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ABSTRACT

The space and time dependence of decaying resonant oscillations excited in a collisionless plasma by an infinitesimally small pulsed dipole is determined, first in the absence and then in the presence of an external magnetic field.

In the presence of a magnetic field only the "quasi-electrostatic" resonances are treated. These occur at the plasma frequency Π, the electron cyclotron frequency Ω, the upper hybrid frequency $(\Pi^2 + \Omega^2)^{1/2}$, the lower hybrid resonant frequencies and at the frequencies $n\Omega$ where n is a positive integer. The quasi-electrostatic approximation is used in the treatment of these resonances; the limits of its validity are examined in some detail for the $(\Pi^2 + \Omega^2)^{1/2}$ resonance. No similar approximations are made for plasma oscillations in the absence of a magnetic field; the more accurate results so obtained differ little from those derived with the aid of the quasi-electrostatic approximation, within the region of interest.

At a given time in the presence of a magnetic field the electric field of the oscillation is approximately uniform up to a certain critical distance, beyond which the amplitude of the oscillations increases proportionally to some power (different for the different resonant frequencies) of the distance and eventually decreases again. The above mentioned critical distance increases as the square root of the time. At a given point the oscillations first build

up and then decay proportionally to some inverse power of the time which in the presence of a magnetic field is different within, and beyond the critical distance (the decay is faster beyond the critical distance) and is also different for the different resonances.

The phase of the oscillations also varies with the distance and therefore the resonant frequency observed with a top-side sounder from a space vehicle is affected by vehicular motion. For typical satellite velocities the percentage frequency shift is insignificant for the electronic resonances; for the lower hybrid resonance, however, the percentage frequency shift is very large and may rule out the use of the local excitation of this resonance in satellite or rocket investigations of the ionosphere.

Vehicular motion generally causes an increase in the observed oscillation amplitudes; the lower hybrid resonance is, however, again a very strong exception from this rule.

The change of phase of the oscillations with position is believed to be responsible for the complicated interference effects observed with the aid of large satellite-borne antennas in the ionosphere.

1. Introduction

There have been several theoretical discussions[1-3] of the resonance effects observed with top-side sounders in the ionosphere[4-6]. Fejer and Calvert[1] showed that resonant quasi-electrostatic oscillations of long persistence can occur in directions approximately parallel to the magnetic field at the plasma frequency Π and at the electron cyclotron frequency Ω, and in directions approximately normal to the magnetic field at the frequency $(\Pi^2 + \Omega^2)^{1/2}$ and at whole multiples of Ω. These are the frequencies at which resonances are consistently observed[7]. Sturrock[3] considered the excitation of resonant quasi-electrostatic oscillations by an infinitesimally small pulsed dipole. In his treatment of these oscillations he uses a combination of the collisionless Boltzmann equation with the equations of electrostatics, hereafter called the quasi-electrostatic approximation. He calculates the oscillations only at the position where the original dipole impulse occurred. In the consideration of the resonance at the harmonics of the cyclotron frequency Sturrock has to change his model from an infinitesimal dipole to an infinitesimal line charge to avoid divergent intergrals. In the treatment of certain other, weaker electromagnetic type resonances not discussed in this paper and not predicted by the quasi-electrostatic approximation Sturrock[3] uses the cold plasma approximation.

[illegible]ll[2] discusses the resonance at $(\Pi^2 + \Omega^2)^{1/2}$ more generally, without using the quasi-electrostatic approximation (i.e. by using the full set of Maxwell's equations instead of those of electrostatics), and obtains a t^{-1} asymptotic time dependence whereas Sturrock's work results in a t^{-2} time dependence.

In the present paper in the absence of an external magnetic field the resonance at the plasma frequency is treated without the restriction of the quasi-electrostatic approximation, using a combination of Maxwell's equations (not just those of electrostatics) with the collisionless Boltzmann equation. It is shown that the more correct results thus obtained represent only a minor correction to the quasi-electrostatic approximation in practice. ~~In the presence of an~~ external magnetic field the quasi-electrostatic approximation is used but both the space and the time dependence of the field is calculated. This part is thus an extension of Sturrock's[3] work and for the resonances at Π and $(\Pi^2 + \Omega^2)^{1/2}$ the results at the position of the exciting dipole impulse agree with those of Sturrock. At other positions the amplitude of the oscillations is larger and not smaller as was anticipated by Sturrock in his remarks on the effects of the satellite motion and the resulting motion of the receiver away from the point of excitation.

The oscillations at the lower hybrid resonant frequencies are also discussed in the present paper and the difficulties anticipated in the use of these resonances in ionospheric investigations from space vehicles are pointed out.

The importance of Landau damping in the treatment of the resonances at harmonics of the electron cyclotron frequency is stressed. This damping limits the angular range of the waves participating in the oscillations and its neglect leads to divergent integrals as is apparent from Sturrock's work. In the present paper, Landau damping is taken into account in only a relatively rough manner and therefore the treatment of the resonances near the harmonics of the cyclotron frequency and particularly near the cyclotron frequency itself is less accurate than the treatment of the other resonances.

Finally it is shown that if $(\Pi^2 + \Omega^2)^{1/2} < 2\Omega$ then the quasi-electrostatic approximation still describes the initial decay of the oscillations at the source near the frequency $(\Pi^2 + \Omega^2)^{1/2}$ according to a t^{-2} law correctly but that after a few milliseconds under typical ionospheric conditions the decay law at the source assumes the t^{-1} form obtained by Nuttall[2]. The field at a point moving away with the satellite velocity is, however, still well described by the quasi-electrostatic approximation.

The treatment of the present paper is restricted to sources that are infinitesimally small, both in space and in time. The limitations implied by this restriction are considered in the body of the paper and in the concluding discussion.

2. Excitation of Plasma Resonances in the Absence of a Magnetic Field.

Assuming an $\exp(i\omega t - i\underset{\sim}{k}\cdot\underset{\sim}{r})$ space-time dependence for the quantities in Maxwell's equations results in the equation

$$i\mu_o^{-1}\omega^{-1}\underset{\sim}{k}\times\underset{\sim}{k}\times\underset{\sim}{E}(\underset{\sim}{k},\omega) + i\omega\varepsilon_o\underset{\sim}{E} + \underset{\sim}{j}_{int}(\underset{\sim}{k},\omega) + \underset{\sim}{j}_{ext}(\underset{\sim}{k},\omega) = 0 \quad , \qquad (1)$$

for the electric field $\underset{\sim}{E}(\underset{\sim}{k},\omega)$, where $\underset{\sim}{j}_{int}$ is the current carried by the plasma particles and $\underset{\sim}{j}_{ext}$ is the source current and where rationalized mks units have been used. Equation (1) will be interpreted as a relation between Fourier components in the following discussion.

The external charge density is taken to be

$$\rho_{ext}(\xi,\eta,\zeta,t) = \delta(t)\delta(\xi)\delta(\eta)\,[\delta(\zeta+\ell)\int_{-\infty}^{\infty} q\,dt - \delta(\zeta-\ell)\int_{-\infty}^{\infty} q\,dt] \qquad (2)$$

where the charges $-\delta(t)\int_{-\infty}^{\infty} q\,dt$ and $\delta(t)\int_{-\infty}^{\infty} q\,dt$, situated at the points $(0,0,\ell)$ and $(0,0,-\ell)$ of the cartesian coordinate system ξ,η,ζ form a dipole whose axis is parallel to the ζ axis. If ℓ is very small (and $\int q\,dt$ correspondingly large) then an expansion of the delta functions in ζ in powers of ℓ results, to first order in

$$\rho_{ext}(\xi,\eta,\zeta,t) = P\delta(t)\delta(\xi)\delta(\eta)\,\frac{d\delta(\zeta)}{d\zeta} \quad , \qquad (3)$$

where $P = 2\ell\int q\,dt$ is the time integral of the dipole moment of the infinitesimally pulsed dipole radiator situated at the origin. A combination of (3) with the equation of continuity div $\underset{\sim}{j}_{ext} = -\partial\rho_{ext}/\partial t$ gives

$$\underset{\sim}{j}_{ext} = P\delta(\xi)\delta(\eta)\delta(\zeta)\frac{d\delta(t)}{dt}\underset{\sim}{u} \quad , \tag{4}$$

with $\underset{\sim}{u}$ being the unit vector in the direction of the dipole axis.

By Fourier's integral theorem

$$\rho_{ext}(\underset{\sim}{r},t) = \int \rho_{ext}(\underset{\sim}{k},\omega)\exp(i\omega t - i\underset{\sim}{k}\cdot\underset{\sim}{r})\, d\underset{\sim}{k}\, d\omega \quad , \tag{5}$$

where

$$\rho_{ext}(\underset{\sim}{k},\omega) = -(2\pi)^{-4} ik_{\zeta}P = (2\pi)^{-4} i\underset{\sim}{k}\cdot\underset{\sim}{u}P \quad . \tag{6}$$

Similarly,

$$\underset{\sim}{j}_{ext}(\underset{\sim}{k},\omega) = (2\pi)^{-4} i\omega P\underset{\sim}{u} \quad . \tag{7}$$

Equations (6) and (7) are independent of the coordinates ξ, η, ζ which will not be used in subsequent parts of the paper.

The plasma current density, obtained from the collisionless Boltzmann equation, neglecting the motion of ions, is[9]

$$\underset{\sim}{j}_{int}(\underset{\sim}{k},\omega) = \frac{ie^2}{m}\underset{\sim}{E}(k,\omega)\cdot\int \frac{(\partial f_o/\partial\underset{\sim}{v})\underset{\sim}{v}}{\omega - \underset{\sim}{k}\cdot\underset{\sim}{v}}\, d\underset{\sim}{v} \tag{8}$$

where the integration path is above the pole and where e is the magnitude of the charge and m the mass of an electron. The distribution $f_o = N(m/2\pi KT)^{3/2} \exp(-mv^2/2KT)$ of the particle velocities $\underset{\sim}{v}$ is assumed to be Maxwellian. Without loss of generality the vector $\underset{\sim}{k}$ is taken parallel to the z axis and the vector $\underset{\sim}{E}$ is assumed to lie in the x-z plane in the following calculation. The component equations of (8) are therefore

$$j_{x\,int}(\underset{\sim}{k},\omega) = -\frac{ie^2}{KT} E_x(\underset{\sim}{k},\omega)\int \frac{f_o v_x^2}{\omega - kv_z}\, d\underset{\sim}{v} \quad ,$$

$$j_{y\,int}(\underset{\sim}{k},\omega) = 0 \quad , \tag{9}$$

$$j_{z\,int}(\underset{\sim}{k},\omega) = -\frac{ie^2}{KT} E_z(\underset{\sim}{k},\omega)\int \frac{f_o v_z^2}{\omega - kv_z}\, d\underset{\sim}{v} \quad .$$

These equations relate the components of the plasma current density $j_{z\,int} = j_{||\,int}$ and $j_{x\,int} = j_{\perp\,int}$, parallel and perpendicular to $\underset{\sim}{k}$, to the parallel and perpendicular components $E_z(k,\omega) = E_{||}$ and $E_x(k,\omega) = E_{\perp}$ of the electric field. Using the identities

$$\int_{-\infty}^{\infty} \frac{\exp(-bv^2)}{a - v}\, dv = 2\pi^{1/2}\, e^{-ba^2}\int_o^{ab^{1/2}} e^{\tau^2}\, d\tau + i\pi e^{-ba^2} \tag{10}$$

and

$$\int_{-\infty}^{\infty} \frac{v^2\exp(-bv^2)}{a - v}\, dv = 2\pi^{1/2} a^2\, e^{-ba^2}\int_o^{ab^{1/2}} e^{\tau^2}\, d\tau - \pi^{1/2} a/b^{1/2} + i\pi a^2 e^{-ba^2}, \tag{11}$$

where the integration path is above the pole, equation (9) may be written in the form

$$j_{\perp\,int} = E_{\perp}\varepsilon_o h^{-2} k^{-1}\left(\frac{KT}{2m}\right)^{1/2}\left[-i2e^{-\frac{m\omega^2}{2KTk^2}}\int_o^{\frac{\omega}{k}\left(\frac{m}{2KT}\right)^{1/2}} e^{\tau^2} d\tau + \pi^{1/2} e^{-\frac{m\omega^2}{2KTk^2}}\right] \tag{12}$$

$$j_{||\,int} = E_{||}\varepsilon_o h^{-2} k^{-3}\omega^2\left(\frac{m}{2KT}\right)^{1/2}\left[-2i\, e^{-\frac{m\omega^2}{2KTk^2}}\int_o^{\frac{\omega}{k}\left(\frac{m}{2KT}\right)^{1/2}} e^{\tau^2}\, d\tau + i\frac{k}{\omega}\left(\frac{2KT}{m}\right)^{1/2} + \pi^{1/2}\, e^{-\frac{m\omega^2}{2KTk^2}}\right] \quad , \tag{13}$$

where $h = (\varepsilon_o KT/Ne^2)^{1/2}$. If $m\omega^2/2KTk^2 \gg 1$, the asymptotic expansion

$$2e^{-x^2}\int_0^x e^{\tau^2}\,d\tau = \frac{1}{x} + \sum_{\nu=1} \frac{1}{2}\cdot\frac{3}{2}\cdots\frac{2\nu-1}{2}\,x^{-(2\nu+1)} \tag{14}$$

may be used and the real parts of the expressions in the square brackets on the right hand side of (12) and (13) may be neglected in comparison with the imaginary parts. This is equivalent to the neglect of Landau damping and to the restriction of attention to the decay rather than to the building up of the oscillations at a given point; the complete solution has been considered by Weitzner[10]. Substitution of (7), (12) and (13) into (1), written in terms of components parallel and perpendicular to $\underset{\sim}{k}$, then yields the following vector equations for the parallel and perpendicular components of the Fourier transform $\underset{\sim}{E}(\underset{\sim}{k},\omega)$ of the electric field $\underset{\sim}{E}(\underset{\sim}{r},t)$:

$$\underset{\sim}{E}_{||}(\underset{\sim}{k},\omega) = (2\pi)^{-4}\varepsilon_o^{-1}\,P\left(\frac{\Pi^2}{\omega^2} - 1 + \frac{3k^2h^2\Pi^4}{\omega^4}\right)^{-1} (\underset{\sim}{k}\cdot\underset{\sim}{u}/k^2)\underset{\sim}{k} \quad , \tag{15}$$

$$\underset{\sim}{E}_{\perp}(\underset{\sim}{k},\omega) = -(2\pi)^{-4}\varepsilon_o^{-1}\,P\left(\frac{\Pi^2}{\omega^2} - 1 + \frac{c^2k^2}{\omega^2} + k^2h^2\Pi^4/\omega^4\right)^{-1} \underset{\sim}{k}\times\underset{\sim}{k}\times\underset{\sim}{u}/k^2 \,, \tag{16}$$

where $\Pi = (Ne^2/\varepsilon_o m)^{1/2}$ is the electron plasma frequency.

If attention is confined to sufficiently small values of k, then the reciprocals of the bracketed expressions assume large values only when $\omega \sim \Pi$ and therefore ω may be replaced by Π when the difference $\Pi - \omega$ is not involved. Equations (15) and (16) then assume the forms

$$\underset{\sim}{E}_{||}(\underset{\sim}{k},\omega) = -(2\pi)^{-4}\varepsilon_o^{-1}\;P\Pi^2(\omega^2-\Pi^2-3k^2V^2)^{-1}\;(\underset{\sim}{k}\cdot\underset{\sim}{u}/k^2)\underset{\sim}{k} \quad , \tag{17}$$

$$\underset{\sim}{E}_{\perp}(\underset{\sim}{k},\omega) = (2\pi)^{-4}\varepsilon_o^{-1}\;P\Pi^2[\omega^2-\Pi^2-k^2(c^2+V^2)]^{-1}\;\underset{\sim}{k}\times\underset{\sim}{k}\times\underset{\sim}{u}/k^2 \quad , \tag{18}$$

where $V^2 = KT/m$.

The Fourier inversions of (17) and (18) may be most easily accomplished by using the relations

$$\underset{\sim}{u}\cdot\underset{\sim}{k}\underset{\sim}{k}e^{-i\underset{\sim}{k}\cdot\underset{\sim}{r}} = -\underset{\sim}{u}\cdot\nabla\nabla e^{-i\underset{\sim}{k}\cdot\underset{\sim}{r}} \quad , \quad \underset{\sim}{k}\times\underset{\sim}{k}\times\underset{\sim}{u}e^{-i\underset{\sim}{k}\cdot\underset{\sim}{r}} = -\nabla\times\nabla\times\underset{\sim}{u}e^{-i\underset{\sim}{k}\cdot\underset{\sim}{r}} \quad . \tag{19}$$

The space-time longitudinal and transverse electric fields are found to be

$$\underset{\sim}{E}_{||}(\underset{\sim}{r},t) = \frac{p\Pi^2}{(2\pi)^4\varepsilon_o}\,\underset{\sim}{u}\cdot\nabla\nabla\int\frac{d\underset{\sim}{k}d\omega}{k^2}\,\frac{\exp[i\omega t-i\,\underset{\sim}{k}\cdot\underset{\sim}{r}]}{\omega^2-\Pi^2-3k^2V^2} \tag{20}$$

$$\underset{\sim}{E}_{\perp}(\underset{\sim}{r},t) = -\frac{p\Pi^2}{(2\pi)^4\varepsilon_o}\,\nabla\times\nabla\times\underset{\sim}{u}\int\frac{d\underset{\sim}{k}d\omega}{k^2}\,\frac{\exp[i\omega t-i\,\underset{\sim}{k}\cdot\underset{\sim}{r}]}{\omega^2-\Pi^2-k^2(c^2+V^2)} \tag{21}$$

Thus, the electric field is determined by integrals having the form

$$A = \int\frac{d\underset{\sim}{k}d\omega}{k^2}\,\frac{\exp[i\omega t-i\,\underset{\sim}{k}\cdot\underset{\sim}{r}]}{\omega^2-\Pi^2-w^2k^2} \tag{22}$$

where $w = 3^{1/2}V$ for the longitudinal field and $w = (c^2+V^2)^{1/2}$ for the transverse field. The integration over frequencies is given by

$$\int_{-\infty}^{\infty}\frac{e^{i\omega t}}{\omega^2-\omega_o^2}\,d\omega = -2\pi\,\frac{\sin\omega_o t}{\omega_o} \quad , \tag{23}$$

with the contour passing below the singularities. Also, the integrations over the directions of $\underset{\sim}{k}$ are easily performed and the result is

$$\int_0^{2\pi}d\phi\int_0^{\pi}d\theta e^{-ikr\cos\theta}\sin\theta = 4\pi\,\frac{\sin kr}{kr} \tag{24}$$

Equation (22) for A therefore reduces to

$$A = \frac{8\pi^2}{wr}\int_0^{\infty}dk\,\frac{\sin kr}{k}\,\frac{\sin\left[wt\left(\frac{\Pi^2}{w^2}+k^2\right)^{1/2}\right]}{\left(\frac{\Pi^2}{w^2}+k^2\right)^{1/2}} \tag{25}$$

The last factor in the integrand of (25) may be written[11] in the form

$$\frac{\sin\left[wt\left(\frac{\Pi^2}{w^2}+k^2\right)^{1/2}\right]}{\left(\frac{\Pi^2}{w^2}+k^2\right)^{1/2}} = \int_o^{wt} d\beta \cos k\beta J_o\left[\frac{\Pi}{\omega}\left(w^2t^2-\beta^2\right)^{1/2}\right] \quad , \qquad (26)$$

in which J_o is the Bessel function of first kind and zero order. Using (26) in (25), the k-integration may be carried out[12] and A becomes

$$A = \frac{4\pi^3}{wr}\int_o^{wt} H(r-\beta)\; J_o\left[\frac{\Pi}{w}\left(w^2t^2-\beta^2\right)^{1/2}\right] d\beta \quad , \qquad (27)$$

where $H(x) = 1$ if $x > 0$, $H(x) = 0$ if $x < 0$.

If $r < wt$, (27) becomes

$$A = \frac{4\pi^3}{wr}\int_o^{r} J_o\left[\Pi t\left(1-\frac{\beta^2}{w^2t^2}\right)^{1/2}\right] d\beta \quad . \qquad (28)$$

Whenever $r << wt$, the radical in J_o may be replaced by $1-\beta^2/2w^2t^2$; furthermore, for times of interest here $\Pi t >> 1$ and J_o may be replaced by its large argument approximation. The result of these approximations on J_o is

$$J_o\left[\Pi t\left(1-\frac{\beta^2}{w^2t^2}\right)^{1/2}\right] \cong \left(\frac{2}{\pi\Pi t}\right)^{1/2} \cos\left(\Pi t-\frac{\Pi\beta^2}{2w^2t}-\frac{\pi}{4}\right) \quad , \qquad (29)$$

and A now reduces to

$$A \cong \left(\frac{32\pi^5}{\Pi t}\right)^{1/2} \frac{1}{wr} \operatorname{Re}\left\{e^{i(\Pi t-\frac{\pi}{4})}\int_o^r e^{-\frac{i\Pi\beta^2}{2w^2t}} d\beta\right\} \qquad (30)$$

(1) If $r^2\Pi/2w^2t << 1$, the exponential in the integrand of A may be expanded. The result of the integration in (30) is

$$A = \left(\frac{32\pi^5}{\Pi t}\right)^{1/2} \frac{1}{w} \left[\cos(\Pi t - \frac{\pi}{4}) + \frac{1}{3}\frac{\Pi r^2}{2w^2 t} \sin(\Pi t - \frac{\pi}{4})\right] . \tag{31}$$

From (20), (21) and (31), the electric fields are

$$\underset{\sim}{E}_{||}(\underset{\sim}{r},t) = \frac{P}{12\epsilon_o V^3}\left(\frac{2\Pi^5}{\pi^3 t^3}\right)^{1/2} \underset{\sim}{u} \sin(\Pi t - \frac{\pi}{4}) \tag{32}$$

and

$$\underset{\sim}{E}_{\perp}(\underset{\sim}{r},t) = \frac{P}{6\epsilon_o (c^2+V^2)^{3/2}}\left(\frac{2\Pi^5}{\pi^3 t^3}\right)^{1/2} \underset{\sim}{u} \sin(\Pi t - \frac{\pi}{4}) . \tag{33}$$

(2) If $r^2\Pi/2w^2t >> 1$ the asymptotic expansion

$$\int_o^r e^{-i\Pi\beta^2/2w^2t} d\beta \cong \left(\frac{\pi w^2 t}{2\Pi}\right)^{1/2} e^{-i\frac{\pi}{4}} + \frac{w^2 t}{r\Pi} \exp\left[-i\left(\frac{\Pi r^2}{2w^2 t} - \frac{\pi}{2}\right)\right] \tag{34}$$

may be used in (30) and then A reduces to

$$A \cong \frac{4\pi^3}{r\Pi} \sin\Pi t + \left(\frac{32t\pi^5}{\Pi^3}\right)^{1/2} \frac{w}{r^2} \cos\left(\Pi t - \frac{\Pi r^2}{2w^2 t} + \frac{\pi}{4}\right) . \tag{35}$$

The electric fields at distances r, which satisfy $2w^2t^2/\Pi t << r^2 << w^2t^2$, are given by forming the appropriate derivatives of the second term in (35). The results (for $c >> V$) are

$$\underset{\sim}{E}_{||}(\underset{\sim}{r},t) \cong \frac{P}{6\epsilon_o V^3}\left(\frac{\Pi^5}{6\pi^3 t^3}\right)^{1/2} \underset{\sim}{e}_r \cos\theta \cos\left(\Pi t - \frac{3\pi}{4} - \frac{\Pi r^2}{6V^2 t}\right) \tag{36}$$

$$\underset{\sim}{E}_{\perp}(\underset{\sim}{r},t) \simeq \frac{p}{\varepsilon_o c^3}\left(\frac{\Pi^5}{8\pi^3 t^3}\right)^{1/2} \underset{\sim}{e}_\theta \sin\theta \cos\left(\Pi t + \frac{\pi}{4} - \frac{\Pi r^2}{2c^2 t}\right) \qquad (37)$$

where θ is the angle between the radius vector $\underset{\sim}{r}$ and the dipole axis.

Equations (32), (33), (36), (37) show that $\underset{\sim}{E}_{||}$ is almost everywhere very much larger than $\underset{\sim}{E}_{\perp}$ provided that the velocity V, characteristic of the thermal motion of the electrons, is much smaller than the velocity of light c.

Attention may now be restricted to equations (32) and (36) whose respective regions of validity are separated roughly by a sphere of radius $r = 2^{1/2}V\Pi^{-1/2}t^{1/2}$; the radius of this sphere increases proportionally to the square root of the time. Well inside the sphere the field given by (32) is uniform and parallel to the dipole axis. Well outside the sphere but for distances much smaller than Vt the field is almost radial and proportional to cos θ; the relation of the field in this region to group propagation is discussed in Section 3.2. At distances not much smaller than Vt it may be shown that the approximations of the present theory (kh << 1 and the neglect of Landau damping) are invalid; presumably the field starts decreasing long before the distance becomes equal to Vt.

The phase term $\Pi r^2/2w^2t$ in (36) should be noted; it differs from the usual phase term occurring in wave propagation which varies linearly with r and t, rather than with r^2 and t^{-1}. The existence of this phase term can give rise to complicated interference patterns if large antennas are used; such interference patterns have been observed with top-side sounders.

3. Electrostatic Oscillations in a Magnetic Field

3.1 Basic Concepts

The relationship between the electric field and the current density in a plasma is complicated considerably by the presence of an external magnetic field. In the last section it became clear that, in the absence of a magnetic field, the electric field of the dipole was almost entirely determined by $E_{||}$, at least for reasonable values of the distance and the time delay. The results obtained there would have remained practically unchanged if the speed of light c had been set equal to infinity in Maxwell's equations. The complications introduced by the external magnetic field are greatly reduced if the speed of light c is set equal to infinity or, differently expressed, if the equations of electrostatics

$$-i\epsilon_o \underset{\sim}{k}\cdot\underset{\sim}{E}(\underset{\sim}{k},\omega) = \rho_{int}(\underset{\sim}{k},\omega) + \rho_{ext}(\underset{\sim}{k},\omega) \tag{38}$$

$$\underset{\sim}{k} \times \underset{\sim}{E} = 0 \quad , \tag{39}$$

are substituted for Maxwell's equations. The electric field is then always parallel to $\underset{\sim}{k}$. It will be shown in section 4 that the approximation $c = \infty$ is not always a good one in the presence of a magnetic field although it is probably adequate in practice.

The external charge density ρ_{ext} and its Fourier transform are taken to be given by equations (5) and (6). The charge density ρ_{int} of the plasma may be shown[13] to be given by

$$\rho_{int}(\underset{\sim}{k},\omega) = E(\underset{\sim}{k},\omega)\sum_r \beta_r \tag{40}$$

where

$$1+i\frac{kh_r^2}{\varepsilon_o}\beta_r = i\omega \exp\left(-\frac{k^2h_r^2\Pi_r^2\sin^2\theta}{\Omega_r^2}\right)\sum_{m=-\infty}^{\infty} I_m\left(\frac{k^2h_r^2\Pi_r^2\sin^2\theta}{\Omega_r^2}\right) \times$$

$$\times \int_o^\infty \exp\left[-\frac{k^2h_r^2\Pi_r^2\cos^2\theta}{2}\tau^2 - i(\omega+m\Omega_r)\tau\right] d\tau \tag{41}$$

and where $\Omega_r = eZ_rB/m_r$ is the cyclotron frequency, $\Pi_r = (Ne^2Z_r^2/m_r\varepsilon_o)^{1/2}$ the plasma frequency and $h_r = (\varepsilon_o KT/Ne^2Z_r^2)^{1/2}$ is the Debye length associated with r-th type charged particles with mass m_r and charge eZ_r. The particles with $r = 1$ are taken to be electrons, for which $Z_1 = -1$. From (38), (39), and (40) one obtains the equation

$$\underset{\sim}{E}(\underset{\sim}{k},\omega) = \frac{\rho_{ext}(\underset{\sim}{k},\omega)}{-i\varepsilon_o k-\sum_r \beta_r}\frac{\underset{\sim}{k}}{k} = \frac{i\rho_{ext}(\underset{\sim}{k},\omega)}{\varepsilon_o k^2}\frac{\underset{\sim}{k}}{1-i\frac{1}{\varepsilon_o k}\sum_r \beta_r} . \tag{42}$$

The electric field $\underset{\sim}{E}(\underset{\sim}{r},t)$ of the pulsed dipole is obtained by taking the inverse Fourier transform of equation (42), with β_r substituted from equation (41).

Formally the solution of the problem has thus been obtained in terms of infinite sums of multiple integrals. In practice it has been shown[1] that, in the presence of a magnetic field, resonant plasma oscillations contain only waves with propagation vectors $\underset{\sim}{k}$ nearly parallel or nearly perpendicular to the magnetic field ($\theta \sim 0$ or $\theta \sim \frac{\pi}{2}$) and with wave numbers $\underset{\sim}{k}$ much smaller than the reciprocal Debye length (or the reciprocal mean cyclotron radius in some cases).

In the following sections the approximations $k\sim0$ and $\theta\sim0$ or $\theta\sim\Pi/2$ (as the case may be) will be used to simplify the evaluation of the relevant integrals which represent individual oscillating contributions to the total field in the vicinity of certain resonant frequencies. In each case the self-consistency of these approximations is demonstrated.

3.2 The Resonance at the Plasma Frequency

The propagation vectors of the waves participating in this resonance are nearly parallel to the magnetic field and their wave lengths are large compared to the Debye length[1]. If the inequality

$$kh \ll |(m\Omega + \omega)/\Pi\cos\theta| \tag{43}$$

is satisfied for all integral values of m then Landau damping may be neglected and equation (41) may be expanded to yield [cf. the derivation of equation (15) in Ref. 1].

$$\frac{\beta_r}{i\varepsilon_o k} = \frac{\Pi_r^2\sin^2\theta}{\Omega_r^2-\omega^2} - \frac{\Pi_r^2\cos^2\theta}{\omega^2} - \frac{k^2h_r^2\Pi_r^4}{\Omega_r^4}\left[\frac{3\sin^4\theta}{(q_r^2-1)(q_r^2-4)} + \frac{6q_r^4-3q_r^2+1}{q_r^2(q_r^2-1)^3}\times \sin^2\theta\cos^2\theta + \frac{3\cos^4\theta}{q_r^4}\right] . \tag{44}$$

where the abbreviations $q_r = \omega/\Omega_r$ is used.

Only the electronic terms are appreciable near the plasma frequency. The use of the approximations $\sin^2\theta\sim\theta^2$ and $\omega^2\sim\Pi^2$ (except where the difference $\omega^2-\Pi^2$ occurs) in a combination of equations (42) and (44) then leads to the equation

$$\underset{\sim}{E}(\underset{\sim}{k},\omega) = \frac{i\rho_{ext}\underset{\sim}{k}\Pi^2}{\varepsilon_o k^2}\left[\omega^2-\Pi^2 + \frac{\Pi^2\Omega^2}{\Omega^2-\Pi^2}\theta^2 - 3k^2V^2\right]^{-1} \tag{45}$$

If the z axis of the coordinate system is taken parallel to the magnetic field and to the axis of the pulsed dipole, then the inverse Fourier transform of the z-component of $\underset{\sim}{E}$ is

$$E(\underset{\sim}{r},t) \approx E_z(\underset{\sim}{r},t) = \int E_z(\underset{\sim}{k},\omega)\, \exp(i\omega t - i\underset{\sim}{k}\cdot\underset{\sim}{r})\, d\underset{\sim}{k}\, d\omega \tag{46}$$

Since the $\underset{\sim}{k}$ vectors are nearly parallel to the z-axis, the prominent spatial variation will be in the z-direction. After the necessary substitutions from (6) and (45) and completion of the ω integration with the aid of (23), the field along the z-axis is given by the equation

$$E(z,t) = (2\pi)^{-2}\, \varepsilon_o^{-1}\, P\Pi[F(t) - F(-t)] \quad , \tag{47}$$

where

$$F(t) = \frac{i}{2} e^{i\Pi t} \frac{\partial^2}{\partial z^2} \int_{k=0}^{\infty} \int_{\theta=0}^{\infty} \exp\left\{ i\left[-\frac{1}{2} \frac{\Omega^2 \Pi t \theta^2}{\Omega^2 - \Pi^2} + \frac{3}{2} \frac{V^2 t k^2}{\Pi} \pm kz \right] \right\} \theta d\theta dk \tag{48}$$

and where a summation of the integrals for the two alternative signs is meant, corresponding to the two θ integrals near $\theta=0$ and $\theta=\pi$.

In the subsequent calculations use will be made of the identities for integral n,

$$\lim_{\varepsilon\to 0} \int_0^{\infty} x^{2n+1}\, e^{(ia-\varepsilon)x^2}\, dx = \frac{i^{n+1}}{2}\, n!\; a^{-(n+1)} \tag{49}$$

and

$$\lim_{\varepsilon\to 0} \int_0^{\infty} x^{2n}\, e^{(ia-\varepsilon)x^2}\, dx = \frac{\pi^{1/2}}{2}\, i^n\, e^{i\frac{\pi}{4}} (1 - \tfrac{1}{2})\cdots(n - \tfrac{1}{2})\; a^{-(n+\frac{1}{2})} \quad , \tag{50}$$

There are corresponding integrals between the limits $-\infty$ and ∞ in the case of (50). The θ integration can be performed using (49) with $n = 0$. The k integrand in (48) can be written as $2\cos[kz]\exp(3iV^2tk^2/2\Pi)$ and the k-integral has the value[12]

$$2\int_0^\infty \exp(iak^2)\cos kz\, dk = \left(\frac{\pi}{a}\right)^{1/2} e^{i[\frac{\pi}{4} - \frac{z^2}{4a}]} \tag{51}$$

where $a = 3V^2 t/2\Pi$. The result for F(t) is

$$F(t) = \left(\frac{\pi}{4a}\right)^{1/2} e^{i(\Pi t+\frac{\pi}{4})} \frac{\partial^2}{\partial z^2} \frac{(\Omega^2-\Pi^2)e^{-i\frac{z^2}{4a}}}{\Omega^2 \Pi t} \tag{52}$$

and for F(-t)

$$F(-t) = -\left(\frac{\pi}{4a}\right)^{1/2} e^{-i(\Pi t+\frac{\pi}{4})} \frac{\partial^2}{\partial z^2} \frac{(\Omega^2-\Pi^2)e^{i\frac{z^2}{4a}}}{\Omega^2 \Pi t} \tag{53}$$

where t is taken as positive.

Use of (47), (52) and (53) gives the electric field as

$$E(z,t) = \frac{P\Pi}{4\pi^2\epsilon_o}\left(\frac{\pi}{a}\right)^{1/2} \frac{\partial^2}{\partial z^2} \frac{(\Omega^2-\Pi^2)\cos(\Pi t+\frac{\pi}{4} - \frac{z^2}{4a})}{\Omega^2 \Pi t} \tag{54}$$

If $z^2/4a << 1$, (54) reduces to the uniform field

$$E(z,t) = \frac{P(\Omega^2-\Pi^2)}{\epsilon_o \Omega^2 t^{5/2}} \left(\frac{\Pi}{6\pi V^2}\right)^{3/2} \sin\left(\Pi t+\frac{\pi}{4}\right) \tag{55}$$

If $z^2/4a >> 1$, (54) reduces to

$$E(z,t) = -\frac{P(\Omega^2-\Pi^2)}{(2\pi)^{3/2}\epsilon_o \Omega^2} \left(\frac{\Pi}{3V^2}\right)^{5/2} \frac{z^2}{t^{7/2}} \cos\left(\Pi t+\frac{\pi}{4} - \frac{z^2 \Pi}{6V^2 t}\right) \tag{56}$$

This expression indicates that the field actually increases as the square of the distance for $z >> (6V^2t/\Pi)^{1/2}$. It may be easily shown (by an approximate calculation of the inverse Fourier transform) that in the x-y plane the field is uniform to the much larger distance of about Vt. Since space vehicles move with speeds much less than V, the variation of the field for the resonance at the plasma frequency in a direction normal to the magnetic field (or, for any resonance in a direction nearly perpendicular to the k vectors of the prominent constitutent waves) will be neglected.

If the cosine in (51) is written as a sum of exponentials then it may be seen that for $z^2/4a \gg 1$ the wave numbers contributing substantially to the field are in the neighborhood of $z/2a$; it is very significant to note that the value of the group velocity for $k = z/2a$ is z/t as would be expected. The present approximations are then only valid if $z/2a \ll h^{-1}$. The combination of the conditions $z^2/4a \gg 1$ and $z/2a \ll h^{-1}$ leads to the inequalities

$$\frac{z}{h} \ll \Pi t \ll \left(\frac{z}{h}\right)^2 \tag{57}$$

which imply $z/h \gg 1$ and $\Pi t \gg 1$; that is, z must be many Debye lengths and the decay described by (56) occurs after many plasma periods. Furthermore, the time interval over which (56) is valid is considerable since $z/h \gg 1$. The first of the inequalities (57) is equivalent to $z \ll Vt$; thus, for a fixed value of time, equation (56) which indicates that E is proportional to z^2 is only correct if $z \ll Vt$. For $z > Vt$, the present results are invalid and the field is probably quite small on account of Landau damping. The value of the field given by (56) for $t = z/V$ may be regarded as a sort of upper bound (larger than the maximum) of the field at a given point z; it is proportional to $z^{-3/2}$ and thus decreases with increasing distance. It should be remarked here that the condition $z \ll Vt$ would be automatically satisfied for a vehicle that moves away from the source point with a velocity much smaller than V.

Equation (56) shows that the phase of the oscillations depends on position (on the z coordinate). This has the consequence that the field of a large antenna has a complicated interference pattern. It is therefore not surprising that the oscillations received after pulse excitation by a large moving antenna do not decay steadily, but fluctuate in intensity with a quasi-period much longer than the oscillation period[8]. No detailed theoretical investigation of these interference effects will be carried out here. However, an effective wavelength λ for these effects may be defined for given values of z and t by the condition $[(z+\lambda)^2 - z^2]\Pi/6V^2t = 2\pi$. If $z >> \lambda$ is assumed then this equation yields $\lambda/h = 6\pi\Pi t/(z/h)$. This equation states that the wavelength, measured in Debye lengths, is equal to 6π times the ratio of the time measured in plasma periods to the distance z, also measured in Debye lengths. It is interesting to note that if the satellite travels at a velocity v_s, parallel to the magnetic field and reaches a distance z large compared to the dimensions of the antenna so that $z \sim v_s t$ then λ is independent of the time and has typical values near 10 meters at a height of about 1000 km in the ionosphere. At these large distances there should be no fluctuations in received amplitude.

3.3 The Hybrid Resonances

3.3.1 General Equations

The propagation vectors of the waves participating in these resonances are very nearly perpendicular to the magnetic field and their wave lengths tend to be large compared to the Debye length [or more precisely satisfy the inequality (43)]. If the inequality (43) is satisfied then, after the introduction of the complimentary angle $\psi = \pi/2-\theta$ and the approximations $\cos^2\theta = \psi^2$, $\sin^2\theta = 1-\psi^2$, a combination of equations (42) and (44) leads to the expression

$$\underset{\sim}{E}(\underset{\sim}{k},\omega) = \frac{i\rho_{ext}\underset{\sim}{k}}{\varepsilon_o k^2}\left\{1 + \sum_r \left[-\frac{\Pi_r^{\,2}}{\omega^2-\Omega_r^{\,2}}(1-\psi^2) - \frac{\Pi_r^{\,2}}{\omega^2}\psi^2 - \frac{3k^2h_r^{\,2}\Pi_r^{\,4}}{\Omega_r^{\,4}(q_r^{\,2}-1)(q_r^{\,2}-4)}\right]\right\}^{-1} \quad (58)$$

In ρ_{ext}, given by equation (6), $\underset{\sim}{u}$ is taken to be perpendicular to the magnetic field.

Resonance occurs when the expression in the curly bracket vanishes for $k = 0$ and $\psi = 0$; this condition leads to the equation

$$1 - \frac{\Pi_e^2}{\omega^2-\Omega_e^2} - \frac{\Pi_i^2}{\omega^2-\Omega_i^2} = 0 \quad , \tag{59}$$

if only one type of ion is assumed to be present. Since the ionic mass is much greater than the electronic mass, the two solutions of the quadratic equation obtained from (59) may be written approximately as

$$\omega_1^2 = \Pi_e^2 + \Omega_e^2 \quad , \tag{60}$$

$$\omega_2^2 = \frac{\Omega_e\Omega_i(\Omega_e\Omega_i+\Pi_e^2)}{\Omega_e^2 + \Pi_e^2} \quad , \tag{61}$$

where ω_1 and ω_2 are the upper and lower hybrid resonant frequencies.

If several ions are present then equation (60) remains approximately valid but there are as many lower hybrid resonances as there are types of ions. If the ion cyclotron frequencies are $\Omega_{i1} > \Omega_{i2} > \Omega_{i3}\cdots$ then the various hybrid resonant frequencies lie between $(\Omega_e\Omega_{i1})^{1/2}$ and Ω_{i1}, between Ω_{i1} and Ω_{i2}, between Ω_{i2} and Ω_{i3}, etc. In the vicinity of each of these resonant frequencies B_j, the term $1-\sum_r \Pi_r^2(\omega^2-\Omega_r^2)^{-1}$ in (58) may be expanded about $\omega^2 = B_j^2$ (where it vanishes by definiton) to give $(\omega^2 - B_j^2) \sum_r \Pi_r^2/(B_j^2-\Omega_r^2)^2 \equiv A_j(\omega^2-B_j^2)$. Thus $\underset{\sim}{E}(\underset{\sim}{k},\omega)$, in the vicinity of each hybrid resonance, may be written in the form

$$\underset{\sim}{E}(k,\omega) = \frac{i\rho_{ext}\underset{\sim}{k}}{\varepsilon_o k^2 A_j} (\omega^2-B_j^2+C_j\psi^2+D_jk^2)^{-1} \quad , \tag{62}$$

where, from equation (58),

$$C_j = A_j^{-1} \sum_r \left(\frac{\Pi_r^2}{B_j^2-\Omega_r^2} - \frac{\Pi_r^2}{B_j^2}\right) \tag{63}$$

and

$$D_j = -A_j^{-1} \sum_r 3h_r^2 \Pi_r^4 (B_j^2 - \Omega_r^2)^{-1} (B_j^2 - 4\Omega_r^2)^{-1} \qquad (64)$$

For the upper hybrid resonance the constants are given by $A_1 = \Pi_e^{-2}$, $B_1^2 = \Omega_e^2 + \Pi_e^2$, $C_1 = \Omega_e^2 \Pi_e^2 (\Omega_e^2 + \Pi_e^2)^{-1}$ and $D_1 = 3h^2 \Pi_e^4 (3\Omega_e^2 - \Pi_e^2)^{-1}$.

If there is only a single ionic constituent present and if the inequality $\Pi_e^2 \gg \Omega_e \Omega_i$ is satisfies (as in ionospheric applications) then the constants A_2, B_2, C_2 and D_2 for the lower hybrid resonance are given by

$$A_2 = (\Omega_e^2 + \Pi_e^2)^2 \Omega_e^{-4} \Pi_i^{-2} \quad , \quad B_2 = \left(\frac{\Omega_e \Omega_i \Pi_e^2}{\Pi_e^2 + \Omega_e^2} \right)^{1/2}$$

$$C_2 = -\Omega_e^2 \Pi_e^2 / (\Omega_e^2 + \Pi_e^2) \quad , \quad D_2 = -3h_e^2 \Pi_i^2 \left[\frac{T_i}{T_e} + \frac{\Pi_e^4}{(\Pi_e^2 + \Omega_e^2)^2} \right]$$

$$= -3h_e^2 \Pi_i^2 f$$

where f is a numerical factor not too different from unity.

Using (6), (19), (23) and (62) the electric field is given by

$$\underset{\sim}{E}(\underset{\sim}{r}, t) = \frac{P \underset{\sim}{u} \cdot \nabla\nabla}{(2\pi)^3 \varepsilon_o A_j B_j} \int \frac{d\underset{\sim}{k}}{k^2} e^{-i\underset{\sim}{k}\cdot\underset{\sim}{r}} \sin\left(B_j t - \frac{C_j t}{2B_j} \psi^2 - \frac{D_j t}{2B_j} k^2 \right) \qquad (65)$$

where $\underset{\sim}{u}$ is perpendicular to the magnetic field. The scalar product in the exponential equals $kr[\sin\theta_o \cos\psi \cos(\phi - \phi_o) + \cos\theta_o \sin\psi]$ where the coordinates of $\underset{\sim}{r}$ are (r, θ_o, ϕ_o). The ϕ integration can be written

$$\int_0^{2\pi} e^{-ikr\sin\theta_o \cos\psi \cos(\phi-\phi_o)}\, d\phi \cong 2\pi J_o(kr\sin\theta_o) \tag{66}$$

since ψ is small, and (65) becomes

$$\underset{\sim}{E}(\underset{\sim}{r},t) = \frac{P\underset{\sim}{u}\cdot\nabla\nabla}{(2\pi)^2\varepsilon_o A_j B_j}\int_0^\infty dk\, J_o(kr\sin\theta_o)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\psi e^{ikr\psi\cos\theta_o}$$
$$\sin\left(B_j t - \frac{C_j t}{2B_j}\psi^2 - \frac{D_j t}{2B_j}k^2\right) \quad . \tag{67}$$

The ψ integration may be extended to $\pm\infty$ and easily evaluated. The result is

$$\underset{\sim}{E}(\underset{\sim}{r},t) = \frac{P\underset{\sim}{u}\cdot\nabla\nabla}{(2\pi)^{3/2}\varepsilon_o A_j(B_j C_j t)^{1/2}}\int_0^\infty dk\, J_o(kr\sin\theta_o)\sin\left(B_j t - \frac{D_j k^2 t}{2B_j}\right.$$
$$\left. + \frac{k^2 r^2\cos^2\theta_o}{2C_j t/B_j} - \frac{\pi}{4}\right) \tag{68}$$

The k-integral is given in Ref. 12. If the vector operator in (68) is expanded the final result for $\underset{\sim}{E}(\underset{\sim}{r},t)$ is

$$\underset{\sim}{E}(\underset{\sim}{r},t) = \frac{P}{4\pi\varepsilon_o A_j}\left(\underset{\sim}{e}_\rho \cos\phi_o\frac{\partial^2}{\partial\rho^2} - \underset{\sim}{e}_\phi\frac{\sin\phi_o}{\rho}\frac{\partial}{\partial\rho} + \underset{\sim}{e}_z\cos\phi_o\frac{\partial^2}{\partial\rho\partial z}\right) F(\rho,z,t) \tag{69}$$

where the function F is given by

$$F(\rho,z,t) = \frac{J_o\left(\dfrac{\rho^2 B_j C_j t}{4(D_j C_j t^2 - B_j^{\,2} z^2)}\right)}{\overset{-}{+}\ |D_j C_j t^2 - B_j^{\,2} z^2|^{1/2}} \begin{matrix}\cos\\ \sin\end{matrix}\left(B_j t + \frac{B_j C_j t\rho^2}{4(D_j C_j t^2 - B_j^{\,2} z^2)}\right) \tag{70}$$

where the upper symbols apply if $(D_j X_j t^2 - B_j z^2)$ is positive and the lower symbols if it is negative. The two alternative expressions only differ in phase. In the previous two equations ρ is the cylindrical radius and z is measured along the direction of the magnetic field. Equation (69) becomes particularly simple in the extreme cases $\rho^2 B_j << 4D_j t$ and $\rho^2 B_j >> 4D_j t$ in the plane $z = 0$; the expressions so obtained apply for values of z satisfying $z^2 B^2 << D_j C_j t^2$; the field is then independent of z.

In the former case a small-argument expansion of J_o leads to the spatially uniform field,

$$\underset{\sim}{E}(\rho,\phi_o,z=0,t) = \underset{\sim}{u}\ \frac{PB_j}{8\pi\varepsilon_o A_j |C_j D_j^{\,3}|^{1/2}}\ \frac{\sin\ (B_j t + \text{phase})}{t^2} \tag{71}$$

where the constant phase term, which is not of great interest, is not written in detail.

In the case $\rho^2 B_j >> 4D_j t$ the asymptotic expansion for J_o may be used to obtain a predominantly radial field; the radial component is given by

$$E_\rho(\rho,\phi_o,z=0,t) = \frac{P\rho\cos\phi_o\ B_j^{\,3/2}}{\pi)^{3/2}|C_j|^{1/2}\varepsilon_o A_j\ D_j^{\,2}}\ \frac{\sin\left(B_j t + \dfrac{B_j\rho^2}{2D_j t} + \text{phase}\right)}{t^{5/2}} \tag{72}$$

and its magnitude thus only depends on the coordinate $\rho\cos\phi_o$ parallel to the dipole axis.

A limit on this second type of approximation may be obtained by introducing it at an earlier stage; if the Bessel function in the integrand on the right side of (68) is approximated by its asymptotic value for large argument then, upon completing the squares in the resulting exponentials, terms of the form $\exp[-ia(k \pm \rho/2a)^2]$ are obtained under the integral sign, just as in equation (51) for the resonance at the plasma frequency. The values of k contributing to the resonance therefore satisfy $k \sim \rho/2a = B\rho/Dt$. Since the expansion (62) is only valid when $Dk^2 << B^2$, substitution of the above value of k into this inequality yields for the validity of (72) the condition

$$t >> \rho D^{-1/2} \tag{73}$$

for ρ which is similar to the condition for z obtained above. Both conditions would be satisfied for a satellite that moves away from the source point.

The expressions so far derived are completely general and can be applied to all of the hybrid resonances. In two special cases explicit expressions for A, B, C, D have been listed previously and those will now be substituted.

3.3.2 The Upper Hybrid Resonance

Putting the previously derived values A_1, B_1, C_1, and D_1 into (71) gives the uniform electric field

$$\underset{\sim}{E}(\rho, z=0, t) = \underset{\sim}{u} \frac{P(\Pi_e^2+\Omega_e^2)(3\Omega_e^2-\Pi_e^2)^{3/2}}{8\pi 3^{3/2}\epsilon_o \Omega_e \Pi_e^2 V^3} \frac{\sin[(\Pi_e^2+\Omega_e^2)^{1/2} t + \text{phase}]}{t^2} \tag{74}$$

for distances ρ that satisfy the inequality

$$\rho^2 << 12h^2 \frac{\Pi_e^4 t}{|3\Omega_e^2-\Pi_e^2|(\Pi_e^2+\Omega_e^2)^{1/2}} \tag{75}$$

For distances [illegible]fying the reverse inequality, application of (72) yields

$$E_\rho(\rho,\phi_o,z=0,t) = \frac{P(\Omega_e^{\,2}+\Pi_e^{\,2})^{5/4}(3\Omega_e^{\,2}-\Pi_e^{\,2})\,\rho\,\cos\phi_o}{9\varepsilon_o(2\pi)^{3/2}\,\Omega_e\Pi_e^{\,3}\,V^4} \times$$

$$\times \frac{1}{t^{5/2}} \cos\left[(\Omega_e^{\,2}+\Pi_e^{\,2})^{1/2}t + \frac{\rho^2(3\Omega_e^{\,2}-\Pi_e^{\,2})(\Pi_e^{\,2}+\Omega_e^{\,2})^{1/2}}{6\,h^2\,\Pi_e^{\,4}t} + \text{phase}\right] \tag{76}$$

The inequality (73) becomes

$$t \gg 3^{-1/2}\,\frac{\rho}{h}\,\frac{|3\Omega_e^{\,2}-\Pi_e^{\,2}|^{1/2}}{\Pi_e^{\,2}} \tag{77}$$

or $\rho \ll 3^{1/2}\,Vt(\Pi_e^{\,2}/|3\Omega_e^{\,2}-\Pi_e^{\,2}|)^{1/2}$. Equation (77) is similar to the corresponding condition in the case of the resonance at the plasma frequency. The qualitative arguments about the time, given by (73), that must elapse at a given point ρ before equation (76) becomes valid, will not be repeated here; the results are similar to the oscillations near the plasma frequency.

Equation (76) like equation (56) contains a phase term which must lead to interference effects if the radiators are large. Depending on the sign of $3\Omega_e^{\,2}-\Pi_e^{\,2}$ the oscillations are delayed or advanced in phase at larger distances. An effective wavelength for given values of ρ and t could again be defined, as in the case of oscillations near the plasma frequency; equation (76) yields for the wavelength λ,

$$\frac{\lambda}{h} = 6\pi\,\frac{t\,\Pi_e^{\,4}}{|3\Omega_e^{\,2}-\Pi_e^{\,2}|(\Pi_e^{\,2}+\Omega_e^{\,2})^{1/2}}\left(\frac{\rho}{h}\right)^{-1}$$

3.3.3 The Lower Hybrid Resonance for a Single Ionic Constituent

Substitution of the previously derived values of A_2, B_2, C_2, and D_2 into (71) leads to the uniform field

$$E(\rho,\phi,z=0,t) = \frac{P\Omega_e^{4}\,\Pi_e^{2}}{8\pi\epsilon_o(3f)^{3/2}(\Omega_e^{2}+\Pi_e^{2})^{2}V^{3}}\;\frac{1}{t^{2}}\sin(B_2 t + \text{phase}) \tag{78}$$

which for typical ionospheric parameters is of the same order of magnitude as the corresponding field (74). Equation (78) is valid for

$$\rho^{2} << 4Dt/B = 12h^{2}f\frac{(\Omega_e^{2}+\Pi_e^{2})^{1/2}}{\Omega_e}\;\Pi_i t \tag{79}$$

and this inequality is seldom satisfied in practice.

If the reverse inequality of (79) holds then (72) yields

$$E_\rho(\rho,\phi_o,z=0,t) = \frac{P\Omega_e^{9/2}\Pi_e^{3}\;\rho\cos\phi_o}{9\epsilon_o(2\pi^{3})^{3/2}(\Omega_e^{2}+\Pi_e^{2})^{9/4}\Pi_i^{1/2}V^{4}f^{2}}\,t^{-5/2}\cos\left[B_2 t + \frac{\rho^{2}\Omega_e\,(\Pi_i t)^{-1}}{12h^{2}(\Omega_e^{2}+\Pi_e^{2})^{1/2}} + \text{phase}\right] \tag{80}$$

which is not very much smaller than the corresponding field given by (76). It should also be noted that the "Doppler" frequency shift caused by the phase term in (80), for typical satellite velocities $v_s = \rho/t$ can be greater than the lower hybrid frequency, B_2, itself. It appears therefore that this type of resonance is unsuitable for ionospheric investigations by the methods which have been used for the electronic resonances.

It is interesting to consider the further condition $\Pi_i t >> 3^{-1/2}\rho h^{-1}f$ following from (73) since it must be satisfied, in addition to the reverse inequality of (79), before (80) becomes valid. This condition is not satisfied for typical satellite velocities and it would seem (although no explicit predictions are made by the present analysis) that the satellite leaves the lower hybrid resonant oscillations behind before they have a chance to build up. The situation is just the reverse in the case of the various electronic resonances; they have decayed considerably by the time the satellite arrives at a given point.

3.4 Oscillations near Integral Multiples of the Electron Cyclotron Frequency.

The angular frequency of these oscillations is in the vicinity of $n\Omega_e$, where n is any positive integer greater than unity. Equation (44) is then not a valid approximation for these oscillations. Although the k^2 term can be safely neglected in (44), the electronic terms m = n and m = -n in (41) must be taken into account since they make large contributions near $\omega \sim -n\Omega_e$ and $\omega \sim n\Omega_e$ respectively; the corresponding terms must be included on the right of (44). If the inequality (43) is satisfied then Landau damping is negligible, the integral on the right of (41) may be replaced by the first term of its asymptotic expansion[1] and the modified Bessel function is well represented by its approximate value for small argument. Equation (44) for electrons then takes the form

$$i\epsilon_o^{-1} k^{-1} \beta = \frac{\Pi^2}{\omega^2-\Omega^2} \sin^2 \theta + 2^{-n+1} \left(\frac{kh}{\Omega}\right)^{2n-2} \frac{n \, \Pi^{2n}}{(n-1)!(\omega^2-n^2\Omega^2)} + \frac{\Pi^2}{\omega^2} \cos^2 \theta \quad (81)$$

In the integration over $\psi = \pi/2-\theta$ the effective angular range is roughly determined by the condition (43) since Landau damping increases rapidly when it is not satisfied. As a crude estimate of the inverse Fourier transform, the angular range of the ψ integration will be taken as

$$\Delta\psi = \left|\frac{|\omega|-n\Omega}{4kh\Pi}\right| , \quad (82)$$

and the value of θ in (81) will be taken as exactly $\pi/2$. With the value of given by (81) the Fourier component $\Phi(\underset{\sim}{k},\omega)$ of the potential $\Phi(\underset{\sim}{r},t)$ is found from the relation $\underset{\sim}{E}(\underset{\sim}{k},\omega) = i\underset{\sim}{k}\Phi(\underset{\sim}{k},\omega)$ and from the equations (6) and (42) to be

$$\Phi(k,\omega) = -\frac{iP\cos\phi}{(2\pi)^4\varepsilon_o k}\left[-1+\frac{\Pi^2}{\omega^2-\Omega^2}+2^{-n+1}\left(\frac{kh}{\Omega}\right)^{2n-2}\frac{n\,\Pi^{2n}}{(n-1)!(\omega^2-n^2\Omega^2)}\right]^{-1}. \quad (83)$$

The important variation with ω is contained in the last term of the square bracket and therefore the replacement of ω in the second term by $n\Omega$ is a good approximation. After re-arrangement equation (83) takes the form

$$\Phi(k,\omega) = -\frac{iP\cos\phi}{(2\pi)^4\varepsilon_o k}\;\frac{\Omega^2(n^2-1)}{\Pi^2-\Omega^2(n^2-1)}\;\frac{N}{\omega^2-n^2\Omega^2-N} \quad (84)$$

where in the numerator $\omega^2-n^2\Omega^2$ has been replaced by N and where

$$N = 2^{-n+1}\frac{n(n+1)}{(n-2)!}(kh)^{2n-2}\,\Omega^{-2n+4}\,\Pi^{2n}\,[\Omega^2(n^2-1)-\Pi^2]^{-1} \quad . \quad (85)$$

In the inverse Fourier transform of (84) the ω integration can be carried out with the aid of (23) and the approximate result of the ψ integration is assumed to be $\Delta\psi$ given by (82), with $|\omega|-n\Omega = N/2n\Omega$ [the condition for the vanishing of the last denominator on the right of (84)]. The resulting expression for $\Phi(\rho,\phi_o,\, z=0,\, t)$ is

$$\Phi(r_o,\phi_o,\, z=0,\, t) = \frac{i(n+1)^3 P 2^{-2n-4}\Omega^{-4n+8}\Pi^{4n-1}h^{4n-5}\cos\phi_o}{\pi^3\varepsilon_o[(n-2)!]^2[\Omega^2(n^2-1)-\Pi^2]^3}\;\frac{G(t)-G(-t)}{2i}\;, \quad (86)$$

where

$$G(t) = e^{in\Omega t}\lim_{\varepsilon\to 0}\int k^{4n-4}\exp[(ia-\varepsilon)k^{2n-2}]\;e^{-ik\rho\cos(\phi-\phi_o)}\cos(\phi-\phi_o)\,d\phi\,dk, \quad (87)$$

and where

$$a = \frac{n+1}{2^n(n-2)!}\;\frac{h^{2n-2}\Omega^{-2n+3}\Pi^{2n}}{\Omega^2(n^2-1)-\Pi^2}\;t \quad (88)$$

After the equation $2\pi i J_o'(k\rho) = \int_o^{2\pi} d\phi \exp[-ik\rho \cos(\phi-\phi_o)] \cos(\phi-\phi_o)$ is used to carry out the ϕ integration, equation (87) takes the form

$$G(t) = 2\pi i e^{in\Omega t} \int J_o'(k\rho)\, k^{4n-4} \exp(iak^{2n-2})\, dk \tag{89}$$

where the integration path indicated by the limiting process of (87) is understood.

The integral in (89) will only be considered in the extreme cases in which the Bessel function is well approximated by its asympotic value for small or large argument. For small arguments (89) becomes

$$G(t) = -2\pi i e^{in\Omega t} \int_o^\infty \frac{\rho}{2} k^{4n-3} \exp(iak^{2n-2})\, dk \quad . \tag{90}$$

After the substitution $x = k^{n-1}$ equation (90) has the form

$$G(t) = -\pi i e^{in\Omega t} \frac{\rho}{n-1} \int_o^\infty x^{\frac{3n-1}{n-1}} e^{iax^2} dx \tag{91}$$

Use of the identity

$$\int_o^\infty y^m e^{i\alpha y^2} dy = \frac{1}{2} i^{\frac{m+1}{2}} \Gamma\left(\frac{m+1}{2}\right) \alpha^{-\frac{m+1}{2}}$$

in (91) and subsequent substitution into (86) leads to the equation

$$E = \frac{\pi^{-2}[(n-2)!]^{\frac{1}{n-1}} (n+1)^{\frac{n-2}{n-1}} \Gamma\left(\frac{2n-1}{n-1}\right) P\Omega^{\frac{4n-5}{n-1}} \sin(n\Omega t+\text{phase})}{2^{\frac{4n-5}{n-1}} (n-1)\epsilon_o V^3 \Pi^{\frac{2}{n-1}} t^{\frac{2n-1}{n-1}} |\Omega^2(n^2-1)-\Pi^2|^{\frac{n-2}{n-1}}} \tag{92}$$

For large arguments of the Bessel function (89) becomes

$$G(t) = 2\pi i e^{in\Omega t} \left(\frac{2}{\pi}\right)^{1/2} \rho^{-1/2} \int k^{4n-9/2} \; \frac{i}{2} \left[e^{i(k\rho - \frac{\pi}{4})} - e^{-i(k\rho - \frac{\pi}{4})} \right] e^{iak^{2n-2}} \, dk \qquad (93)$$

Under the present assumptions only those values of k will contribute to (93) for which the exponent $ak^{2n-2} \pm k\rho$ is near a stationary value. After approximating this exponent by the first and third term of the Taylor series about this stationary value (since the second term vanishes) the resulting integration is easily carried out. In this manner the rather complicated expression

$$\begin{aligned} E_\rho(\rho,\phi_o,t) = {} & 2^{-\frac{7n-12}{2n-3}} \pi^{-2} (n^2-1)^{-\frac{4n-3}{2n-3}} (2n-3)^{-1/2} [(n-2)!]^{\frac{3}{2n-3}} \\ & \times P\varepsilon_o^{-1} \cos\phi_o \, \rho^{\frac{2n}{2n-3}} V^{-\frac{8n-9}{2n-3}} \Pi^{-\frac{6}{2n-3}} \Omega^5 |\Omega^2(n^2-1) - \Pi^2|^{\frac{-2n+6}{2n-3}} \\ & \times t^{-\frac{4n-3}{2n-3}} \sin\left\{ n\Omega t \pm \frac{(2n-3)}{2^{\frac{n-2}{2n-3}}} \frac{\rho^{\frac{2n-2}{2n-3}}}{(n-1)^{\frac{2n-2}{2n-3}}} \left[\frac{(n-2)!}{n+1}\right]^{\frac{1}{2n-3}} V^{-\frac{2n-2}{2n-3}} \Omega \right. \\ & \left. \times |\Omega^2(n^2-1) - \Pi^2|^{\frac{1}{2n-3}} t^{-\frac{1}{2n-3}} + \text{phase} \right\} \end{aligned} \qquad (94)$$

is obtained for the radial electric field. These oscillations will not be discussed here in detail; the arguments used for the oscillations at the upper hybrid frequency remain valid. For large values of n and practical values of ρ and t equation (94) rather than (92) must be used; it has the asymptotic form

$$E_\rho = \frac{\pi^2 P\rho \cos\phi_o}{2^3 e^{3/2} n^3 \varepsilon_o V^4} \Omega^5 \left|\Omega^2(n^2-1)-\Pi^2\right|^{-1} t^{-2} \sin\left(n\Omega t \pm \frac{2^{1/2}\rho\Omega}{eV} + \text{phase} \right) \quad (95)$$

where the sign of the phase term in both (94) and (95) has the sign of $[\Omega^2(n^2-1)-\Pi^2]$. It should be noted that for not too large values of n the amplitudes predicted by (95) could be considerably larger than those predicted by equation (76) for the upper hybrid frequency.

3.5 Oscillations near the Electron Cyclotron Frequency

The propagation vectors of the waves participating in this resonance are nearly, but never entirely, parallel to the magnetic field. As in the case of the oscillations at harmonics of the cyclotron frequency, Landau damping plays an important part. If a crude estimate equivalent to equation (82) is made then the expression

$$\theta_e = 4\Pi^{-1/2}\,\Omega^{-1/2}\,(hk)^{1/2}\,\left|\Pi^2-\Omega^2\right|^{1/2} \quad (96)$$

[c.f. equation (22) of Ref. 1] is obtained for the angle θ_e which θ must exceed before Landau damping becomes unimportant; smaller angles than this will be excluded in the θ integration. A combination of (42) and (44) leads to the equation

$$E(\underset{\sim}{k},\omega) = \frac{i\rho_{ext}}{\varepsilon_o k}\left(1 - \frac{\Pi^2}{\Omega^2} - \frac{\Pi^2\theta^2}{\omega^2-\Omega^2} - 3k^2h^2\Pi^4\Omega^{-4}\right)^{-1} \quad (97)$$

for small angles θ and for small values of kh. If equation (97) is written in the form

$$E(\underset{\sim}{k},\omega) = \frac{i\rho_{ext}\,\Omega^2}{\varepsilon_o k(\Omega^2-\Pi^2)}\;\frac{\omega^2-\Omega^2}{\omega^2-\Omega^2-\Pi^2\Omega^2(\Omega^2-\Pi^2)^{-1}\theta^2 - 3k^2h^2(\omega^2-\Omega^2)\Pi^4\Omega^{-2}(\Omega^2-\Pi^2)^{-1}} \quad (98)$$

then for values of θ larger than θ_e given by (96), for values of kh much smaller than unity and for values of ω close to Ω the k^2 term of the denominator inside the curly bracket of (98) is much smaller than the θ^2 term and will hence forth be neglected. In the calculation of the inverse Fourier transform the ω integration is carried out first; the contribution comes from values of ω for which the denominator nearly vanishes and the numerator may then be taken equal to $\Pi^2\Omega^2(\Omega^2-\Pi^2)^{-1}\theta^2$. Application of (23) and substitution of ρ_{ext} from (6) then leads to the equation

$$E(z,t) = \frac{P}{4\pi^2\varepsilon_o} \frac{\Omega^3\Pi^2}{(\Omega^2-\Pi^2)^2} \exp(i\Omega t) \frac{M(t)-M(-t)}{2} \tag{99}$$

where

$$M(t) = \lim_{\varepsilon\to 0} \int_{\theta=\theta_e}^{\infty} \int_{k=0}^{\infty} k^2\theta^3 \exp\left(\frac{i\Pi^2\Omega t\theta^2}{2(\Omega^2-\Pi^2)}\right) e^{(\pm iz-\varepsilon)k}\, dk d\theta \tag{100}$$

where a sum of the integrals corresponding to the alternative signs is meant, corresponding to the two θ integrals near $\theta = 0$ and $\theta = \pi$. Execution of the trivial θ integrations results in the expression

$$M(t) = \frac{2(\Pi^2-\Omega^2)}{\Pi^4\Omega^2t^2} \int_o^{\infty} [-1+8ikh\Pi t]\, k^2 e^{ikh\Pi t\pm ikz}\, dk \tag{101}$$

It is clear from the form of (101) that for typical satellite velocities the dependence of the field on the $z = v_s t$ coordinate is unimportant and will therefore be neglected. The k integration for E(z,t) gives the expression

$$E(z,t) = \frac{P\Omega}{2^7\pi^2\varepsilon_o\Pi^2V^3} \frac{1}{t^5} \sin \Omega t \quad . \tag{102}$$

The decay of the oscillations predicted by (102) is very rapid. On the basis of equation (102) alone the observed duration of the oscillations at the cyclotron frequency should therefore be very much shorter than the durations observed at the other resonant frequencies. This prediction does not, however, take into account the size of the antenna. It is clear from the form of (101) that the integrals occurring in it are functions of $8h\Pi t-z$ and thus represent travelling waves which are multiplied by factors of t^{-2} and t^{-3} in the two terms instead of the t^{-5} decay predicted by (102). If the antenna is large then the waves do not immediately leave the antenna and the oscillations decay more slowly than equation (102) predicts. A more quantitative treatment of the problem would be required for more accurate predictions; moreover the use of the electrostatic approximation in the treatment of this resonance is questionable.

4. Validity of the Quasi-Electrostatic Approximation

In the absence of an external magnetic field it was shown that the quasi-electrostatic approximation represents the oscillations in the vicinity of a pulsed point source rather well. The same may still be true for some of the resonances, even in the presence of an external magnetic field. Moreover, the quasi-electrostatic approximation probably provides an adequate description of the initial decay of the oscillations at the point source itself for all the resonant frequencies predicted by it. It may be shown, however, by a rather crude semi-quantitative argument that near the upper hybrid frequency the quasi-electrostatic approximation does not describe correctly the asymptotic decay of the oscillations at the position of the point source after a very long time.

A glance at Fig. 3 of reference 1 shows that if the upper hybrid frequency is less than twice the electron cyclotron frequency then for $\theta = 90°$ (in a direction perpendicular to the magnetic field) the frequency is increasing with increasing wave number according to the Appleton-Hartree (cold plasma) approximation but is decreasing with increasing wave number according to the electrostatic approximation (which applies to much larger values of k). For some intermediate value k_o of k somewhere between the usually widely different values of $k = h^{-1}$ and $k = \Pi/c$ the frequency ω must reach a maximum value ω_o. In the vicinity of k_o and for $k > k_o$ the square of the frequency ω_D, resulting from the dispersion relation, must be given by a relation not unlike

$$\omega_D^2 = \omega_o^2 - C\psi^2 - D(k-k_o)^2 \tag{103}$$

In analogy with the rest of this paper the inverse Fourier transform of the electric field is proportional to integrals of the form $\int e^{i\omega_D t} e^{-i\underset{\sim}{k}\cdot\underset{\sim}{r}} dk$ which for r = 0 may be written as $\int e^{i\omega_D t} k^2 dk d\psi$. The ψ integral results in a factor with a $t^{-1/2}$ time dependence. The k integral for r = 0

$$\int_o^\infty dk\, k^2 \exp\left[i\frac{D}{2\omega_o}(k-k_o)^2 t\right] = \frac{\pi^{1/2}}{4}\, e^{-i\frac{3\pi}{4}} \left(\frac{Dt}{2\omega_o}\right)^{-3/2} - ik_o \left(\frac{2\omega_o}{Dt}\right) + \tag{104}$$

$$+ \frac{1}{2} e^{-i\frac{\pi}{4}} \pi^{1/2} k_o^2 (Dt/2\omega_o)^{-1/2}$$

has a $t^{-3/2}$ time dependence for $t << 2\omega_o k^{-2} D^{-1}$ and therefore the electric field has a t^{-2} time dependence which is the time dependence obtained in Section 3.4. For $t >> 2\omega_o k_o^{-2} D^{-1}$ the last term on the right of (104) predominates and results in a $t^{-1/2}$ time dependence for the integral and therefore a t^{-1} time dependence for the electric field, as shown by the less crude analysis of Nuttall[2]. If k_o is crudely taken to be the geometric mean of h^{-1} and Π/c, so that $k_o^2 \sim \Pi^2 V^{-1} c^{-1}$, and if the value $D = 3V^2\Pi^2(3\Omega^2-\Pi^2)^{-1}$ derived in Section 3.4 is used then the quasi-electrostatic approximation is valid for

$$t \ll 2\left(\frac{\Omega^2}{\Pi^2} + 1\right)\left(\frac{\Omega^2}{\Pi^2} - \frac{1}{3}\right)\Pi^{-1}\frac{c}{V} \tag{105}$$

If for example Π and Ω are both equal to about 1 Mc and if $c/V = 2.10^3$ then $t \ll 5$ millisecond is obtained as the crude condition for the validity of the quasi-electrostatic approximation at the position of the source. At a point moving away from the source with satellite velocity the quasi-electrostatic approximation remains valid much longer since the field predicted by it decays more slowly.

5. Discussion

The present calculations were restricted by the assumption of an infinitesimally small dipole source whose moment is a δ function of the time. In principle the field of any external charge distribution can be expressed as an integral over space and time of the electric fields (which are the Green's functions of the problem) calculated here, provided that the plasma is uniform and that the external charge distribution in space and time is known. In reality the plasma is not uniform but is bounded by the ion sheath and the antenna, and the determination of the charge and its distribution on the antenna is difficult. Although a complete solution of the problem is therefore not given, the results of this paper are nevertheless believed to provide a useful insight as well as a convenient starting point for making quantitative predictions of the resonant oscillations excited by an antenna in a plasma.

The calculated fields are accurate within the limits of their respective approximations for the oscillations near the plasma frequency and the hybrid frequencies. Less accurate results have been obtained for the oscillations near the electron cyclotron frequency and near its harmonics because Landau damping, which plays a more important part in these resonances, has only been taken into account in a rather rough manner.

A common feature of the present results is the existence of a surface that roughly separates two regions in which different approximations are valid. The critical distance to this surface (from the point of occurrence of the exciting impulse) increases proportionally to the square root of the time. Within the critical distance the field of the oscillations is approximately uniform but beyond the critical distance the field is non-uniform and contains only those waves which have nearly the correct group velocity corresponding to a given time and position.

In the absence of an external magnetic field the electric field (of the oscillations near the plasma frequency) is uniform and parallel to the dipole axis within the critical distance. The field is approximately radial beyond the critical distance; its magnitude at any given time is proportional to the cosine of the angle between the radius vector and the dipole axis and is independent of the magnitude of the radius vector up to a distance somewhat less than Vt. In this region only those waves contribute to the field at any given distance and time whose group velocity is nearly equal to the ratio of the distance to the time. Still further the field probably decreases rather rapidly with distance on account of Landau damping. Both inside and outside the critical surface (but for distances much smaller than Vt) the field decreases as $t^{-3/2}$. Outside the critical surface the phase of the oscillations changes substantially with position and interference effects can occur.

In the presence of an external magnetic field the electric field of the oscillation near the plasma frequency is approximately parallel to the magnetic field and it's magnitude beyond the critical distance is proportional to the square of the distance up to distances somewhat less than Vt. Since space vehicles move with much smaller velocities than V (which is roughly the mean thermal velocity of the electrons) they can never reach a distance close to Vt. The space vehicle is initially inside the critical surface and eventually overtakes it (since the distance to the surface is only proportional to the square root of the time) but can never approach the distance Vt closely. This argument is somewhat modified by the finite size of the antenna. Initially one section of the antenna will be outside the critical distance of the field generated by another section, and even outside the distance Vt, and it is only after some time that the approximations used here become applicable (for the decaying phase).

The time dependence is $t^{-5/2}$ inside the critical distance and $t^{-7/2}$ outside the critical distance at a fixed point. At a point that moves away from the source with a constant velocity, the time dependence is $t^{-5/2}$ inside the critical surface and $t^{-3/2}$ outside. While a fixed point is first outside and then inside the critical surface, the reverse is true for a moving point. Thus at a fixed point the time dependence is first $t^{-7/2}$ and then $t^{-5/2}$; at a moving point first $t^{-5/2}$ and then $t^{-3/2}$.

The phase, as well as the amplitude of the oscillating field depends on position outside the critical distance at a given time, as indicated by the phase term $z^2/4a$ in (56). Interference effects are thus possible (both with or without an external magnetic field) when a source of finite size (i.e. a practical antenna) is used. The apparent frequency of the oscillations is, moreover, modified by the motion of the vehicle (as the result of the phase term) but this effect is only of practical importance at the lower hybrid frequencies.

The approximations used in this paper break down at the lower hybrid frequencies for typical satellite and even rocket velocities. The frequency change caused by the vehicle motion would be too large for useful interpretation of the observations, even if the velocity of the space vehicle were a little less than the mean thermal velocity of ions and the approximations of this paper were to remain valid. Moreover, it appears that since the satellite or rocket moves away almost immediately from the oscillating regions where the present approximations are valid, these resonances may not even be observable by the techniques used for the other resonances; a considerably more sophisticated theory, or an experiment would be required, however to confirm such a tentative conclusion.

The resonance at the upper hybrid frequency resembles more closely the resonance at the plasma frequency but the electric field is approximately normal and not parallel to the magnetic field for these oscillations, at least within the limits of the electrostatic approximation. It was pointed out that the electrostatic approximation is certainly valid initially for this resonance but breaks down at the position of the source (but not at a point moving with typical satellite velocity at right angle to the magnetic field) after a time of the order of milliseconds for typical ionospheric conditions if the upper hybrid frequency is less than twice the cyclotron frequency. The detailed behavior of these oscillations inside and outside the critical surface is described in the text. Inside the critical surface the uniform field decreases with time as t^{-2}; outside the critical surface at a fixed point the field decreases as $t^{-5/2}$, at a moving point as $t^{-3/2}$. At a fixed time outside the critical distance the magnitude of the field is proportional to $\rho \cos \phi_o$ where ρ is the distance from the magnetic field line passing through the source and ϕ_o is the angle between the component of the radius vector normal to magnetic field and the dipole axis (assumed normal to the magnetic field).

Of all the oscillations those near the ~~relatively~~ low ~~order harmonics~~ of the electron cyclotron frequency have the largest calculated amplitudes. At a fixed point the amplitude is proportional to the [-(2n-1)/(n-1)] - th power of the time inside the critical surface and to the [-(4n-3)/(2n-3)] - th power of the time outside the critical surface. At a fixed time outside the critical surface the amplitude is proportional [illegible] [2n/(2n-3)] - th power of the distance from the field line passing th[illegible] the source. The amplitude of the oscillations observed from a point moving away from the source with a constant velocity has a t^{-1} time dependence. Outside the critical surface the phase of the oscillations depends strongly on position and interference effect can therefore occur when large radiators are used.

While no calculations of the field have been carried out for large radiators, it is quite clear that many of the conclusions will be modified for them. Thus the Fourier transform of the charge distribution of the point source assumed in this paper has the largest absolute value, for a given magnitude of k, along the dipole axis and vanishes for directions normal to the dipole axis. This conclusion is sharply modified when a finite source is considered. The transform of the charge distribution for a finite thin dipole antenna still vanishes for directions normal to the dipole axis but for values of k much larger than the reciprocal length the largest absolute value of the transform is found in directions nearly normal, rather than parallel, to the dipole axis [as may be seen, for example, from equation (2.2.10) of Balmain[14]]. This is in good agreement with Lockwood's[15] observations of the dependence of the oscillation amplitude on the orientation of the satellite antenna near the harmonics of the cyclotron frequency.

Since the completion of this paper the authors became acquainted with the work of Dougherty and Monaghan[16] on the same type of resonance effects. Their aim is not a prediction of the amplitudes of the resonances but rather a thorough investigation of the conditions for resonant behavior of the medium, not restricted by the quasi-electrostatic approximation. A direct comparison of their results with ours is therefore difficult and will not be attempted here.

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