Destabilizing Effect of Velocity-Dependent Forces in Nonconservative Continuous Systems

by S. Nemat-Nasser, S. N. Prasad and G. Herrmann

National Aeronautics and Space Administration
Grant NsG 605

December, 1965
Technical Report No. 65-4

DESTABILIZING EFFECT OF VELOCITY-DEPENDENT FORCES IN NONCONSERVATIVE CONTINUOUS SYSTEMS

by

S. Nemat-Nasser

Post-doctoral Research Fellow
Department of Civil Engineering
The Technological Institute
Northwestern University
Evanston, Illinois

S. N. Prasad

Graduate Student
Department of Civil Engineering
The Technological Institute
Northwestern University
Evanston, Illinois

and

G. Herrmann

Professor of Civil Engineering
The Technological Institute
Northwestern University
Evanston, Illinois

*This research was supported by the National Aeronautics and Space Administration under Grant NsG 605.

ABSTRACT

A cantilevered, continuous pipe conveying fluid at a constant velocity is studied analytically. It is shown that sufficiently small velocity-dependent forces, such as internal and external damping, as well as Coriolis forces, may have a destabilizing effect. It is also demonstrated that the Galerkin method with a two-term approximation may lead to erroneous results when velocity-dependent forces exist in the system.

1. Introduction

Some technological demands of recent years have brought into focus the importance of stability investigations of elastic systems subjected to nonconservative forces, i.e. forces which do not possess a potential. Missiles, aircraft and various elements of high-performance launch vehicles may give rise to this type of problems. For example, a flexible missile under end thrust must be investigated for possible overall flutter-type instability. In this case one may idealize the system by a free-free, slender beam, subjected at one end to a follower force, and determine the critical thrust.

One of the principal features of these problems is that the static approach is usually inadequate for stability analysis, and a dynamic stability criterion must be employed, i.e. the actual motion of these systems must be studied [1,2,3,4,5,6]*.

In 1952 Ziegler [7], by means of an example, indicated that linear viscous damping forces in a nonconservative elastic system with two degrees of freedom may have a destabilizing effect. This remarkable discovery provided an impetus for further studies, and subsequently several investigators [4,8,9,10,11] explored this phenomenon in more detail. For example, Nemat-Nasser and Herrmann [11] have proved that the critical flutter load of an undamped (no velocity-dependent forces exist) nonconservative, discrete, linear system is an upper bound for the critical flutter load of the same system when sufficiently small velocity-dependent forces are also present. Therefore, it was concluded that in a general nonconservative system with N degrees of freedom not only slight viscous damping but all sufficiently

*Numbers in brackets refer to the Bibliography at the end of this paper.

small velocity-dependent forces, such as Coriolis forces on vibrating pipes conveying fluid, or other gyroscopic forces, may have a destabilizing effect.

For a continuous system, however, which possesses an infinite number of degrees of freedom, no such theorems are as yet established. To study the effect of viscous damping forces in such systems most investigators, in general, reduce first the continuous system to a discrete one by means of, for example, the Galerkin method, and then study the reduced, discrete system [4,12,13]. But, as was shown in [11], a discrete system does, in fact, always have this property, except in very particular cases. Therefore, by this approach one does not know whether the original continuous system also exhibits the same behavior or whether it is produced only through the reduction procedure.

The purpose of the present study is to show that the presence of sufficiently small velocity-dependent forces in a continuous elastic system subjected to follower forces does, indeed, have a destabilizing effect. To this end, a cantilevered, continuous pipe conveying fluid at a constant velocity is considered. The internal and external viscous damping forces are also included, and then it is proved that the critical flutter load of the system may be reduced by almost 50% for some combinations of these velocity-dependent forces.

Although the problem of a cantilevered viscoelastic pipe conveying fluid has numerous practical applications, in the present study it is also used as a model to demonstrate the remarkable phenomenon of destabilizing effect of sufficiently small velocity-dependent forces in a nonconservative, continuous system without reducing the system to a discrete one. (The destabilizing effect of Coriolis forces in torsional flutter of a cantilevered continuous bar was first discussed by Nemat-Nasser and Herrmann in [14].) Moreover, the method of analysis, which effectively reduces a complicated non-self-adjoint boundary value problem to a simple frequency analysis by utilizing fully the fact that the velocity-dependent forces are sufficiently small, is itself of some value and may be employed for the stability analysis of one-, two-, or three-dimensional continuous nonconservative systems.

In the final portion of this work, we shall use the opportunity to test the accuracy of the widely used Galerkin method with a two-term approximation. It is to be noted that no such analysis of this approximate method, for the case when the equations of motion of the system also contain mixed time and space derivatives, has as yet been carried out. Therefore, there exists no a priori certainty that the method of Galerkin, especially with a two-term approximation, should necessarily yield sufficiently accurate results, particularly when the effect of Coriolis forces, in the reduced discrete system, has to be included through a nonsymmetric matrix.

In the present study, critical flutter loads of the system, for small velocity-dependent forces, and also for large values of Coriolis forces, are obtained by using the Galerkin method with a two-term approximation. The results are then compared with the exact solution. It is then shown that the two-term approximation yields sufficiently accurate values for the critical flutter load only if the velocity-dependent forces are small. That is, for large values of Coriolis forces the critical load obtained by the Galerkin method with a two-term approximation may be greatly in error.

2. Statement of the Problem

We consider a cantilevered, uniform pipe of length L and internal cross-sectional area A, conveying fluid at a constant velocity U. A nozzle whose opening is n times smaller than A is placed at the free end of the system, as is shown in Figure 1.

We shall assume that the material of the pipe obeys a stress-strain relationship of the Kelvin type, i.e.

$$\sigma = E\epsilon + \eta\dot{\epsilon} \tag{1}$$

where E is the modulus of elasticity and η is the coefficient of viscosity. Under the assumption of plane sections remaining plane, the moment-curvature relationship, for small deformations, is

$$\frac{M}{EI} = - \frac{\partial^2 y}{\partial x^2} \tag{2}$$

where M is the resultant moment at section x and at time t, I the moment of inertia, and y the transverse deflection of the pipe. With u denoting the displacement in the x direction, and z the distance of each fiber from the neutral axis, we also have

$$\sigma = \frac{Mz}{I}\,, \qquad \epsilon = \frac{\partial u}{\partial x}\,, \qquad u = - z\,\frac{\partial y}{\partial x}\,. \tag{3}$$

The equation of motion may now be stated as

$$\frac{\partial^2 M}{\partial x^2} = p \tag{4}$$

where p is the resultant lateral force exerted on the pipe. This lateral force may be decomposed into three parts. The first part is due to the inertia forces and is given by $+(m + m_1)\,\frac{\partial^2 y}{\partial t^2}$, where m is the mass of the pipe per unit of length, and m_1 the mass of the fluid contained within the pipe. The second part is due to Coriolis acceleration and is given by $+2m_1 U \frac{\partial^2 y}{\partial x \partial t}$, and finally, the third part, which is due to equivalent compressive force induced by the flux of momentum out of the pipe, and is given by $+m_1 U^2 n \frac{\partial^2 y}{\partial x^2}$. Therefore, the equation of motion, (4), becomes

$$\frac{\partial^2 M}{\partial x^2} = (m + m_1) \frac{\partial^2 y}{\partial t^2} + 2m_1 U \frac{\partial^2 y}{\partial x \partial t} + m_1 U^2 n \frac{\partial^2 y}{\partial x^2} , \tag{5}$$

and substitution from (1), (2), and (3) into (5) finally yields

$$EI \frac{\partial^4 y}{\partial x^4} + \eta I \frac{\partial^5 y}{\partial x^4 \partial t} + m_1 U^2 n \frac{\partial^2 y}{\partial x^2} + 2m_1 U \frac{\partial^2 y}{\partial x \partial t} + (m + m_1) \frac{\partial^2 y}{\partial t^2} = 0 .$$

If we include also the effect of external damping in the form $K \frac{\partial y}{\partial t}$, where K is a constant, and introduce the following dimensionless quantities:

$$\xi = \frac{x}{L} , \qquad t = \tau \sqrt{\frac{(m + m_1)L^4}{EI}} , \qquad \frac{m_1}{m + m_1} = \beta' ,$$

$$\frac{m_1 U^2 n L^2}{EI} = F^2 , \qquad \sqrt{\frac{\eta^2 I}{E(m + m_1)L^4}} = \delta' , \qquad \sqrt{\frac{K^2 L^4}{EI(m + m_1)}} = \gamma' ,$$

then we obtain

$$\frac{\partial^4 y}{\partial \xi^4} + \delta' \frac{\partial^5 y}{\partial \xi^4 \partial \tau} + F^2 \frac{\partial^2 y}{\partial \xi^2} + 2 \sqrt{\frac{\beta'}{n}}\, F \frac{\partial^2 y}{\partial \xi \partial \tau} + \gamma' \frac{\partial y}{\partial \tau} + \frac{\partial^2 y}{\partial \tau^2} = 0 . \tag{6}$$

To study the effect of small viscous damping forces and Coriolis forces, we now let

$$\delta' = \nu\delta , \qquad \gamma' = 2\nu\gamma, \qquad \text{and} \qquad \sqrt{\frac{\beta'}{n}} = \nu\beta ,$$

where ν is a small parameter. The equation of motion, (6), and the boundary conditions at $\xi = 0, 1$, may then be written as

$$\frac{\partial^4 y}{\partial \xi^4} + F^2 \frac{\partial^2 y}{\partial \xi^2} + \frac{\partial^2 y}{\partial \tau^2} + \nu \left[\delta \frac{\partial^5 y}{\partial \xi^4 \partial \tau} + 2\beta F \frac{\partial^2 y}{\partial \xi \partial \tau} + 2\gamma \frac{\partial y}{\partial \tau} \right] = 0 ,$$

$$y = \frac{\partial y}{\partial \xi} = 0 ; \quad \text{at} \quad \xi = 0 ,$$

$$\frac{\partial^2 y}{\partial \xi^2} = \frac{\partial^3 y}{\partial \xi^3} = 0 \; ; \quad \text{at} \quad \xi = 1 \; . \tag{7}$$

In the sequel, we shall study the stability of system (7) when ν is sufficiently small.

3. Stability Analysis

We let $y = \psi(\xi)e^{i\omega\tau}$; $i = \sqrt{-1}$, and reduce (7) to the following boundary value problem

$$\frac{d^4\psi}{d\xi^4} + F^2 \frac{d^2\psi}{d\xi^2} - \omega^2\psi + i\omega\nu \left[\delta \frac{d^4\psi}{d\xi^4} + 2\beta F \frac{d\psi}{d\xi} + 2\gamma\psi \right] = 0 ,$$

$$\psi = \psi' = 0 ; \quad \text{at} \quad \xi = 0 ,$$

$$\psi'' = \psi''' = 0 ; \quad \text{at} \quad \xi = 1 , \tag{8}$$

where prime denotes differentiation with respect to ξ.

We then set $\psi = e^{\bar{\lambda}\xi}$; $\bar{\lambda} = \lambda + i\nu a$, and obtain

$$(\lambda + i\nu a)^4 + F^2(\lambda + i\nu a)^2 - \omega^2 + i\nu\omega \left[\delta(\lambda + i\nu a)^4 + \right.$$

$$\left. + 2\beta F(\lambda + i\nu a) + 2\gamma\right] = 0 , \tag{9}$$

which is the characteristic equation of system (8). Expanding (9) in a series of powers of ν, we are led to

$$\left\{\lambda^4 + F^2\lambda^2 - \omega^2\right\} + (i\nu)\left\{4a\lambda^3 + 2F^2\lambda a + \omega(\delta\lambda^4 + 2\beta F\lambda + 2\gamma)\right\} +$$

$$+ (i\nu)^2\left\{6a^2\lambda^2 + F^2a^2 + \omega(4\delta a\lambda^3 + 2\beta Fa)\right\} + (i\nu)^3\left\{4\lambda a^3 + 6\delta\omega a^2\lambda^2\right\} +$$

$$+ (i\nu)^4\left\{a^4 + 4\omega a^3\lambda\right\} + (i\nu)^5(\omega a^4) = 0 .$$

Next, we equate terms of like powers in ν, neglecting $O(\nu^2)$ and higher, and finally arrive at

$$\lambda^2 = -\frac{F^2}{2} \pm \sqrt{\left(\frac{F^2}{2}\right)^2 + \omega^2} ,$$

$$a = -\omega \frac{\delta\lambda^4 + 2\beta F\lambda + 2\gamma}{2\lambda(2\lambda^2 + F^2)} ; \quad \bar{\lambda} = \lambda + i\nu a . \tag{10}$$

The solution to system (8) may now be written as $\psi(\xi) = \sum_{j=1}^{4} A_j e^{\bar{\lambda}_j\xi}$, where

A_j ; j = 1,2,3,4, are constants which can be obtained from the boundary conditions at $\xi = 0,1$. That is, they must satisfy the following four linear, homogeneous equations:

$$\sum_{j=1}^{4} A_j = 0 \ ,$$

$$\sum_{j=1}^{4} \bar{\lambda}_j A_j = 0 \ ,$$

$$\sum_{j=1}^{4} \bar{\lambda}_j^2 A_j e^{\bar{\lambda}_j} = 0 \ ,$$

$$\sum_{j=1}^{4} \bar{\lambda}_j^3 A_j e^{\bar{\lambda}_j} = 0 \ . \tag{11}$$

System (11) has non-trivial solutions if and only if the determinant of the coefficients is identically zero, i.e. the frequency equation is

$$\begin{aligned}\Delta \equiv\ & \bar{\lambda}_1^2\bar{\lambda}_2^2(\bar{\lambda}_4 - \bar{\lambda}_3)(\bar{\lambda}_2 - \bar{\lambda}_1)\ e^{(\bar{\lambda}_1 + \bar{\lambda}_2)} + \\ & + \bar{\lambda}_1^2\bar{\lambda}_3^2(\bar{\lambda}_2 - \bar{\lambda}_4)(\bar{\lambda}_3 - \bar{\lambda}_1)\ e^{(\bar{\lambda}_1 + \bar{\lambda}_3)} + \\ & + \bar{\lambda}_1^2\bar{\lambda}_4^2(\bar{\lambda}_3 - \bar{\lambda}_2)(\bar{\lambda}_4 - \bar{\lambda}_1)\ e^{(\bar{\lambda}_1 + \bar{\lambda}_4)} + \\ & + \bar{\lambda}_2^2\bar{\lambda}_3^2(\bar{\lambda}_4 - \bar{\lambda}_1)(\bar{\lambda}_3 - \bar{\lambda}_2)\ e^{(\bar{\lambda}_2 + \bar{\lambda}_3)} + \\ & + \bar{\lambda}_2^2\bar{\lambda}_4^2(\bar{\lambda}_1 - \bar{\lambda}_3)(\bar{\lambda}_4 - \bar{\lambda}_2)\ e^{(\bar{\lambda}_2 + \bar{\lambda}_4)} + \\ & + \bar{\lambda}_3^2\bar{\lambda}_4^2(\bar{\lambda}_2 - \bar{\lambda}_1)(\bar{\lambda}_4 - \bar{\lambda}_3)\ e^{(\bar{\lambda}_3 + \bar{\lambda}_4)} = 0 \ .\end{aligned} \tag{12}$$

This expression may be rewritten with the aid of (10) as follows, after expanding it in terms of powers of ν, and neglecting $O(\nu^3)$,

$$\Delta \equiv \left\{F^4 + 2\omega^2 + 2\omega^2 \text{ ch } \lambda_1 \cos \lambda_3 + F^2\omega \text{ sh } \lambda_1 \sin \lambda_3\right\} -$$

$$- i\nu \left\{ \left(\frac{2\beta F\omega}{2\lambda_1^2 + F^2} \right) \left[(\lambda_1^4 - \lambda_3^4) + (\lambda_3^3 - 3\lambda_1^2\lambda_3) \text{ ch } \lambda_1 \sin \lambda_3 + \right. \right.$$

$$\left. + (3\lambda_1\lambda_3^2 - \lambda_1^3) \text{ sh } \lambda_1 \cos \lambda_3 \right] + \left(\frac{\delta\lambda_1^4 + 2\gamma}{2\lambda_1(2\lambda_1^2 + F^2)} \right) \left[(\lambda_3^5 + 5\lambda_1^4\lambda_3) + \right.$$

$$+ 6\lambda_1^2\lambda_3^3 \text{ ch } \lambda_1 \cos \lambda_3 - 4\lambda_1^3\lambda_3^2 \text{ sh } \lambda_1 \sin \lambda_3 + 2\lambda_1\lambda_3^4 \text{ sh } \lambda_1 \sin \lambda_3 +$$

$$\left. + 2\lambda_1^3\lambda_3^3 \text{ sh } \lambda_1 \cos \lambda_3 + \lambda_1^2\lambda_3^4 \text{ ch } \lambda_1 \sin \lambda_3 - \lambda_1^4\lambda_3^2 \text{ ch } \lambda_1 \sin \lambda_3 \right] +$$

$$+ \left(\frac{\delta\lambda_3^4 + 2\gamma}{2\lambda_3(2\lambda_1^2 + F^2)} \right) \left[(\lambda_1^5 + 5\lambda_1\lambda_3^4) + 6\lambda_1^3\lambda_3^2 \text{ ch } \lambda_1 \cos \lambda_3 + \right.$$

$$+ 4\lambda_1^2\lambda_3^3 \text{ sh } \lambda_1 \sin \lambda_3 - 2\lambda_1^4\lambda_3 \text{ sh } \lambda_1 \sin \lambda_3 - 2\lambda_1^3\lambda_3^3 \text{ ch } \lambda_1 \sin \lambda_3 +$$

$$\left. \left. + \lambda_1^2\lambda_3^4 \text{ sh } \lambda_1 \cos \lambda_3 - \lambda_1^4\lambda_3^2 \text{ sh } \lambda_1 \cos \lambda_3 \right] \right\} = 0 , \qquad (13)$$

where $\lambda_1^2 = -\frac{F^2}{2} + \sqrt{\left(\frac{F^2}{2}\right)^2 + \omega^2}$, $\lambda_3^2 = \frac{F^2}{2} + \sqrt{\left(\frac{F^2}{2}\right)^2 + \omega^2}$.

The first term in braces, in equation (13), is the frequency equation when $\nu = 0$, and the second term, to the first order of approximation in ν, indicates the effect of small viscous damping forces and Coriolis forces. For $\nu = 0$, we obtain the frequency equation of a purely elastic cantilevered beam subjected to a compressive force which stays tangent to the axis at the free end [1,4]. The critical value of the load, in this case, is $F_e^2 = 20.05$, which was first computed by Beck [1].

For non-zero but sufficiently small values of ν and for small F, all the roots of equation (13) are located to the left of the imaginary axis in the complex $i\omega$ plane. As we increase F, at least one of these roots approaches the imaginary axis, and for a certain value of F, say F_d, equation (13) yields one purely imaginary root $i\omega = i\omega_c$. If we now increase F beyond this critical value F_d, one of the roots of (13) becomes complex with negative imaginary part, and the system oscillates with an exponentially increasing amplitude. Therefore, for given values of δ, β and γ, we shall seek critical values of $\omega = \omega_c$ (real), and $F = F_d$ which identically satisfy (13). This is illustrated in Figure 2 where, for $\delta = 1$, $\beta = 1$, and $\gamma = 0$, real (Δ_1) and imaginary ($-\Delta_2$) parts of Δ are plotted against the values of ω^2. Similar results may be obtained for other values of δ, β, and γ.

It may also be of interest to establish the destabilizing effect of Coriolis forces, internal viscous damping forces, and external viscous damping forces independently.

To this end, we let $\delta = \gamma = 0$, $\beta = 1$, and with $\gamma_d = \frac{F_d^2}{\pi^2}$ obtain, from equation (13), $\gamma_d = 1.78$. Similarly, for $\beta = \gamma = 0$ and $\delta = 1$, the critical load is obtained to be $\gamma_d = 1.107$. However, for $\beta = \delta = 0$ and $\gamma = 1$ we get $\gamma_d = 2.035$, which is equal to the critical load of the system when no velocity-dependent forces are present. That is, although sufficiently small Coriolis forces and internal viscous damping forces have a destabilizing effect in this continuous system, external viscous damping forces do not have the same effect.

The combined effect of velocity-dependent forces on the value of the critical parameter $\gamma_d = \frac{F_d^2}{\pi^2}$ is shown in Figures 3 and 4. In these figures the parameter γ_d is plotted against β/δ for various values of γ. The horizontal dashed line in these figures represents the critical value of γ_d when no velocity-dependent forces exist and the cantilevered column is subjected to a compressive follower force at the free end [1].

It is important to note that the stability curves shown in Figures 3 and 4 have a finite discontinuity at $\nu = 0$. That is, although for $\nu = 0$ we have $F^2 = F_e^2 = 20.05$, for $\nu = 0^+$, the critical value of F^2 is, in general, less than 20.05.

It may also be of interest to explore the order of magnitude of ν for which the destabilizing effect of velocity-dependent forces still exists. This may be accomplished by considering ν large and seeking values of ω and F for which equation (12) is identically satisfied. We note that, in equation (12), $\bar{\lambda}_j$; $j = 1,2,3,4$, are defined as functions of ω and the other parameters of the system through equation (9). In order to circumvent the difficulty of solving polynomials with complex coefficients, we let $\delta = \gamma = 0$ and put $\bar{\lambda} = i\eta$ in equations (9) and (12).

The critical values of ω and F may now be evaluated employing computers. The computer may be instructed to obtain the roots of equation (9) for given parameters, and then calculate Δ, (equation (12)).* These results are shown in Figure 5, where $\gamma_d = \frac{F_d^2}{\pi^2}$ is plotted against values of $\sqrt{\frac{\beta'}{n}}$, by a solid line. The dashed line in this figure corresponds to the critical γ_d when the Galerkin method with a two-term approximation is employed for the analysis. We shall discuss this in the following section.

*In reference [15] this problem is solved using an indirect method.

4. Approximate Method of Galerkin

We now employ the Galerkin method [4] for the analysis of system (7). To this end, we consider a set of orthonormal [16] eigenfunctions, $\{\varphi_n(\xi)\}$, obtained by solving the following eigenvalue problem

$$\frac{d^4\varphi_n}{d\xi^4} - \omega_n{}^2\varphi_n = 0 ,$$

$$\varphi_n = \frac{d\varphi_n}{d\xi} = 0 \quad ; \quad \text{at} \quad \xi = 0 ,$$

$$\frac{d^2\varphi_n}{d\xi^2} = \frac{d^3\varphi_n}{d\xi^3} = 0 \; ; \quad \text{at} \quad \xi = 1 . \tag{14}$$

We then let $y = \sum_{n=1}^{\infty} q_n(\tau)\, \varphi_n(\xi)$, substitute it into the first equation in (7), multiply both sides of this equation by $\delta y = \sum_{m=1}^{\infty} \varphi_m(\xi)\, \delta q_m(\tau)$, and integrate the result from zero to 1 with respect to ξ to obtain

$$\frac{d^2 q_n}{d\tau^2} + \sum_{m=1}^{\infty} (\omega_m{}^2\delta_{mn} + F^2 b_{mn})q_m +$$

$$+ \nu \sum_{m}^{\infty} (\delta\omega_m{}^2\delta_{mn} + 2\beta a_{mn} + 2\gamma\delta_{mn})q_m = 0 \; , \; n = 1,2,\ldots,\infty \; , \tag{15}$$

where

$$\varphi_m = \cosh \lambda_m \xi - \cos \lambda_m \xi - \sigma_m(\sinh \lambda_m \xi - \sin \lambda_m \xi) ,$$

$$\sigma_m = \frac{\sinh \lambda_m - \sin \lambda_m}{\cosh \lambda_m + \cos \lambda_m} ,$$

$$a_{mn} = \int_0^1 \frac{d\varphi_m}{d\xi} \varphi_n \, d\xi ,$$

$$\lambda_m{}^2 = \omega_m ,$$

$$b_{mn} = \int_0^1 \frac{d^2\varphi_m}{d\xi^2} \varphi_n \, d\xi ,$$

$$\delta_{mn} = \int_0^1 \varphi_m \varphi_n \, d\xi = \begin{cases} 1 \; ; \text{ for } m = n \\ 0 \; ; \text{ for } m \neq n \; . \end{cases} \tag{16}$$

System (15) is a set of non-self-adjoint, linear, second order, homogeneous, ordinary differential equations which admit solutions of the form $q_m = A_m e^{i\omega\tau}$. To obtain the critical values of F^2, we seek conditions under which ω becomes complex with negative imaginary part. System (15), however, consists of infinite number of equations each with infinite number of terms. This, therefore, leads to a determinant which possesses an infinite number of rows and columns.

In practice, it is quite common to let m,n = 1,2 in equations (15) and reduce this system to only two linear, homogeneous differential equations [4]. Hence, the characteristic equation becomes a polynomial of degree four, which can easily be solved. The values of F^2, which renders at least one real root and all the other roots complex with positive imaginary parts, are then taken to be approximation to the critical flutter loads.

In the present case, using the above approximation, we obtain the approximate characteristic equation as

$$\omega^4 - i\nu \left\{2F\beta(a_{11} + a_{22}) + [\delta(\omega_1{}^2 + \omega_2{}^2) + 4\gamma]\right\} \omega^3 - \left\{[(\omega_1{}^2 + \omega_2{}^2) + \right.$$

$$+ \nu^2(\delta\omega_1{}^2 + 2\gamma)(\delta\omega_2{}^2 + 2\gamma)] + F\nu^2[2\beta a_{22}(\delta\omega_1{}^2 + 2\gamma) +$$

$$\left. + 2\beta a_{11}(\delta\omega_2{}^2 + 2\gamma)] + F^2[(b_{11} + b_{22}) + \nu^2(4\beta^2 a_{11} a_{22} - 4\beta^2 a_{12} a_{21})]\right\} \omega^2 +$$

$$+ i\nu \left\{[\omega_1^2(\delta\omega_2^2 + 2\gamma) + \omega_2^2(\delta\omega_1^2 + 2\gamma)] + F[2\beta\omega_1^2 a_{22} + 2\beta\omega_2^2 a_{11}] + \right.$$

$$+ F^2[b_{11}(\delta\omega_2^2 + 2\gamma) + b_{22}(\delta\omega_1^2 + 2\gamma)] + F^3[(2\beta a_{22} b_{11} + 2\beta a_{11} b_{22}) -$$

$$\left. - 2\beta(a_{12} b_{21} + a_{21} b_{12})]\right\} \omega + \left\{F^4[b_{11} b_{22} - b_{21} b_{12}] + \right.$$

$$\left. + F^2[\omega_1^2 b_{22} + \omega_2^2 b_{11}] + \omega_1^2 \omega_2^2 \right\} = 0 \ , \tag{17}$$

where

$a_{11} = 1.99999$; $a_{12} = 0.759489$; $a_{21} = -4.75939$; $a_{22} = 1.99989$

$b_{11} = 0.858243$; $b_{12} = 1.87386$; $b_{21} = -11.7429$; $b_{22} = -13.2945$

$\omega_1^2 = 12.3624$; $\omega_2^2 = 485.519$;

For sufficiently small values of ν, we may neglect terms associated with ν^2 in equation (17), and using Routh-Hurwitz criteria [17], calculate approximate values of the critical load $F^2 = \gamma_d \pi^2$. In Table I these approximate flutter loads are compared with the exact values obtained in the previous section. From this table we observe that, for sufficiently small ν, the Galerkin method with a two-term approximation yields very accurate results. We note also that, for $\nu = 0$, this approximate method gives $F^2 = 20.15$ as compared with the exact critical load, $F^2 = 20.05$.

The above conclusion, however, does not imply that, for ν finite, the approximate method should necessarily give sufficiently accurate results. In fact, as is shown in Figure 5, for $\delta = \gamma = 0$, the critical flutter load obtained by the approximate method (dashed line in Figure 5) can be greatly in error for relatively large values of the Coriolis forces. We note that, for $\sqrt{\frac{\beta'}{n}}$ smaller than 0.25, the resulting error, when the Galerkin method with a two-term approximation is used, is less than 5% and decreases as the value of ν decreases.

It may be of interest to determine whether similar discrepancies may also result for large values of the internal and external damping forces. This analysis, however, is postponed to a future study.

REFERENCES

[1] M. Beck, "Die Knicklast des einseitig eingespannden," Zeitschrift für angewandte Mathematik und Physik, Vol. 3, No. 3, 1952.

[2] H. Ziegler, "Linear Elastic Stability," Zeitschrift für angewandte Mathematik und Physik, Vol. 4, 1953, pp. 89-121, 168-185.

[3] H. Ziegler, "On the Concept of Elastic Stability," Advances in Applied Mechanics, Vol. 4, edited by H. L. Dryden and T. von Karman, Academic Press, Inc., New York, N.Y., 1956, pp. 351-403.

[4] V. V. Bolotin, Nonconservative Problems of the Theory of Elastic Stability, Moscow, 1961; English translation published by Pergamon Press, Inc., New York, N.Y., 1963.

[5] G. Herrmann and R. W. Bungay, "On the Stability of Elastic Systems Subjected to Nonconservative Forces," Journal of Applied Mechanics, Vol. 31, 1964, pp. 435-440.

[6] S. Nemat-Nasser and G. Herrmann, "On the Stability of Equilibrium of Continuous Systems," Tech. Report No. 65-1, March 1965, Northwestern University, The Tech. Inst., Depart. of C.E., Structural Mech. Lab., Evanston, Ill., to be published in Ingenieur Archiv.

[7] H. Ziegler, "Die Stabilitätskriterien der Elastomechanik," Ingenieur-Archiv, Vol. 20, 1952, pp. 49-56.

[8] G. Herrmann and I. C. Jong, "On the Destabilizing Effect of Damping in Nonconservative Elastic Systems," J. Appl. Mech., Vol. 32, 1965, pp. 592-597.

[9] G. Herrmann and I. C. Jong, "On Nonconservative Stability Problems of Elastic Systems with Slight Damping," J. Appl. Mech., in press.

[10] H. Leipholz, "Über den Einfluss der Dämpfung bei nichtkonservativen Stabilitätsproblem elastischer Stäbe," Ingenieur-Archiv, Vol. 33, 1964, pp. 308-321.

[11] S. Nemat-Nasser and G. Herrmann, "Some General Considerations Concerning the Destabilizing Effect in Nonconservative Systems," Zeitschrift für angewandte Mathematik und Physik, in press.

[12] J. M. Hedgepeth, B. Budiansky and R. W. Leonard, "Analysis of Flutter in Compressible Flow of a Panel on Many Supports," Journal of Aeronautical Sciences, Vol. 21, No. 7, 1954, pp. 475-486.

[13] D. J. Johns and P. C. Parks, "Effect of Structural Damping on Panel Flutter," Aircraft Engineering, Oct. 1960, pp. 304-308.

[14] G. Herrmann and S. Nemat-Nasser, "Instability Modes of Cantilevered Bars Induced by Fluid Flow through Attached Pipes," to be published.

[15] R. W. Gregory and M. P. Paidoussis, "Unstable Oscillation of Tubular Cantilevers Conveying Fluid," unpublished manuscript.

[16] R. Courant and D. Hilbert, Methods of Mathematical Physics, Interscience Publishers, Inc., New York, N.Y., 1953.

[17] E. J. Routh, Advanced Dynamics of a System of Rigid Bodies, Dover Publications, New York, N.Y., 1955.

Table 1

β	γ	δ	$\gamma_d = F^2/\pi^2$	
			Exact	Galerkin Method
0.0	1.0	0.0	2.035	2.035
1.0	0.0	0.0	1.780	1.768
0.0	0.0	1.0	1.107	1.082
1.0	0.0	1.0	1.462	1.447
2.5	0.0	1.0	1.73	1.729
5.0	0.0	1.0	1.92	1.924
0.0	1.0	1.0	1.155	1.133
1.0	1.0	1.0	1.483	1.469
2.5	1.0	1.0	1.735	1.738
5.0	1.0	1.0	1.925	1.926
0.0	10.0	1.0	1.426	1.414
1.0	10.0	1.0	1.618	1.611
2.5	10.0	1.0	1.795	1.794
5.0	10.0	1.0	1.935	1.940
0.0	100.0	1.0	1.895	1.902
1.0	100.0	1.0	1.926	1.930
2.5	100.0	1.0	1.960	1.964
5.0	100.0	1.0	1.996	2.000

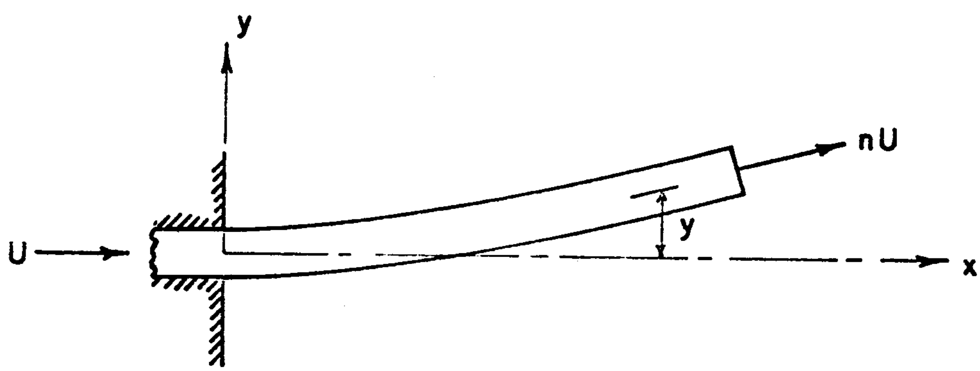
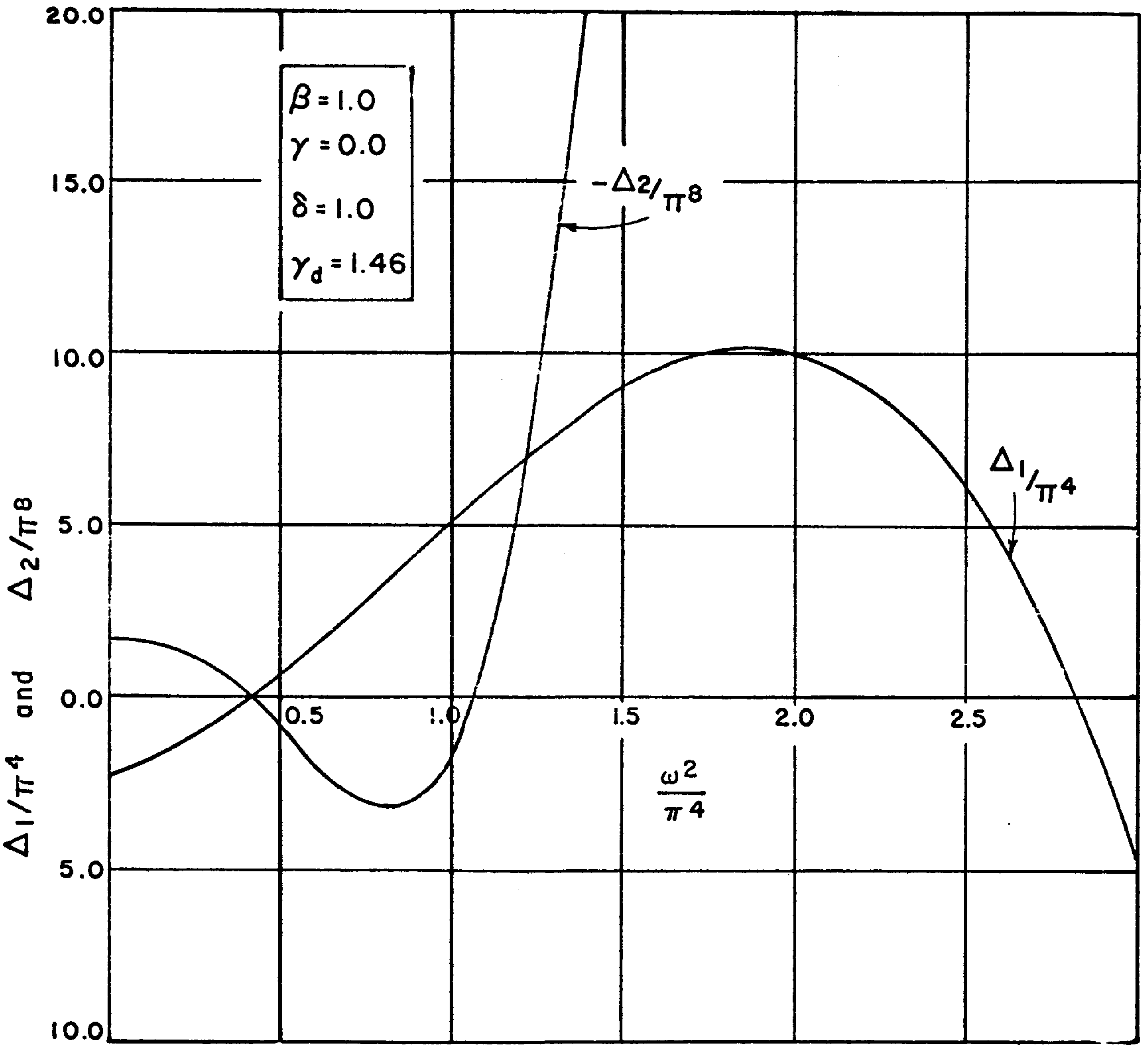


Figure 1



20.0
15.0
10.0
5.0
0.0
5.0
10.0
$\beta = 1.0$
$\gamma = 0.0$
$\delta = 1.0$
$\gamma_d = 1.46$
$-\Delta_2/\pi^8$
Δ_1/π^4
Δ_1/π^4 and Δ_2/π^8
0.5
1.0
1.5
2.0
2.5
$\frac{\omega^2}{\pi^4}$

Figure 2

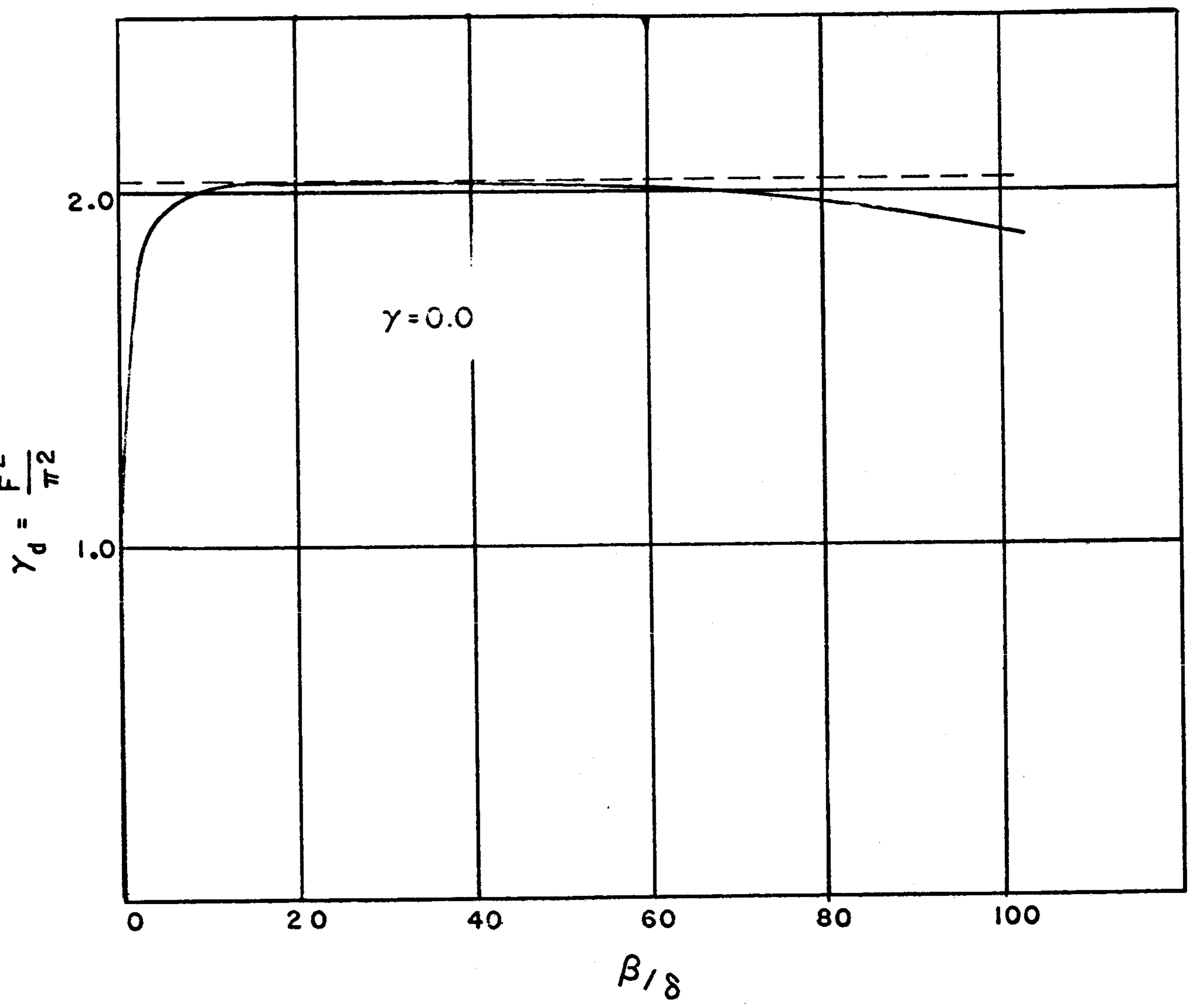
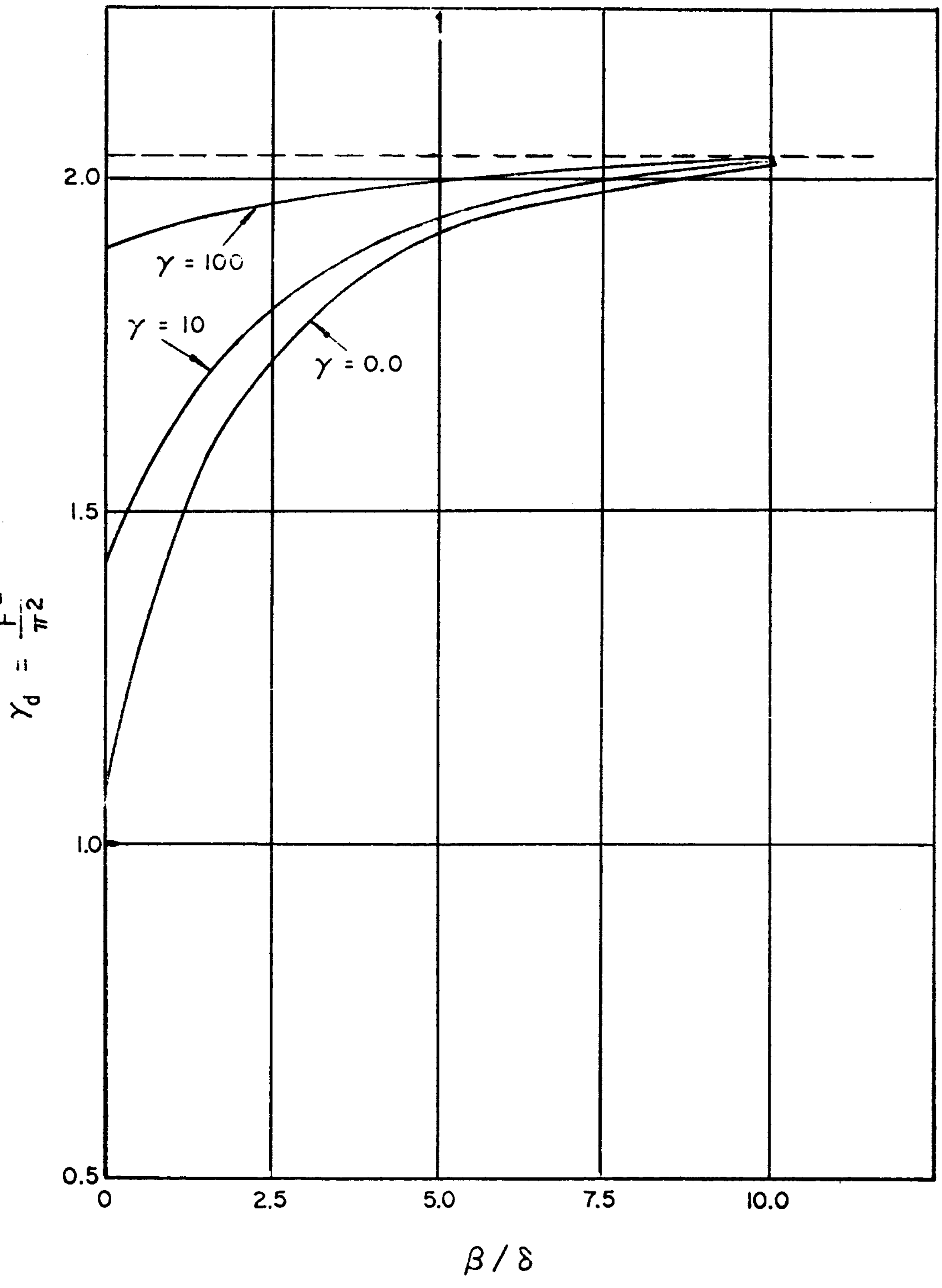


Figure 3



2.0
1.5
1.0
0.5
0
2.5
5.0
7.5
10.0
γ = 100
γ = 10
γ = 0.0
$\gamma_d = \frac{F^2}{\pi^2}$
β / δ

Figure 4

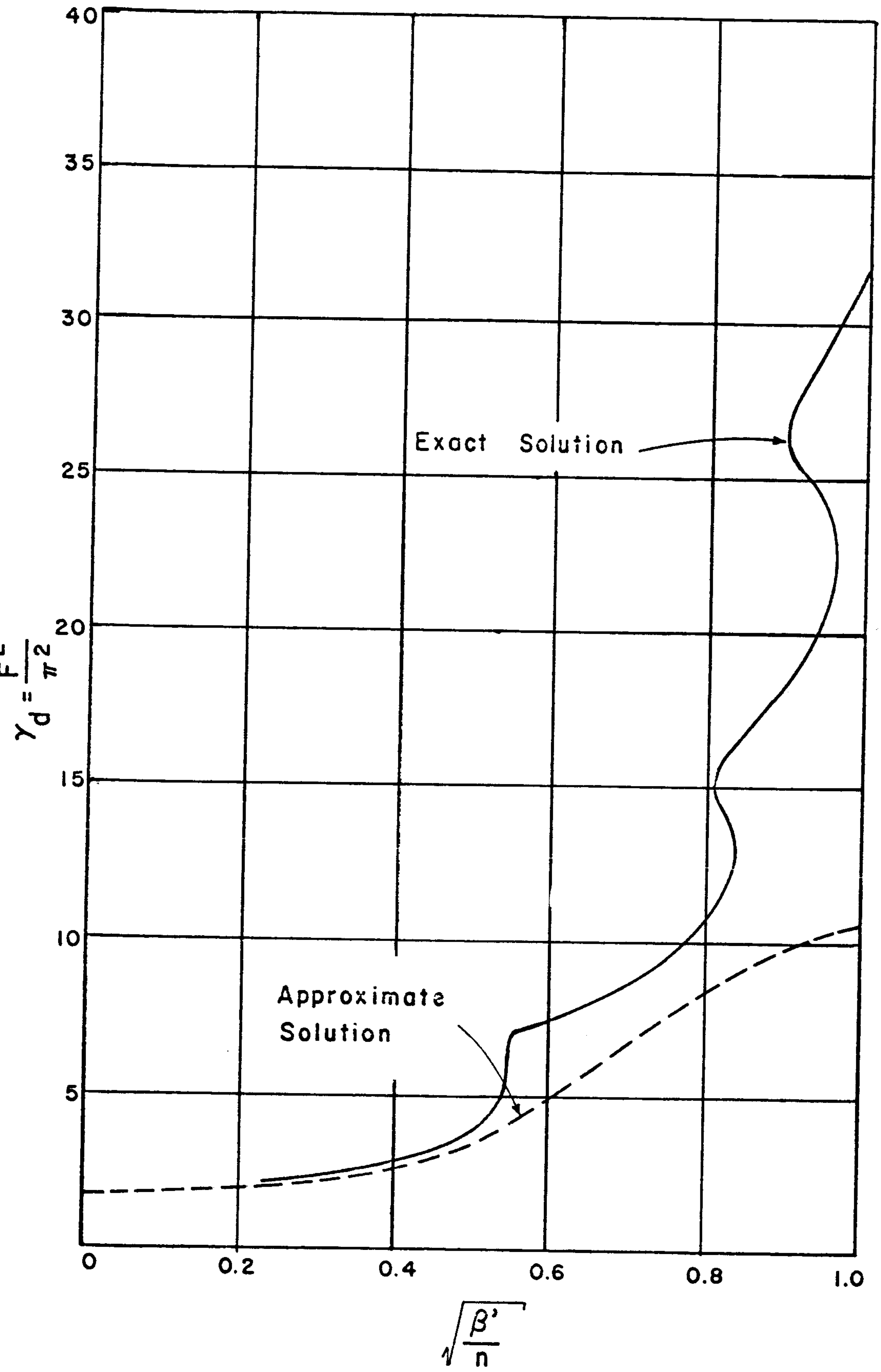

40
35
30
25
20
15
10
5
0
0.2
0.4
0.6
0.8
1.0
$\gamma_d = \frac{F^2}{\pi^2}$
$\sqrt{\frac{\beta'}{n}}$
Exact Solution
Approximate Solution

Figure 5