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on

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APPLICATIONS OF CALCULUS OF VARIATIONS

TO TRAJECTORY ANALYSIS

Submitted to

National Aeronautics and Space Administration

Marshall Space Flight Center

Huntsville, Alabama

by

M. G. Boyce, Principal Investigator

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INTRODUCTION AND SUMMARY

During the year 1965 two reports were made at the two contractor conferences on guidance and space flight theory held at Huntsville in February and August. They were entitled "The Multistage Weierstrass and Clebsch Conditions with Some Applications to Trajectory Optimization," by M. G. Boyce, and "Applications of Multistage Calculus of Variations Theory to Two and Three Stage Rocket Trajectory Problems," by G. E. Tyler. Also a paper was contributed to the Progress Report No. 7 on Studies in the Fields of Space Flight and Guidance Theory, NASA TM X - 53292, pp. 7 - 31, on "Necessary Conditions for a Multistage Bolza - Mayer Problem Involving Control Variables and Having Inequality and Finite Equation Constraints," by M. G. Boyce and J. L. Linnstaedter. A summary of the results in these reports, omitting proofs of theorems, is given in this annual report. Additional work has been done recently on gradient methods and on sufficient conditions, but results are as yet incomplete.

The first section of this report contains a modification of the Denbow multistage calculus of variations theory, allowing for discontinuities at stage boundaries. This modification is an intermediate step in passing from the classical theory to a form readily applicable to trajectory optimization.

The second section summarizes extensions of the multistage theory to problems involving control variables and having inequality and finite equation constraints. The Mayer formulation is used, and differential constraints are taken in normal form since in trajectory problems the equations of motion are in such form.

A three stage re-entry problem is treated in Section III. This example serves to partially illustrate the theory of Section II. To avoid computational complexity, simple intermediate point constraints are assumed and first order approximation to gravitational attraction is used.

SECTION I. ON MULTISTAGE PROBLEMS HAVING DISCONTINUITIES AT STAGE BOUNDARIES

Discontinuities will be allowed in the functions appearing in the differential equation constraints and in the dependent variable coordinates defining admissible paths. Let t be the independent variable. For fixed p, define a set of variables $(t_o, t_1, \ldots, t_p)$ to be a partition set if and only if $t_o < t_1 < \ldots < t_p$. Let I denote the interval $t_o \leq t \leq t_p$ and I_a the subinterval $t_{a-1} \leq t < t_a$ for $a = 1, \ldots, p-1$ and $t_{a-1} \leq t \leq t_a$ for $a = p$. Let z(t) denote the set of functions $(z_1(t), \ldots, z_N(t))$, where each $z_\alpha(t)$, $\alpha = 1, \ldots, N$, is continuous on I except possibly at partition points $t_1, \ldots t_{p-1}$. At these points right and left limits $z_\alpha(t_1^-)$, $z_\alpha(t_1^+)$, $\ldots z_\alpha(t_{p-1}^+)$ are assumed to exist and we let $z_\alpha(t_b) = z_\alpha(t_b^+)$, $b = 1, \ldots, p-1$.

The problem will be to find in a class of admissible arcs

$$z(t), \qquad (t_o, \ldots, t_p), \qquad t_o \leq t \leq t_p,$$

satisfying differential equations

$$(1) \qquad \phi_\beta^a\,(t,z,\dot{z}) = 0, \qquad t \text{ in } I_a, \qquad \beta = 1,\ldots,M < N,$$

and end and intermediate point conditions

$$(2) \qquad f_\gamma(t_o,\ldots,t_p,z(t_o),z(t_1^-),z(t_1^+),\ldots,z(t_p)) = 0, \qquad \gamma = 1, \ldots, K \leq (N+1)(p+1),$$

$$(3) \qquad z_\alpha(t_b^+) - z_\alpha(t_b^-) - d_{\alpha b} = 0$$

one that will minimize

$$f_o(t_o,\ldots,t_p,\ z(t_o),\ z(t_1^-),\ z(t_1^+),\ \ldots,\ z(t_p)).$$

Let R_a' be an open connected set in the 2N+1 dimmensional $(t,z,\dot{z})$ space whose projection on the t-axis contains I_a. The functions ϕ_β^a are required to have continuous third partial derivatives in R_a' and each matrix $\left\| \phi^a_{\beta \dot{z}_\alpha} \right\|$ is assumed of rank M in R_a'.

Let S' denote an open connected set in the 2Np+p+1 dimmensional space of points $(t_o,\ldots,t_p,z(t_o),z(t_1^-),z(t_1^+),\ldots,z(t_p))$ in which the functions f_ρ, $\rho = 0, 1, \ldots, K$ have continuous third partial derivatives and the matrix

$$(4) \qquad \left\| f_{\rho t_o} \quad f_{\rho t_b} \quad f_{\rho t_p} \quad f_{\rho z_\alpha(t_o)} \quad f_{\rho z_\alpha(t_b^-)} \quad f_{\rho z_\alpha(t_b^+)} \quad f_{\rho z_\alpha(t_p)} \right\|$$

is of rank K+1.

An <u>admissible set</u> is a set $(t,z,\dot z)$ in R_a' for some a=1,...,p. An <u>admissible subarc</u> C_a' is a set of functions z(t), t on I_a, with each $(t,z,\dot z)$ an admissible set and such that z(t) is continuous and $\dot z(t)$ is piecewise continuous on I_a. An admissible arc E' is a partition set $(t_o,\ldots,t_p)$ together with a set of admissible subarcs C_a', a = 1, ..., p, such that the set $(t_o,\ldots,t_p,z(t_o),z(t_1^-),z(t_1^+),\ldots,z(t_p))$ is in S'.

<u>Multiplier Rule</u>. <u>An admissible arc</u> E' <u>that satisfies equations</u> (1), (2), (3) <u>is said to satisfy the multiplier rule if there exist constants</u> e_ρ <u>not all zero and a function</u>

$$F(t,z,\dot z,\lambda) = \lambda_\beta \phi_\beta^a(t,z,\dot z), \ t \ \text{in} \ I_a,$$

<u>with multipliers</u> $\lambda_\beta(t)$ <u>continuous except possibly at corners or discontinuities of</u> E', <u>where left and right limits exist</u>, <u>such that the following equations hold</u>:

$$(5) \qquad F_{\dot z_\alpha} = \int_{t_{a-1}}^{t} F_{z_\alpha}\,dt + c_\alpha^a, \quad t \ \text{in} \ I_a,$$

$$e_\rho f_{\rho t_o} + \Big[\dot z_\alpha F_{\dot z_\alpha}\Big]^{t_o} = 0,$$

$$e_\rho f_{\rho t_b} + \Big[\dot z_\alpha F_{\dot z_\alpha}\Big]_{t_b^-}^{t_b^+} = 0,$$

$$e_\rho f_{\rho t_p} + \Big[\dot z_\alpha F_{\dot z_\alpha}\Big]_{t_p} = 0,$$

$$e_\rho f_{\rho z_\alpha(t_o)} - \Big[F_{\dot z_\alpha}\Big]^{t_o} = 0,$$

$$e_\rho (f_{\rho z_\alpha}(t_b^+) + f_{\rho z_\alpha}(t_b^-) - \Big[F_{\dot z_\alpha}\Big]_{t_b^-}^{t_b^+} = 0,$$

$$e_\rho f_{\rho z_\alpha}(t_p) - \Big[F_{\dot z_\alpha}\Big]_{t_p} = 0.$$

<u>Every minimizing arc must satisfy the multiplier rule.</u>

An <u>extremal</u> is defined to be an admissible arc and set of multipliers

$$z_\alpha(t),\ (t_o,\dots,t_p),\ \ \lambda_\beta(t),\ \ t_o \le t \le t_p,$$

satisfying equations (1) and (5) and such that the functions $\dot z_\alpha(t)$, $\lambda_\beta(t)$ have continuous first derivatives except possibly at partition points, where finite left and right limits exist. An extremal is <u>non-singular</u> in case the determinant

$$\begin{Vmatrix} F_{\dot z_\alpha \dot z_\eta} & \phi_{\delta \dot z_\alpha} \\ \phi_{\beta \dot z_\eta} & 0 \end{Vmatrix} \qquad \begin{matrix} \alpha, \eta = 1, \dots, N \\ \beta, \delta = 1, \dots, M \end{matrix}$$

is different from zero along it. An admissible arc with a set of multipliers satisfying the multiplier rule is called <u>normal</u> if $e_o = 1$. With this value of e_o the set of multipliers is unique.

<u>Weierstrass Condition</u>. <u>An admissible arc</u> E' <u>with a set of multipliers</u> $\lambda_\beta(t)$ <u>is said to satisfy the Weierstrass condition if</u>

$$(t,z,\dot z,\lambda,\dot Z) = F(t,z,\dot Z,\lambda) - F(t,z,\dot z,\lambda) - (\dot Z'_\alpha - \dot z_\alpha)F_{\dot z_\alpha}(t,z,\dot z,\lambda) \ge 0$$

<u>holds at every element</u> $(t,z,\dot z,\lambda)$ <u>of</u> E' <u>for all admissible sets</u> $(t,z,\dot Z)$ <u>satisfying the equations</u> $\phi_\beta^a = 0$. <u>Every normal minimizing arc must satisfy the Weierstrass condition</u>.

<u>Clebsch Condition</u>. <u>An admissible arc</u> E' <u>with a set of multipliers</u> $\lambda_\beta(t)$ <u>is said to satisfy the Clebsch condition if</u>

$$F_{\dot z_\alpha \dot z_\eta}(t,z,\dot z,\lambda)\,\pi_\alpha \pi_\eta \ge 0$$

holds at every element $(t,z,\dot{z},\lambda)$ *of* E' *for all sets* $(\pi_1,\ldots,\pi_N)$ *satisfying the equations*

$$\phi^a_{\beta\dot{z}_\alpha}(t,z,\dot{z})\,\pi_\alpha = 0.$$

Every normal minimizing arc must satisfy the Clebsch condition.

SECTION II. ON MULTISTAGE PROBLEMS INVOLVING CONTROL VARIABLES AND HAVING INEQUALITY AND FINITE EQUATION CONSTRAINTS

By the introduction of new variables and by notational transformations the theory of Section I can be utilized to establish necessary conditions for the more general formulation of this section. As before, let t be the independent variable and define a set of variables $(t_o, \ldots, t_p)$ contained in the range of t to be a partition set if and only if $t_o < t_1 < \ldots < t_p$. Let I denote the interval $t_o \leq t \leq t_p$, and let I_a denote the sub-interval $t_{a-1} \leq t < t_a$ for $a = 1, \ldots, p - 1$ and $t_{a-1} \leq t \leq t_a$ for $a = p$.

Let $x(t)$ denote the set of functions $(x_1 (t), \ldots, x_n (t))$. For each i, $i = 1, \ldots, n$, assume $x_i(t)$ to be continuous on I except possibly at partition points t_b, $b = 1, \ldots, p - 1$, where finite left and right limits exist; denote these limits by $x_i(t_b^-)$ and $x_i(t_b^+)$, respectively. The amount of discontinuity of each member of $x(t)$ at each partition point will be assumed known, and we write

$$x_i(t_b^+) - x_i(t_b^-) - d_{ib} = 0,$$

with each d_{ib} a known constant. Also let $x_i(t_b) = x_i(t_b^+)$. Thus $x_i(t)$ is continuous at t_b if and only if $d_{ib} = 0$.

Let $y(t)$ denote the set $(y_1(t), \ldots, y_m(t))$, where $y_j(t)$ is piecewise continuous on I, $j = 1, \ldots, m$, finite discontinuities being allowed between, as well as at, partition points. In the formulation of the problem the $y_j(t)$ will occur only as undifferentiated variables and will not occur in the function to be minimized nor in the end and intermediate point constraints. Such variables are called control variables, while the $x_i(t)$ are called state variables.

The problem is to find in a class of admissible arcs

$$x(t), \qquad y(t), \qquad (t_o, \ldots, t_p), \qquad t_o \leq t \leq t_p,$$

which satisfy differential equations

$$\dot{x}_i = L_i^a(t,x,y,), \quad t \text{ in } I_a, \quad a = 1, \ldots, p, \quad i = 1, \ldots, n,$$

finite equations

$$M_g^a(t,x,y) = 0, \qquad g = 1, \ldots, q,$$

inequalities

$$N_h^a(t,x,y) \geq 0, \qquad h = 1, \ldots, r, \qquad q + r \leq m,$$

and end and intermediate point conditions

$$J_k(t_o, \ldots, t_p, x(t_o), x(t_1^-), x(_1^+), \ldots, x(t_p)) = 0,$$

$$k = 1, \ldots, s < (n + 1) \quad (p + 1),$$

$$x_i(t_b^+) - x_i(t_b^-) - d_{ib} = 0, \quad b = 1, \ldots, p - 1,$$

one that will minimize

$$J_o(t_o, \ldots, t_p, x(t_o), x(t_1^-), x(t_1^+), \ldots, x(t_p)).$$

In order to state precisely the properties of the functions involved in the problem, let R_a be an open connected set in the $m + n + 1$ dimensional (t,x,y) space whose projection on the t-axis contains the interval I_a, and let S be an open connected set in the $2np + p + 1$ dimmensional space of points

$$(t_o, \ldots, t_p, x(t_o), x(t_1^-), x(t_1^+), \ldots, x(t_p)).$$

The functions L_i^a, M_g^a, N_h^a are assumed continuous with continuous partial derivatives through those of third order in R_a, and J_o, J_k are to have such continuity properties in S. For each a, the matrix

$$\begin{Vmatrix} M^a_{gy_j} & 0 \\ N^a_{hy_j} & D_1^a \end{Vmatrix}$$

is assumed of rank $q + r$ in R_a, where D_1^a is an r by r diagonal matrix with $N_1^a, \ldots, N_r^a$ as diagonal elements. The matrix

$$\begin{Vmatrix} J_{ct_o} & J_{ct_b} & J_{ct_p} & J_{cx_i(t_o)} & J_{cx_i(t_b^-)} & J_{cx_i(t_b^+)} & J_{cx_i(t_p)} \end{Vmatrix}, \quad c = 0, \ldots, s,$$

is assumed of rank $s + 1$ in S.

An <u>admissible set</u> is a set (t,x,y) in R_a for some $a = 1, \ldots, p$. An <u>admissible sub-arc</u> C_a is a set of functions $x(t)$, $y(t)$, t on I_a, with each (t,x,y) admissible, and such that $x(t)$ is continuous and $\dot{x}(t)$, $y(t)$ are piecewise continuous on I_a. An <u>admissible arc</u> is a partition set $(t_o, \ldots, t_p)$ together with a set of admissible sub-arcs C_a, $a = 1, \ldots, p$, such that the set $(t_o, \ldots, t_p, x(t_o), x(t_1^-), x(t_1^+), \ldots, x(t_p))$ is in S.

On introducing a generalized Hamiltonian function H as defined below and utilizing the normal form of the differential equation constraints, one can now apply the theory of Section I to obtain the following multiplier rule.

<u>The Multiplier Rule</u>

<u>An admissible arc E for which</u>

$$J_k(t_o, \ldots, t_p, x(t_o), x(t_1^-), x(t_1^+), \ldots, x(t_p)) = 0,$$

$$x_i(t_b^+) - x_i(t_b^-) - d_{ib} = 0,$$

<u>is said to satisfy the multiplier rule if there exists a function</u>

$$H(t,x,y,\lambda,\mu,\nu) = \lambda_i L_i^a - \mu_g M_g^a + \nu_h N_h^a,$$

<u>with multipliers</u> $\lambda_i(t)$, $\mu_g(t)$, $\nu_h(t)$ <u>continuous except possibly at partition points or corners of E, where finite left and right limits exist, such that for each t in</u> I_a, <u>$a = 1, \ldots, p$,</u>

$$(1) \quad \lambda_i = -\int_{t_{a-1}}^{t} H_{x_i}\,dt + c_i^a,\ H_{y_j} = 0,\ \dot{x}_i = L_i^a,\ M_g^a = 0,\ N_h^a \geq 0,$$

<u>and such that the transversality matrix</u>

$$(2) \quad \left\| \begin{matrix} H(t_o) & H(t_b^+) - H(t_b^-) & -H(t_p) & -\lambda_i(t_o) & -\lambda_i(t_b^+) + \lambda_i(t_b^-) & \lambda_i(t_p) \\ J_{ct_o} & J_{ct_b} & J_{ct_p} & J_{cx_i(t_o)} & J_{cx_i(t_b^+)} + J_{cx_i(t_b^-)} & J_{cx_i(t_p)} \end{matrix} \right\|$$

<u>is of rank $s + 1$</u>. <u>The multipliers</u> ν_h <u>are zero when</u> $N_h > 0$. <u>Every minimizing arc E must satisfy the multiplier rule</u>.

Between corners of a minimizing arc E the equations

$$\dot{x}_i = H_{\lambda_i}, \quad \dot{\lambda}_i = -H_{x_i}, \quad H_{y_j} = 0, \quad \nu_h H_{\nu_h} = 0 \quad \text{(not summed)}$$

hold and hence also

$$\frac{dH}{dt} = H_t .$$

Transversality Conditions for Normal Arcs

Under the usual normality assumptions, the transversality matrix can be put into a form having one fewer rows. This leads to the following statement of transversality conditions.

For a normal minimizing arc the transversality matrix

$$\left\| \begin{matrix} H(t_o)+J_{ot_o} & H(t_b^+)-H(t_b^-)+J_{ot_b} & -H(t_p)+J_{ot_p} & -\lambda_i(t_o)+J_{ox_i}(t_o) & -\lambda_i(t_b^+)+\lambda_i(t_b^-)+J_{ox_i}(t_b^+)+J_{ox_i}(t_b^-) & \lambda_i(t_p)+J_{ox_i}(t_p) \\ J_{kt_o} & J_{kt_b} & J_{kt_p} & J_{kx_i}(t_o) & J_{kx_i}(t_b^+)+J_{kx_i}(t_b^-) & J_{kx_i}(t_p) \end{matrix} \right\|$$

is of rank s.

Since the matrix is of order $s + 1$ by $(n+1)(p+1)$, the requirement that the rank be s imposes $(n+1)(p+1) - s$ conditions. This is one more condition than was imposed by (2), which was sufficient to determine the multipliers up to an arbitrary proportionality factor.

Weierstrass Condition

For a normal minimizing arc E the inequality

$$\lambda_i L_i(t,x,y) \geq \lambda_i L_i(t,x,Y)$$

must hold at each element (t,x,y,λ,μ,ν) of E for all admissible sets (t,x,Y) satisfying $M_g(t,x,Y) = 0$ and $N_h(t,x,Y) \geq 0$.

Clebsch Condition

For a normal minimizing arc E the inequality

$$H_{y_j y_e} \pi_j \pi_e \leq 0$$

must hold at each element (t,x,y,λ,μ,ν) *of* E *for all sets* $\pi_1, \ldots, \pi_m$ *satisfying* $M_{gy_j}(t,x,y)\pi_j = 0$ *and* $N_{hy_j}(t,x,y)\pi_j = 0$, *where in the last equation* h *ranges only over the subset of* $1, \ldots, r$ *for which* $N_h(t,x,y) = 0$.

For a normal minimizing arc the multipliers ν_h *are all non-negative*.

SECTION III. A THREE STAGE RE-ENTRY OPTIMIZATION PROBLEM

In this section the theory of Section II is applied to a three stage re-entry problem. Since it is primarily an illustrative example, certain simplifying assumptions are made. In particular, the vehicle is assumed to be a particle of variable mass, with thrust magnitude proportional to mass flow rate and thrust direction subject to instantaneous change. Moreover, external forces are required to be functions of position only, while the earth is assumed spherically symmetrical and nonrotating with respect to the coordinate system of the vehicle. Finally, motion is restricted to two dimensions, gravitational acceleration is approximated by first order terms, and air resistance is neglected.

The foregoing conditions allow the motion of the vehicle to be described by the following equations:

$$\dot{u} = \begin{cases} -a^2x + cB_1m^{-1}\cos\theta, & t_o \leq t < t_1, \\ -a^2x, & t_1 \leq t < t_2, \\ -a^2x + cB_3m^{-1}\cos\theta, & t_2 \leq t \leq t_3, \end{cases}$$

$$\dot{v} = \begin{cases} -g_o + 2a^2y + cB_1m^{-1}\sin\theta, & t_o \leq t < t_1, \\ -g_o + 2a^2y, & t_1 \leq t < t_2, \\ -g_o + 2a^2y + cB_3m^{-1}\sin\theta, & t_2 \leq t \leq t_3, \end{cases}$$

$$\dot{x} = u, \quad t_o \leq t \leq t_3,$$

$$\dot{y} = v, \quad t_o \leq t \leq t_3,$$

$$\dot{m} = \begin{cases} -B_1, & t_o \leq t < t_1, \\ 0, & t_1 \leq t < t_2, \\ -B_3, & t_2 \leq t \leq t_3, \end{cases}$$

where t_o is initial time, t_3 is final time, and t_1, t_2 are intermediate staging times. The symbols a, g_o represent gravitation constants, and B_1, B_3 denote constant mass flow rates. This description implies a burning arc, a coast arc, and finally a burning arc, with B_3 not necessarily different from B_1.

The following end and intermediate point conditions will be imposed.

$$J_1 \equiv t_o = 0,$$
$$J_2 \equiv u\,(t_o) - u_o = 0,$$
$$J_3 \equiv v\,(t_o) = 0,$$
$$J_4 \equiv x\,(t_o) = 0,$$
$$J_5 \equiv y\,(t_o) - y_o = 0,$$
$$J_6 \equiv x\,(t_1) - x_1 = 0,$$
$$J_7 \equiv y\,(t_2) - y_2 = 0,$$
$$J_8 \equiv x\,(t_3) - x_3 = 0,$$
$$J_9 \equiv y\,(t_3) - y_3 = 0,$$
$$J_{10} \equiv m\,(t_3) - m_3 = 0,$$

and

$$m\,(t_1^-) - m\,(t_1^+) = d_1,$$

with u_o, y_o, x_1, y_2, x_3, y_3, m_3, d_1, B_1, and B_3 known constants.

The function to be minimized is taken to be the sum of the times of the powered stages, that is,

$$J_o \equiv t_1 - t_o + t_3 - t_2.$$

If $B_1 = B_3$, this is equivalent to requiring that the fuel used be minimized, or $J_o \equiv m\,(t_o)$. The conditions J_6 and J_7 insure the existence of three stages.

The Multiplier Rule of Section II allows the following Hamiltonian to be written:

$$H = \begin{cases} \lambda_1 \; (-a^2x + cB_1m^{-1}\cos\,\theta) + \lambda_2 \; (-g_o + 2a^2y + cB_1m^{-1}\sin\,\theta) \\ \qquad + \lambda_3u + \lambda_4v + \lambda_5 \; (-B_1), \; t_o \leq t < t_1, \\ \lambda_1 \; (-a^2x) + \lambda_2 \; (-g + 2a^2y) + \lambda_3 \; u + \lambda_4v, \quad t_1 \leq t < t_2, \\ \lambda_1 \; (-a^2x + cB_3m^{-1}\cos\,\theta) + \lambda_2 \; (-g_o + 2a^2y + cB_3m^{-1}\sin\,\theta) \\ \qquad + \lambda_3u + \lambda_4v + \lambda_5 \; (-B_3), \quad t_2 \leq t \leq t_3. \end{cases}$$

The Euler equations for this Hamiltonian are:

$$\dot{\lambda}_1 + \lambda_3 = 0,$$
$$\dot{\lambda}_2 + \lambda_4 = 0,$$
$$\dot{\lambda}_3 - a^2\lambda_1 = 0,$$
$$\dot{\lambda}_4 + 2a^2\lambda_2 = 0,$$
$$\dot{\lambda}_5 - cB_1m^{-2} \; (\lambda_1 \cos\,\theta + \lambda_2 \sin\,\theta) = 0,$$
$$cB_1m^{-1} \; (\lambda_1 \sin\,\theta - \lambda_2 \cos\,\theta) = 0,$$

for t in $[t_o, t_1)$;

$$\dot{\lambda}_1 + \lambda_3 = 0,$$
$$\dot{\lambda}_2 + \lambda_4 = 0,$$
$$\dot{\lambda}_3 - a^2\lambda_1 = 0,$$
$$\dot{\lambda}_4 + 2a^2\lambda_2 = 0,$$

for t in $[t_1, t_2)$; and

$$\dot{\lambda}_1 + \lambda_3 = 0,$$
$$\dot{\lambda}_2 + \lambda_4 = 0,$$
$$\dot{\lambda}_3 - a^2\lambda_1 = 0,$$
$$\dot{\lambda}_4 + 2a^2\lambda_2 = 0,$$
$$\dot{\lambda}_5 - cB_3m^{-2} \; (\lambda_1 \cos\,\theta + \lambda_2 \sin\,\theta) = 0,$$
$$cB_3m^{-1} \; (\lambda_1 \sin\,\theta - \lambda_2 \cos\,\theta) = 0,$$

for t in $[t_2, t_3]$.

Simple techniques for integration allow these equations to be expressed in integrated form as follows:

$$\lambda_1 = A_1 \sin a(t + C_1),$$
$$\lambda_2 = A_2 \sinh a\sqrt{2}(t + C_2),$$
$$\lambda_3 = -aA_1 \cos a(t + C_1) \;,$$
$$\lambda_4 = -aA_2 \sqrt{2} \cosh a\sqrt{2}\,(t + C_2),$$

for $t_o \leq t < t_1$;

$$\lambda_1 = A_1' \sin a(t + C_1'),$$
$$\lambda_2 = A_2' \sinh a\sqrt{2}(t + C_2'),$$
$$\lambda_3 = -aA_1' \cos a(t + C_1'),$$
$$\lambda_4 = -aA_2'\sqrt{2} \cosh a\sqrt{2}\,(t + C_2'),$$

for $t \leq t < t_2$; and

$$\lambda_1 = A_1'' \sin a(t + C_1''),$$
$$\lambda_2 = A_1'' \sinh a\sqrt{2}(t + C_2''),$$
$$\lambda_3 = -aA_1'' \cos a(t + C_1''),$$
$$\lambda_4 = -aA_2'' \sqrt{2} \cosh a\sqrt{2}(t + C_2''),$$

for $t_2 \leq t \leq t_3$. It is clear in expressing λ_1, λ_2, λ_3, λ_4 as functions of time with two constants of integration, that the last two Euler equations in stage 1 and stage 3 have been ignored. These equations together with the Weierstrass condition will be used to expressed the control angle as a function of the multipliers λ_1 and λ_2. From the last Euler equation of stage 1 and stage 3 we have (for $\lambda_1 \neq 0$, $\cos\theta \neq 0$)

$$\tan\theta = \lambda_2 / \lambda_1$$

and hence

$$\sin\theta = \pm \lambda_2 \Big/ \sqrt{\lambda_1^2 + \lambda_2^2}$$

and

$$\cos\theta = \pm \lambda_1 \Big/ \sqrt{\lambda_1^2 + \lambda_2^2} \;.$$

From the Weierstrass condition of section II,

$$cB_1 m^{-1}(\lambda_1 \cos\theta + \lambda_2 \sin\theta - \lambda_1 \cos\alpha - \lambda_2 \sin\alpha) \geq 0$$

for $t_o \leq t < t$. Here θ is the control angle that actually optimizes, and α ranges over all possible control angles for which the original equations of motion are satisfied. This expression being non-negative is equivalent to maximizing the following function (with respect to α):

$$W = \lambda_1 \cos\alpha - \lambda_2 \sin\alpha.$$

Thus $\frac{\partial W}{\partial \alpha} = 0$ and $\frac{\partial^2 W}{\partial \alpha^2} \leq 0$ which gives

$$-\lambda_1 \sin\alpha + \lambda_2 \cos\alpha = 0$$

and $$-\lambda_1 \cos\alpha - \lambda_2 \sin\alpha \leq 0.$$

thus $\tan\alpha = \lambda_2 / \lambda_1$ and

$$\lambda_1 \left(\pm \lambda_1 \Big/ \sqrt{\lambda_1^2 + \lambda_2^2}\right) + \lambda_2 \left(\pm \lambda_2 \Big/ \sqrt{\lambda_1^2 + \lambda_2^2}\right) \geq 0$$

which implies that $\cos\alpha = \lambda_1 \Big/ \sqrt{\lambda_1^2 + \lambda_2^2}$ and similarly that $\sin\alpha = \lambda_2 \Big/ \sqrt{\lambda_1^2 + \lambda_2^2}$. Hence the control angle θ is expressed as follows:

$$\tan\theta = \lambda_2 / \lambda_1, \quad \lambda_1 \neq 0, \ \cos\theta \neq 0,$$

$$\cos\theta = \lambda_1 \Big/ \sqrt{\lambda_1^2 + \lambda_2^2},$$

$$\sin\theta = \lambda_2 \Big/ \sqrt{\lambda_1^2 + \lambda_2^2}$$

for stage 1. The same expressions for control angle θ hold for stage 3. The fifth Euler equation on stage 1 and stage 3 becomes

$$\dot{\lambda}_5 = \begin{cases} cB_1 m^{-2}\sqrt{\lambda_1^2 + \lambda_2^2}, & t_o \leq t < t_1, \\ cB_3 m^{-2}\sqrt{\lambda_1^2 + \lambda_2^2}, & t_2 \leq t \leq t_3. \end{cases}$$

The transversality matrix which is given at the end of this section has eleven rows and twenty-four columns and is of rank ten. From this matrix fourteen end and intermediate conditions are found. These conditions imply that all multipliers, except possibly λ_3 at t_1 and λ_4 at t_2, are continuous across staging times.

Also the following condition holds at t_1:

$$cB_1 m^{-1} (\lambda_1 \cos\theta + \lambda_2 \sin\theta) + \lambda_5 (B_1) + (\lambda_3^+ - \lambda_3^-) u + 1 = 0$$

where $\lambda_3^+ = \lambda_3 (t_1^+)$.

A similar condition that holds at t_2 is:

$$cB_3 m^{-1} (\lambda_1 \cos\theta + \lambda_2 \sin\theta) - \lambda_5 (B_3) + (\lambda_4^+ - \lambda_4^-) v - 1 = 0.$$

The other four conditions implied by the transversality condition are:

$$\lambda_5(t_o) = 0,$$

$$\lambda_1(t_3) = 0,$$

$$\lambda_2(t_3) = 0,$$

$$- H(t_3) + 1 = 0$$

An optimal trajectory for this problem requires the finding of fifteen constants of integration from the equations of motion, a like number from the Euler equations, and the four times t_o, t_1, t_2, and t_3. Fourteen transversality conditions, ten end and intermediate conditions, and ten requirements on state variables at staging points provide the necessary number of conditions for the determination of these constants.

It is possible to start at the last stage to determine the integration constants for the Euler equations in terms of multiplier values. The constants for the third stage are:

$$A_1'' = -\lambda_{33}/a, \quad (\lambda_{33} \text{ is the final value of } \lambda_3),$$

$$A_2'' = -\lambda_{43}/a\sqrt{2},$$

$$C_1'' = C_2'' = -t_3.$$

Because of the continuity of λ_1 and λ_3 at t_2,

$$A_1'' = A_1',$$

$$C_1'' = C_1'.$$

The values for A_2'' and C_2'' only hold for the third stage. To proceed from the third stage back into the second stage we need the value of the difference

$\lambda_4(t_2^+) - \lambda_4(t_2^-)$. This can be found from the transversality condition above which holds at t_2. Supposing this equation solved, the determination of constants A_2', C_2' for the second stage can proceed, and these values also hold for the first stage for λ_2 and λ_4. An analogous procedure is applied to λ_1 and λ_3 for the first stage.

TRANSVERSALITY MATRIX FOR THREE STAGE PROBLEM

$H(t_o)-1$	$H(t_1)\Big\vert_-^+ +1$	$H(t_2)\Big\vert_-^+ -1$	$-H(t_3)+1$	$-\lambda_1(t_o)$	$-\lambda_2(t_o)$
1	0	0	0	0	0
0	0	0	0	1	0
0	0	0	0	0	1
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0

$-\lambda_3(t_o)$	$-\lambda_4(t_o)$	$-\lambda_5(t_o)$	$\lambda_1(t_1)\Big\vert_+^-$	$\lambda_2(t_1)\Big\vert_+^-$	$\lambda_3(t_1)\Big\vert_+^-$
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
1	0	0	0	0	0
0	1	0	0	0	0
0	0	0	0	0	1
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0

$\lambda_4(t_1)\Big\vert_+^-$	$\lambda_5(t_1)\Big\vert_+^-$	$\lambda_1(t_2)\Big\vert_+^-$	$\lambda_2(t_2)\Big\vert_+^-$	$\lambda_3(t_2)\Big\vert_+^-$	$\lambda_4(t_2)\Big\vert_+^-$
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	1
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0

$\lambda_5(t_2)\Big\vert_+^-$	$\lambda_1(t_3)$	$\lambda_2(t_3)$	$\lambda_3(t_3)$	$\lambda_4(t_3)$	$\lambda_5(t_3)$
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	1	0	0
0	0	0	0	1	0
0	0	0	0	0	1

SECTION IV. CONCLUSIONS AND RECOMMENDATIONS

The multistage theory summarized in Section II and illustrated in Section III involves multipoint boundary value problems. Closed form solutions are to be expected only in problems with simple constraints. However, considerable practical information is obtainable from the general theorems. In particular, the transversality matrix may yield usable switching functions for determining coasting and powered stages, and the Weierstrass condition can be interpreted as the Maximum Principle of Hestenes and Pontryagin, which is especially important for optimal paths lying partially along region boundaries.

The procedure in Section III of beginning with the final stage and successively determining the constants of integration for the several stages in terms of the Lagrange multipliers would seem applicable to more complicated problems. Trial estimates of the multipliers at the final point would determine an optimal trajectory which in general would not satisfy all intermediate and initial conditions. New estimates could then be made and the process repeated. This suggests attempting to obtain a sequence of optimal trajectories converging to one that would satisfy all the multipoint boundary conditions.

Another type of sequential process consists of using non-optimal solutions of the equations of motion, each satisfying the multipoint conditions, with the sequence converging to an optimal trajectory satisfying all conditions. Gradient, or steepest descent, methods would apply to this procedure, and further study is recommended.

Some study has been made under this contract of analogues to the Jacobi conjugate point conditions for multistage problems. Further effort toward obtaining such conditions in computationally useful form is recommended. Also the combination of necessary conditions, suitably strengthened, into a set of sufficient conditions is desirable.

Application of multistage calculus of variations theory to the optimization of six stage earth-moon trajectories is in progress under this contract. Special attention to the reduction of computational difficulties is needed.

BIBLIOGRAPHY

Balakrishnan, A. V., and Neustadt, L. W. (editors). Computing Methods in Optimization Problems. New York: Academic Press, 1964

Bliss, Gilbert Ames. Lectures on the Calculus of Variations. Chicago: University of Chicago Press, 1946.

Boyce, M. G. and Linnstaedter, J. L. "Necessary Conditions for a Multistage Bolsa-Mayer Problem Involving Control Variables and Having Inequality and Finite Equation Constraints," Progress Report No. 7 on Studies in the Fields of Space Flight and Guidance Theory. Huntsville, Alabama: NASA-MSFC, Aero-Astrodynamics Laboratory, 1965.

Denbow, C. H. "A Generalized Form of the Problem of Bolza," Contributions to the Calculus of Variations 1933-1937. Chicago: University of Chicago Press, (1937), pp.449 - 484.

Hesteness, Magnus R. A General Problem in the Calculus of Variations with Application to Paths of Least Time. U. S. Air Force Project Rand Research Memorandum RM - 100. Santa Monica, Calif.,: Rand Corporation, March 1, 1950

Leitmann, G. "Variational Problems with Bounded Control Variables," Optimization Techniques. New York: Academic Press, 1962, pp. 171 - 204.

Pontryagin, L. S., Boltyanskii, V. G., Gamkrelidze, R. V., and Mishchenko, E. F. The Mathematical Theory of Optimal Control Processes. New York: Interscience Publishers, 1962

Tyler, G. E. Applications of Multistage Calculus of Variations Theory to Rocket Trajectory Problems. Master's thesis, Vanderbilt University, Nashville, Tennessee, 1965.