SYNTHESIS OF ACTIVE DISTRIBUTED RC NETWORKS

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ABSTRACT

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This work presents a general method for the synthesis of any open-circuit transfer function, which is expressed as a rational function with real coefficients. It is shown that such functions ᴄᴀɴ be realized by using two distributed RC elements (modified by active units if necessary), one negative impedance converter and one lumped capacitor. The distributed RC element used here is a three-layer-structure, a pure dielectric sandwiched in between a perfect conductor and a resistive film.

I. Introduction

This paper presents a method to synthesize the open-circuit transfer function of a two-port network. The distributed RC elements described in the previous papers [1]-[6] are used. It is shown that any rational transfer function can be realized with some active elements imbedded in two units of such distributed RC elements and one lumped capacitor. The procedure is extremely simple. The difference in the realizations for different functions lies only in the curve cut on the conductor sheet of each distributed RC element. This is quite different from the situation confronted in the lumped RC active synthesis where the number of elements may increase tremendously as the complexity of the transfer function is increased.

II. Summary of Previous Works on Synthesis

Barker [5] first presented experimental data for an active filter by using two units of such distributed elements coupled with some lumped RC elements. He also claimed that any given rational open-circuit transfer function can be synthesized by a Yanagisawa's configuration containing two networks A and B and an NIC (Fig. 1). Each one of the networks A and B is in general composed of cascaded distributed elements and some lumped RC elements. Obviously, his
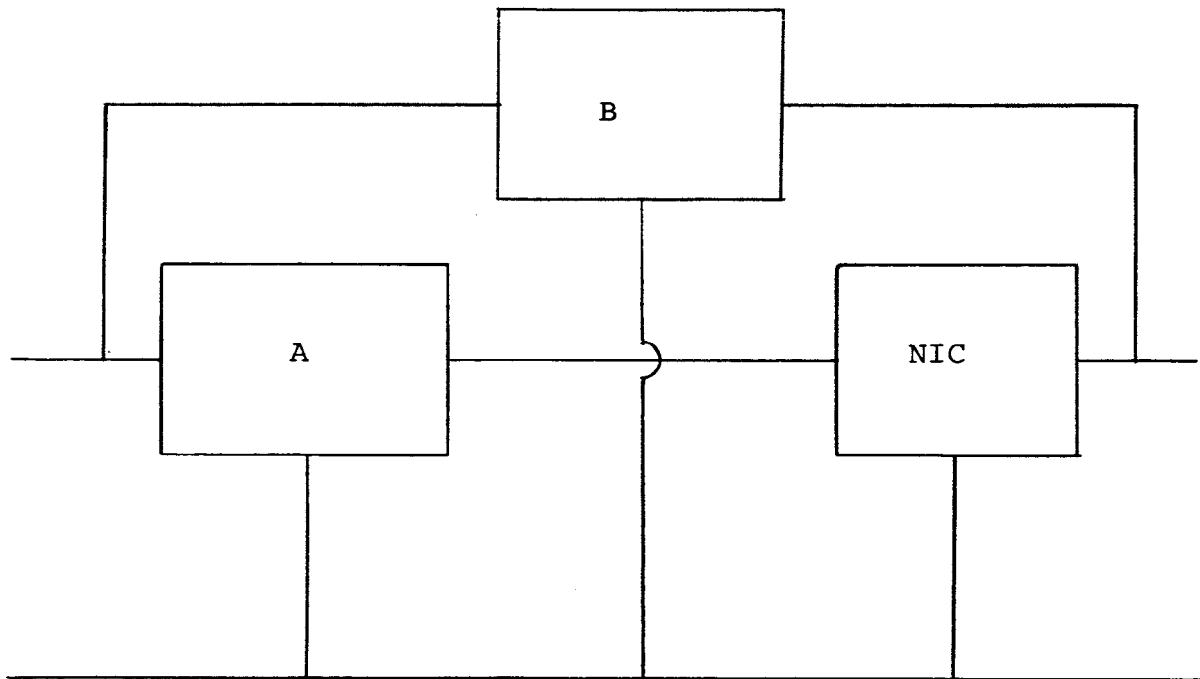
Fig. 1.— Yanagisawa's Configuration

method again suffers the drawback that the number of either distributed elements or lumped elements required will become very large when the order of the given transfer function is high.

Before Barber, Heizer [3] has also discussed synthesis method by two numerical examples. He realized a low-pass filter with open-circuit transfer function

$$G(s) = \frac{K(s^2 + 5.15)}{(s + 0.695)(s^2 + 0.54s + 1.151)}$$

by starting with Yanagisawa's configuration and ending up with a network composed of two distributed elements, one NIC, and two lumped resistors and one capacitor. He also realized one biquardratic functions

$$G(s) = K \frac{s^2 + 1}{s^2 + as + 1} \qquad (a > 2)$$

with a distributed element, a phase invertor and a resistor.

Later, Woo and Hove [6] have shown that a second order or third order open-circuit transfer function can be synthesized by two distributed elements (the special kind they call single pole partitioned capacitance network) and an NIC and some lumped R, C elements. They further tabulated some prototype networks corresponding to these kinds of transfer functions and showed the advantages by comparison with networks realized with all lumped RC elements.

III.   Review of the Distributed RC Elements

Fig. 2 represents a unit of the distributed RC element considered as a two-port network. Its schematic diagram with both normalized dimensions and coordinates is shown in Fig. 3. The short-circuit parameters are [4]:

$$y_{11} = [\frac{s'C_1}{R_1}]^{\frac{1}{2}} \tanh\sqrt{R_1 C_1 s'} \tag{1a}$$

$$-y_{21} = \frac{\pi s'}{2R_1} \sum_{m=0}^{M} \frac{(2m+1) \beta_{1m}}{s' + (2m+1)^2 \lambda_1} \tag{1b}$$

$$y_{22} = \frac{2C_1 s'}{\pi} \sum_{m=0}^{M} \frac{\beta_{1m}}{2m+1} - \frac{s'^2 C_1}{2} \sum_{m=0}^{M} \frac{\beta_{1m}^2}{s' + (2m+1)^2 \lambda_1} \tag{1c}$$

where $\lambda_1 = \frac{\pi^2}{4R_1 C_1}$, s' is the complex frequency and R_1 is the total resistance measured between x = 0 and x = 1 in the resistive film and C_1 is the total capacitance between the resistive film and two conducting plates when they are at the same potential. The β_{1m}'s are the parameters which govern the variation of the capacitances by a cut $f_1(x)$ on the conductor sheet, where

$$f_1(x) = \sum_{m=0}^{M} \beta_{1m} \sin\frac{(2m+1)\pi x}{2} \leq 1 \qquad (0 \leq x \leq 1) \tag{2}$$
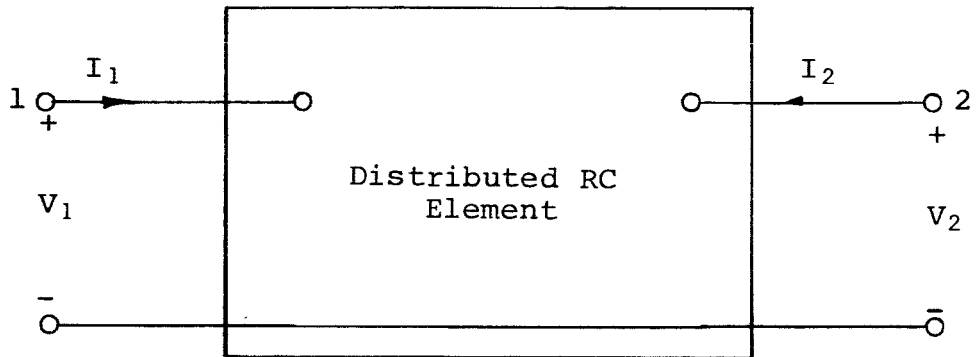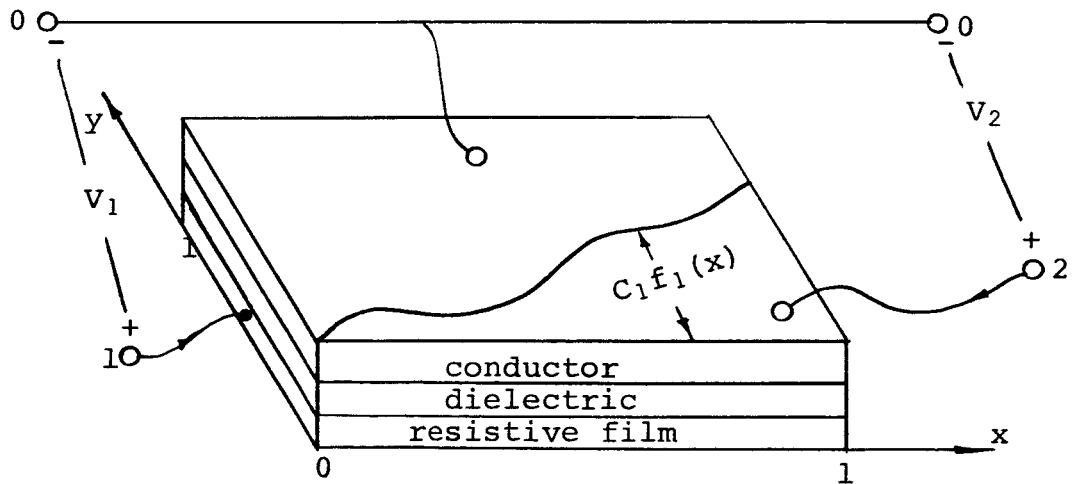
Fig. 2 - A Two Port Network

Fig. 3 - A Schematic Diagram of a Distributed RC Element

Note that the distributed capacitance $C_1 f_1(x)$ are physically realizable only if $f_1(x)$ is positive for $0 \le x \le 1$.

In the synthesis procedure, we have to test this condition after we find the β_m's. The test is, in general, a very tedious work. Besides, it may have negative result and the whole problem has to be started over again. However, if a negative impedance converter is introduced, we may replace the cut $f_1(x)$ by $|f_1(x)|$ and have a similar result. The proof is as follows:

Consider the basic differential equation (6) of [4] when applied to the circuit of Figure 2.

$$\frac{d^2 v(x)}{dx^2} - \gamma^2 v(x) = - \gamma^2 f_1(x) V_2 \quad , \tag{3}$$

where $v(x)$ is the voltage at x on the resistive film with respect to the ground and $\gamma^2 = R_1 C_1 s'$. Replace $f_1(x)$ by $|f_1(x)|$ and V_2 by $\phi(x)$, we have

$$\frac{d^2 v(x)}{dx^2} - \gamma^2 v(x) = - \gamma^2 |f_1(x)| \phi(x) \quad . \tag{4}$$

If we let $\phi(x) = V_2$ when $f_1(x) \ge 0$ and $\phi(x) = - V_2$ when $f_1(x) < 0$, then (4) will have the same solution as (3).

This suggests that we can cut the conductor sheet according to $|f_1(x)|$ and use an active unit connected to the distributed RC elements. This active unit is composed of a current negative impedance converter (INIC) and a voltage negative impedance converter (VNIC). (Fig.4).
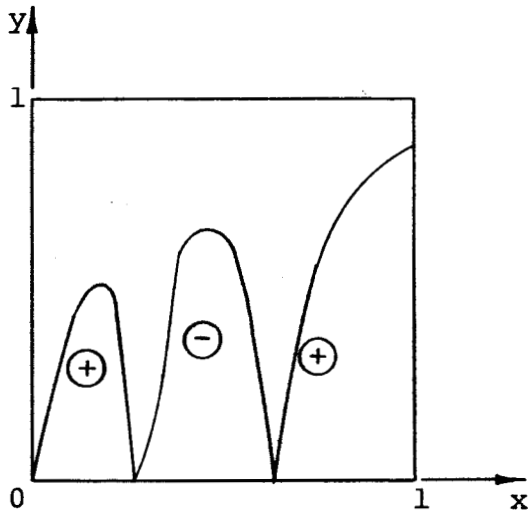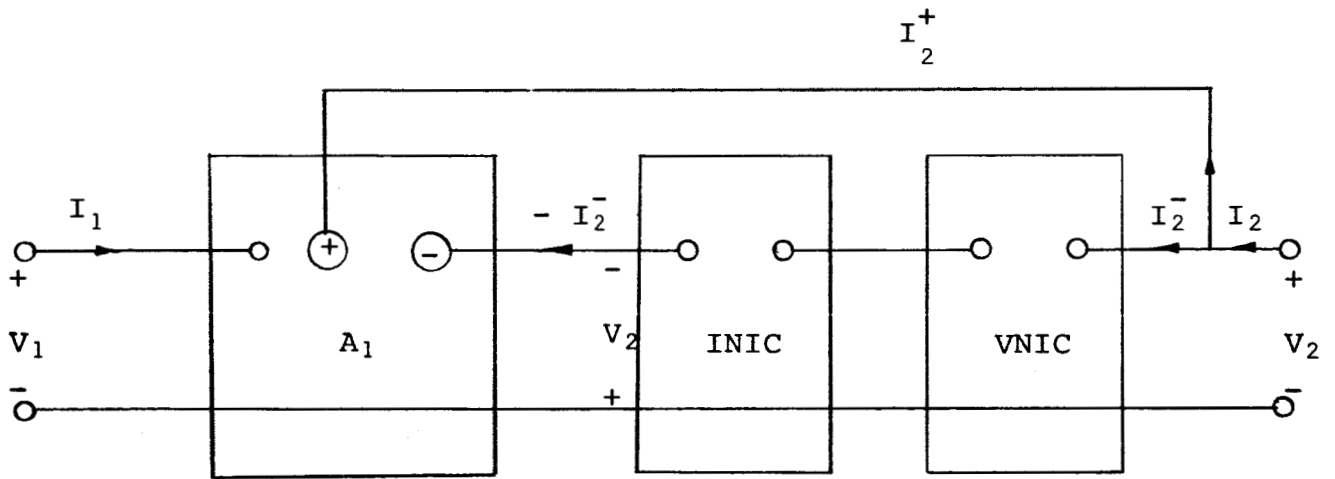
Fig. 4. - Modified Connection of the Distributed
RC Element

To find the short circuit parameters, by reference [4], we know that y_{11}, y_{21} are still of the form (1a) and (1b). But y_{22} will be changed. Consider the connection in Fig. 4 with port 1 shorted and (11) of [4]

$$I_2 = I_2^+ + I_2^- = \int_0^1 [\phi(x) - v(x)] s' C_1 |f_1(x)| dx$$

$$= s' C_1 V_2 \int_0^1 |f_1(x)| dx - s' C_1 \int_0^1 v(x) f_1(x) dx$$

$$= s' C_1 V_2 \int_0^1 \left| \sum_{m=0}^{M} \beta_{1m} \sin\frac{(2m+1)\pi x}{2} \right| dx - \frac{s'^2 C_1 V_2}{2} \sum_{m=0}^{M} \frac{\beta_{1m}^2}{s' + (2m+1)^2 \lambda_1}$$

and

$$y_{22} = s' C_1 \int_0^1 \left| \sum_{m=0}^{M} \beta_{1m} \sin\frac{(2m+1)\pi x}{2} \right| dx - \frac{s'^2 C_1}{2} \sum_{m=0}^{M} \frac{\beta_{1m}^2}{s' + (2m+1)^2 \lambda_1} \quad (1c')$$

Compare (1c) and (1c'), the second summations of both are the same but the first terms are different. Either the summation in (1c) or the integral in (1c') are constant and independent of s'. For the convenience of calculation we let

$$K_1 = \int_0^1 \left| \sum_{m=0}^{M} \beta_{1m} \sin\frac{(2m+1)\pi x}{2} \right| dx \quad (5)$$

which is exactly the first summation of (1c) when $f_1(x) \geq 0$ for $0 \leq x \leq 1$. Further from this point on, we normalize the frequency by λ_1 or $s = s'/\lambda_1$ and rewrite the equation in the normalized frequency for the circuit in Fig. 4 as

$$Y_{11} = - \frac{\pi \sqrt{s}}{2R_1} \tanh \frac{\pi \sqrt{s}}{2} \tag{6a}$$

$$Y_{21} = - \frac{\pi s}{2R_1} \sum_{m=0}^{M} \frac{(2m+1)\beta_{1m}}{s + (2m+1)^2} \tag{6b}$$

$$Y_{22} = \frac{\pi^2 K_1 s}{4R_1} - \frac{\pi^2 s^2}{8R_1} \sum_{m=0}^{M} \frac{\beta_{1m}^2}{s + (2m+1)^2} \tag{6c}$$

This distributed element shall be called A_1 in the sequel.

Another unit (element) to be used is a distributed two-terminal element formed by short-circuiting the terminals of port 1 of the element shown in Fig. 2. This element shall be called element A_2 and it has admittance in the same form as (6c)

$$y = \frac{\pi^2 K_2 s}{4R_2} - \frac{\pi^2 s^2}{8R_2} \sum_{m=0}^{M} \frac{\beta_{2m}^2}{s + (2m+1)^2} \tag{7}$$

where s is the normalized complex frequency in the unit of $\lambda_2 = \frac{\pi^2}{4R_2 C_2} = \lambda_1$ and R_2 is the total resistance measured between x = 0 and x = 1 in the resistive film and C_2 is the total capacitance between the resistive film and the conductor plates when they are at the same potential. The β_{2m}'s are parameters which govern the variation of the capacitances by a cut $|f_2(x)|$ on the conductor sheet, where

$$f_2(x) = \sum_{m=0}^{M} \beta_{2m} \sin\frac{(2m+1)\pi x}{2} \tag{8}$$

and

$$K_2 = \int_0^1 \left| \sum_{m=0}^{M} \beta_{2m} \sin\frac{(2m+1)\pi x}{2} \right| dx \tag{9}$$

IV.   The Transfer Function

Suppose the given open-circuit transfer function is

$$G(s) = \frac{N(s)}{D(s)} = \frac{a_0 + a_1 s + \ldots + a_{N_1} s^{N_1}}{d_0 + d_1 s + \ldots + d_{N_2} s^{N_2}} \tag{10}$$

where $N(s)$ and $D(s)$ are polynomials with real coefficients in the complex frequency s.   Consider a two-port network which contains elements A_1 and A_2 and a lumped capacitor C_0 as shown in Fig. 5. The open-circuit transfer function of such a connection is

$$G(s) = \left.\frac{V_2(s)}{V_1(s)}\right|_{I_2(s) = 0} = \frac{-y_{21}}{y_{22} - (y + C_0 s)} \tag{11}$$

where the y_{21}, y_{22} and y are defined in (6),(7),and C_0 is the capacitance of the lumped element.   One way to realize (10) by (11) is to let

$$\frac{sN(s)}{B(s)} = -y_{21} \tag{12}$$
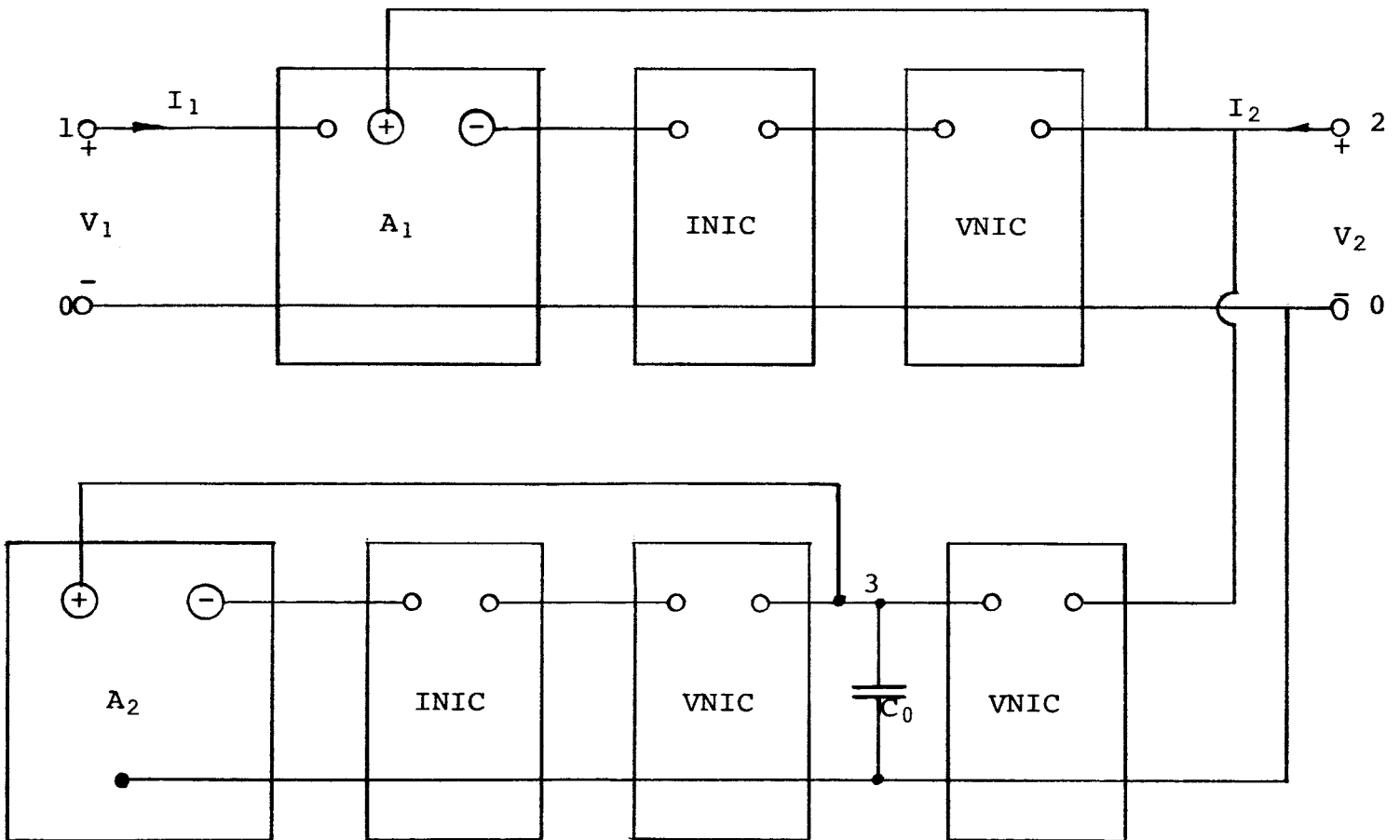
and

$$\frac{sD(s)}{B(s)} = y_{22} - y - sC_0 \tag{13}$$

Fig. 5. - The Realization of G(s)

where B(s) is a polynomial having only negative real roots. Thus, the necessary and sufficient condition for physical realizability of G(s) is that both (12) and (13) are satisfied. The following section will show that generally any G(s) can be realized by a net-work shown in Fig. 5. This network contains a lumped capacitor, a VNIC, and two distributed units modified by active elements. The proof is stated in the following realization procedure.

V. The Realization Procedure

Knowing the conditions (12), (13) and the partial fraction expansion form of (6), (7), we may choose a suitable polynominal for B(s). This requires that N(s)/B(s) is a proper fraction and the degree of D(s) is not greater than B(s). Let

$$B(s) = \prod_{m=0}^{M} [s + (2m+1)^2] \tag{14}$$

where

$$M = N_1 \qquad \text{if } N_1 \geq N_2$$

$$M = N_2 - 1 \qquad \text{if } N_1 \leq N_2$$

Expand both $\frac{N(s)}{B(s)}$ and $\frac{D(s)}{B(s)}$ in partial fractions as follows:

$$\frac{N(s)}{B(s)} = \sum_{m=0}^{M} \frac{h_m}{s + (2m+1)^2} \tag{15}$$

and

$$\frac{D(s)}{B(s)} = \sum_{m=0}^{M} \frac{k_m}{s + (2m+1)^2} + k' \tag{16}$$

where $k' = 0$ if $N_1 \geq N_2$.

Substituting (6b), (6c) and (7) into (11) and simplifying the result, we have

$$G(s) = \cfrac{\displaystyle\sum_{m=0}^{M} \cfrac{(2m+1)\beta_{1m}}{s + (2m+1)^2}}{\cfrac{\pi}{2}(K_1-K_2\rho) - \displaystyle\sum_{m=0}^{M} \cfrac{\pi}{4}(\beta_{1m}^2-\rho\beta_{2m}^2) + \cfrac{\pi}{4}\displaystyle\sum_{m=0}^{M} \cfrac{(2m+1)^2(\beta_{1m}^2-\rho\beta_{2m}^2)}{s + (2m+1)^2} - \cfrac{C_0 2R_1}{\pi}} \tag{17}$$

where $\rho = \dfrac{R_1}{R_2}$. Let G_1 and G_2 be two positive constants. To realize $\dfrac{G_1N(s)}{G_2D(s)}$ by (17); it is necessary that the following equations are satisfied:

$$G_1h_m = (2m+1)\beta_{1m} \qquad m = 0, 1, \ldots, M \tag{18}$$

$$G_2k_m = \frac{\pi}{4}\left[(G_1h_m)^2 - \rho(2m+1)^2\beta_{2m}^2\right] \qquad m = 0, 1, \ldots, M \tag{19}$$

and

$$G_2k' = \frac{\pi}{2}(K_1 - K_2\rho) - \frac{\pi}{4}\sum_{m=0}^{M}(\beta_{1m}^2 - \rho\beta_{2m}^2) - \frac{2C_0R_1}{\pi} \tag{20}$$

The two constants G_1 and G_2 and the ratio ρ are introduced to ensure that the β_{1m}'s and the β_{2m}'s are real numbers and that $|f_1(x)|$ and

$|f_2(x)|$ both have their value less than unity for all $0 \leq x \leq 1$. In (18), since all h_m's are real, so are the $\frac{\beta_{1m}}{G_1}$'s. After $\frac{\beta_{1m}}{G_1}$'s are determined we may form

$$\frac{1}{G_1} \sum_{m=0}^{M} \beta_{1m} \sin\frac{(2m+1)\pi x}{2} = \frac{1}{G_1} f_1(x)$$

and determine a value of G_1 such that

$$|f_1(x)| \leq 1. \qquad (0 \leq x \leq 1) \tag{21}$$

to fulfill the condition of normalization.

Rearrange (19), we have

$$\rho(2m+1)^2 \beta_{2m}^2 = (G_1 h_m)^2 - \frac{4}{\pi} G_2 k_m \qquad (m = 0, 1, \ldots, M) \quad .$$

Among these (M+1) equations we can find a value for G_2 such that the right side of all these equations are positive. Then we obtained all the $\sqrt{\rho} \, \beta_{2m}$'s, either of the two possible value (positive and negative) can be taken arbitrarily. Again, we form

$$\sqrt{\rho} \sum_{m=0}^{M} \beta_{2m} \sin\frac{(2m+1)\pi x}{2} = \sqrt{\rho} \, f_2(x)$$

and determined a value for $\sqrt{\rho}$ such that

$$|f_2(x)| \leq 1 \qquad (0 \leq x \leq 1)$$

to fulfill the condition of normalization. After $f_1(x)$ and $f_2(x)$ are determined, we can make cuts on the conductor sheets of A_1 and A_2.

By aid of (5) and (9) we may calculate K_1 and K_2 and then C_0R_1 can be determined from (20). Here, we have a freedom to decide the values of C_0 and R_1. If C_0R_1 is positive then according to (11) the lumped capacitor C_0 is connected as in Fig. 5. If C_0R_1 is negative then the C_0 should be connected between terminal 2 and terminal 0 in Fig. 5 instead.

The above procedure shows that for any rational function $G(s)$ we can always find a $f_1(x)$ and a $f_2(x)$ and a C_0. No matter how complicated $G(s)$ is, $f_1(x)$ and $f_2(x)$ are two single-valued curves cut on A_1 and A_2 respectively, since they are summations of finite terms of Fourier series.

It should be noted that the network shown in Fig. 5 is the most general case for which both $f_1(x)$ and $f_2(x)$ are negative in some subintervals of the interval $0 \leq x \leq 1$. The number of NIC's can be reduced for the following cases:
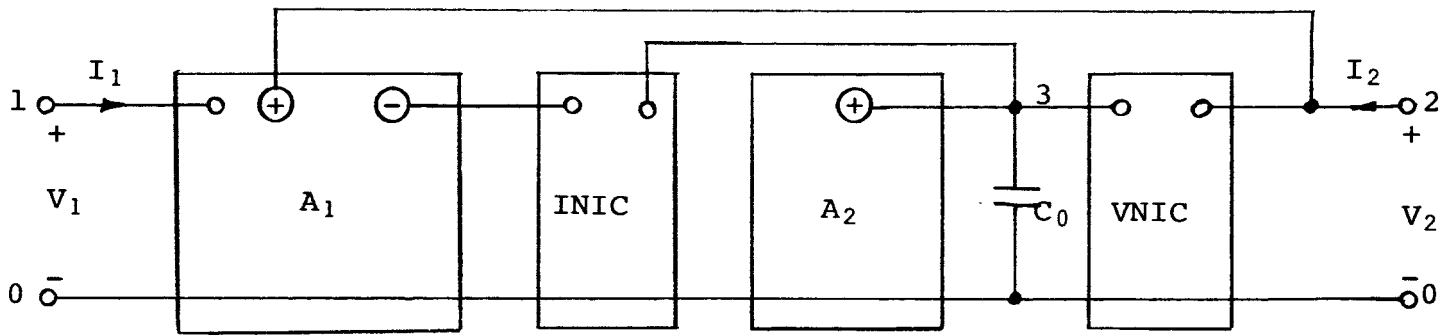
(i) $f_1(x) \geq 0$ for all $0 \leq x \leq 1$; $f_2(x) < 0$ for some $0 \leq x \leq 1$, then only three NIC's are needed (Fig. 6a).

(ii) $f_1(x) < 0$ for some $0 \leq x \leq 1$; $f_2(x) \geq 0$ for all $0 \leq x \leq 1$, then only two NIC's are needed (Fig. 6b)
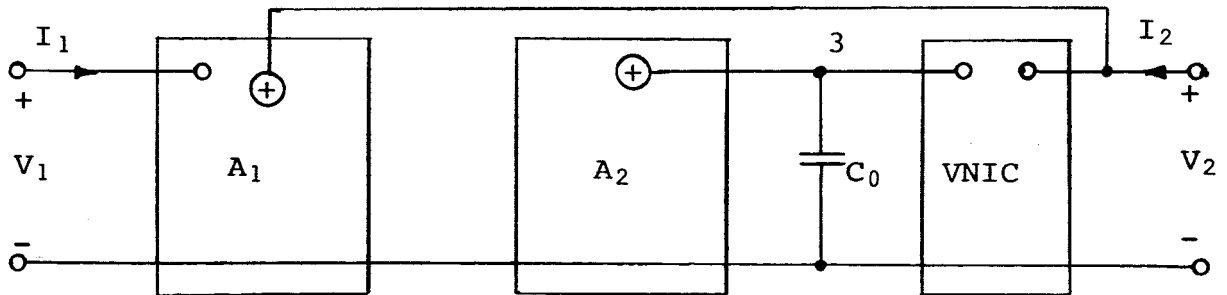
(iii) $f_1(x) \geq 0$ and $f_2(x) \geq 0$ for all $0 \leq x \leq 1$, only one NIC is needed (Fig. 6c).
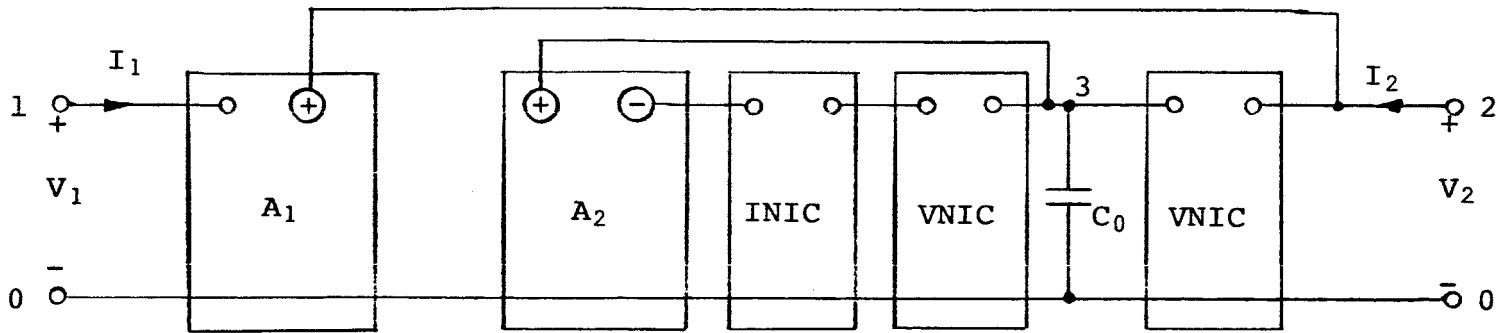
(a)

(b)

(c)

Fig. 6 - Simplifications for Special Cases of
the Configuration in Fig. 5

VI.   Further Remarks

(i)   In choosing A_1 and A_2, we require that both elements should have the same λ, that is $\lambda_1 = \lambda_2$, or $R_1 C_1 = R_2 C_2$, so that the poles of y_{21}, y_{22} and y can be made identical to cancel in the transfer function.

(ii)   In the previous section, we have chosen

$$B(s) = \prod_{m=0}^{M} [s + (2m+1)^2] \quad .$$

However, it is not necessary to have the $(M+1)$ zeros of $B(s)$ to be $-(2m+1)^2$ consecutively, this is, $m = 0, 1, \ldots, M$. The $B(s)$ may have any $(M+1)$ zeros of the form $-(2m+1)^2$, where the m's are any distinctive integers. For instance, if $M = 3$, we may choose

$$B(s) = (s + 1)(s + 9)(s + 25)(s + 49)$$

or $$B(s) = (s + 1)(s + 25)(s + 49)(s + 121)$$

or $$B(s) = (s + 49)(s + 121)(s + 169)(s + 225)$$

etc.

Besides, the value for M given after (14) can be broadened as

$$M \geq N_1 \qquad \text{if } N_1 \geq N_2$$

and $$M \geq N_2 - 1 \qquad \text{if } N_1 \leq N_2$$

(iii)   Though the above procedure is for the realization of open-circuit transfer function, it can be applied to a transfer function G' with a load Y_L at the port 2 where

$$G' = \frac{-y_{21}}{y_{22} - (y + C_0 s) + Y_L} \ . \tag{21}$$

Compare (21) with (11), if Y_L is a capacitance load, $Y_L = C_L s$, we can follow the exactly same procedure, just replacing C_0 in (11) by $C_0' = C_0 - C_L$.

If Y_L is a conductive load, then we can proceed similar formulation with slight modifications. One simplest way for all kinds of Y_L is to use the same formula as the open-circuit transfer function but connect an identical Y_L to the left of the VNIC between terminals 3 and 0 (Fig. 5).

VII. Examples

For the explanation of the synthesis procedure, we realize third order and sixth order Butterworth low-pass filters. The only difference between the two realizations is the cuts for the two distributed RC elements in each case.

The third order Butterworth low-pass filter with 3db cut-off frequency of 30λ or $s = 30$ has the magnitude at $s = j\omega$

$$|G(j\omega)| = \frac{1}{(30)^3 [1 + (\frac{\omega}{30})^6]^{\frac{1}{2}}}$$

we find

$$G(s) = \frac{1}{s^3 + 60s^2 + 1800s + 27000} \quad .$$

According to (14), M = 3 for the present case, so we choose

$$B(s) = (s + 1)(s + 9)(s + 25)$$

Let N(s) = 1 and D(s) = $s^3 + 60s^2 + 1800s + 27000$, then we have

$$\frac{N(s)}{B(s)} = \frac{.0052083333}{s + 1} - \frac{.0078125000}{s + 9} + \frac{.0026041667}{s + 25}$$

and

$$\frac{D(s)}{B(s)} = \frac{131.557292}{s + 1} - \frac{116.648437}{s + 9} + \frac{10.0911458}{s + 25} + 1$$

From (18), we determine β_{1m}'s as follows:

$$\beta_{10} = 0.0052083333 \ G_1$$

$$\beta_{11} = -0.0026041667 \ G_1$$

$$\beta_{12} = 0.00052083333 \ G_1$$

The maximum value that the constant G_1 can assume is fixed by the condition (21). Choose the maximum value $G_1 = 120$, then we have $\beta_{10} = 0.625, \beta_{11} = -0.3125$ and $\beta_{12} = 0.0625$. The next step is to determine β_{2m}'s from (19) by

$$\rho \beta_{2m}^2 = \frac{(G_1 h_m)^2 - \frac{4}{\pi} G_2 k_m}{(2m+1)^2} \quad , \ m = 0, \ 1, \ 2 \quad .$$

As all the $\rho\beta_{2m}^2$ (m = 0, 1, 2) have to be positive, the upper limit of G_2 is thus fixed. For simplicity, we choose this limit as G_2 in this problem, that is

$$G_2 = 2.332034608 \times 10^{-3}$$

and then we have

$$\sqrt{\rho}\beta_{20} = 0 \ , \ \sqrt{\rho}\beta_{21} = 0.368972041$$

and

$$\sqrt{\rho}\beta_{22} = 0.0520358451 \ .$$

The factor ρ is now used to scale the second curve

$$|f_2(x)| = |\sum_{m=0}^{2} \beta_{2m}\sin\frac{(2m+1)\pi x}{2}|$$

to be less than unity for all x. For this example we take $\rho = .16098092$ arbitrarily which makes max $|f_2(x)| = 1$ and $\beta_{20} = 0$, $\beta_{21} = .91961543$, $\beta_{22} = .12969266$. Finally, from (20), we obtain

$$C_0R_1 = .16436742$$

The two curves $f_1(x)$ and $f_2(x)$ are shown in Fig. 7.

Next, we realize

$$G(j\omega) = \frac{1}{(30)^6[1 + (\frac{\omega}{30})^{12}]^{\frac{1}{2}}}$$

or

$$G(s) = \frac{1}{s^6+115.911099s^5+6717.6914s^4+246823.74s^3+6045922.3s^2+93887990s+729\times10^6}$$
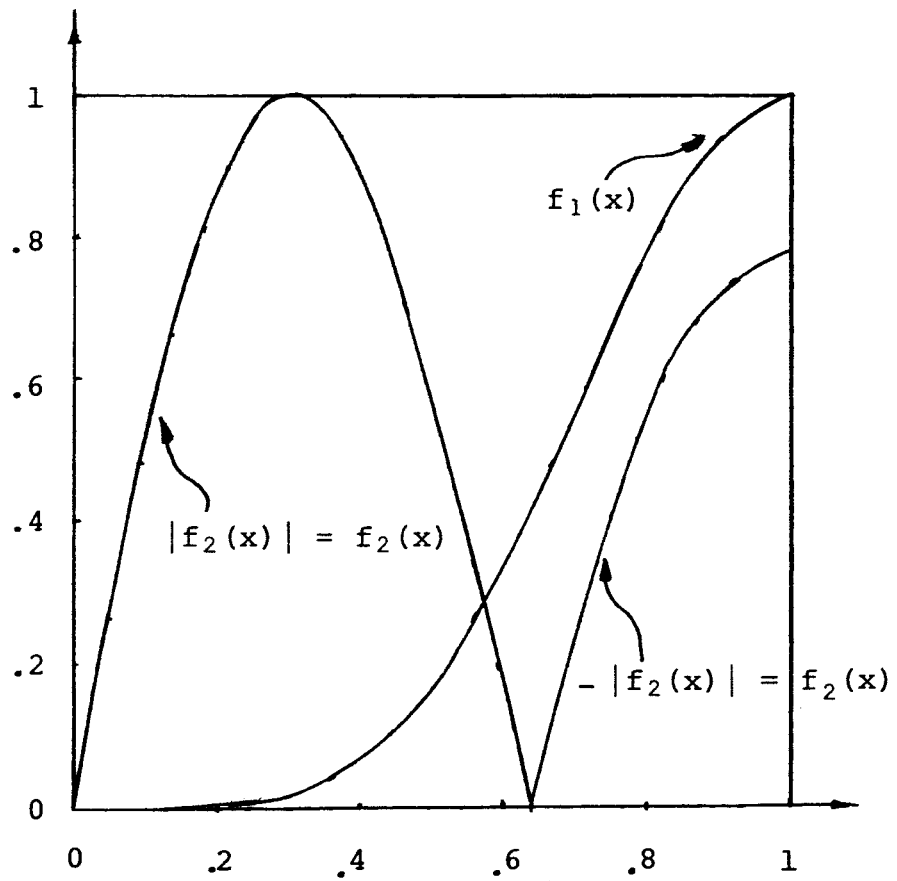
Fig. 7 - $f_1(x)$ and $f_2(x)$ for the third order Butterworth low-pass filter

By the same procedure described above, we determine the parameters to be

$$G_1 = 3.9916800 \times 10^7 \qquad G_2 = 4.55647653 \times 10^{-5}$$

$$\rho = .395177065 \qquad C_0R_1 = -.01184772$$

and

$$\beta_{10} = .45117187 \qquad \beta_{20} = .71700039$$

$$\beta_{11} = -.32265625 \qquad \beta_{21} = .51262685$$

$$\beta_{12} = .16113281 \qquad \beta_{22} = .25649567$$

$$\beta_{13} = -.05371094 \qquad \beta_{23} = .08542315$$

$$\beta_{14} = .0107421875 \qquad \beta_{24} = 0.$$

$$\beta_{15} = -.0009765625 \qquad \beta_{25} = .01992741$$

The two curves to be cut on the conductors $f_1(x)$ and $f_2(x)$ are shown in Fig. 8.

VIII. Conclusion

The rational short-circuit parameters of the class of distributed RC elements used in this paper were pointed out early in 1962 [1] . No general synthesis of this kind of problem has to date been studied and potentiality of this approach has been essentially overlooked. Only recently, some very special functions were realized with good experimental results [5], [6]. The present paper furnishes a general method for the synthesis of any given open-circuit transfer
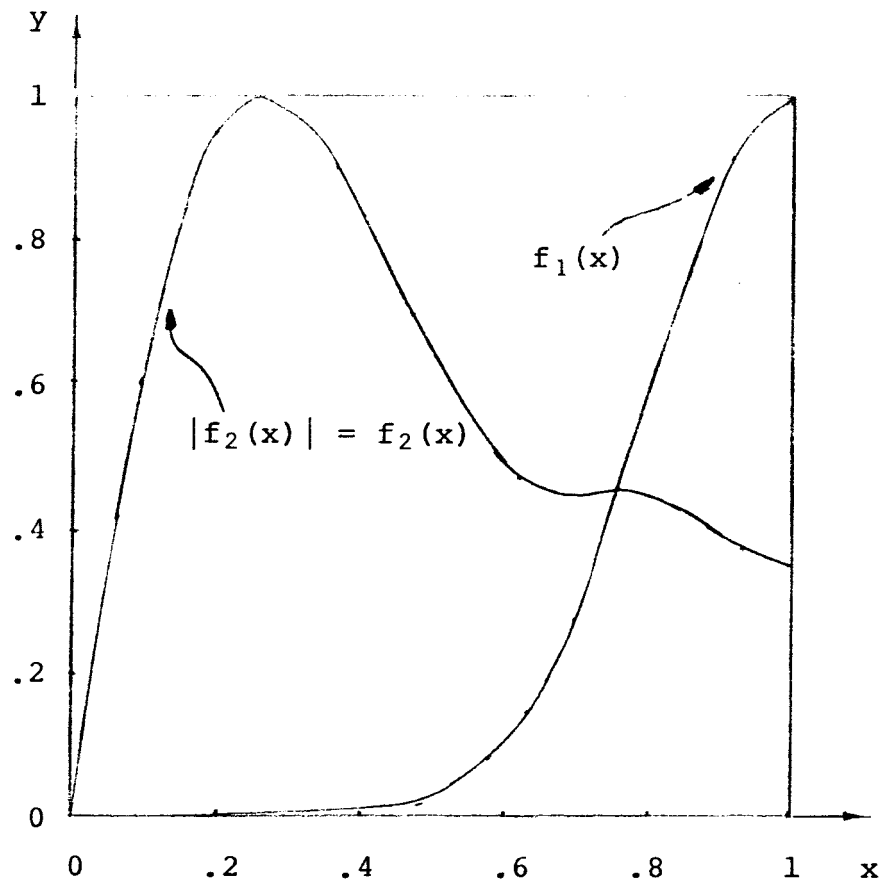
Fig. 8 - $f_1(x)$ and $f_2(x)$ for the sixth order Butterworth low-pass filter

function by using such distributed RC elements. Examples presented are realized only theoretically due to the lack of facilities. However, we believe that there should be no difficulty in manufacturing since two previous works have been carried out satisfactorily at the Boeing Company [3], [4], and that its simplicity in the synthesis procedure and compactness in physical structure should merit practical interest.

The authors wish to thank Professor K.W. Heizer for many helpful discussions.

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