$$\Delta\dot{\underline{\gamma}} = -\, kW'H' \left\{ \frac{\partial F(\underline{e}_r)}{\partial \underline{e}_r} \right\} \tag{A-18}$$

$W'H' \left\{ \frac{\partial F(\underline{e}_r)}{\partial \underline{e}_r} \right\}$ is the approximate sensitivity of the criterion function to changes in the model parameters.

In using the variational expansion of the model equations to indicate the approximate sensitivity of the criterion function to changes in the model parameters, we have formed the ratios $\delta z_{ik}/\delta\gamma_k$ and $\delta z_{ik}/\delta\gamma_k$ in the matrix, W, subject to the following restrictions:

1. The variations in the parameters have been specified to be constants. This is permissible since in testing the criterion function for stationarity the variations may be chosen as arbitrary nonzero functions.

2. The variations in the parameters have been allowed to become arbitrarily small.

The necessity for adopting restriction 1 arose because the variation of the parameter of interest (in the case of a particular weighting filter) always appears in the forcing signal path of the weighting filter, and its inverse always appears in the output path of the weighting filter by virtue of Eq A-17. Hence by choosing the coefficient variation function as constant with time, the variation in the input path divides out with its inverse in the output path because the weighting filter is always a linear system. As a practical matter, we should like to avoid considering higher order effects of the variations, and from a theoretical viewpoint, we would like to deal with a limiting form of the approximate sensitivity, wherein the model parameters are also parameters of the weighting filter, hence restriction 2.

As a direct result of these two restrictions we must also restrict, or at least carefully define what is meant by "approximate sensitivity." The interpretation is as follows.

> The weighting filter equations under restrictions 1 and 2 generate the first order sensitivity of the time-varying linear or nonlinear model responses to changes in the parameters made at $t \leq 0$ (a time before the application of an input to the plant and model) and held constant during the response of the model.

In other words, at some time greater than $t = 0$ the output of the variational equations represents the present sensitivity of the model response to an infinitesimal constant parameter change made at or before $t = 0$. It is not the sensitivity of the model response to parameter changes made subsequent to $t = 0$ although the effects of any such change will be correctly included in the weighting filter structure.

STRUCTURE OF THE PARAMETER ADJUSTMENT LOOPS

Having developed the equations for a general response error parameter tracking system, let us now turn our attention to the structure of the system. A vector block diagram, Figure A-2, provides the most concise means for showing the interrelationships of the equations describing the system, Eq A-4, 5, 7, 8, 9, 11, 14, 17 and 18.

Figure A-2 clearly shows that the parameter adjustment loop dynamics consist of the integration for generating $\Delta \underline{\gamma}$ and the weighting filter dynamics. This observation has been stated previously in References 7, 8 and 12. However, in each case the analysis was approximate. Here, the

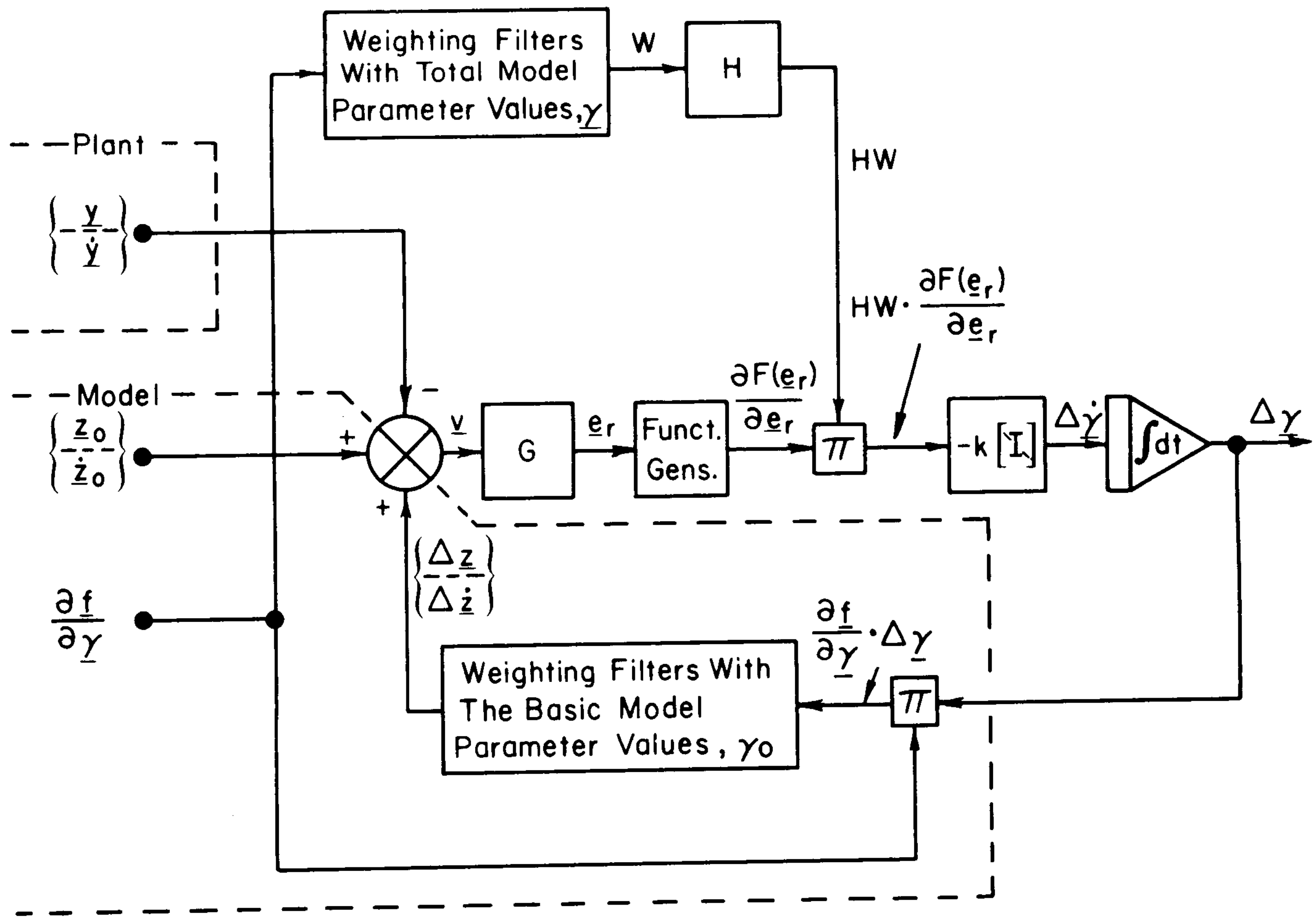


Figure A-2. Vector Block Diagram of General Response Error Parameter Tracking System analysis is exact for linear models and is a first order approximation for a broad class of nonlinear models with simple coefficients. This loop also contains a nonlinear element and two time-varying gains. The G matrix provides the means for selecting different response error vectors through specification of the matrix elements. The constant, k, merely controls the open parameter adjustment loop gain. The characteristics of the nonlinear element are determined by the choice of criterion function. For example, if

$$F(\underline{e}_r) = \underline{e}_r{}'\underline{e}_r/2 \qquad \text{(A-19)}$$

then:

$$\frac{\partial F(\underline{e}_r)}{\partial \underline{e}_r} = \underline{e}_r \qquad \text{(A-20)}$$

That is, for the special case where the criterion function is the sum of the squares of the response error vector components, the nonlinear characteristics reduce to linear, unity gain characteristics.

APPENDIX B

RELATION OF EQUATION ERROR TO RESPONSE ERROR

Response error is defined by Eq A-5 of Appendix A as:

$$\underline{e}_r = C(\underline{z}-\underline{y}) + D(\dot{\underline{z}}-\dot{\underline{y}}) \tag{B-1}$$

Let us now make the following specifications and assumptions.

1. Assume a linear model with simple coefficients of the form

$$\dot{\underline{z}} = \left[\frac{\partial f_i}{\partial z_j}\right]\underline{z} + \left[\frac{\partial f_i}{\partial x_k}\right]\underline{x} \tag{B-2}$$

   which is a special case of Eq A-4.

2. Let C and D be n x n matrices; that is, let the order of the response error vector equal the order of the model, and assume all state variables are measurable.

3. Let the matrices C and D be:

$$C = \left[-\frac{\partial f_i}{\partial z_j}\right] \qquad D = \begin{bmatrix} \diagdown & & \\ & 1 & \\ & & \diagdown \end{bmatrix} \tag{B-3}$$

Evaluating the expression for the response error vector using Eqs B-1 and B-3 we obtain:

$$\underline{e}_r = \left[-\frac{\partial f_i}{\partial z_j}\right](\underline{z}-\underline{y}) + \begin{bmatrix} \diagdown & & \\ & 1 & \\ & & \diagdown \end{bmatrix}(\dot{\underline{z}}-\dot{\underline{y}}) \tag{B-4}$$

Using Eq B-2

$$\underline{e}_r = \left[\frac{\partial f_i}{\partial x_k}\right]\underline{x} + \left[\frac{\partial f_i}{\partial z_j}\right]\underline{y} - \dot{\underline{y}} \tag{B-5}$$

Equation B-5 defines a response error vector, the components of which are the equation errors associated with each first order differential equation of the (implied) n^{th} order model. This may be appreciated by comparing the right hand side of Eq B-5 with Eq B-2.

This analysis demonstrates that the equation error vector is a special case of the response error vector for linear models with simple coefficients. For nonlinear models with simple coefficients, the same statement holds true although the matrices, $\left[\partial f_i/\partial z_j\right]$ and $\left[\partial f_i/\partial x_k\right]$, have elements which are algebraic functions of the components of $\underline{z}$ and $\underline{x}$.

Next, we shall examine the component of this special response error vector ascribable to the manipulated component of the parameter vector. Making use of Eqs B-1 and A-10:

$$\Delta\underline{e}_r = \sum_{k=1}^{m} \left\{ \left[-\frac{\partial f_i}{\partial z_j} \right] \Delta z_k + \left[\begin{matrix} \diagdown & & \\ & 1 & \\ & & \diagdown \end{matrix} \right] \Delta \dot{z}_k \right\} = \left[\frac{\partial f_{0i}}{\partial \gamma_k} \right] \Delta\underline{\gamma} \qquad \text{(B-6)}$$

Because of the simple coefficient assumption, $\left[\partial f_{0i}/\partial \gamma_k\right]$ will have elements which are algebraic functions of the components of $\underline{z}_0$ and $\underline{x}$ but <u>not</u> of $\underline{\gamma}$, nor will time derivatives of $\underline{z}_0$ be involved. Or equivalently, making use of Eq B-5:

$$\Delta\underline{e}_r = \left[\sum_{k=1}^{l} \frac{\partial^2 f_i}{\partial x_k \partial \gamma_j} x_k + \sum_{k=1}^{n} \frac{\partial^2 f_i}{\partial z_k \partial \gamma_j} y_k \right] \Delta\underline{\gamma} \qquad \text{(B-7)}$$

For linear models with simple coefficients, $\partial^2 f_i/\partial x_k \partial \gamma_j$ and $\partial^2 f_i/\partial z_k \partial \gamma_j$ will always be zero or one. Hence $\Delta\underline{e}_r$ can alternatively be an algebraic function of $\underline{y}$ and $\underline{x}$.

This demonstrates that the weighting filters with the <u>basic</u> model parameter values in Figure A-2 have transmittances of unity (no dynamics) for this special case of the response error vector.

Finally, we shall examine the dynamics of the weighting filters with total model parameter values, $\underline{\gamma}$, in Figure A-2. To accomplish this we must evaluate the matrix, W'H', in Eq A-18. We shall restrict our attention to linear models with simple coefficients. From Eq A-14

$$H = \left[C \,\vdots\, D \,\vdots\, E \right] \tag{B-8}$$

where

$$E_{ik} = \sum_{j=1}^{n} \left\{ \frac{\partial C_{ij}}{\partial \gamma_k} (z_j - y_j) + \frac{\partial D_{ij}}{\partial \gamma_k} (\dot{z}_j - \dot{y}_j) \right\} \tag{B-9}$$

For the specified C and D matrices:

$$\begin{aligned} \frac{\partial C_{ij}}{\partial \gamma_k} &= - \frac{\partial^2 f_i}{\partial z_j \partial \gamma_k} \\ \frac{\partial D_{ij}}{\partial \gamma_k} &= 0 \end{aligned} \tag{B-10}$$

From Eq A-17

$$W = \begin{bmatrix} \delta z_{ik} / \delta \gamma_k \\ \text{-----} \\ \delta \dot{z}_{ik} / \delta \gamma_k \\ \text{-----} \\ \diagdown 1 \diagdown \end{bmatrix} \tag{B-11}$$

Then

$$HW = \left[-\frac{\partial f_i}{\partial z_k} \;\vdots\; \diagdown 1 \diagdown \;\vdots\; -\sum_{j=1}^{n} \frac{\partial^2 f_i}{\partial z_j \partial \gamma_k} (z_j - y_j) \right] \begin{bmatrix} \delta z_{ik} / \delta \gamma_k \\ \text{-----} \\ \delta \dot{z}_{ik} / \delta \gamma_k \\ \text{-----} \\ \diagdown 1 \diagdown \end{bmatrix}$$

$$HW = -\left[\sum_{k=1}^{m} \frac{\partial f_i}{\partial z_k} \frac{\delta z_{ik}}{\delta \gamma_k}\right] + \left[\frac{\delta \dot{z}_{ik}}{\delta \gamma_k}\right] - \left[\sum_{j=1}^{n} \frac{\partial^2 f_i}{\partial z_j \partial \gamma_k} z_j\right]$$

$$+ \left[\sum_{j=1}^{n} \frac{\partial^2 f_i}{\partial z_j \partial \gamma_k} y_j\right] \qquad \text{(B-12)}$$

Since $\frac{\partial^2 f_i}{\partial z_j \partial \gamma_k} z_j = \frac{\partial f_i}{\partial \gamma_k}$, we can see

that each column of the first three terms satisfies Eq A-7 (assuming that $\delta\gamma_k$ is constant and dividing through by it in Eq A-7). Therefore:

$$W'H' = \left[\sum_{j=1}^{n} \frac{\partial^2 f_i}{\partial z_j \partial \gamma_k} y_j\right]' \qquad \text{(B-13)}$$

For linear models with simple coefficients, $\partial^2 f_i/\partial z_j \partial \gamma_k$ will always be zero or one. Here then, we may consider the weighting filters to have transmittances of unity. These equations for W'H' are first order approximations for nonlinear models with simple coefficients.

Equations B-5, B-13 and A-18 define an equation error parameter tracking system. The linear model with simple coefficients implied by these equations is given by Eq B-2. For linear simple coefficient models all the equations given above are exact.

APPENDIX C

DEFINITIONS AND THEOREM PROOFS

This appendix contains definitions and proofs of the theorems contained in Section II.

It is assumed that we have a linear system of differential equations of the form:

$$\dot{\underline{\gamma}} = - k\, A(t)\, \underline{\gamma} \tag{C-1}$$

(See Eq 2-23)

The solution of Eq C-1 which satisfies the initial condition $\underline{\gamma} = \underline{\gamma}^0$ at $t = t_0$ will be denoted by

$$\underline{\gamma}(t;\ \underline{\gamma}^0,\ t_0)$$

Since Eq C-1 is linear, the only equilibrium point is the origin.

<u>Definition 1</u> The system described by Eq C-1 is said to be stable if for some $r > 0$, there exists an $R(r,\ t_0) > 0$, such that $\| \underline{\gamma}^0 \| < r$ implies

$$\| \underline{\gamma}(t;\ \underline{\gamma}^0,\ t) \| \leq R$$

for all $t \geq t_0$.

<u>Definition 2</u> The system described by Eq C-1 is said to be asymptotically stable if

(1) it is stable

(2) there exists a region, $r(t_0) > 0$ such that $\| \underline{\gamma}^0 \| < r(t_0)$ implies:

$$\lim_{t \to \infty} \| \underline{\gamma}(t;\ \underline{\gamma}^0,\ t_0) \| = 0$$

Therefore, to prove a system is asymptotically stable, it must first be shown that it is stable.

Definition 3 A system is said to be completely (asymptotically) stable if it is (asymptotically) stable for all initial states, $\underline{\gamma}^0$.

Since Eq C-1 is linear, any stability proofs will hold for all $\underline{\gamma}^0$; i.e. if Eq C-1 is asymptotically stable it will be completely asymptotically stable.

Using these definitions, it is possible to prove the complete stability of the system describing the parameter differences, Eq C-1, using the first theorem of Liapunov. This is done by finding a Liapunov function, $V(\underline{\gamma})$, which is positive definite, i.e.

(1) $V(\underline{\gamma}) \geq \xi \, \| \underline{\gamma} \|$

where ξ is some positive constant.

(2) $V(0) = 0$

and then showing that the derivative $\dot{V}$ is never positive.

To prove asymptotic stability, using Liapunov's theorem requires that $\dot{V}$ be always negative. The system given by Eq C-1 does not satisfy this requirement. However, it is intuitively clear since $\dot{V}$ is zero only at isolated instants and is negative otherwise, that the system must be asymptotically stable. To put this on a rigorous basis requires the analog of a theorem of LaSalle (Reference 15) for non-autonomous systems. The following definition is required.

Definition 4 $\Gamma(t)$ is defined as the set of all points, $\underline{\gamma}$, such that $\dot{V} = 0$ at time t. By $\Gamma(\infty)$ we denote the limiting value of this set, $\lim_{t \to \infty} \Gamma(t) = \Gamma(\infty)$. It is important to note that the Liapunov functions, V, considered here are not explicitly dependent on t. Hence, contours of V are time-invariant.

We can now establish the following theorem:

Theorem C-1 Given: A linear differential system whose coefficients are continuous functions of t. If a Liapunov function, $V(\underline{\gamma})$, differentiable with respect to $\underline{\gamma}$, can be found such that

(1) $V(\underline{\gamma}) \geq \xi \; \| \underline{\gamma} \|$

(2) $V(0) = 0$

(3) $\dot{V} \leq 0$ for all $\underline{\gamma}$, t

then the solution will approach the set $\Gamma(\infty)$ as $t \to \infty$.

Proof. Since $\dot{V} \leq 0$, V is a non-increasing, continuous function which is bounded from below. Therefore $V(\underline{\gamma}(t))$ has a limit, c, as $t \to \infty$. Since

$$V(\underline{\gamma}(t)) = V(\underline{\gamma}^0) + \int_{t_0}^{t} \dot{V}(\underline{\gamma}(\tau), \tau) \, d\tau \qquad \text{(C-2)}$$

this implies that

$$\dot{V} \to 0 \text{ as } t \to \infty.$$

Since $\Gamma(\infty)$ is the set of all points such that $\dot{V} \to 0$, the solution must approach $\Gamma(\infty)$.

Theorem 1 of Section II follows directly since it restricts the elements of $\Gamma(\infty)$ which can be approached by any solution of the differential equation, to the origin.

Theorem 2 of Section II applies to cases where the elements of A(t) in Eq C-1 are random functions of t. For this problem, a new concept of stability is required.

Definition 5 The system given by Eq C-1 said to be stable in the mean if for some $r > 0$, there exists an $R(r, t_0) > 0$ such that $\| \underline{\gamma}^0 \| < r$ implies

$$E(\| \underline{\gamma}^0 \|) < R$$

for all $t \geq t_0$.

Definition 6 The system given by Eq C-1 is said to be asymptotically stable in the mean if

(1) it is stable in the mean

(2) there exists a region, $r(t_0) > 0$ such that $\| \underline{\gamma}^0 \| < r(t_0)$ implies

$$\lim_{t\to\infty} E(\| \underline{\gamma}(t) \|) = 0.$$

Definition 7 A system is said to be completely (asymptotically) stable in the mean if it is asymptotically stable in the mean for all initial states, $\underline{\gamma}^0$.

These definitions are exact analogs of Definitions 1, 2 and 3.

Stability in the mean of the system described by Eq C-1 can be proved using the following theorem of Bertram and Sarachik (Reference 27); specialized here to the linear case:

Theorem C-2 Given: A linear differential system whose coefficients are random functions of time. If a Liapunov function, $V(\underline{\gamma})$, differentiable in $\underline{\gamma}$, can be found such that

(1) $V(\underline{\gamma}) \geq \xi \; \| \underline{\gamma} \|$

(2) $V(0) = 0$

(3) $E(\dot{V}) \leq 0$

then the system is stable in the mean.

(The theorem proved in Reference 27 is applicable also to nonlinear systems and with time dependent Liapunov functions, $V = V(\underline{\gamma}, t)$. We have specialized here in the interests of brevity).

Clearly the system of Eq C-1 satisfies these conditions. To show asymptotic stability in the mean, Bertram and Sarachik have proved another theorem which imposes the additional condition that

$$E(\dot{V}) \leq \eta \; \| \underline{\gamma} \|$$

where η is some constant. This condition is <u>not</u> satisfied by Eq C-1, because of the hyperplane where $e = 0$.

Asymptotic stability in the mean can, however, be proved using an analog of Theorem C-1 for the stochastic case:

Theorem C-3 Given: A linear differential system whose coefficients are random functions of time. If a Liapunov function $V(\underline{\gamma})$, differentiable in $\underline{\gamma}$, can be found such that

(1) $V(\underline{\gamma}) \geq \xi \, \|\underline{\gamma}\|$

(2) $V(0) = 0$

(3) $E(\dot{V}) \leq 0$

(4) the only set such that $E(\dot{V}) \to 0$ as $t \to \infty$ is the origin

then the system is asymptotically stable in the mean.

Proof. The argument is identical to that used in Theorem C-1. $E(V)$ is now a non-increasing function of t, continuous and bounded from below. Therefore it approaches a limit, c. Taking the expected value of Eq C-2 yields

$$E(V(\underline{\gamma}(t))) = V_0 + \int_{t_0}^{t} E(\dot{V}(\underline{\gamma}(\tau), \tau)) \, d\tau$$

Since $E(V) \to c$ as $t \to \infty$, it is necessary that $E(\dot{V}) \to 0$. However $E(\dot{V}) \to 0$ only if $\|\underline{\gamma}\| \to 0$. $\therefore$ $E(V)$ must $\to 0$ since $E(\dot{V})$ is always ≤ 0.

Theorem C-3 is identical to Theorem 2 of Section II.

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