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CASSINI'S SECOND AND THIRD LAWS

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The author dedicates this Report to
the memory of Imre Izsak

CASSINI'S SECOND AND THIRD LAWS[1]

G. Colombo[2]

Introduction

In 1693 G. D. Cassini published the three following empirical laws on the Moon's rotational motion:

1) The Moon rotates uniformly about its polar axis with a rotational period equal to the mean sidereal period of its orbit about the earth.

2) The inclination of the Moon's equator to the ecliptic is a constant angle approximately $1^\circ.5$.

3) The ascending node of the lunar orbit on the ecliptic coincides with the descending node of the lunar equator on the ecliptic.

These laws describe fairly well the main rotational motion of the Moon, as well as the behavior of some of the lunar orbital elements, in particular the effect of solar perturbations on the motion of the nodal line.

Physical libration of the Moon means the departure of the Moon's rotational motion from that defined by Cassini's (1693) three laws.

In the literature on the subject, Cassini's laws are taken as a reference motion, and the perturbations due to the gravitational torque acting on a rigid body are studied. Naturally the parameters (principal moments of inertia) are adjusted in such a way that the motion can be a dynamical solution, after which the small oscillations about this solution due to the different

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torques are investigated, taking advantage of the fact that the equation can be simplified if one assumes that the actual motion of the Moon is close to that described by Cassini's laws. To my knowledge no specific investigation has been made of all other possible motions of the Moon permitted by the actual gravitational dynamics, in order to understand the properties of the actual motion.

I will be concerned only with Cassini's second and third laws, since the first one seems to be quite clear and easily understandable if we assume the inertial ellipsoid of the Moon to be triaxial. With this in mind, I have studied the motion for a model representing the real case to an extent that I judged sufficient for the main purpose of understanding the second and third laws. I will show that the second and third laws are independent of the first one, at least qualitatively, in the sense that even if the Moon's inertial ellipsoid were rotationally symmetric, the second and third laws could still be satisfied (provided the rotational angular velocity of the Moon is the same) since they represent a motion corresponding to the minimum dissipation of energy by internal friction. For a different rotational angular velocity, only the inclination of the Moon's equator on the ecliptic would change.

This very simple theory applies to any satellite or planet whose nodal line on the invariable plane shifts because of perturbations, that may be of different origin (e. g., oblateness of the primary body, a perturbing third body, or both). We will treat the case of Mercury, the case of an artificial Earth satellite (gravitational torque only), and finally the case of Iapetus since G. D. Cassini was the discoverer of this, for many reasons peculiar, satellite of Saturn.

1. Equations of motion

We study first the general problem of the motion of the rotational axis of a rigid satellite due to a gravitational torque. We deduce the equation of motion in a closed, very simple form since in our model a reduction to one degree of freedom is possible and also a first integral is easily found. The motion may also be represented in a very simple geometrical form.

Let Σ be a satellite of the primary E; G the center of mass of Σ; $\vec{i}$, $\vec{j}$, $\vec{k}$ the unit vectors along the principal axes of the inertia ellipsoid of Σ; and A, B, C the principal moments of inertia ($A < B < C$). A permanent rotation of Σ about C is a stable motion, and internal dissipation minimizes any possible wobbling of the spin axis in relation to the external torque. We assume that only the gravitational torque is present and that the body is rigid. Later on we consider the effects of internal dissipation. The gravitational torque acting on the satellite may be written

$$\vec{T} = \frac{3\mu}{r^3}\Big\{(C - B)(\vec{r}\cdot\vec{j})(\vec{r}\cdot\vec{k})\,\vec{i} + (A - C)(\vec{r}\cdot\vec{k})(\vec{r}\cdot\vec{i})\,\vec{j} + (B - A)(\vec{r}\cdot\vec{j})(\vec{r}\cdot\vec{i})\,\vec{k}\Big\} . \tag{1}$$

Here μ is the product of the gravitational constant and the mass of E, $\vec{r}$ is the unit vector from G to E, and r is the distance GE.

a) Suppose A = B; then the torque reduces to

$$\vec{T} = \frac{3\mu}{r^3}(C - A)\,(\vec{r}\cdot\vec{k})(\vec{r}\times\vec{k}) . \tag{2}$$

b) Suppose the rotational angular velocity ω_3 is much larger than the orbital angular velocity. We may then consider $\vec{r}$ and $\vec{k}$ constant during one rotational period; and averaging we find

$$\vec{T} = \frac{3\mu}{r^3}\left(C - \frac{A + B}{2}\right)(\vec{r}\cdot\vec{k})(\vec{r}\times\vec{k}) . \tag{3}$$

c) Suppose the body is moving in such a way that the axis of minimum moment of inertia is nearly pointing to E (stable relative equilibrium position); and suppose that B-A is small compared to C-A. Thus, neglecting second-order terms, we may write

$$\vec{T} = \frac{3\mu}{r^3}(C - A)(\vec{r}\cdot\vec{k})(\vec{r}\times\vec{k}) . \tag{4}$$

The perturbing torque, being generally small, may be averaged over an orbital period.

Let us call $\vec{\Omega}$ the unit vector along the ascending node of the orbit of E with respect to G; $\vec{n}$ the unit vector normal to the orbital plane; $\vec{m} = \vec{n} \times \vec{\Omega}$; and $\vec{c}_1$, $\vec{c}_2$, $\vec{N}$ an inertial reference frame centered at G. We have

$$\left.\begin{aligned} \vec{r} &= \cos(\omega + v)\,\vec{\Omega} + \sin(\omega + v)\,\vec{m} \\ r &= \frac{a(1 - e^2)}{1 + e\cos v} \end{aligned}\right\} , \tag{5}$$

where a, e, ω, and v are the semimajor axis, eccentricity, argument of perigee, and true anomaly, respectively, of the orbit of E with respect to G. Now let us consider the dynamical equation (with $\vec{K}$ the angular momentum vector):

$$\left.\begin{aligned} \frac{d\vec{K}}{dt} &= \frac{3\mu}{r^3}(C - A)(\vec{r} \cdot \vec{k})(\vec{r} \times \vec{k}) \\ \frac{\Delta\vec{K}}{P} &= \frac{1}{P}\int_0^P \frac{3\mu}{r^3}(C - A)(\vec{r} \cdot \vec{k})(\vec{r} \times \vec{k})\, dt \end{aligned}\right\} , \tag{6}$$

where P is the orbital period of E about G. We may also write

$$\frac{\Delta\vec{K}}{P} = \frac{1}{P}\int_0^{2\pi} \frac{3\mu}{r^3}(C - A)(\vec{r} \cdot \vec{k})(\vec{r} \times \vec{k}) \frac{dv}{\dot{v}} . \tag{7}$$

Since

$$r^2\dot{v} = h = \frac{2\pi a^2(1 - e^2)^{1/2}}{P} ,$$

we have

$$\frac{1}{\dot{v}} = \frac{Pr^2}{2\pi\, a^2(1 - e^2)^{1/2}} \quad ,$$

and consequently

$$\frac{\Delta\vec{K}}{P} = \frac{1}{2\pi a^2(1 - e^2)^{1/2}} \int_0^{2\pi} \frac{3\mu}{r} (C - A)(\vec{r} \cdot \vec{k})(\vec{r} \times \vec{k})\, dv. \tag{8}$$

Let us now introduce the unit vector $\vec{a}$ from G to the periapse and define the unit vector $\vec{b}$ as $\vec{b} = \vec{n} \times \vec{a}$. We have

$$\vec{r} = \vec{a} \cos v + \vec{b} \sin v \quad ,$$

and

$$\begin{aligned}
\frac{\Delta\vec{K}}{P} &= \frac{3\mu(C - A)}{2\pi a^3(1 - e^2)^{3/2}} \int_0^{2\pi} (1 + e \cos v)(k_a \cos v + k_b \sin v) \\
&\qquad [\vec{a} \sin v\, k_n - \vec{b} \cos v\, k_n + \vec{n}(k_b \cos v - k_a \sin v)]\, dv \\
&= \frac{3\mu(C - A)}{2\pi a^3(1 - e^2)^{3/2}}\, 2\pi \frac{k_n}{2} \left(k_b\, \vec{a} - k_a\, \vec{b}\right) \\
&= \frac{3}{2} \frac{\mu(C - A)}{a^3(1 - e^2)^{3/2}} (\vec{k} \cdot \vec{n})(\vec{k} \times \vec{n}) = \frac{3}{2} \frac{n^2}{(1 - e^2)^{3/2}} (C - A)(\vec{k} \cdot \vec{n})(\vec{k} \times \vec{n}) ,
\end{aligned} \tag{8'}$$

where n is the mean motion of E.

Now let us write the angular momentum vector

$$\vec{K} = A\omega_1 \vec{i} + B\omega_2 \vec{j} + C\omega_3 \vec{k} \quad .$$

As is usual in the treatment of gyrophenomena, we shall now assume as a first approximation that

$$\vec{K} = C\omega_3 \vec{k} \quad . \tag{9}$$

For the Moon A ≅ B ≅ C, and we may write

$$\vec{K} = (A - C)\omega_1 \vec{i} + (B - C)\omega_2 \vec{j} + C(\omega_1 \vec{i} + \omega_2 \vec{j} + \omega_3 \vec{k}) \quad .$$

Neglecting small terms of the order of 3×10^{-6} of the main term leaves

$$\vec{K} = C\omega \vec{k}_1 \quad .$$

By writing the equation of motion in the form

$$\frac{d}{dt}\left(C\omega \vec{k}_1 \right) = \frac{3}{2} \frac{n^2(C - A)}{(1 - e^2)^{3/2}} (\vec{k}_1 \cdot \vec{n})(\vec{k}_1 \times \vec{n}) \quad , \tag{10}$$

we neglect only very small terms. From now on we shall call $\vec{k}_1 = \vec{k}$ and go back to the fundamental equation

$$C\omega \frac{d\vec{k}}{dt} = \frac{3}{2} \frac{n^2(C - A)}{(1 - e^2)^{3/2}} (\vec{k} \cdot \vec{n})(\vec{k} \times \vec{n}) \quad . \tag{11}$$

The derivative $\frac{d\vec{k}}{dt}$ is naturally evaluated with respect to an invariable system.

It is much simpler to study the motion of $\vec{k}$ with respect to a moving reference system, that is, the rotating system $\vec{N}$, $\vec{\Omega}$, $\vec{N} \times \vec{\Omega} = \vec{M}$. Let us call $\dot{\Omega}\vec{N}$ the angular velocity of this system around $\vec{N}$. The sign of $\dot{\Omega}$ is positive if the nodal line is advancing and negative if (as for a low-inclination Earth satellite* or for the Moon) the nodal line is regressing.

The preceding equation may be rewritten as

$$C\omega\left(\frac{d\vec{k}}{dt} + \dot{\Omega}\vec{N} \times \vec{k}\right) = \frac{3}{2}\,\frac{n^2(C - A)}{(1 - e^2)^{3/2}}\,(\vec{k}\cdot\vec{n})(\vec{k} \times \vec{n})\ .$$

Dividing by $C\omega$ we may write

$$\frac{d\vec{k}}{dt} = -\dot{\Omega}\vec{N} \times \vec{k} + \alpha(\vec{k}\cdot\vec{n})(\vec{k} \times \vec{n})\ , \tag{12}$$

where

$$\alpha = \frac{3}{2}\,\frac{n^2}{(1 - e^2)^{3/2}}\,\frac{C - A}{C\omega}\ .$$

Let us now introduce the inclination i of the orbit. We have

$$\vec{n} = -\sin i\,\vec{M} + \cos i\,\vec{N}$$

$$\vec{k} = k_x\,\vec{\Omega} + k_y\,\vec{M} + k_z\,\vec{N}$$

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*For an Earth satellite we have

$$\Delta\Omega/\mathrm{rev} = -3\pi\frac{J_2}{p^2}\cos i$$

where $p = a(1 - e^2)$, $J_2 = 1.083 \times 10^{-3}\,R_0$, and $R_0 \approx 6,378$ km.

and the following three equations:

$$\left.\begin{aligned}
\frac{dk_x}{dt} &= \dot{\Omega} k_y + a k_y k_z \cos 2i + \frac{a}{2} \sin 2i \,(k_z^2 - k_y^2) \\
\frac{dk_y}{dt} &= -\dot{\Omega} k_x - a k_x k_z \cos^2 i + \frac{a}{2} \sin 2i \, k_x k_y \\
\frac{dk_z}{dt} &= -\frac{a}{2} k_x k_z \sin 2i + a k_x k_y \sin^2 i
\end{aligned}\right\} \quad . \tag{13}$$

The path of the vertex of $\vec{k}$ on the unit sphere

$$k_x^2 + k_y^2 + k_z^2 = 1$$

is determined by the quadratic integral

$$a \cos^2 i \, k_z^2 + a \sin^2 i \, k_y^2 - 2a \sin i \cos i \, k_y k_z + 2\dot{\Omega} k_z = H. \tag{14}$$

2. Relative equilibrium configuration of the $\vec{k}$ axis

Let us first consider the relative equilibrium configuration. It is easily shown that the only equilibrium configurations are in the $\vec{M}$, $\vec{N}$ plane. That is

$$\left.\begin{aligned}
& \overset{\circ}{k}_x = 0 \, , \\
& \dot{\Omega} \overset{\circ}{k}_y + a \overset{\circ}{k}_y \overset{\circ}{k}_z \cos 2i + \frac{a}{2} \sin 2\, i \,(\overset{\circ}{k}{}_z^2 - \overset{\circ}{k}{}_y^2) = 0 .
\end{aligned}\right\} \tag{15}$$

Let us put

$$\overset{\circ}{k}_y = \sin \lambda , \quad \overset{\circ}{k}_z = \cos \lambda \quad . \tag{16}$$

We have

$$2\dot{\Omega} \sin\lambda + a \sin 2(i + \lambda) = 0 \quad . \tag{17}$$

This relative equilibrium configuration for $\vec{k}$, naturally corresponds to a regular precessional motion of the body, the precessional angle being λ, and the angular velocity of precession being $\dot{\Omega}$. The two Poinsot cones (the body-fixed and the space-fixed) are easily computed.

3. Precessional and nutational motion

Equation (14) may be written in the following form

$$a\,(k_y \sin i - k_z \cos i)^2 + 2\dot{\Omega} k_z = H \, . \tag{18}$$

It represents a family of parabolas in the $\vec{M}$-$\vec{N}$ plane determined by the value of H. The axis of these parabolas is independent of H as is the latus rectum. The equation of the axis is

$$k_y \sin i - k_z \cos i - \cos i \frac{\dot{\Omega}}{a} = 0 \quad . \tag{19}$$

The intersection with the z axis is at $y = -\frac{\dot{\Omega}}{a}$; the inclination is i. The length 2p of the latus rectum is

$$2p = \left| \frac{2 \sin i \, \dot{\Omega}}{a} \right| . \tag{20}$$

The convexity of the parabola is determined by the sign of $\frac{\dot{\Omega}}{\alpha}$; if $\frac{\dot{\Omega}}{\alpha}$ is positive, the convexity of the parabola is directed toward negative y; if $\frac{\Omega}{\alpha}$ is negative (as for the Moon), the convexity is directed toward positive y. The relative motion is in some cases a pure regular precessional motion but, in many cases, it may be very different from regular precessional motion.

4. The Moon

For the Moon (case c) of p. 3) we have, assuming the unit of time to be the sidereal period of the Moon's orbital motion,

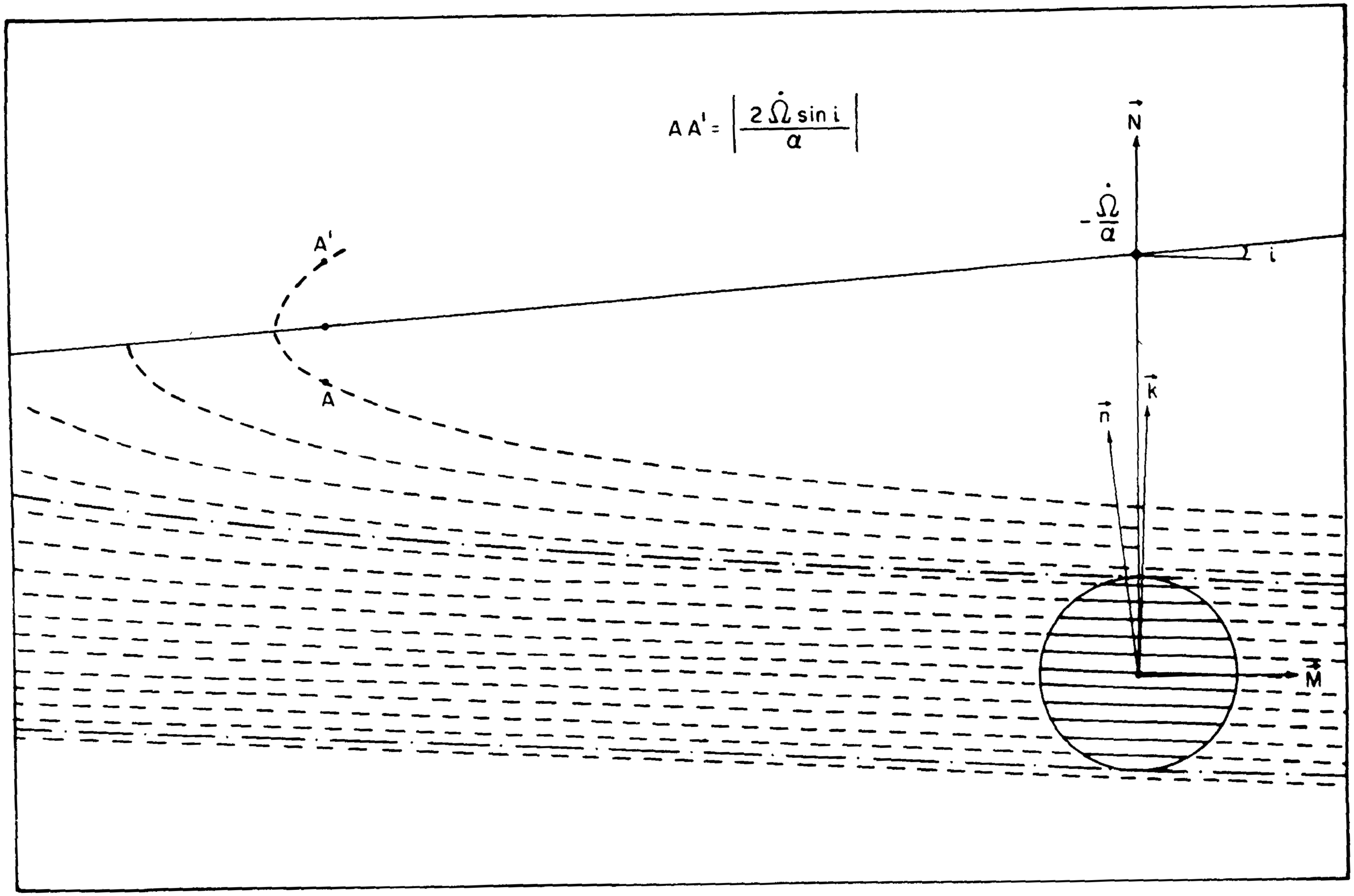
$$\begin{cases} \Omega = -2.525 \times 10^{-2} \text{ rad/rev} \quad , \\ \alpha = 10.188 \dfrac{C - A}{C} \text{ rad/rev} \quad , \\ i = 5^\circ 8' 43'' \quad . \end{cases}$$

From equation (17), assuming the Moon's axis of rotation to be in the relative equilibrium configuration, we have

$$10.188 \frac{C - A}{C} = -2.525 \times 10^{-2} \frac{\sin \lambda}{\sin 2(i + \lambda)} \ .$$

From this relation, using the observed values of λ and i, we have for $\frac{C - A}{C}$ the usual value 6.3×10^{-4}.

In Figure 1 we have shown the unit sphere and the projection on the y-z plane of the possible motion of the $\vec{k}$ axis. In this case ($\widehat{\vec{k}, \vec{N}}$ small and also i small) we may very easily compute the angular velocity of the relative precessional motion about this relative equilibrium configuration, that is, the angular velocity of nutation about the absolute precessional motion of the Moon defined



MOON

Figure 1. Paths of the vertex of the unit vector along the spin axis as intersection of parabolic cylinders with the unit sphere (in the case of the Moon).

by Cassini's second and third laws. Neglecting small quantities, we find for this angular velocity $\dot{\Omega} + \alpha$. The absolute velocity $\frac{d\vec{k}}{dt}$ will be to a first approximation

$$\frac{d\vec{k}}{dt} = \dot{\Omega}\vec{N} \times \vec{k} + \theta_1 \left\{ \cos\left[\Phi_0 + (\dot{\Omega} + \alpha)\,t\right]\vec{c}_1 + \sin\left[\Phi_0 + (\dot{\Omega} + \alpha)\,t\right]\vec{c}_2 \right\},$$

θ_1 and Φ_0 being the amplitude and the phase of the nutation. We may also write

$$\frac{d\vec{k}}{dt} = [\lambda \cos(\dot{\Omega} t + \Psi_0) + \theta_1 \cos[\Phi_0 + (\dot{\Omega} + \alpha)\,t]\,\vec{c}_1 +$$

$$[\lambda \sin(\dot{\Omega} t + \Psi_0) + \theta_1 \sin[\Phi_0 + (\dot{\Omega} + \alpha)\,t]]\,\vec{c}_2 \quad .$$

We have consequently

$$\left|\frac{d\vec{k}}{dt}\right| = [\lambda^2 + \theta_1^2 + 2\lambda\,\theta_1 \cos(\alpha t + \Psi_0 - \Phi_0)]^{1/2} \quad .$$

For small θ_1, we have the average value of $\frac{d\vec{k}}{dt}$ roughly equal to λ, but the maximum value is $\lambda + \theta_1$ and the minimum value is $\lambda - \theta_1$.

We may here observe that internal friction produces a dissipation of energy, because both the magnitude of the angular velocity and the instantaneous axis of rotation change. In fact, in a simple way, let us consider two neighboring points P and Q, in the body, and let P′ and Q′ be the projection of P and Q on the instantaneous axis of rotation about the center of mass. The accelerations of the two points in the motion relative to the center of mass, are

$$- \omega^2 \overrightarrow{P'P} + \frac{d\vec{\omega}}{dt} \times \overrightarrow{OP} ; \qquad - \omega^2 \overrightarrow{Q'Q} + \frac{d\vec{\omega}}{dt} \times \overrightarrow{OQ} \quad .$$

When the motion is a uniform rotational motion, the second term in each expression goes to zero and the inertial forces correspond to a static field of forces, and no relative motion occurs. When the angular velocity vector $\vec{\omega}$ changes, internal forced elastic vibration takes place under the action of internal friction. If the internal friction dissipation is linear with the amplitude of the forced vibration, any regular precession-nutation will dissipate the same amount of energy as in the pure precessional motion; but if the dissipation is an increasing function of the amplitude with a positive second derivative, the energy dissipation will also increase if nutation is present. For any nonrigid body (elastic) any precessional motion will be damped out and a pure rotational motion will finally result. Since here a pure rotational motion is dynamically impossible, the forced precessional motion will correspond to a minimum dissipation of energy by internal friction. We conclude by saying that the second and third laws correspond to a motion that, among all possible dynamical solutions, minimizes the internal dissipation.

5. An artificial Earth satellite

We shall consider a 24-hour satellite (case b) of p. 3) at an inclination of 30°. We suppose the inertial ellipsoid to be rotationally symmetric about the axis of maximum moment of inertia. In addition, we suppose that the magnetic torque is minimized by suitable design. We have

$$\dot{\Omega} = -2.5 \times 10^{-3} \text{ rad/day} \qquad \alpha = 6\pi^2 \frac{C - A}{C\omega} \text{ rad/day}$$

where ω is the spin velocity about the symmetric axis.

Suppose $\frac{C - A}{C} = \frac{2}{3}$ and $\omega = 10^4$ rad/day; we then have

$$-\frac{\dot{\Omega}}{a} = 0.7, \quad \left| \frac{2\dot{\Omega}}{a} \sin i \right| = 0.7 \quad .$$

The path of the spin axis is represented in Figure 2. We note that there are three stable and one unstable relative equilibrium configurations, as for the case of Mercury. We considered the satellite case in order to give a very clear picture of the behavior of the path of $\vec{k}$ on the sphere.

6. Mercury

For Mercury, assuming B-A negligible with respect to C-A, we have (case a) of p. 3)

$$\left.\begin{aligned} i &\approx 7^\circ \\ \dot{\Omega} &= 6''\!.2/\text{year} \cong 8.1 \times 10^{-8} \text{ rad/day} \\ a &= 8.1 \times 10^{-2} \frac{C - A}{C} \text{ rad/day} \end{aligned}\right\} ,$$

and hence the following result:

$\frac{C - A}{C}$	10^{-3}	10^{-4}
$-\frac{\dot{\Omega}}{a}$	-10^{-3}	-10^{-2}
$\left\lvert \frac{2\dot{\Omega} \sin i}{a} \right\rvert$	2.6×10^{-4}	2.6×10^{-3}

The configurations of the possible paths of the rotational axis of Mercury are represented in Figure 3 . If Mercury is locked-in as the Moon is, Cassini's laws make it possible to infer the position of the axis of rotation. In this case, three equilibrium configurations are compatible, but for one of them, even if actually stable, the region of stability is very narrow. Moreover, the argument given above for the Moon needs to be revised from the point of view of the effects of internal dissipation. Remembering that Mercury rotates in a direct

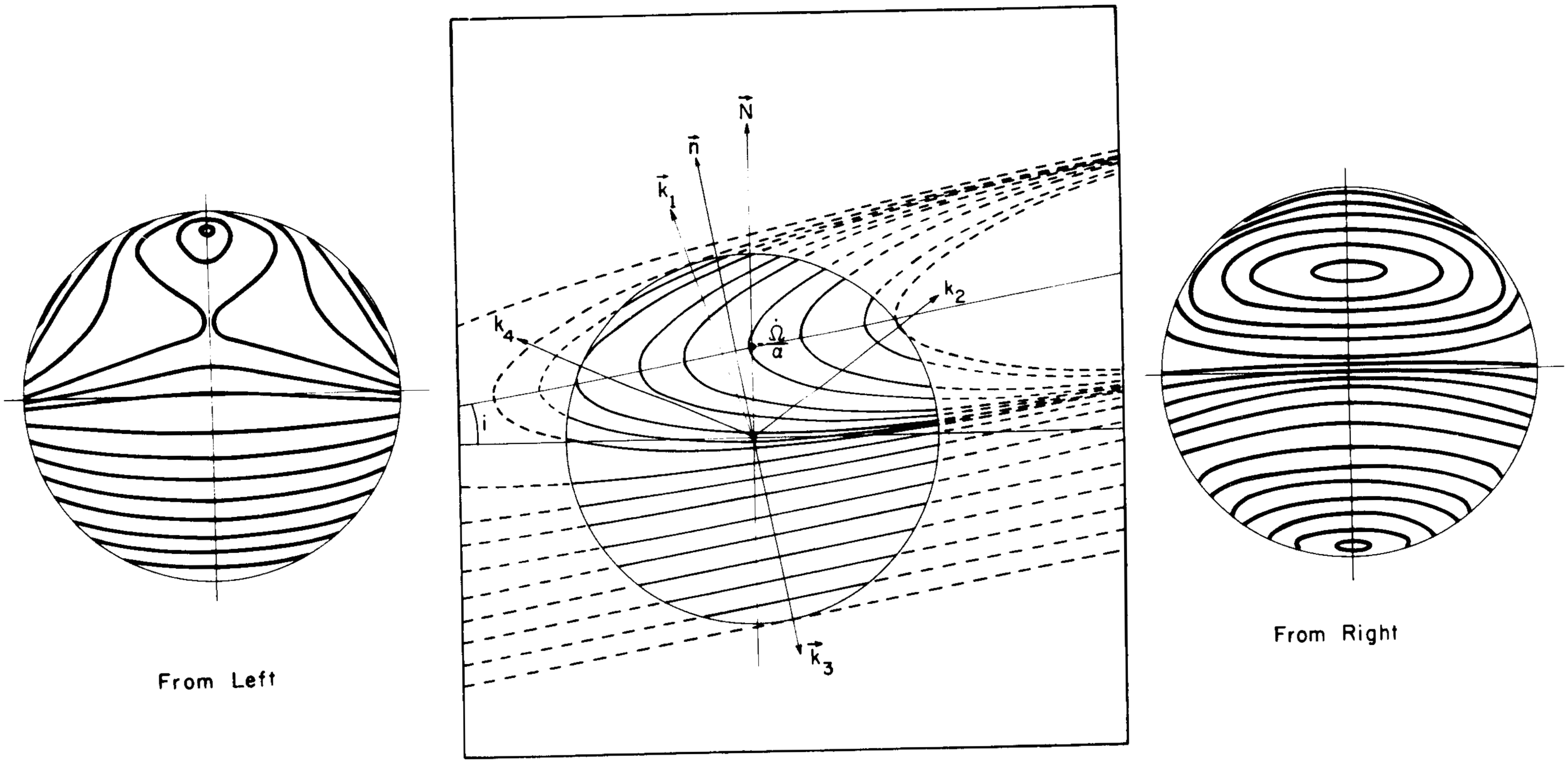
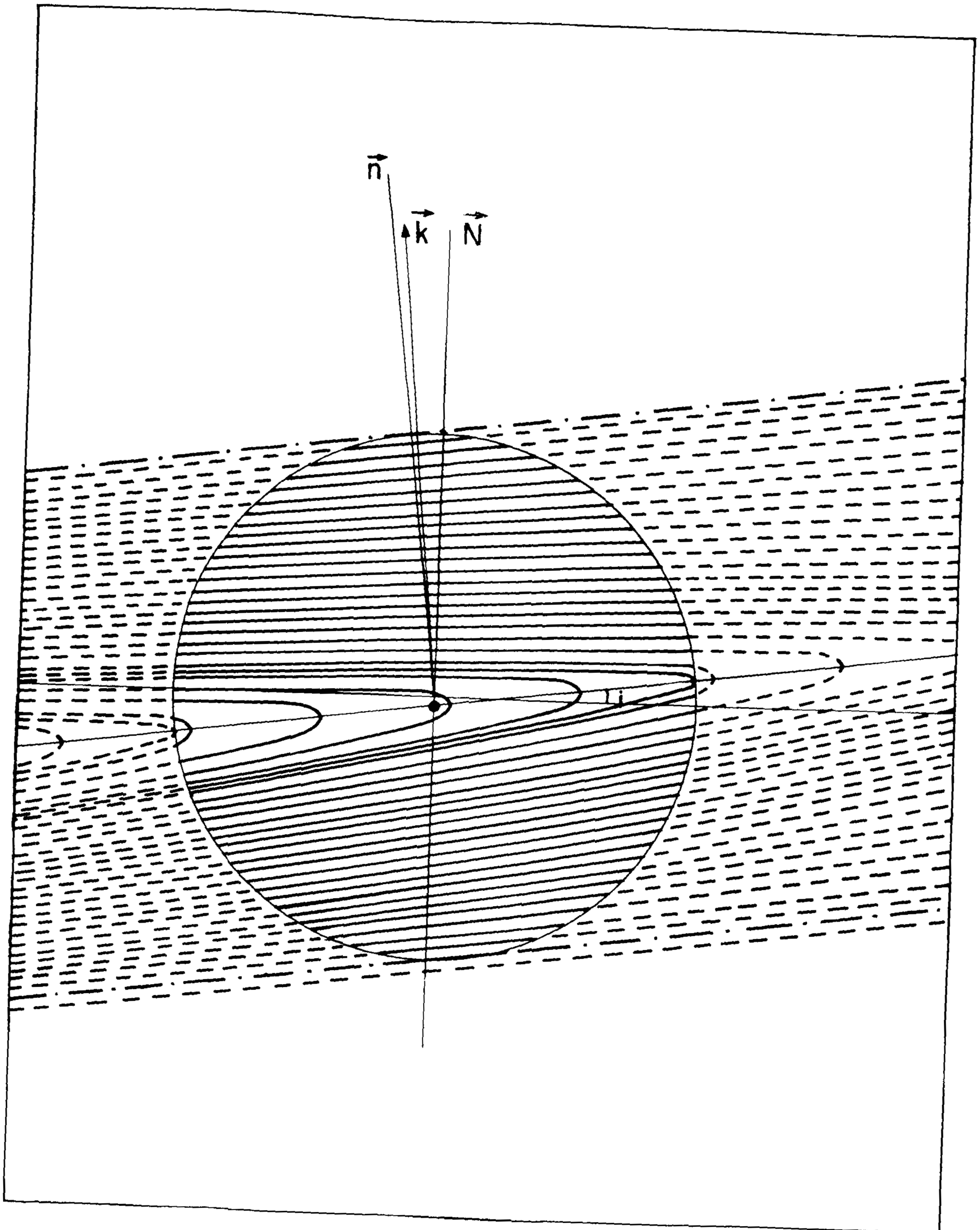


Figure 2. Projection of the relative paths of the vertex of the unit vector along the spin axis for a particular 24-hour artificial Earth satellite.



MERCURY

Figure 3. Paths of the vertex of the unit vector along the spin axis as intersection of parabolic cylinders with the unit sphere (in the case of Mercury).

fashion, we shall compute only the relative equilibrium configuration with $\vec{k}$ very near $\vec{n}$. In fact, we find for the angle between $\vec{n}$ and $\vec{k}$, the result 4'28" with $\frac{C - A}{C} = 10^{-4}$ and the result 26"8 for $\frac{C - A}{C} = 10^{-3}$.

Since the precession of the node with respect to an invariable system is at present very slow, we cannot be sure that Mercury has had enough time to settle down into the final configuration in accordance with the first as well as with the second and third of Cassini's laws. Naturally this discussion is true only if A - B is small compared to C - A and C - B since otherwise none of the three cases we have considered at the beginning is consistent with the Mercury case.

There is no need to emphasize the importance of the determination of the orientation of the rotational axis of Mercury. We may determine this orientation within the next few years by obtaining accurate delay-doppler radar maps of Mercury's surface.

7. Iapetus

This very simple theory may be applied to many of the satellites that are in some way locked-in to the primary. Since G. D. Cassini discovered Iapetus (Oct. 1671), we want to end this paper by devoting a few words to this particular inhabitant of the sky. Actually Iapetus is peculiar from two points of view. Revolving with the same face toward Saturn, Iapetus' brightness varies during a rotation by a factor of 6 (Whipple, 1963). On the other hand, the perturbations due to the Sun and the oblateness of Saturn cause the pole of the orbit to describe a circular path about a mean pole that lies in the great circle passing through the pole of Saturn's orbital plane and of Saturn's equator. It takes the pole of Iapetus about 3, 000 years to describe its curve with a mean radius about 8° (Tisserand, 1891; Jeffreys, 1953). Taking for granted all these data, we will find out what the theory gives in this case assuming case c)of p. 3. Thus, we have

$$i = 8^\circ , \quad \dot{\Omega} = \frac{2\pi}{3000} \text{ years}^{-1} , \quad n \approx \frac{2\pi(366)}{79.33} , \quad \omega = n$$

$$\alpha = \frac{3}{2} n \frac{C - A}{C} = \frac{3}{2} \frac{2\pi(366)}{79.33} \frac{C - A}{C} = 2\pi(6.9) \frac{C - A}{C} ,$$

and

$$\frac{\dot{\Omega}}{\alpha} = - \frac{1}{3000 \times 6.9} \frac{C}{C - A} = - 4.83 \times 10^{-5} \frac{C}{C - A} ,$$

and

$$\left| \frac{2\dot{\Omega} \sin i}{\alpha} \right| = 1.3 \times 10^{-5} \frac{C}{C - A}$$

Suppose, for instance,that, $\frac{C}{C - A} = 10^4$; we then have

$$\frac{\dot{\Omega}}{\alpha} = - 0.483 , \quad \left| \frac{2\dot{\Omega} \sin i}{\alpha} \right| = 0.13 .$$

We examine only the relative equilibrium configuration $\vec{k}_1$. As a first approximation we have $\lambda = -2i$. If the body, as suggested by Whipple, is very irregular, we should have $\frac{C}{C - A}$ much smaller and $\vec{k}_1$ will be close to n.

Acknowledgments

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