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Design of Optimized Three-Dimensional Thrust Nozzle Contours

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LAFAYETTE, INDIANA

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ABSTRACT

The objectives of the investigation described herein are two fold: (1) to establish a method of designing three-dimensional (non-axisymmetric) thrust nozzle contours for maximum thrust with prescribed inlet conditions and constraint relations, and (2) to illustrate the methodology for the optimum design by considering two examples.

Design procedures for maximizing the thrust of axisymmetric rocket motor nozzles under various isoparametric conditions have been developed within the last decade and are widely used currently. It is well known, however, that rocket motor nozzles may be required to have flow geometries that cannot be adequately approximated by a simple two-dimensional or axisymmetric shape. Hence, the need has been felt for a method of designing optimum three-dimensional nozzles. Assuming an irrotational, homentropic flow (of a perfect gas) with given initial conditions, the following constraint relations are specified: (a) the shape of the nozzle exit, (b) the variation of nozzle length with respect to the angular coordinate (r, ϕ, z coordinates), and (c) a streamline which coincides with the z-axis in the flow regime.

The general problem of optimizing the contour of a three-dimensional nozzle is formulated by postulating a three-dimensional control surface which is constrained to pass through the exit contour of the

nozzle and intersect the core region (kernel) but is otherwise an arbitrary three-dimensional surface.

The standard (r, ϕ, z)-cylindrical coordinate system is used to describe the problem. The axial thrust and mass flow rate are written as integrals over the control surface. The solution to the optimization problem is obtained by applying the techniques of the calculus of variations. The variational integral is formed by summing the integral equation for axial thrust, the integral equation for the mass flow rate times a Lagrange multiplier, and the irrotationality condition times another Lagrange multiplier. The constraints on length and nozzle exit geometry are included by substitution in the variational integral. From the variational problem a set of four design equations which relate the flow variables on the control surface is derived.

The design equations together with the boundary conditions in a particular problem, are sufficient to locate the control surface and determine the flow properties on it.

In order to ensure that a shock free flow field exists which will produce the optimum flow on the control surface and also match the flow in the kernel, it is shown that the control surface is a characteristic surface; that is, the control surface is shown to be oriented in a characteristic direction and the compatibility equations for a characteristic surface in three-dimensional flow are shown to apply on the control surface.

The methodology for the application of the solution of the optimization problem is discussed with reference to two examples as follows:

1. a nozzle in which the initial and ambient conditions and the length of the nozzle are prescribed; the shape of the nozzle at the exit plane is required to be an ellipse of given eccentricity but with variable area; and the exit contour is on a plane normal tc a given axis;
2. a nozzle in which the initial and ambient conditions are prescribed (in particular the throat section is required to be circular), and the nozzle length and shape at the exit plane are the same as that obtained by arbitrarily truncating an optimized axisymmetric nozzle.

1. INTRODUCTION

One of the interesting problems in fluid dynamics is the optimization of flow geometries under specified constraints. Such problems arise in the determination of the shapes of wing bodies and ships, of the moving parts of a turbo-machine,and of the reaction nozzle of a rocket motor. In general, the problem of optimization should include the properties of the medium of flow, both with respect to the equation of state and the stress-strain relationships. Even when the medium of flow is assumed to obey the perfect gas law and all viscous effects are neglected, the optimization of a flow, three-dimensional in character, presents many interesting features. The research reported here pertains to the optimization of the geometry of a thrust nozzle (in a rocket motor) under the assumption that the medium of flow is an inviscid mixture of gases obeying the perfect gas law.

Several different types of constraints may be considered for the thrust nozzle, such as (a) geometrical constraints, (b) weight-based constraints,or (c) constraints based upon the loss of momentum or energy in the flow. The objective of the investigation* described herein is the establishment of a method for designing three-dimensional (non-axisymmetric) thrust nozzle contours for maximum thrust with prescribed inlet conditions and specified constraints on the overall length and exit geometry.

* Thompson, H. D., "Design of Optimized Three-Dimensional Thrust Nozzle Contours," Ph.D. Thesis, Purdue University, June 1965.

Furthermore, it may be pointed out at the outset that the procedure for a complete design will involve the determination of the flow geometry in (a) the subsonic region, (b) the transonic region, and (c) the supersonic region. The research under report is concerned exclusively with the optimization of the flow geometry in the supersonic region of the thrust nozzle.

* * *

The thrust of a propulsion device operating with chemical propellants is developed primarily by imparting momentum to the products of combustion by discharging them through the nozzle. The gases are accelerated from low subsonic velocities in the converging (initial) portion of the nozzle; they pass from subsonic to supersonic velocities in the minimum area section (throat)*, and are further accelerated in the diverging portion of the nozzle to achieve the required supersonic velocity at the exit. An analysis of the performance of a thrust nozzle or the development of a method of design requires the determination of the flow field in the nozzle and demands a separate method of analysis for each of the three sections of the nozzle, namely the subsonic converging section, the transonic (throat) section, and the supersonic diverging section.

The total thrust achieved by the nozzle depends upon the rate of mass flow through the nozzle, the velocity (in the axial direction) of

* The nozzle throat, as defined herein, is the intersection of the nozzle contour with the plane which is normal to the general direction of flow and at the point of minimum cross sectional area of the nozzle. The plane through the nozzle throat is used as a fixed reference for the coordinate system and hence for measuring the nozzle length.

the combustion gases at the nozzle exit, and the pressure difference between the exhaust flow and the ambient conditions. The specific impulse of a nozzle is the total axial thrust divided by the weight flow rate of propellant and is a measure of the nozzle efficiency.

One of the objectives of a nozzle designer is to obtain the maximum thrust from a nozzle under a given set of operating conditions. In general, this is accomplished by increasing the exhaust velocity in the desired direction of thrust and by decreasing the difference in pressure between the combustion gases at the nozzle exit and the ambient value.

The mass flow rate through the nozzle is determined by the throat area and the operating conditions in the combustion chamber. For a given set of operating conditions in the combustion chamber and a fixed throat area, the flow geometry or the design of the subsonic portion of a nozzle contour may influence only the flow up to the throat section and will have no effect on the flow field beyond the throat. For this reason the performance of a thrust nozzle may be considered to depend almost entirely on the design of the supersonic diverging portion of the nozzle contour.

A complete analysis of the flow in a nozzle should account for (a) the state of the medium of flow, (b) the stress-strain relations governing the flow, and (c) the mass, momentum, and energy transfer processes associated with the flow. While the changes in performance from introducirg the effects of such parameters may prove to be of great importance in practice, initial comparisons of gross performance parameters can be obtained by assuming an adiabatic expansion of an inviscid, ideal gas with a constant ratio of specific heats; that is, it is both thermally and calorically perfect. The value of the thrust obtained from any of the flow geometries considered under such approximations may be vastly different from the actual value obtained in practice.

For a fixed set of operating conditions (initial gas conditions and the ambient conditions), the best performance in terms of thrust in a given direction is obtained from a nozzle which isentropically expands the (combustion) gases to a uniform (supersonic) speed at the exit plane of the nozzle under the conditions that (a) the entire stream is oriented at the exit plane in the desired direction of the thrust, and (b) the pressure at the exit plane is no different from the ambient pressure. Such a nozzle contour is referred to as a perfect nozzle. When the entire flow is axisymmetric, it is clear that a perfect nozzle with the aforementioned constraints provides the maximum thrust.

Two-dimensional perfect nozzles are widely used in wind tunnel construction. However, perfect nozzles tend to become very long; consequently, they are not employed as thrust nozzles because of their excessive length and weight.

On the other hand, in a practical rocket motor several other constraints may be imposed while still requiring the maximum value of thrust to be generated in a particular direction. Then the conditions specified for determining the flow geometry of the nozzle become (a) the initial conditions of flow, say at the throat section, (b) the ambient conditions, and (c) the other conditions constraining the flow.

Those other constraining conditions may be related to (i) the geometry of the flow, such as the length* of the nozzle, the shape of the throat, and/or the shape of the exit plane, (ii) the surface area of the flow geometry—governing the heat transfer (or momentum loss), and/or (iii) some other conditions related to an aspect of the fabrication or overall-system design objectives. In short, a number of alternative conditions may be imposed as constraints in the design of a thrust nozzle; in each case, the objective of the designer may remain the same, namely the determination of the nozzle contour which will yield the best value of thrust and satisfy all of the constraint conditions imposed. The problem of optimization arises precisely in that situation; for a given set of initial and constraining conditions, the flow geometry which yields the maximum value of one performance parameter (for example, the thrust) is to be determined. Mathematically, the determination of such a flow geometry requires showing that such a geometry exists and is unique for a given set of initial and constraint conditions.

The application of optimization techniques to the design of rocket motor nozzle contours to obtain the maximum thrust under various constraining conditions has been the subject of considerable interest over the past decade[1-8**]. All of those analyses pertain to axisymmetric

* The length of the nozzle is the axial distance between the fixed reference plane at the nozzle throat and a point on the exit boundary of the nozzle; therefore unless the exit boundary of the nozzle is on a plane parallel to the fixed reference plane (at the throat) the length will be different at various points on the exit contour. For the general three-dimensional nozzle contour the length may thus be a function of the angular coordinate, namely ϕ.

** Superscripted numerals refer to references listed in the Bibliography.

flow geometries, and the methods developed are widely used currently for the practical design of thrust nozzles. It is becoming increasingly apparent, however, that thrust nozzles may be required to have flow geometries that cannot be approximated adequately by a simple two-dimensional or axisymmetric shape[9].

1.1 Survey of Literature

In the formulation[1,2] of the optimization problem, the operating conditions in the combustion chamber, the subsonic and the transonic parts of the nozzle contour and the mass flow rate have been considered to be known and to remain fixed. One is therefore concerned with the determination of the supersonic portion of the contour that will maximize the thrust and satisfy the constraints imposed. The most common constraint is ordinarily related to the length of the nozzle.

The original formulation of the problem is due to Guderley and Hantsch[1] and utilizes the optimization methods based upon the calculus of variations. The essential elements of the problem formulation are best explained by referring to the axisymmetric nozzle contour illustrated in Fig. 1.1. In addition to the subsonic contour A T, an initial expansion arc T B B' is considered to be given. The flow field in the core region of the nozzle, T B C D O, denoted as the kernel, is then uniquely determined by the fixed initial conditions in the throat region and the prescribed initial expansion contour. The essence of the formulation then consists in postulating and introducing a control surface, C E, in order to determine the axial momentum and the other quantities of interest.

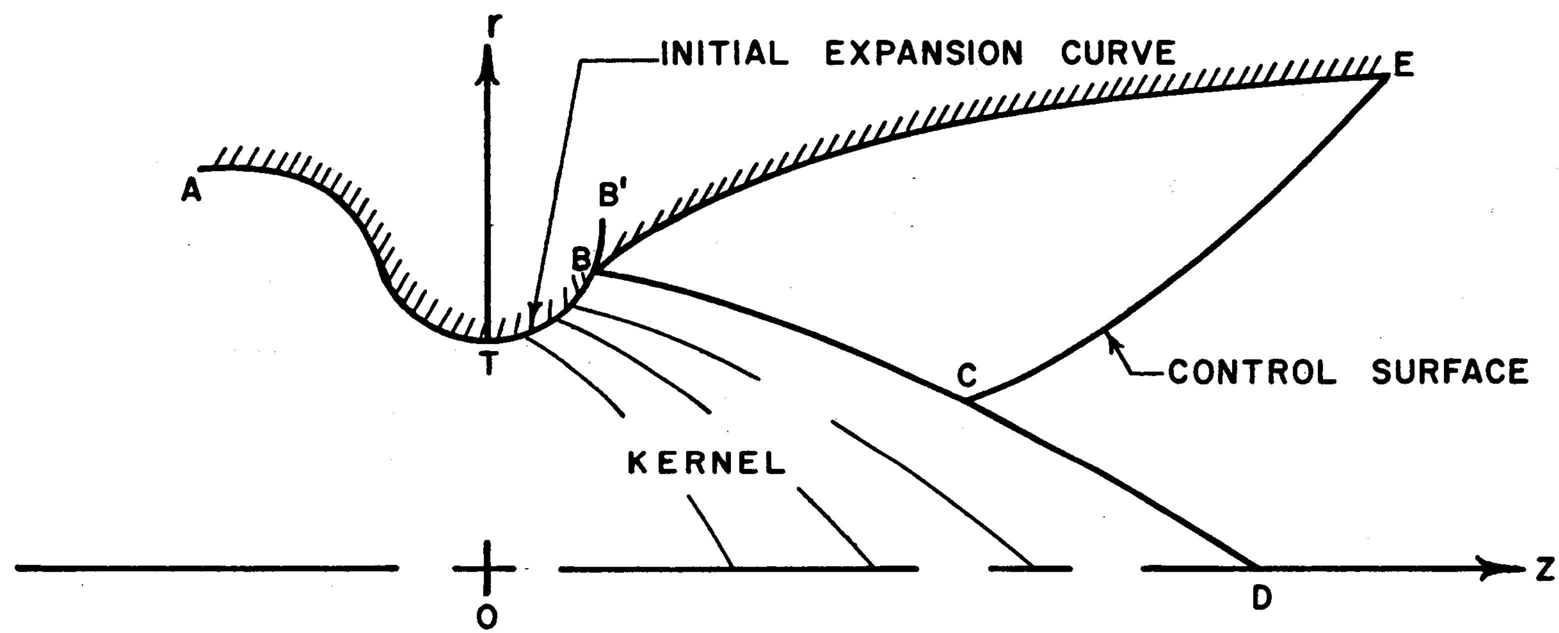

FIGURE I.1

AXISYMMETRIC NOZZLE CONTOUR ILLUSTRATING THE INITIAL EXPANSION CURVE, THE KERNEL AND THE CONTROL SURFACE

The particular control surface which, along with the kernel, constitutes the final solution is that control surface across which total axial momentum is the maximum, subject to the constraints that are imposed. Guderley and Hantsch[1] imposed the constraints that

(a) the mass flow rate through the nozzle remains a given and constant value,

(b) the length of the nozzle may not vary but remains at a given value, and

(c) the control surface is a characteristic surface.*

Condition (c) in reality imposes two constraints on the control surface; namely, that it be oriented in the characteristic direction and that the compatibility equations for a characteristic be satisfied on the control surface. The optimization problem is formulated using Lagrange multipliers to impose the constraint conditions. By employing the calculus of variations, a set of design equations is obtained from which the control surface may be located and the flow properties determined on it. It then only remains to find the flow between the kernel and the control surface. The details of the procedure required for laying out that portion of the flow are given in Ref. 5.

Sometime later Rao[2] discovered that it is not necessary to impose specifically the constraint that the control surface be a characteristic

* Guderley and Hantsch[1] imposed the condition that the control surface be a characteristic surface to ensure that the derived flow variables on the control surface could be matched to the flow in the kernel. Although this appears to be the most plausible method of formulating the problem it has proven to be less desirable than the formulation due to Rao[2].

surface. Instead, by allowing the control surface direction to be arbitrary, the resulting design equations could, in fact, be proved to require the control surface to be a characteristic surface. Thus, it has been established that once a control surface is postulated it will be uniquely determined as a characteristic surface when the constraints are appropriately chosen. The choice of the constraining relations is critical in the formulation and solution of the problem.

Both Guderley and Hantsch[1] and Rao[2] pose the problem of obtaining the flow geometry which will produce the maximum value of momentum in the desired direction of thrust (axial direction for an axisymmetric nozzle) under the constraints that (a) the entire flow geometry is axisymmetric, (b) the axial length of the region of flow over which the acceleration occurs is fixed, and (c) the initial conditions, including the mass flux, are fixed. However, each of the authors presents a different formulation of the problem and as a result they obtain different forms of the solution which are not clearly demonstrated to be equivalent to each other.

An excellent comparison of the two methods of posing the same problem has been presented by Guderley[3] who also derived the axisymmetric design equations for the non-homentropic flow case (i.e., the case of constant total enthalpy and constant entropy on a streamline but with allowable variations in entropy between streamlines).

It may be pointed out that the design equations derived by Rao[2] are considerately simpler when compared to those derived by Guderley and Hantsch[1]. The relative simplicity of Rao's solution for the optimized nozzle design has resulted in the wide practical use of the method.

A procedure for optimizing the thrust of axisymmetric nozzles subject to other geometric constraint conditions, such as a prescribed surface area, has been developed by Guderley and Armitage[6]. The procedure is again based on the optimization methods utilizing the calculus of variations but the problem is formulated in a different manner so as to make the nozzle boundary the control surface.

A more extensive discussion on the methods of design of axisymmetric nozzles is presented in Ref. 5.

1.2 Three-Dimensional Nozzle Flow

A general three-dimensional rocket motor nozzle is completely determined when the following data are prescribed:

(i) initial conditions,

(ii) total thrust,

(iii) nozzle shape in the transverse plane,

(iv) variation of the shape in the transverse plane along the meridional axis,

(v) exit flow conditions, and

(vi) the shape of at least one streamline.

Such data have to be given before a nozzle contour can be determined by analytical means.

For the purposes of analysis of the performance of a rocket motor nozzle, the following data are required:

(i) initial conditions and

(ii) complete nozzle contour.

In any case, it is first necessary to set up the governing equations for the state and the motion of the gas.

The procedure for the computation of a three-dimensional supersonic flow utilizing the method of characteristics is still under development. Several alternative procedures and their relative merits are discussed in Ref. 10, wherein two of the five recommended methods are presented in some detail.

The procedure for the optimum design of a three-dimensional thrust nozzle contour can be considered as a generalization of the method developed by Guderley and Hantsch[1] and by Rao[2] for the optimum design of axisymmetric thrust nozzles. However the problem of three-dimensional nozzle flows is considerably complicated by the following features:

1. a homentropic* flow is not necessarily an irrotational flow;
2. the control surface which is postulated for the purpose of determining the flow parameters is required to match the flow properties both on the boundary of the kernel (which it intersects) and on the exit contour of the nozzle (with which it coincides);
3. no streamline may be present in the flow regime which can be described simply with reference to a chosen coordinate system;
4. no planes of symmetry may be present in the flow regime and there may be no axis of symmetry at the "throat" or the "exit" boundary; and
5. the thrust may be computed with respect to an arbitrary direction.

* The term homentropic refers to a flow with constant specific entropy throughout the physical domain of flow. See Ref. 11, page 3.

If one considers an axisymmetric nozzle and examines the five aforementioned complicating features, it is obvious that they correspond to the following features pertaining to the axisymmetric case:

1. a homentropic flow is necessarily irrotational;
2. the control surface is axisymmetric thus requiring that the exit plane of the nozzle as well as the plane of intersection of the control surface with the kernel be planes normal to the axis of the nozzle. Further, the velocity and flow properties on the control surface as well as the length of the nozzle are independent of the angular coordinate;
3. the axis of the nozzle is a streamline;
4. not only are there planes of symmetry, but there is axial symmetry by definition throughout the flow regime; and
5. while the thrust may be calculated with respect to an arbitrary direction, the axial direction is generally the natural choice.

It is clear, therefore, that both in the formulation of the problem and in the development of a methodology for the application of the solution the three-dimensional nozzle will present entirely new features. Nevertheless, the general principles involved are the same as those employed for axisymmetric flows.

1.3 The Optimization Problem

The formulation of the optimum design problem consists of considering the flow across a three-dimensional control surface which is constrained to pass through the nozzle exit contour and to intersect the three-dimensional kernel, but otherwise, it is an arbitrary three-dimensional surface. As in the design of optimized axisymmetric nozzle contours, the emphasis is on the supersonic portion of the contour. Consequently, the operating conditions in the combustion chamber as well as the contours of the subsonic, transonic, and initial expansion portions of the nozzle wall are determined by design criteria other than thrust optimization and are considered to be known for purposes of thrust maximization. In addition, the flow field in the kernel (which is uniquely determined by those initial conditions and the prescribed portion of the nozzle contour) is considered to be known in the optimization problem.

The formulation of the problem is restricted by the following assumptions:

1. the flow is homentropic and irrotational throughout the flow regime;
2. the flow regime includes one straight streamline which coincides with the coordinate direction representing the general direction of flow; and
3. the desired direction of maximum thrust is the direction represented by the straight streamline mentioned under 2 in the foregoing.

The constraints imposed are the following:

1. the initial conditions at the throat section are given in a region where at every point the flow is supersonic; so also are the ambient conditions given;
2. the mass flux through the control surface is given and may not be varied;

3. the length of the nozzle is given as a specific function of the angular coordinate and may not be varied; and

4. the control surface is a continuous smooth surface.

It may be stated that the three basic restrictions and the four constraint relations are the only limitations under which the optimization problem is formulated.

The flow on the control surface is described in terms of five dependent variables V, θ, ψ, α and β.* The variables, V, θ, and ψ describe the velocity vector, $\vec{V}$, and the variables α and β define the direction of the unit normal to the control surface. The axial thrust and mass flow rate are expressed in terms of V, θ, ψ, α and β as integral equations over the control surface. The irrotationality condition on the control surface is derived in terms of a partial differential equation involving derivatives of V, θ, and ψ. The condition for maximum thrust, with a fixed mass flow and with irrotational flow, then requires that the variational integral, I, be stationary where the variational integral is formed using Lagrange multipliers to form a linear combination of the axial thrust, mass flow rate, and irrotationality constraint. The conditions imposed on the length and exit shape of the nozzle contour are imposed by substituting into the variational integral.

In order to solve the problem, additional relations are required in in the form of boundary conditions. Such boundary conditions will pertain to some or all of the following:

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* All symbols are defined later in Chapter 2 and an alphabetical listing of all symbols used with definition for each is included as Appendix A.

1. the functional relation between the length of the nozzle and the angular coordinate;
2. the shape of the physical boundaries at the throat section and at the exit plane of the nozzle;
3. the existence of planes of symmetry in the flow regime; and
4. the functional relation governing the variation of the velocity with the angular coordinate at the exit plane of the nozzle.

1.4 The Outline of the Thesis

As mentioned earlier, the determination of the optimized contour of a thrust nozzle involves essentially the supersonic portion of the nozzle. In the absence of viscous effects and under the assumption that the flow medium is a thermally and calorically perfect gas the problem of optimization becomes the determination of the flow contours for obtaining the maximum value of momentum in a particular direction, within a certain length of the flow regime, and with given initial and ambient conditions.

The coordinate system employed for the formulation of the problem and other significant features related to the flow are presented in Chapter 2.

Chapter 3 is devoted to the formulation of the problem under the restrictions of homentropic, irrotational flow in which one streamline coincides with the coordinate representing the general direction of flow and the length of the nozzle is given as a function of the angular coordinate. The solution of the problem is based upon the optimization procedures of the calculus of variations and results in a set of seven design equations which apply on a postulated control surface. The proof that such a control surface is a physical and uniquely-determined surface is obtained by showing that the control surface complies with all of the conditions required on a characteristic surface.

The seven design equations can be reduced to a set of four partial differential equations. In order to solve them one needs boundary conditions. Such boundary conditions may be obtained in many different forms and in relation to different geometric and flow parameters.

In Chapter 4, a set of boundary conditions is discussed which are related to (a) the initial flow conditions at the throat section of the nozzle, (b) the length of the nozzle as a function of the angular coordinate (in the particular example cited independent of the angular coordinate) and, (c) the minimum number of conditions for defining the variation of velocity on a prescribed shape at the exit plane of the nozzle. It should be emphasized that if the length of the nozzle is prescribed, the only boundary conditions that may be prescribed in relation to the geometry of the nozzle are the shapes of the nozzle at the throat section and at the exit plane of the nozzle. One then obtains the location and shape of the control surface which, along with the kernel of the flow, determines the entire flow field.

The methodology for the application of the solution of the optimization problem is discussed in Chapter 5 with reference to two examples as follows:

1. a nozzle in which the initial and ambient conditions and the length of the nozzle are prescribed; the shape of the nozzle at the exit plane is required to be an ellipse of given eccentricity but with variable area; and the exit contour is on a plane normal to a given axis;
2. a nozzle in which the initial and ambient conditions are prescribed; in particular the throat section is required to be circular; and the nozzle length and shape at the exit plane are the same as that obtained by arbitrarily truncating an optimized axisymmetric nozzle.

According to the theory there are ten boundary conditions to be satisfied considering both the inner boundary at the intersection of the control surface with the kernel and the outer boundary at the nozzle exit; however, the manner in which the boundary conditions are prescribed and the fact that each boundary is a curve on which the problem variables may not be constant makes it extremely difficult to ascertain just what constitutes one boundary condition. It is therefore necessary to rely on the formalism of the theory to provide the needed number of boundary conditions. The number of iteration procedures for solving a problem are inseparably connected to the manner in which the boundary conditions are prescribed and the number of boundary equations which are known on each boundary. The procedure for iteration in discussed briefly in Chapter 5.

The determination of the final methodology which may prove suitable under given conditions is an open problem, both in regard to finding a computational procedure as well as in regard to obtaining the desired degree of convergence of numerical solutions; nevertheless, it may be concluded that the existence of an optimized solution for a three-dimensional internal flow under appropriate constraints has been demonstrated and the broad outlines of the methodology required for the application of the solution have been established.

2. FORMULATION OF THE MATHEMATICAL PROBLEM

The optimum design of a thrust nozzle has usually been divided into three separate, though admittedly not independent, problems of design related to

1. the subsonic converging contour,
2. the transonic contour near the throat, and
3. the supersonic diverging contour.

In the region employed for matching the transonic contour with the supersonic contour of the nozzle, the initial expansion contour also needs to be determined. The initial state of the gas and the wall contour for the subsonic, transonic, and initial expansion regions of the nozzle are to be determined on the basis of design criteria other than thrust optimization (maximization), and, consequently, will be treated as known quantities with respect to the thrust maximization problem. Thus, the problem of optimum design of a nozzle to be discussed in this research report concerns only the determination of the supersonic portion of the nozzle contour beyond the initial expansion contour.

Figure 2.1 is a schematic representation of a general three-dimensional nozzle contour. The zone of influence of the initial expansion contour of the nozzle is denoted as the kernel and is the portion of the supersonic flow for which the flow variables are completely determined by the initial conditions and the prescribed subsonic, transonic and

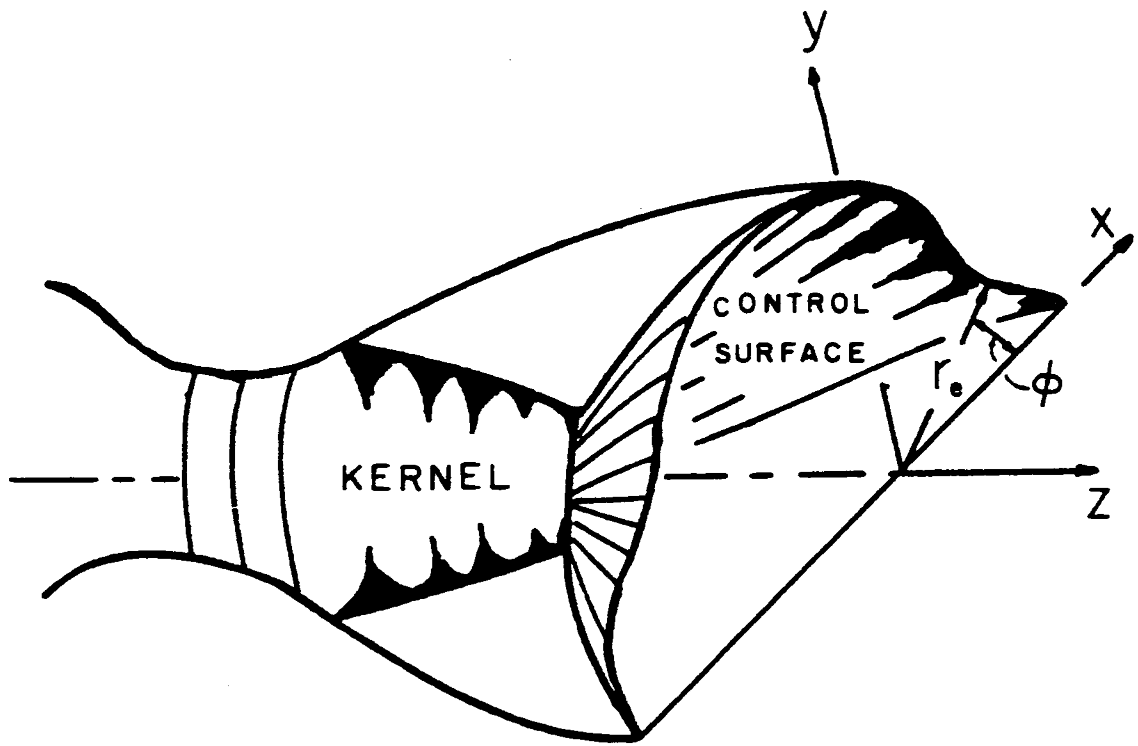

FIGURE 2.1

THREE-DIMENSIONAL NOZZLE

initial expansion contour. It is assumed that the flow variables in the kernel are determinable by applying the three-dimensional method of characteristics[10] and that the flow variables on the outer surface of the kernel, which are necessary for further analysis, are available. Consequently, the initial flow conditions must be specified as part of the problem and thus will act as constraints on the mathematical optimization problem.

In this report the only initial conditions which will be considered are the conditions of inviscid irrotational flow of a perfect gas with constant total enthalpy. Thus the constraints imposed by the initial conditions are constant entropy throughout the flow, constant total enthalpy and the irrotationality condition for the vorticity component along a streamline. These constraints are discussed in more detail in Section 2.2.3.

In order to formulate the optimization problem mathematically, a control surface, also illustrated in Fig. 2.1, is introduced. The control surface is constrained to pass through the nozzle exit contour and to intersect the kernel but is otherwise an arbitrary, three-dimensional surface. The axial momentum and mass flow rate are expressed as integral equations over the control surface. One of the essential steps in the solution of the mathematical problem is the establishment of the uniquely determinable character of the control surface. That is done by showing that the control surface is a characteristic surface. For convenience, therefore, a summary of the relationships governing characteristic surfaces in three-dimensional flow is included in Section 2.1.3 as part of the discussion of nomenclature.

In Section 2.2 the mathematical relationships pertaining to the control surface are deduced. The transformation relations which are employed to transform from the control surface to the two-dimensional (r, ϕ)-plane and the reverse transformations are also derived therein. In Section 2.3 the variational integral is formed and the variational relations are derived. Section 2.4 summarizes the overall mathematical problem.

2.1 Nomenclature

For convenience the symbols employed are listed alphabetically in Appendix A. All symbols are defined when first introduced but will be without definition thereafter. Standard notation has been employed as far as possible.

2.1.1 Coordinate System

The standard (r, ϕ, z)-cylindrical coordinate system illustrated in Fig. 2.2, is used throughout as the spatial reference. The z-axis is oriented along the straight nozzle axis, the r-coordinate is measured radially from the z-axis to the r-axis.

The velocity vector, $\vec{V}$, at any point is defined in terms of its magnitude, V, and the spherical angles θ and ψ as illustrated in Fig. 2.3. The angle θ is measured counterclockwise from z to $\vec{V}$. The angle ψ is measured in the (r, ϕ)-plane, counterclockwise from r to the projection of $\vec{V}$ on the (r, ϕ)-plane. The direction cosines of the velocity vector with respect to the (r, ϕ, z)-coordinates are

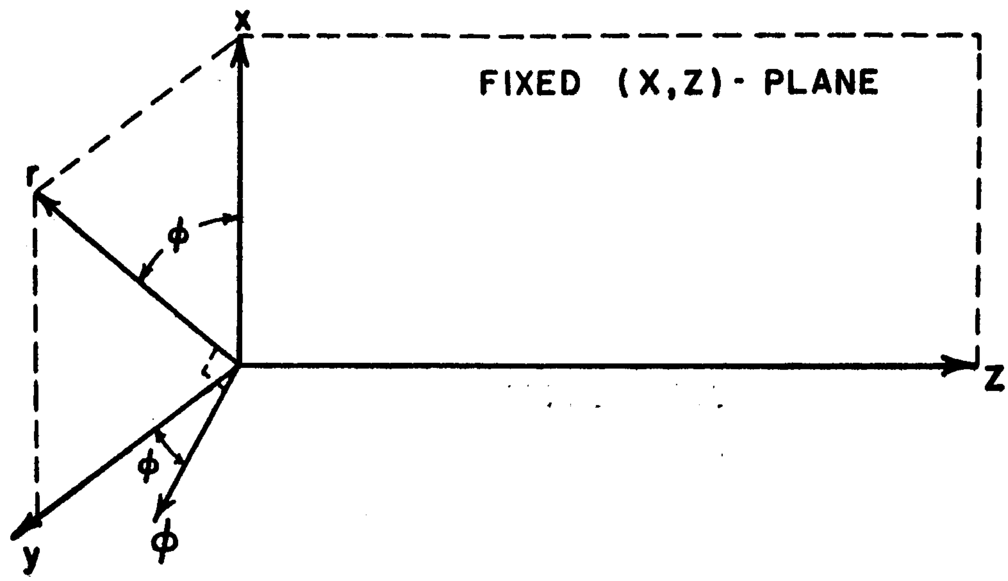

FIGURE 2.2

(r, ϕ, z) CYLINDRICAL COORDINATE SYSTEM

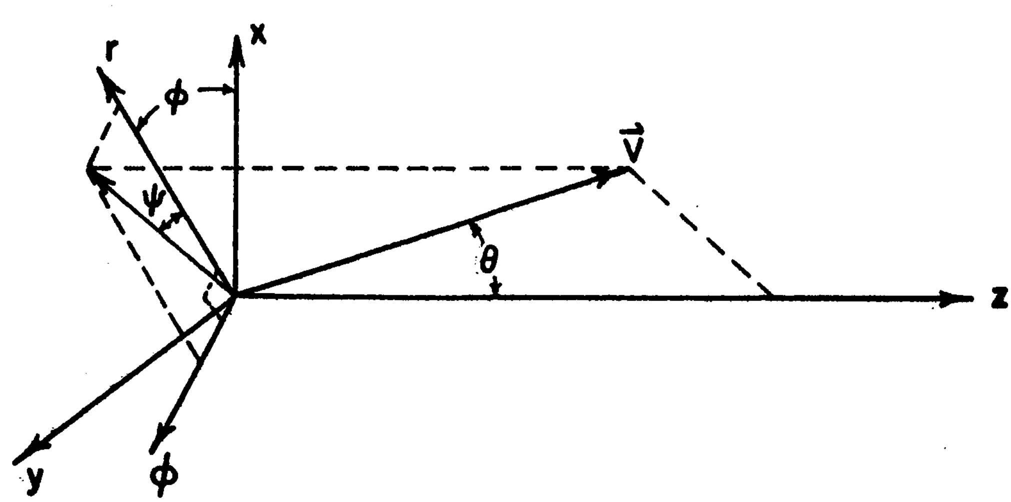

FIGURE 2.3

COMPONENTS OF THE VELOCITY VECTOR, $\vec{V}$

$$V_r = \sin\theta \cos\psi$$
$$V_\phi = \sin\theta \sin\psi \qquad (2.1)$$
$$V_z = \cos\theta$$

The control surface is defined in terms of the unit normal to the surface, $\vec{n}$. The direction of $\vec{n}$ is determined by the angles α and β as illustrated in Fig. 2.4. The angle β is measured counterclockwise from the z-axis to $\vec{n}$, and the angle α is measured in the (r, ϕ)-plane, counterclockwise from r to the projection of $\vec{n}$ on the (r, ϕ)-plane. Thus, the direction cosines of $\vec{n}$ are

$$n_r = -\sin\beta \cos\alpha$$
$$n_\phi = -\sin\beta \sin\alpha \qquad (2.2)$$
$$n_z = \cos\beta$$

2.1.2 Pressure, Density, Entropy and Vorticity Relationships

The problem is limited to the inviscid, irrotational, homentropic flow of a perfect gas. Using the perfect gas relationship, viz.

$$P = \rho R T \qquad (2.3)$$

the equation of state for a homentropic flow is

$$\frac{P}{\rho^\gamma} = \frac{P_o}{\rho_o^\gamma} = \text{constant} \qquad (2.4)$$

where P is the pressure, T is the temperature, ρ is the density, R is the gas constant, γ is the specific heat ratio, and the subscript o denotes initial (total) conditions.

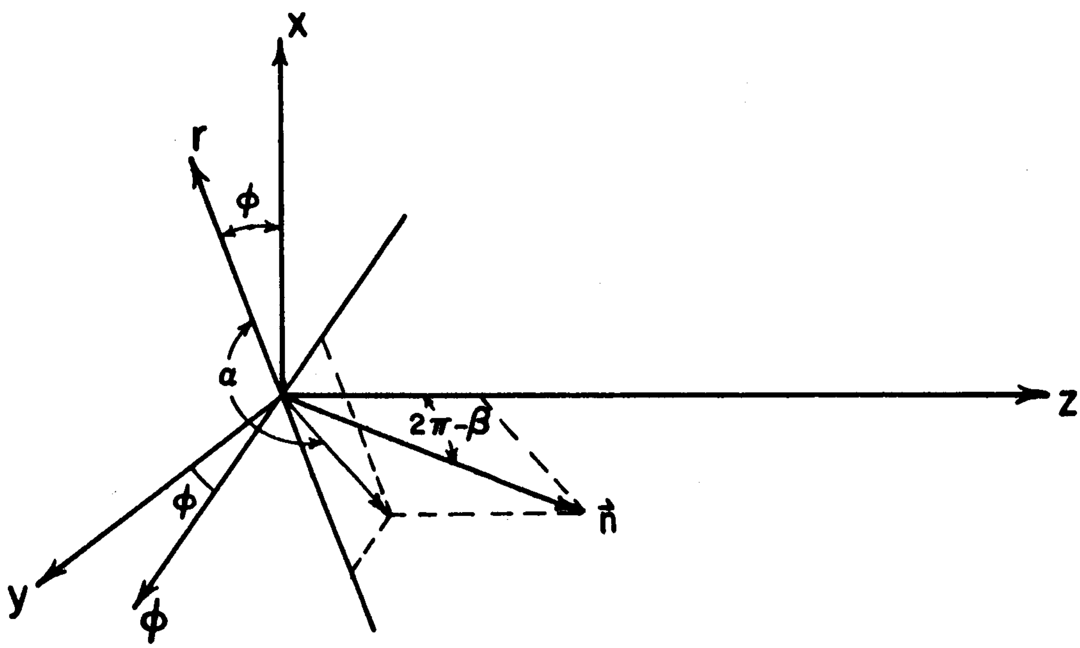

FIGURE 2.4

COMPONENTS OF THE NORMAL TO THE CONTROL SURFACE, $\vec{n}$

The sound speed, c, at any point is defined by the relation

$$c^2 = \left(\frac{\partial P}{\partial \rho}\right)_s \tag{2.5}$$

where the subscript s denotes a constant entropy process. Thus, from eqns. (2.4) and (2.5)

$$c^2 = \left(\frac{\partial P}{\partial \rho}\right)_s = \frac{\gamma P}{\rho} \tag{2.6}$$

Bernoulli's equation can be written

$$\frac{c^2}{\gamma - 1} + \frac{V^2}{2} = \frac{c_o^2}{\gamma - 1} = \text{constant} \tag{2.7}$$

Consequently, the pressure, density, and sound speed are functions of V and the initial conditions so that the following differential relationships are valid:

$$dP = -\rho V\, dV \tag{2.8}$$

$$d\rho = -\frac{\rho V}{c^2}\, dV \tag{2.9}$$

$$dc = -\left(\frac{\gamma - 1}{2}\right)\frac{V}{c}\, dV \tag{2.10}$$

For future use, the Mach angle, μ, is defined by the equation

$$\mu = \sin^{-1}\frac{1}{M} = \sin^{-1}\frac{c}{V} \tag{2.11}$$

where M is the Mach number. Hence the differential $d\mu$ can be written in terms of the differential dV as

$$d\mu = \frac{-\left(\frac{\gamma - 1}{2} + \sin^2\mu\right)}{V \sin\mu \cos\mu}\, dV \tag{2.12}$$

In the formulation of the optimization problem the homentropic flow constraint is imposed by eliminating the pressure and density derivatives by substitution from eqns. (2.8) and (2.9). In a general three-dimensional flow, however, a constant entropy does not assure an irrotational flow*. That is, the entropy gradients in a flow are related only to the components of vorticity which are normal to the streamline, and, therefore, a vorticity vector can exist along a streamline even in a homentropic flow.[12]

The vortex vector, $\vec{\omega}$, is defined as the curl of the velocity vector. Hence,

$$\vec{\omega} = \nabla \times \vec{V} = \text{Curl } \vec{V} \tag{2.13}$$

In terms of the cylindrical coordinates r, ϕ, and z the components of $\vec{\omega}$ are

$$\omega_r = \frac{\partial(V \cos \theta)}{r\partial\phi} - \frac{\partial(V \sin \theta \sin \Psi)}{\partial z} \tag{2.14}$$

$$\omega_\phi = \frac{\partial(V \sin \theta \cos \Psi)}{\partial z} - \frac{\partial(V \cos \theta)}{\partial r} \tag{2.15}$$

$$\omega_z = \frac{1}{r}\left[\frac{\partial(r\, V \sin \theta \sin \Psi)}{\partial r} - \frac{\partial(r\, V \sin \theta \cos \Psi)}{r\partial\phi}\right] \tag{2.16}$$

For irrotational flow all three components of the vorticity vector must be identically zero.

2.1.3 Three-Dimensional Characteristic Relationships

One of the essential steps in the solution of the mathematical problem is the establishment of the uniquely determinable character of the control surface. The proof of such uniqueness rests here on showing that the control surface is a characteristic surface. Accordingly, the terminology and equations governing characteristic surfaces are summarized here for future reference.

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* The vorticity vector is related to the entropy gradient by Crocco's equation which for a steady, three-dimensional flow with constant total enthalpy is $\vec{V} \times \vec{\omega} = -T\,\nabla s$. If $\vec{\omega}$ is zero then ∇s must be zero, but if ∇s is zero then either $\vec{\omega}$ is zero or $\vec{V}$ is parallel to $\vec{\omega}$. In two-dimensional or axisymmetric flows the latter possibility, namely $\vec{V}$ parallel to $\vec{\omega}$, does not exist.

The governing partial differential equations for supersonic flow are of hyperbolic type. It is a feature of hyperbolic equations that they possess unique directions called characteristic directions. The solution of hyperbolic partial differential equations utilizing the special properties of the equations along characteristic directions is commonly referred to as the method of characteristics.

The application of the method of characteristics to three-dimensional flow has received considerable attention over the past decade. The basic concepts and the fundamental equations required for applying the method of characteristics to compute the three-dimensional flow in nozzles are given in detail in Ref. 10. No attempt will be made here to duplicate that work, but it may be pointed out that the application of the techniques of the method of characteristics to three-dimensional supersonic flow fields is an essential part of the overall design problem. Considerable care is required in the choice of the technique, particularly from the point of view of convergence and non-occurrence of singularities.

Terminology: Each point Q in a supersonic flow field is associated with a Mach conoid or characteristic conoid as illustrated in Fig. 2.5. The right circular cone formed by the tangents to the Mach conoid at Q is the characteristic cone or Mach cone. The rays of the characteristic cone make the angle μ with the velocity vector when μ is the Mach angle defined by eqn. (2.11).

Associated with each point on a non-characteristic line such as T T' (illustrated in Fig. 2.6) is a characteristic conoid. The two surfaces tangent to the characteristic conoids and containing the line

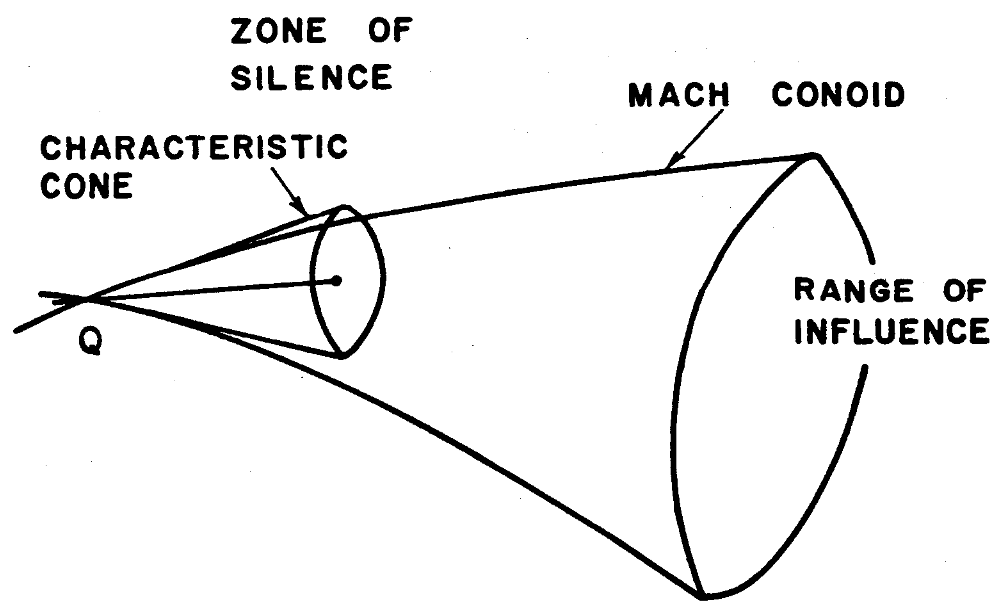

FIGURE 2.5

THREE - DIMENSIONAL MACH CONOID

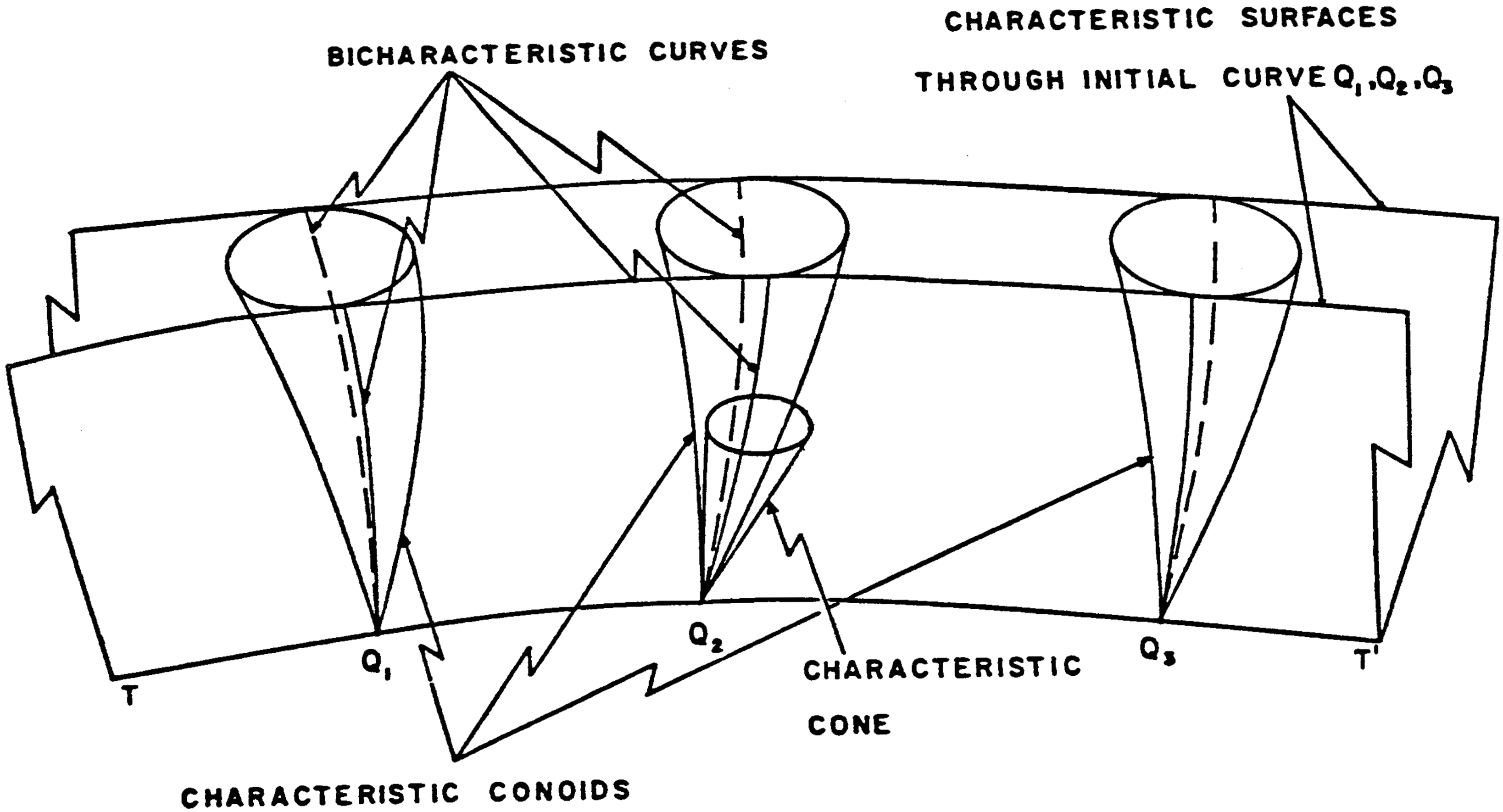

FIGURE 2.6

RELATIONSHIP BETWEEN BICHARACTERISTICS AND CHARACTERISTIC CONOIDS, CONES AND SURFACES

T T' are characteristic surfaces. The intersection of a characteristic conoid and a characteristic surface is a bicharacteristic curve. The relationships between bicharacteristics, characteristic conoids, characteristic cones, and characteristic surfaces are illustrated in Fig. 2.6.

Characteristic Surfaces: In a three-dimensional supersonic flow the characteristic surfaces are of fundamental importance since the governing equations for the three-dimensional flow reduce to two equations on each characteristic surface. For a surface to be a characteristic surface it is necessary and sufficient that

(a) the surface be oriented in a characteristic direction, and

(b) the so-called compatibility equations apply on the surface.

Characteristic Directions: A surface is oriented in a characteristic direction if it is tangent to the characteristic conoids associated with each point on the surface. Thus, if at every point on a surface the velocity vector, $\vec{V}$, and the unit normal to the surface, $\vec{n}$, satisfy the relation

$$\frac{\vec{V} \cdot \vec{n}}{V} = \pm \sin \mu \qquad (2.17)$$

the surface is oriented in a characteristic direction.

Compatibility Equations: The compatibility equations associated with the method of characteristics for supersonic flow are the governing equations (i.e., the equations of conservation of mass, momentum and energy) for the flow transformed to a coordinate system on a characteristic surface. The compatibility equations (for several coordinate systems) are derived in Ref. 10.

Figure 2.7 illustrates the coordinate relationships at a point in a supersonic, three-dimensional flow. The direction L is along a bicharacteristic. The direction N is normal to L and in the tangent plane to the characteristic cone at the point Q. Consequently, the (L, N)-coordinates lie on the characteristic surface. The direction of the velocity vector, $\vec{V}$, is defined by the angles θ and ψ which are measured in accordance with the previously established convention (see Fig. 2.3). The angle δ is the angle between the ($\vec{V}$, z)-plane and the ($\vec{V}$, L)-plane.

Denote the unit vectors in the L and N directions by $\vec{L}$ and $\vec{N}$ respectively, and let the components of $\vec{L}$ and $\vec{N}$ in the r, ϕ and z directions be denoted by L_r, L_ϕ, L_z, N_r, N_ϕ, and N_z. These components are related to the components of the unit normal to the characteristic surface, $\vec{n}$, and to $\vec{V}$ by the equations $\vec{V} \cdot \vec{N} = 0$; $\vec{n} \cdot \vec{N} = 0$; $N_r^2 + N_\phi^2 + N_z^2 = 1$; $\vec{L} \cdot \vec{n} = 0$; $\vec{L} \cdot \vec{N} = 0$; $L_r^2 + L_\phi^2 + L_z^2 = 1$ and $(\vec{V} \cdot \vec{L})/V = \cos\mu$. Solving these vector equations for N_r, N_ϕ, N_z, L_r, L_ϕ, and L_z yields the relationships

$$N_r = \pm \frac{n_z V_\phi - n_\phi V_z}{\cos\mu} \tag{2.18}$$

$$N_\phi = \pm \frac{n_r V_z - n_z V_r}{\cos\mu} \tag{2.19}$$

$$N_z = \pm \frac{n_\phi V_r - n_r V_z}{\cos\mu} \tag{2.20}$$

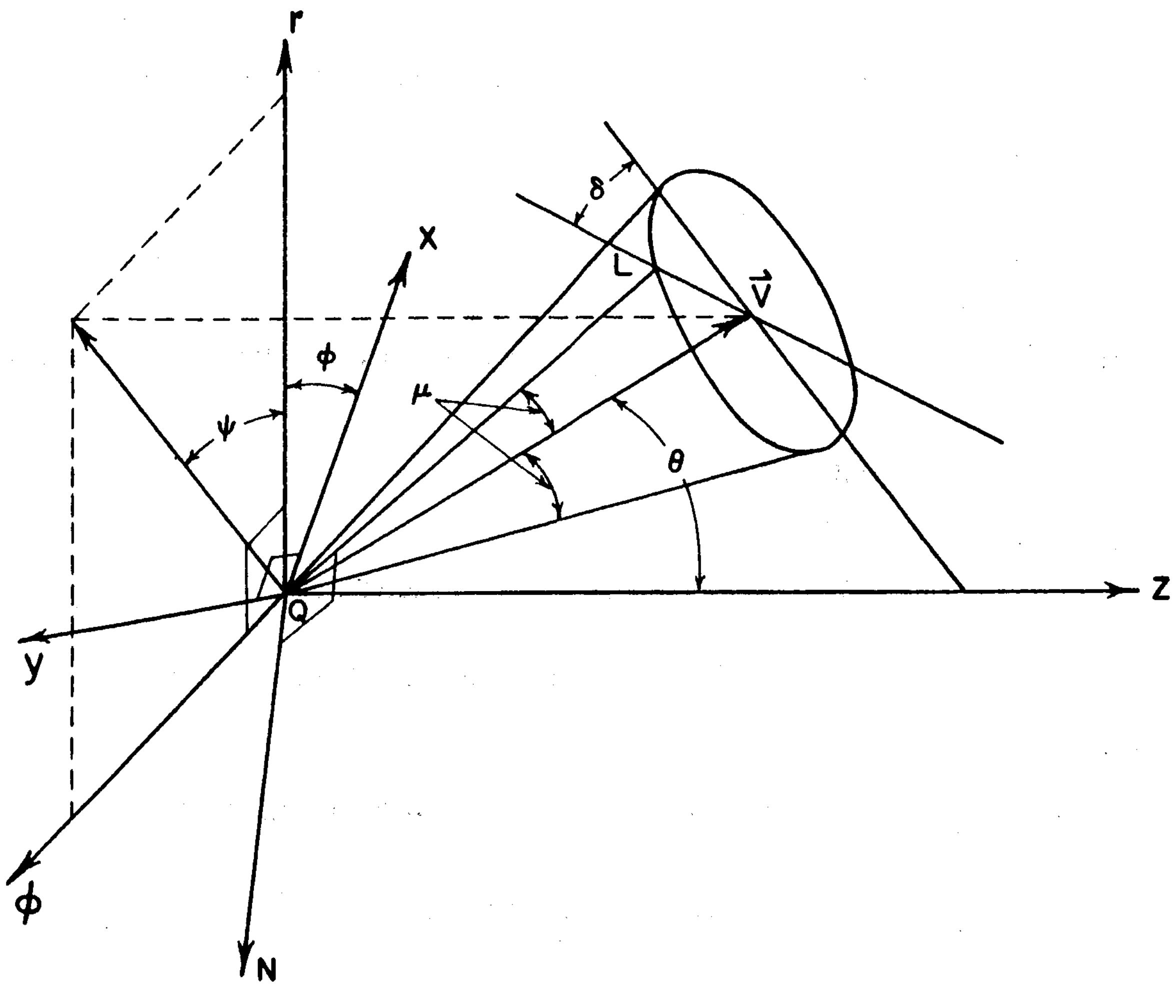

FIGURE 2.7

COORDINATE SYSTEM FOR
THREE DIMENSIONAL CHARACTERISTICS

$$L_r = \pm \frac{n_r \sin \mu - V_r}{\cos \mu} \quad (2.21)$$

$$L_\phi = \pm \frac{n_\phi \sin \mu - V_\phi}{\cos \mu} \quad (2.22)$$

$$L_z = \pm \frac{n_z \sin \mu - V_z}{\cos \mu} \quad (2.23)$$

Now one can determine the angle δ from the vector relationships

$$\cos \delta = \frac{\vec{N} \cdot (\vec{z} \times \vec{V})}{|\vec{z} \times \vec{V}|} \quad (2.24)$$

and

$$\sin \delta = \frac{\vec{N} \times (\vec{z} \times \vec{V})}{|\vec{z} \times \vec{V}|} \quad (2.25)$$

where $\vec{z}$ denotes the unit vector along the z-axis. Evaluating eqns. (2.24) and (2.25) gives

$$\sin \delta = \mp \frac{\sin\beta \quad \sin(\psi - \alpha)}{\cos \mu} \quad (2.26)$$

and

$$\cos \delta = \pm \frac{\cos\beta \quad \sin \theta + \sin\beta \quad \cos \theta \cos (\psi - \alpha)}{\cos \mu} \quad (2.27)$$

It may then be shown[10,13] that the compatibility equations for homentropic irrotational flow are

$$\frac{1}{V}\left(\frac{dV}{dL}\right)_N - \tan \mu \left[\cos \delta \left(\frac{d\theta}{dL}\right)_N + \sin \delta \sin \theta \left(\frac{d(\psi + \phi)}{dL}\right)_N\right]$$

$$- \sin \mu \tan \mu \left[\cos \delta \sin \theta \left(\frac{d(\psi + \phi)}{dN}\right)_L - \sin \delta \left(\frac{d\theta}{dN}\right)_L\right] = 0 \quad (2.28)$$

and

$$\frac{1}{V}\left(\frac{dV}{dN}\right)_L + \tan \mu \left[\cos \delta \left(\frac{d\theta}{dN}\right)_L + \sin \theta \sin \delta \left(\frac{d(\psi + \phi)}{dN}\right)_L\right]$$

$$+ \frac{\sin \delta}{\cos \mu}\left(\frac{d\theta}{dL}\right)_N - \frac{\sin \theta \cos \delta}{\cos \mu}\left(\frac{d(\psi + \phi)}{dL}\right)_N = 0 \quad (2.29)$$

where subscripted parentheses or brackets enclosing a derivative denote differentiation on the characteristic surface in the direction which holds the subscripted variable constant.

2.2 Relationships on the Control Surface

The optimization problem is formulated by expressing the mass flow rate, axial thrust, and other quantities of significance in terms of their values either across or on the control surface.

The control surface for the nozzle, illustrated in Fig. 2.1, is an arbitrary three-dimensional surface constrained by the exit contour of the nozzle and the kernel of the flow. The unit normal vector, $\vec{n}$, with direction cosines given by eqns. (2.2) defines the control surface.

Since the control surface is a three-dimensional surface it is convenient to transform the pertinent equations to the two-dimensional (r, ϕ)-plane. The transformation corresponds physically to projecting the control surface onto the (r, ϕ)-plane.* Partial derivatives in the transformed (r, ϕ)-plane will be denoted by $(d/dr)_\phi$ and $(d/rd\phi)_r$ to differentiate them from the partial derivatives with respect to the three-dimensional spatial coordinates.

2.2.1 Transformation Equations

The control surface can be described parametrically by the equation

$$F\ (r, \phi, z) = 0 \qquad (2.30)$$

Equation 2.30) can be solved for z in terms of r and ϕ to give

$$z = f\ (r, \phi) \qquad (2.31)$$

which can be used to locate the control surface if the function $f(r, \phi)$ is known. The projection (transformation) of the control surface onto the (r, ϕ)-plane permits f(or z) to be considered as a dependent variable. The transformation equations are

* The vector quantities on the control surface are not projected in this transformation. For example, the quantities V, θ, and ψ are the same for corresponding points on the two surfaces but their gradients will transform in accordance with the transformation equations.

$$\left(\frac{d}{dr}\right)_\phi = \frac{\partial}{\partial r} + \frac{\partial f}{\partial r}\ \frac{\partial}{\partial z} \tag{2.32}$$

$$\left(\frac{d}{rd\phi}\right)_r = \frac{\partial}{r\partial\phi} + \frac{\partial f}{r\partial\phi}\ \frac{\partial}{\partial z} \tag{2.33}$$

The partial derivatives of f can be evaluated in terms of the angles α and β (which define the normal to the control surface) as

$$\frac{\partial f}{\partial r} = \tan\beta \quad \cos\alpha \tag{2.34}$$

$$\frac{\partial f}{r\partial\phi} = \tan\beta \quad \sin\alpha \tag{2.35}$$

Equations (2.32) - (2.35) are used for transformation from the control surface to the (r, ϕ)-plane. The optimization problem is solved in the (r, ϕ)-plane; however, as previously mentioned, the physical compatibility of the flow is assured by proving that the control surface is a characteristic surface. That proof involves a transformation of the design equations from the (r, ϕ)-plane back to the control surface. To determine the equations for that reverse transformation, the unit vector on the control surface in the plane defined by $\vec{n}$ and $\vec{V}$ is denoted by $\vec{L}$, and the unit vector on the control surface and normal to $\vec{L}$ is denoted as $\vec{N}$. The angle between $\vec{L}$ and $\vec{V}$ is ξ, so that

$$\frac{\vec{L}\cdot\vec{V}}{V} = \cos\xi \tag{2.36}$$

or, in a more convenient form

$$\frac{\vec{V}\cdot\vec{n}}{V} = \sin\xi \quad = \cos\beta \quad \cos\theta - \sin\beta \quad \sin\theta\cos(\Psi - \alpha) \tag{2.37}$$

where eqns. (2.1) and (2.2) have been used to evaluate the scalar product.

The equations for the transformation from the (r, ϕ)-plane to the (L, N)-plane (control surface) are

$$\left(\frac{d}{dr}\right)_\phi = a_1 \left(\frac{d}{dL}\right)_N + a_2 \left(\frac{d}{dN}\right)_L \tag{2.38}$$

$$\left(\frac{d}{rd\phi}\right)_r = b_1 \left(\frac{d}{dL}\right)_N + b_2 \left(\frac{d}{dN}\right)_L \tag{2.39}$$

where

$$a_1 = \pm \frac{n_r V_z - n_z V_r}{n_z \cos \varsigma} \tag{2.40}$$

$$a_2 = \pm \frac{V_\phi - n_\phi \sin \varsigma}{n_z \cos \varsigma} \tag{2.41}$$

$$b_1 = \pm \frac{n_\phi V_z - n_z V_\phi}{n_z \cos \varsigma} \tag{2.42}$$

$$b_2 = \mp \frac{V_r - n_r \sin \varsigma}{n_z \cos \varsigma} \tag{2.43}$$

2.2.2 Integral Equations

The axial thrust, T_z, and the mass flow rate, $\dot{m}$, are written as integral equations over the control surface.

<u>Axial Thrust</u>: The element of the axial momentum flux, dT_z, across the area element dA is

$$dT_z = V\, V_z\, d\dot{m} \tag{2.44}$$

where V_z is defined by eqn. (2.1), $V\, V_z$ is the axial component of velocity, and $d\dot{m}$ is the differential element of mass flow across the element of area dA shown in Fig. 2.8. Hence,

$$d\dot{m} = \rho\, \vec{V} \cdot \vec{n}\, dA \tag{2.45}$$

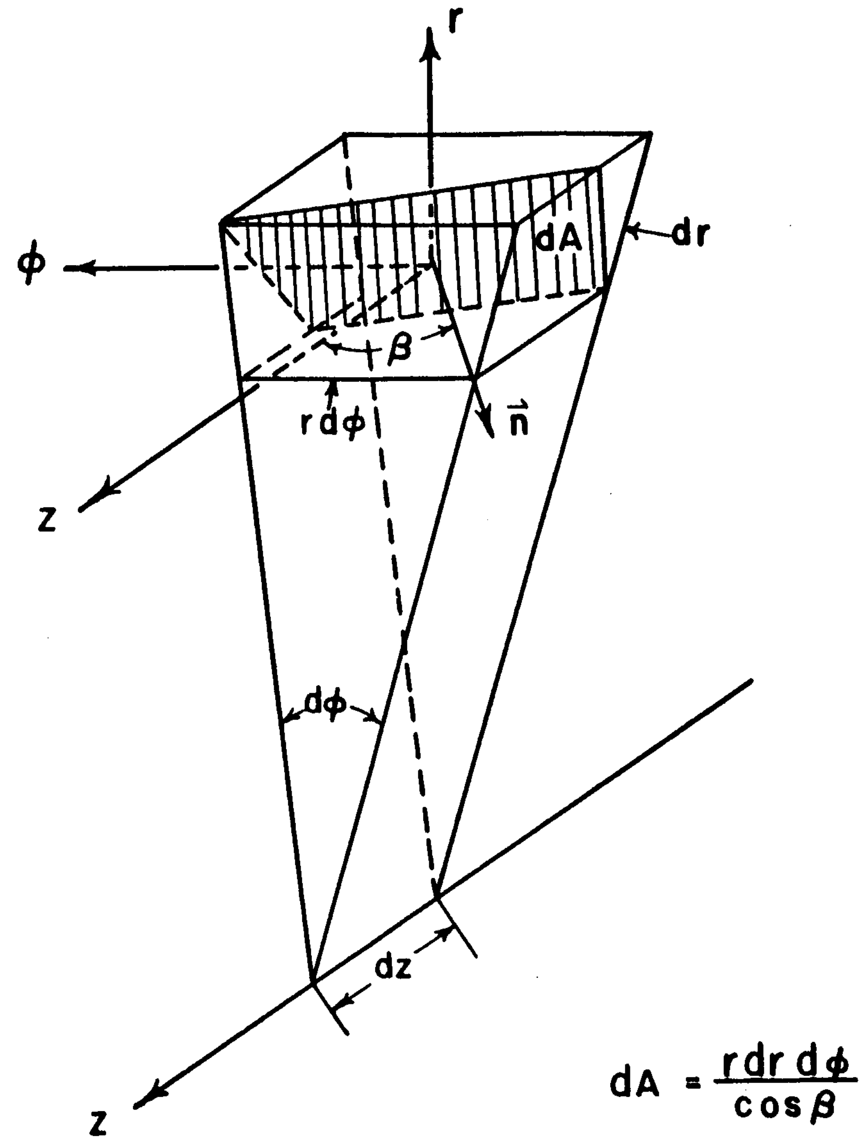

FIGURE 2.8

ELEMENT OF AREA dA ON THE CONTROL SURFACE

The scalar product $\vec{V} \cdot \vec{n}$ is evaluated from eqn. (2.37). The area element dA is given in terms of dr and dϕ in the transformed (r, ϕ)-plane by the equation

$$dA = \frac{r\,dr\,d\phi}{\cos\beta} \tag{2.46}$$

The axial thrust is the sum of the pressure differential and the axial momentum flux. Thus,

$$T_z = \iint_S F_1\,dr\,d\phi \tag{2.47}$$

where S is the area of integration in the (r, ϕ)-plane and

$$F_1 \equiv r\,(P - P_a) + \frac{r\rho\,V^2 \sin\xi\,\cos\theta}{\cos\beta} \tag{2.48}$$

The term $(P - P_a)$ represents the difference between the gas pressure at the control surface and the ambient pressure.

<u>Mass Flow Rate</u>: The mass flow rate is to be held constant in the optimization problem. Using eqns. (2.45) and (2.46), the mass flow rate is expressed as an integral over the area S by the equation

$$\dot{m} = \iint_S F_2\,dr\,d\phi = \text{constant} \tag{2.49}$$

where

$$F_2 \equiv \frac{r\rho\,V\sin\xi}{\cos\beta} \tag{2.50}$$

2.2.3 Constraint Equations

The constraints imposed on the problem are of two types, namely

(a) gas dynamic constraints, and

(b) geometric constraints.

Gas Dynamic Constraints: The gas dynamic constraints are imposed to ensure that the flow on the control surface can be matched to the flow in the kernel without violating the laws of gas dynamics (mechanics). To enumerate, firstly, the assumption of steady flow requires that the mass flow rate through the nozzle be a constant value which is determined by the prescribed initial conditions. A second gas dynamic constraint is related to the irrotational flow conditions which, in reality, is comprised of three separate conditions for the components of the vorticity vector. The vorticity components normal to a streamline are zero if the homentropic flow relations of Section 2.1.2 are imposed. In addition, however, the component of the vorticity vector along the streamline must be zero. That constraint condition will be satisfied, it may be observed, if the component of $\vec{\omega}$ in any direction other than the direction normal to $\vec{V}$ is zero. Therefore, the irrotationality condition will be satisfied by a homentropic flow if the component of the vorticity vector normal to the control surface is zero, provided, of course, that the velocity vector does not lie on the control surface.

Equations (2.42) - (2.45) are used to transform the components of the vorticity vector, eqns. (2.14) - (2.16), onto the (r, ϕ)-plane giving rise to the following definitive equation which merely states the assumption of irrotationality.

$$F_3 = A_1 \frac{1}{V} \left(\frac{dV}{dr}\right)_\phi + A_2 \frac{1}{V} \left(\frac{dV}{rd\phi}\right)_r + A_3 \left(\frac{d\theta}{dr}\right)_\phi$$

$$+ A_4 \left(\frac{d\theta}{rd\phi}\right)_r + A_5 \left(\frac{d\Psi}{dr}\right)_\phi + A_6 \left(\frac{d\Psi}{rd\phi}\right)_r + \frac{A_6}{r} = 0 \qquad (2.51)$$

where

$$A_1 = \sin\theta \cos\beta \quad \sin\psi \quad + \sin\beta \quad \cos\theta \sin\alpha \tag{2.52}$$

$$A_2 = - \sin\theta \cos\beta \quad \cos\psi \quad - \sin\beta \quad \cos\theta \cos\alpha \tag{2.53}$$

$$A_3 = \cos\theta \cos\beta \quad \sin\psi \quad - \sin\beta \quad \sin\theta \sin\alpha \tag{2.54}$$

$$A_4 = - \cos\theta \cos\beta \quad \cos\psi \quad + \sin\beta \quad \sin\theta \cos\alpha \tag{2.55}$$

$$A_5 = \sin\theta \cos\beta \quad \cos\psi \tag{2.56}$$

$$A_6 = \sin\theta \cos\beta \quad \sin\psi \tag{2.57}$$

Equation (2.51) requires the component of the vorticity vector normal to the control surface to be zero.

A third gas dynamic constraint is imposed on the boundary and requires the component of the velocity vector normal to the surface to be zero. This constraint is discussed in Chapter 4.

In summary the gas dynamic constraints are the constancy of the rate of mass flow, eqn. (2.49), the homentropic flow conditions, eqns. (2.8) and (2.9), the irrotationality condition, eqn. (2.51), and the boundary condition. The condition of constant mass flux and the irrotationality condition are imposed using the Lagrange multiplier technique, the homentropic flow conditions are imposed by substitution in the variational problem, and the boundary condition is imposed on the boundary.

Geometric Constraints: The geometric constraints are the conditions imposed on the nozzle geometry and are of primary interest to the design engineer. Since the solution to the problem under consideration is dependent on the flow variable relationships on the control surface, the geometric constraints must be expressed in terms of the variables on the control surface and its boundaries. For the most part the geometric constraints will be imposed on the outer boundary of the control surface, that is, the intersection with the nozzle exit, since the relationship between the constraint equations and the nozzle geometry can be interpreted more readily on this boundary.

Geometric constraints involving the control surface boundaries are discussed in Chapter 4 and include conditions on both the nozzle length and the exit geometry. Thus, for example, for an axisymmetric nozzle the condition of axial symmetry also can be considered to be a geometric constraint.

It should be noted that different boundary conditions can be imposed on the problem without affecting the design equations.

2.3 Variational Relations

The mathematical problem of optimization related to three-dimensional nozzle design consists of the formation of a variational integral and the application of the calculus of variations to determine the relationships among the flow variables on the control surface which will optimize the thrust.

2.3.1 Formation of the Variational Integral

The variational integral, I, is formed by using Lagrange multipliers to form a linear combination of the axial thrust and the constraint equations on the control surface. Thus,

$$I = \iint_S (F_1 + \lambda_2 F_2 + \lambda_3 F_3) \, dr \, d\phi \tag{2.58}$$

where F_1, F_2, and F_3 are defined by eqns. (2.48), (2.50), and (2.51) respectively, and λ_2 and λ_3 are Lagrange multipliers. Note that λ_2 must be a constant whereas λ_3 can be a function of the independent variables r and ϕ. For convenience the term G is defined as

$$G = F_1 + \lambda_2 F_2 + \lambda_3 F_3 \tag{2.59}$$

Upon expanding,

$$G = r\,(P - P_a) + \frac{(V\cos\theta + \lambda_2)\, r\rho V \sin\xi}{\cos\beta}$$

$$+ \lambda_3 \left[\frac{A_1}{V}\left(\frac{dV}{dr}\right)_\phi + \frac{A_2}{V}\left(\frac{dV}{rd\phi}\right)_r + A_3\left(\frac{d\theta}{dr}\right)_\phi + A_4\left(\frac{d\theta}{rd\phi}\right)_r + A_5\left(\frac{d\Psi}{dr}\right)_\phi \right.$$

$$\left. + A_6\left(\frac{d\Psi}{rd\phi}\right)_r + \frac{A_6}{r} \right] \tag{2.60}$$

where the coefficients A_1 - A_6 are defined by eqns. (2.52) - (2.57).

The control surface projected onto the (r, ϕ)-plane is illustrated in Fig. 2.9. The outer boundary, Γ, encloses the area of integration, S, and represents the nozzle exit contour. The variables evaluated on Γ are assigned the subscript e to denote the values which apply at the exit lip of the nozzle. For example, the exit radius of the nozzle is denoted by r_e, where r_e may be a function of ϕ. Thus eqn. (2.58) can be rewritten by including the limits of integration as

$$I = \int_0^{2\pi} \int_0^{r_e(\phi)} G\, dr\, d\phi \tag{2.61}$$

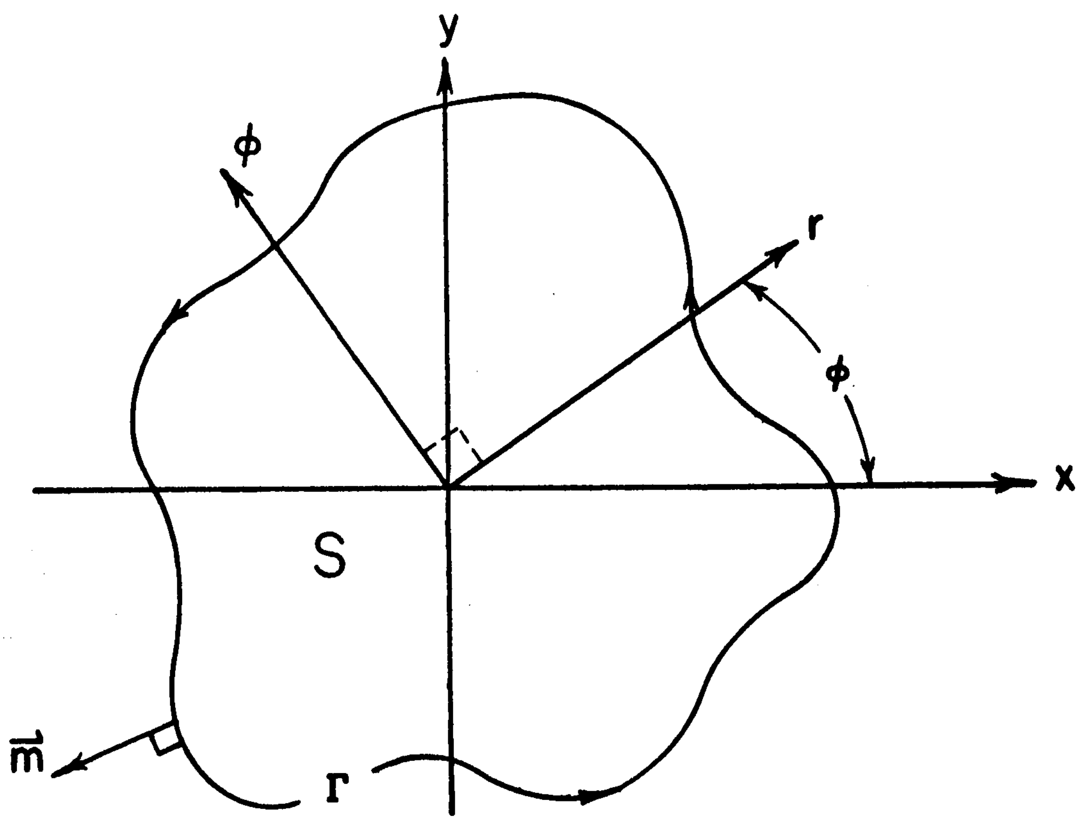

FIGURE 2.9

INTEGRATION AREA, S, OF THE NOZZLE CONTROL SURFACE PROJECTED ONTO THE (r,ϕ)- PLANE

2.3.2 Application of the Calculus of Variations

The next step in the solution of the optimization problem is to apply the calculus of variations to the variational integral. A didactic derivation of the variational relations which are required to solve the variational problem is given in Appendix B. The results of that analysis can be stated as follows:

Consider the integral

$$I(\epsilon) = \int_0^{2\pi} \int_0^{r_e(\epsilon)} G(r, \phi, w_i(\epsilon), R_i(\epsilon), T_i(\epsilon))\, dr\, d\phi \qquad (2.62)$$

where ϵ is the variational parameter. The variables $w_i (i = 1, 2, ..p)$ represent p dependent variables (such as the problem variables V, ϕ, Ψ , etc.); the terms R_i and T_i are defined by the equations

$$\text{(a) } R_i = \frac{\partial w_i}{\partial r} \; ; \qquad \text{(b) } T_i = \frac{\partial w_i}{r \partial \phi} \qquad (2.63)$$

and the domain of integration is illustrated in Fig. 2.9.

If the boundary Γ is permitted to vary, that is, if the integration limit r_e is a function of the variational parameter ϵ, then the variation of I in eqn. (2.62) is

$$\delta I = \int_0^{2\pi} \int_0^{r_e} E_i\, \delta w_i\, dr\, d\phi$$

$$+ \oint_{\Gamma} \left(\frac{1}{r} \frac{\partial G}{\partial R_i} \frac{\partial r}{\partial m} + \frac{\partial G}{\partial T_i} \frac{\partial \phi}{\partial m}\right) \delta w_i\, dl$$

$$+ \int_0^{2\pi} G \Big|_{r_e} \frac{dr_e}{d\epsilon} \Big|_{\epsilon = 0} d\epsilon\, d\phi \qquad (2.64)$$

In eqn. (2.64), the element dl is along the boundary Γ in a positive sense (keeping the area S always on the left); the vector $\vec{m}$ is the unit outward normal to Γ as illustrated in Fig. 2.9; the term E_i is the Euler-Lagrange equation for the $i^{\underline{th}}$ variable. From eqn. (B-18)

$$E_i = \frac{\partial G}{\partial w_i} - \frac{\partial}{\partial r}\left(\frac{\partial G}{\partial R_i}\right) - \frac{\partial}{r\partial \phi}\left(\frac{\partial G}{\partial T_i}\right) \tag{2.65}$$

The repeated variable index i in eqn. (2.64) indicates a summation on the index in accordance with standard convention. Hence,

$$E_i\ \delta w_i = E_1\ \delta w_1 + E_2\ \delta w_2 + \cdots + E_n\ \delta w_n \tag{2.66}$$

etc.

Now the variational integral I, eqn. (2.61), is of the same form as eqn. (2.62) where the dependent variables $w_i (i = 1,\ldots 5)$ are V, θ, ψ, α, and β. The variables P and ρ are functions of V alone. Thus the variation of eqn. (2.61) is given by eqn. (2.64) where

$$E_1\ \delta w_1 = \left\{\frac{\partial G}{\partial V} + \frac{\partial G}{\partial P}\frac{dP}{dV} + \frac{\partial G}{\partial \rho}\frac{d\rho}{dV} - \left[\frac{d}{dr}\left(\frac{\partial G}{\partial R_1}\right)\right]_\phi - \left[\frac{d}{rd\phi}\left(\frac{\partial G}{\partial T_1}\right)\right]_r\right\}\delta V \tag{2.67}$$

$$E_2\ \delta w_2 = \left\{\frac{\partial G}{\partial \theta} - \left[\frac{d}{dr}\left(\frac{\partial G}{\partial R_2}\right)\right]_\phi - \left[\frac{d}{rd\phi}\left(\frac{\partial G}{\partial T_2}\right)\right]_r\right\}\delta\theta \tag{2.68}$$

$$E_3\ \delta w_3 = \left\{\frac{\partial G}{\partial \psi} - \left[\frac{d}{dr}\left(\frac{\partial G}{\partial R_3}\right)\right]_\phi - \left[\frac{d}{rd\phi}\left(\frac{\partial G}{\partial T_2}\right)\right]_r\right\}\delta\psi \tag{2.69}$$

$$E_4\ \delta w_4 = \frac{\partial G}{\partial \beta}\ \delta\beta \tag{2.70}$$

$$E_5\ \delta w_5 = \frac{\partial G}{\partial \alpha}\ \delta\alpha \tag{2.71}$$

$$R_1 = \left(\frac{dV}{dr}\right)_\phi \quad ; \quad T_1 = \left(\frac{dV}{rd\phi}\right)_r \tag{2.72}$$

$$R_2 = \left(\frac{d\theta}{dr}\right)_\phi \quad ; \quad T_2 = \left(\frac{d\theta}{r d\phi}\right)_r \tag{2.73}$$

and

$$R_3 = \left(\frac{d\Psi}{dr}\right)_\phi \quad ; \quad T_3 = \left(\frac{d\Psi}{r d\phi}\right)_r \tag{2.74}$$

The variations $\delta\alpha$ and $\delta\beta$ are not independent but are both related to variations in f by eqns. (2.34) and (2.35). Let

$$f_r \equiv \frac{\partial f}{\partial r} = \left(\frac{df}{dr}\right)_\phi \tag{2.75}$$

and

$$f_\phi \equiv \frac{\partial f}{\partial \phi} = \left(\frac{df}{d\phi}\right)_r \tag{2.76}$$

where the transformation eqns. (2.32) and (2.33) are employed to obtain the derivatives in the (r, ϕ)-plane.

The variations of f_r and f_ϕ, obtained from eqns. (2.34) and (2.35), are

$$\delta f_r = \frac{\cos\alpha}{\cos^2\beta} \delta\beta - \tan\beta \quad \sin\alpha\, \delta\alpha \tag{2.77}$$

$$\delta f_\phi = \frac{r \sin\alpha}{\cos^2\beta} \delta\beta + r\tan\beta \quad \cos\alpha\, \delta\alpha \tag{2.78}$$

Solving for $\delta\alpha$ and $\delta\beta$ in terms of δf_r and δf_ϕ gives

$$\delta\beta = \cos^2\beta \quad \left(\cos\alpha\, \delta f_r + \frac{\sin\alpha}{r} \delta f_\phi\right) \tag{2.79}$$

$$\delta\alpha = -\operatorname{ctn}\beta \quad \left(\sin\alpha\, \delta f_r - \frac{\cos\alpha}{r} \delta f_\phi\right) \tag{2.80}$$

Hence, substituting eqns. (2.79) and (2.80) into eqns. (2.70) and (2.71) and adding yields

$$E_4\, \delta w_4 + E_5\, \delta w_5 = \frac{\partial G}{\partial\beta} \delta\beta + \frac{\partial G}{\partial\alpha} \delta\alpha = E_6\, \delta f_r + E_7\, \delta f_\phi \tag{2.81}$$

where

$$E_6 = \cos^2\beta \quad \cos\alpha \frac{\partial G}{\partial \beta} - \text{ctn}\,\beta \quad \sin\alpha \frac{\partial G}{\partial \alpha} \tag{2.82}$$

and

$$E_7 = \frac{\cos^2\beta \quad \sin\alpha}{r} \; \frac{\partial G}{\partial \beta} + \frac{\text{ctn}\,\beta \quad \cos\alpha}{r} \; \frac{\partial G}{\partial \alpha} \tag{2.83}$$

Whether δf_r and δf_ϕ can be related now depends on the continuity and smoothness of the control surface. To ensure that the control surface is a continuous, smooth surface one can impose the integrability condition on f(or z). The integrability condition is

$$\frac{\partial f_r}{\partial \phi} = \frac{\partial f_\phi}{\partial r} \tag{2.84}$$

which can be considered as an additional constraint equation. Using eqns. (2.32) - (2.35), eqn. (2.84) becomes

$$\left[\frac{d}{dr} \quad (r \tan\beta \quad \sin\alpha)\right]_\phi = \left[\frac{d}{rd\phi} \quad (r \tan\beta \quad \cos\alpha)\right]_r \tag{2.85}$$

in which form it will be used later as a design equation. Imposing the integrability condition permits interchanging the order of the variation and the partial differentiation in eqn. (2.81) so that

$$\delta f_r = \delta \left(\frac{df}{dr}\right)_\phi = \left(\frac{d \;\delta f}{dr}\right)_\phi \tag{2.86}$$

and

$$\delta f_\phi = \delta \left(\frac{df}{d\phi}\right)_r = \left(\frac{d \;\delta f}{d\phi}\right)_r \tag{2.87}$$

Hence the terms $E_6 \;\delta f_r + E_7 \;\delta f_\phi$ in eqn. (2.81) can be expanded into the form

$$E_6 \;\delta f_r + E_7 \;\delta f_\phi = - \;\delta f\left[\left(\frac{dE_6}{dr}\right)_\phi + \left(\frac{dE_7}{d\phi}\right)_r\right] + \left[\frac{d(E_6 \;\delta f)}{dr}\right]_\phi + \left[\frac{d(E_7 \;\delta f)}{d\phi}\right]_r \tag{2.88}$$

Applying Stokes' Theorem in the form of eqn. (B-9) (see Appendix B) to the last two terms of eqn. (2.88) and writing

(a) $F_r = - E_7 \, \delta f$ and

(b) $rF_\phi = E_6 \, \delta f$ (2.89)

the following result is obtained.

$$\iint_S (E_6 \, \delta f_r + E_7 \, \delta f_\phi) \, dr \, d\phi = - \iint_S \left[\left(\frac{dE_6}{dr}\right)_\phi + \left(\frac{dE_7}{d\phi}\right)_r \right] \delta f \, dr \, d\phi$$

$$+ \oint_\Gamma \left(E_7 \, r \frac{\partial \phi}{\partial m} + \frac{E_6}{r} \frac{\partial r}{\partial m}\right) \delta f \, dl \qquad (2.90)$$

Substituting eqns. (2.81) and (2.90) into eqn. (2.64) for the variation of I yields the result

$$\delta I = \iint_S H_1 \, dr \, d\phi + \oint_\Gamma H_2 \, dl = 0 \qquad (2.91)$$

where

$$H_1 = E_1 \, \delta V + E_2 \, \delta\theta + E_3 \, \delta\psi - \left[\left(\frac{dE_6}{dr}\right)_\phi + \left(\frac{dE_7}{d\phi}\right)_r \right] \delta f \qquad (2.92)$$

and

$$H_2 = G \Big|_{r_e} \frac{d\phi}{dl} \frac{dr_e}{d\epsilon} \Big|_{\epsilon = 0} d\epsilon + \left(\frac{\partial G}{\partial R_1} \frac{\partial r}{\partial m} \frac{1}{r} + \frac{\partial G}{\partial T_1} \frac{\partial \phi}{\partial m}\right) \delta V$$

$$+ \left(\frac{\partial G}{\partial R_2} \frac{\partial r}{\partial m} \frac{1}{r} + \frac{\partial G}{\partial T_2} \frac{\partial \phi}{\partial m}\right) \delta\theta + \left(\frac{\partial G}{\partial R_3} \frac{\partial r}{\partial m} \frac{1}{r} + \frac{\partial G}{\partial T_3} \frac{\partial \phi}{\partial m}\right) \delta\psi$$

$$+ \left(E_7 \, r \frac{\partial \phi}{\partial m} + \frac{E_6}{r} \frac{\partial r}{\partial m}\right) \delta f \qquad (2.93)$$

To satisfy eqn. (2.91), the integral of H_1 over the control surface must be identically zero and the integral of H_2 over the boundary also must be identically zero. On satisfying the condition that the integral of H_1 is zero on the control surface one obtains the design equations for the control surface which are deduced in Chapter 3. The condition that the integral of H_2 is zero over the boundary together with the constraint relations on the boundaries lead to boundary conditions which are discussed in Chapter 4.

2.4 Summary of the Problem

The overall objective is to determine the three-dimensional thrust nozzle contour that will produce the maximum axial thrust when subjected to constraints of fixed mass flow rate, limited overall length, shock free irrotational flow, and a prescribed exit section contour. The design of the subsonic, transonic, and initial expansion contour is determined on the basis of design criteria other than thrust maximization and, therefore, is expected to be independent of the optimization criteria. Consequently, all portions of the nozzle contour except the supersonic contour are considered to be known for purposes of maximizing the thrust. It is further assumed that the flow variables in the kernel of the nozzle (see Fig. 2.1), which are necessary for further analysis, are available. The objective of the research presented in this report consists in the determination of the supersonic contour of a three-dimensional nozzle of given exit shape, limited length, and fixed mass flow rate, which will produce the maximum axial thrust and maintain a shock free irrotational flow.

It is convenient to divide the procedure for the solution of the problem into three parts, namely

(a) the mathematical formulation and solution of the optimization problem,

(b) proof of the compatibility of the mathematical solution with physically possible flow fields, and

(c) the development of a methodology for determining the optimum supersonic contour.

The optimization problem is mathematically formulated by introducing a control surface which is constrained by the exit contour and the kernel. The axial thrust which is to be maximized as well as the constraint relationships are expressed in terms of the problem variables on the control surface and its boundaries.

The solution of the optimization problem consists of applying the optimization techniques (utilizing the calculus of variations) to determine the relationships among the problem variables on the control surface which will produce the maximum axial thrust under the constraints imposed. The relationships on the control surface are the design equations and are derived in Chapter 3.

In deriving the design equations, it is necessary to ensure that the flow field which produces the optimum thrust conditions on the control surface can be matched with the flow field already established in the kernel. This may be achieved by showing that the control surface is a characteristic surface in the flow and, therefore, is uniquely determined and thus exists. The proof of the existence is contained in Section 3.2.

In Section 3.3 the design equations for axisymmetric flow are shown to be a special case of the three-dimensional design equations.

In Chapter 4 the boundary equations and their relationship to the geometric constraints imposed on the nozzle exit are discussed. Finally, in Chapter 5 a methodology for applying the design equations to determine the optimum supersonic three-dimensional nozzle contour is discussed. As is most often the case when considering supersonic flow problems, the methodology involves numerical techniques and trial and error solutions. The problems involved in the methodology for determining the design contour are twofold: (a) those associated with the solution of the mathematical equations by numerical means, and (b) those associated with computerizing the equations. Those problems are discussed briefly in Chapter 5.

3. DESIGN EQUATIONS

The design equations are the equations relating the flow variables on the control surface and, together with the boundary equations, provide the relationships needed to locate the control surface and to calculate the flow variables on it. The design equations include the equations which arise from considering possible variations in the variables V, θ, ψ, and f on the control surface and the constraint equations which must hold on the control surface.

In this chapter the design equations are derived in Section 3.1. In Section 3.2 the design equations are employed to show that the control surface must be a characteristic surface and is therefore unique. And in Section 3.3 the design equations for axisymmetric flow are shown to be a special case of the general three-dimensional design equations derived in Section 3.1.

The boundary equations are discussed separately in Chapter 4.

3.1 Derivation of the Design Equations

To obtain the maximum axial thrust, the integral

$$\delta I_1 = \iint_S H_1 \, dr \, d\phi \tag{3.1}$$

must be zero where H_1 is defined by eqn. (2.92). That is

$$H_1 = E_1 \, \delta V + E_2 \, \delta\theta + E_3 \, \delta\psi - \left[\left(\frac{dE_6}{dr}\right)_\phi + \left(\frac{dE_7}{d\phi}\right)_r \right] \delta f \tag{3.2}$$

where E_1, E_2, E_3, and E_7 are defined by eqns. (2.67), (2.68), (2.69), (2.82), and (2.83), respectively.

In eqn. (3.1) the area of integration, S, is the projection of the control surface onto the (r, ϕ)-plane and is represented by the area enclosed by the curve Γ_e in Fig. 3.1. The curve Γ_k represents the projection of the intersection of the control surface with the outer boundary of the kernel onto the (r, ϕ)-plane and divides the area S into an inner area S_k common to the kernel and an outer area S_e external to the kernel. Hence eqn. (3.1) can be written as the sum of two integrals in the form

$$\delta I_1 = \iint_{S_k} H_1 \, dr \, d\phi + \iint_{S_e} H_1 \, dr \, d\phi = 0 \qquad (3.3)$$

Over the area S_k the variables V, θ, Ψ are determined by the initial conditions. Further, the variation δf can be made zero on S_k by requiring the surface S_k to be a characteristic surface and continuous with the surface S_e. Consequently, the design equations apply only on the portion of the control surface external to the kernel. Hence, the term "control surface" hereafter will apply only to that portion of the surface external to the kernel.

The problems involved with matching the control surface with the kernel of the flow are discussed as part of the methodology in Chapter 5.

Consider now the variation over the area S_e, namely

$$\delta I_1 = \iint_{S_e} H_1 \, dr \, d\phi = 0 \qquad (3.4)$$

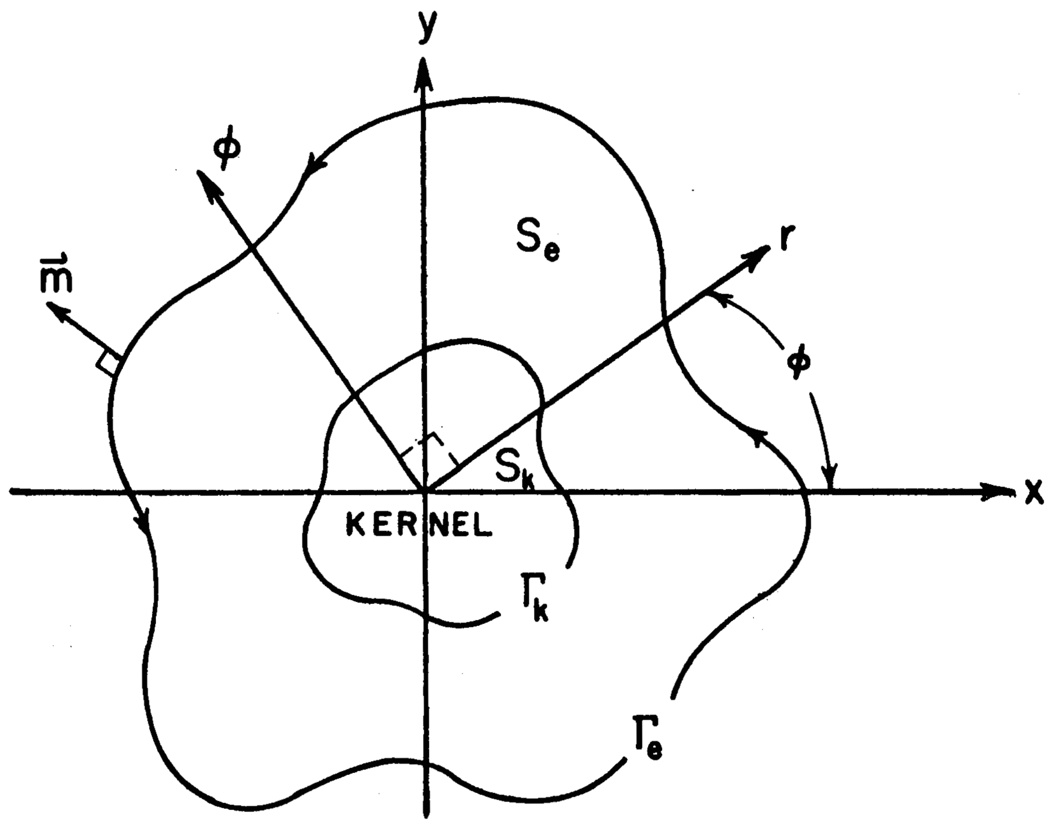

FIGURE 3.1

CONTROL SURFACE PROJECTION ON THE (r, ϕ) - PLANE SHOWING THE BOUNDARY OF THE KERNEL, Γ_k

On the area S_e only two of the four variables V, θ, ψ, and f are actually independent; however, the heretofore unspecified Lagrange multipliers make it possible to consider all of the four variables as independent. Consequently, invoking the classical arguments of variational calculus it can readily be seen that the coefficients of the variations δV, $\delta\theta$, $\delta\psi$, and δf in eqn. (3.2) must each be identically zero. That is,

$$E_1 = \frac{\partial G}{\partial V} + \frac{\partial G}{\partial P}\frac{dP}{dV} + \frac{\partial G}{\partial \rho}\frac{d\rho}{dV} - \left[\frac{d}{dr}\left(\frac{\partial G}{\partial R_1}\right)\right]_\phi - \left[\frac{d}{rd\phi}\left(\frac{\partial G}{\partial T_1}\right)\right]_r = 0 \tag{3.5}$$

$$E_2 = \frac{\partial G}{\partial \theta} - \left[\frac{d}{dr}\left(\frac{\partial G}{\partial R_2}\right)\right]_\phi - \left[\frac{d}{rd\phi}\left(\frac{\partial G}{\partial T_2}\right)\right]_r = 0 \tag{3.6}$$

$$E_3 = \frac{\partial G}{\partial \psi} - \left[\frac{d}{dr}\left(\frac{\partial G}{\partial R_3}\right)\right]_\phi - \left[\frac{d}{rd\phi}\left(\frac{\partial G}{\partial T_3}\right)\right]_r = 0 \tag{3.7}$$

and

$$-\left(\frac{dE_6}{dr}\right)_\phi - \left(\frac{dE_7}{d\phi}\right)_r = 0 \tag{3.8}$$

where G is defined by eqn. (2.60), R_i and T_i $(i = 1,2,3)$ are defined by eqns. (2.72) - (2.74), and E_6 and E_7 are defined by eqns. (2.82) and (2.83).

The indicated differentiations in eqns. (3.5) - (3.8) are long and tedious but straight forward. To reduce the presentation of the algebra, certain recurring groups of terms are redefined as follows.

$$X_2 \equiv \frac{V\cos\theta + \lambda_2}{V} \tag{3.9}$$

$$X_3 \equiv \frac{\lambda_3 \cos\beta}{rV} \tag{3.10}$$

$$D_r \equiv \left(\frac{dX_3}{dr}\right)_\phi \tag{3.11}$$

$$D_\phi \equiv \left(\frac{dX_3}{rd\phi}\right)_r \tag{3.12}$$

$$A_7 \equiv \sin\theta\cos\beta + \sin\beta\cos\theta\cos(\Psi - \alpha) \tag{3.13}$$

In terms of those definitions, eqn. (3.5) can be expanded as

$$- E_1 = h_1 + r\sin\theta\,(\sin\Psi\; D_r - \cos\Psi\; D_\phi)$$
$$+ \cos\theta \left\{ \left[\frac{d\left(\frac{\lambda_3 \sin\beta\sin\alpha}{V}\right)}{dr} \right]_\phi - \left[\frac{d\left(\frac{\lambda_3 \sin\beta\cos\alpha}{V}\right)}{rd\phi} \right]_r \right\} = 0 \tag{3.14}$$

where

$$h_1 = \frac{r\rho V}{\cos\beta}\,(X_2 \sin\xi\;\mathrm{ctn}^2\mu + A_7\sin\theta) \tag{3.15}$$

In eqn. (3.14) the term

$$\frac{\lambda_3 \sin\beta\sin\alpha}{V} = \left(\frac{\lambda_3\cos\beta}{rV}\right) r\tan\beta\sin\alpha = X_3\, r\tan\beta\sin\alpha \tag{3.16}$$

and the term

$$\frac{\lambda_3 \sin\beta\cos\alpha}{V} = X_3\, r\tan\beta\cos\alpha \tag{3.17}$$

Thus,

$$\left[\frac{d}{dr}\left(\frac{\lambda_3\sin\beta\sin\alpha}{V}\right)\right]_\phi - \left[\frac{d}{rd\phi}\left(\frac{\lambda_3\sin\beta\cos\alpha}{V}\right)\right]_r =$$
$$r\tan\beta\,(\sin\alpha\, D_r - \cos\alpha\, D_\phi)$$
$$+ X_3\left\{\left[\frac{d}{dr}(r\tan\beta\sin\alpha)\right]_\phi - \left[\frac{d}{rd\phi}(r\tan\beta\cos\alpha)\right]_r\right\} \tag{3.18}$$

The last line of eqn. (3.18) is zero due to eqn. (2.85); thus, eqn. (3.14) becomes

$$- E_1 = h_1 + r \sin\theta\ (\sin\psi\ D_r - \cos\psi\ D_\phi)$$

$$+ r \cos\theta \tan\beta\ (\sin\alpha\ D_r - \cos\alpha\ D_\phi) = 0 \tag{3.19}$$

In an analogous manner eqn. (3.6) is expanded to give

$$- E_2 = h_2 + rV \cos\theta\ (\sin\psi\ D_r - \cos\psi\ D_\phi)$$

$$- rV \sin\theta \tan\beta\ (\sin\alpha\ D_r - \cos\alpha\ D_\phi) = 0 \tag{3.20}$$

where

$$h_2 \equiv \frac{r\rho V^2}{\cos\beta}\ (X_3\ A_7 + \sin\theta \sin\xi\ ) \tag{3.21}$$

Equation (3.7) expands into the equation

$$E_3 = h_3 - rV \sin\theta\ (\cos\psi\ D_r + \sin\psi\ D_\phi) = 0 \tag{3.22}$$

where

$$h_3 \equiv r\rho V^2\ X_2 \sin\theta \tan\beta\ \sin(\psi - \alpha) \tag{3.23}$$

Equation (3.8) becomes

$$\left(\frac{dh_4}{dr}\right)_\phi + \left(\frac{dh_5}{rd\phi}\right)_r + r\ D_r \left(\frac{d(V \cos\theta)}{rd\phi}\right)_r$$

$$- r\ D_\phi \left(\frac{d(V \cos\theta)}{dr}\right)_\phi = 0 \tag{3.24}$$

where

$$h_4 \equiv r\rho V^2\ X_2 \sin\theta \cos\psi \tag{3.25}$$

$$h_5 \equiv r\rho V^2\ X_2 \sin\theta \sin\psi \tag{3.26}$$

and the condition

$$\left[\frac{d}{dr} \left(\frac{d(V \cos\theta)}{d\phi}\right)_r\right]_\phi = \left[\frac{d}{d\phi} \left(\frac{d(V \cos\theta)}{dr}\right)_\phi\right]_r \tag{3.27}$$

has been used to reduce the result.

Equations (3.19), (3.20), (3.22) and (3.24) correspond to eqns.(3.5) - (3.8), respectively. Further reduction is now possible. Solving eqns. (3.20) and (3.22) for D_r and D_ϕ the following equations are obtained.

$$\frac{D_r}{\rho V} = - \frac{X_2}{\sin \xi} \frac{A_8}{\cos \beta} - \sin \theta \sin \Psi \tag{3.28}$$

and

$$\frac{D_\phi}{\rho V} = \frac{X_2}{\sin \xi} \frac{A_9}{\cos \beta} + \sin \theta \cos \Psi \tag{3.29}$$

where

$$A_8 \equiv \sin \theta \sin \Psi + \sin \beta \sin \xi \sin \alpha \tag{3.30}$$

and

$$A_9 \equiv \sin \theta \cos \Psi + \sin \beta \sin \xi \cos \alpha \tag{3.31}$$

By substituting eqns. (3.28) and (3.29) into eqn. (3.19) and simplifying the result as far as possible one obtains the result, namely,

$$\tan^2 \xi = \tan^2 \mu \tag{3.32}$$

From the problem geometry it is readily deduced that $\mu = \xi$. This result is significant in that it requires the control surface to be in a characteristic direction.

Equation (3.24) can be simplified by using eqns. (3.28), (3.29), and (3.32) and expanding the partial derivatives. The following equation is obtained as a result of the simplification.

$$\frac{r\rho V^2 X_2}{\sin \mu \cos \beta} \left[\frac{B_1}{V} \left(\frac{dV}{dr}\right)_\phi + \frac{B_2}{V} \left(\frac{dV}{rd\phi}\right)_r + B_3 \left(\frac{d\theta}{dr}\right)_\phi \right.$$

$$\left. + B_4 \left(\frac{d\theta}{rd\phi}\right)_r + B_5 \left(\frac{d\Psi}{dr}\right)_\phi + B_6 \left(\frac{d\Psi}{rd\phi}\right)_r + \frac{B_6}{r} \right] = 0 \tag{3.33}$$

where

$$B_1 \equiv A_9 \cos\theta + \cos\mu \,\mathrm{ctn}\,\mu \sin\theta \cos\beta \quad \cos\psi \tag{3.34}$$

$$B_2 \equiv A_8 \cos\theta + \cos\mu \,\mathrm{ctn}\,\mu \sin\theta \cos\beta \quad \sin\psi \tag{3.35}$$

$$B_3 \equiv - A_9 \sin\theta - \sin\mu \cos\beta \quad \cos\theta \cos\psi \tag{3.36}$$

$$B_4 \equiv - A_8 \sin\theta - \sin\mu \cos\beta \quad \cos\theta \sin\psi \tag{3.37}$$

$$B_5 \equiv \sin\mu \sin\theta \cos\beta \quad \sin\psi \tag{3.38}$$

and

$$B_6 \equiv \sin\mu \sin\theta \cos\beta \quad \cos\psi \tag{3.39}$$

In summary, the four equations derived by employing variational techniques on the control surface, namely eqns. (3.28), (3.29), (3.32), and (3.33) plus the constraint equations, eqns. (2.49), (2.51), and (2.85), constitute a set of seven design equations with the seven unknowns V, θ, ψ, α, β, λ_2, and λ_3. A methodology for the solution of the design equations is discussed in Chapter 5.

3.2 Proof of the Existence of the Solution

The design equations derived in Section 3.1, along with the boundary equations, are sufficient for locating the control surface and for determining the flow properties on it. However, there is no assurance that it is possible to produce the optimum flow conditions on the control surface with a shock free irrotational flow which will match the given flow in the kernel. That is, the constraints employed in deriving the design equations do not explicitly require that a shock free flow field exist which will produce the desired flow conditions and also match the flow in the kernel. In general, the realization of such matching is highly unlikely unless the control surface is a characteristic surface. If, however, the control surface can be shown to be a characteristic surface, a matching of the flows is possible and the compatibility of the derived solution is assured. It, therefore, is sufficient to show that the control surface is a characteristic surface.* Thus the approach used here is first to assume that the constraints implicitly require the control surface to be a characteristic surface and next to show that the foregoing assumption is valid by demonstrating that the design equations define the control surface as a characteristic surface.

In Section 2.1.3 the necessary and sufficient conditions for a surface to be a characteristic surface are discussed. Briefly a characteristic surface must satisfy two requirements. First, it must be oriented in a characteristic direction as required by eqn. (2.17); and second, the compatibility equations, eqns. (2.28) and (2.29), must be valid on the surface. Those conditions, in fact, will be demonstrated to be valid.

It is evident from the design eqn. (3.32) that the control surface should be oriented in a characteristic direction. Therefore, the first of the conditions required for the control surface to be a characteristic surface is assured. It, then, remains to show that the design equations and the constraint equations may be employed to derive the compatibility equations, eqns. (2.28) and (2.29).

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* It may be observed that one may have also imposed the condition that the control surface be a characteristic surface. While that constraint would also lead to a physically possible solution, it does not assure that the design equations by themselves would lead to the condition that the control surface be a characteristic surface. Furthermore, it is found in practice[1] that such a constraint leads to rather large difficulties in the methodology for the final computation of the flow.

The procedure for deducing the compatibility eqn. (2.29) is to transform the constraint eqn. (2.51) from the (r, ϕ)-plane to the control surface. Since the control surface is oriented in a characteristic direction the transformation eqns. (2.38) and (2.39) can be employed to transform to the (L, N)-coordinates on the control surface. The transformed constraint eqn. (2.51) becomes

$$F_3 = C_1 \frac{1}{V} \left(\frac{dV}{dL}\right)_N + C_2 \frac{1}{V} \left(\frac{dV}{dN}\right)_L + C_3 \left(\frac{d\theta}{dL}\right)_N + C_4 \left(\frac{d\theta}{dN}\right)_L$$

$$+ C_5 \left(\frac{d(\Psi + \phi)}{dL}\right)_N + C_6 \left(\frac{d(\Psi + \phi)}{dN}\right)_L + C_7 = 0 \qquad (3.40)$$

where

$$C_1 \equiv A_1 a_1 + A_2 b_1 \qquad (3.41)$$

$$C_2 \equiv A_1 a_2 + A_2 b_2 \qquad (3.42)$$

$$C_3 \equiv A_3 a_1 + A_4 b_1 \qquad (3.43)$$

$$C_4 \equiv A_3 a_2 + A_4 b_2 \qquad (3.44)$$

$$C_5 \equiv A_5 a_1 + A_6 b_1 \qquad (3.45)$$

$$C_6 \equiv A_5 a_2 + A_6 b_2 \qquad (3.46)$$

and

$$C_7 \equiv \frac{A_6}{r} - C_5 \left(\frac{d\phi}{dL}\right)_N - C_6 \left(\frac{d\phi}{dN}\right)_L \qquad (3.47)$$

It is then necessary to evaluate the various terms and coefficients appearing in eqn. (3.40). The coefficients $A_1 - A_6$ are defined in eqns. (2.52) - (2.57) and the coefficients a_1, a_2, b_1, and b_2 are defined by eqns. (2.40) - (2.43). From eqns. (2.38) and (2.39) the relationships

$$\left(\frac{d\phi}{dL}\right)_N = - \frac{a_2}{r(a_1 b_2 - a_2 b_1)} \tag{3.48}$$

and

$$\left(\frac{d\phi}{dN}\right)_L = \frac{a_1}{r(a_1 b_2 - a_2 b_1)} \tag{3.49}$$

are readily obtained.

In evaluating the coefficients $C_1 - C_7$ the angle δ is introduced as defined by eqns. (2.26) and (2.27). The choice of sign in eqns. (2.26) and (2.27) is made by defining δ as the angle measured counter clockwise from the $(\vec{V}, \vec{z})$-plane to the $(\vec{V}, \vec{L})$-plane. Then, on the basis of geometry, eqn. (2.26) becomes

$$\sin \delta = - \frac{\sin \beta \quad \sin(\Psi - \alpha)}{\cos \mu} \tag{3.50}$$

and eqn. (2.27) becomes

$$\cos \delta = \frac{\cos \beta \quad \sin \theta + \sin \beta \quad \cos \theta \cos(\Psi - \alpha)}{\cos \mu} = \frac{A_7}{\cos \mu} \tag{3.51}$$

where A_7 is defined by eqn. (3.13).

The coefficients $C_1 - C_7$ in eqns. (3.41) - (3.47) are obtained in terms of the angles μ, θ, and δ as follows.

$$C_1 = C_7 = 0 \tag{3.52}$$

$$C_2 = \cos \mu \tag{3.53}$$

$$C_3 = \sin \delta \tag{3.54}$$

$$C_4 = \sin \mu \cos \delta \tag{3.55}$$

$$C_5 = - \sin \theta \cos \delta \tag{3.56}$$

$$C_6 = \sin \mu \sin \theta \sin \delta \tag{3.57}$$

Substituting in eqn. (3.40) for C_1 - C_7 from eqns. (3.52) - (3.57) and dividing throughout by cos μ, the following equation is obtained.

$$\frac{1}{V}\left(\frac{dV}{dN}\right)_L + \frac{\sin\delta}{\cos\mu}\left(\frac{d\theta}{dL}\right)_N + \tan\mu\cos\delta\left(\frac{d\theta}{dN}\right)_L$$

$$-\frac{\sin\theta\cos\delta}{\cos\mu}\left(\frac{d(\Psi+\phi)}{dL}\right)_N + \tan\mu\sin\theta\sin\delta\left(\frac{d(\Psi+\phi)}{dN}\right)_L = 0 \tag{3.58}$$

which is precisely the compatibility eqn. (2.29). It is apparent that eqn. (2.29) arises solely from the irrotationality constraint.

Next, in order to derive the compatibility eqn. (2.28) from the design equations, eqn. (3.33) is transformed to the (L, N)-coordinates using eqns. (2.38) and (2.39) to obtain

$$K_1\frac{1}{V}\left(\frac{dV}{dL}\right)_N + K_2\frac{1}{V}\left(\frac{dV}{dN}\right)_L + K_3\left(\frac{d\theta}{dL}\right)_N + K_4\left(\frac{d\theta}{dN}\right)_L$$

$$+ K_5\left(\frac{d(\Psi+\phi)}{dL}\right)_N + K_6\left(\frac{d(\Psi+\phi)}{dN}\right)_L + K_7 = 0 \tag{3.59}$$

where

$$K_1 \equiv B_1\, a_1 + B_2\, b_1 \tag{3.60}$$

$$K_2 \equiv B_1\, a_2 + B_2\, b_2 \tag{3.61}$$

$$K_3 \equiv B_3\, a_1 + B_4\, b_1 \tag{3.62}$$

$$K_4 \equiv B_3\, a_2 + B_4\, b_2 \tag{3.63}$$

$$K_5 \equiv B_5\, a_1 + B_6\, b_1 \tag{3.64}$$

$$K_6 \equiv B_5\, a_2 + B_6\, b_2 \tag{3.65}$$

$$K_7 \equiv \frac{B_6}{r} - B_5\left(\frac{d\phi}{dL}\right)_N - B_6\left(\frac{d\phi}{dN}\right)_L \tag{3.66}$$

To evaluate K_1 - K_7 eqns. (3.34) - (3.39) for B_1 - B_6, eqns. (3.30) and (3.31) for A_8 and A_9, eqns. (2.40) - (2.43) for a_1, a_2, b_1, and b_2, and eqns. (3.50) and (3.51) for sin δ and cos δ are employed. After some algebraic manipulation the following result is obtained.

$$K_1 = - \frac{\cos \mu \cos \beta}{\sin \mu} \tag{3.67}$$

$$K_2 = \cos^2 \mu \sin \theta \sin \delta \tag{3.68}$$

$$K_3 = \cos \beta \ \cos \delta + \sin \theta \sin^2 \delta \cos \mu \tag{3.69}$$

$$K_4 = - \sin \delta \cos \theta \sin^2 \mu \tag{3.70}$$

$$K_5 = \sin \delta \sin \mu \sin \theta \cos \theta \tag{3.71}$$

$$K_6 = \sin \mu \sin \theta \ (\cos \delta \cos \beta \ + \cos \mu \sin \theta \sin^2 \delta) \tag{3.72}$$

and

$$K_7 = 0 \tag{3.73}$$

Equations (3.40) and (3.59) can now be combined to eliminate the term containing $(dV/dN)_L$. It is easily verified that the resulting equation is

$$- \frac{\cos^2 \mu \cos \beta}{\sin \mu} \left\{ \frac{1}{V} \left(\frac{dV}{dL}\right)_N - \tan \mu \left[\cos \delta \left(\frac{d\theta}{dL}\right)_N - \sin \mu \sin \delta \left(\frac{d\theta}{dN}\right)_L \right.\right.$$
$$\left.\left. + \sin \theta \sin \delta \left(\frac{d(\psi + \phi)}{dL}\right)_N + \sin \mu \sin \theta \cos \delta \left(\frac{d(\psi + \phi)}{dN}\right)_L \right]\right\} = 0 \tag{3.74}$$

If eqn. (3.74) is mulitiplied by $\left[- \sin \mu / (\cos^2 \mu \cos \beta \ ) \right]$, one obtains precisely the compatibility eqn. (2.28) as the resulting equation.

It has now been established that (a) the control surface is oriented in the direction of the characteristic, (b) the compatibility eqn. (2.29) may be obtained starting from the irrotationality condition, and (c) the compatibility eqn. (2.28) may be obtained starting from design eqn. (3.33) and the constraint eqn. (2.51). Thus the control surface defined by the design equations is a characteristic surface, and the compatibility of the flow on the control surface with the flow in the kernel is assured.

3.3 The Special Case of Axisymmetric Flow

The three-dimensional design eqns. (3.24), (3.28), (3.29), and (3.32) reduce to the axisymmetric design equations[2] as a special case. The conditions for axisymmetric flow are $\psi = 0$, $\alpha = \pi$, and $(dV/d\phi)_r = (d\theta/d\phi)_r = (d\beta/d\phi)_r = (d\lambda_3/d\phi)_r = 0$. Substituting these conditions into eqn. (3.32) yields the relationship

$$\tan^2 \xi = \tan^2 \mu = \mathrm{ctn}^2 (\beta - \theta) \tag{3.75}$$

From geometric considerations eqn. (3.75) becomes

$$\mu = \beta - \theta - \frac{3\pi}{2} \tag{3.76}$$

Imposing the axisymmetric flow conditions on eqn. (3.28) yields

$$D_r = \left[\frac{d}{dr}\left(\frac{\lambda_3 \cos\beta}{rV}\right)\right]_\phi = 0 \tag{3.77}$$

and, since λ_3, β, and V are not functions of ϕ, eqn. (3.77) reduces to

$$\frac{\lambda_3 \cos\beta}{rV} = \text{constant} \tag{3.78}$$

a relationship which adds nothing to the problem solution.

Equation (3.29) becomes

$$\frac{D\phi}{r\rho} = 0 = \frac{(V \cos\theta + \lambda_2)(\sin\theta - \sin\beta \ \sin\mu)}{V \sin\mu \cos\beta} + \sin\theta \quad (3.79)$$

which yields the result

$$V \cos\theta + \lambda_2 = - V \sin\theta \tan\mu \quad (3.80)$$

when eqn. (3.76) is employed to eliminate β . Finally, eqn. (3.24) becomes

$$\frac{d}{dr}\left[r\rho V (V \cos\theta + \lambda_2) \sin\theta\right] = 0 \quad (3.81)$$

under the axisymmetric flow conditions. Integrating eqn. (3.81) and substituting from eqn. (3.80) the following equation is obtained.

$$r\rho V^2 \sin^2\theta \tan\mu = k_2 \quad (3.82)$$

where k_2 is the integration constant.

Equations (3.76), (3.80), and (3.82) are equivalent to the design equations derived by Rao[2] for the optimum thrust design of an axisymmetric nozzle.*

* The design eouations (3.76), (3.80), and (3.82) differ from those obtained by Rao due to the notation differences and the method of describing the control surface. When these differences are taken into account the two sets of equations are identical.

4. BOUNDARY CONDITIONS

The design eqns. (3.28), (3.29), (3.33) and (3.34) which are derived in Chapter 3, together with the constraint relations (2.49), (2.51), (2.85), and either (2.34) or(2.35), establish relations among the dependent variables on the control surface, namely V, θ, ψ, α, β, f, X_3, and λ_2. However, in order to solve the design equations it is necessary to know the values of the dependent variables on the boundaries of the control surface. The boundaries of the control surface are: (a) the boundary at the intersection of the kernel with the control surface (the boundary Γ_k in the (r, ϕ)-plane, illustrated in Fig. 3.1) and (b) the boundary at the intersection of the control surface with the nozzle exit contour (the boundary Γ_e in the (r, ϕ)-plane, illustrated in Fig. 3.1). The latter, for any nozzle, pertains to the conditions at the exit section of the nozzle. The boundary conditions accordingly may be divided into two parts.

1. <u>At the inner boundary</u>: The conditions of flow at the inner boundary must match the flow conditions in the kernel which in turn depend upon the known or prescribed conditions in the wholly supersonic region (downstream of the throat) and the initial turning of the nozzle wall. As stated earlier, the flow conditions immediately downstream of the throat are fully prescribed, while the extent of the initial turning of the nozzle wall may become part of the final process of iteration required for determining the optimized flow geometry.

2. At the outer boundary: The boundary conditions at the outer boundary relate the flow variables and the variation of the flow variables around the contour forming the exit section of the nozzle.

The primary interest in evolving boundary conditions therefore should rest at the outer boundary. A set of relationships must be prescribed which relate the flow variables among themselves which are valid explicitly at the exit section of the nozzle.

These relationships or boundary equations arise from three requirements. The first requirement is the condition that is imposed on the flow by the fact that the flow boundary is a continuous stream surface and will be referred to as the natural boundary condition. The natural boundary condition requires that the component of the velocity normal to the nozzle wall be zero.

The second source of boundary equations is the geometric constraint relationships which are imposed on the shape of the flow contour at the exit plane. There is some flexibility in the number and form of the geometric constraints which can be imposed. For example, only nozzles with exit contours which lie on a plane normal to the z-axis may be considered, as is in fact done in this chapter. It is possible, however, to consider cases in which the exit contour may be a function of ϕ either in a prescribed or in an arbitrary manner. In relation to the variational problem it is important to distinguish among boundary contours that are prescribed, partially prescribed, or left free to seek their optimum value. For example the length may be prescribed to be a given constant for all points on the nozzle exit boundary or it may be prescribed to be a given function of ϕ as is done in the second problem example in Chapter 5. In both cases the length is prescribed. If, however, the length is required to be the same for all points on the boundary but no restriction is placed on its value, then the length is partially prescribed. Finally, there may be no restrictions whatever placed on the nozzle length in which case it is considered arbitrary with respect to the optimization problem and the problem requirements will then dictate what value the length must take to satisfy the requirement of maximum axial thrust. It is noted that allowing the length to be arbitrary will result in the design of a perfect nozzle and, therefore, since the objective here is to design a shorter than perfect nozzle which will produce maximum thrust, it is always necessary to prescribe the nozzle length. Other geometric constraint relationships which may be employed are discussed in Section 4.2.

The third source of boundary conditions is the transversality equation which involves variations of the dependent variables on the boundaries. It will be observed that in eqn. (2.91) the variation of I is written in two parts, namely the variation on the control surface from which the design equations of Chapter 3 are derived and the variation on the control surface boundary. Considering the latter variation, namely the variation on the boundary, and setting it to zero, one obtains the equation

$$\oint_{\Gamma_e} H_2 \, dl = 0 \qquad (4.1)$$

which is the transversality equation of optimization theory.

The integral H_2 is defined by eqn. (2.93). That is,

$$H_2 = G\Big|_{r_e} \frac{d\phi}{dl} \frac{dr_e}{d\epsilon}\Big|_{\epsilon=0} d\epsilon + \left(\frac{\partial G}{\partial R_1}\frac{\partial r}{\partial m}\frac{1}{r} + \frac{\partial G}{\partial T_1}\frac{\partial \phi}{\partial m}\right)\delta V$$

$$+ \left(\frac{\partial G}{\partial R_2}\frac{\partial r}{\partial m}\frac{1}{r} + \frac{\partial G}{\partial T_2}\frac{\partial \phi}{\partial m}\right)\delta\theta + \left(\frac{\partial G}{\partial R_3}\frac{\partial r}{\partial m}\frac{1}{r} + \frac{\partial G}{\partial T_3}\frac{\partial \phi}{\partial m}\right)\delta\psi$$

$$+ \left(E_1\, r \frac{\partial \phi}{\partial m} + \frac{E_6}{r}\frac{\partial r}{\partial m}\right)\delta f \tag{4.2}$$

where $\vec{m}$ is the unit outward normal to the boundary, E_6 and E_7 are defined by eqns. (2.82) and (2.83), G is given by eqn. (2.60), and R_i and T_i ($i = 1,2,3$) are defined by eqns. (2.72) - (2.74). Thus eqn. (4.1) may be employed as a boundary condition and requires that I be stationary on the boundary. Boundary equations are obtained from eqn. (4.1) by considering variations in V, θ, ψ, f, and r.

It is important that the boundary conditions themselves should be self-consistent. Thus, the variations introduced in the third of the aforementioned boundary conditions should satisfy the natural boundary condition and the geometric constraints imposed.

In the following sections the boundary equations (arising from each of the three aforementioned sources) are considered in detail. The discussion is related to the design of a nozzle under the following conditions:

1. the z-axis is a streamline; thus there exists in the flow one straight streamline, namely the z-axis; and
2. the exit plane of the nozzle is normal to the z-axis. Thus, the length of the nozzle (measured from a reference plane which is also normal to the z-axis at the throat) is independent of ϕ.

The implications of the boundary geometry in the exit plane are explored for the particular example by deriving the boundary equations for both an arbitrary geometric shape at the exit and a prescribed elliptic contour at the nozzle exit.

4.1 Natural Boundary Conditions

The natural boundary condition constrains the velocity vector to lie in the plane tangent to the nozzle wall at the exit section. The following development of the constraint equation from the natural boundary condition applies, as mentioned in the preceding paragraph, to the design of a nozzle with the exit contour on a plane normal to the z-axis. Consequently, the exit contour and its projection on the (r, ϕ)-plane (the curve Γ_e in Fig. 3.1) are identical.

The unit vector tangent to Γ_e is designated by $\vec{l}$ and the unit outward normal to the nozzle boundary at the exit plane by $\vec{p}$. The direction cosines of $\vec{l}$ and $\vec{p}$ in the r, ϕ, and z-directions are denoted as l_r, l_ϕ, l_z, p_r, p_ϕ, and p_z. Now, since $\vec{l}$ lies on the control surface, $l_z = 0$ and $\vec{l} \cdot \vec{n} = 0$. Thus, it can be shown that

$$l_r = \sin\alpha \quad \text{and} \quad l_\phi = -\cos\alpha \tag{4.3}$$

From the relationships $\vec{p} \cdot \vec{V} = 0$ and $\vec{p} \cdot \vec{l} = 0$ the direction cosines of $\vec{p}$ can be shown to be

$$p_r = \frac{-\cos\theta\cos\alpha}{\left[1 - \sin^2\theta\,\sin^2(\Psi - \alpha)\right]^{1/2}} \tag{4.4}$$

$$p_\phi = \frac{-\cos\theta\sin\alpha}{\left[1 - \sin^2\theta\,\sin^2(\Psi - \alpha)\right]^{1/2}} \tag{4.5}$$

$$p_z = \frac{\sin\theta\cos(\Psi - \alpha)}{\left[1 - \sin^2\theta\sin^2(\Psi - \alpha)\right]^{1/2}} \tag{4.6}$$

The unit vector normal to $\vec{l}$, oriented in the positive z-direction, and in the plane tangent to the nozzle wall is denoted as $\vec{t}$. The direction cosines of $\vec{t}$ are readily shown to be

$$t_r = \frac{\sin\theta\cos\alpha\cos(\Psi - \alpha)}{\left[1 - \sin^2\theta\sin^2(\Psi - \alpha)\right]^{1/2}} \tag{4.7}$$

$$t_\phi = \frac{\sin\theta\sin\alpha\cos(\Psi - \alpha)}{\left[1 - \sin^2\theta\sin^2(\Psi - \alpha)\right]^{1/2}} \tag{4.8}$$

$$t_z = \frac{\cos\theta}{\left[1 - \sin^2\theta\sin^2(\Psi - \alpha)\right]^{1/2}} \tag{4.9}$$

The velocity vector is now resolved into its components in the $\vec{p}$, $\vec{l}$, and $\vec{t}$ directions as

$$V_p = 0 \tag{4.10}$$

$$V_l = -V\sin\theta\sin(\Psi - \alpha) \tag{4.11}$$

$$V_t = V\left[1 - \sin^2\theta\sin^2(\Psi - \alpha)\right]^{1/2} \tag{4.12}$$

Equations (4.11) and (4.12) are used to relate the variations δV_l and δV_t to $\delta\theta$, $\delta\Psi$, δV, and $\delta\alpha$. Their use is illustrated in Section 4.3.

4.2 Geometric Constraints

The geometric constraints can be imposed by a design engineer to require that the thrust nozzle contour conform to specified geometric conditions. One constraint already imposed is that the exit contour lie on a plane normal to the z-axis. Other geometric constraints may include one or more of the following.

4.2.1 Fixed Length

The function f defined by eqn. (2.31) defines the length measured along the z-axis to a point on the control surface. Therefore, f_e represents the nozzle length. The nozzle length will be a fixed quantity if the total variation in f_e expressed by the equation

$$\left.\frac{df_e}{d\epsilon}\right|_{\epsilon=0} d\epsilon = 0 \tag{4.13}$$

is zero.* Since the exit contour is in a plane normal to z,

$$f_e = f_e\ (\epsilon;\ r(\epsilon,\ \phi)) \tag{4.14}$$

and since r is a dependent variable on the boundary, the total variation in the length f_e can be written in terms of δf_e and δr as

$$\left.\frac{df_e}{d\epsilon}\right|_{\epsilon=0} d\epsilon = \delta f_e + \left.\frac{\partial f}{\partial r}\frac{dr}{d\epsilon}\right|_{\epsilon=0} d\epsilon = \delta f_e + \frac{\partial f}{\partial r}\,\delta r \tag{4.15}$$

4.2.2 Prescribed Exit Contour

A geometric constraint may be imposed on the shape of the exit contour. As an example, the boundary curve Γ_e, illustrated in Fig. 3.1, may be required to be elliptic in shape, in which case the equation for Γ_e could be written in the form

$$r^2\ (e^2 \cos^2 \phi + 1) - a^2 = 0 \tag{4.16}$$

where e is the eccentricity of the ellipse and a is the length of the semi-major axis. If both e and a are prescribed, then there is no allowable variation in r on Γ_e and $\delta r = 0$. If e is fixed but a is allowed to vary then

* It may be noticed that eqn. (4.13) does not require the length to be the same for all values of ϕ. This restriction must be imposed separately.

$$\left.\frac{dr}{d\epsilon}\right|_{\epsilon=0} d\epsilon \equiv \delta r = \frac{r}{a} \delta a \tag{4.17}$$

If a is fixed but e is allowed to vary then

$$\delta r = - \frac{er^3}{a^2} \cos^2 \phi \, \delta e \tag{4.18}$$

And if both a and e are allowed to vary then

$$\delta r = \frac{r}{a} \delta a - \frac{er^3}{a^2} \cos^2 \theta \, \delta e \tag{4.19}$$

Other geometric shapes can be prescribed for the exit curve Γ_e in a similar manner.

4.2.3 Other Constraints

Other geometric constraints, if introduced, must be expressible in terms of the boundary curve Γ_e. In general, each set of geometric constraints constitutes a separate problem and will require a separate analysis of the boundary equations.

4.3 Variational Relationships on the Boundary

Finally, the variations of the dependent variables on the boundary Γ_e, namely δV, $\delta\theta$, $\delta\Psi$, δf, and δr are considered. In considering those variations in eqn. (4.2), eqn. (4.2) is reduced to an equation containing only variations which can be considered as independent and arbitrary.

The procedure for reducing eqn. (4.2) depends upon the geometrical constraint relations which are introduced as boundary conditions. The procedure is discussed first without prescribing the exit geometrical shape; later, the particular example of a nozzle with an elliptic exit section is considered to clarify the implications of the several variations involved.

The nozzle under consideration is assumed to have a length independent of ϕ.* Furthermore, for the present, no restriction will be placed on the exit shape. The unit normal to the boundary, $\vec{m}$, is related to the angle α on the boundary by the relations

$$r \frac{\partial \theta}{\partial m} = - \sin \alpha \qquad \text{and} \qquad \frac{\partial r}{\partial m} = - \cos \alpha \tag{4.20}$$

The derivatives in the l-direction along Γ_e are related to α by the equations

$$\frac{d\phi}{dl} = - \frac{\cos \alpha}{r} \qquad \text{and} \qquad \frac{dr}{dl} = \sin \alpha \tag{4.21}$$

Hence eqn. (4.2) can be rewritten in the form

$$\begin{aligned} H_2 &= g_1 \ \delta r + g_2 \ \delta V + g_3 \ \delta\theta \\ &\quad + g_4 \ \delta \psi + g_5 \ \delta f_e \end{aligned} \tag{4.22}$$

where the coefficients $g_1 - g_5$ can be evaluated by differentiating eqn. (2.60), using the definitions given by eqns. (2.52) - (2.57), (2.72) - (2.74), (4.20) and (4.21) to give

$$g_1 = \frac{G \cos \alpha}{r} \tag{4.23}$$

$$g_2 = - \frac{\lambda_3}{rV} \sin \theta \cos \beta \ \sin (\psi - \alpha) \tag{4.24}$$

$$g_3 = - \frac{\lambda_3}{r} \cos \theta \cos \beta \ \sin (\psi - \alpha) \tag{4.25}$$

$$g_4 = - \frac{\lambda_3}{r} \sin \theta \cos \beta \ \cos (\psi - \alpha) \tag{4.26}$$

and

$$g_5 = - E_7 \sin \alpha - E_6 \frac{\cos \alpha}{r} \tag{4.27}$$

where E_6 and E_7 are defined by eqns. (2.82) and (2.83).

* The condition that the length be independent of ϕ is equivalent to the condition that the exit contour lie in a plane normal to the z-axis.

Equation (4.15) is employed to eliminate δf_e from eqn. (4.22) and the fixed length restriction, eqn. (4.13), is imposed. Equation (4.22) then becomes

$$H_2 = (g_1 - \frac{\partial f}{\partial r} g_5)\ \delta r + g_2\ \delta V + g_3\ \delta\theta + g_4\ \delta\psi \qquad (4.28)$$

The term $g_2\ \delta V + g_3\ \delta\theta + g_4\ \delta\psi$ can be evaluated on the boundary in terms of variations in the velocity components V_p, V_e and V_t derived in Section 4.1. The variation δV_1 is deduced in terms of δV, $\delta\theta$, $\delta\psi$, and $\delta\alpha$ from eqn. (4.11) as

$$\delta V_1 = - \sin\theta \sin(\psi - \alpha)\ \delta V - V\cos\theta \sin(\psi - \alpha)\ \delta\theta$$
$$- V\sin\theta\cos(\psi - \alpha)\ \delta\psi + V\sin\theta\cos(\psi - \alpha)\ \delta\alpha \qquad (4.29)$$

Thus the term

$$g_2\ \delta V + g_3\ \delta\theta + g_4\ \delta\psi = \frac{\lambda_3 \cos\beta}{rV}\ \delta V_1 - \lambda_3 \frac{\sin\theta\cos\beta\ \cos(\psi - \alpha)}{r}\ \delta\alpha$$
$$= X_3\ \delta V_1 - X_3\ V\sin\theta\cos(\psi - \alpha)\ \delta\alpha \qquad (4.30)$$

where X_3 is defined by eqn. (3.10). Substituting from eqn. (4.30) into eqn. (4.28) gives

$$H_2 = (g_1 - \frac{\partial f}{\partial r} g_5)\ \delta r + X_3\ \delta V_1 - X_3\ V\sin\theta\cos(\psi - \alpha)\ \delta\alpha \qquad (4.31)$$

If no further restrictions are imposed on the boundary, the variations δr, δV_1, and $\delta\alpha$ can be considered as independent variations. Therefore, to satisfy the transversality eqn. (4.1) the individual coefficients of δr, δV_1, and $\delta\alpha$ must be identically zero on the boundary. The resulting boundary equations are:

$$g_1 - \frac{\partial f}{\partial r} g_5 = 0 = - \cos \alpha \Big[(P - P_a) + (V \cos \theta + \lambda_2) \rho V \cos \theta + \frac{\lambda_3 \sin \beta}{rV} \frac{d}{dl} (V \cos \theta) \Big] \tag{4.32}$$

and

$$X_3 = 0 \tag{4.33}$$

which must hold on the entire boundary curve Γ_e.

It may be observed that eqn. (4.33) can be written in terms of the variables V, θ, ψ, α, β, and λ_2 on the boundary by employing the design eqns. (3.28) and (3.29). That is, the eqns. (3.28) and (3.29) which hold on the boundary as well as the control surface can be combined to obtain the derivative of X_3 along the boundary in the l-direction as

$$\frac{dX_3}{dl} = - \cos \alpha \left(\frac{dX_3}{rd\phi}\right)_r + \sin \alpha \left(\frac{dX_3}{dr}\right)_\phi \tag{4.34}$$

Now from eqn. (4.33) the value of X_3 is zero over the entire exit boundary so that dX_3/dl is zero on the boundary; therefore, substituting for $(dX_3/rd\phi)_r$ and $(dX_3/dr)_\phi$ from eqns. (3.28) and (3.29), the result

$$(V \cos \theta + \lambda_2) = \frac{- V \sin \theta \sin \mu \cos (\psi - \alpha)}{\sin \beta \cos \theta + \cos \beta \sin \theta \cos (\psi - \alpha)} \tag{4.35}$$

is obtained which applys on the boundary Γ_e and can be used in place of eqn. (4.33).

On the other hand, if one wishes to restrict the exit contour of the nozzle the variations δr, δV_1, and $\delta\alpha$ in eqn. (4.31) cannot remain independent. For example, one may require the exit contour to be elliptic with a fixed eccentricity and a variable area. Then eqn. (4.17) relates δr

to δa at the exit. Further, the angle α is fixed by the eccentricity of the ellipse so that $\delta\alpha \equiv 0$. The transversality eqn. (4.1) therefore reduces to

$$\oint_{\Gamma_e} \left\{ \left[(g_1 - \frac{\partial f}{\partial r} g_5) \frac{r}{a} \right] \delta a + X_3 \, \delta V_1 \right\} dl$$

$$= \oint_{\Gamma_e} X_3 \, \delta V_1 \, dl + \delta a \oint_{\Gamma_e} (g_1 - \frac{\partial f}{\partial r} g_5) \frac{r}{a} \, dl = 0 \qquad (4.36)$$

where it has been noted that the variation δa is independent of the integration around the boundary and has therefore been taken outside the integral. With no further restrictions, the variations δV_1 and δa are considered independent and arbitrary. The resulting boundary equations are

$$\oint_{\Gamma_e} (g_1 - \frac{\partial f}{\partial r} g_5) \frac{r}{a} \, dl = 0 \qquad (4.37)$$

and

$$X_3 = 0 \qquad (4.38)$$

where eqn. (4.35) can be used in place of eqn. (4.38) if desired. If, in addition, the initial conditions are such that planes of symmetry exist which contain the major and minor axes of the ellipse, the problem can be reduced to computing the flow in one quadrant of the ellipse. Also, the velocity component V_1 cannot now be considered arbitrary at the planes of symmetry. The boundary equations in this instance are eqn. (4.37) plus the equation for X_3,

$$X_3 = 0 \quad (0 < \phi < \frac{\pi}{2}) \qquad (4.39)$$

which is restricted to the portion of the exit contour between the planes of symmetry.

5. METHODOLOGY FOR DESIGN

The overall problem of design of a thrust nozzle may be divided conveniently into the following problems:

1. design of the subsonic portion of the nozzle;
2. design of the transonic region of the nozzle; and
3. design of the supersonic region of the nozzle.

If it can be assumed that the conditions obtained in the transonic region where the flow speed is definitely supersonic are the initial conditions in the design problem, the only region of interest, whether the nozzle is optimized or not, is the supersonic region of the nozzle. The design of that portion of the nozzle depends upon (a) the boundary conditions required to be satisfied at the throat section and at the exit plane of the nozzle and (b) any other constraining relations imposed upon the flow regime. If one of the requirements in the design problem is, for example, that the thrust from the nozzle should be the maximum for given initial, boundary,and constraint conditions, the problem becomes one of determining an optimized solution.

When such an optimized solution is attempted, it has been shown in Chapter 3 that a control surface may be postulated which intersects the kernel and coincides with the designed geometry at the exit plane of the nozzle; that such a control surface is unique in that it satisfies all of the requirements for a characteristic surface; and lastly, that a set of design equations may be obtained to determine the control surface.

The manner in which the boundary conditions, necessary for obtaining the solution, may be developed is described in Chapter 4. Considering the intersection of the control surface with the kernel and the exit plane of the nozzle, there are essentially ten boundary conditions related to the problem variables on the boundaries. These conditions may be given in terms of quantities associated with the nozzle geometry and flow parameters, but must be equivalent to the ten boundary conditions mentioned earlier.

Utilizing the design equations in conjunction with a set of assumed and given boundary conditions on the initial boundary, one can solve for the control surface. The control surface so obtained must satisfy the given terminal boundary conditions. When a discrepancy arises, it becomes necessary to apply iterative methods of solution.

In broad outline, therefore, the determination of an optimized nozzle contour involves the following tasks which must be performed in the order indicated:

1. determine the initial conditions. That is, determine the subsonic, transonic and initial expansion contours of the nozzle for given inlet conditions. It will be assumed in the present discussion that adequately detailed procedures are available for carrying this out;
2. calculate the flow field in the kernel. The procedure for computing three-dimensional supersonic flows by employing the method of characteristics is given in Ref. 10;

3. choose the initial value boundary and assume values of the problem variables not given on that boundary;

4. solve the design equations with the applicable boundary conditions in order to locate the corresponding control surface and to determine the flow variables on it;

5. compare the calculated boundary conditions on the terminal boundary with given boundary conditions on that boundary. If a discrepency arises perform the necessary iterations; and

6. compute the flow field between the kernel and the control surface and determine the supersonic boundary of the optimized nozzle by following the boundary streamline at the throat section of the nozzle. The procedure for this is given in Ref. 10.

It is apparent that several alternatives may be possible in regard to the following, even for a particular formulation of the optimization problem,

1. the manner in which the boundary conditions are developed;
2. the choice of initial and terminal boundaries; and
3. the method employed for solving the set of design equations.

A detailed discussion of those aspects of the problem is beyond the scope of the present thesis. It is merely noted here that the nature of the problem forbids even a firm recommendation in regard to the procedure for the mathematical solution of the design equations except to point out that numerical methods may be tried within the limitations of possible non-uniformities in the convergence of solutions. However a general discussion of the methods and the procedures that may prove suitable and that are presently available is attempted in the remainder of this chapter. In Section 5.1 the methods available for the establishment of the initial conditions and for the calculation of the flow field in the kernel are discussed.

Regarding the possible methods for the solution of the design equations, two illustrative examples are discussed in Section 5.2. In both cases the design equations derived in Chapter 3 are applicable. That is the conditions of the homentropic irrotational flow of a perfect gas, a constant mass flow rate, and a smooth continuous control surface are requisites of the flow. The examples differ from each other in the following respects.

Example one: (a) no specific contour is prescribed for the throat contour and initial expansion contour; they are not variable with respect to the optimization problem, however; (b) the length is prescribed and is independent of ϕ_2^*; and (c) the exit contour is elliptic with a given eccentricity and a variable area.

Example two: (a) an axisymmetric contour is prescribed for the subsonic, transonic, and initial expansion contour; (b) the length is a prescribed function of ϕ_2^*; and (c) the exit contour is fixed and corresponds to the exit contour of a truncated axisymmetric nozzle.

Those two examples will clarify many aspects of the application of the design equations. Example one, discussed in Section 5.2.1, is utilized to illustrate in particular the principal features of a methodology that may be developed to design a nozzle under prescribed conditions. While example two will serve a similar purpose, it is discussed in Section 5.2.2 primarily from the point of view of illustrating when three-dimensional optimization procedures are unavoidable; for example, in a modification of an apparently axisymmetric nozzle.

A method for calculating the intermediate flow field between the kernel and the control surface and thus determining finally the optimized nozzle contour is discussed in Section 5.3.

5.1 The Initial Conditions

According to the formulation of the problem as described in Chapter 3, the initial conditions of flow are to be prescribed or obtained by calculation before the optimization problem can be taken up. They include the initial state of the gas and the wall contour for the subsonic, the transonic, and the initial expansion portions of the nozzle. The subsonic, transonic, and initial expansion contours will generally be determined by such factors as the combustion chamber design requirements, heat transfer requirements, fabrication limitations, and special geometric requirements. The contours chosen determine the flow properties in the kernel; however, the calculation of those flow properties depends upon a solution of the transonic flow problem and also upon a part of the supersonic flow solution. At present the solution to the transonic flow problem in nozzles is limited to approximate solutions for axisymmetric or two-dimensional flows. For non-circular throat cross sections it appears that the approximate methods of Sauer[17] or of Oswatitsch and Rothstein[18] could possibly be modified to apply to some simple non-circular throat cross sections.

The flow field in the kernel is calculated using the method of characteristics beginning on an initial value surface which is the product of the transonic flow solution. If the throat geometry is three-dimensional, either the three-dimensional method of characteristics or an approximate solution technique is required. Although the applications of the three-dimensional method of characteristics have been limited, a few solutions to three-dimensional flow fields which have been obtained recently[14,15], together with advances in digital computer size and technology, indicate that solutions of three-dimensional flow fields based on the three-dimensional method of characteristics may become more satisfactory in the future. Procedures for application of the three-dimensional method of characteristics are given in Refs. 10, 13, 14, 15, and 16.

5.2 Solution Methods for the Design and Boundary Equations

The design equations in terms of the (r, ϕ)-coordinates are eqns.(2.49), (2.51), (2.85), (3.28), (3.29), (3.32), and (3.33). They are rewritten here for immediate reference.

Equation (2.49) expresses the mass conservation constraint in integral form, namely

$$\iint_S F_2 \, dr \, d\phi = \dot{m} = \text{constant} \tag{5.1}$$

where S is the area of integration in the (r, ϕ)-plane and F_2 is defined by eqn. (2.50).

Equation (2.51) is the irrotationality constraint, viz.

$$F_3 = A_1 \frac{1}{V} (\frac{dV}{dr})_\phi + A_2 \frac{1}{V} (\frac{dV}{rd\phi})_r + A_3 (\frac{d\theta}{dr})_\phi + A_4 (\frac{d\theta}{rd\phi})_r + A_5 (\frac{d\psi}{dr})_\phi$$

$$+ A_6 (\frac{d\psi}{rd\phi})_r + \frac{A_6}{r} = 0 \qquad (5.2)$$

where the coefficients A_1 - A_6 are defined by eqns. (2.52) - (2.57).

Equation (2.85) assures the continuity of the control surface. It is,

$$(\frac{d(r \tan \beta \sin \alpha)}{dr})_\phi = (\frac{d(r \tan \beta \cos \alpha)}{rd\phi})_r \qquad (5.3)$$

Design eqns. (3.28), (3.29), (3.32), and (3.33) are

$$(\frac{dX_3}{dr})_\phi = - \rho V (\frac{X_2 A_8}{\sin \mu \cos \beta} + \sin \theta \sin \psi) \qquad (5.4)$$

$$(\frac{dX_3}{d\phi})_r = r\rho V (\frac{X_2 A_9}{\sin \mu \cos \beta} + \sin \theta \cos \psi) \qquad (5.5)$$

$$\sin \mu = \cos \beta \quad \cos \theta - \sin \beta \quad \sin \theta \cos (\psi - \alpha) \qquad (5.6)$$

and

$$\frac{r\rho V^2 X_2}{\sin \mu \cos \beta} \Big[B_1 \frac{1}{V} (\frac{dV}{dr})_\phi + B_2 \frac{1}{V} (\frac{dV}{rd\phi})_r + B_3 (\frac{d\theta}{dr})_\phi + B_4 (\frac{d\theta}{rd\phi})_r$$

$$+ B_5 (\frac{d\psi}{dr})_\phi + B_6 (\frac{d\psi}{rd\phi})_r + \frac{B_6}{r} \Big] = 0 \qquad (5.7)$$

where X_2 and X_3 are defined by eqns. (3.30) and (3.31), and B_1 - B_6 are defined by eqns. (3.34) - (3.39).

Equations (5.1) - (5.7) constitute a set of seven equations for the variables V, θ, ψ , α, β , X_2, and X_3. The variable X_2 is defined by eqn. (3.30) as

$$X_2 = \frac{V \cos \theta + \lambda_2}{V} \qquad (5.8)$$

where λ_2 is the constant Lagrange multiplier and may be determined by satisfying the integral eqn. (5.1). Thus, one is left with the six eqns. (5.2) - (5.7) for the six variables, V, θ, Ψ, α, β, and X_3.

In addition to the aforementioned variables, it is necessary to determine the functional relation defining the length coordinate to a point on the control surface given by eqn. (2.31), namely $z = f(r, \phi)$, which can be obtained by utilizing the eqns. (2.34) and (2.35). Furthermore, at any point on the control surface the Mach angle, μ, and the thermodynamic variables P, ρ, and T can be calculated from the known initial conditions and the calculated value of V at any point.

Before seeking and attempting a (largely) trial and error approach for the solution of the design equations (5.2) - (5.7), some simplification can be achieved initially by combining the design equations to reduce the number of dependent variables from six to four as follows.

(a) Expand eqn. (5.3) by carrying out the indicated partial differentiation to give

$$q_1 \left(\frac{d\beta}{dr}\right)_\phi + q_2 \left(\frac{d\beta}{rd\phi}\right)_r + q_3 \left(\frac{d\alpha}{dr}\right)_\phi + q_4 \left(\frac{d\alpha}{rd\phi}\right)_r + q_5 = 0 \qquad (5.9)$$

where q_1 - q_5 are functions of r, α and β.

(b) Differentiate eqn. (5.6) with respect to r and solve the resulting partial differential equation for $(d\alpha/dr)_\phi$.

(c) Differentiate eqn. (5.6) with respect to ϕ and solve the resulting partial differential equation for $(d\alpha/rd\phi)_r$.

(d) Eliminate partial derivatives of α from eqn. (5.9) utilizing the results of (b) and (c) above to obtain the following partial differential equation in V, θ, Ψ, and β.

$$Q_1 \left(\frac{dV}{dr}\right)_\phi + Q_2 \left(\frac{dV}{rd\phi}\right)_r + Q_3 \left(\frac{d\theta}{dr}\right)_\phi + Q_4 \left(\frac{d\theta}{rd\phi}\right)_r$$

$$+ Q_5 \left(\frac{d\psi}{dr}\right)_\phi + Q_6 \left(\frac{d\psi}{rd\phi}\right)_r + Q_7 \left(\frac{d\beta}{dr}\right)_\phi + Q_8 \left(\frac{d\beta}{rd\phi}\right)_r + Q_9 = 0 \tag{5.10}$$

Equation (5.10) constitutes the first of the four desired equations.

(e) Eliminate X_3 between eqns. (5.4) and (5.5) by differentiating eqn. (5.4) with respect to ϕ, differentiating eqn. (5.5) with respect to r, and equating the right hand sides. The result is a partial differential equation in V, θ, ψ, α and β.

(f) Utilize the results of (b) and (c) above to eliminate derivatives of α in step (e) to give a partial differential equation in V, θ, ψ, and β, namely

$$U_1 \left(\frac{dV}{dr}\right)_\phi + U_2 \left(\frac{dV}{rd\phi}\right)_r + U_3 \left(\frac{d\theta}{dr}\right)_\phi + U_4 \left(\frac{d\theta}{rd\phi}\right)_r$$

$$+ U_5 \left(\frac{d\psi}{dr}\right)_\phi + U_6 \left(\frac{d\psi}{rd\phi}\right)_r + U_7 \left(\frac{d\beta}{dr}\right)_\phi + U_8 \left(\frac{d\beta}{rd\phi}\right)_r + U_9 = 0 \tag{5.11}$$

(g) The third and fourth equations are eqns. (5.2) and (5.7) which may be left unaltered.

Those four equations in the variables, V, θ, ψ, and β constitute the final set of design equations on the control surface. They are common to both of the examples to be discussed under Section 5.2.1 and 5.2.2.

Before discussing those examples, the four eqns. (5.2), (5.7), (5.10), and (5.11) may be examined from the point of view of the mathematical methods available for their solution. Such methods may be summarized as follows.

1. It will be observed that the set of four equations is presented in the (r, ϕ)-plane of the chosen coordinate system. By a further suitable transformation, it may be possible to modify the equations such that they are further simplified. Such transformations must be examined both from the point of view of the complexities that may arise in numerical computation in the transformed plane as well as from the point of view of establishing the necessary reverse transformations.

2. Whether or not simplification can be obtained, the errors likely to arise in the application of numerical methods of analysis must be carefully examined. In general, an attempt should be made to determine if the equations may be classified as hyperbolic, parabolic or elliptic depending upon the relationships among the coefficients of the partial derivatives.* If the equations are hyperbolic, then characteristic directions exist on the (r, ϕ)-plane and a numerical solution utilizing the properties of characteristics is possible. If the system of equations is elliptic or parabolic, then numerical techniques are generally unsatisfactory.

In spite of the computational difficulties that may arise, it is possible to clarify many aspects of the application of the design equations by considering the following illustrative examples.

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* Procedures for classifying systems of partial differential equations can be found in a number of good references on applied mathematics. See, for example, Ref. 19, Chapter 3.

5.2.1 Illustrative Example One

The design eqns. (5.2), (5.7), (5.10), and (5.11) for the variables V, θ, Ψ, and β are applicable in this example. Those equations relate the variables on the control surface, but are presented in the (r, ϕ)-plane. They have been derived under the following conditions:

1. the flow is homentropic and irrotational throughout the flow regime, and the working fluid is a mixture that can be represented by a perfect gas;
2. the mass flow rate through the nozzle is prescribed;
3. the initial conditions at the throat section are prescribed in a region where every point in the flow is supersonic;
4. the shape of the initial expansion contour for the nozzle is prescribed and, therefore, the flow variables in the kernel are known;
5. the z-axis is straight and coincides with a streamline in the flow and with the direction of desired thrust maximization;
6. the design ambient pressure is known; and
7. the control surface is a continuous smooth surface.

The boundary conditions and prescribed quantities (which define the special features of the example) are the following:

1. the length of the nozzle is fixed and is independent of ϕ; therefore, the exit contour of the nozzle must lie on a prescribed plane normal to the z-axis; and
2. the exit contour of the nozzle is elliptic with fixed eccentricity and variable area.

In order to locate the control surface, one must know in detail the flow variables in the kernel of the flow. The location of the boundary defined by the intersection of the control surface and the kernel is not known initially and, in fact, the location of that boundary constitutes part of the solution.

The boundary conditions at the exit contour of the nozzle are eqns. (4.35) and (4.37) plus the geometric constraints of a fixed length independent of ϕ and an elliptic exit shape given by eqn. (4.16). Initially, it is necessary to choose either the inner boundary at the extent of the kernel or the outer boundary at the exit plane as an initial boundary for purposes of the numerical calculation. The choice will, in general, depend on the number of known boundary conditions on each boundary since each unknown condition on the initial boundary will require an iteration loop to determine the correct value on that boundary.

In this example the inner boundary is chosen as the initial boundary. It is then necessary to assume (a) the coordinates of the inner boundary and (b) the value for λ_2 on the control surface. Since the variable X_3 has been eliminated both from the design and boundary equations, it is not necessary to include X_3 in the solution procedure. The initial value curve must be a continuous closed curve encircling the z-axis and must be symmetric with respect to planes of symmetry that may exist in the flow. No other restrictions are placed on the initial choice of the initial value curve. Since the flow variables in the kernel are known, the choice of an initial curve fixes the initial values of V, θ, ψ, α, β, r, and f. The additional choice of a value for λ_2 (which is constant over the entire control surface) then permits the control surface to be calculated using a properly chosen numerical method of computation for solving the design equations. It then remains to examine the resulting boundary values on the terminal boundary from the point of view of compliance with the known, final boundary conditions. The desired exit boundary conditions are obtained by performing iterations on the initial choice of coordinates for the initial value curve and on the initial choice of the constant λ_2. With regard to the iteration procedure the following items are noted.

1. The constant mass flow requirement on the total mass flow in the nozzle may provide a stopping condition for the calculation of the control surface if an exit shape is assumed; however, it is, in addition, essential that the mass flow also be constant in each small segment $\Delta\ \phi$ bounded by stream surfaces passing through the z-axis. The satisfaction of the constant mass flow requirements for each segment requires the computation of the intermediate flow field between the kernel and the control surface and, therefore, that calculation becomes part of at least one iteration cycle.

2. By satisfying the mass flow condition and iterating the value of λ_2 to satisfy eqn. (4.35) a control surface is obtained. The corresponding nozzle contour is the optimized contour for the boundary conditions obtained on the terminal boundary and the ambient pressure calculated from eqn. (4.37) (using eqn.(4.32) to evaluate the integrand in terms of P_a). This iteration

procedure has the disadvantage that it requires the calculation of the entire nozzle contour for each new set of initial data but has the advantage that families of optimized nozzles are determined; thereby valuable information regarding the relationships between the initial and terminal boundary conditions is obtained which can be used to improve the calculation technique.

5.2.2 Illustrative Example Two

The design eqns. (5.2), (5.7), (5.10), and (5.11) for the variables V, θ, ψ, and β are applicable in this example. Those equations relate the variables on the control surface, but are presented in the (r, ϕ)-plane. They have been derived under the following conditions:

1. the flow is homentropic and irrotational throughout the flow regime, and the working fluid is a mixture that can be represented by a perfect gas;
2. the mass flow rate through the nozzle is prescribed;
3. the initial conditions at the throat section are prescribed in a region where every point in the flow is supersonic;
4. the shape of the initial expansion contour for the nozzle is prescribed and, therefore, the flow variables in the kernel are known;
5. the z-axis is straight and coincides with a streamline in the flow and with the direction of desired thrust maximization;
6. the design ambient pressure is known; and
7. the control surface is a continuous smooth surface.

The boundary conditions and prescribed quantities (which define the special features of the example) are the following:

1. the subsonic and throat contours are axisymmetric and the initial expansion contour in any (r, z)-plane is a circular arc of fixed radius; and
2. the exit contour and length are prescribed in the following manner.
   (a) An optimized axisymmetric nozzle of a given length with initial conditions corresponding to those prescribed in 1 in the foregoing is designed to the given design ambient pressure.
   (b) The optimized axisymmetric nozzle is truncated so that the exit contour is on the two-dimensional surface which is normal to the (y, z)-plane, slightly concave toward the nozzle throat (in the shape of a parabola, say) and intersects the axisymmetric exit contour at two points (in the (y, z)-plane).
   (c) The contour and length prescribed for the example are those obtained for the truncated axisymmetric nozzle described in (b).

It is noted that the nozzle length in this example is a prescribed function of the angular coordinate ϕ and, therefore, the problem cannot be solved using the axisymmetric optimization solution since no provision is made in the axisymmetric formulation for a length which varies with the angular coordinate. Apart from the complications which arise in the computation of the flow for the example under consideration, some interesting features of a theoretical nature may be observed. Those are as follows.

1. The control surface which intersects the kernel of the flow and coincides with the exit plane of the nozzle in this example is not axisymmetric; the control surface in an axisymmetric nozzle is axisymmetric. In the present example, therefore, all three of the components of the unit normal to the control surface may have non-zero values.
2. The flow in the present example is not confined to the (r, z)-plane for every value of ϕ; the flow in an axisymmetric nozzle is entirely independent of the angular coordinate.
3. When the axisymmetric flow nozzle is to be deduced from the general three-dimensional flow, it is necessary to impose both of the conditions, $\psi \equiv 0$ and $\sin \alpha \equiv 0$.

Statements 2 and 3 result from the fact that it is not possible, in general, to reduce the design equations (or their equivalent in some other optimization problem solved by variational techniques) by imposing restrictions on the dependent variables. That is, the restriction $\psi = 0$ imposed on the three-dimensional design equations will not produce the design equations for an optimized nozzle in which $\psi \equiv 0$. This aspect of the problem may be seen clearly by reference to the variational integral I_1 defined by eqn. (3.1). Using the definition of eqn. (3.2) the variation of I_1 can be written

$$\delta I_1 = \iint_S \left[E_1 \, \delta V + E_2 \, \delta\theta + E_3 \, \delta\psi + (\ ) \, \delta f \right] dr \, d\phi = 0 \quad (5.12)$$

where E_1, E_2, and E_3 are defined by eqns. (2.67), (2.68), and (2.69) respectively and the coefficient of δf is of no consequence in the present discussion. The design equations were derived by setting the coefficients of the variations δV, $\delta\theta$, $\delta\psi$, and δf to zero which is justified only if the variations can be considered independent and arbitrary. If the constraint

$$\psi = 0 \tag{5.13}$$

is to be imposed the variation $\delta\psi$ can no longer be considered independent and arbitrary but is, instead, identically zero. Equation (5.13) is then used instead of eqn. (2.69) ($E_3 = 0$) in deriving the design equations. It is clear that the resulting design equations are not equivalent to the equations derived by imposing the constraint on the general solution except under very special circumstances.

The point is further emphasized by deriving the design equations with the constraint $\psi = 0$ but without restricting the control surface to be axisymmetric.

In this example the irrotationality constraint is no longer required. The problem formulation follows essentially that already presented in Chapters 2 and 3, and the equations $h_1 = 0$, $h_2 = 0$, and $(dh_4/dr)_\phi = 0$ are obtained where h_1, h_2, and h_4 are defined by eqns. (3.15), (3.21), and (3.25) respectively. These three equations are combined to yield the design equations

$$V \cos\theta + \lambda_2 = V \sin\theta \tan\mu \tag{5.14}$$

$$\tan^2\mu = \frac{\sin^2\varrho}{\cos^2\varrho - \sin^2\beta \sin^2\alpha} \tag{5.15}$$

and

$$\frac{d}{dr} (r\rho V^2 \sin^2 \theta \tan \mu)_{\phi} = 0 \tag{5.16}$$

Equations (5.14) - (5.16) are not the same equations as are obtained from setting $\psi = 0$ in the three-dimensional design equations. Specifically, the direction of the control surface as defined by eqn. (5.15) is no longer in a characteristic direction except in the special case $\sin \alpha = 0$ (an axisymmetric control surface).

In summary, it may be observed that (a) a truncated axisymmetric nozzle, however truncated, is not an optimized nozzle even when the original axisymmetric nozzle is an optimized nozzle, and (b) in whatever manner the length of the nozzle and the shapes at the inlet section and at the exit plane are specified, unless the flow is of a lower dimension over the entire flow regime, the optimization problem must be posed as a problem in three-dimensional flow.

The remaining tasks for obtaining the nozzle contour in this example are the same as those described under the illustrative example in Section 5.2.1.

5.3 Intermediate Flow Field Calculation

The intermediate flow field between the kernel and the control surface may be needed to impose the constant mass flow restrictions on the control surface as a function of ϕ and, in any event, will be required for the determination of the final optimized contour. This three-dimensional flow field can be calculated using the three-dimensional method of characteristics by modifying existing procedures to permit the use of the kernel and the control surface as initial value surfaces. Procedures which can be readily adapted to this calculation are described in detail in Ref. 10.

The final contour is determined by computing the stream tube passing through the nozzle throat.

6. CONCLUSIONS

The problem of optimizing a flow geometry under given initial conditions and constraint relations has been solved by the use of the calculus of variations. In particular a three-dimensional, irrotational, homentropic, internal flow problem pertaining to the optimization of the supersonic portion of the contour of a thrust nozzle is posed on the assumption that the initial conditions and the exit flow geometry are fixed while requiring that the value of thrust (or momentum) obtained be a maximum within a specified length of flow. In view of the three-dimensional nature of the flow, the length of the nozzle is, in general, a function of the angular coordinate.

Two illustrative examples are discussed to demonstrate the nature of the problems which arise in the actual application of the general solutions (of the optimization problem) to particular cases. The following are the principal conclusions which may be derived from the investigation.

1. A three-dimensional supersonic flow geometry, such as is obtained in the supersonic portion of a nozzle, can be optimized with respect to a given set of initial conditions and a set of constraint relations.

2. One formulation of the optimization problem may be based upon a postulated control surface which can be shown under the proper

constraint conditions to be a characteristic surface and thus uniquely determined.

3. While the design equations so obtained under certain general requirements, such as integrability of the control surface, irrotationality, and homentropicity of flow, will apply in all problems (governed by such requirements) on the control surface, each problem becomes specialized in regard to the boundary conditions specified and the manner in which such boundary conditions are chosen to be employed for analysis.

4. The nature of the equations determining the control surface and the boundary conditions required for their solution make it imperative that numerical methods of solution be employed. Furthermore, a series of iterative procedures is required in relation to the constraints imposed on the boundary. Often the iterative procedures may involve the computation of the entire flow field, unless great simplifications are obtained in the basic design equations.

5. When a variational problem is formulated for a three-dimensional flow, there is no direct method of deducing a set of design equations applicable to a lower dimensional flow (e.g. an axi-symmetric flow). The conditions under which the higher dimensional flow is reduced to the lower dimensional flow may be seen clearly, but the solution to the problem of the lower dimensional flow has to be formulated and solved by itself.

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7. BIBLIOGRAPHY

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APPENDICES

APPENDIX A

NOTATION

A	=	area
A_i	=	(i = 1-9) coefficients defined by eqns. (2.52) - (2.57) (3.13), (3.30), (3.31) respectively
a	=	length of semi-major axis of ellipse
a_1, a_2	=	coefficients defined by eqns. (2.40) and (2.41)
B_i	=	(i = 1-6) coefficients defined by eqns. (3.34) - (3.39) respectively
b_1, b_2	=	coefficients defined by eqns. (2.42) and (2.43)
C_i	=	(i = 1-7) coefficients defined by eqns. (3.41) - (3.47) respectively
c	=	local sound speed
D_r, D_ϕ	=	derivatives defined by eqns. (3.11) and (3.12)
d	=	derivative operator
E_i	=	functional representation of Euler-Lagrange equations defined by eqn. (2.65)
e	=	eccentricity of an ellipse
F	=	function defined by eqn. (2.30)
F_1	=	function defined by eqn. (2.48)
F_2	=	function defined by eqn. (2.50)
F_3	=	function defined by eqn. (2.51)
f	=	function describing the control surface defined by eqn. (2.31)

f_r = partial derivative of f with respect to r as defined by eqn. (2.75)

f_ϕ = partial derivative of f with respect to ϕ as defined by eqn. (2.76)

G_i = function defined by eqn. (2.59)

g_i = (i = 1-5) functions defined by eqns. (4.23) - (4.27) respectively

H_1, H_2 = functions defined by eqns. (2.92) and (2.93)

h_i = (i= 1-5) functions defined by eqns. (3.15), (3.21), (3.23), (3.25), and (3.26) respectively

I = variational integral defined by eqn. (2.61)

I_1 = variational integral defined by eqn. (3.1)

J = integral defined by eqn. (B-6)

K_i = (i = 1-7) coefficients defined by eqns. (3.60) - (3.66) respectively

$\vec{L}$ = unit vector along a bicharacteristic on a characteristic surface

L_r = r-component of unit vector $\vec{L}$ as defined by eqn. (2.21)

L_ϕ = ϕ-component of unit vector $\vec{L}$ as defined by eqn. (2.22)

L_z = z-component of unit vector $\vec{L}$ as defined by eqn. (2.23)

$\vec{l}$ = unit vector along the boundary of the control surface

l_r = r-component of $\vec{l}$ as defined by eqn. (4.3)

l_ϕ = ϕ-component of $\vec{l}$ as defined by eqn. (4.3)

$\dot{m}$ = mass flow rate

$\vec{m}$ = unit vector normal to control surface boundary (see Fig. 2.9)

$\vec{N}$ = unit vector on a characteristic surface normal to the bicharacteristic direction

N_r = r-component of $\vec{N}$ defined by eqn. (2.18)

N_ϕ = ϕ-component of $\vec{N}$ defined by eqn. (2.19)

N_z	=	z-component of $\vec{N}$ defined by eqn. (2.20)
$\vec{n}$	=	unit vector normal to the control surface
n_r	=	r-component of $\vec{n}$ defined by eqn. (2.2)
n_ϕ	=	ϕ-component of $\vec{n}$ defined by eqn. (2.2)
n_z	=	z-component of $\vec{n}$ defined by eqn. (2.2)
P	=	pressure
P_a	=	ambient pressure
$\vec{p}$	=	unit vector normal to the nozzle wall at the exit
p_r	=	r-component of $\vec{p}$ defined by eqn. (4.4)
p_ϕ	=	ϕ-component of $\vec{p}$ defined by eqn. (4.5)
p_z	=	z-component of $\vec{p}$ defined by eqn. (4.6)
Q_i	=	(i = 1-9) coefficients in eqn. (5.10)
q_i	=	(i = 1-5) coefficients in eqn. (5.9)
R	=	gas constant
R_i	=	(i = 1-3) derivatives defined by eqn. (2.72) - (2.74)
r	=	coordinate of (r, ϕ, z)-cylindrical coordinates
S	=	area of projected control surface on (r, ϕ)-plane
s	=	entropy
T	=	temperature
T_i	=	derivatives defined by eqns. (2.72) - (2.74)
T_z	=	axial thrust
$\vec{t}$	=	unit vector in tangent plane to nozzle boundary
t_r	=	r-component of $\vec{t}$ defined by eqn. (4.7)
t_ϕ	=	ϕ-component of $\vec{t}$ defined by eqn. (4.8)
t_z	=	z-component of $\vec{t}$ defined by eqn. (4.9)

U_i = (i = 1-9) coefficients in eqn. (5.11)

V = magnitude of the velocity

$\vec{V}$ = velocity vector

V_r = r-component of $\vec{V}$ as defined by eqn. (2.1)

V_ϕ = ϕ-component of $\vec{V}$ as defined by eqn. (2.1)

V_z = z-component of $\vec{V}$ as defined by eqn. (2.1)

V_p = p-component of $\vec{V}$ as defined by eqn. (4.10)

V_l = l-component of $\vec{V}$ as defined by eqn. (4.11)

V_t = t-component of $\vec{V}$ as defined by eqn. (4.12)

w_i = independent variable

X_2, X_3 = function defined by eqns. (3.9) and (3.10)

x = coordinate in rectangular (x, y, z)-coordinates

y = coordinate in rectangular (x, y, z)-coordinates

z = coordinate in (r, ϕ, z)-cylindrical coordinate system

$\vec{z}$ = unit vector in z-direction

Greek Symbols

α = angle partially defining the direction of the unit normal to the control surface, see Fig. 2.3

β = angle partially defining the direction of the unit normal to the control surface, see Fig. 2.3

Γ = boundary of a domain of integration

γ = ratio of specific heat capacities

Δ = incremental change

δ = variational operator

δ = angle relating bicharacteristic direction to (r, ϕ, z)-cylindrical cooruinates, see Fig. 2.7

ϵ = variational parameter

θ = angle partially defining the direction of the velocity vector, see Fig. 2.2

λ_2 = Lagrange multiplier

λ_3 = Lagrange multiplier

μ = Mach angle defined by eqn. (2.10)

ξ = angle defined by eqn. (2.37)

π = ratio of the circumference to the diameter of a circle

ρ = density

ϕ = angular coordinate of (r, ϕ, z)-cylindrical coordinates

ψ = angle partially defining the direction of the velocity vector, see Fig. 2.2

$\vec{\omega}$ = vorticity vector defined by eqn. (2.13)

ω_r = r-component of $\vec{\omega}$ defined by eqn. (2.14)

ω_ϕ = ϕ-component of $\vec{\omega}$ defined by eqn. (2.15)

ω_z = z-component of $\vec{\omega}$ defined by eqn. (2.16)

Subscripts

a = ambient conditions

e = conditions at the exit plane

i = variable index

k = conditions in the kernel or on the boundary of the kernel

o = total conditions of thermodynamic variables

s = constant entropy

<u>Operators</u>

∂ = partial derivative operator

d = differential operator

δ = variational operator

$\left(\frac{d(\ )}{dL}\right)_N$ = partial derivative on the control surface in the direction which holds N constant

$\left(\frac{d(\ )}{dN}\right)_L$ = partial derivative on the control surface in the direction which holds L constant

$\left(\frac{d(\ )}{dr}\right)_\phi$ = partial derivative in (r, ϕ)-plane in the direction which holds ϕ constant

$\left(\frac{d(\ )}{rd\phi}\right)_r$ = partial derivative in (r, ϕ)-plane in the direction which holds r constant

∇ = Del operator of vector calculus

$\overrightarrow{(\ )}$ = vector

APPENDIX B

DERIVATION OF VARIATIONAL RELATIONSHIPS

The object of this Appendix is to derive the variational relationships which are needed for the solution of the optimum thrust nozzle design problem. An understanding of the basic concepts of variational calculus is assumed.

Consider a function G defined over the domain A in the (r, ϕ)-plane illustrated in Fig. B-1. G may be an explicit function of the two independent variables r and ϕ, the p dependent variables $w_i (i=1,2,\ldots p)$, and the partial derivatives of w_i denoted as R_i and T_i where

$$\text{(a)} \quad R_i \equiv \frac{\partial w_i}{\partial r} \quad ; \qquad \text{(b)} \quad T_i \equiv \frac{\partial w_i}{r \partial \phi} \tag{B-1}$$

To treat the variational problem, the variational parameter ϵ is introduced. The dependent variables of the system are considered as functions of ϵ so that

$$G = G(r, \phi, w_i(\epsilon), R_i(\epsilon), T_i(\epsilon)) \tag{B-2}$$

In accordance with standard notation the first variation of $w_i(\epsilon)$ is defined as

$$\delta w_i = \left. \frac{\partial w_i}{\partial \epsilon} \right|_{\epsilon = 0} d\epsilon \tag{B-3}$$

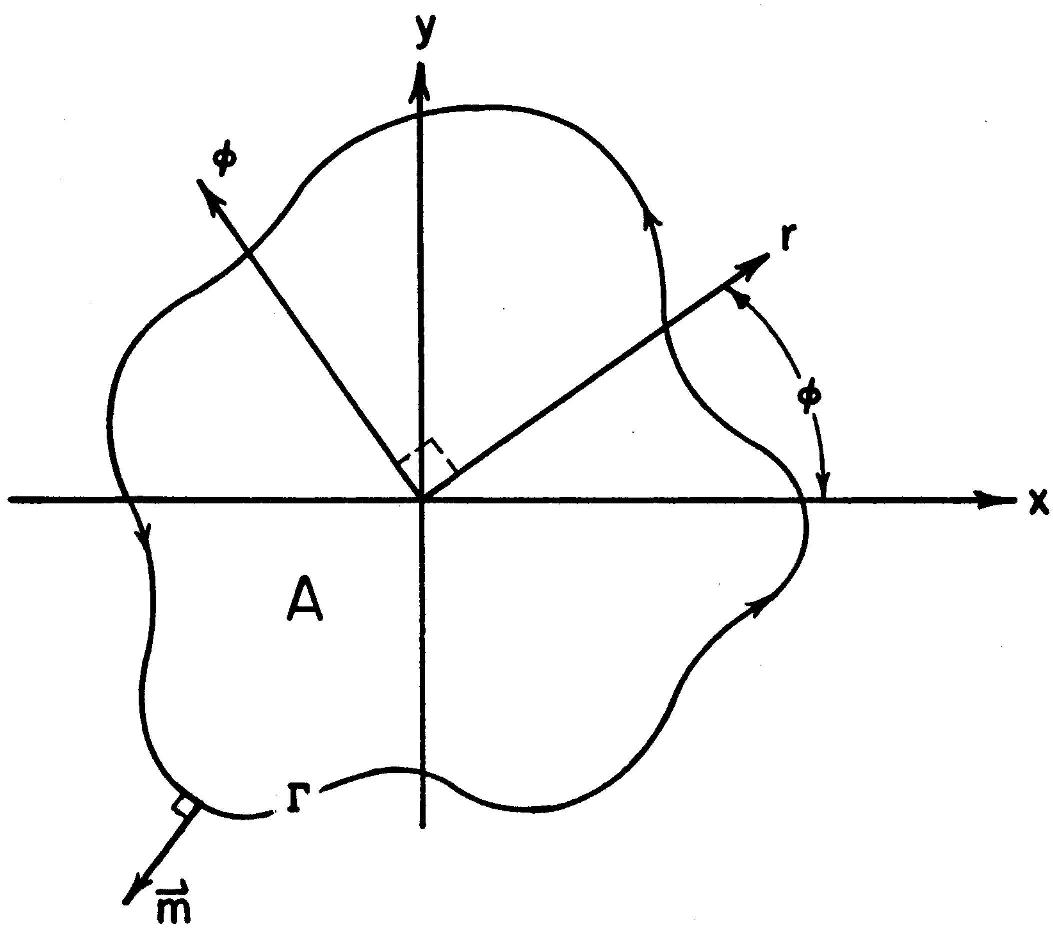

FIGURE B-1

DOMAIN OF INTEGRATION, A,
IN THE (r,ϕ) - PLANE

Consider now the integral I defined by the equation

$$I(\epsilon) \equiv \iint_{A(\epsilon)} G(r, \phi, w_i(\epsilon), R_i(\epsilon), T_i(\epsilon))\, dr\, d\phi \qquad \text{(B-4)}$$

where the area of integration, A, may be a function of the variational parameter ϵ. To determine the first variation of I, namely

$$\delta I = \left.\frac{dI}{d\epsilon}\right|_{\epsilon=0} d\epsilon \qquad \text{(B-5)}$$

requires the application of two well known concepts from calculus. The first is Liebnitz' rule for differentiation of an integral with variable limits and the second is integration by parts which for an area integral is equivalent to Stokes' Theorem. These two concepts can be stated in equation form as follows:

<u>Liebnitz' Rule</u>[20]:

If

$$J(\epsilon) = \int_{a(\epsilon)}^{b(\epsilon)} f(t, \epsilon)\, dt \qquad \text{(B-6)}$$

where a and b are differentiable functions of ϵ and both $f(t, \epsilon)$ and $\partial f(t, \epsilon)/\partial t$ are continuous in both t and ϵ, then

$$\frac{dJ}{d\epsilon} = \int_{a(\epsilon)}^{b(\epsilon)} \frac{\partial f(t,\epsilon)}{\partial t}\, dt + f(b(\epsilon),\epsilon)\frac{db(\epsilon)}{d\epsilon} - f(a(\epsilon),\epsilon)\frac{da(\epsilon)}{d\epsilon} \qquad \text{(B-7)}$$

<u>Stokes' Theorem</u>: Stokes' Theorem in vector form is[21]

$$\iint_A \nabla \times \vec{F} \cdot \vec{dA} = \oint_\Gamma \vec{F} \cdot \vec{dl} \qquad \text{(B-8)}$$

where the element dl is along the boundary Γ in a positive sense, keeping the area A always on the left. In terms of polar coordinates r and ϕ and the components of $\vec{F}$ in the r and ϕ directions (denoted as F_r and F_ϕ), eqn. (B-8) becomes

$$\iint_A \left[\frac{\partial}{\partial r} (rF_\phi) - \frac{\partial}{\partial \phi} (F_r) \right] dr\, d\phi = \oint_\Gamma \left(-F_r\, r \frac{\partial \phi}{\partial m} + F_\phi \frac{\partial r}{\partial m}\right) dl \qquad \text{(B-9)}$$

where $\vec{m}$ is the unit outward normal to Γ as illustrated in Fig. B-1.

Now eqn. (B-4) can be rewritten to include the limits of integration as

$$I(\epsilon) = \int_0^{2\pi} \int_0^{r(\epsilon)} G(r, \phi, w_i(\epsilon), R_i(\epsilon), T_i(\epsilon))\, dr\, d\phi \qquad \text{(B-10)}$$

so that Liebnitz' Rule can be applied to calculate δI. Thus,

$$\delta I \equiv \left.\frac{dI}{d\epsilon}\right|_{\epsilon=0} d\epsilon = \iint_{A(\epsilon)} \left(\frac{\partial G}{\partial w_i} \delta w_i + \frac{\partial G}{\partial T_i} \delta T_i + \frac{\partial G}{\partial R_i} \delta R_i\right) dr\, d\phi$$

$$+ \int_0^{2\pi} G\Big|_{r_\Gamma} \left.\frac{dr}{d\epsilon}\right|_{\epsilon=0} d\epsilon\, d\phi \qquad \text{(B-11)}$$

where the standard summation convention for repeated variable indicies is employed. For example, the term

$$\frac{\partial G}{\partial w_i} \delta w_i = \frac{\partial G}{\partial w_1} \delta w_1 + \frac{\partial G}{\partial w_2} \delta w_2 + \cdots + \frac{\partial G}{\partial w_p} \delta w_p \qquad \text{(B-12)}$$

In the terms δR_i and δT_i, the order of the variation δ and the partial derivative operations can be interchanged as follows:

$$\text{(a)}\quad \delta R_i = \delta\left(\frac{\partial w_i}{\partial r}\right) = \frac{\partial(\delta w_i)}{\partial r} \quad ; \qquad \text{(b)}\quad \delta T_i = \delta\left(\frac{\partial w_i}{r\partial\phi}\right) = \frac{\partial}{r\partial\phi}(\delta w_i) \qquad \text{(B-13)}$$

Therefore, the terms involving δR_i and δT_i in eqn. (B-11) can be expanded to give

$$\frac{\partial G}{\partial R_i}\,\delta R_i = \frac{\partial G}{\partial R_i}\,\frac{\partial(\delta w_i)}{\partial r} = \frac{\partial}{\partial r}\left(\frac{\partial G}{\partial R_i}\,\delta w_i\right) - \delta w_i\,\frac{\partial}{\partial r}\left(\frac{\partial G}{\partial R_i}\right) \qquad \text{(B-14)}$$

and

$$\frac{\partial G}{\partial T_i}\,\delta T_i = \frac{\partial G}{\partial T_i}\,\frac{\partial(\delta w_i)}{r\partial\phi} = \frac{\partial}{r\partial\phi}\left(\frac{\partial G}{\partial T_i}\,\delta w_i\right) - \delta w_i\,\frac{\partial}{r\partial\phi}\left(\frac{\partial G}{\partial T_i}\right) \qquad \text{(B-15)}$$

Equations (B-14) and (B-15) are substituted into eqn. (B-11) which can then be partially integrated using Stokes' Theorem in the form of eqn. (B-9) where

$$\text{(a)}\quad F_r = -\frac{\partial G}{\partial T_i}\,\frac{\delta w_i}{r} \qquad \text{and (b)}\quad F_\phi = \frac{\partial G}{\partial R_i}\,\frac{\delta w_i}{r} \qquad \text{(B-16)}$$

Thus,

$$\delta I = \iint_A E_i\,\delta w_i\,dr\,d\phi + \oint_\Gamma \left(\frac{1}{r}\frac{\partial G}{\partial R_i}\frac{\partial r}{\partial m} + \frac{\partial G}{\partial T_i}\frac{\partial\phi}{\partial m}\right)\delta w_i\,dl$$

$$+ \int_0^{2\pi} G\Big|_{r_\Gamma}\,\frac{dr}{d\epsilon}\Big|_{\epsilon=0}\,d\epsilon\,d\phi \qquad \text{(B-17)}$$

where

$$E_i = \frac{\partial G}{\partial w_i} - \frac{\partial}{\partial r}\left(\frac{\partial G}{\partial R_i}\right) - \frac{\partial}{r\partial\phi}\left(\frac{\partial G}{\partial T_i}\right) \qquad \text{(B-18)}$$

is the well-known Euler-Lagrange equation of variational calculus.

Notice that eqn. (B-17) for the variation of I can be separated into two parts, namely a part that arises for a fixed area A and a part that is ascribed to the variation of the domain of A.