Drexel Institute of Technology
Project 243

Contract Number
NAS8-11196

FUNDAMENTAL CONCEPTS OF STABILITY THEORY

Supplement to: New Methods for Systematic Generation of Liapunov Functions

(I) INTRODUCTION

In this section we list some of the basic theorems and definitions which are used in the analysis of dynamic systems. The proofs of the theorems will not be given but can be found in the references which are listed at the end of this section. Wherever possible, examples will be given to facilitate the understanding of the theorems and definitions.

The first part of this section will be a "naive" discussion of non-linear phenomena. The second part will be concerned with the properties of the dynamic systems whose stability is desired. Finally, the last (and main) part of the section will contain a list of the important theorems and definitions in the "stability" and "boundedness" fields.

(II) THE NONLINEAR WORLD (1) to (8)*

Before giving various examples of linear and nonlinear phenomena in the physical world, let us define in mathematical terms a linear and a nonlinear operator. Suppose that θ is an arbitrary mathematical operator which maps a given space into another space. Suppose further, that $f=f(x)$ and $g=g(x)$ are arbitrary functions of a variable x, and a and b are arbitrary constants. (In our work we will usually think of θ as mapping the reals into the reals, or an n-dimensional real vector into another n-dimensional real vector.) Thus, we say that θ is a linear operator if it satisfies

$$\theta\,(a\, f(x)) = a\,\theta\,(f(x)),$$
$$\theta\,(f(x) + g(x)) = \theta\,(f(x)) + \theta\,(g(x)),$$

for all constants a and functions f,g under consideration. If θ is a linear

---

*The numbers in the parentheses refer to the references at the end of the report.

operator and g is not identically zero, then $\Theta(f(x)) = g(x)$ is a linear, nonhomogeneous equation; and $\Theta(f(x)) = 0$ is a linear, homogeneous equation. If Θ is invariant under a translation in x, then it is an autonomous operator; if it is not, Θ is called a nonautonomous operator.

Some examples of linear operators are:

(1). $\Theta(f(x)) = f(x)$, identity operator,

(2). $\Theta(f(x)) = 3\, f(x-5)$,

(3). $\Theta(f(x)) = \dfrac{d\, f(x-2)}{dx}$,

(4). $\Theta(f(x)) = \int_0^\infty f(x-s)e^{-s}ds$,

(5). $\Theta(f(x)) = x\, f(x)$, nonautonomous operator.

In the physical world, the governing equations of motion for certain phenomena are often linearized. The physical assumptions which are invoked in some of these cases are:

(1). Ideal, homogeneous, uniform fluids,

(2). perfect insulation,

(3). isotropic media,

(4). weightless members,

(5). infinitesimal waves,

(6). small deflections.

The deficiency of an operator Θ is defined as, (8),

$$D = \Theta(f(x) + g(x)) - \Theta(f(x)) - \Theta(g(x)).$$

If D is not identically zero for all the functions under consideration, then Θ is a nonlinear operator. An example is:

$$\theta\,(f(x)\,) = f(x)\,\frac{df(x)}{dx},$$

$$D\,(f(x) + g(x)\,) = (f(x) + g(x)\,\frac{d(f(x) + g(x))}{dx} - f(x)\frac{df(x)}{dx} - g\frac{dg}{dx}$$

$$= f\,\frac{dg}{dx} + g\,\frac{df}{dx},$$

which is not in general zero. Some other nonlinear operators are,

(1). $\theta\,(f) = f^2(x),$

(2). $\theta\,(f) = \int_0^\infty f(x-s)\,f(s)ds,$

(3). $\theta\,(f) = \exp\,(f(x)\,).$

Some examples of physical phenomena which are nonlinear are,

(1). certain springs and oscillators,

(2). gases at high pressures,

(3). finite displacement theory of elasticity,

(4). rigid body dynamics,

(5). movement of flood waves in rivers.

Reasons for studying nonlinear systems in physics and engineering are:

(1). physical measurements have grown more refined and thus nonlinear effects are exhibited in the collected data;

(2). certain systems are by their very nature nonlinear, such as rigid body motion in three-space;

(3). nonlinear devices are sometimes far superior to linear ones; i.e., nonlinear controls can often far outperform linear controls.

The basic mathematical tool used for expressing change and its causes in certain mechanical, biological, economic, electronic, and control systems is the ordinary differential equation. This equation may be linear or nonlinear and autonomous or nonautonomous. Some examples of nonlinear equations which describe certain physical phenomena are given below:

(1). from Newton's 2nd Law of Motion, we have the equation of the oscillating pendulum

$$\ddot{\theta} + g/L\,\sin\theta = 0,$$

where θ is the angular displacement and L is the length of the pendulum;

(2). Van der Pol's equation

$$\ddot{X} - \mu \dot{x} \; (1-x^2) + x = 0,$$

$$\mu > 0$$

describes the variation of the current in a radio circuit, certain periodic biological processes, and business cycles;

(3). the theory of finite waves progressing over the surface of a water mass of infinite depth reduces to

$$\dot{X} = A \exp(-3x) \; f(t);$$

(4). the theory of large elastic deflections produces the following equation

$$W'' + AW\left[1-(W')^2\right]^{\frac{1}{2}} = 0.$$

The question might be asked, how do linear and nonlinear systems differ in behavior? If a system is linear and homogeneous, then any linear combination of solutions of the system is also a solution; this is called the principle of superposition. Nonlinear systems, in general, do not obey this principle.

The global behaviour of a linear, autonomous system can be predicted by local behavior, which is not the case for nonlinear systems. In some nonlinear systems, local properties can be determined by linear approximation, but usually the global properties require that the nonlinear terms be investigated. There are nonlinear systems in which local phenomena can not even be determined by the linear approximation.

In a linear, autonomous system the phenomenon of resonance can occur if there is no damping in the systemand a certain bounded forcing function is impressed upon the system. Thus, a bounded imput can cause an unbounded output in a linear system. If damping is present, bounded imputs produce bounded outputs. In nonlinearsystems, linear resonance does not occur because the periods of oscillations are amplitude dependent and thus, in some cases, the nonlinearities produce stable conditions in a system even though damping may be absent. But there are <u>nonlinear resonance</u> effects which go under the names of subharmonic resonance, jump phenomenon, parametric excitation, and hysteresis effects.

In nonlinear systems (such as Van der Pol's equation) there may exist self-sustained oscillations. These oscillations may occur without the influence of external forces, but simply arise from the internal structure of the system and the manner in which the system's energy is transformed from one state to another state. This phenomenon can only be explained by considering the nonlinearities.

Two great names in the theory of differential equations are Poincare of France and Liopunov of Russia. Poincare studied the geometric properties of solutions and invented certain techniques for the computation of solutions. His work stimulated the development of certain topics of modern abstract mathematics. Some of the ideas of Poincare are considered in the next part of the section.

Liopunov studied the stability properties of the solutions of ordinary differential equations by generalizing the work of Lagrange. In his second or direct method, Liopunov analyzed the stability of systems without obtaining the actual solutions. His work stimulated the development of quantitative and qualitative information about the stability of various types of systems. In the last part of this section we will list many of these results.

(III) The Systems Under Investigation

(A) 2nd Order Systems (3), (8), (9), (12)

Consider the second order system defined by

$$\dot{X} = P\ (x,y), \qquad \dot{Y} = Q\ (x,y). \tag{1}$$

The XY-plane is called the phase plane or state space. The equations in (1) give the flow in the state space, and the corresponding velocity of flow is given by the two-dimensional vector

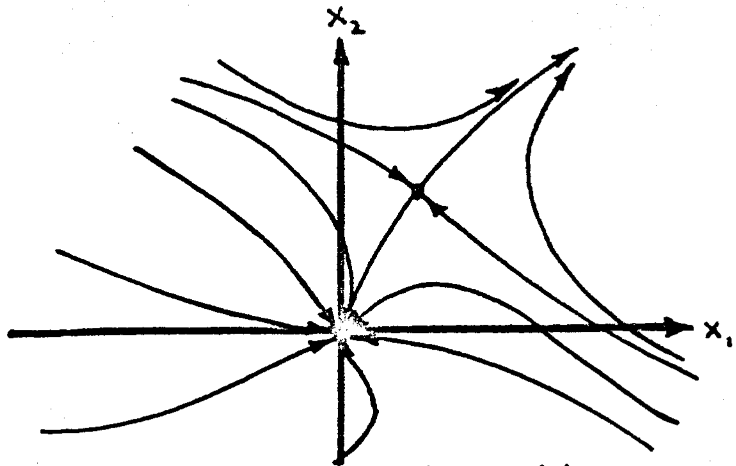
$$\underline{V} = (\dot{x},\dot{y}) = (\ P(x,y),\ Q(x,y)\ ). \tag{2}$$

If P and Q are not both zero, then $\underline{V}$ is not $\underline{0}$, or the state of the system is changing at each point in the xy-plane (for which $\underline{V} \neq \underline{0}$). The points where $\underline{V} = \underline{0}$ are called rest points, equilibrium points, singular points, or points of no flow. An ordinary point is any point where $\underline{V} \neq \underline{0}$. The flow lines in the xy-plane are the solutions of (1) or the trajections of (1). We assume that P and Q are such that through every ordinary point there passes one and only one flow line, and that each singular point is isolated.

From reference (8), we have the following example of a second order system. The system is a simplified model of Volterro's "struggle for existence between two species", where one species preys exclusively on the other. The defining equations are

$$\dot{X}_1 = -2x_1 + x_1x_2, \qquad \dot{X}_2 = -x_2 + x_1x_2, \tag{3}$$

having singular points located at (0,0) and (1,2). In the phase plane plot given below, the arrows point in the direction of increasing time. It can be shown by Poincare's singular-point analysis that (0,0) is asymptolically stable and (1,2) is unstable.



Phase - Plane Plot for Eq.(3)

By assuming that equation (1) could be expanded in series form,

$$\dot{X} = a_1x + b_1y + c_1x^2 + d_1y^2 + e_1xy + \ldots, \qquad (4)$$
$$\dot{Y} = a_2x + b_2y + c_2x^2 + d_2y^2 + e_2xy + \ldots,$$

Poincare was able to classify the singular points of (1) according to their stability properties, and he also described the trajections in the neighborhoods of these points by considering the linear terms in (4). This work is repeated in many texts, such as references (8), (9), (10), (12), and (17). For certain critical cases the linear approximation fails, and the nonlinear terms must be considered, see reference (17).

Another topic of interest in second order systems is the search for periodic solutions or limit cycles. Some results of this work for autonomous and nonautonomous systems can be found in references (8) to (14) and (17). In references (15) and (16), the geometric properties of third, fourth, and higher order systems are investigated, in analogy with the second order systems. These references also list extended bibliographies in this area. In reference (14), a survey of non-autonomous systems, as applied to mechanics, is presented for second order systems.

(B) Higher Order Systems (12), (18), (19), (20).

The state of a system, described by the n-dimensional vector $\underline{X}$, is the minimum amount of information needed about a "physical dynamic" system at some past time in order to predict its future behavior. The dynamic system may be described by the following n-dimensional vector differential equation:

$$\dot{\underline{X}} = \underline{f}\ (t,\underline{x},\underline{u}). \qquad (5)$$

The vector $\underline{U}$ is a specified m-dimensional control vector or forcing function; $\underline{X}$ is the state vector; t is a scalar variable (usually taken to be time); and $\underline{f}$ is an n-dimensional function which satisfies sufficient conditions for existence and uniqueness of solutions as required by whatever problem is under consideration.

If the control vector $\underline{U}$ is identically zero, then (5) becomes a free, nonlinear, nonautonomous system

$$\dot{\underline{X}} = \underline{f}\ (t,\underline{x}). \quad (6)$$

A free, nonlinear, autonomous system is defined by

$$\dot{\underline{X}} = \underline{f}\ (\underline{x}). \quad (7)$$

In all of our problems, we will assume that t and the components of $\underline{X}$ are real numbers.

Suppose our nonlinear, nonautonomous system is given by

$$\dot{\underline{Y}} = \underline{g}\ (t,\underline{y}), \quad (8)$$

where $\underline{y} = \underline{\eta}$ is a particular solution of (8) whose stability properties are desired. The first step in the analysis is to derive the <u>equation of perturbed motion</u> about $\underline{y} = \underline{\eta}$. Thus, the perturbation about is defined as $\underline{X} = \underline{y} - \underline{\eta}$, and the corresponding equation of perturbed motion is

$$\dot{\underline{X}} = \underline{f}\ (t,\underline{x}), \quad (9)$$

where $\underline{f}\ (t,\underline{x}) = \underline{g}\ (t,\underline{x} + \underline{\eta}) - g\ (t,\underline{\eta})$.

The unperturbed solution of (9) is $\underline{X} = \underline{0}$. Throughout the remaineder of this report equation (9), with an isolated singular solution at $\underline{X} = \underline{0}$, will be the subject of our investigation. An example of a perturbed equation is given below; it was obtained from reference (21).

<u>Example</u> : <u>Atmospheric Reentry System</u>.

Consider the simple reentry problem of a "point-mass" space vehicle reentering the atmosphere. We assume that the motion is two-dimensional, and the earth is spherical (radius R_1) and non-rotating. The required notation is defined as follows:

θ = angle of latitude

γ = flight path angle

α = angle of attack

S = reference area of vehicle

L = lift force on the vehicle

D = drag force on the vehicle

$\underline{V}$ = velocity vector, $\underline{V} = V$

$\underline{R}$ = radius vector from the center of the earth to the vehicle

h = altitude of the vehicle

g_o = gravitational acceleration at

$\underline{R} = R = R_1$

ρ = density of the air

$C_L(\ )$, $C_D(\ )$ = lift and drag coefficients,

m = mass of the vehicle.

From Newton's second law, the state of the system is given by

$$\dot{\underline{Y}} = \underline{g}\,(\underline{Y},\underline{U}),$$

where the state vector is defined as $\underline{Y}_T = [h, V, \gamma]$ and the control, $\underline{u}$, is the scalar quantity α. The corresponding components of $\underline{g}$ are $g_1 = V \sin\gamma$, $g2 = -g \sin\gamma - \frac{D}{m}$, and $g_3 = \frac{V}{R} \cos\gamma - \frac{3}{V} \cos\gamma + \frac{L}{mV}$, where $L = C_L(\alpha)\, \frac{\rho(h)V^2 S}{2}$, $R = R_1 + h$, and $g = g_o \left[\frac{R_1}{R_1 = h}\right]^2$.

If we now assume that $\underline{\eta}$ is a nominal solution of $\dot{\underline{Y}} = \underline{g}\,(\underline{Y}, \alpha)$, then the first-order perturbation equation is given by

$$\begin{bmatrix} \delta\dot{h} \\ \delta\dot{V} \\ \delta\dot{\gamma} \end{bmatrix} = \begin{bmatrix} \frac{\partial g_1}{\partial h} & \frac{\partial g_1}{\partial V} & \frac{\partial g_1}{\partial \gamma} \\ \frac{\partial g_2}{\partial h} & \frac{\partial g_2}{\partial V} & \frac{\partial g_2}{\partial \gamma} \\ \frac{\partial g_3}{\partial h} & \frac{\partial g_3}{\partial V} & \frac{\partial g_3}{\partial \gamma} \end{bmatrix}_{@\underline{\eta}} \begin{bmatrix} \delta h \\ \delta V \\ \delta \gamma \end{bmatrix} + \begin{bmatrix} \frac{\partial g_1}{\partial \alpha} \\ \frac{\partial g_2}{\partial \alpha} \\ \frac{\partial g_3}{\partial \alpha} \end{bmatrix}_{\underline{\eta}}$$

where δh, δV, $\delta\gamma$ and $\delta\alpha$ are the perturbations with reference to the nominal trojectory. The matrix coeffients of these variations are evaluated along the nominal trajectory and these are known functions of time. These, the first-order perturbation equation is a linear, nonautonomous system for δh, δV, $\delta\gamma$ along with a specified control $\delta\alpha$.

Now, let us consider some of the necessary local properties of the right hand side of (9), $\underline{f}\,(t,\underline{X})$, and some definitions of terms needed in the following discussion. Any solution $\underline{X}\,(t)$ of the system in (9) is called a trajectory in the n-dimensional Euclidean space. Apoint solution or-singular solution of (9) can be expressed as $\underline{X} = \underline{C}$, constant, and satisfies $\underline{f}\,(t,\underline{C}) = \underline{0}$ for all t in some interval. In the remainder of this section we will consider various norms of the vector $\underline{X}$, such as,

$$\|\underline{x}\|_1 = \sum_{i=1}^{n} |x_i|$$

$$\|\underline{x}\|_2 = \sup\,(|x_1|, \ldots, |x_n|),$$

$$\|\underline{x}\|_3 = \sqrt{x_1^2 + \cdots + x_n^2}$$

Since in our work these norms are equivalent and the form used is usually dictated by proofs of the theorems in which they appear, we will denote any of the above norms of $\underline{x}$ by $\|\underline{x}\|$. The particular norm wich is used will be dictated by the convenience of its use in a particular problem.

As in the above example, we canconsider the linear approximation of (9) about $\underline{X} = \underline{0}$ if $\underline{f}$ satisfies $\dot{\underline{x}} = \underline{f}(t,\underline{x}) = A(t)\,\underline{x} + \underline{g}(t,\underline{x})$, (10)
where $\|\underline{g}\|/\|\underline{x}\| \longrightarrow 0$ as $\|\underline{x}\| \longrightarrow 0$.
Of course, this linear approximation $\dot{\underline{X}} = \underline{A}(t)\underline{X}$ for (9) is only valid in the neighborhood of $\underline{X} = \underline{0}$.

We say that $\underline{f}$ is a Lipschitz function in $\underline{X}$ in a region R* of the $(t,\underline{x})$ space if

$$\| \underline{f}(t,\underline{X}) - \underline{f}(t,\underline{y})\| \leq M\| \underline{x} - \underline{y}\| \qquad (11)$$

for all $\underline{x}$ and $\underline{y}$, and t in R*. The constant M is called the Lipschitz constant. Some theorems dealing with Lispschitz function are the following:

Theorem 1

(H) (i) If $\underline{F}(t,\underline{x})$ is continuous in $(t,\underline{x})$ - space,

(ii) $\underline{f}(t,\underline{x})$ is linear in $\underline{x}$,

(c) then $\underline{f}(t,\underline{x})$ is a Lipschitz function for all t in some interval.

Theorem 2

(H) (i) If region S is defined in two-dimensional space by $0 \leq |x-x_0| \leq a$ and $0 \leq |y-y_0| \leq b$ or by $0 \leq |x-x_0| \leq a$ and $0 \leq |y| < \infty$

(ii) f (x,y) is real-valued on S,

(iii) $\partial f/\partial y$ exists and is continous on S,

(iv) $\left|\frac{\partial f}{\partial y}\right| \leq K$ for all (x,y) in S,

(C) then f(x,y) is a Lipschitz function in S and K is a Lipschitz constant.

Examples

(1) Consider $f(x,y) = xy^2$ in the region S:

thus, we have $\left|\frac{\partial f}{\partial y}\right| = |2xy| \leq 2 = K$ in S

and $|f(x,y_1) - f(x,y_2)| = |x|\,|y_1^2 - y_2^2|$

$= |x|\,|y_1+y_2|\,|y_1-y_2| \leq 2|y_1-y_2|$

for all x in S.

(2) Consider $f(x,y) = y^{2/3}$ in the region S: $|x| \leq 1$ and $|y| \leq 1$

then we have $|f(x,y_1) - f(x,0)| = |y_1^{2/3}| = \frac{|y_1 - 0|}{|y_1^{1/3}|}$

which has an unbounded "Lipschitz constant" as $y_1 \to 0$. Thus, $y^{2/3}$ is not a Lipschitz function in the neighborhood of (0,0).

(3) Consider $f(x,y) = x^2 |y|$ in the region S: $|x| \leq 1$ and $|y| \leq 1$

Even though $\frac{\partial f}{\partial y}$ does not exist at y = 0, f is a Lipschitz function because

$$|f(x,y_1) - f(x,y_2)| = |x|^2 \, \big| |y_1| - |y_2| \big| \leq |y_1 - y_2|$$

for all x in S.

(4) Consider the second order, time-varying system defined by

$$\underline{f}\,(t,\underline{x}) = \begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 3t \\ 0 \end{bmatrix}$$

where S: $|t| < \infty$ and $\|\underline{x}\| < \infty$. Thus, we have that $\underline{f}$ is a Lipschitz function because

$$\|\underline{f}\,(t,\underline{x}) - \underline{f}\,(t,\underline{y})\| = \| 2(x_1-y_1),\ (x_1-y_1) - (x_2-y_2) \|$$

$$\leq 2|x_1-y_1| + |x_1-y_1| + |x_2-y_2|$$

$$\leq 3|x_1-y_1| + 3|x_2-y_2| = 3\|\underline{x} - \underline{y}\|$$

(C) Existence and Uniqueness Theorems /Reference (12)/

We now consider the initial value problem (autonomous and forward in time) defined by

$$\begin{aligned} \dot{\underline{X}} &= \underline{f}(\underline{x}), \\ \underline{X}(o) &= \underline{C}, \end{aligned} \qquad (12)$$

where $\underline{X}$ beongs to a region $R \equiv \{\underline{x} : \|\underline{x} - \underline{c}\| \leq a\}$ and $0 \leq t \leq b$, $(b>0)$.
The function $\underline{f}$ is continuous in R. Corresponding to the initial value problem (I.V.P.) in (12) is the integral equation given below:

$$\underline{X}(t) = \underline{C} + \int_0^t \underline{f}\,(\underline{x}(s))\,ds \ , \ 0 \leq t \leq b \qquad (13)$$

In seeking a unique solution to (12), the following nonlinear integral transformation is very important. For any $\underline{X}(t)$, continuous and defined on $0 \leq t \leq b$ with values in R, the transformation is

$$\underline{y} = T(\underline{x}) = \underline{c} + \int_0^t \underline{f}\,(x(s))\,ds \qquad (14)$$

Since $\underline{X}$ and $\underline{f}$ are continuous functions of their respective arguments, then $\underline{Y}$ is a continuous function of t. Also, for sufficiently small t, $\underline{Y}$ is in R since $\underline{X}$ is in R and T is a continuous transformation. Since R is compact and $\underline{F}$ is continuous, $\|\underline{f}(\underline{x})\| \leq K$ for any $\underline{X}$ in R. An invariant set S of the transformation T is defined as the set of all continuous functions $\underline{X}$ in R for $0 \leq t \leq b$ such that for any $\underline{X}$ in S, $T(\underline{x})$ is in S.

The existence and uniqueness of the solution of the I.V.P. is proved by the Method of successive approximations, making use of the fixed point theorems for contraction mappings. The sequence of transformations is defined by

$$
\begin{aligned}
\underline{X}_o &= \underline{C}, \\
\underline{X}_1 &= T(\underline{X}_o), \\
\underline{X}_2 &= T(\underline{X}_1), \\
&\cdots \\
\underline{X}_{k+1} &= T(\underline{X}_{k+1}),
\end{aligned}
$$

where the fixed point $\underline{X}$ is the unique solution of (12) and is defined by

$\lim_{n\to\infty} \underline{x}_{(n)} = \underline{x}$,provided T is a contraction mapping.

Theorem: /reference (12)/

(H) (i) If $\underline{X}_o = \underline{C}$ is in S

(ii) $\underline{X}_{k+1} = T(\underline{X}_k) \equiv \underline{C} + \int_0^t \underline{f}(\underline{X}_k(s))ds$, k= 0,1,2,...)

(iii) $\underline{f}(\underline{x})$ is a Lipschitz function in R for $0 \leq t \leq b$

(C) then, (1). $X_o, X_1, X_2, \ldots, X_k, \ldots$ converges to a solution of the I.V.P.:

(2). this solution is unique;

(3). this solution is a continuous function of the initial vector $\underline{C}$; and

(4). if $\underline{f} = \underline{f}(\underline{x},k)$, where K is a system parameter, then the unique solution of the corresponding I.V.P. is a continuous function of K.

Examples

Let us now consider examples illustrating the various aspects of the above theorem.

(1) Consider the I.V.P. defined by $|\dot{x}| + |x| = 0$, $X(o) = 1$. Since the only solution of the equation is $X(t) \equiv 0$, then no solution of the I.V.P. exists.

(2) For the I.V.P. described by $\dot{X} = x^{2/3}$, $X(o) = 0$, there exists two solutions; they are $X = 0$ and $X = \frac{t^3}{27}$. We note that $X^{2/3}$ is not a Lipschitz function in the neighborhood of $t = 0$.

(3) In the system defined by

$$\dot{X} = f(t,x), \quad X(0) = 0,$$

where

$$f(t,x) = \begin{cases} \dfrac{4t^3x}{t^4 + X^2}, & (t,x) \neq (0,0) \\ 0, & (t,x) = (0,0), \end{cases}$$

we obtain an infinite number of solutions,

$$X = C^2 - \sqrt{t4 + c4}, \quad c \neq 0.$$

Again, $f(t,x)$ does not satisfy Lipschitz's condition in the neighborhood of $t = 0$.

(4). The above theorem verifies the following two statements:

a. if $\dot{x} = \log(1-x^2)$, then the corresponding I.V.P. will have a unique solution for all (t,x) in the strip $-1 < x < +1$

b. if $\dot{x} = \begin{cases} t + \sin X, & x \geq 0 \\ t + x^2, & x < 0 \end{cases}$,

then the corresponding I.V.P. will have a unique solution for all (t,x) in the whole plane.

(5) Consider the linear, first order equation $\dot{x} = g(t)X + h(t)$, where g and h are continuous in some interval I. The solution of the corresponding I.V.P. is

$$\phi(t) = e^{Q(t)} \int_{t_0}^{t} e^{-Q(s)} h(s)\,ds + \phi(t_0)\,e^{Q(t)}$$

where $t_0 \in I$ and $Q(t) = \int_{t_0}^{t} g(t)\,dt$.

The interesting result is that $\phi(t)$ is unique and exists for all t in I. This is not generally true of nonlinear systems. For example, consider the system $\dot{X} = X^2$ and $X(o) = +1$. The $\underline{F}(t,\underline{x})$ in this case is analytic for all t, but the solution is $x = -1/t-1$. This solution blows up at $t = 1$. The conclusion is that any general existence theorem for (12) can only assert the existence of a solution in some interval near the initial value of t.

(6) As an example of the application of successive approximations, consider:

$$\dot{X}_1 = X_2 \quad , \quad X_1(o) = 0$$

$$\dot{X}_2 = -X_1 \quad , \quad X_2(o) = 1.$$

The integral transformation is

$$T(\underline{x}) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \int_0^t \begin{bmatrix} x_2(s) \\ -x_1(s) \end{bmatrix} ds \, ,$$

or

$$\underline{x}_{k+1} \equiv \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \int_0^t x_k(s)\, ds \, , \qquad k = 0, 1, 2, \ldots\ldots$$

Thus

$$\underline{x}_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} , \qquad \underline{x}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \int_0^t \begin{bmatrix} 1 \\ 0 \end{bmatrix} ds = \begin{bmatrix} t \\ 1 \end{bmatrix}$$

$$\underline{x}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \int_0^t \begin{bmatrix} 1 \\ -s \end{bmatrix} ds = \begin{bmatrix} t \\ 1 - t^2/2 \end{bmatrix}$$

$$\underline{x}_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \int_0^t \begin{bmatrix} 1 - s^2/2 \\ -s \end{bmatrix} ds = \begin{bmatrix} t - t^3/6 \\ 1 - t^2/2 \end{bmatrix}$$

and so on.

It can be shown that $\underline{x}_K \to \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}$

(7) Finally, we give an example of the fourth conclusion of the above theorem. Consider the satellite equation $\ddot{X} + X = K x^2$, where K is a system parameter. To analyse this problem let $X_1 = X$, $X_2 = \dot{X}$ and $X_3 = K$. Thus, the state-variable form of the problem is given by

$$\dot{\underline{x}} = \begin{bmatrix} x_2 \\ x_1^2 x_3 - x_1 \\ 0 \end{bmatrix} \equiv \underline{f}(\underline{x}) \ , \ \underline{x}(0) = \begin{bmatrix} c_1 \\ c_2 \\ K \end{bmatrix}$$

Consider a second system defined by

$$\dot{\underline{y}} \quad \begin{bmatrix} y_2 \\ y_1^2 y_3 - y_1 \\ 0 \end{bmatrix} \equiv \underline{f}(\underline{y}) \ , \ \underline{y}(0) = \begin{bmatrix} c_1 \\ c_2 \\ K_1 \end{bmatrix}$$

From the equation (14) we have

$$\| \underline{x} - \underline{y} \| \leq |K - K_1| + m \int_0^t \| \underline{x}(s) - \underline{y}(s) \| \, ds \ ,$$

where

$$\| \underline{f}(\underline{x}) - \underline{f}(\underline{y}) \| = |x_2 - y_2| + |x_1^2 x_3 - x_1 - y_1^2 y_3 + y_1|$$

$$= |x_2 - y_2| + |x_1^2 x_3 - x_1^2 y_3 + y_3(x_1^2 - y_1^2) + y_1 - x_1|$$

$$\leq |x_1 - y_1| \left\{ 1 + |y_3| \, |x_1 + y_1| \right\} + |x_2 - y_2| + |x_1|^2 \, |x_3 - y_3|$$

$$\leq m \| \underline{x} - \underline{y} \|$$

and where M is the max $\left\{ |x_1|^2 , 1 + |y_3| \, |x_1 + y_1| \right\}$ in some bounded region in the state space. From Gronwall's Lemma, reference (12), we have

$$\| \underline{x} - \underline{y} \| \leq |K - K_1| e^{mt}$$

or

$$|x - y| + |\dot{x} - \dot{y}| \leq |K - K_1| \left\{ e^{mt} - 1 \right\}$$

Thus, if t is restricted to some interval, $\underline{Y} \longrightarrow \underline{X}$ as $K_1 \longrightarrow K$, thus proving the continuity of the solution with respect to K.

The next theorem relaxes the Lipschitz's condition on $\underline{F}$ (x) in (12); thus we get an existence theorem but not the property of uniqueness.

<u>Cauchy - Peano Existence Theorem: reference (12)</u>.

(H) (i) if $\underline{f}$ ($\underline{x}$), is continuous in the neighborhood of $\underline{x}(o) = \underline{c}$,

(C) then there exists a solution of the I.V.P., defined by equation (12).

<u>Example</u>

Consider the equation $\dot{X} = |x|^{\alpha}$, $0<\alpha<1$, $X(o) = 0$. This I.V.P. has an infinite number of solutions. The function defined by $|x|$ is continuous at X = 0 but does not satisfy a Lipschitz condition.

<u>Theorem (Generalization of the above Example)</u>

(H) (i) If $\dot{X} = f(x)$, $X(o) = 0$,

(ii) f(x) is continuous if $X \neq 0$,

(iii) $f(x) > 0$ if $X \neq 0$,

(iv) $f(o) = 0$,

(v) $\dfrac{dx}{f(x)}$ has integrable singularities at X = 0

(C) then the I.V.P. has infinitely many solutions.

We now consider two theorems for nonautonomous systems. The second theorem is more general than the first.

<u>Theorem: reference (20)</u>

(H) (i) If $\dot{\underline{X}} = \underline{f}\,(t,\underline{x})$, $\underline{X}(to) = \underline{X}o$, is an n-dimensional system,

(ii) R is some region of the n + 1 -dimensional Euclidean space of $\underline{X}$ and t,

(iii) $\underline{f}$ has continous first partial derivatives in R,

(C) then (1) there exists a unique solution

$\underline{X}(t)$ of $\dot{\underline{X}} = \underline{f}(t,\underline{x})$ such that $\underline{X}(to) = \underline{X}o$;

(2) $\underline{X}$ can be extended throughout R;

(3) $\underline{X}$ is a continous function of $\underline{X}o$ and to in R.

Theorem: reference (12).

(H) (i) If $\underline{f}$ is continous in $\underline{X}$ and t for

$\| \underline{x} - \underline{x}_o \| \leq a$ and $o < t \leq b$,

(ii) $\| \underline{f} \| \leq K(t)$ for $\| \underline{x} - \underline{x}_o \| \leq a$ and $o < t \leq b$,

(iii) $\| \underline{f}(t,\underline{x}) - \underline{f}(t,\underline{y}) \| \leq m(t) \| \underline{x} - \underline{y} \|$

for $\| \underline{x} - \underline{x}_o \| \leq a$, $o < t \leq b$,

(iv) each K(t) and m(t) are integrable on $o < t \leq b$

(v) for any b_1, such that $o < b_1 \leq b$, there exists a and δ such that $\int_o^{b_1} K(t)\,dt \leq a$ and $\int_o^{b_1} m(t)\,dt = \delta < 1$,

(vi) $\underline{X}^{(o)}(t)$ is continous and $\| \underline{X}^{(o)} - \underline{X}_o \| \leq a$ for $o < t \leq b_1$

(vii) $\underline{X}^{(K+1)} = \underline{X}_o + \int_o^t \underline{f}(s, \underline{X}^k(s))\,ds$, $K = 0, 1, 2, \ldots$,

(C) than the sequence $\underline{X}^{(o)}, \underline{X}^{(1)}, \ldots, \underline{X}^{(k)}, \ldots$ converges uniformly to a unique solution. $\underline{X}$ of the I.V.P. on the interval $0 \leq t \leq b_1$

Examples

(1) the system defined by $\dot{X} = (1/\sqrt{t})\, X^2$, $X(o) = 1$,

satisfies all the hypotheses of the theorem.

(2) the system defined by $\dot{X} = (2/t^3)\, X$, $X(o) = 0$, has an infinite number of solutions; hypotheses (iv) and (v) are violated.

(3) the system defined by $\dot{X} = \sqrt{|t|}\,\sqrt{|x|}$, $X(o) = 0$, has an infinite number of solutions; hypotheses (iii) is violated.

(4) this next example has a unique solution but it does not follow from the above theorem since hypothesis (iii) is violated. The system is $\dot{X} = t\, x^{-2}$, $X(o) = 0$, whose unique solution is $X = \sqrt[3]{3/2}\; t^{2/3}$.

(D) Types of Trajectories and Their Continuity Properties

Let us consider the system defined by

$$\dot{\underline{X}} = \underline{f}\ (t,\underline{x},\underline{\alpha}), \qquad (15)$$

where $\underline{X}$ is the n-dimensional state vector dnd $\underline{\alpha}$ is an m-dimensional state vector whose components are the parameters of the system. First, we list several theorems dealing with the continuity of the solutions of the I.V.P. corresponding to (15).

Theorem: reference (12)

(H) (i) If $\dot{\underline{X}} = \underline{f}\ (\underline{x})$ and $\underline{X}\ (o) = \underline{C}$,

(ii) $\underline{F}\ (\underline{x})$ satisfies a Lipschitz condition,

(iii) the system passesses Kth order continous derivatives,

(C) then the solution $\underline{X}(t)$ of the I.V.P. possesses a (K + 1)-st order continuous derivative with respect to t.

Theorem: reference (12)

(H) (i) If $\underline{f}$ in (15) is analytic in $\underline{X}$, $\underline{\alpha}$ and t,

(C) then $\underline{X}$ is analytic in $\underline{C}$, $\underline{\alpha}$ and t.

Theorem: reference (12)

(H) (i) If $\underline{f}$ in (15) is analytic in $\underline{X}$, $\underline{\alpha}$ and continuous in t,

(C) then $\underline{X}$ is analytic in $\underline{C}$ and $\underline{\alpha}$ and is continuously differentiable in t.

Theorem: reference (12)

(H) (i) If $\underline{f}$ possesses Kth order continuous derivatives in $\underline{\alpha}$, $\underline{X}$, and t,

(C) then, (1) the solution $\underline{X}$ of the I.V.P. possesses Kth order continuous derivatives in $\underline{\alpha}$, $\underline{C}$, and t;

(2) $\underline{\dot{X}}$ possesses K-th order continuous derivatives in $\underline{\alpha}$, $\underline{C}$, and t;

Two im portant notes about the above theorems are: one, the theorems are the basis of present day perturbation theory in mechanics; two, the theorems are local and thus t is restricted to a finite interval. Therefore, the theorems are not directly applicable to perturbation problems concerning periodic solutions.

We are now turn to a short discussion concerning the possible trajectories of the I.V.P. corresponding to the system $\underline{\dot{X}} = \underline{f}\ (t,\underline{x})$. The existence theorems discussed in this report were local theorems. But by a stepwise application of the theorems, the unique solution of a problem can be uniquely extended in the backward or forward direction to some finite value of t, or to $\pm\infty$.

We will dismiss the case of the trajectory being discontinuous after finite time by giving two examples. The remainder of the report will then be concerned with infinite time extensions of the trajectories.

Examples

(1) Consider the system $\dot{X} = 1/1-x$, $X(o) = 0$. The unique solution is $X = 1 - \sqrt{1-2t}$, which is well-defined for $t \in [o,\frac{1}{2}]$, but undefined for the "reals" beyond $t = 1/2$. The solution is bounded throughout the interval

$$0 \leq t \leq 1/2$$

(2) Consider, again, the system $\dot{X} = X^2$, $X(o) = 1$. The unique solution, $X = 1/1-t$, is well-defined for $t \in [0,1)$. But X is unbounded over the interval $0 \leq t < 1$. We say that this system has a finite-escape time (at $t = 1/2$).

Let us now consider only systems having unique solutions to the I.V.P. and which possess unique extension in both the backward (t to) and forward ($t > to$) directions. Let Γ be this uniquely extended trajectory. As $t \rightarrow +\infty$ or $-\infty$, the trajectory, Γ, will exhibit one of the following properties or combinations of these properties:

(1) Γ will approach a singular point;

(2) Γ will approach a closed trajectory which contains no singular points (periodic solution of the system);

(3) Γ will not approach any particular set of points;

(4) Γ will approach a <u>separatrix</u>; that is, a closed curve made up of singular points and connecting paths.

We know give several examples to illustrate above limiting conditions for Γ.

(1) the system defined by $\ddot{X} = -g$, or $\dot{X}_1 = X_2$ and $\dot{X}_2 = -g$ has parabolic trajectories which do not approach any finite set of points as $t \rightarrow \infty$ or $t \rightarrow -\infty$.

(2) the system defined by $\ddot{X} + 3\dot{X} \pm 2X = 0$, or $\dot{X}_1 = X_2$ and $\dot{X}_2 = -3X_1 - 2X_2$ has trajectories which approach (0,0) as $t \rightarrow +\infty$

(3) In the following system the trajectories approach the closed curve $X_1^2 + X_2^2 = 1$ as $t \rightarrow -\infty$. As $t \rightarrow +\infty$, the trajectories starting inside $X_1^2 + X_2^2 = 1$ appraoch (0,0) and the trajectories outisde $X_1^2 + X_2^2 = 1$ approach infinity. The system is

$$\dot{X}_1 = X_2 + X_1 \left(X_1^2 + X_2^2 - 1\right),$$

$$\dot{X}_2 = -X_1 + X_2 \left(X_1^2 + X_2^2 - 1\right),$$

whose solution in polar coordinates is given by

$$r = \left[1 - A e^{2t} \right]^{-1/2}$$

$$\theta = -t + \theta_0$$

(4) The system defined by

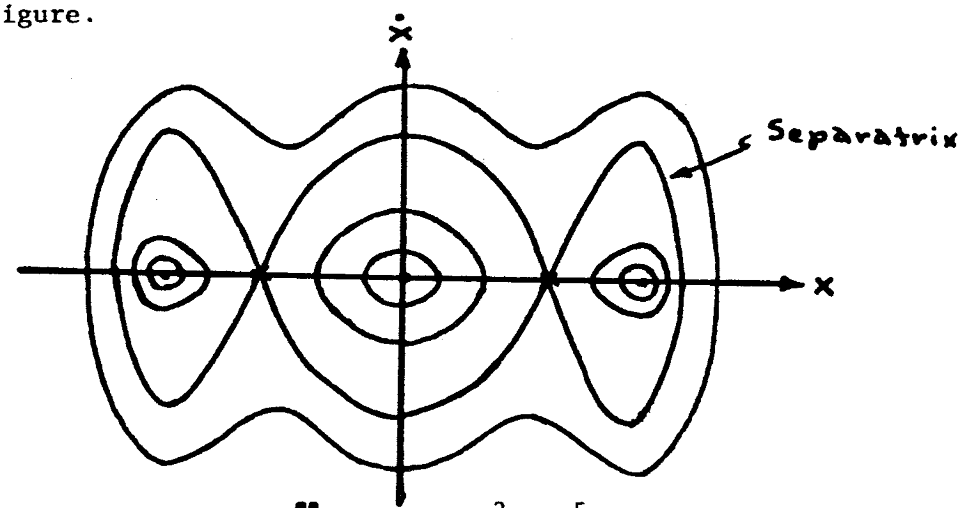
$$\ddot{X} + f(\underline{x}) = 0,$$

where

$$f(x) = 3x - 4x^3 + X^5 = x(x-1)\ (x+1)\ (x+3)\ (x-3).$$

has singular points at $(0,0)$, $(\pm 3, 0)$, $(\pm 1,0)$.

It can be shown that $(0,0)$, $(\pm 3, 0)$ are stable centers and $(\pm 1,0)$ are unstable saddle points. In the $\dot{x}x$- phase plane, the equation of the separtrix is given by $X^6 - 6X^4 + 9X^2 + 3\dot{X}^2 = 4$. All of the trajectories in the phase plane are closed curves (except the singular points) as shown below in the figure.



Phase-Plane Plot of $\ddot{X} + 3X = 4X^3 + X^5 = 0$.

(IV) Definitions and Theorems of Stability Theory

The Second Method of Liopunov is currently the best known method of analyzing the stability of dynamic systems whose laws of motion are described by ordinary linear and nonlinear differential equations. This part of the report presents a brief review of the baisc theorems and definitions of the Second Method, along with illustrative examples. We will keep the discussion about these theorems and definitions to a minimum, letting the "examples" do most of the talking. For completion, we will make a few remarks concerning Liopunov's First Method, and the concept of boundedness.

(A) Liopunov's First Method : reference (22)

Liopunov's First Method consists of the construction of the general solution of

$$\dot{\underline{X}} = \underline{f}\ (t,\underline{x}) \qquad (1)$$

in the form of a series and then to investigate the problem of stability of the homogeneous solution of (1) directly from its outer form. Assume that (1) can be written as

$$\dot{X}_s = \sum_{i=1}^{n} P_{si}(t) X_i + \sum_{m_1+\dots+m_n>1} P_s^{(m_1,\dots,m_n)}(t)\, X_1^{m_1} \dots X_n^{m_n} \qquad (2)$$

where X_s is the s-component of $\underline{X}$. The P's in (2) are real continuous bounded functions for $t \geqslant 0$, or they are piecewise continuous. The series, $\sum\limits_{m_1+\dots+m_n>1}$,converges for all $t \geqslant 0$ and $|X_s| \leq \delta \leq$ constant. The linear approximation of (2) is given by

$$\dot{\underline{X}} = \underline{P}\ (t)\ \underline{X}, \qquad (3)$$

and λ_i are the characteristic numbers of the linear system.

The assumed form of the solutions for (1) are given by:

$$X_s = \sum_{m_1+\dots+m_n \geqslant 1} L_s^{(m_1,\dots,m_n)}(t) \exp\left[-\sum_{i=1}^{n} m_i \lambda_i t\right] C_1^{m_1} C_2^{m_2} \dots C_n^{m_n} \qquad (4)$$

where the L's are continuous functions for $t \geqslant 0$ and

$$L_s^{(m_1,\dots,m_n)} e^{-\alpha t} \to 0 \quad \text{as} \quad t \to \infty$$

for any $\alpha > 0$. The $C_i^{m_i}$'s are constants. The results derived from the solutions in (4) are as follows:

(1) If the characteristic numbers λ_i of the linear system in (3) are all positive, then the series in X_s converges for all $t \geqslant 0$ and $|C_s| < \gamma$

γ a positive constant. Thus from the outer form of these series (4), we have that the homogeneous solution of (1) is locally asymptotically stable (the definition is given in the following discussion.).

(2) If one of the characteristic numbers is negative, then the nonlinear solution is unstable.

The disadvantage of the First Method is that the characteristic numbers of (3) must be determined; this is a difficult job if $\underline{P}$ is time-varying. Also, the First Method only gives information locally. Thus, Liopunov developed the Second Method.

(B) Definitions of Stability and Boundedness

We consider the following nonlinear, nonautonomous systems:

$$\dot{\underline{X}} = \underline{f}\,(t,\underline{x}). \qquad (5)$$

The conditions placed on $\underline{f}$ are given in the following statements, reference (23).

(1) $\underline{f}$ has values in R^n, the n-dimensional Euclidean Space.

(2) $\underline{f}$ is defined and continuous on some set $I \times S = \left\{(t,\underline{x}) \in R \times R^n \mid t \geq T \geq 0, \|\underline{x}\| < r\right\}$

(3) $\underline{f}$ is sufficiently smooth on I x S such that given any (to, $\underline{Xo}$) there exists for all $t \geq t_0$ a unique solution in S. This solution is denoted by $\underline{X} = \underline{F}$ (t;to,$\underline{Xo}$) and depends continuously upon (to, $\underline{Xo}$) and equals $\underline{X}$o at to.

(4) $\underline{f}\,(t,\underline{o}) = \underline{0}$ on I. Thus, $\underline{X} = \underline{0}$ is a null solution of (5). We assume that $\underline{X} = \underline{0}$ is an isolated equilibrium solution.

Note: In the above discussion S is the set of all points in R^n satisfying $\|\underline{x}\| < r$, and I is the set of all values of t satisfying $t \geq T \geq 0$, T being fixed.

The first nine definitions of the various types of stability are taken from reference (23). These definitions are local properties of the system, in (5), in neighborhood of $\underline{X} = \underline{0}$.

Definition 1, (23)

The solution $\underline{X} = \underline{0}$ is stable if for any $\epsilon > 0$ and any $t_0 \in I$ there exists a $\delta = \delta(\epsilon, t_0) > 0$ such that $\| \underline{X}_0 \| < \delta$ implies $\| \underline{F}(t; t_0, \underline{X}_0) \| < \epsilon$ for $t \geq t_0$.

Definition 2, (23)

The solution $\underline{X} = \underline{0}$ is uniformly stable if for any $\epsilon > 0$, there exists a $\delta(\epsilon) > 0$ such that $t_0 \in I$ and $\| \underline{X}_0 \| < \delta$ imply $\| E(t, t_0, \underline{X}_0) \| < \epsilon$ for $t \geq t_0$.

Definition 3, (23)

The solution $\underline{X} = \underline{0}$ is quasi-asymptotically stable if for any $t_0 \in I$ there exists a $\delta(t_0) > 0$ such that $\| \underline{X}_0 \| < \delta$ implies

$$\underline{F}(t, t_0, \underline{X}_0) \longrightarrow \underline{0} \quad \text{as} \quad t \rightarrow \infty$$

Definition 4, (23)

The solution $\underline{X} = \underline{0}$ is asymptotically stable if it is both stable and quasi-asymptotically stable.

Definition 5, (23)

The solution $\underline{X} = \underline{0}$ is quasi-equiasymptotically stable if for any $t_0 \in I$ there exists a $\delta(t_0) > 0$ such that $\| \underline{X}_0 \| < \delta$ implies $\underline{F}(t, t_0, \underline{X}_0) \longrightarrow \underline{0}$ as $t \longrightarrow \infty$ uniformly on $\| \underline{X}_0 \| < \delta$.

Definition 6, (23)

The solution $\underline{X} = \underline{0}$ is equiasymptotically stable if it is both stable and quasi-equiasymptotically stable.

Definition 7, (23)

The solution $\underline{X} = \underline{0}$ is quasi-uniform-asymptotically stable if there exists a $\delta_0 > 0$ such that $t_0 \in I$ and $\|x_0\| < \delta_0$ imply $\underline{F}(t, t_0, \underline{x}_0) \to 0$ as $t \to \infty$ uniformly for $t_0 \in I$,and, on $\|x_0\| < \delta_0$.

Definition 8, (23)

The solution $\underline{x} = \underline{0}$ is uniform-asymptotically stable if it is both uniformly stable and quasi-uniform-asymptoticallly stable.

Definition 9, (23)

The solution $\underline{X} = \underline{0}$ is exponential-asymptotically stable if there exists a $\lambda > 0$, and for any $\epsilon > 0$, there exists a $\delta(\epsilon) > 0$ such that $t_0 \in I$ + $\|x_0\| < \delta$ imply $\|\underline{F}(t; t_0, \underline{x}_0)\| < \epsilon \exp[-\lambda(t - t_0)]$ for all $t \geq t_0$

From reference (23) we obtain the following relationships between the above definitions ;

(1) Definition 9 implies all the other definitions.

(2) Definition 7 implies definition 5, and it in turn implies definition 3.

(3) Definition 6 implies definition 4 and 4 implies definition 1.

(4) Definition 2 implies definition 1.

(5) If $\underline{f}$ (t,$\underline{x}$) is Lipschitzion on some set I x H, H being a subset of S, with a time-varying "Lipschitzion Constant" $K(t) \geq 0$, and if K(t) is defined and piecewise continuous on I, then definiton 5 implies definition 6.

(6) If $\underline{f}$ (t,$\underline{x}$) is Lipschitzion on the set I x H, with a constant $(k > 0)$ "Lipschitian Constant", then definition 7 implies definition 8.

(7) If $\underline{f}$ (t,$\underline{x}$) is independent of t or periodic in t on I x S, then definition 1 implies definition 2 and definition 4 implies definition 8.

(8) If $\underline{f}$ (t,$\underline{x}$) is linear in $\underline{X}$ on I x S, then definition 4 implies definition 6 and definition 8 implies definition 9.

(9) If $\underline{f}$ $(t,\underline{x})$ is a scalar function, then definition 4 implies definition 6.

In the next few definitions (these are also from reference (23)) we assume that $\underline{f}$ $(t,\underline{x})$ in (5) is defined over I x R^n, where R^n is the entire n-dimensional Euclidean space.

Definition 10, (23)

Every solution of $\dot{\underline{X}} = \underline{f}\ (t,\underline{x})$ is bounded if for any $t_o \in I$ and any $r_o > o$, there exists an $r(t_o, r_o) > o$ such that $\| \underline{X}_o \| < r_o$ implies $\| F(t; t_o, \underline{X}_o) \| < r$ for $t \geqslant t_o$.

Definition 11, (23)

Every solution of $\dot{\underline{X}} = \underline{f}\ (t,\underline{x})$ is uniformly bounded if for any $r_o > o$ there exists an $r(r_o) > o$ such that $t_o \in I$ and $\| \underline{X}_o \| < r_o$ imply $\| \underline{F}(t; t_o, \underline{X}_o) \| < r$ for $t \geqslant t_o$.

Definition 12, (23)

The solutions of $\dot{\underline{x}} = \underline{f}\ (t,\underline{x})$ are ultimately bounded if for any r_o and r_1 where $r_o > r_1 > o$, there exists an $r(r_1) > o$ and a $\tau(r_o, r_1) > o$ such that $t_o \in I$ and $\| \underline{X}_o \| < r_o$ imply $\| \underline{F}(t; t_o, \underline{X}_o) \| < r$ for' $t \geqslant t_o + \tau$.

Definition 13, (23)

The solution $\underline{X} = \underline{0}$ of (5) is asymptotically stable in the large if it is stable and if $(t_o, \underline{X}_o) \in I \times R^n$ implies $\underline{F}(t; t_o, \underline{X}_o) \longrightarrow o$ as $t \longrightarrow \infty$.

Definition 14, (23)

The solution $\underline{X} = \underline{0}$ of (5) is uniform-asymptotically stable in the large if every solution is uniformly bounded and if for any positive r_o and r_1, there exists a $\tau(r_o, r_1) > o$ such that $t_o \in I$ and $\| \underline{X}_o \| < r_o$ imply $\| \underline{F}(t; t_o, \underline{X}_o) \| < r_1$ for all $t \geqslant t_o + \tau$.

A relationship exists between definitions (13) and (14) if $\underline{f}$ satisfies certain conditions. "If $\underline{f}$ $(t,\underline{x})$ is indepent of t or periodic in t on I x R^n, then definition 13 implies definition 14."

To aid in the understanding of the above definitions, we will consider several examples illustrating these concepts. From reference (18) we obtained several elementary examples from physics. They are:

(1) a solid homogeneous sphere resting on a horizontal flat table is <u>stable</u>.

(2) A small solid ball resting in a large rough spherical cup is <u>asymptotically stable</u>.

(3) If the size of the cup in example two is constant for all time, than the ball is <u>uniform-asymptotically stable.</u>

(4) If the ball is in a rough cup whose inside surface is defined by $z = x^2 + y^2$, then the ball is <u>asymptotically stable in the large</u>, or <u>globally-asymptotically stable.</u>

(5) If the ball is in a rough hemispherical cup of radius 1/t, then the ball is <u>asymptoticallystable</u> but <u>not uniformly in to.</u>

We now consider examples of equations which illustrate the above definitions.

<u>Example 1</u>

From reference (25) , we have the system defined by $\dot{X} = -(1/1+t)\ x$, whose solution is $x = x_o \left(\frac{to + 1}{t + 1} \right)$. If $t_0 \geq 0$, then $X = 0$ is <u>equiasymptotically stable</u>.

Example 2

From reference (26), we consider the system which exhibits on impûlse response with growing peaks. The system is defined by $\dot{X} = (4\ t \sin t - 2t)x$. The solution is given by

$$x = \exp\{4\sin t - 4t\cos t - t^2 - 4\sin t_0 + 4t_0\cos t_0 + t_0^2\}$$
$$< \exp\{4(2 + |t| + |t_0|) - t^2 + t_0^2\}$$

thus, $\| x \| \longrightarrow 0$ as $t \longrightarrow \infty$,uniformly in to and $\| x_0 \| \leq r$. But the peaks of the impluse resonse increase indefinitely as $t_0 \longrightarrow \infty$, as seen from the following $x = \exp[\pi(4-\pi)(4n+1)]$ when $t_0 = 2\pi n$ and $t_0 = (2n+1)\pi$ Therefore, the motion is not uniformly stalbe or uniformly bounded, but relies greatly on the value of to.

Example 3

From reference (26), we consider the second order system defined by

$$\dot{r} = \frac{\dot{g}(t,\theta)}{g(t,\theta)}\ r,$$

$$\theta = 0,$$

where : $0 \leq r < \infty$, $0 \leq \theta \leq 2\pi$, and

$$g(t,\theta) = \frac{\sin^2\theta}{\sin^4\theta + (1 - t\sin^2\theta)^2} + \frac{1}{1+t^2}$$

The solution of this system is given by

$$r \equiv r(t;t_0,r_0,\ \theta_0) = \frac{g(t,\theta)}{g(t_0,\theta_0)}\ r_0$$

$$\theta \equiv \theta(t;t_0,r_0,\theta_0) = \theta_0$$

The motion (r, θ) is continuous with respect to (ro,θo) and the null solution is quasi-asymptotically stable. But the system is not stable and not quasi-equiasymptotically stable because at $t_1 = (\sin\theta_0)^{-2}$ we have that $g(t,\theta_0) > (\sin\theta_0)^{-2}$ and as $\theta_0 \longrightarrow \pm\pi$, $t_1 \longrightarrow \infty$.

Example 4

From reference (27), we consider the system $\dot{X} = 1 - X^2$. The singular solution $X = -1$ is unstalbe as $t \rightarrow +\infty$ since the general solution, as defined by

$$X = \tanh (t - to + K),$$

$$K = \tanh^{-1} xo, \quad -1 < Xo < 1,$$

approaches + 1 as $t \longrightarrow +\infty$. For the same reason, the singular solution $X = +1$ is asymptotically stable for all Xo in (-1,+1). The idea of an unstable solution in this example is that the singular solution is not stable in the sense of Definition 1. The stability in Definition 1 is Liopunov stability. A formal definition of an unstable solution in the sense of Liopunov is given below:

Definition 15, (19)

The solution $\underline{x} = \underline{0}$ of (15) is unstable if there exists a number $\epsilon > 0$ with the following property: there exists a sequence of numbers t_1, t_2, ,,, ...,t_n, ... and a null sequence of initial points $\underline{X}_1, \underline{X}_2, \ldots, \underline{X}_n, \ldots \rightarrow \underline{0}$ sucht that $\| \underline{F}(t_0 + t_n ; t_0, \underline{X}_n) \| \geq \epsilon$, $n = 1, 2, \ldots$.

A special case of instability occurs when every motion tends away from the equilibrium. This case is defined by:

Definition 16, (19)

The solution $\underline{X} = \underline{0}$ of (15) is completely unstable if there exists a number $\epsilon > 0$ with the following property: after finite time, t_1, each motion $\underline{F}$ (t, to, $\underline{X}$o) reaches the sphere $\| \underline{X} \| = \epsilon$,where $0 < \| \underline{X}_0 \| < \epsilon$ and $t_1 \geq t_0$. (Actually, in the previous example X = -1 was completely unstable.)

The next example shows that boundedness an d stability are different concepts.

Example 5

From reference (27), we first consider the system defined by

$$\ddot{x} = -\frac{1}{2}\left[x^2 + (x^4 + 4\dot{x}^2)^{1/2}\right]x.$$

Every solution ofthis system can be expressed $X = C \sin (Ct + d)$, where C and D are constants. It is obvious that $X = 0$ is stable as $t \to \pm\infty$. But any nonzero solution is stable neither as $t \to +\infty$ or as $t \to -\infty$. This can be proved as follows: consider the two solutions $X_1 = C_1 \sin (c_2 t + d_2)$ and $X_2 = C_2 \sin (c_2 t + d_2)$, where $C_1 \neq C_2$, C_2/C_1 is an irrational number, but C_1 is sufficiently close to C_2. Then it can be shown that the upper limit of $/X_1 - X_2/$ satisfies

$\overline{\lim}\ |x_1 - x_2| = |C_1| + |C_2|$ as $t \to \pm\infty$. Therefore, only the null solution is stable while all solutions are bounded as $t \to \pm\infty$.

The converse situationcan be shown by the folloing system, reference (27):

$$\dot{X} = 1,$$

where any solution is given by $X = C + t$, C a constant. All solutions are unbounded as $T \to \pm\infty$; but every solution is stable as $t \to \pm\infty$ since if $X_1 = C + \delta C + t$ and $X_2 = C + t$, then $/X_1 - X_2/ = \delta c$ for all t.

Example 6

Weconsider several linear equations from reference (27) whose solutions exhibit vatious properties of boundedness. All the solutions of $\ddot{X} + X = 0$ are bounded. All the nontrivial solutions of $\ddot{X} - 2/t\,\dot{X} + X = 0$ are unbounded since the general solution has the form

$$X = C_1\left[\sin t - \cos t\right] + C_2\left[\cos t + t \sin t\right]$$

In the interval $1 \leq t < \infty$, all the nontrivial solutions of $\ddot{X} + 2/t\dot{X} + X = 0$ are bounded and asymptotically stable because any solution can be written as

$$X = C_1\,\frac{\sin t}{t} + C_2\,\frac{\cos t}{t}.$$

Example 7

Consider the system, reference (27), defined by

$$\dot{r} = \frac{\dot{h}}{h} r,$$

$$\dot{\phi} = 0,$$

where $h = (1 + t^3 \sin^2\phi)(1 + t + t^4 \sin^4\phi)^{-1}$;

and where $r_0 \geq 0$ and ϕ_0 are the initial conditions the unique solution is given by

$$r(t) = r_0 h(t, \phi_0), \quad t \geq t_0 = 0.$$

For any $r_0 \geq 0$ and ϕ_0 we have that $r(t) \to 0$ as $t \to \infty$. But for $0 < \phi < \frac{\pi}{2}$, $r_0 > 0$, and $t_1 = (\sin\phi)^{-3/2}$ we have

$$r(t_1) = \frac{r_0 [1 + (\sin\phi)^{-5/2}]}{[1 + (\sin\phi)^{-3/2} + (\sin\phi)^{-2}]}$$

where $r(t_1) \to \infty$ as $\phi \to 0$. Thus, $(r,\phi) \equiv (0,0)$ is <u>not stable</u> in the sense of Liapunov.

Example 8

Consider the following system from reference (12),

$$\dot{x} + 2x - e^t x^2 = 0, \quad x_0 = x(0)$$

the solution is given by

$$x = \left\{ \frac{x_0}{x_0(1-e^t) + 1} \right\} e^{-t}.$$

thus, if $X_0 \leq 1$, then the system is <u>exponentially stable</u>; if $X_0 > 1$, then there exists a <u>finite escape time</u>, or $X \to \infty$ as $t \to \log\left(\frac{x_0}{x_0 - 1}\right)$ from below.

The next several definitions are concerned with some practical unstable motions (that is, unstable in the sense of Liapunov) such as orbital stability, stability in a neighborhood of an equilibrium point which is itself unstable in the strict Liapunov sense, and stability with respect to a certain subset of the components of $\underline{X}$. We also mention the practical type of stability where bounded imputs produce bounded outputs from a physical system. The stability of the differential equation is considered, that is to say we consider the Structural Stability of the system with respect to the parameters and physical constants in the system. Amore general definition of this type of stability required that the system be stable withrespect to persistent disturbances.

Orbital Stability

In reference (18), Krasovskii gives the following intuitive picture of orbital stability. We say that the null solution $\underline{X} = \underline{o}$ of a system is orbitally stable if some function of the dependent variables changes only by a small amount as $t \rightarrow \infty$ when $\underline{X}o \equiv \underline{X}o(t)$ is restricted to be in a sufficiently small neighborhood of $\underline{X} = \underline{0}$. Weshould note that this is not in general Liapunov stability since some $\underline{f}(\underline{x})$ may change only a small amount $\underline{X}$ itself might change a great deal as $t \rightarrow \infty$.

An example of this type of stability is a planet constrained to move under the universe square law of a central force field. A slight change in the position or velocity of the planet may perturb it to another orbit with a different period. Thus, two planets with their initial positions and velocities nearby the same may eventually be very far apart while their angular momenta, ecentricities, and certain other parameters that describe the orbits remain close together for all time. (In the previous paragraph $\underline{X}$ represents the velocities and positions and $\underline{f}(\underline{x})$ represents momenta, eccentricities, etc.)

In the taking of pictures of a fixed region of the earth's surface by an artificial satellite, the concept of orbital stability is more important than Liopunov stability. For rendezvous problems in space, Liapunov stability is required. In a meteor shower the orbits of the particles are close to one another due to common source, but the particles are not stable in the sense of Liapunov.

Definition 17, reference (18)

Let there be (n - 1) or fewer independent continuous functions $f_k(\underline{x})$ of the arguments X_i, where $K = 1,\ldots, n-1$ and $f_k(\underline{0}) = 0$. Then $\underline{X} = \underline{0}$ is called orbitally stable with respect to the orbit functions f_k provided that for every $\epsilon > 0$ there exists a $\delta = \delta(\epsilon) > 0$ such that $|f_k| < \epsilon$ for all K, $t \geq t_o$, and all $\underline{X}o$ satisfying $\| \underline{X}_o \| < \delta$.

In reference (28), Hochstadt considers the nth order system, $\dot{\underline{X}} = \underline{f}(t,\underline{x})$. Let C* be aclosed orbit of this system in the state space. Thus, C* is a trajectory of $\dot{\underline{X}} = \underline{f}(t,\underline{x})$.

Definition 18, reference (28)

The distance between a fixed point $\underline{X}$ and the closed orbit C* is defined by

$$d\,(\underline{x},C^*) = \min_{\underline{Y} \text{ on } C^*} \| \underline{X} - \underline{Y} \| .$$

Definition 19, reference (28)

The orbit C* is orbitally stable if for every $\epsilon > 0$, there exists a $\delta(\epsilon) > 0$ such that for $\underline{X}o$ satisfying $d(\underline{X}o,C^*) < \delta$, then $d(\underline{X},C^*) < \epsilon$ for all $t > t_o$.

Definition 20, reference (28)

The orbit C* is asymptotically orbitally stable if $\lim_{t \to \infty} d\,(\underline{X},C^*) = 0$. and if C* is orbitally stable.

Example 9

Consider the system defined by

$$\dot{X} = r^2 y,$$

$$\dot{Y} = - r^2 X,$$

$$r^2 = X^2 + y^2.$$

Rewriting these equations gives us the following:

$$\dot{r} = 0,$$

where $r = c$ constant are the integral curves. On a given orbit, $r = c$, the parametric equations for x and y are

$$X = \sin c^2 t \;, \quad y = \cos c^2 t.$$

Thus, the periods for themotion are $2\pi/c^2$, which vary from orbit to orbit. In conclusion, we say that the system (a particular orbit) is orbitally stable but not asymptotically orbitally stable.

Example 10

Consider the system defined by

$$\dot{X}_1 = X_2 + x_1 \; (1 - r^2),$$

$$\dot{X}_2 = - X_1 + X_2 \; (1 - r^2),$$

where $r^2 = X1^2 + X2^2$ and $\theta = \tan^{-1} X2/x_1$. In terms of r and θ, our system is

$$\dot{r} = r \; (1 - r^2),$$

$$\dot{\theta} = - 1.$$

The equilibrium solutions are defined by $r = 0$ and 1, $r = 0$ being a singular point and $r = 1$ being a closed curve or orbit. The general solution is $r = [1 + c^2 e^{-2t}]^{-1/2}$. Thus, for any $c \neq 0$, $r \rightarrow 1$ as $t \rightarrow \infty$; therefore, $r = 1$ is asymptotically orbitally stable and $r = 0$ is unstable.

Example 11, reference (28)

We now consider a second order system containing a discontinuous function:

$$\ddot{X} + \mathrm{Sgn} x = 0,$$

where $\quad Sgnx = 1, x \quad 0$

$$= 0, x = 0$$

$$= -1, x \quad 0.$$

The corresponding first order system is

$$\dot{X}_1 = X_2 ,$$

$$\dot{X}_2 = -sgnx_1.$$

For the initial conditions $x_1 = 0$ and $X_2 = A > 0$, we have the following solution, which is periodic with period 4A:

$$X_1 = At - \tfrac{1}{2} t^2, \quad 0 \leq t \leq 2A,$$

$$= (t - 2A)\ (t/2 - 2A),\ 2A \leq t \leq 4A,$$

$$X_2 = A - t, \quad 0 \leq t \leq 2A,$$

$$= t - 3A, \quad 2A \leq t \leq 4A.$$

the orbit in the phase plane is defined by $|x_1| = 1/2\ (A^2 - X_2^2)$. Thus the solution is a periodic solution whose period depends on the initial conditions, A. A small change in the initial condition given by A produces only a small change in the orbit, but it produces a large change in the "particle's position" after a certain length of time. The system is orbitally stable but not stable in the sense of Liopunov.

Practical Stability, references (29), (30), (31).

There are many physical systems where the desired state of the system is not stable but yet the system always tends to return sufficiently close to the desired stateso that the performance of the system is acceptable and thus possesses a practical stability.

Definition 21, reference (30)

If all solutions $\underline{X}$ (t; $\underline{X}$o) of $\dot{\underline{X}} = \underline{f}\ (\underline{x})$ appraoch a neighborhood N of $\underline{X} = \underline{0}$, N being a measure of the satisfactory performance of the system, as $t \to \infty$, then the system $\dot{\underline{X}} = \underline{f}(\underline{x})$ has a practical stability.

The notion of a practical stability is closely related to Yosbizowa's ultimate boundedness, Definition (12).

Definition 22

If all the disturbed motions ($\underline{X}(t,to,\underline{X}o) \not\equiv 0$). of $\dot{\underline{X}} = \underline{f}(t,\underline{x})$ are bounded, the system is Lagrange Stable.

Definition 23

If all the solutions in definition 21 start in the complement N^c of N and tend to N as $t \to \infty$, then the system is called asymptotically stable in the sense of Lagrange.

The important of Definitions 12, 21, 22, 23 are in systems which possess a dead-zone andwhich may ultimately be bounded but never asymptotically stable in the sense of Liapunov.

Example 12, reference (29)

Consider the system $\dot{X} = E^2X - X^3$, where $E > 0$ is a small constant and the equilibrium points are at $X = 0, \pm E$. The point $X = 0$ is unstable and $\pm E$ are stable; that is, a moving point displaced from the origin will remain in $(-E, +E)$. A moving point to the right of $+E$ approaches $+E$ as $t \to \infty$, similarly for points to the left of $-E$. Since E is small, the origin is stable for practical purposes.

Example 13, reference (29)

Consider the system $\dot{X} = -E^2X + X^3$, $E > 0$. In this case $X = 0$ is stable and $\pm E$ are unstable. Thus a mvoing point in $(-E, E)$ moves to 0 as $t \to \infty$; and ponits in (E, ∞) or $(-\infty, -E)$ move to infinity as $t \to \infty$. Thus, $X = 0$ is unstable for practical purposes.

Example 14, reference (22) First Order Time-Varying System

Consider the system defined by $\dot{Y} = g(t)\, y$, whose general solution is

$$y = \exp\left[\int_{t_0}^{t} g(\tau)\, d\tau\right] y_0$$

(1) This system is Stable if and only if $\int_{t_0}^{t} g(\tau)\,d\tau$ is bounded from above for all $t \geq t_o$. But stability may still be "bad" from a practical point of view. Let

$$g(t) = \ln 10 \quad \text{for } 0 \leq t \leq 10,$$
$$= 0 \quad \text{for } t > 10.$$

Thus,
$$y(t) = 10^t y_o \quad , t \in (0,10),$$
$$= 10^{10}\, y_o, \quad t \in (10,\infty).$$

The system is stable in the matematical sense; but if @ $t = 0$, $y = y_o = 10^{-5}$, when $t > 10$ the value of y is 10^5!

(2) This system is asymptotically stable if $\int_{t_0}^{t} g(\tau)\,d\tau \longrightarrow -\infty$ as $t \longrightarrow +\infty$. But asymptotic stability may still give trouble in a practical way. Suppose

$$g(t) = \ln 10 \quad \text{for } t \in (0,10),$$
$$= -1 \quad \text{for } t \in (10,\infty).$$

Thus,
$$y = 10^t y_o \quad \text{for } t \in (0,10),$$
$$= 10^{10} \exp(10 - t)\, Y_o \text{ for } t \in (10,\infty).$$

If $Y_o = 10^{-5}$ at $t = 0$, then at $t = 10$, $Y = 10^5$ and at $t = 20$, $y \approx 3$, even though as $t \rightarrow \infty$, $y \rightarrow 0$.

(3) This system is unstable if and only if $\int_{t_0}^{t} g(\tau)\,d\tau$ is not bounded from above as $t \longrightarrow +\infty$. But instability as $t \longrightarrow +\infty$, need not be "bad" practically speaking.

Suppose
$$g(t) = -1 \text{ for } t \in [0,100],$$
$$= 10 \text{ for } t \in (100,\infty).$$

Thus
$$y(t) = e^{-t} y_o \quad \text{for } t \in [0,100]$$
$$= Y_o \exp\left[-100 + 10\,(t - 100)\right] \text{ for } t \in (100,\infty).$$

Therefore, if the time of action is limited in this system, the instability as $t \longrightarrow +\infty$ can be neglected in a finite time interval.

In reference (31), the author compares Liapunov stability with bounded input- bounded output concepts, the latter concepts being very closely related to Lagrange stability. Suppose that the input vector of a system is given by the m-dimensional vector $\underline{u}$ and the output vector by the q -dimensional vector $\underline{z}$.

Definaition 23, reference (31)

A physical system is stable if every bounded input, $\|\underline{u}\| <$ constant, produces a bounded output, $\|\underline{z}\| <$ constant.

The concept of stability in given in Definition 23 is a gross or global phenomena of a system, whereas Liopunov stability is a specific or local phenomena about a particular system response or motion. A system may be unstalbe in the Liopunov sense, but stable according to definition 23. And a particular motion may be asymptotically stable in-the-large and the system will not obey Definition 23. For linear systems the two concepts are equivalent, but not for nonlinear systems.

Example 15, reference (31)

Consider the system defined by $\dot{X} = \tanh X + U$; X is the output. The equilibrium of the unforced system, U = 0, is totally stable in the large. But if we consider the bounded input $U = (1 + E) \tanh X$, $E > 0$, the output is defined by $X = E \tanh X$. This leads to an unbounded output. Thus, the system does not preserve bounded outputs for all bounded inputs.

Structural Stability

Structural stability, not the topological dynamics variety, is the insensitivity to disturbances in the parameter space of the aystem. For example, consider the second order system

$$\dot{X} = P\ (x,y),$$

$$\dot{Y} = Q\ (x,y).$$

Suppose that P and Q are approximated by polynomials. Therefore the coefficients of the terms are somewhat in error. If the system is structually stable, the qualitative picture of the exact system and the polynomial approximation must be the same. Structual stability need not only be with respect to parameter variations, but may also be with respect to changes in the matematical model, functional dependence, etc. Thus, a more general definition requires that the system be stable with respect to persistent disturbances or constantly acting perturbations.

Let the system defined by

$$\dot{\underline{X}} = \underline{f}(t,\underline{x}) + \underline{g}\ (t,\underline{x}), \qquad (6)$$

be a real, physical system upon which certain small perturbation forces act, described by $\underline{g}\ (t,\underline{x})$. It must be realized that often these forces are not accurately known; thus, $\underline{g}\ (t,\underline{x})$ represents an estimate for the true perturbations. For this reason we can not assume that $\underline{g}\ (t,\underline{0}) = \underline{0}$; but we do assume that both $\underline{f}$ and $\underline{g}$ in (6) satisfy the conditions of $\underline{f}$ in equation (5). Also, we assume that the equilibrium solution of the unperturbed system is $\underline{X} = \underline{0}$, that is, $\underline{f}\ (t,0) = \underline{0}$. for all $t \geq t0$.

Definition 24, reference (19)

The equilibrium solution $\underline{X} = \underline{0}$ of $\dot{\underline{X}} = \underline{f}(t,\underline{x})$ is called stable under constantly acting perturbations if for every $E > 0$, there exists two constants $\delta_1(\epsilon)$ and $\delta_2(\epsilon)$, such that for every solution $\underline{F}\ (t;to,\underline{X}o)$ of (6), the inequality

$$\|\underline{F}\ (t;to,\underline{X}o)\| < E \qquad (t > to)$$

holds, provided that

$$\|\underline{Xo}\| < \delta_1$$

and $$\|\underline{g}\ (t,\underline{X})\| < \delta_2$$

in the domain $\|\underline{X}\| < E$ and $t \geq t_o$.

If the magnitude of the disturbing terms $\underline{g}\,(t,\underline{X})$ are measured in a different way, that is if $\|\underline{g}\,(t,\underline{X})\| < \delta_2$ is replaced by $\int_0^\infty \sup_{\underline{X}} \|\underline{g}\,(t,\underline{X})\|\,dt < \delta_2$ one obtains the <u>definition of integral stability</u>, reference (35). In the case of integral stability the perturbations may be large in a small interval, whereas in the stability under constantly acting perturbations they have to be small, but they may be persistent. The properties of both these types are possessed by stability under persistent perturbations in the mean value, for short, call this <u>stability in the mean</u>, reference (35). This type of stability is obtained if $\underline{g}\,(t,\underline{X}) < \delta_2$ is replaced by $\int_t^{t+T} \sup_{\underline{X}} \|\underline{g}\,(t,\underline{X})\|\,dt < \delta_2(T)$ where both δ_1 and δ_2 depend on T.

<u>Definition 25, reference (35)</u>

The solution $\underline{X} = \underline{0}$ is <u>asymptotically stable under constantly acting perturbations,</u> if it is stable under constantly acting perturbations and if to any sufficiently small numbers $\delta > 0,\ \eta > 0$ (i.e., $\delta \leq K,\ \eta \leq K$) there exists numbers $T(\delta,\eta) > 0,\ \gamma(\delta,\eta) > 0$ such that for every solution E (t;to;$\underline{X}$0) of (6) $\|\underline{E}\,(t;t_o,\underline{X}_o)\| < \eta$ for $t \geq t_o + T(\delta,\eta)$ whenever $\|\underline{X}_o\| < \delta,\ \ \|\underline{g}\,(t,\underline{X})\| < \gamma(\delta,\eta)$.

If one requires that the perturbations $\underline{g}\,(t,\underline{X})$ are smaller than γ in the sense of the inequalities corresponding to integral stability and stability in the mean, one obtains the asymptotic analogues of the respective above mentioned concepts of stability. The following list is a comparison between the different stabilities affecting equation (6):

(1) Asymptotic integral stability implies asymptotic stability in the mean, and conversely.

(2) Asymptotic integral stability implies asymptotic stability under persistent perturbations.

(3) Asymptotic stability under persistent perturbations implies stability under persistent perturbations.

(4) Asymptotic stability in the mean implies stability in the mean.

(5) Stability in the mean implies both integral stability and stability under persistent perturbations.

(6) In the autonomous case, all three asymptotic types of stability aee equivalent to each other. Furthermore, stability under persistent perturbations is equivalent to stability in the mean. Finally, each of the above types of stability implies integral stability.

Example 16, reference (44) Linear, Variable-Parameter System

Consider the system defined by

$$\dot{X}_1 = X_3,$$

$$\dot{X}_2 = - 3 X_1,$$

$$\dot{X}_3 = \alpha X_1 + 2X_2 - X_3.$$

The characteristic equation is

$$K^3 + K^2 - \alpha K + 6 = 0.$$

If $\alpha < -6$, the system is asymptotically stable as $t \to +\infty$. If $\alpha = -6$, the system is stable as $t \to +\infty$. If $\alpha > -6$, the system is unstable as $t \to +\infty$.

Example 17, reference (27)

Consider the system $\dot{X} = 0$, with the perturbation X^2. Thus, in the unperturbed system $X = 0$ is stable as $t \to \pm \infty$. In theperturbed system, $\dot{X} = X^2$, $X = 0$ is unstable as $t \to \pm \infty$.

Example 18, reference (27)

Consider the system $\dot{X} = X$, with the perturbation $- e^t X^3$. Thus, in the unperturbed system $X = 0$ is unstable as $t \to \infty$. Since the general solution is $X = C e^t$. In the perturbed system, $\dot{X} = X - e^t X^3$, $X = 0$ is asymptotically stable as $t \to +\infty$ since the solution is given by

$$X = C e^t \left[1 + \frac{2}{3} C^2 (e^{3t} - 1)\right]^{-1/2}$$

where $X \geqslant 0$ and $X \to 0$ as $t \to \infty$.

<u>Example 19, reference (27)</u>

Let the unperturbed system be defined by

$$\dot{\underline{X}} = \underline{A}(t)\,\underline{X} \qquad \text{(2nd order)},$$

where $a_{11} = -a$, $a_{12} = a_{21} = 0$, and

$$a_{22} = \sin(\log t) + \cos(\log t) - 2a, \; t > 0.$$

The general solution is

$$X_1 = C_1 \exp(-a t)$$

$$X_2 = C_2 \exp(t \sin(\log t) - 2at),$$

where $X_1 \to 0$, $X_2 \to 0$ as $t \to \infty$ for $a > 1/2$. Thus, the unperturbed system is stable (not uniform). Now, let the perturbed system be defined by

$$\dot{\underline{X}} = (\underline{A}(t) + \underline{B}(t))\,\underline{X},$$

where $b_{11} = b_{12} = b_{22} = 0$, $b_{21} = \exp(-at)$.

We will restrict "a" to the interval $1/2 < a < 1/2 + 1/4 \exp(-\pi)$. This perturbed system is <u>not stable</u>.

The general solution of the perturbed system is

$$X_1 = C_1 \exp(-at),$$

$$X_2 = \left[\exp(t \sin(\log t) - 2at)\right]\left[C_2 + C_1 \int_0^t \exp(-\alpha \sin(\log \alpha))\, d\alpha\right].$$

The upper limit of $|X_2|$ is infinite as $t \to \infty$, if $C_1 \neq 0$. We can prove this if we let $t = t_n \equiv \exp\left[(2n + 1/2)\pi\right]$, $n = 1, 2 \ldots$.

Hence we can show that the integral in X_2 is larger than

$$t_n \left[\exp\left(-\frac{2\pi}{3}\right) - \exp(-\pi)\right] \exp\left[\frac{t_n \exp(-\pi)}{2}\right].$$

Thus, we can finally derive the following inequality:

$$|x_2| > |C_1| \exp\left[1 - 2\lambda + \frac{\exp(-\pi)}{2}\right] t_n + O(1)$$

as $n \to \infty$

Conditionally Stable Systems

A conditionally stable system is a system which is in general unstable but one in which under certain initial conditions or relations, which limit the choice of disturbance, is stalbe. Suppose we consider the n=th order system in (5); and suppose we define $\underline{y}_o$ by $(\underline{y}_o)_T = (y_1, \ldots, y_m)$, where $y_1, \ldots, y_m$ are m (m$<$n) of the variables or components of $\underline{X}o$. We can now define a type of conditional stability; that is, a stability depended on a subset of the components of X_e

Definition 26, reference (19)

The equilibrium solution $\underline{X} = \underline{0}$ of $\dot{\underline{X}} = \underline{f}\ (t,\underline{x})$ is said to be stable with respect to a subset of the variables $X_1, \ldots . X_n$ if there exists for each $\epsilon > 0$ a number $\delta > 0$ such that the inequality

$$\| \underline{y}_o \| < \delta$$

implies $\| \underline{F}(t; \underline{y}_o, t_o) \| < \epsilon \qquad (t \geq t_o)$

From reference (27), Cesari defines a conditional stability with respect to a given manifold, M, of solutions of the system being investigated. The equilibrium solution of the system is only stable with respect to the manifold M.

Example 20, reference (27)

The system defined by $\ddot{X} - X = 0$ hasd an asymptotocally stable equilibrium solution, X = 0, as $t \to +\infty$ with respect to the manifold of solutions of the form $X = C\ e^{-t}$. The solution X = 0 is asymptotically stable as $t \to -\infty$ with respect to the manifold of solutions of the form $X = c\ e^{t}$.

The system $\dot{X} - |X| = 0$ has an equilibrium solution $X = 0$ which is asymptotically stable with respect to the manifold of solutions which are nonpositive.

<u>A few Other Types of Stability</u>

We will now list a few other types of stability which arise in the study of deterministic systems described by ordinary differential equations. If one was to considered stochostic systems and functional differential equations, the list of definitions would grow considerably. In passing, we make not of Ingverson's fine article, reference (24), in which he basically clothes the definitons found in Antosiewicz's article in the language of control theory.

<u>Definition 27, reference (27)</u>

The linear system $\dot{\underline{X}} = \underline{A}(t)\, \underline{X}$ is <u>restrictively stable</u> if the system itself is stable and if the corresponding adjoint system is stable. The adjoint system is defined by $\dot{\underline{Y}} = -\underline{A}^*(t)\, \underline{Y}$, where $\underline{A}^*(t)$ is the conjugate transpose of $\underline{A}$.

Restrictively stable systems are uniformly stable; the converse is not necessarily true.

<u>Definition 28, reference (20)</u>

If the null solution of $\dot{\underline{X}} = \underline{f}(\underline{x})$ is asymptotically stable over the whole state space (asymptotically stable in-the-large), then the system is called <u>completely stable</u>.

The next two stability definitions arose out of the work of G.D. Binkhoff, Consider the autonomous system defined by $\dot{\underline{X}} = \underline{f}(\underline{x})$, where $\underline{X}_1$ is is the equilibrium solution. That is, $\underline{f}(\underline{x}_1) = \underline{0}$. The following definitions deal with the stability of $\underline{X}_1$ of mth order.

Definiton 29, reference (27)

The equilibrium solution $\underline{X}_1$ of $\dot{\underline{X}} = \underline{f}(\underline{x})$ is parturbatively stable of order m if for any $\epsilon > 0$, there exists two positive constants K and L such that any solution $\underline{F}$ (t;to,$\underline{X}$o) satisfies $\| \underline{F} - \underline{X}_1 \| < K\epsilon$ when $\| \underline{X}_0 - \underline{X}_1 \| \leq \epsilon$ and for all t with $|t - to| \leq L\,\epsilon^{-m}$.

Defintion 30, reference (27)

The equilibrium solution $\underline{X}_1$ of $\underline{X} = \underline{F}(\underline{x})$ is trigonometrically stable of order m if for any solution $\underline{F}$ (t;to,$\underline{X}$o) in definition (29), for all fixed T and for all polynomials P($\underline{x}$) whose terms have degrees $\geq$ S, the function $P[\underline{X}(t);\ to,\ \underline{X}o]$ cand be represented in $[to - T,\ to + T]$ by a trigonometric series

$$A_0 + \sum (A_i \cos \lambda_i t + B_i \sin \lambda_i t)$$

where $|\lambda_i - \lambda_j| \geq \lambda > 0$, of not more than N + 1 terms with an error less than or equal to $Q\,\epsilon^{m+s}$. (Q,λ,N are constants and S = 1,2...)

Some notes about these last two defintions are given below:

(1) Stability of order m implies stability of orders 1,2,..., m - 1.

(2) Complete stability im;lies stability of all orders.

(3) Perturbative stability is similar to Liopunov stability.

(4) If $\underline{f}(\underline{x})$ is an analytic function of $\underline{X}$ in the neighborhood of the equilibrium solution $\underline{X}_1$, then ananalytic transformation $\underline{X}$ = ($\underline{Y}$) can be applied to $\dot{\underline{X}} = \underline{f}(\underline{x})$, the result being $\dot{\underline{Y}} = \underline{g}(\underline{y})$.

Both perturbative and trigonometric stabilities are invariant with respect to their analytic transformations.

(5) Neither Liopunov stability nor boundedness have invariant characteristics, in general, with respect to a change of coordinates, as the following examples will verify.

Example 21, reference (27)

A mechanical system may have a stable solution with respect to a given system of Lagrargian coordinates and be unstable with respect to another system. Consider the system.

$$\dot{x} = -y\sqrt{x^2+y^2}$$
$$\dot{y} = x\sqrt{x^2+y^2}$$

The solution $(x,y) \equiv (0,0)$ is stable, and all other solutions are unstable in the Liapunov sense since the general solution has the form:

$$X = C\cos(ct + d),$$

$$Y = C\sin(ct + d),$$

C, d constants.

Now introduce the new coordinate r and d by the equations

$$X = r\cos\theta,$$

$$Y = r\sin\theta,$$

$$\theta = rt + d.$$

The new system is defined by

$$\dot{r} = 0,$$

$$\dot{d} = 0.$$

The solutions $r = c$, $d = c_g$ are all stable in the Liapunov sense.

Example 22, reference (27)

Consider the equantion of the pendulum

$$\ddot{X} + \sin X = 0,$$

whose general solution is

$$X = C\sin\left[\phi(c)\, t + d\right],$$

where C and d are constants and (c) is a function of c given in terms of elliptic functions. In the Liapunov sense, X = 0 is stable and all other solutions are unstable. Transfrom the above system by introducing the equation

$$X = r \sin\left[\phi(r)\, t + d\right],$$
$$Y = r \cos\left[\phi(r)\, t + d\right],$$

The new system is $\dot{r} = 0$ and $\dot{d} = 0$, and all of the solutions are stable in the sense of Liopunov.

Let us now consider the dynamic process (could be a control process) defined by

/

$$\dot{x}_i = \sum_{j=1}^{n} a_{ij} x_j + b_i y \qquad i = 1, 2, \ldots, n$$

$$y = \phi(\sigma)$$

$$\sigma = \sum_{k=1}^{n} c_k x_k$$

In these equations, a_{ij}, b_i and c_k are real constants. The "characteristic" $\phi(\sigma)$ is an arbitrary, single-valued, piecewise, continuous, real function, defined for all real values of σ and satisfying the conditon $\phi(0) = 0$. The function ϕ also satisfies the conditon

$$0 \leq \frac{\phi(\sigma)}{\sigma} \leq K$$

where K is a positive constant or infinity. Thus, $\underline{X} = \underline{0}$ is a null solution of the above system.

Definition 31, reference (35)

We say that for a given K, the above system is absolutely stable if for any $\phi(\sigma)$ satisfying all of the above conditions, the zero solution $\underline{X} = \underline{0}$ is asymptotically stable in-the-large, or completely stable.

(C.) Definitions of Closedness, Definiteness, and Liapunov Functions

Consider the system defined by'

$$\dot{\underline{X}} = \underline{f}(t, \underline{x}) \qquad (7)$$
$$\underline{f}(t, \underline{0}) = \underline{0}, \quad t \geqslant t_0,$$

where $\underline{f}$ satisfies properties sufficient for the existence of unique solutions of (7). We denote the spherical neighborhood of $\underline{X} = \underline{0}$, $\|\underline{x}\| \leq h$ by R(h), and we denote the half-cylindrical neighborhood $\|\underline{x}\| \leq h,\ t \geq t_0 \geq 0$ by R(h,to). We will consider real scalar functions defined in R(h) and R(h,to) and will denote these fucntions by $V(t,\underline{x})$, $W(\underline{x})$, etc. Usually, we will assume these functions are continuous and possess continuous first partial derivatives with respect to all of their arguments.

Definition 32, reference (19)

$V(\underline{x})$ is positive (negative) semi-definite if $V(\underline{0}) = 0$ and if $V(\underline{x}) \geq 0 (\leq 0)$ in R(h).

Definition 33, reference (19)

$V(\underline{x})$ is positive (negative) definite if $V(\underline{0}) = 0$ and if $V(\underline{x}) > 0\ (< 0)$ in R(h) for $\underline{X} \neq \underline{0}$.

Dkfinition 34, reference (19)

$V(t,\underline{x})$ is positive (negative) semi-definite if $V(t,\underline{0}) = 0$ for $t \geq$ to and if for some suitable $h_1 \leq h$, $V(t,\underline{x}) \geq 0\ (\leq 0)$ in $R(h_1,$to$)$.

Definition 35, reference (19)

$V(t,\underline{x})$ is positive (negative) definite if $V(t,\underline{0}) = 0$ and $V(t,\underline{x}) \geq W(\underline{x})$ $(\leq -W(\underline{x}))$ in $R(h_1)$, where $h_1 < h$ and $W(\underline{x})$ is positive definite.

Definitdon 36, reference (19)

$V(t,\underline{x})$ is radially unbounded if for any $\alpha > 0$, there exists $\beta > 0$ such that $V(t,\underline{x}) > \alpha$ whenever $\|\underline{x}\| > \beta$ and $t >$ to. (that is, V becomes infinitely large with $\|\underline{x}\|$).

Definition 37, reference (19)

$V(t,\underline{x})$ is decrescent or admits an infinitely small upper bound in $R(h)$ at $\underline{X} = \underline{0}$ if the limit of $V(t,\underline{x})$ is zero as $\|\underline{x}\| \to 0$ uniformly in t. An equivalent definiton is that there exists a positive definite function $W(\underline{x})$ such that $|V(t,\underline{x})| \leq W(\underline{x})$ in R(h,to).

Definiton 38, reference (18)

The derivative of $V(t,\underline{x})$ along the trajectories of the system in (7) is the basic relationship in the Liapunov theory between the differential equation and the Liapunov function. The derivative is defined by

$$\dot{V}(t,\underline{x}) = \frac{\partial V(t,\underline{x})}{\partial t} + (\underline{\nabla V})_T \, \underline{f}(t,\underline{x})$$

where $\underline{\nabla V}$ is the gradient of $V(t,\underline{x})$ with respect to $\underline{X}$/

Defintion 39, reference (19)

If the total derivative of V along the trajectories of (7) does nto exist, then the following expression is defined to be $\dot{V}$,

$$\dot{V} = \lim_{\Delta t \to 0^+} \sup \left\{ \frac{V(t+\Delta t, \underline{x}(t+\Delta t)) - V(t, \underline{x}(t))}{\Delta t} \right\}$$

Definition 40, reference (23)

Antosiewics defines V in the following way when the total derivative does not exist. The generalized (upper right-hand)total derivative of $V(t,\underline{x})$ with respect to $\dot{\underline{X}} = \underline{f}(t,\underline{x})$ defined on R(h,to) is the function defined by

$$\dot{V}(t,\underline{x}) = \overline{\lim_{K \to 0^+}} \left\{ \frac{V(t+K, \underline{x} + K\underline{f}(t,\underline{x})) - V(t,\underline{x})}{K} \right\}$$

We now consider several theorems dealing withthe above definitons.

If $V(\underline{x})$ is positive definite, then $V(\underline{x}) = C$ (positive constant) represents a family of closed curves or surfaces about $\underline{X} = \underline{0}$. As $C \rightarrow 0$, $V = C$ contracts to the origin. The following "closedness" theorem is due to Letov.

Theorem 1

(H) (i) If $V(\underline{x})$ is positive definite in the entire space, and

(ii) $V(\underline{x}) \rightarrow \infty$ as $\|\underline{x}\| \rightarrow \infty$

(C) then $V(x) = C$ is closed as $c \rightarrow \infty$.

The next three theorems deal with the "decrescent" property of $V(\underline{x})$ and $V(t,\underline{x})$.

Theorem 2, reference (19)

(H) (i) If $V(\underline{0}) = 0$, and

(ii) $V(\underline{x})$ is continuous at $\underline{X} = \underline{0}$

(C) then $V(\underline{x})$ is decrescent at $\underline{X} = \underline{0}$.

Theorem 3, reference (19)

(H) (i) If $V(t,0) = 0$, and

(ii) V has bounded first partial derivatives in R(h,to) with respect to X_i

(C) then $V(t,\underline{x})$ is decrescent.

Theorem 4, reference (19)

(H) (i) If $V(t,\underline{x})$ has a power series expansion in X_i in R(h,to),

(ii) the series has no constant terms, and,

(iii) the time dependent coefficients are uniformly bounded in t,

(C) then $V(t,\underline{x})$ is decrescent.

The next three theorems deal with the positive definiteness of quadratic forms.

Theorem 5, reference (32) (Bhatia)

The quadratic form $\underline{X}_T \underline{D}(t) \underline{X}$ is positive definite if

$$|\underline{D}_k(t)| > 0 \quad , \quad k = 1, \ldots, n-1 \qquad \text{for } t \geq t_0$$

$$|D(t)| \geq \delta > 0 \qquad \text{for } t \geq t_0$$

where $|D_k(t)|$ are the principal minors of the determinant $|\underline{D}(t)|$ and δ is an arbitrayy constant.

Theorem 6, reference (32) (Zurmuhl)

The necessary and sufficient conditions that $\underline{X}_T \underline{D}(t) \underline{X}$ is positive semi-definite for $t \geq t_0$ is that

$$|\underline{D}_k(t)| \geq 0 \qquad (k = 1, \ldots, n-1) \qquad \text{for } t \geq t_0$$

$$|\underline{D}(t)| = 0 \qquad \text{for } t = t_0$$

Theorem 7, reference (32) (Malkin)

The sum of a positive (negative) definite quadratic form and a positive (negative) semi-definite quadratic form is a positive (negative) definite quadratic form.

From references (18), (19), (33), and (34), we get the following examples of definite and semi-definite functions.

Example 23 -----$X_1 X_2$ - plane

(1) $V = X_1^2$ is positive semi-definite.

(2) $V = X_1^2 - 2X_1X_1^2$ is indefinite.

(3) $V = X_1^2 - 2X_1X_2^2 + X_2^4$ is positive semi-definite.

(4) $V = X_1^2 - 2X_1X_2^2 + X_2^4 + X_1^4$ is positive definite.

(5) $V = X_1^2 - 2X_1X_2^2 + X_2^4 + X_1^4 + X_1X_2^5$ is indefinite.

(6) $V = X_1^2 + X_2^2 + X_1X_2^2 + X_2^3$ is positive definite

(7) $V = X_1^2 - X_2^2 + X_1X_2^2 + X_2^3$ is indefinite.

(8) $V = X_1^2 + X_2^2/1 + t$ is not positive definite for all t 0.

(9) $V = X_1^2 + X_2^2 + 1/2X_1X_2 \sin t$ is positive definite in (x_1,X_2)-space for all t.

(10) $V = X_1^2 + \dfrac{X_2^2}{1 + X_2^2}$ is positive definite in the X_1X_2-space but is only closed for V = C 1.

(11) $V = \int_0^{x_1} f(x_1)dx_1 + X_2^2$, where $f(x,)$ x, 0 for x, $\neq$ 0, and is only closed if $V = C < b$.

(12) $V = X_1^2 + X_2^2\, t$ is not decrescent because it violates uniformity.

(13) $V = X_1^2 + X_2^2 \sin t$ is decrescent, and indefinite.

Example 24, (LaSalle)

Consider the function defined by

$$V(\underline{x}) = x_n^2 + 2b(x_1,\dots,x_{n-1})x_n + a(x_1,\dots,x_n)$$

$$= x_n^2 + 2bx_n + b^2 + (a-b^2) = (x_n+b)^2 + (a-b^2)$$

where $(a - b^2)$ is a function of $X_1,\dots,X_{n-1}$. If $(a - b^2)$ is quadratic in X_{n-1}, then repeat the above "cpmpleting the sqaare" process, and so on for lower orders. As an example of this procedure , consider:

$$V = x_3^2 + 6x_1^2x_2x_3 + 2x_2^2 + 9x_1^4x_2^2 + 2x_1^3x_2 + x_1^6$$

$$= x_3^2 + 2(3x_1^2x_2)x_3 + (2x_2^2 + 9x_1^4x_2^2 + 2x_1^3x_2 + x_1^6)$$

$$= (x_3 + 3x_1^2x_2)^2 + 2(x_2^2 + 2[x_1^3/2]x_2 + \tfrac{1}{4}x_1^6) - \tfrac{1}{2}x_1^6 + x_1^6$$

$$= (x_3 + 3x_1^2x_2)^2 + 2(x_2 + \tfrac{1}{2}x_1^3)^2 + \tfrac{1}{2}x_1^6$$

which is positive definite.

Example 25, reference (34)

Consider the quartic form in two variables given by

$$V_4 = q_{40}x_1^4 + q_{41}x_1^3x_2 + q_{42}x_1^2x_2^2 + q_{43}x_1x_2^3 + q_{44}x_2^4$$

A necessary condition for positive definiteness is that $q_{40} > 0$ and $q_{44} > 0$. Thus, we can divide V_4 by $q_{40} X_2^4$ and get

$$V_4^* = X^4 + q_1 X^3 + q_2 X^2 + q_3 X + q_4$$

where $X = X_1/X_2$ and $q_4 = q_{44}/q_{40} > 0$. The positive definiteness of V_4^* and V_4 are equivalent. The following Newton Sums must be evaluated:

$$\begin{aligned}
S_0 &= 4 \\
S_1 &= -q_1 \\
S_2 &= q_1^2 - 2q_2 \\
S_3 &= -q_1^3 + 3q_1 q_2 - 3q_2 \\
S_4 &= q_1^4 - 4q_1^2 q_2 + 4q_1 q_3 + 2q_2^2 - 4q_4 \\
S_5 &= -q_1^5 + 5q_1^3 q_2 - 5q_1 q_2^2 - 5q_1^2 q_3 + 5q_1 q_4 + 5q_2 q_3 \\
S_6 &= q_1^6 - 6q_1^4 q_2 + 9q_1^2 q_2^2 + 6q_1^3 q_3 - 6q_1 q_4 - 12 q_1 q_2 q_3 - 2q_2^3 + \\
&\quad + 6q_2 q_4 + 3q_3^2
\end{aligned}$$

From these sums the matrix $\underline{S}_4$ is formed: $S_4 = \begin{bmatrix} S_0 & S_1 & S_2 & S_4 \\ S_1 & S_2 & S_3 & S_4 \\ S_2 & S_3 & S_4 & S_5 \\ S_3 & S_4 & S_5 & S_6 \end{bmatrix}$

The principal minors of $\underline{S}_4$ are denoted by D_1, D_2, D_3, D_4, D_1 being So and D_4 being $|\underline{S}_4|$. The signature of the $\underline{S}_4$ matrix is denoted by σ and defined as $\sigma = r - 2V$, where r is the rank of $\underline{S}_4$ and V is the number of sign variations in the sequence 1, $D_1, D_2, D_3, \ldots D_r$. Therefore, $V_4(\underline{x})$ is positive definite if $q_{40} > 0$, $q_{44} > 0$, and $\sigma = 0$. As on example, consider

$$V_4 = x_1^4 + x_1^3 x_2 + 2x_1^2 x_2^2 + x_1 x_2^3 + x_2^4$$

thus, $q_{40} = q_{44} = 1 > 0$ and

$S_0 = 4$, $S_1 = -1$, $S_2 = -3$, $S_3 = -2$, $S_4 = 1$,
$S_5 = -1$, $S_6 = 0$ therefore, $r = 4$, $D_1 = 4$,
$D_2 = -13$, $D_3 = 10$, $D_4 = 12$, and $\sigma = 4 - 2 \cdot 2 = 0$. $V_4(\underline{x})$ is positive definite.

(D). Stability Theorems & Boundedness Theorems

In this part of the report we will consider the system defined by

$$\dot{\underline{X}} = \underline{F}\ (t,\underline{x}) \qquad (7)$$

where $\underline{f}$ satisfies the same conditons as in equation (5), except we may want to replace set S, $\|\underline{x}\| < \rho$, by the entire space R^n on occasion. Let the function $V(t,\underline{x})$ be a real scalar function defined and locally Lipschitzian on some set $I_0 \times S_0 = \{ (t,\underline{x}) \in R \times R^n \mid t \geq To \geq 0, \|\underline{X}\| < r_0 \}$ and such that, given any $\underline{X}$ in S_0, V is continuous in I_0, and $V(t,\underline{0}) \equiv 0$. for all t in I_0. It is assumed that the intersection of S_0 and S contains a neighborhood N of $\underline{X} = \underline{0}$, say $N = \{ \underline{x} \in S_0 \cap S \mid \|\underline{x}\| < \rho \}$,where $\rho > 0$ is some fixed constant. Also, we assume that $I_0 \cap I = I = [T, \infty)$. To be consistent with the notation in Part C, let us denote T by to, and ρ by h, and I x N by the "half-cylinder" $R\ (h; t_0)$.

There are many ways in which one might define a Liapunov function but the definition we use here comes from reference (23).

Definition 41, reference (23)

$V(t,\underline{x})$ is a Liapunov function on R(h,to) for the equation $\dot{\underline{X}} = \underline{f}(t,\underline{x})$ if it is defined, locally Lipschitzian and positive definite on R(h,to); if, given any $\underline{X}$ in $\|\underline{x}\| < h$, V is continuous for all $t \geq$ to and $V(t,\underline{0}) = 0$. for all t; and if $\dot{V}(t,\underline{x}) \leq 0$ on R(h,to).

Definition 42

The vector function $\underline{f}(t,\underline{x})$ is said to be, on R(h,to), of class C^k (K positive integer) with respect to $\underline{X}$ if, on R(h,to), $\underline{f}$ is continuous and has continuous K-th order partial derivatives with respect to $\underline{X}$.

Theorem 8, reference (23) (Liapunov)

(H) (i) If there exists a Liapunov function $V(t,\underline{x})$ on $R(h,to)$,

(C) Then $\underline{X} = \underline{0}$ of (7) is stable.

Theorem 9, reference (23) (Converse Theorem of Persidskii)

(H) (i) If $\underline{f}(t,\underline{x})$ is of class C^k with respect to $\underline{X}$ on $R(h,to)$ and of class C^{k-1} with respect to t for any $t \geq to$ and any fixed $\underline{X}$, $\|\underline{X}\| < h$,

(ii) for any initial point in $R(h,to)$ the solution of (7) is continuable for all $t \geq to$,

(iii) $\underline{X} = \underline{0}$ of (7) is stable,

(C), then there exists in some half-cylindrical subset of $R(h,to)$ a Liapunov function $V(t,\underline{x})$ of class C^k in $\underline{X}$ and such that $\dot{V}$ is negative definite for any finite time interval in this subset.

In Theorem 9, if $\underline{f}(t,\underline{x})$ is linear in $\underline{X}$, then there exists a real quadratic form in $\underline{X}$ with time-varying coefficients of class C^1 which is a Liapunov function.

Theorem 10, reference (23) (Persidskii)

(H) (i) If there exists on $R(h,to)$ a Liapunov function,

(ii) $V(t,\underline{x})$ is decrescent,

(C) $\underline{X} = \underline{0}$ of (7) is uniformly stable.

Theorem 11, reference (23) (Converse of Kurzweil)

(H) (i) If $\underline{f}(t,\underline{x})$ is of class C^1 on $R(h,to)$ with respect to $\underline{X}$.

(ii) for any initial point in $R(h,to)$, the solution of (7) is continuable for all $t \geq to$,

(iii) $\underline{X} = \underline{0}$ of (7) is uniformly stable,

(C) there exists on some half-cylindrical subset of R(h,to) a Liapunov function V(t,$\underline{x}$) of class C^1 such that V is decrescent in this subset.

Theorem 12, reference (23) (Massera)

(H) (i) If $\underline{f}$(t,$\underline{x}$) is linear in X on R(h,to),

(ii) $\underline{f}$ (t,$\underline{x}$) is independent of t on R(h,to),

(iii) $\underline{X} = \underline{0}$ of (7) is stable and thus uniformly stable,

(C) then for any even positive integer m there exists a real algebraic form V($\underline{x}$) of degree m which is a Liapunov function on R(h,to).

The above theorems dealt with the local phenomena of stability and uniform stability. The next several theorems will be concerned with the study of asymptotic stability as a local phenomemon.

Theorem 13, reference (23) (Marachkov)

(H) (i) If $\underline{f}$(t,$\underline{x}$) is bounded on R(h,to),

(ii) there exists a Liapunov function V(t,$\underline{x}$) on R(h,to) such that $\dot{V}$ is negative definite on R(h,to);

(C) then $\underline{X} = \underline{0}$ is asymptotically stable.

Corollary, reference (23)

(H) (i) If $\underline{f}$ (t,$\underline{x}$) is independent of t or periodic in t on R(h,to),

(ii) there exists a Liapunov function V(t,$\underline{x}$) on R(h,to) such that V is negative definite on R(h,to).

(C) then $\underline{X} = \underline{0}$ of (7) is uniform-asymptotically stable.

Thereom 14, reference (23) (Converse of Massera)

(H) (i) If $\underline{f}(t,\underline{x})$ is linear in $\underline{X}$ on R(h,to), and for every $\underline{X}$ in $\underline{f}(t,\underline{x})$ is of class C^{k-1} with respect to t ($t \geqslant$ to), $K \geqslant 1$, ,

(ii) X = 0 of (7) is asymptotically stable (hence, equiasymptotically stable),

(C) then there exists a Liapunov function $V(t,\underline{x})$ defined and of class C^k On R(h,to) such that $\dot{V}(t,\underline{x})$ is negative definite on R(h,to).

Theorem 15, reference (23) (Massera)

(H) (i) If there exists on R(h,to) a real scalar function $V(t,\underline{x})$, locally Lipschitzian, and positive definite,

(ii) there exists a real scalar function W(s), defined, continuous and increasing for t 0, where W(0) = 0 and,

$\dot{V}(t,\underline{x}) \leq -W(V(t,\underline{x}))$ on R(h,to).

(C) then $\underline{X} = \underline{0}$ is asymptotically stable.

Theorem 16, reference (23) (Massera)

(H) (i) If there exists on R(h,to) a Liapunov function $V(t,\underline{x})$ such that $\dot{V}(t,\underline{x})$ is negative definite on R(h,to), and

(ii) $V(t,\underline{x})$ is such that for any $\xi > 0$, $0 < \xi < h$, and any $\sigma \geqslant t_0$, there are constants $\eta(\xi)$, $0 < \eta \leq \xi$, and $\tau(\sigma,\xi) \geqslant \sigma$ such that for any $S \in [to, \sigma]$ and any $\underline{Y}$ in $\|\underline{Y}\| < \eta$, the inequalities $t \geqslant \tau$ and $V(t,\underline{x}) \leq V(s,\underline{Y})$ imply $\|\underline{X}\| < \xi$,

(C) then $\underline{X} = \underline{0}$ of (7) is equiasypmtotically stable.

Theorem 17, reference (23) (Malkin and Massera)

(H) (i) If there exists on R(h,to) real scalar positive definite functions $U(t,\underline{x})$ and $V(t,\underline{x})$ such that Vis continuous on R(h,to) and decrescent, and $\dot{V} + U \to 0$ as $t \to \infty$ uniformly on $\rho_1 \leq \|\underline{x}\| \leq \rho_2$ for every ρ_1 and ρ_2 less than h,

(C) then $\underline{X} = \underline{0}$ of (7) is equiasymptotically stable.

Corollary, reference (23)

(H) (i) If $\underline{f}(t,\underline{x})$ is Lipschitzian for some constant $K > \theta$ on R(h,to),

(ii) hypothesis of Theorem 17 is satisfied,

(C) then $\underline{X} = \underline{0}$ of (7) is uniform-asymptotically stable.

Theorem 18, reference (23)

(H) (i) If $\underline{X} = \underline{0}$ of (7) is uniformly stable,

(ii) there exists on R(h,to) a Liapunov function $V(t,\underline{x})$ such that $\dot{V}$ is negative definite on R(h,t0),

(C) then $\underline{X} = \underline{0}$ is equiasymptotically stable.

Theorem 19, reference (23) (Liapunov and Persidskii)

(H) (i) If there exists on R(h,to) a Liapunov function $V(t,\underline{x})$ such that $V(t,\underline{x})$ is decrescent and $\dot{V}$ is negative definite on R(h,to),

(C) then $\underline{X} = \underline{0}$ is uniform-asymptotically stable.

The next theorem, theorem 20, deals with a "differential Inequality" property of Liapunov functions. (We dealt with this topic in one of the sections in the main part of our report.)

Theorem 20, reference (23)

(H) (i) If $V(t,\underline{x})$ is a Liapunov function on R(h,to) such that V is decrescent and $\dot{V}$ is negative definite,

(C) then given any constants ρ_1 and ρ_2 , $0 < \rho_2 < \rho_1 < h$,there exist constants $\lambda(\rho_1, \rho_2) > 0$, $\nu(\rho_2) > 0$ such that $W(t,\underline{x})$ $W(t,\underline{x}) = \exp(\lambda t)\, V(t,\underline{x})$ satisfies $W(t,\underline{x}) \leq -\nu$ for $t \geq to$, $\rho_2 \leq \|\underline{x}\| < \rho_1$

Theorem 21, reference (23) (Malkin's Partial Converse of Theorem 19)

(H) (i) If $\underline{f}(t,\underline{x})$ is of class C^k with respect to $\underline{X}$ on R(h,to) and for every $\underline{X}$ in $\|\underline{x}\| < h$, $\underline{f}(t,\underline{x})$ is of class C^{k-1} with respect to $t \geq$ to,

(ii) $\underline{X} = \underline{0}$ is uniform-asymptotically stable,

(C) thenthere exists on some half-cylinder subset of R(h,t0) a Liapunov function $V(t,\underline{x})$ of class C^k with respect to $\underline{X}$such that V is decrescent and $\dot{V}$ is negative definite.

Theorem 22, reference (23) (Converse Theorem of Malkin)

(H) (i) If $\underline{f}(t,\underline{x})$ is linear in $\underline{X}$ and bounded on R(h,to),

(ii) $\underline{X} = \underline{0}$ of (7) is uniform-asymptotically stable (hence, exponential-asymptotically stable),

(iii) $W(t,\underline{x})$ is a real scalar function defined and continuous on R(h,to) and is a positive definite form in $\underline{X}$ of degree $m > 0$,

(C) then there exists on R(h,to) a real scalar function $V(t,\underline{x})$ of class C^1 which is a positive definite form in $\underline{X}$ of degree m such that V is decrescent and $\dot{V} = -W$ on R(h,to).

Theorem 23, reference (23) (Massera, Converse of the Corollary of Theorem 13)

(H) (i) If $\underline{f}$ (t$\underline{x}$,) is locally Lipschitzian on R(h,t0),

(ii) $\underline{X} = \underline{0}$ is uniform-asymptotically stable,

(C) then (1) there exists a Liapunov function $V(t,\underline{x})$ on some half-cylindrical subset of R(h,to) possessing partial derivatives with respect to t and $\underline{X}$ of any order, such that V is decrescent and $\dot{V}$ is negative definite.

(2) If $\underline{f}$ is Lipschitizian on R(h,to), the partial derivatives of V are

bounded on the subset of R(h,to).

(3) If $\underline{f}$ is independent of t or periodic in t in R(h,to), then V is independent of t or periodic in t on the subset.

The next three theorems, 24, 25, 26, are very useful in applications. The topic being treated by them is still local asymptotic stability. Theorems 27 and 28 are concerned with uniform asymptotic stability in the large for $\underline{X} = \underline{0}$ of (7).

Theorem 24, reference (23)

(H) (i) If $\underline{X} = \underline{0}$ in (7) is uniformly stable,

(ii) there exists on R(h,to) a bounded Liapunov function such that $\dot{V}$ is negative definite on R(h,to),

(C) then $\underline{X} = \underline{0}$ of (7) is uniform-asymptotically stable.

Theorem 25, reference (23) (Massera)

(H) (i) If $\underline{f}$ $(t,\underline{x})$ is Lipschitzian on R(h,to),

(ii) there exists a real scalar function $V(t,\underline{x})$ defined, locally Lipschitzian and positive definite on R(h, to),

(iii) there exists a real scalar function W(s) defined, continuous and increasing for $s \geq 0$, and $W(0) = 0$, such that $\dot{V}(t,\underline{x}) \leq W(v(t,\underline{x}))$ on R(h,to),

(C) then $\underline{X} = \underline{0}$ of (7) is uniform-asymptotically stable.

Theorem 26, reference (23) (Massera)

(H) (i) If there exists a real scalar function $V(t,\underline{x})$ defined, locally Lipschtzian and positive definite on R(h,to), and there exists real scalar functions $W(s)$, $g(s)$ defined and continuous for $s \geq 0$, and $W(s)$ is an increasing function for $s \geq 0$, $w(0) = 0$, $g(s)$ is positive for $s \geq 0$, and $\int_0^{\infty} ds/g(s) = \infty$ such that $\dot{V}(t,\underline{x}) \leq -W(V(t,\underline{x}))$ and $(\underline{X})_T\, \underline{f}(t,\underline{x}) \leq \|\underline{X}\|\, g(\|\underline{X}\|)$ on $R(h,to)$,

(C) then $\underline{X} = \underline{0}$ of (7) is uniform-asymptotically stable.

Theorem 27, reference (23) (Massera)

(H) (i) If there exists for $t \geq to$ and $\|\underline{x}\| < \infty$, a Liapunov fucntion $V(t,\underline{x})$ such that V is decrescent and radially unbounded,

(ii) $\dot{V}$ is negative definite for $t \geq to$ and $\|\underline{x}\| < \infty$,

(C) then $\underline{X} = \underline{0}$ of (7) is uniform-asymptotically stable in-the-large.

Theorem 28, reference (23) (Converse Theorem of Massera)

(H) (i) If, $\underline{f}(t,\underline{x})$ is locally Lipschitzianfor $t \geq to$ and $\|\underline{x}\| < \infty$,

(ii) $\underline{X} = \underline{0}$ is uniform-asymptotically stable in-the-large,

(C) then there exists (1) for $t \geq to$, $\|\underline{x}\| < \infty$ a Liapunov function possessing partial derivatives with respect to t, $\underline{x}$ of any order such that V is decrescent and radially unbounded and $\dot{V}$ is negative definite.

(2) If $\underline{f}$ is Lipschitzian for $t \geq to$, $\|\underline{x}\| < \infty$, the partial derivatives of V are bounded in every bounded subset of the state space for $t \geq to$.

(3) If $\underline{f}$ is independent of t or periodic in t, then so is $V(t,\underline{x})$.

In the next set of theorems, we state some of the more important "instability" results.

Theorem 29, reference (19) (Liapunov's First Theorem on Instability

The equilibrium solution ($\underline{x} = \underline{0}$) of (7) is unstable if there exists a decrescent function $V(t,\underline{x})$ which has a domain where $V < 0$, and whose derivative $\dot{V}$ is negative definite.

The next two theorems give both necessary and sufficient conditions for instability, and thus are equivalent. Theorem 31 is usually more convienent for applications than Theorem 30.

Theorem 30, reference (19) (Liapunov's Second Theorem on Instability

The equilibrium solution of (7) is unstable if the following holds: In the domain $R(h,to)$, there exists a bounded function $V(t,\underline{x})$ with the properties:

(a) Its total derivative for (7) is of the form $\dot{V} = gV + W(t,\underline{x})$ where g is a positive constant, and where W is a semi-definite function;

(b) If $W(t,\underline{x})$ does not vanish identically, there exists in each domain $R(h_1,t_1)$ with arbitrarily large t1 and arbitrarily small $h_1 \leq h$ such points $\underline{X}$ that V and W have the same sign for $t > t_1$.

Theorem 31, reference (19) (Chetaev's Instability Theorem)

Given that the differential equation (7) and a function $V(t,\underline{x})$ with the following properties:

(a) In every domain $R(\epsilon)$, $\epsilon > 0$ is arbitrarily small, there exists points $\underline{X}$ such that $V(t,\underline{x})$ is negative for all $t \geq to$, to bring sufficiently large. The totality of points $(t,\underline{x})$ with $\| \underline{x} \| < h$ and $V(t,\underline{x}) < 0$ shall be denoted as the "domain $V < 0$". This domain is bounded by the hypersurfaces $\| \underline{x} \| = h$ and $V = 0$, and is possibly separated into several subdomains, U1, $U_2, \ldots$;

(b) V is bounded below in a certain subdomain U of the "domain $V < \theta$";

(c) In the domain U of $(t,\underline{x})$ space in (b) the $\dot{V}$ for (7) is negative; in particular, $\dot{V} \leq -\phi(\|V\|) < 0$, where $\phi(r)$ is continuous, monotorically increasing, and $\phi(0) = 0$. The existence of such a function $V(t,\underline{x})$ implies that $\underline{X} = \underline{0}$ of (7) is unstable.

In Theorem 32, Chetaev uses two functions to establish that $\underline{X} = \underline{0}$ of (7) is ustable.

Theorem 32, reference (19) (Chetaev)

If there eixts a decrescent function $V(t,\underline{x})$ and a function $W(t,\underline{x})$ such that (1) the "domain $V\dot{V} > 0$" is not empty for any t in $t_0 \leq t \leq \infty$ (which has to be considered a closed interval), and (2) for arbitrarily small $\|\underline{x}\|$ there exists a "subdomain $W > 0$" of the "domain $V\dot{V} > 0$" where $\dot{W}$ has constant sign on the boundary of the "domain $W > 0$", (the boundary is $W = 0$.), then the equilibrium ($\underline{x} = \underline{0}$) of (7) is unstable.

Theorem 33 considers the concept of complete instability.

Theorem 33, reference (19) (Persidskii)

The equilibrium is completely unstable in (7) if a function $V(t,\underline{x})$ exists which has the following properties in R (h,to):

(1) $V > 0$ for $\underline{X} \neq \underline{0}$;

(2) $\dot{V} \geqslant \theta$;

(3) the function V tends uniformly toward zero as t increases.

The next two theorems are the time-invariant results corresponding to Theorems 29 and 31.

Theorem 34, reference (20) (Liapunov's First Instability Theorem)

(H) (i) If $V(\underline{x})$, with $V(\underline{0}) = 0$, has continuous first partials in some neighborhood N of $\underline{X} = \underline{0}$,

(ii) V is positive definite arbitrarily near $\underline{X} = \underline{0}$,

(iii) V assumes positive values arbitrarily near $\underline{X} = \underline{0}$,

(C) then $\underline{X} = \underline{0}$ of $\underline{X} = \underline{f}(\underline{x})$ is unstable.

Theorem 35, reference (20) (Chetaev's Instability Theorem)

Let N be a neighborhood of $\underline{X} = \underline{0}$ and let there be given $V(\underline{x})$ and a region N_1 in N with the following properties:

(1) $V(\underline{x})$ has continuous first partials in N_1,

(2) $V(\underline{x})$ and $V(\underline{x})$ are positive in N_1,

(3) At the boundary points of N_1 inside N, $V(\underline{x}) = 0$.

(4) the origin is a boundary point of N_1. Then, under these conditons the origin is unstable.

We nextconsider LaSalle's theorems on the extent of asymptotic stability for autonomous systems. A more thorough discussion of this work can be found in references (20), (30), (37), and (38). The autonomous system is defined by $\dot{\underline{X}} = \underline{f}(\underline{x})$, $\underline{f}(\underline{0}) = \underline{0}$, where the usual properties required for existence and uiniqueness of solutions are assigned to $\underline{f}$. We must first introduce two notions from the work of G.D. Birkhoff.

Definition 43, reference (38)

Let $\underline{X}(t)$ be a solution of the autonomous system. A point $X_1$is said to be in the positive limiting set Γ^+ of X(t), if for every $\epsilon > 0$ and each $T > 0$ there is a $t > T$ such that $\| \underline{x}(t) - X_1 \| < \epsilon$.

One of the important properties of limiting sets is the following; "If $\underline{X}(t)$ is bounded for $t \geq 0$, then Γ^+ is a nonempty, compact, invariant set."

Definition 44, reference (38)

A set M is said to be invariant, if each solution of $\dot{X} = f(x)$ starting in M remains in M for all $t \geqslant t_o$.

Another important property of limiting sets is: "If $\underline{X}(t)$ is bounded for $t \geqslant 0$ and if a set M contains Γ^+, then $\underline{X}(t) \longrightarrow$ set M as $t \rightarrow \infty$."

Theorem 36 gives criteria for determining the extent of asymptotic stability.

Theorem 36, reference (38) (LaSalle)

(H) (i) If N is a bounded closed setwhich is also an invariant set of the system $\dot{\underline{X}} = \underline{f}(\underline{x})$,

(ii) $V(\underline{x})$ has continuous first partials in N and $\dot{V} \leq 0$ in N,

(iii) set E is the set of all points in Nfor which V = 0,

(iv) set M is the largest invariant set in E,

(C) than every solution starting in N approaches M as $t \rightarrow \infty$.

In Theorems 37, and 38, the set N is defined by the Liapunov function $V(\underline{x})$.

Theorem 37, reference (38) (LaSalle)

(H) (i) If N_L denotes the closed region defined by $V(\underline{x}) \leq L$,

(ii) $V(\underline{x})$ has continuous first partials in N_L and is postive definite in N_L,'

(iii) N_L is bounded and $\dot{V}(\underline{x}) \leq 0$ in N_L,

(iv) E is the set in N_L where $\dot{V} = 0$ and M is the largest invariant set in E,

(C) then every solution of $\dot{\underline{X}} = \underline{f}(\underline{x})$ starting in N_L tends to M as $t \rightarrow \infty$.

Theorem 38, reference (20) (LaSalle)

If $\dot{V}(\underline{x}) < 0$, $\underline{X} \neq \underline{0}$ in N_L, replace $V(\underline{x}) \leq 0$ in Theorem 37, then $\underline{X} = \underline{0}$ is asymptotically stable and every solution in N_L tends to $\underline{0}$ as $t \rightarrow +\infty$.

The next two theorems deal with the concept of stability in-the-large for the null sollution of $\underline{X} = \underline{f}(\underline{x})$.

Theorem 39, reference (38) (LaSalle)

(H) (i) If $V(\underline{x})$ has continuous first partials for all $\underline{X}$,

(ii) $V(\underline{x}) > 0$ for all $\underline{X} \neq \underline{0}$,

(iii) $\dot{V}(\underline{x}) \leq 0$ for all $\underline{X}$,

(iv) E is the set of all $\underline{X}$ such that $\dot{V}(\underline{x}) = 0$,

(v) M is the largest invariant set contained in E,

(C) then every solution of $\dot{\underline{X}} = \underline{f}(\underline{x})$ bounded for $t \geq 0$ appraoches M as $t \to \infty$.

Theorem 40, reference (38) (LaSalle)

(H) (i) If $V(\underline{x})$ has continuous first partials for all $\underline{X}$,

(ii) $V(\underline{x}) > 0$ for all $\underline{X} \neq \underline{0}$,

(iii) $\dot{V}(\underline{x}) \leq 0$ for all $\underline{X}$,

(iv) $V(\underline{x}) \longrightarrow \infty$ as $\| \underline{x} \| \to \infty$.

(v) $\dot{V}$ is not identically zero along any solution other than $\underline{0}$,

(C) then $\dot{\underline{X}} = \underline{f}(\underline{x})$ is completely stable.

THe next set of theorems deal with practical stabiliy and Lagrange stability.

Theorem 41, reference (38) (LaSalle)

(H) (i) N is a bounded neighborhood of $\underline{X} = \underline{0}$,

(ii) N^c is the complement of N,

(iii) $W(\underline{x})$ is a scalar function with continuous first partials in N^c,

(iv) $W > 0$ for all $\underline{X}$ in N^c,

(v) $\dot{W} \leq 0$ for all $\underline{X}$ in N^c,

(vi) $W \to \infty$ as $\|\underline{X}\| \to \infty$,

(C) then each solution of $\dot{\underline{X}} = \underline{f}(\underline{x})$ is bounded for all $t \geq 0$.

Theorem 42, reference (39) (Rekasius)

(H) If (i) N is a bounded region containing $\underline{0}$,

(ii) $V(\underline{x})$ is a scalar function with continuous first partials in N^c,

(iii) $V(\underline{x}) > 0$ for all $\underline{X}$ in N^c, and is locally Lipshitzian in N^c,

(iv) $V \to \infty$ as $\|\underline{X}\| \to \infty$,

(v) $\dot{V} < 0$ for all $\underline{X}$ in N^c,

(C) then the system $\dot{\underline{X}} = \underline{f}(\underline{x})$ is asymptotically stable in the sense of Lagrange; that is, every solution of $\underline{X} = \underline{f}(\underline{x})$ starting in N^c approaches N asymptotically as $t \to \infty$.

A Theorem similar to Theorem 42 is the following theorem from reference (30).

Theorem 43, reference (30) (LaSalle)

(H) If (i) N is the set defined by $V(\underline{x}) \leq \alpha$ and N^c is the complement ($V(\underline{x}) > \alpha$), with V having the usual continuity properties,

(ii) $\dot{V}(\underline{x}) \leq 0$ for all $\underline{X}$ in N^c,

(iii) $\dot{V}$ does not vanish identically along any trajectory that starts in N^c,

(iv) $\dot{\underline{X}} = \underline{F}(\underline{x})$ is Lagrange stable,

(C) then every solution of $\dot{\underline{X}} = \underline{f}(\underline{x})$ approaches N as $t \to \infty$.

From the reference (29) we have the following theorem dealing dealing with the problem of practical stability of an autonomous system.

Theorem 44, reference (29)

(H) If (i) there exisss a scalar function $V(\underline{x})$ with continuous first partials such that $V > 0$ and $\dot{V} < 0$ in the whole state space except in a small neighborhood N of $\underline{X} = \underline{0}$, and

(ii) $V \rightarrow \infty$ as $\| \underline{x} \| \rightarrow \infty$,

(C) then there exists another small neighborhood N_o of the origin such that N_o contains N and such that any moving point $\underline{X}$ having departed from N_o will return toward N_o as $t \rightarrow \infty$. (that is, the origin is stable in the large if small unpredictable oscillations within a small neighborhood N_o are neglected). Theorems 42 and 44 are basically the same.

The next two theorems aer concerned with the regions of eventual asymptotic stability for the nonautonomous system $\dot{\underline{X}} = \underline{f}(t,\underline{x})$, where $\underline{f}$ has continuous first partials. We will denote the solution to the system by $\underline{F}$ $(t;\ t_o,\ \underline{X}o)$. In Theorem 45, regions of eventual asymptotic stability are discussed. In Theorem 46, a method for determining a region of eventual asymptotic stability is given.

Theorem 45, reference (40) (LaSalle & Rath)

(H) If (i) N is a bounded closed set containg $\underline{0}$,

(ii) N_o is a subset of N such that solutions starting in N_o at time t_o T_o remauin for all $t \geq t_o$ in N,

(iii) $V(t,\underline{x})$ is a scalar function such that $V(t,\underline{x}) \rightarrow U(\underline{x})$ as $t \rightarrow \infty$ uniformly for $\underline{X}$ in N,

(iv) $\dot{V}(t,\underline{x}) \longrightarrow -W(\underline{x})$ as $t \longrightarrow \infty$ uniformly for $\underline{X}$ in N,

(v) $U(\underline{x})$ and $W(\underline{x})$ are positive definite for $\underline{X}$ in N,

(C) Then there exists a $T_o > 0$ with the property that $\underline{F}(t;t_o,\underline{X}_o) \rightarrow \underline{0}$ as $t \longrightarrow \infty$ for all $\underline{X}_o$ in N_o and all $t_o \geq To$.

Theorem 46, reference (40) (LaSalle & Rath)

(H) If (i) N is a closed bounded set defined by $U(\underline{x}) \leq L$ $(L > 0)$,

(ii) conditons (iii), (iv), and (V) in Theorem 45 are satisfied,

(iii) for any $\delta > 0$, N_δ is the set defined by $U(\underline{x}) \leq L - \delta$,

(C) then there is a $T_\delta > 0$ such that $\underline{F}(t;t_0,\underline{X}_0) \longrightarrow 0$ as $t \longrightarrow \infty$ for all X_0 in N_δ and all $t_0 \geqslant T_\delta$.

In Theorem 46, a sufficientcondition for set N, defined by $U(\underline{x}) \leq L$, to be bounded for all L is that $U(x) \longrightarrow \infty$ as $\| \underline{X} \| \rightarrow \infty$. Moreover, if

$$\lim_{\|\underline{x}\| \to \infty} \{\inf U(\underline{x})\} = L_0 ,$$

then N is bounded for all fixed $L < L_0$. Thus, Theorems 45 and 46 are useful in determing how large a region of stability exits around $\underline{X} = \underline{0}$.

The next two theorems are extensions of LaSalle"s work on the "extent of asymptotic stability." Theorem 47 is due to Yoshizowa and is concerned with what LaSalle calls the "pertubbed autonomous system," references (41) and (42),. The basic system $\mathbf{x}$ is defined by

$$\dot{\underline{X}} = \underline{F}(\underline{x}) + \underline{P}(t,\underline{x}) + \underline{q}(t,\underline{x}), \qquad (8)$$

where $\underline{P}$ and $\underline{q}$ are different types of perturbations. We assume the following:

(1) If $\underline{X}(t)$ is continuous and bounded for all $t \geqslant 0$, then $\int_0^\infty \| \underline{P}(t,\underline{x}(t)) \| dt < \infty$

(2) The function $\underline{q}(t,\underline{x}) \longrightarrow 0$ as $t \rightarrow \infty$ uniformly for $\underline{X}$ in any compact set.

Theorem 47, references (41) and (42) (Yoshizawa)

(H) If (i) all solutions of (8) which start in a compact set N remain in N,

(ii) there exits a scalarfunction $V(t,\underline{x})$ which is nonnegative for all $t \geqslant 0$, and all $\underline{X}$ in N,

(iii) $\dot{V}(t,\underline{x}) \leq W(\underline{x}) \leq 0$ for all $t \geqslant 0$ and all $\underline{X}$ in N,

(iv) set E in N is defined by $W(\underline{x}) = 0$ and M is the largest invariant set of the system $\dot{\underline{X}} = \underline{F}(\underline{x})$ in E,

(C) then every solution of (8) starting in N approaches M as $t \rightarrow \infty$.

In Theorem 48, Matrosov, in references (41) and (43), considers a more general situation; he gives sufficient conditions for the asymptotic stability of an equilibrium state of a nonautonomous system. Two "Liapunov functions" are used; thus he is able to relax the condition of positive definiteness of the Liapunov functions by relating the second function to the set where the time derivative of the first function vanishes. The system considered is given by

$$\dot{\underline{X}} = \underline{E}(t,\underline{x}), \qquad \underline{E}(t,\underline{0}) = \underline{0} \quad \text{for all } t. \tag{9}$$

The two "Liapunov functions" are $V(t,\underline{x})$ and $W(t,\underline{x})$. Relative to a set E, $\dot{W}$ is defined to be <u>definitely not equal to 0 in the set E</u> if given numbers α andd A there exist r and δ such that $|\dot{W}| > \delta$ for all $t \rightarrow 0$ and all $\underline{X}$ within a distance r of set E and such that $\alpha \leq \|\underline{x}\| \leq A$

<u>Theorem 48, references (41) and (43)</u> (<u>Matrosov</u>)

(H) If (i) $V(t,\underline{x})$ and $W(t,\underline{x})$ are defined in some neighborhood of $\underline{X} = \underline{0}$,

(ii) $0 \leq U(\underline{x}) \leq V(t,\underline{x}) \leq v(\underline{x})$, where U and v are continous and $v(\underline{0}) = 0$,

(iii) $\dot{V}(t,\underline{x}) \leq - w(\underline{x})$ 0, where w is continuous,

(iv) $\dot{W}(t,\underline{x})$ is uniformly bounded for $t \geq 0$,

(v) $\dot{W}(t,\underline{x})$ is definitely not equal to zero in E, where E is the set of points $\underline{X}$ for which $w(\underline{x}) = 0$,

(C) then $\underline{x} = \underline{0}$ of (9) is asymptotically stable.

<u>Note by LaSalle reference (41)</u>

The introduction of $W(t,\underline{x})$ assumes that the solutions cannot remain too long near E but must approach $\underline{0}$.

The next four theorems are concerned with the perturbed system given by equation (6), which is repeated below:

$$\dot{\underline{X}} = \underline{f}(t,\underline{x}) + \underline{g}(t,\underline{x}), \qquad (6)$$

where $\underline{f}(t,\underline{0}) = \underline{0}$ for all $t \geq t_0$ and $\underline{g}$ is the perturbation term.

Theorem 49, reference (19) (Gorsin and Malkin)

(H) If (i) $\underline{X} = \underline{0}$ of $\dot{\underline{X}} = \underline{f}(t,\underline{x})$ is uniformly asymptotically stable,

(C) then $\underline{X} = \underline{0}$ of (6) is also stable under constantly acting perturbations.

Theorem 50, reference (19) (Malkin)

(H) If (i) a positive definite Liapunov functiion $V(t,\underline{x})$ exists whose partial derivatives are bounded in a domain, and

(ii) $\dot{V}$ with respect to $\dot{\underline{X}} = \underline{f}(t,\underline{x})$ is negative definite,

(C) then $\underline{X} = \underline{0}$ of (6) is stable under constantly acting perturbations.

Theorem 51, reference (35) (Vrkoc)

The solution $\underline{X} = \underline{0}$ of (6) is stable under constantly acting perturbations, if and only if, there exists a function $V(t,\underline{x})$ with continuous partial derivatives and satisfying the following:

(1) $V(t,\underline{x})$ is positive definite;

(2) $V(t,\underline{x})$ is bounded uniformly with respect to t;

(3) there exists a continuous function $U(\underline{x})$ which is positive except at the point $\underline{X} = \underline{0}$, and the function

$$Q(t,\underline{x}) = \dot{V} + U(x)\sqrt{(\nabla \underline{V})_T (\nabla V)} \leq 0$$

Theorem 52, reference (35) (Vrkoč)

The solution $\underline{X} = \underline{0}$ of (6) is integrally stable, if and only if, there exists a function $V(t,\underline{x})$ with continuous first partials fulfilling the following conditions:

(1) $V(t,\underline{x})$ is psoitive definite;

(2) $V(t,\underline{x})$ fulfills a Lipschitz condition with a constant independent of t;

(3) $\dot{V} \leq 0$.

Note by Vrkoč, reference (35)

In the case of stability in the mean only sufficient conditions are known at present.

The last set of theorems which we will list are three due to Yoshizowa dealing with boundedness, namely, theorems dealing with Definitions 10, 11, and 12.

Theorem 53, reference (23) (Yoshizawa)

(H) If (i) there exists for $t \geq t_o$ and $\underline{X}$ in $\|\underline{x}\| \geq h$, a real scalar function $W(t,\underline{x})$ defined, locally Lipschitzian and positive definite, such that $W(t,\underline{x}) \to \infty$ with $\underline{X}$ uniformly for $t \geq t_o$, and

(ii) $\dot{W}(t,\underline{x}) \leq \theta$ for $t \geq t_o$ and $\|\underline{x}\| \geq h$,

(C) then every solution of $\dot{\underline{X}} = f(t,\underline{x})$ is bounded. ($\underline{f}$ having the usual "nóce" properties).

Theorem 54, reference (23) (Yoshizawa)

(H) If (i) there exists for $t \geq t_o$, $\|\underline{x}\| \geq h$, a real scalar function $W(t,\underline{x})$, defined, locally Lipschitzian and positive definite, such that for any open sphere S containing $\|\underline{x}\| < h$, $W(t,\underline{x})$ is bounded on the intersection of S and $\|\underline{x}\| \geq h$, for all $t \geq t_o$,

(ii) $W(t,\underline{x}) \to \infty$ as $\|\underline{x}\| \to \infty$ uniformly for $t \geq t_o$,

(iii) $\dot{W}(t,\underline{x}) \leq \theta$ for $t \geq t_o$ and $\|\underline{x}\| \geq h$,

(C) then every solution of $\dot{\underline{X}} = \underline{f}(t,\underline{x})$ is uniformly bounded.

Theorem 55, reference (23) (Yoshizawa)

(H) If (i) conditons (i) and (ii) are satisfied in Theorem 54, and

(ii) $\dot{W}(t,\underline{x})$ is negative definite for $t \geq to$, and $\| \underline{x} \| \geq h$,

(C) then every solution of $\dot{\underline{X}} = \underline{f}(t,\underline{x})$ is ultimately bounded.

(E). Examples of Some of The Theorems in Part D.

The last part of this report contains examples to help illustrate some of the theorems in Part D.

Example 26, reference (21)

In the systemdefined below we consider the Liapunov function as a distance in phase space. The system is given by

$$\dot{X}1 = X_2 - aX_1 (X_1^2 + X_1^2),$$
$$\dot{X}_2 = - X_1 - aX_2 (X_1^2 + X_2^2),$$
$$a = \text{constant}.$$

Thus, let $V = X_1^2 + X_2^2$, where

$$\dot{V} = 2X_1(X_2 - aX_1V) + 2X_2(- X_1 - aX_2V) = - 2aV^2.$$

Solving for V, where $V = V_o > 0$ for $t = to$, given

$$V = \frac{V_o}{1 = 2aV_o (t = to)}$$

Three possibilities exist; namely,

(1) for $a > 0$ and any $V_o > 0$, $V \rightarrow 0$ as $t \rightarrow \infty$, thus the system is completely stable;

(2) for $a = 0$ and any $V_o > 0$, $V = V_o$ for all $t \geq 0$, thus the system is stable;

(3) for $a < 0$ and any $V_o > 0$, $V \rightarrow +\infty$ as $t \rightarrow$ to $-1/2aV_o$, thus the system has a finite escape time and is obviously unstable.

Example 27 (First Method Vs. Second Method)

(A) From reference (44), we have the system defined by

$$\dot{X} = - Y - X^3,$$
$$\dot{Y} = X - Y^3.$$

Let $V = X^2 + Y^2$; thus $\dot{V} = -2 (X^4 + Y^4)$.

Therefore by Liapunov's Second Method, the system is compeetely stable.

But the linear approximation only says that the origin is locally stable.

(B) From reference (44), consider the system defined by

$$\dot{X} = Y + \alpha X - X^5,$$
$$\dot{Y} = - X - Y^5,$$

where α is a system parameter. From the linear approximation the origin is locally asymptotically stable if $\alpha < 0$ and unstable of $\alpha > 0$. By the Second Method, if $V = X^2 + Y^2$, then $\dot{V} = -2(X^6 + Y^6 - \alpha X^2)$. Thus, if $\alpha < 0$, the system is completely stalbe.

When $\alpha = 0$, we have the critical case. The First Method says (0,0) is locally stable. The Second Method says that the system is completely stable because if $V = X^2 + Y^2$, then $\dot{V} = -2(X^6 + Y^6)$.

Example 28 Unstable Cases

(A) From reference (12), consider the system defined by

$$\dot{X}_1 = - X_1 + X_2^3 ,$$
$$\dot{X}_2 = X_2 + X_1^2 .$$

The linear approximation shows that (0,0) is unstable. We can also prove instability by the Second Method if we let $V = X_2 - X_1^2$. Then, we have

$$\dot{V} = 2(X_1^2 + X_2^2 + X_1^2 X_2 - X_1 X_2^3).$$

Thus, in any neighborhood of (0,0), V can be both positive and negative, but if $\| \underline{x} \|$ is sufficiently small ($\| \underline{x} \| \neq 0$), then $\dot{V} > 0$.

By the instability theorems (0,0) is unstable.

(B) From reference (44), we obtain the following example:

$$\dot{X}_1 = X_2^3 + X_1^5 ,$$
$$\dot{X}_2 = X_1^3 + X_2^5 .$$

Let $V = X_1^4 - X_2^4$, then $\dot{V} = 4(X_1^8 - X_2^8)$. If $|x_1| > |x_2|$, V and $\dot{V}$ are both positive. Therefore, (0,0) is unstable.

Example 29, refernce (45) Time-Dependent Domain of Asymptotic Stability

Consider the first order system defined by $\dot{X} = -X - \frac{X}{t}(1 - t^3X^2)$. For $t_0 > 0$, the solution of the linear approximation is given by $X = X_0 t_0 \frac{e^{-(t-t_0)}}{t}$ Thus, as $t \to \infty$, $X \to 0$ and the origin is locally asymptotically stable for any $t_0 > 0$. By Zubov's method we can verify that the domain of asymptotic stability for the nonlinear system is given by $|x| < 1/t_0$. Thus, as t_0 becomes large, the domain becomes small; and for large t_0, the system is unstable for practical purposes.

Example 30, reference (39) Practically Stable System

Consider the forced Duffing equation defined by

$$\dot{X}_1 = X_2,$$
$$\dot{X}_2 = - X_1 - 2X_2 - bX_1^3 + g(t),$$

where $b > 0$ and $|g(t)| < m$. Let $V = 1/2(x_1 + X_2)^2 + b/4\, X_1^4$.
Thus, $\dot{V} = -(X_1 + X_2)^2 - bX_1^4 + g(t)(X_1 + X_2) \leq -(X_1 + X_2)^2 - bX_1^4 + m|X_1 + X_2|$. Choose the neighborhood N to be N: $|X_1 + X_2| \leq m$. Therefore, in the complement of N, N^c, we have $V > 0$, $V \to \infty$ as $\| \underline{X} \| \to \infty$, and $\dot{V} < 0$. Thus, by Theorem 42, the system is ultimately bounded and is "practically stable".

Example 31. reference (39) Asymptotically Stable Non-null Solution

Consider the following equations which define a nonlinear regular:

$$\dot{X}_1 = -X_1 + X_2^3 ,$$
$$\dot{X}_2 = X_1 = 1/2X_2 .$$

By linear approximation we see that (0,0) is locally asymptotically stable. Also, as can be easily seen, $2X_1^2 - X_2^4 = 0$ is a solution of the system. By Liapunov type arguments, we want to prove that this solution is asymptotically stable.

Let $V = 2X_1^2 - X_2^4$. Then, $\dot{V} = -2V$. The solution of this V-equation is

$$V(t) = V(0)\, e^{-2t}, \quad V(0) = 2X_1^2(0) - X_2^4(0) \neq 0$$

Thus, any initial point $(X,(0), X_2(0))$ not on the curve $2X_1^2 - X_2^4 = 0$ at $t = 0$, will approach the curve asymptotically as $t \to 0$.

Example 32, Some Examples of LaSalle"s Theorems

(A) From reference (12), consider the "vibrating spring" problem with a nonlinear damping factor, $f(x,\dot{x})$, defined by

$$\ddot{X} + f(x,\dot{x})\dot{X} + W^2X = 0,$$

or
$$\dot{X}_1 = X_2,$$
$$\dot{X}_2 = -w^2X_1 - f(x_1,X_2)\, X_2 .$$

Let $V = w^2X_1^2 + X_2^2$, $w \neq 0$. Thus, $\dot{V} = -2f(X_1,X_2)\, X_2^2$. Therefore, the system is completely stable if $f(x_1,x_2) \geq 0$ but not identically zero, and if $f(x_1,x_2) \not\equiv 0$ on any non-null trajectories of the system.

(B) From reference (20), we have the famous Van der Pol system which is defined by

$$\dot{X}_1 = X_2 - E\left(\frac{x_1^3}{3} - X_1\right).$$

$$\dot{X}_1 = - X_1 , \quad E < 0.$$

Let $V = 1/2(x_1^2 + x_2^2)$, giving $\dot{V} = -E\, X_1^2 \left(\frac{x_1^2}{3} - 1 \right)$.

Thus $\dot{V} \leq 0$ for $X_1^2 \leq 3$. Therefore theregion $X_1^2 + X_2^2 < 3$ is interior to the rgion of asymptotic stability for (0,0). Thus every solution starting inside $X_1^2 + X_2^2 = 3$, approaches (0,0) as $t \to +\infty$.

(C) Example due to LaSalle

The following example shows that Theorem 39 can not be used for nonautonomous systems. Consider

$$\ddot{X} + (2 + e^t)\, \dot{X} + X = 0,$$

or

$$\dot{X}_1 = X_2,$$

$$\dot{X}_2 = -X_1 - (2 + e^t)\, X_2.$$

Let $V = X_1^2 + X_2^2$; thus, $\dot{V} = -(2 + e^t)\, X_2^2$. The set E, $\dot{V} = 0$, is the X_1 - axis. The largest invariant set in E is (0,0). Therefore, all the hypotheses of Theorem 39 are satisfied; but for the initial values $(2b, -b)$, $X = b(1 + e^{-t})$. Thus as $t \to +\infty$, $X \to b$ and $\dot{X} \to 0$. The origin is not asymptotically stable as the incorrect use of Theorem 39 would indicate.

(D) From references (41) and (42), we see that the Theorem of Yoshizawa (Theorem 47) covers the problem in Case C. Suppose, the system is defined by

$$\dot{X}_1 = X_2,$$

$$\dot{X}_2 = -X_1 - a(t)\, X_2,$$

where $a(t)$ is bounded by $M > a(t) > m > 0$ and $a(t) \to a_0 > 0$ as $t \to \infty$. Thus, let $2V = X_1^2 + X_2^2$, giving $\dot{V} = -a(t)\, X_2^2$. By theorem 47, the system is completely stable. (the trouble in case C is that $2 + e^t$ is unbounded as $t \to \infty$.)

Example 33, reference (42) Example of Theorem 47

Consider the system defined by $\ddot{X} + f(x)\, \dot{X} + g(x) = e(t)$ where $f(x)$ and $g(x)$ are continuous for all x, e(t) is continuous for

$0 \leq t < \infty$, $x\,g(x) > 0$ for $x \neq 0$, $g(0) = 0$, $f(x) > 0$ for $x \neq 0$ and $f(0) \geq 0$, $F(x) = \int_0^x f(x)\,dx \rightarrow \pm\infty$ as $x \rightarrow \pm\infty$, and $E(t) = \int_0^t |e(t)|\,dt < \infty$.

Now consider the equivalent system

$$\dot{X} = Y,$$
$$\dot{Y} = -f(x)\,y - g(x) + e(t)$$

We define the set N:

$$|x| < c \text{ and } |y| < c, \quad c \text{ a positive constant.}$$

Let $V(t,x,y) = \exp\left[-2E(t)\right] \cdot \left[G(x) + Y^2/2 + 1\right]$,

where $G(x) = \int_0^x g(x)\,dx \geq 0$. Thus, $\dot{V} \leq -\exp\left[-2E(\infty)\right] f(x)\,y^2$.

Thus set E is made up of the points on the x-axis satisfying $|x| < c$, and the points satisfying $f(x) = 0$, $|x| < c$ and $|y| < c$. The largest invariant set, M, in E is (0,0). Thus (0,0) is asymptotically stable by Theorem 47.

Example 34, reference (39) Example of Theorem 42

Consider the system defined by

$$\dot{X}_1 = X_2 + X_1\,(1 - X_1^2 - X_2^2),$$
$$\dot{X}_2 = -X_1 + X_2\,(1 - X_1^2 - X_2^2).$$

Let $V = 1/2\,(X_1^2 + X_2^2)$; thus,

$$\dot{V} = (x_1^2 + X_2^2)\,(1 - X_1^2 - X_2^2)$$
$$= 2V\,(1 - 2V)$$

Thus the origin is usntalbe. The region N in Theorem 42 is defined by $X_1^2 + X_2^2 \leq 1$. Thus, the hypotheses of Theorem 42 aee satisfied in N^c, and any solution starting in N^c approaches N asymptotically as $t \rightarrow \infty$. Any solution starting insde N approaches $X_1^2 + X_2^2 = 1$, therefore, $X_1^2 + X_2^2 = 1$ is a limit cycle.

Example 35, reference (29) Example of Theorem 44

Consider the system defined by

$$\dot{X}_1 = -X_2 + a E X_1 - a X_1^3,$$
$$\dot{X}_2 = X_1 + a E X_2 - a X_2^3,$$

where a and E are positive constants and E is small. Let $V = X_1^2 + X_2^2$. Thus, $\dot{V} = 2a\left[E(X_1^2 + X_2^2) - (X_1^4 + X_2^4)\right]$. $\dot{V}$ is less than or equal to zero when $X_1^2 + X_2^2 \geq 4E$. Thus inside the neighborhood, N, defined by $X_1^2 + X_2^2 \leq 4E$, the system is ustable; but any solution starting in N^c approaches $X_1^2 + X_2^2 = 4E$. Therefore, the origin may be regarded as stable in-the-large if small oscillations of amplitude less than or equal to $2\sqrt{E}$ around the origin are neglected.

Example 36, reference (43) Example of Theorem 48

Let us consider a symmetrical, heavy, rigid body with one point fixed, and under the presence of resistance forces of the medium. The equations of motion are

$$A\dot{p} + (C-A)qr = P z_0 \gamma_2 - \partial R/\partial p$$
$$A\dot{q} + (A-C)pr = -P z_0 \gamma_1 - \partial R/\partial q$$
$$C\dot{r} = M z(t, r)$$
$$\dot{\gamma}_1 = r\gamma_2 - q\gamma_3$$
$$\dot{\gamma}_2 = p\gamma_3 - r\gamma_1$$
$$\dot{\gamma}_3 = q\gamma_1 - p\gamma_2$$
$$\gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1$$

The function R is a homogeneous function of P, q of order $m \geq 2$ with coefficients which are continuous and limited functions of time t. M_z is the moment with respect to the axis of symmetry z of the driving and resistance forces. The equations of motion admit the solution.

$$p = q = 0 \;, \quad r = r(t, r_0, t_0)$$
$$\gamma_1 = \gamma_2 = 0 \;, \; \gamma_3 = 1$$

describing the irregular notation of the body around the vertical axis of symmetry.

The V and W functions in Theorem 48 are defined by

$$V = \frac{1}{2} A (p^2 + q^2) - \frac{1}{2} P z_0 (\gamma_1^2 + \gamma_2^2 + \gamma^4)$$
$$W = A (p \gamma_2 - q \gamma_1)$$
$$\gamma^2 = 1 - \gamma_3 \geqslant 0$$

The corresponding time derivatives are

$$\dot{V} = -mR \leq 0,$$
$$\dot{W} = \frac{1}{2} P z_0 \left[\gamma_1^2 + \gamma_2^2 + \gamma^2 (2 - \gamma^2) \right] - \gamma_1 \left(C p r - \partial R / \partial q \right) +$$
$$- \gamma_2 \left(C q r + \partial R / \partial p \right) + A \gamma_3 (p^2 + q^2)$$

The set E ($\dot{V} = 0$) corresponds to $p = 0$, $q = 0$. For this set E,

$$\dot{W} = \frac{P z_0}{2} \left[\gamma_1^2 + \gamma_2^2 + \gamma^2 (2 - \gamma^2) \right]$$

If $z_0 \neq 0$, then

$$|\dot{W}| > \frac{1}{2} P |z_0| \alpha^2 \quad \text{for} \quad p = q = 0$$
$$0 < \alpha^2 < \gamma_1^2 + \gamma_2^2 + \gamma^2 < A^2 < H^2.$$

Therefore, because of the continuity of $\dot{W}$ and the bounds on the coefficients, it is possible to find $r_1 > 0$, such that

$$|\dot{W}| > \frac{1}{4} P |z_0| \alpha^2 \qquad \text{when } t \geqslant 0, \quad \begin{array}{l} \alpha^2 < \gamma_1^2 + \gamma_2^2 + \gamma^2 < A^2 \\ p^2 + q^2 < r_1^2 \end{array}$$

Thus, $\dot{W}$ is definitely not equal to zero in set E ($\dot{V} = 0$).

If $z_0 < 0$ (center of gravity is lower than the point of support) then Theorem 48 is satisfied and the autonomous motion ($p = q = 0$, $r = r(t, r_0, t_0)$, $\gamma_1 = \gamma_2 = 0$, $r_3 = 1$) is asymptotically stable.

REFERENCES

(1) Kovach, L.D.; "Life Can Be So Nonlinear",
Amer. Sc. , Vol. 48, pp 218 - 225, 1960.

(2) Cunningham, W.J.; "Concept of Stability",
Amer. Sc., Vol 51, pp 425 - 436, 1963

(3) Hale, J.K. and J.P. LaSalle; "Differential Equations:
Linearity V.S. Nonlinearity,"
Siam Review , Vol. 5, No. 3, July, 1963

(4) Ficken, F.A.; "Linear and Nonlinear Equations: Local and Global Properties",
Siam Review, 1960

(5) Clauser, F. ; "Behavior of Nonlinear Systems,"
Jour. of Aero. Sc., Vol. 23, 1956

(6) Von Karman, T.; "Engineer Grapples with Nonlinear Problems,"
Coll. Works of Von Karman, Paper NO. 81, 1940.

(7) Morrey, C.B., Jr.; "Nonlinear Methods,"
Modern Mathematics for the Engineer, Vol I,
Ed. by E.F. Beckenbach, pp 389-427, 1956

(8) David, H.T.; Introduction to Nonlinear Differential and Integral Equations,
Dover Publ, Inc., New York, N.Y., 1962

(9) Minorsky, N.; Nonlinear Oscillations, D. Von Nostrand Co, Inc,
Princeton, N.J., 1962

(10) Stoker, J.J., Nonlinear Vibrations in Mechanical and Electrical Systems, Interscience Publ. Inc, New York, 273 pp, 1950.

(11) Hale, J., Nonlinear Oscillations, McGraw-Hill Book Co., Inc,. New York, 1963

(12) Struble, R.A., Nonlinear Differential Equations, McGraw-Hill Book Co., Inc., New York, 1963.

(13) Lifschitz, S., Differential Equations: Geometric Theory 2nd Ed. Interscience Publ., New York, 1962

(14) Bellin, A.I., "Non-Autonomous Systems", Adv. in Applied Mech., Vol. III, Academic Press, 1953

(15) Nemyckii, V.V., " On Some Qualitative Methods for Systems of Ordinary Differential Equations in the Large", Diff. Eq. and their Application, Proc. of a Conf. held in Prague, Sept., 1962, Academic Press, New York, pp 105-120, 1963.

(16) Sansone, G., "Nonlinear Differential Systems of the Third and Fourth Order", Diff. Eq. and their Appl., Proc. of a Conf. held in Prague, Sept. 1962, Academic Press, New York, pp 143-165.

(17) Kaplan, W., Ordinary Differential Equations Addison-Welsey Publ. Co, Inc., Cambridge, Mass, 1961

(18) Krasouskii, N.N., Stability of Motion (trans. by J.L. Brenner), Stanford Univ. Press, 1963.

(19) Hahn, W., Theory and Application of Liapunov's Direct Method, Englewood Cliffs, N.J., 1963

(20) LaSalle, J. and S. Lefschitz, Stability by Liapunov's Direct Method, New York, Academic Press, 1961.

(21) Kovatch, G., Unpublished Class Notes on Modern Control Theory, 1963-64, Drexel Inst. of Tech., Phila., Pa.

(22) Zubov, V.I., Mathematical Methods for the Study of Automatic Control Systems, trans. by Y. Schoor-Kan, Pergamon Press Book Co., New York, 1963

(23) Antosiewics, H.A., "A Survey on Liapunov's Second Method", Ann. Math. Studies 41, (contr. to Theory of Nonlinear Oscill. 4), pp 141-166, 1958.

(24) Ingwenson, D.R., "Principal Definitions of Stability", Proc. of Workshop Session in Liapunov's Second Method, Ed. by L. Kazda, U. of Mich. Press, Ann Arbor, Mich., pp 53-75, 1962.

(25) Massera, J.L. and J.L. Schaffer, " Linear Differential Equations and Functional Analysis, I", Annals of Mathematics, Vol. 67, No. 3, May., 1958, pp 517-573.

(26) Kalmon, R.E. and J.E. Bartram, " Control Analysis and Design VIA the Second Method of Liapunov, I, Continuous Time Systems", ASME Jour. of Baisc Eng , pp 371-393, June, 1960.

(27) Cessari, L., Asymptotic Behavior and Stability Problems in Ordinary Differential Equations, Acad. Press. Inc., New York, 1963

(28) Hochstadt, H., Differential Equations - A Modern Approach, Holt, Rinehart and Winston, 1963.

(29) Cesari, L. " Problem of Asymptotic Behavior and Stability", AIEE Trans. - Part II, Appl. and Industry, Vol. 80, pp 161-166, 1961.

(30) LaSalle, J.P. "Asymptotic Stability Criteria", Proc. of Sym. in Appl. Math. Vol. 13, Hydrodynamics Instability, Amer. Soc., pp 299-307, 1962

(31) Johnson, G.W. " On Liapunov Stability V.S. Bounded Input - Bounded Output", IEEE Trans. on Auto. Control., Apr. 1964, pp 178-179.

(32) Szego, G.P., "Method for Constructing Liapunov Functions for Linear Non-stationary Systems," Jour. of Electronics and Control, Vol. 13, No. 1, pp 69-88, July, 1962.

(33) Malkin, I.G., Theory of Stability of Motion AEC- tr-3352 (trans. Series), 1952.

(34) Gaderz, R.N. and C.C. Li, " On Positive Definiteness of Quartic Forms of Two Variable", I.E.E.E. trans. on Auto. Control, Apr. 1964, pp 187-188.

(35) Vrkoc, I. "On Some Stability Problems" Differential Equations and their Applications Proc. of a Conf. held in Prague, Sept. 1962, Academic Press, New York pp 217-221, 1963.

(36) Aizerman, M.A. and F.R. Gantmasher, Absolute Stability of Regulator Systems, (trans. by E. Polak), Holden-Day Inc, San Francisco, 1964

(37) LaSalle, J.P., "The Extent of Asymptotic Stability", Proc. of Nat. Acad. of Sc., Vol. 26, No.3, pp 363,365, March, 1960.

(38) LaSalle, J.P., "Some Extensions of Liapunov's Second Method," I.R.E. trans. on Circuit Theory, pp 520-527, Dec. 1960/

(39) Geis, G.R., The Synthesis of Nonlinear Control Systems Via Liapunov Stability Theory --- An Example , Grumman Aircraft Eng. Corp. Res. Note RN-153, Feb. 1963.

(40) LaSalle, J.P. and R.J. Rath, A New Concept of Stability, Paper No. 415, RIAS, Baltimore, Md., Aug. 29, 1962.

(41) LaSalle, J.P., "Recent Advances in Liapunov Stability Theory", Siam Review Vol.6, No. 1, pp 1-11, Jan., 1964.

(42) Yoshizawa, T., " Asymptotic Behavior of Solutions of a System of Differential Equations", Contr. to Dif. Eq., Vol. I, pp 371-387, 1963.

(43) Matrosov, V.M., "On the Stability of Motion", P.M.M., Vol. 26, pp 885-895, 1962.

(44) El"sgol'ts, L.E., Differential Equations, Hindustan Publ. Corp., India, 1961.

(45) Hahn, W., "The Present State of Liapunov's Direct Method", Nonlinear Problems, (Ed. by R.E. Langer), Univ. of Wis. Press, 1963 pp. 1[illegible]-205.