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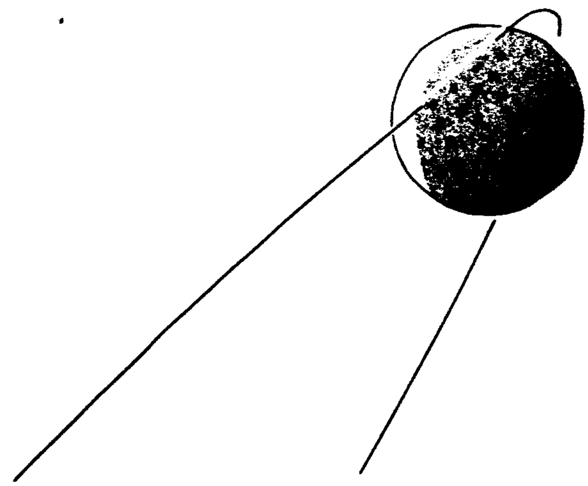


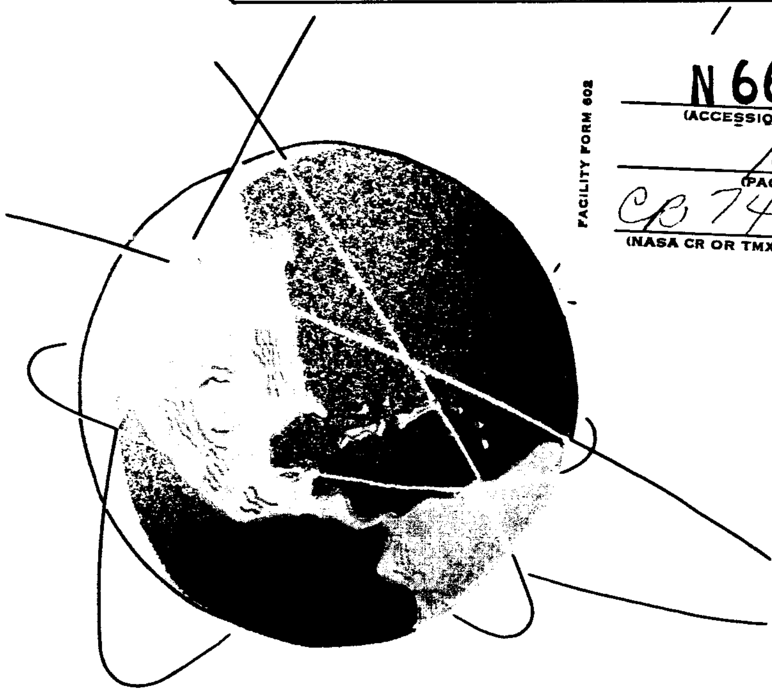
FIRST ORDER CONTROL FOR LOW-THRUST INTERPLANETARY VEHICLES

BY

WALLACE T. FOWLER

EMRL TR 1001

MAY 1965



ENGINEERING MECHANICS RESEARCH LABORATORY

THE UNIVERSITY OF TEXAS · AUSTIN, TEXAS

FIRST ORDER CONTROL FOR LOW-THRUST INTERPLANETARY VEHICLES

A

DISSERTATION

Presented to the Faculty of the Graduate School of
The University of Texas in Partial Fulfillment
of the Requirements
For the Degree of
DOCTOR OF PHILOSOPHY

By

Wallace Thomas Fowler, B. A., M. S.

Austin, Texas

May, 1965

This report was prepared under

Grant NsG-551

for the

National Aeronautics and Space Administration

by the

Engineering Mechanics Research Laboratory

The University of Texas

Austin, Texas

under the direction of

Dr. Byron D. Tapley

Associate Professor of Aerospace Engineering

and Engineering Mechanics

ERRATA

for the report entitled

FIRST ORDER CONTROL FOR LOW-THRUST INTERPLANETARY SPACE VEHICLES

Page	
xiii	Line 2 - sumbol should be symbol
12	Line 2 - Newtonion should be Newtonian
15	Equation (2.6) - upper limit on integral should be t
17	Section 2.4 line 7 - periohelion should be perihelion
24	Below Equation (3.1) - u is an m-vector of control variables
27	Line 6 - delete the words "one-half of"
27	The first line of Equation (3.14) should read

$$\delta''V = [\Delta\bar{x}^T [P_{\bar{x}\bar{x}}\,\Delta\bar{x} + P_{\bar{x}\bar{\nu}}\,\Delta\bar{\nu} + P_{\bar{x}t}\,\Delta t - \Delta\bar{\lambda}]]_{t_f} + \ldots$$

29	Equations (3.16) and (3.17) - upper limit on integral should be t. Lower limit on integral in (3.17) should be 0.
30	Equation (3.18) should read

$$V_1 = -\frac{\beta c^2}{m_o}\left[1 - \frac{m_o}{m_f}\right]$$

41	Equation (4.11) should read

$$(\Lambda\delta\bar{x})_{t_f} = (\Lambda\delta\bar{x})_{t_1} + \int_{t_1}^{t_f} \Lambda B\,\delta\,\bar{u}\,dt$$

43	Equation (4.16) should read

$$\Delta\overline{M} = \Delta\overline{M}\,(\bar{x}^*(t_f),\ t_f) = 0$$

52	Line 2 - he should be the
52	Section 4.6 - line 1 - qualitatvely should be qualitatively
115	First and third equations should read

$$\frac{\partial^2 H}{\partial u_2 \partial u_3} = \left(\frac{\partial H}{\partial u_3}\right)/u_2$$

$$\frac{\partial^2 H}{\partial u_3^{\,2}} = -\left(\frac{\partial H}{\partial u_2}\right) u_2$$

PREFACE

An important problem in astrodynamics is that of performing trajectory corrections in such a way that certain terminal conditions necessary to the completion of the mission are satisfied. The total process of determining how the corrections in the trajectory should be made is called guidance. The guidance problem can be separated into two parts; the navigation problem and the control problem. Navigation determines the current values of the position and the velocity of the vehicle. These values are compared to a stored set of reference (state) values resulting in a current state error. The control procedure then determines the changes which must be made in the reference control program so as to correct the error. Generally, the control process is executed in such a way that some quantity associated with the control maneuver will be optimized.

The current investigation considers the control problem for a low-thrust interplanetary vehicle. Three members of a first order control scheme family are compared using a mathematical model of the vehicle. The comparison takes the form of a numerical experiment on a digital computer as it is impossible at present to undertake a mission of the type proposed. The investigation seeks to determine from the numerical data, whether a new first order control scheme introduced here is superior to existing first order control schemes.

The author wishes to express his gratitude to Dr. B. D. Tapley, Associate Professor of Aero-Space Engineering and Engineering Mechanics at The University of Texas for his many helpful suggestions during the course of this investigation and to Mr. Dan Colunga and Mr. Gary J. Lastman for their ideas and advice in all phases of this investigation. Especially, the author wishes to thank his wife, Madeline Kay, for her inspiration and understanding.

The author is pleased to acknowledge the support of the National Aeronautics and Space Administration under Grant NsG-551.

W. T. F.

March 19, 1965

FIRST ORDER CONTROL FOR LOW-THRUST

INTERPLANETARY VEHICLES

Publication No. ________

Wallace Thomas Fowler, Ph. D.
The University of Texas, 1965

Supervising Professor: Byron D. Tapley

A new control scheme based on the first variation of the calculus of variations and the Weierstrass E-Function is presented. It is found that this control scheme is a member of a family of first order control schemes. In general, a member of the control scheme family is characterized by an arbitrary weighting matrix. The choice of the arbitrary weighting matrix determines the effects of the control scheme on the system being controlled. The new control scheme, E-Function control, provides a criterion for choosing the weighting matrix.

A numerical comparison of three members of the family of first order control schemes is made. Two of the three control schemes are characterized by arbitrarily chosen constant weighting matrices. The weighting matrix for the third control scheme is chosen in the manner prescribed by the E-Function Control scheme.

A general method of predicting the state of a disturbed system in which not all of the state variables are terminally constrained is introduced. This state prediction method, used in conjunction with each control scheme, generates data which is used in the numerical comparison of the control schemes.

The system model used for the control scheme comparison is a low-thrust vehicle moving along a three-dimensional Earth-Mars trajectory. A reference trajectory is established and then each control scheme is forced to correct numerically introduced state errors.

It is found that the new control scheme provides more satisfactory control than either of the other two control schemes. It appears that the new control scheme gives the low-thrust analog of the impulsive correction characteristic of optimal high-thrust guidance.

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LIST OF SYMBOLS

The following list tabulates all significant symbols used in the main body of this report. Each sumbol is accompanied by a brief description and the number of the equation where the symbol is introduced. A definition of each symbol is given where the symbol is introduced.

Matrices:

The size of a matrix is indicated in the square bracket immediately following the symbol. For example, a matrix K having j rows and k columns would be noted K [j x k]. Let the following general indices be used in this section and in the section on vectors.

m	-	the number of control variables
n	-	the number of state variables
p	-	the number of terminal constraint relations

A [n x n]	matrix of second partial derivatives $(H_{\overline{\lambda x}})$	(4.3)
B [n x m]	matrix of second partial derivatives $(H_{\overline{\lambda u}})$	(4.4)
C [p x n]	submatrix of the matrix Λ	(4.13)
$H_{\overline{uu}}$ [m x m]	matrix of second partial derivatives	(4.28)
I [n x n]	the (n x n) identity matrix	(4.14)
J [p x p]	matrix of integrals, a submatrix of ψ	(4.21)
T_1 [3 x 3]	coordinate transformation matrix	(2.18)
T_2 [3 x 3]	coordinate transformation matrix	(2.21)
W [m x m]	arbitrary weighting matrix	(4.15)
W_1 [m x m]	weighting matrix	(5.1)
W_2 [m x m]	weighting matrix	(5.2)

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W_3 [m x m]	weighting matrix	(5.3)
$\Lambda(t)$ [n x n]	matrix of influence functions	(4.11)
$\psi(t)$ [n x p]	matrix of integrals	(4.32)
$\Delta\psi(t, t_1)$ [n x p]	difference between two ψ matrices	(4.34)

<u>Vectors:</u>

All vectors are column vectors unless otherwise noted. The number in brackets indicates the number of elements in the vector.

$\bar{c}$ [3]	propellent exhaust relative velocity	(2.3)
$\bar{D}$ [3]	general drag force vector	(2.2)
$\bar{f}$ [n]	vector of derivatives of the state variables	(2.10)
$H_{\bar{u}}$ [m]	row vector of partial derivatives of H with respect to the control variables	(3.10)
$H_{\bar{x}}$ [n]	row vector of partial derivatives of H with respect to the state variables	(3.9)
$H_{\bar{\lambda}}$ [n]	row vector of partial derivatives of H with respect to the Lagrange multipliers	(3.8)
$\bar{M}$ [p]	terminal constraint relations	(3.3)
$\Delta\bar{M}$ [p]	a change in the terminal constraints	(3.7)
$P_{\bar{u}}$ [m]	row vector of partial derivatives of P with respect to the control variables	(3.7)
$P_{\bar{x}}$ [n]	row vector of partial derivatives of P with respect to the state variables	(3.7)
$P_{\bar{\nu}}$ [p]	row vector of partial derivatives of P with respect to the unknown constant multipliers	(3.7)
$\bar{r}$ [3]	vector defining the position of the vehicle	(2.1)
$\bar{S}$ [3]	general solar radiation force vector	(2.2)

LIST OF SYMBOLS

(CONT'D)

$\bar{T}$ [3]	vehicle thrust vector	(2.2)
$\bar{u}$ [m]	vector of control variables	(2.9)
$\delta\bar{u}$ [m]	vector of control variable changes	(3.7)
$\bar{v}$ [3]	vehicle velocity vector	(2.2)
$\bar{x}$ [n]	vector of system state variables	(2.8)
$\delta\bar{x}$ [n]	vector of state variable deviations	(3.7)
$\bar{y}$ [p]	vector of terminally constrained state variables	(4.7)
$\bar{z}$ [n - p]	vector of terminally free state variables	(4.7)
$\bar{\lambda}$ [n]	vector of adjoint variables	(3.4)
$\delta\bar{\lambda}$ [n]	vector of adjoint variable deviations	(3.7)
$\bar{\nu}$ [m]	vector of unknown constant multipliers	(3.4)

Scalars:

a	thrust acceleration magnitude	(3.16)
a_m	semi-major axis of Mars' orbit	(2.15)
b_m	semi-minor axis of Mars' orbit	(2.15)
c	propellent exhaust speed relative to vehicle	(2.3)
e	eccentricity of Mars' orbit	(2.14)
E	Weierstrass E-function	(4.24)
F	eccentric anomaly of a point on Mars' orbit	(2.14)
G	non-integrated part of performance index	(3.2)
H	variational Hamiltonian	(3.5)

LIST OF SYMBOLS

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L	variable introduced in connection with the E-function	(4.25)
m	mass of the vehicle	(2.2)
m_o	initial value of the vehicle mass	(2.6)
P	augmented performance index	(3.15)
p	orbital parameter of Mars' orbit	(2.16)
Q	integrated portion of performance index	(3.2)
r	distance measured from the origin of the x-y-z coordinate system	(2.5)
t_o	value of time at the start of the mission	(3.2)
t_1	time at which a state disturbance occurs	(4.1)
U	solar gravitational force potential	(2.1)
(u, v, w)	velocity components in the x-y-z coordinate system	(2.5)
(u', v', w')	velocity components in the x'-y'-z' coordinate system	(2.18)
(u'', v'', w'')	velocity components in the x''-y''-z'' coordinate system	(2.16)
V	performance index	(3.2)
V_1	usual performance index for low-thrust studies	(3.16)
(x, y, z)	position components in the x-y-z coordinate system	(2.5)
(x', y', z')	position components in the x'-y'-z' coordinate system	(2.17)
(x'', y'', z'')	position components in the x''-y''-z'' coordinate system	(2.15)
β	propellent ejection rate, u_1	(2.3)

LIST OF SYMBOLS
(CONT'D)

ΔV	total variation of the scalar performance index V	(3.6)
$\delta' V$	the first variation of V	(3.6)
$\delta'' V$	the second variation of V	(3.6)
ε	an arbitrarily small positive number	(4.39)
θ	control angle as defined in Figure 2	(2.4)
ϕ_m	true anomaly of point in Mars' orbit	(2.14)
ψ	control angle as shown in Figure 2	(2.4)

Superscripts:

T	transpose of the superscripted matrix
-1	inverse of the superscripted matrix
*	indicates evaluation of the superscripted quantity along the reference trajectory

Subscripts:

f	indicates evaluation at the end of the mission
m	coordinate and velocity components of Mars
o	indicates evaluation at the beginning of the mission
1	indicates evaluation at some time while the mission is in progress

Miscellaneous Symbols:

$(\dot{\ })$	indicates $\frac{d}{dt}(\ )$

LIST OF SYMBOLS

(CONT'D)

Abbreviations:

AU	Astronomical Unit
EC	E-Function Control
LMC	Lambda Matrix Control

CHAPTER 1

INTRODUCTION

1.1 Preliminary Descriptions

In the period from 1970 to 1990, it is expected that unmanned and/or manned expeditions to Mars will be made. Some of the space vehicles which will carry out these missions may be propelled by low-thrust ion or plasma jet propulsion systems. These types of propulsion systems are characterized by low fuel consumption rates and continuous thrusting capability at very low thrust levels. It is probable that the thrust levels which will be attained will be below 10^{-3} pounds of thrust per pound of initial vehicle weight. This means that the acceleration experienced by the vehicle due to the thrust (thrust acceleration) will be very small and that the thrust will be applied over most or all of the mission.

Such an interplanetary mission can be divided into an escape phase in which the vehicle attains escape velocity relative to the earth, a heliocentric transfer phase in which the vehicle moves from the vicinity of the Earth to the vicinity of Mars, and a terminal maneuver phase during which the dominant gravitational attraction is that due to Mars. The thrust requirements placed on the vehicle by the terminal maneuver phase depend on the mission objectives and could range from a simple fly-by to orbiting or landing on Mars. The first phase of the mission (and the third phase in case the mission objective is an orbit around Mars) can be carried out by either low-thrust or high-thrust devices. However, preliminary results of a study by Lewallen[1*] indicate that the perturbations to any low-thrust escape spiral

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* Numbers appearing in the text as superscripts indicate references listed in the Bibliography.

caused by the Earth's oblateness and the presence of the Moon will be quite severe. The present study will consider only the second phase. That is, it will be concerned only with the problem of the evaluation of schemes for controlling the deviation in the state (position and/or velocity) along a low-thrust heliocentric transfer from Earth to Mars.

Before a study of such a mission is attempted, a nominal (planned) optimum reference trajectory will be computed. Several authors[2, 3, 4, 5] have presented studies of optimal trajectories for low-thrust heliocentric interplanetary missions. The quantity minimized in these studies was either the fuel consumption or the time integral of the square of the thrust acceleration. The minimizations were carried out using various optimization schemes including direct search and gradient methods and both two and three dimensional models have been used in these studies.

Before an interplanetary transfer is initiated, the <u>reference trajectory</u>* and the related time-dependent control settings (<u>control program</u>) will be stored in an onboard computer. During the course of the mission, the reference trajectory can be compared to the actual trajectory in order to detect the presence of any <u>state deviations</u> (state errors). Such deviations are to be expected because of control inaccuracies, navigation errors, errors in the mathematical model used to determine the reference trajectory, etc. For this study the actual trajectory is assumed to be explicitly known. Thus, the deviations from the reference trajectory are known and the problem of controlling the disturbed system is a deterministic control problem.

In order to correct for an error in state (position and/or velocity),

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* Terms which appear underlined on first usage in the text are defined in Appendix A.

the reference control program must be changed. For a given state disturbance, an entire program of subsequent control deviations (control deviation program) must be provided in order to insure rendezvous with Mars. Deviations from the reference trajectory can be tolerated during the early and middle parts of the mission, but the conditions which must be met at the end of the mission restrict the state deviations which can be allowed at the terminal point. At the terminal point, rendezvous must be achieved. This requires that the vehicle arrive at Mars (wherever Mars is in its orbit) with the same velocity as Mars' velocity. Both the position and velocity of Mars are time dependent, so early or late arrival at a point in Mars' orbit cannot be tolerated if the rendezvous conditions are to be satisfied. If the vehicle gets behind schedule, the rendezvous point must be shifted further along Mars' path. The satisfaction of rendezvous conditions should be a primary objective of any control scheme used on a mission of this type.

For the occurrence of a state deviation, there exist many possible control deviation programs which will theoretically insure rendezvous with the target planet, i.e., Mars. Out of the possible control deviation programs, one must be chosen. The criterion by which a specific control deviation program is selected is that some physical quantity evaluated along the deviated trajectory must take on an extreme value. This quantity is referred to as the performance index for the control scheme under consideration. Three similar control schemes evolving from the choice of three different performance indices are considered in the following investigation. One of the objectives of the investigation is to compare and evaluate the adaptability of the control programs resulting from these performance indices to the problem of carrying out the guidance of a low-thrust space vehicle.

A control procedure which corrects for known state deviations and is optimal in some sense is called an optimal deterministic control procedure. In such a procedure, the control deviations are made in such a way that rendezvous is insured and simultaneously the performance index is extremized. In general, the choice of an optimal control procedure is determined by the choice of the quantity chosen as the performance index. Once a performance index is chosen, the form of the control deviations can be determined.

1.2 Previous Studies

A number of studies[6, 7, 8, 9, 10, 11] have been performed in which linear control procedures have been applied to both linear and nonlinear systems. Many quantities have been used as performance indices for control schemes including integrals of quadratic functions of the control deviations and/or state deviations, the indices associated with the change in the performance index used to obtain the reference trajectory, quadratic functions of the terminal state deviations and/or terminal time deviations. The study presented here was motivated by the results of an investigation by Tapley[6] in which the Lambda Matrix Control scheme[7] and the Extremal Field Control scheme[8, 9] were compared employing as the mathematical model a continous thrust vehicle moving in a constant gravitational field. Due to the performance index chosen and the simplicity of the model used for the comparison, the control schemes were found to be very similar and could be made identical with the proper choice of the arbitrary weighting matrix which appears in the Lambda Matrix Control scheme. It is expected that for the low-thrust interplanetary vehicle the differences between the control schemes will be of a more significant nature. The major characteristics of these two control schemes are summarized below.

The Lambda Matrix Control scheme (LMC) is a first order control scheme (based on the first variation) for which the control deviation program is determined so as to minimize a weighted quadratic function of the control deviations while insuring rendezvous with the target planet. There are two troublesome points associated with the use of this scheme. First, the weighting matrix used in the LMC scheme is arbitrary. Different weighting matrices lead to different control deviation programs and correspondingly different state deviations. Furthermore, the scheme proposed in Reference 9 for predicting the state deviation at any time subsequent to the initiation of a control program deviation is applicable only if the final values of all state variables are specified. Some missions (planetary fly-by, hard planetary impact, destructive interception, etc.) do not require specification of the terminal values of all state variables. In planetary fly-by missions, for example, those corrections which insure a proper terminal miss distance from the planet should be made, but the use of propellent to correct small errors in the terminal velocity may be undesirable.

The Extremal Field Control scheme is a second-order scheme (based on the second variation) for which the control deviation program is determined so as to produce a deviated trajectory which is optimal in the same sense as the reference trajectory. For example, if fuel consumption is minimized along the reference trajectory, then the deviation in fuel consumption is minimized along the perturbed path. A disadvantage of second order schemes is that they are quite complex and are difficult to implement.

Friedlander[10] implements the Lambda Matrix Control for a low-thrust two-dimensional Earth-Mars transfer and also presents a study of the sensitivity of terminal conditions to small errors in the state and/or control. Another work worthy of note is that of Mitchell[11]. Mitchell performs operations having results somewhat analogous to those of the state prediction scheme introduced in the present study. Mitchell presents a method for state prediction explicitly designed for the case in which a state variable is the performance index. However, the method which is proposed does not provide an explicit solution to the general state prediction problem created by allowing some of the state variables to be terminally free.

Each of the above-mentioned studies with the exception of that by Bryson[7] has applied some first order control scheme to low-thrust interplanetary flight. The works by Friedlander[10] and Mitchell[11] were concerned also with the effects of navigational errors on the system.

Friedlander used the Lambda Matrix Control scheme with an arbitrary weighting matrix to control a two-dimensional Earth-Mars trajectory. Mitchell suggests that Friedlander's control approximates, but does not attain, optimal first order control. The first order control scheme developed by Mitchell is very similar to that used by Friedlander (Lambda Matrix Control) but contains no weighting matrix (or it could be said to use an identity weighting matrix). The introduction of the report by Mitchell contains a very good survey of important studies in the areas of optimization, trajectory studies, and low-thrust control.

1.3 Scope of the Investigation

In the investigation presented here, a new control scheme similar in form to the Lambda Matrix Control scheme is presented. The new scheme is a first order control scheme as is the LMC scheme, but it is derived by using a performance index similar to that used by the second order Extremal Field Control scheme. An advantage of the new scheme is that it replaces the arbitrary weighting matrix characteristic of the LMC scheme by a uniquely specified weighting matrix. Furthermore, a state prediction scheme is presented which is general enough to handle all terminal constraint requirements.

In the present study, linear control schemes are used in conjunction with a nonlinear model. The nonlinear equations of motion for the vehicle are linearized by expanding them in a Taylor's series about a known reference trajectory at each point in time. The deviations from this reference trajectory are assumed to be small enough so that terms of higher order than the first in the Taylor's series expansion can be neglected.

A control scheme based on the linearized model will be able to correct small errors. In general, a state error will be detected while it is small and can be kept small using the control capabilities of the system. Thus, the linear control scheme appears to be a satisfactory approximation for the low-thrust vehicle.

The subsequent study consists of the development of the new control scheme and its comparison with two other control schemes which belong to the same first order control scheme family. In order to obtain the effects of the control scheme on the vehicle's state deviation history, a general state prediction scheme is developed. This state prediction scheme is designed to handle predictions of state variables which are terminally free as well as state variables which are constrained at the terminal point on the trajectory.

The control schemes are compared by allowing each to correct a specified set of state disturbances. Both control and state deviations are plotted as functions of time and much of the comparison data is taken from these curves. All state deviations are plotted for each initial state disturbance.

The comparison of the control procedures is made using a three-dimensional model of the solar system. In the formulation of the problem, four control variables are used, but two of these are held constant for the study presented here. The four control quantities are fuel mass-ejection rate, relative propellent exhaust speed, and two angles defining the thrust direction relative to a fixed coordinate system. Results are given for the case where fuel mass-ejection rate and the relative propellent exhaust speed are held constant. Hence, the two angles which define the orientation of the thrust vector will be regarded as the control variables. Holding the fuel mass-ejection rate and the relative exhaust speed constant in effect restricts the mission to one characterized by constant thrust magnitude. The effects of allowing these two quantities to vary should be investigated in a later study. The mission considered here is a low-thrust, continously powered transfer from Earth to Mars for which it is required that the vehicle match position and velocity with Mars at the end of the mission.

A near optimal control program which produces such a three-dimensional Earth-Mars trajectory along which fuel consumption is approximately minimized was obtained using the method of steepest descent as described by Bryson and Denham[12, 13]. This nominal control program and the resulting nominal trajectory are used by the guidance control schemes as the reference control program and reference trajectory, respectively. Each control scheme is used to correct small state errors in the vicinity of this reference trajectory. A state perturbation, as used here, is a suddenly discovered difference between the actual state and the reference state. The sudden discovery of a state error is simulated in the subsequent study by mathematical introduction of a state perturbation. State perturbation vectors, $\delta\bar{x}_i$, are introduced in which the perturbation in all state variables except the i^{th} state variable are zero and the perturbation in the i^{th} state variable is 10^{-6} units (the actual units depend on the state variable).

A typical state perturbation vector is used in the following manner. The state of the system at the specified time t_1 is assumed to be displaced from the reference state (as defined by the reference trajectory) by the amount specified by the state perturbation vector. Each control scheme is used to produce a control deviation program which insures satisfaction of the terminal conditions. State deviations are predicted for the remainder of the mission. For the results presented here, the state perturbations are assumed to take place at the start of the mission.

CHAPTER 2

FORMULATION OF THE PROBLEM

2.1 The Mathematical Model

The problem under consideration is the comparison of three members of a family of optimal deterministic control schemes for correcting deviations in a three-dimensional low-thrust Earth-Mars transfer trajectory. This chapter describes the mathematical model assumed for the comparison. The equations governing the motion of the vehicle and the transformations between the coordinate systems used are presented. Matrix notation is used for compactness where it is convenient to do so.

The nominal or reference trajectory is a heliocentric transfer from Earth to Mars. The retarding effect of the Earth on the departing rocket has been taken into account in the manner suggested by Irving[14] in which the departing vehicle is given the velocity of Earth at the start of the mission instead of escape velocity relative to the Earth, and then Earth's gravitational effects are neglected. The details of arrival at Mars will depend on specific mission requirements. In the study presented here, the gravitational effects of Mars and the other planets have been neglected although these effects would have to be included in an actual mission planning study. It is desirable to neglect these effects, since in a preliminary control comparison study, the mathematical model should be as realistic as possible without introducing factors which complicate interpretation of the results or which will make computations more tedious than warranted in a preliminary investigation. Factors omitted in the initial study can be introduced in subsequent studies in such a manner that their effects on the results of the preliminary study are apparent.

The geometry of Earth's orbit does not enter into the problem since only the position and velocity of the Earth at launch are needed for initial conditions. The position and velocity of Earth at noon on May 9, 1971, were used as initial conditions.

The orbit of Mars is assumed to be an ellipse with an eccentricity of 0.093393 lying in a plane which is inclined at an angle of 1.8499 degrees to the ecliptic plane. The semimajor axis of the ellipse is 1.523691 AU (Astronomincal Units). The ascending node and line of perihelion of the model orbit are properly oriented in the coordinate systems used to represent the actual orientation of Mars' orbit at the time of Earth departure, May 9, 1971. Values of the constants used in this study, including the constants dealing with Mars' orbit, are given in Appendix E.

The treatment of Mars' orbit as an ellipse is both desirable and necessary. A circular orbit in Mars' orbital plane which has its radius equal to the semimajor axis of Mars' orbit will deviate from the elliptic orbit by 0.142 AU (13.2×10^6 miles) at aphelion and at perihelion. Also, the inclusion of the inclination of Mars' orbit plane is necessary since this effect will carry Mars over four million miles out of the ecliptic plane (Earth's orbital plane).

2.2 Equations of Motion

The gravitational force potential for a homogeneous and spherical body such as the Sun is given by

$$U = \mu m / r \tag{2.1}$$

where r is the distance between the attracting mass and the position of the attracted mass, m, and μ is the gravitational constant of the attracting mass. If G is the Newtonion gravitational constant and M is the mass of the attracting body, then $\mu = GM$.

The equations of motion for a vehicle of mass m traveling in the field of influence of an attracting body of mass M are

$$m\dot{\overline{v}} = \overline{\nabla} U + \overline{T} + \overline{S} + \overline{D} \tag{2.2}$$

where

$\overline{v}$	is the vehicle velocity
$\overline{T}$	is the vehicle thrust vector
$\overline{S}$	is the solar radiation force
$\overline{D}$	is the drag force.

For this analysis it is assumed that $\overline{S}$ and $\overline{D}$ are zero.

The thrust vector, $\overline{T}$, is given by

$$\overline{T} = -\beta\overline{c} \tag{2.3}$$

where

β	is the propellent mass-ejection rate, and
$\overline{c}$	is the effective propellent exhaust velocity relative to the vehicle.

Let the equations of motion be expressed in a heliocentric right-handed rectangular Cartesian coordinate system. This coordinate system is oriented so that the x-y plane coincides with the ecliptic plane, the positive x-axis points along the line of the ascending node of Mars' orbit, and the positive z-axis coincides with the angular momentum vector of the Earth with respect to the Sun. This coordinate system is shown in

Figure 1.

Components of the thrust vector in the x-y-z coordinate system are specified by introducing two angles θ and ψ as shown in Figure 2. Then, the thrust components are given by

$$
\begin{aligned}
T_x &= T \cos\theta \cos\psi \\
T_y &= T \cos\theta \sin\psi \\
T_z &= T \sin\theta
\end{aligned}
\qquad (2.4)
$$

where $T = |\overline{T}| = \beta \, |\overline{c}|$

Letting (u, v, w) and (x, y, z) be the velocity components and position components respectively in the x-y-z coordinate system, the equations of motion become

$$
\begin{aligned}
\dot{u} &= -\frac{\mu x}{r^3} + \frac{\beta c}{m} \cos\theta \cos\psi \\
\dot{v} &= -\frac{\mu y}{r^3} + \frac{\beta c}{m} \cos\theta \sin\psi \\
\dot{w} &= -\frac{\mu z}{r^3} + \frac{\beta c}{m} \sin\theta \\
\dot{x} &= u \\
\dot{y} &= v \\
\dot{z} &= w
\end{aligned}
\qquad (2.5)
$$

where $r = (x^2 + y^2 + z^2)^{1/2}$ and where the dot indicates differentiation with respect to time.

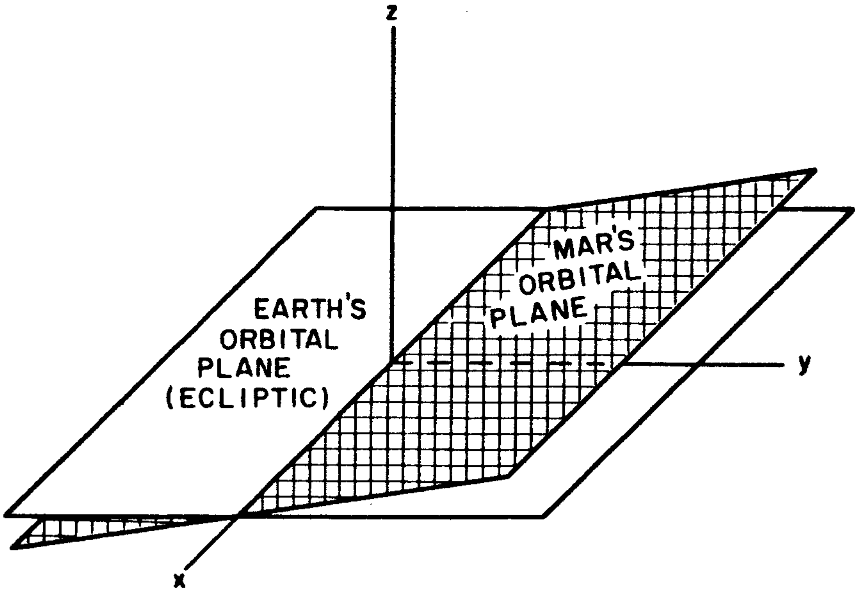

FIGURE 1. MAJOR COORDINATE SYSTEM ORIENTATION

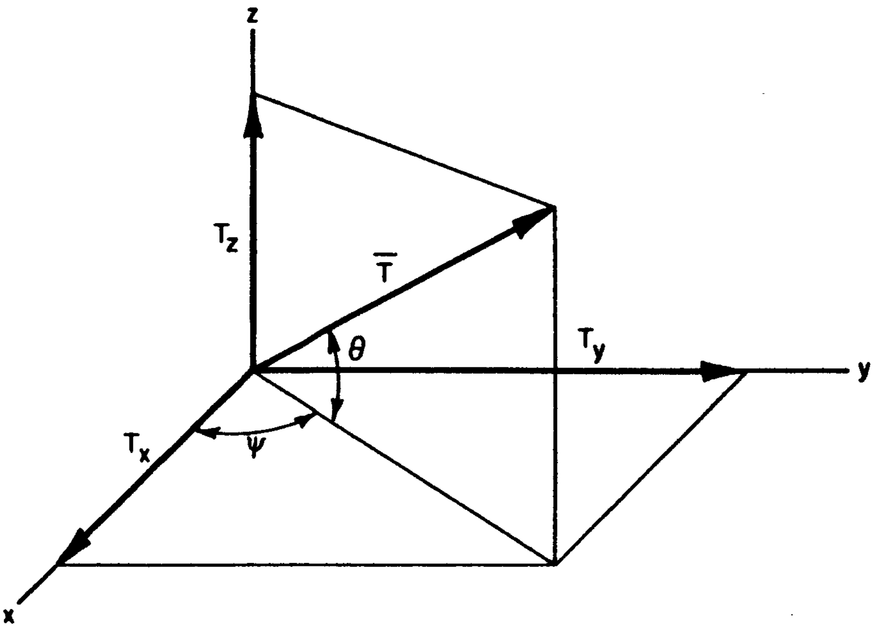

FIGURE 2. THRUST VECTOR COMPONENTS

The mass m of the vehicle at any time t is given by

$$m = m_o - \int_{t_o}^{t-t_o} \beta dt \tag{2.6}$$

The differential equation governing the vehicle mass is

$$\dot{m} = -\beta \tag{2.7}$$

For generalization of the problem, let the variables which appear differentiated on the left hand side of Equations (2.5) and (2.7) be called state variables and be denoted by the vector $\bar{x}$, where

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} u \\ v \\ w \\ x \\ y \\ z \\ m \end{bmatrix} \tag{2.8}$$

Similarly, let β, c, θ, and ψ be called control variables and be denoted by the vector $\bar{u}$, where

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} \beta \\ c \\ \theta \\ \psi \end{bmatrix} \tag{2.9}$$

Also, let the seven expressions on the right hand sides of Equations (2.5) and (2.7) be denoted as f_1 through f_7. Thus, the equations of state, i.e. Equations (2.5) and (2.7), can be expressed in matrix form as

$$\dot{\bar{x}} = \bar{f}(\bar{u}, \bar{x}, t) \tag{2.10}$$

where

$(\dot{\ })$ represents $\frac{d(\ )}{dt}$,

$\bar{x}$ is a 7 x 1 vector of state variables,

$\bar{f}$ is a 7 x 1 vector of known functions,

$\bar{u}$ is a 4 x 1 vector of control variables, and

t is the scalar independent variable, time.

2.3 Notation Conventions

All vectors are denoted by a bar. All vectors are column vectors unless otherwise indicated. The symbol $(\ )^T$ indicates the transpose of the vector or matrix ( ). The symbol $(\ )^{-1}$ indicates the inverse of the square matrix ( ). All first partial derivatives of scalars with respect to vectors, are row vectors, i.e.

$$\frac{\partial H}{\partial \bar{x}} = H_{\bar{x}} = \left[\frac{\partial H}{\partial x_1}\ \frac{\partial H}{\partial x_2}\ \cdots\ \frac{\partial H}{\partial x_n}\right] \tag{2.11}$$

All second partial derivatives of scalars with respect to vectors are matrices formed and denoted in the following manner.

$$\frac{\partial^2 H}{\partial \bar{u} \partial \bar{x}} = H_{\overline{xu}} = \begin{bmatrix} \frac{\partial^2 H}{\partial u_1 \partial x_1} & \cdots & \frac{\partial^2 H}{\partial u_m \partial x_1} \\ \cdots & \cdots & \cdots \\ \frac{\partial^2 H}{\partial u_1 \partial x_n} & \cdots & \frac{\partial^2 H}{\partial u_m \partial x_n} \end{bmatrix} \qquad (2.12)$$

or, in a more compact form,

$$\frac{\partial^2 H}{\partial \bar{u} \partial \bar{x}} = \frac{\partial}{\partial \bar{u}} \left(\frac{\partial H}{\partial \bar{x}}\right)^T \qquad (2.13)$$

2.4 Coordinate Systems

The motion of Mars in its orbit must be specified in order to define the rendezvous condition. This motion can be expressed in terms of a right-handed Cartesian x''-y''-z'' coordinate system in which the x''-y'' plane coincides with the orbital plane of Mars, the positive x''-axis points along the line of perihelion of Mars, and in which the origin lies at the center of the Mars orbit ellipse. This coordinate system is shown in Figure 3. Note that the Sun is at the focus near the periohelion and is not at the origin of the (x''-y''-z'') coordinate system.

If a value of the true anomaly, ϕ_m, is given, then a point on the orbit is determined and the corresponding value of the eccentric anomaly, F, can be found by use of the relation

$$F = 2 \tan^{-1}\left[\sqrt{\frac{1 - e}{1 + e}} \tan \frac{\phi_m}{2}\right] \qquad (2.14)$$

where e is the eccentricity of the Mars orbit ellipse.

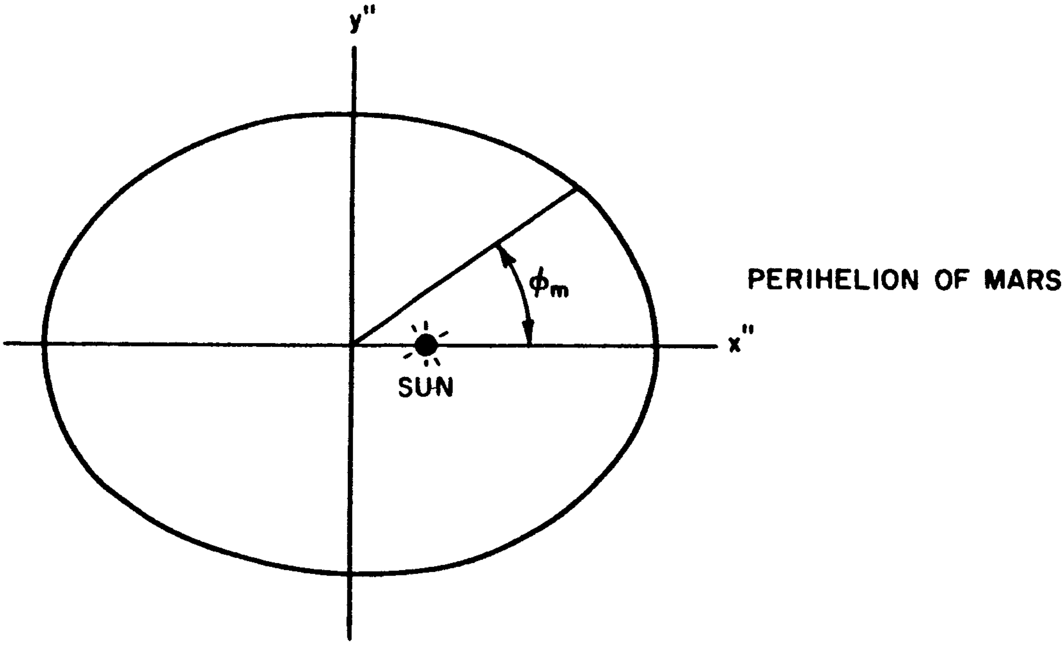

FIGURE 3. RENDEZVOUS COORDINATE SYSTEM

The position (x'', y'', z'') and velocity (u'', v'', w'') of Mars when it is passing through this point in its orbit is given by

$$
\begin{aligned}
x'' &= a_m \cos F \\
y'' &= b_m \sin F \\
z'' &= 0
\end{aligned}
\tag{2.15}
$$

and

$$
\begin{aligned}
u'' &= \sqrt{\frac{\mu}{p}} \sin \phi_m \\
v'' &= \sqrt{\frac{\mu}{p}} (e + \cos \phi_m) \\
w'' &= 0
\end{aligned}
\tag{2.16}
$$

where

a_m	is the semimajor axis of Mars' orbit,
b_m	is the semiminor axis of Mars' orbit ($b_m = a_m (1 - e)$), and
p	is the orbital parameter ($p = a_m (1 - e^2)$).

The coordinate transformation from the x''-y''-z'' system to the x-y-z system is most easily seen if the intermediate x'-y'-z' Cartesian coordinate system is introduced. Let the x'-y' plane coincide with the x''-y'' plane, while the x'-axis coincides with the positive x-axis of the x-y-z coordinate system (See Figure 4). The origin of the x'-y'-z' coordinate system is the Sun. Thus z' and z'' are parallel but distinct.

The transformation of position from the x''-y''-z'' coordinate system to the x'-y'-z' system is given by

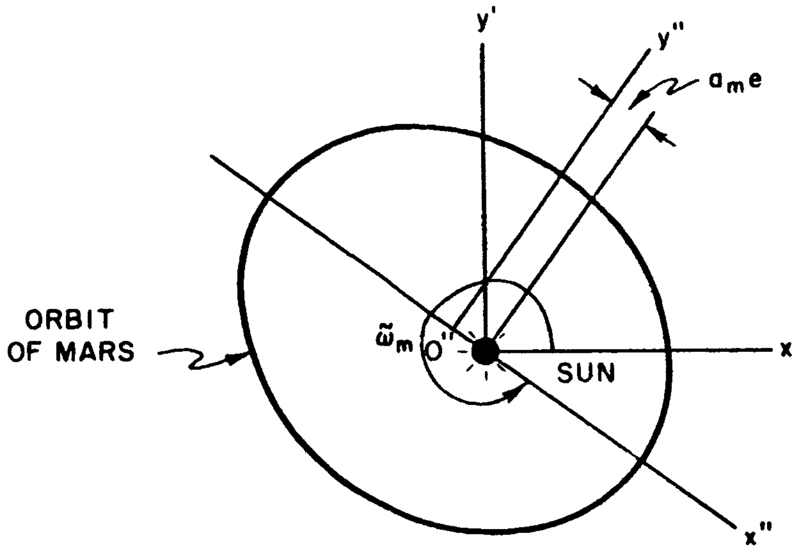

FIGURE 4. RELATIVE ORIENTATIONS OF x'-y'
AND x''-y'' COORDINATE SYSTEM

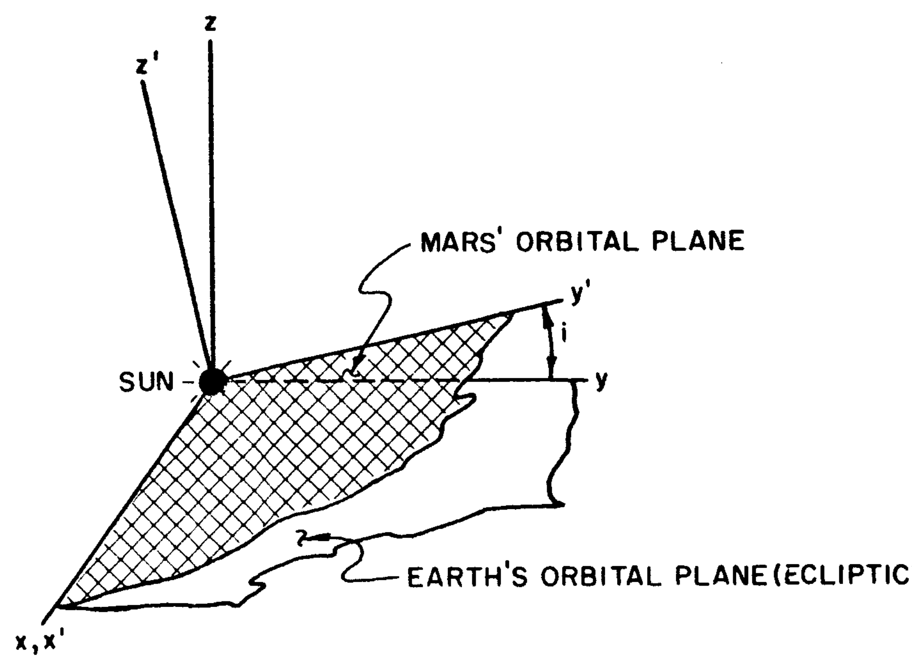

FIGURE 5. RELATIVE ORIENTATIONS OF x-y-z
AND x'-y'-z' COORDINATE SYSTEM

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos \omega_m & -\sin \omega_m & 0 \\ \sin \omega_m & \cos \omega_m & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x'' - a_m e \\ y'' \\ z'' \end{bmatrix} \qquad (2.17)$$

where the translation of origin to the Sun is taken care of by the term $(-a_m e)$, and where ω_m is the argument of the perihelion of Mars.

The transformation of velocities from the x''-y''-z'' system to the x'-y'-z' system does not involve the translation of axes present in the position transformation and is given by

$$\begin{bmatrix} u' \\ v' \\ w' \end{bmatrix} = \begin{bmatrix} T_1 \end{bmatrix} \begin{bmatrix} u'' \\ v'' \\ w'' \end{bmatrix} \qquad (2.18)$$

where T_1 is the 3 x 3 transformation matrix in Equation (2.17). The transformation from the x'-y'-z' coordinate system to the x-y-z coordinate system involves no translation for the position components. Thus, the positions and velocities will transform in exactly the same manner. The position transformation is as follows

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos i & -\sin i \\ 0 & \sin i & \cos i \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} \qquad (2.19)$$

where i is the angle of inclination of Mars' orbit plane measured from the ecliptic plane. The relation of the x'-y'-z' system to the x-y-z system is shown in Figure 5.

The transformation from the x''-y''-z'' coordinate system to the x-y-z coordinate system for positions can now be given by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos \omega_m & -\sin \omega_m & 0 \\ \cos i \sin \omega_m & \cos \omega_m \cos i & -\sin i \\ \sin i \sin \omega_m & \sin i \cos \omega_m & \cos i \end{bmatrix} \begin{bmatrix} x''-a_m e \\ y'' \\ z'' \end{bmatrix} \tag{2.20}$$

If the 3×3 matrix in Equation (2.20) is denoted by T_2 then the velocity transformation from the x''-y''-z'' system to the x-y-z system is given by

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} T_2 \end{bmatrix} \begin{bmatrix} u'' \\ v'' \\ w'' \end{bmatrix} \tag{2.21}$$

The entire geometrical relationships of the various coordinate systems are shown in Figure 6.

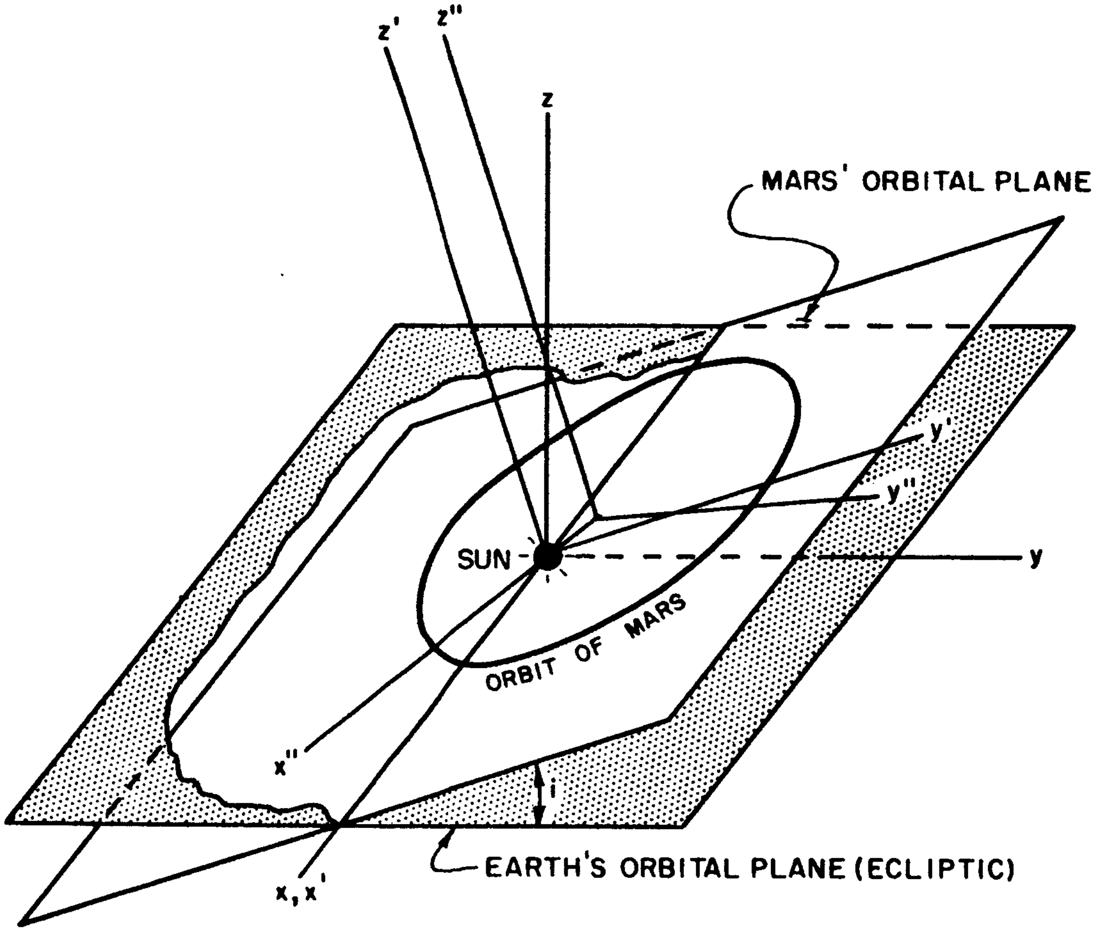

FIGURE 6. COORDINATE SYSTEMS

CHAPTER 3

THE REFERENCE TRAJECTORY

Before discussing the control schemes, a discussion of the mathematical relations necessary for a description of the reference trajectory will be presented. Consider a dynamic system governed by a set of differential equations

$$\dot{\bar{x}} \quad = \quad \bar{f}\,(\bar{x},\ \bar{u},\ t) \qquad (3.1)$$

where

$\bar{x}$ is an n-vector of state variables,

$\bar{u}$ is an n-vector of control variables,

$\bar{f}$ is an n-vector of derivative functions,

t is an independent scalar variable, time.

If the reference trajectory is to be optimal, then a quantity which is to be maximized or minimized along this trajectory must be chosen. This quantity acts as an index of merit by means of which the optimal reference trajectory is chosen from the set of possible reference trajectories. The quantity to be maximized or minimized will be called the performance index for the optimization process. In general, such a performance index could consist of the sum of a function of the terminal values of the state variables and a function which is integrated over the trajectory. Thus, a general performance index can be defined in the following manner.

$$V \quad = \quad G\,(\bar{x}_f,\ t_f) + \int_{t_o}^{t_f} Q\,(\bar{u},\ \bar{x},\ t)\,dt \qquad (3.2)$$

where t_o is the initial value of time, t_f is the final value of time, and $\bar{x}_f$ is the final state vector.

It is required, that V be extremized subject to the conditions that Equations (3.1) be satisfied along the trajectory and the p terminal constraint relations

$$\bar{M}(\bar{x}_f, t_f) = 0 \tag{3.3}$$

where $\bar{M}$ is a p-vector, be satisfied at the terminal point.

Adjoining Equations (3.1) and (3.3) to (3.2) by the unspecified Lagrange multiplier vectors $\bar{\lambda}(t)$ and $\bar{\nu}$, respectively, the quantity to be extremized becomes

$$V = (G + \bar{\nu}^T \bar{M})_{t_f} + \int_{t_o}^{t_f} (Q + \bar{\lambda}^T \bar{f} - \bar{\lambda}^T \dot{\bar{x}})\, dt \tag{3.4}$$

where $\bar{\lambda}$ is an n-vector function of time and $\bar{\nu}$ is a p-vector of unknown constants.

Letting $H = Q + \bar{\lambda}^T \bar{f}$ + and $P = G + \bar{\nu}^T \bar{M}$ and integrating the last term under the integral in Equation (3.4) by parts, we obtain

$$V = (P - \bar{\lambda}^T \bar{x})_{t_f} + (\bar{\lambda}^T \bar{x})_{t_o} + \int_{t_o}^{t_f} (H + \dot{\bar{\lambda}}^T \bar{x})\, dt \tag{3.5}$$

The functional expression V is a function of $\bar{x}$, $\bar{u}$, $\bar{\lambda}$, $\bar{\nu}$, and t. The requirement that V be an extremum demands that the first variation of V, $\delta' V$, be zero while the second variation, $\delta'' V$, be positive

+ The variational Hamiltonian, H, for the model used in this study, and its partial derivatives are given in Appendix B.

(negative) for a minimum (maximum). Following Bliss[15], the total variation, ΔV, of the functional expression, V, is given by

$$\Delta V = \delta' V + \frac{1}{2!}\delta'' V + \frac{1}{3!}\delta''' V + \ldots \tag{3.6}$$

3.1 The First and Second Variations

The first variation, $\delta' V$, is given by

$$\begin{aligned} \delta' V = \; & \left[(P_{\bar{x}} - \bar{\lambda}^T)\,\Delta\bar{x} + P_{\bar{\nu}}\,\Delta\bar{\nu} + (P_t + H)\,\Delta t\right]_{t_f} \\ + \; & \left[\bar{\lambda}^T\,\Delta\bar{x} - H\Delta t\right]_{t_o} \\ + \; & \int_{t_o}^{t_f} \left[(H_{\bar{x}} + \dot{\bar{\lambda}}^T)\,\delta\bar{x} + H_{\bar{u}}\,\delta\bar{u} + (H_{\bar{\lambda}} - \dot{\bar{x}}^T)\,\delta\bar{\lambda}\right]dt \end{aligned} \tag{3.7}$$

For an extremum in V, where $\Delta\bar{x}(t_o) = 0$ and $\Delta t_o = 0$, the condition $\delta' V = 0$ requires that along the trajectory

$$\dot{\bar{x}}^T = H_{\bar{\lambda}} \tag{3.8}$$

$$\dot{\bar{\lambda}}^T = -H_{\bar{x}} \tag{3.9}$$

$$H_{\bar{u}} = 0 \tag{3.10}$$

and at the terminal point

$$(\bar{\lambda}^T - P_{\bar{x}})_{t_f} = 0 \tag{3.11}$$

$$(P_{\bar{\nu}})_{t_f} = 0 \tag{3.12}$$

$$(P_t + H)_{t_f} = 0 \tag{3.13}$$

Equations (3.8) through (3.13) are necessary conditions which must be satisfied if V is to be an extremum. Note that Equation (3.12) is just Equation (3.3) in a different notation.

The second variation of V is one-half of the first variation of $\delta'V$, i.e., $\delta(\delta'V)$. From Equations (2.7), for fixed Δt_o and known $\Delta\bar{x}(t_o)$, the second variation can be obtained as follows,

$$\begin{aligned}
2\delta''V = & \left[\Delta\bar{x}^T \left[P_{\bar{x}\bar{x}}\Delta\bar{x} + P_{\bar{x}\bar{\nu}}\Delta\bar{\nu} + (P_{\bar{x}t} + H_{\bar{x}})^T \Delta t - \Delta\bar{\lambda}\right]_{t_f}\right. \\
& + \left[\Delta\bar{\nu}^T \left[P_{\bar{\nu}\bar{x}}\Delta\bar{x} + P_{\bar{\nu}t}\Delta t\right]\right]_{t_f} + \left[\Delta t\left[P_{t\bar{x}}\Delta\bar{x}\right.\right. \\
& \left.\left. + P_{t\bar{\nu}}\Delta\bar{\nu} + H_{\bar{x}}\Delta\bar{x} + H_{\bar{\lambda}}\Delta\bar{\lambda} + (P_{tt} + H_t)\Delta t\right]\right]_{t_f} \\
& + \int_{t_o}^{t_f} \left[\delta\bar{x}^T (H_{\bar{x}\bar{x}}\delta\bar{x} + H_{\bar{x}\bar{u}}\delta\bar{u} + H_{\bar{x}\bar{\lambda}}\delta\bar{\lambda} - \delta\dot{\bar{\lambda}})\right. \\
& + \delta\bar{u}^T (H_{\bar{u}\bar{x}}\delta\bar{x} + H_{\bar{u}\bar{u}}\delta\bar{u} + H_{\bar{u}\bar{\lambda}}\delta\bar{\lambda}) \\
& \left. + \delta\bar{\lambda}^T (H_{\bar{\lambda}\bar{x}}\delta\bar{x} + H_{\bar{\lambda}\bar{u}}\delta\bar{u} - \delta\dot{\bar{x}})\right] dt
\end{aligned} \tag{3.14}$$

It is possible to arrange the terms in Equation (3.14) into other groupings and to remove some of the terms under the integral through integration by parts. This will not be done as Equation (3.14) is presented here only to show the form of $\delta''V$. The second derivatives of the variational

Hamiltonian found in Equation (3.14) are given in Appendix C.

As was previously stated, $\delta''V$ is positive for a minimum value of V and negative for a maximum. The use of Equation (3.14) to check the sign of $\delta''V$ in order to distinguish maxima and minima is a very tedious task. In most physical problems the nature of the problem and the solution obtained leave no doubt as to the nature of the extremum (maximum or minimum).

3.2 Reference Trajectory Determination

In order to compare the control schemes as proposed, a reference trajectory along which the comparison is to be made must be defined. The reference trajectory is a three-dimensional Earth-Mars transfer which approximately minimizes fuel consumption and is nearly optimal in this sense. The conditions which must be satisfied by an optimal trajectory are given in Equations (3.8) through (3.10) with the terminal conditions given by Equations (3.11) through (3.13)

The explicit forms of the equations of motion, Equation (2.8), are given in Equations (2.5) and (2.7) while the explicit forms of the adjoint equations, Equations (3.9), are given in Appendix D.

The differential equations given in Equations (3.8) and (3.9) form a set of $2n$ simultaneous first order nonlinear differential equations with half of the boundary conditions specified at the initial point, (i.e., $\bar{x}_o$) and half at the final point (i.e., $\bar{\lambda}(t_f)$). A number of iterative numerical methods for the solution of such systems have been proposed. The method of steepest descent as proposed by A. E. Bryson and W. F. Denham[12, 13] was used in the investigation presented here to obtain the near optimum control program and trajectory along which the control schemes were compared. This numerical scheme is outlined in Appendix B.

The performance index used in the study presented here is the value of the mass at the end of the mission. Thus, Equation (3.2) becomes

$$V = x_7(t_f) \tag{3.15}$$

where $G(\bar{x}_f, t_f) = x_7(t_f)$ and $Q(\bar{u}, \bar{x}, t) = 0$.

The maximization of $x_7(t_f)$ is clearly the same as the minimization of fuel consumed since the only way that the vehicle loses mass is by expending fuel.

The equivalence of the performance index expressed in Equation (3.15) to that usually used in low-thrust optimization studies; i.e., minimization of

$$V_1 = \int_{t_o}^{t_f - t_o} a^2 dt \tag{3.16}$$

(where a is thrust acceleration magnitude, T/m), can be demonstrated for a constant propellent mass-ejection rate, β, and a constant relative exhaust speed, c.

If a is replaced in Equation (3.16) by $\beta c/m$, and if β is constant, then $m = m_o - \beta t$. Thus,

$$V_1 = \int_{t_o}^{t_f - t_o} \left(\frac{\beta c}{m_o - \beta t}\right)^2 dt \tag{3.17}$$

If β and c are constant, Equation (3. 17) can be integrated to obtain

$$V_1 = -\frac{\beta c}{m_o}\left[1 - \frac{m_o}{m_f}\right] \tag{3.18}$$

Since $m_f < m_o$, the maximum value of m_f in Equation (3. 18) will lead to a minimum of V_1 as was stated.

Thus, in the case that β and c are constant, the performance indices of Equations (3. 15) and (3. 16) are equivalent. In the cases treated here, β and c will be constants.

The near optimum Earth-Mars trajectory (hereafter called the reference trajectory) adopted for the subsequent study departs from Earth at noon on May 9, 1971 (Julian Date 2441 080. 5). The initial conditions used are given in Appendix D. Arrival near Mars occurs on November 9, 1971, after a trip of 184 days. The mass fraction expended as fuel is 0. 19872.

The projection of the reference trajectory on the ecliptic (x-y) plane, the x-z plane, and the y-z plane are shown in Figures 7, 8, and 9, respectively. The position components as functions of time are shown in Figure 10, and the corresponding velocity-time relationships are shown in Figure 11. The control program which produced the reference trajectory is shown in Figure 12.

From Figure 12 it can be seen that the control variable u_4 undergoes a change in angle of almost 180 degrees during the interval from 75 to 105 days after the start of the mission. This interval will be called reversal. In contrast, the control variable u_3 undergoes a change of only a few degrees during reversal.

Note that the magnitude of u_3 is always quite small. By referring

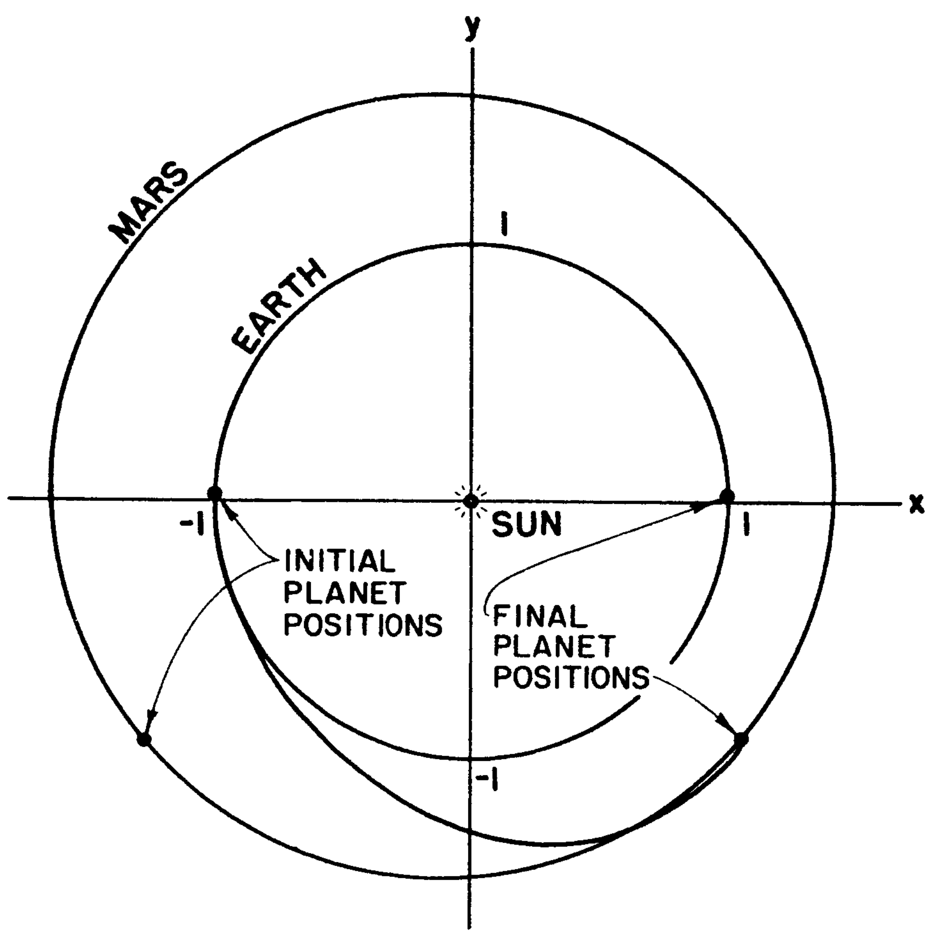

FIGURE 7. PROJECTION OF THE REFERENCE TRAJECTORY ON THE ECLIPTIC (x-y) PLANE

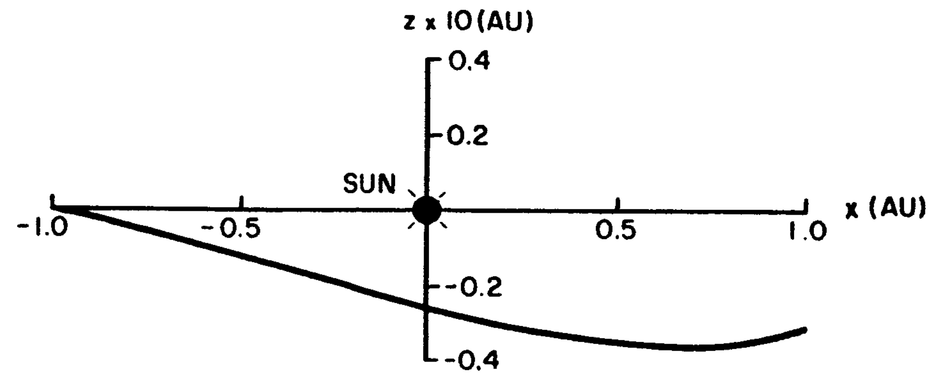

FIGURE 8. PROJECTION OF THE REFERENCE TRAJECTORY ON THE x-z PLANE

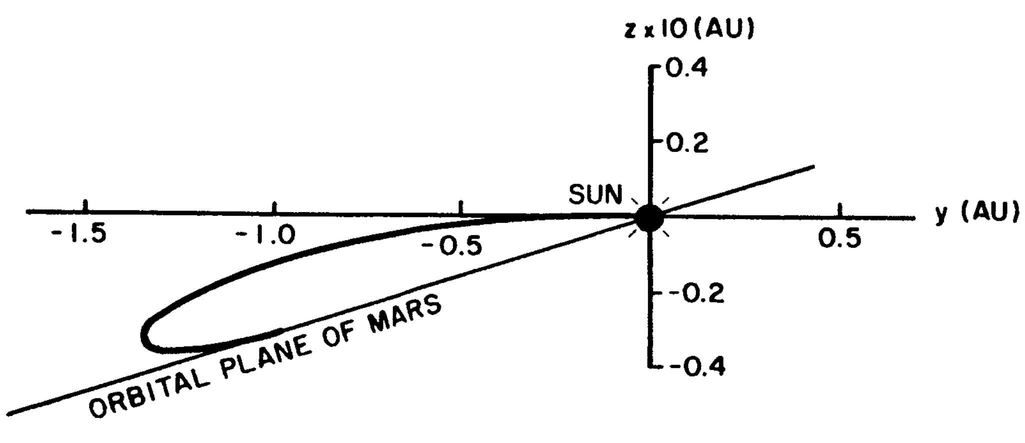

FIGURE 9. PROJECTION OF THE REFERENCE TRAJECTORY ON THE y-z PLANE

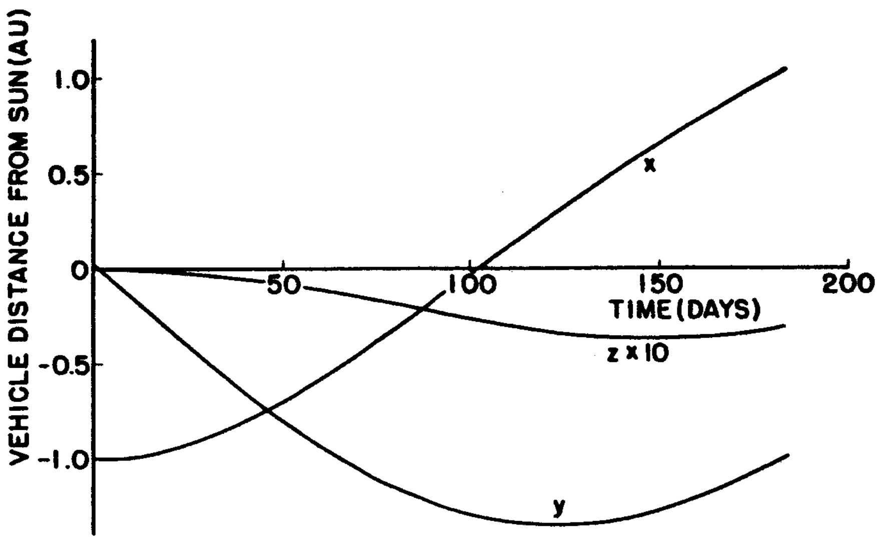

FIGURE 10. REFERENCE TRAJECTORY POSITION-TIME HISTORY

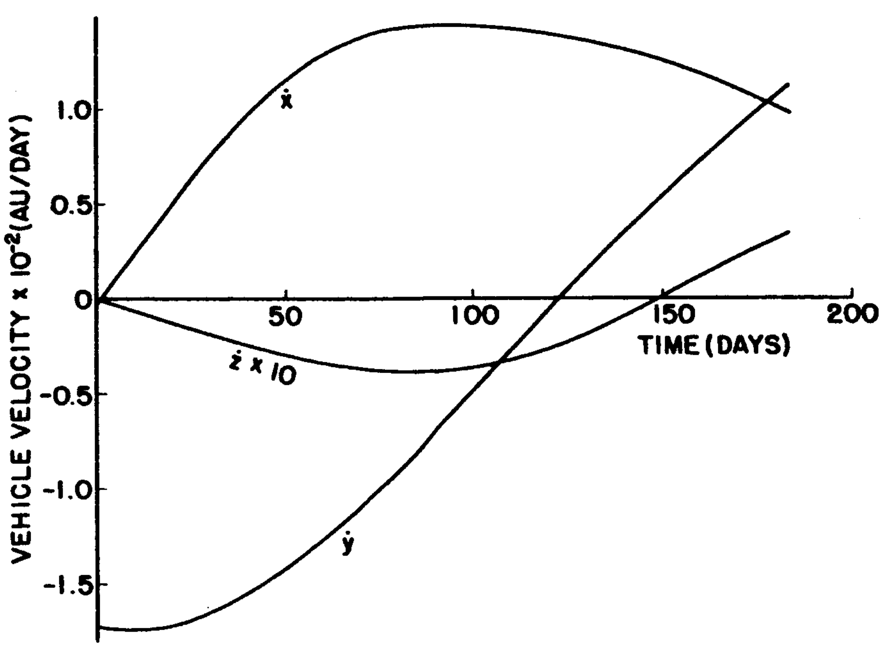

FIGURE 11. REFERENCE TRAJECTORY VELOCITY-TIME HISTORY

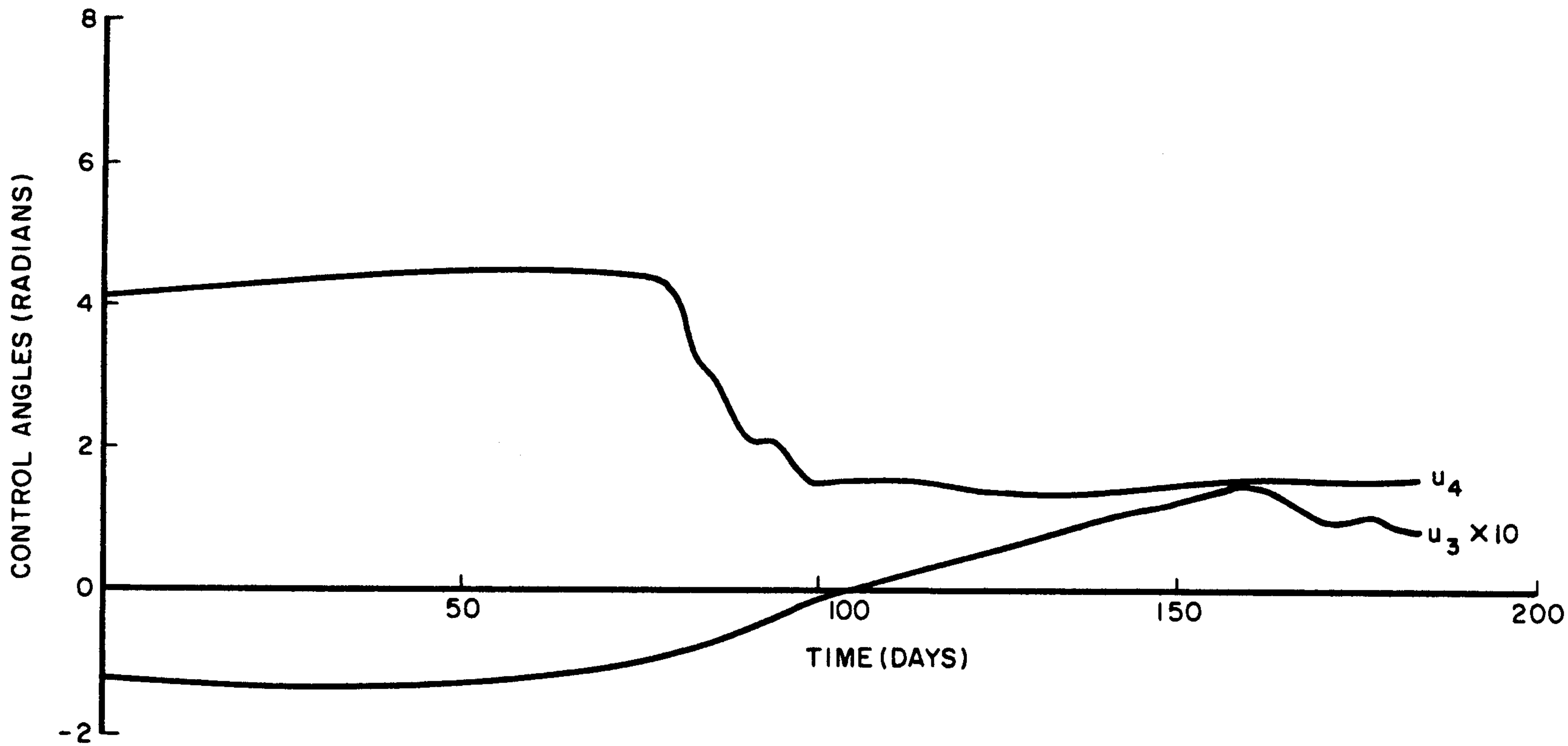

FIGURE 12. NOMINAL TRAJECTORY CONTROL PROGRAM

to Equations (2.5), (2.6), and (2.7) it can be seen that the main effects of u_3 (i.e., θ) on the system will show up in the out-of-plane state variables x_3 and x_6 while the main effects of u_4 (i.e., ψ) will be concentrated in the in-plane variables x_1, x_2, x_4, and x_5. Thus, errors in x_3 and/or x_6 should be controlled mainly by u_3 and control of errors in x_1, x_2, x_4, and/or x_5 should be carried out by u_4.

CHAPTER 4

FIRST ORDER CONTROL

This chapter introduces the first order control schemes which are compared in the subsequent discussion. A general state prediction scheme which can handle terminally constrained and terminally free state variables is introduced to describe the effects of the control schemes on the state of the system.

Consider a system of n first order differential equations which govern the system under consideration

$$\dot{\bar{x}} = \bar{f}(\bar{x}, \bar{u}, t) \tag{4.1}$$

where the symbols are the same as those defined in Equation (3.1). For this system there are given initial conditions $\bar{x}^*(t_o)$ and a reference control program $\bar{u}^*(t)$. Equations (4.1) can be integrated forward in time using the given initial conditions and the reference control program to produce a unique state variable history, $\bar{x}^*(t)$, which is called the reference trajectory. The reference trajectory satisfies a set of terminal conditions which can be expressed as follows:

$$\bar{M}(\bar{x}_f^*, t_f) = 0 \tag{4.2}$$

where t_f is the final time, and where $\bar{M}$ is a p vector of terminal constraint relations (rendezvous conditions).

The general deterministic control problem for such a system can be stated as follows. If a known state variable error, $\delta\bar{x}(t_1)$, is discovered at a known point on the reference trajectory (i.e., at a known time t_1), how must the reference control program be changed in order that the terminal conditions still be met and a performance index associated with the control maneuver be extremized? Also, how will the resultant trajectory differ from the reference trajectory?

4.1 Fundamental Guidance Equation

The consideration of the control of a dynamic system for which an unperturbed reference solution is known leads to the question of predicting the future state deviations after a disturbance has occurred. A general control deviation and state deviation prediction scheme should work when the number of terminal constraint relations, p, is less than the number of state variables, n, as well as when $p = n$. If $p < n$, it would be desirable to have the prediction scheme predict all n of the state deviation histories while the p terminal constraint relations, $\overline{M}$, (Equations (4.2)) determine the control variable deviation program.

The adjoint equations are

$$\dot{\overline{\lambda}} = -A^T\overline{\lambda} = -H_{\overline{x}} \tag{4.3}$$

where

$$A = \begin{bmatrix} \left(\frac{\partial f_1}{\partial x_1}\right)^* & \cdots & \left(\frac{\partial f_1}{\partial x_n}\right)^* \\ \\ \left(\frac{\partial f_n}{\partial x_1}\right)^* & \cdots & \left(\frac{\partial f_n}{\partial x_n}\right)^* \end{bmatrix} = (H_{\overline{\lambda x}})^*$$

$(\ )^*$ indicates that the quantities are evaluated along the reference trajectory and $\overline{\lambda}$ is an n-vector of adjoint variables.

The scalar differential equations, the adjoint equations, and terminal constraint relations for the model treated here are found in Appendix D.

Assuming small perturbations, $\delta\overline{x}$, Equation (4.1) can be expanded in a Taylor's series about the reference trajectory at each point in time to obtain, to first order in $\delta\overline{x}$ and $\delta\overline{u}$,

$$\delta\dot{\overline{x}} = A\,\delta\overline{x} + B\,\delta\overline{u} \tag{4.4}$$

where $\delta\overline{x}$ is an n-vector of state deviations, $\delta\overline{u}$ is an m-vector of control deviations, and

$$B = \begin{bmatrix} \left(\frac{\partial f_1}{\partial u_1}\right)^* & \cdots & \left(\frac{\partial f_1}{\partial u_m}\right)^* \\ \\ \left(\frac{\partial f_n}{\partial u_1}\right)^* & \cdots & \left(\frac{\partial f_n}{\partial u_m}\right)^* \end{bmatrix} = (H_{\overline{\lambda}\overline{u}})^*$$

By premultiplying Equation (4.4) by $\overline{\lambda}^T$ and adding the product to the transpose of Equation (4.4) postmultiplied by $\delta\overline{x}$, the following expression can be obtained.

$$\frac{d}{dt}(\overline{\lambda}^T\overline{x}) = \overline{\lambda}^T B\,\delta\overline{u} \tag{4.5}$$

Integrating Equation (4.5) between limits t_1 and t_f, the Fundamental Guidance Equation can be determined as follows.

$$(\bar{\lambda}^T \delta\bar{x})_{t_f} = (\bar{\lambda}^T \delta\bar{x})_{t_1} + \int_{t_1}^{t_f} \bar{\lambda}^T B \, \delta\bar{u} \, dt \tag{4.6}$$

Equation (4.6) provides a means of combining the state error at time t_1 with the subsequent control changes in order to predict the state at a later time, provided that the values of the $\bar{\lambda}$'s are known. It will be seen that a vector of $\bar{\lambda}$'s is necessary for each state variable, whether terminally constrained or terminally free.

Let the $\bar{x}$ vector be partitioned into a p-vector of terminally constrained state variables, $\bar{y}$, and an (n-p) vector of terminally free state variables, $\bar{z}$, i.e.,

$$\bar{x} = \begin{bmatrix} \bar{y} \\ \cdots \\ \bar{z} \end{bmatrix} \tag{4.7}$$

At times it will be more convenient to use $\bar{x}$ and at other times $\bar{y}$ and $\bar{z}$ must be used. Several other cases of alternate notation will be presently introduced for the same reason.

Equations (4.2) consitute p conditions on the terminal values of the n state variables. In the simplest case, the final time will be fixed and these p equations will contain only p of the state variables. Thus the terminal values of these p state variables can be determined directly. In this case, the Equations (4.2) can be replaced by

$$\bar{y}(t_f) - \bar{y}_f^* = 0 \tag{4.8}$$

where $\bar{y}_f^*$ is the p vector of values given by solution of Equations (4.2).

This replacement, where possible, will make the computation of initial conditions for the adjoint equations a trivial matter as will be shown later.

If Equations (4.2) involve more than p of the state variables, then the involved state variables themselves may not be directly constrained, but there are algebraic relations which must hold between the terminal values of the state variables involved. In many such cases, Equations (4.2) can be manipulated to eliminate all but p of the state variables.

In any case, if more than p state variables are involved in Equations (4.2), then for prediction purposes p of the n state variables, $\bar{x}$, must be chosen as the terminally constrained set of state variables, $\bar{y}$. The remaining (n-p) state variables, $\bar{z}$, are terminally free. The p terminally constrained state variables, $\bar{y}$, will be used to determine the control deviation program.

In order to obtain the control deviation program and to predict the state deviation histories of the p terminally constrained state variables, $\bar{y}$, we start with the Fundamental Guidance Equation in the form of Equation (4.6). For each of the terminally constrained state variables, the conditions

$$\bar{\lambda}_i^T (t_f) = \left[\frac{\partial M_i}{\partial \bar{x}} \right] \Bigg|_{t = t_f} \tag{4.9}$$

must hold at the final time, t_f.

For the (n-p) terminally free state variables,

$$\lambda_{ij} = \delta_{ij} \qquad \begin{matrix} (i = p + 1, n) \\ (j = 1, n) \end{matrix} \tag{4.10}$$

where δ_{ij} is the Kronecker delta function ($\delta_{ij} = 1$ when $i = j$; $\delta_{ij} = 0$ when $i \neq j$).

If an $(n \times n)$ matrix Λ is formed from n solutions $[\bar{\lambda}_1, \bar{\lambda}_2, \bar{\lambda}_3, \ldots \bar{\lambda}_n]$ to Equation (4. 3) with conditions given at t_f by Equations (4. 9) and (4. 10), then an expanded form of Equation (4. 6) can be written as

$$(\Lambda\delta\bar{x})_{t_f} = (\Lambda\delta\bar{x})_{t_1} + \int_{t_1}^{t_f} \Lambda\,\delta\bar{u}\,dt \qquad (4.11)$$

where

$\Lambda^T = [\bar{\lambda}_1, \bar{\lambda}_2, \ldots\ldots\ldots, \bar{\lambda}_n]$ and the $\bar{\lambda}_i$'s are column n-vectors.

Thus, for example, if $n = 7$, and $p = 4$, the matrix Λ^T at t_f (made up of the n $(n \times 1)$ vectors, $\bar{\lambda}$) has the form

$$\Lambda^T(t_f) = \begin{bmatrix} \frac{\partial M_1}{\partial x_1} & \frac{\partial M_2}{\partial x_1} & \frac{\partial M_3}{\partial x_1} & \frac{\partial M_4}{\partial x_1} & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & 1 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & 0 & 1 & 0 \\ \frac{\partial M_1}{\partial x_7} & \frac{\partial M_2}{\partial x_7} & \frac{\partial M_3}{\partial x_7} & \frac{\partial M_4}{\partial x_7} & 0 & 0 & 1 \end{bmatrix} \qquad - (4.12)$$

Note that if C^T is defined as the left-most $(n \times p)$ matrix in Λ^T, i.e., those elements involving $\overline{M}$, then the values of $\Delta\overline{M}$, are given to first order by

$$\Delta\overline{M} = (C\,\delta\overline{x})_{t_f} = (C\,\delta\overline{x})_{t_1} + \int_{t_1}^{t_f} C\,B\,\delta\overline{u}\,dt \tag{4.13}$$

If, as was mentioned above, Equation (4.2) involves only p of the state variables, and the final time is fixed, then Equation (4.9) is equivalent to Equation (4.2). Using Equation (4.9) instead of Equation (4.2), the matrix Λ^T at the final time, t_f, is given by

$$\Lambda^T(t_f) = I, \text{ the } (n \times n) \text{ identity matrix.} \tag{4.14}$$

The rendezvous conditions set up for the model used in the present study fall into this category. These conditions are given in Appendix D. If each of the $n \times 1$ vectors of λ's in $\Lambda^T(t_f)$ is integrated backward in time from t_f to t_o, a C matrix and a Λ matrix can be obtained at each point in time. Then the control deviation program and the resulting state deviations can be found from Equation (4.13) and the appropriate control deviation scheme.

4.2 Lambda Matrix Control Scheme

The control deviation program for the Lambda Matrix Control scheme is obtained by minimizing a quadratic function of the control variable deviations subject to the constraints that Equations (4.2) must hold at the final time, t_f. Thus, the quantity V is minimized where

$$V = \int_{t_1}^{t_f} \delta\bar{u}^T W \delta\bar{u}\, dt \tag{4.15}$$

and where the m x m matrix W (t) is a known arbitrary symmetric positive definite weighting matrix, while requiring that the following p equations hold.

$$\Delta\bar{M} = \Delta\bar{M}(\bar{x},(t_f), t_f) = 0 \tag{4.16}$$

If Equations (4.13) are adjoined to Equations (4.16) with a set of constant Lagrange multipliers, $\bar{\nu}$, then

$$V = \int_{t_1}^{t_f} [\delta\bar{u}^T W \delta\bar{u} - \bar{\nu}^T CB\, \delta\bar{u}]\, dt + \bar{\nu}^T [\Delta\bar{M} - C(t_1)\, \delta\bar{x}(t_1)] \tag{4.17}$$

Consideration of a variation in $\delta\bar{u}$ and requiring that V be an extremum with respect to an arbitrary $\delta(\delta\bar{u})$, i. e. $\delta'V = 0$, leads to the condition

$$\delta'V = 0 = \int_{t_1}^{t_f} [2\delta\bar{u}^T W - \bar{\nu}^T CB]\, \delta(\delta\bar{u})\, dt \tag{4.18}$$

Then, for an arbitrary variation, $\delta(\delta\bar{u})$, Equation (4.18) requires that

$$\delta\bar{u}(t) = \frac{1}{2} W^{-1}(t)\, B^T(t)\, C^T(t)\, \bar{\nu} \tag{4.19}$$

The unknown multipliers $\bar{\nu}$ can be evaluated by substituting Equation

(4.19) into Equation (4.13) and solving for $\bar{\nu}$.

$$\bar{\nu} = 2J^{-1}(t_1)\,[\Delta\overline{M} - C(t_1)\,\delta\bar{x}(t_1)] \tag{4.20}$$

where

$$J(t_1) = \int_{t_1}^{t_f} CBW^{-1}B^T C^T\, dt \tag{4.21}$$

Finally, after substituting Equation (4.20) into Equation (4.19), the $\Delta\overline{M}$ are given by $(C\delta\bar{x})_{t_f}$ and from Equation (4.11),

$$\delta\bar{u}(t) = W^{-1}(t)\,B^T(t)\,C^T(t)\,J^{-1}(t_1)\,[\Delta\overline{M} - C(t_1)\,\delta\bar{x}(t_1)] \tag{4.22}$$

This form is identical to that given by Bryson and Denham in Reference 9 except for notation.

4.3 The E-Function Control Scheme

A new control scheme is developed by requiring that along the perturbed trajectory, the deviation in the value of the original performance index associated with the reference trajectory be a minimum. That is, the control deviation program seeks to optimize the perturbed trajectory by minimizing the change in the performance index associated with the reference trajectory. Since the change in the original performance index caused by a deviation in the reference control program can be approximated by an integral form of the Weierstrass E-Function, the term E-Function Control (EC) is applied to the resulting control procedure. If the reference trajectory was determined by requiring that some mission oriented performance index be a minimum, then the change in this performance index associated with following some trajectory other than the reference trajectory will be an increase in its value. This increase can be approximated by

$$(V - V^*) = \int_{t_1}^{t_f} E(\bar{x}^*, \dot{\bar{x}}^*, \dot{\bar{x}}, t)\, dt \qquad (4.23)$$

where E is the Weierstrass E-function (See Page 148, Reference 16).

The E-function is defined as

$$E = L(\bar{x}^*, \dot{\bar{x}}, t) - L(\bar{x}^*, \dot{\bar{x}}^*, t) - \left(\frac{\partial L}{\partial \dot{\bar{x}}}\right)^* (\dot{\bar{x}} - \dot{\bar{x}}^*) \qquad (4.24)$$

For the system under consideration,

$$L(\bar{x}, \dot{\bar{x}}, t) = H(\bar{x}, \bar{u}, t) - \bar{\lambda}^T \dot{\bar{x}} \qquad (4.25)$$

where $H = \bar{\lambda}^T \bar{f}$.

As before, the starred quantities represent evaluation on the reference trajectory and the nonstarred quantities refer to evaluation on any other trajectory. A necessary condition for a minimum value of the original performance index is that E be nonnegative over the interval $t_o \leq t \leq t_f$. This insures that the integral of Equation (4.23) will be nonnegative.

Now consider the effect on the E-function of changing the control at a point along the reference trajectory. Note that from Equation (2.5) a variation in the control will immediately affect the vector $\dot{\bar{x}}$, (actually only the first three elements of this vector) and will affect the vector $\bar{x}$ only after a finite time has elapsed. Therefore, it will be assumed in the expansion of Equation (4.24) that a variation in the control will be accompanied only by a variation in the vector $\dot{\bar{x}}$, and not by a variation in the vector $\bar{x}$.

In view of Equation (4.25), Equation (4.24) reduces to

$$E = H(\bar{x}^*, \bar{u}, t) - H(\bar{x}^*, \bar{u}^*, t) \tag{4.26}$$

Expanding the first term on the right hand side of Equation (4.26) in a Taylor's series about the reference trajectory at each point in time, the following expression can be obtained.

$$H(\bar{x}, \bar{u}, t) = H(\bar{x}^*, \bar{u}^*, t) + H_{\bar{u}}^* \, \delta\bar{u} + \frac{1}{2}\, \delta\bar{u}^T H_{\bar{u}\bar{u}}^* \, \delta\bar{u} + \ldots \tag{4.27}$$

Noting that along an optimal reference trajectory, $H_{\bar{u}}^* = 0$, Equation (4.26) reduces to

$$E \cong \frac{1}{2}\, \delta\bar{u}^T H_{\bar{u}\bar{u}}^* \, \delta\bar{u} \geq 0 \tag{4.28}$$

Substituting this approximate form for the E-function into Equation (4.23), the function which is to be minimized along the perturbed path becomes

$$2(V - V^*) \cong \int_{t_1}^{t_f} \delta\bar{u}^T H_{\bar{u}\bar{u}}^* \, \delta\bar{u} \, dt \tag{4.29}$$

The factor 2 which multiplies $(V - V^*)$ in Equations (4.29) can be dropped since minimization of $2(V - V^*)$ is equivalent to minimizing $(V - V^*)$.

Note that along the perturbed path a weighted quadratic function of the control deviations is to be minimized in exactly the same manner as in the Lambda Matrix Control scheme. The only difference being that here the weighting matrix is not arbitrary but is uniquely specified as $H_{\overline{uu}}^*$. Thus, if the weighting matrix in the LMC scheme is chosen as $H_{\overline{uu}}^*$, then the E-function Control scheme and the LMC scheme are identical, and the control for the EC scheme is given in Equation (4.22). In order to derive the control for the EC scheme, one need only replace the right hand side of Equation (4.15) by the right hand side of Equation (4.29) and carry out the operations as given in Equations (4.16) through (4.22).

4.4 State Deviation Prediction

In order to predict the state deviations which result from a perturbation of the reference trajectory and the subsequent effect of a control deviation program, a general time t can be substituted for t_f in Equation (4.11). Solving for $\delta\bar{x}(t)$ and manipulating the limits on the integral terms, the following expression can be obtained.

$$\delta\bar{x}(t) = \Lambda^{-1}(t)\left[\Lambda(t_1)\,\delta\bar{x}(t_1) + \int_{t_1}^{t_f} \Lambda B\,\delta\bar{u}\,dt - \int_{t}^{t_f} \Lambda B\,\delta\bar{u}\,dt\right] \tag{4.30}$$

Equation (4.32) is the basic prediction equation. In order to use it with a specific control scheme, the control deviations produced by the specific control scheme must be substituted into the integrals in Equation (4.30).

Now substitute Equation (4.22) into Equation (4.30) to obtain for a general time, t,

$$\delta\bar{x}(t) = \Lambda^{-1}(t)\Big[\Lambda(t_1)\,\delta\bar{x}(t_1) + \Big[\int_{t_1}^{t_f} \Lambda B W^{-1} B^T C^T \, dt\Big] \cdot J^{-1}(t_1)\,[\Delta\overline{M} - C(t_1)\,\delta\bar{x}(t_1)] - \Big[\int_{t}^{t_f} \Lambda B W^{-1} B^T C^T dt\Big] \cdot J^{-1}(t_1)\,[\Delta\overline{M} - C(t_1)\,\delta\bar{x}(t_1)]\Big] \tag{4.31}$$

or letting

$$\psi(t) = \int_{t}^{t_f} \Lambda B W^{-1} B^T C^T \, dt \tag{4.32}$$

then Equation (4.31) becomes

$$\delta\bar{x}(t) = \Lambda^{-1}(t)\Big[\Lambda(t_1)\,\delta\bar{x}(t_1) + [\psi(t_1) - \psi(t)] \cdot J^{-1}\,[\Delta\overline{M} - C(t_1)\,\delta\bar{x}(t_1)]\Big] \tag{4.33}$$

Now by defining $\Delta\psi(t, t_1) = \psi(t) - \psi(t_1)$, Equation (4.33) can be reduced to the following form.

$$\delta\bar{x}(t) = \Lambda^{-1}(t)\,[\Lambda(t_1) + \Delta\psi(t, t_1)\,J^{-1}(t_1)\,C(t_1)]\,\delta\bar{x}(t_1)$$

$$- \Lambda^{-1}(t)\,\Delta\psi(t, t_1)\,J^{-1}(t_1)\,\Delta\overline{M} \qquad - (4.34)$$

And finally, in the special case where $\Delta\overline{M} = 0$, $\delta\bar{x}(t)$ is given by

$$\delta\bar{x}(t) = \Lambda^{-1}(t)\,[\Lambda(t_1) + \Delta\psi(t, t_1)\,J^{-1}(t_1)\,C(t_1)]\,\delta\bar{x}(t_1) \qquad (4.35)$$

Equation (4.34) or (4.35) is the basic state prediction scheme for the family of control schemes, while Equation (4.22) gives the required control variable programs.

Note that in case the final time and all of the terminal values of the state variables are constrained, (i.e., $n = p$), $\Lambda(t)$ and $C(t)$ are identical and

$$\psi(t) = J(t) = \int_t^{t_f} \Lambda B W^{-1} B^T \Lambda^T \, dt \qquad (4.36)$$

In addition, if $\Delta\overline{M} = 0$, Equation (4.34) becomes

$$\delta\bar{x}(t) = \Lambda^{-1}(t)\,J(t)\,J^{-1}(t_1)\,\Lambda(t_1)\,\delta\bar{x}(t_1) \qquad (4.37)$$

This is precisely the form obtained by Friedlander in Reference 2.

The state deviation prediction scheme described in Reference 7 as a part of LMC is a limited special case of the more general state deviation prediction scheme which is given by Equation (4.35). The state prediction scheme described in Reference 7 is as follows. Consider that an error has been discovered at t_1 and is being corrected. Equation (4.22) generates a unique control deviation for each time after t_1. The state deviation after t_1 will be a function of the original state error and the subsequent control deviation. At a time $t > t_1$, the state deviations are measured. If $\Delta\overline{M} = 0$ and no state errors have been introduced after t_1, then the control deviations generated by Equations (4.22) using the state error at t_1 should be the same as the control deviations generated using the state deviations measured at t. Thus, from Equation (4.22),

$$J^{-1}(t)\, C(t)\, \delta\overline{x}(t) = J^{-1}(t_1)\, C(t_1)\, \delta\overline{x}(t_1) \tag{4.38}$$

In order to predict $\delta\overline{x}(t)$, Reference 9 suggests that both sides of Equation (4.38) now be multiplied by $C^T(t)$ and the matrix $[C^T(t)\, J^{-1}(t)\, C(t)]$ be inverted. This inversion cannot be carried out unless $n = p$ because $[C^T(t)\, J^{-1}(t)\, C(t)]$ is an $n \times n$ matrix of rank p where $p \leq n$. Thus, this state prediction scheme cannot be used unless the number of terminal constraint equations, p, equals the number of state variables, n.

4.5 A Family of First Order Control Schemes

Let any time-dependent sequence of weighting matrices, W (introduced in Equation (4.15)), be called a weighting function. The fact that a change in the arbitrary weighting matrix associated with the LMC scheme changes the subsequent control and state deviations means that in reality a change in the weighting function $W(t)$ is a change of control law. Thus, the first order control schemes having control deviations of the form given in Equation (4.22) and state deviations of the form given in Equation (4.36) form a family of control schemes. The various members of this family can be identified by their weighting functions $W(t)$.

Since the LMC scheme is derived with an arbitrary choice of weighting function, it is not restricted to control about an optimal reference trajectory. The LMC scheme requires only that the reference trajectory, the reference control program, and the initial and terminal states be known.

In contrast, the development leading to the EC scheme with its unique specification of the weighting function in Equation (4.29) is based on the assumption that the reference trajectory is optimal in addition to being well known. The weighting function is found to be $H_{\underline{uu}}^{*}$, the matrix of second partial derivatives of the variational Hamiltonian evaluated along the optimal reference trajectory.

The use of a control scheme from this family of control schemes requires the selection of a weighting function, $W(t)$. The choice of $W(t)$ determines the control law. If the reference trajectory is optimal, then a good choice for $W(t)$ would be that given by the E-Function Control scheme, i.e., $W(t) = H_{\underline{uu}}(t)^{*}$. If however, the reference trajectory is only approximately optimal (the usual case when the reference trajectory is determined using a gradient method of optimization), or is not optimal at all, what choice should be made for the weighting function?

In the case of the approximately optimal reference trajectory, it seems likely that the weighting function specified by the E-Function Control scheme would again be desirable. If this choice is rejected, then the user is again faced with the problem of choosing a weighting function arbitrarily from an infinite number of possible weighting functions. In any case, it would seem that a weighting function which reflects the changing sensitivities of the vehicle system to he controls would be desirable. This again suggests $W(t) = H_{\overline{uu}}(t)^*$.

It should be noted here that the use of $W(t) = H_{\overline{uu}}^*$ implies that some optimization process has been used in the generation of the reference trajectory since the Lagrange multipliers, $\overline{\lambda}(t)$ (See Equation (3.4)), associated with the optimal trajectory are needed in order to compute $H_{\overline{uu}}^*$. In general, the reference trajectory will be at least a crude approximation to an optimal trajectory. Thus, $\overline{\lambda}(t)$ along the reference path will be known and $H_{\overline{uu}}^*$ can be used as the weighting function.

In the unusual case where control about a trajectory which is not even approximately optimal is desired, the $\overline{\lambda}(t)$'s obtainable are not the Lagrange multipliers associated with an optimal process and $H_{\overline{uu}}^*$ should not be used as the weighting function. In this case an arbitrary choice of $W(t)$ is necessary.

4.6 Approximately Optimal Trajectories

Let us qualitatvely consider the case of the approximately optimal reference trajectory. Since the reference trajectory is not truly optimal, there exist nearby trajectories along which $V < V^*$ (for V^* approximately minimal). Thus, it is possible that a trajectory on which a disturbance is being corrected could be more nearly optimal than the approximately optimal reference trajectory itself.

Assume that a control problem is stated in which the reference trajectory is known to be approximately optimal. Assume further that the weighting function $W_3 = H_{\overline{uu}}^*$ is known to be positive definite on some intervals of the reference trajectory and negative definite in others. (This cannot happen on a true optimal trajectory since Equation (4.29) must hold there). Thus, W_3 is singular at the points between intervals of positive and negative definiteness. Singularity of W_3 implies unbounded control in the neighborhood of a singular point (See Equation (4.22)).

Let us assume that in regions near a singularity W_3 is redefined as the m x m identity matrix. This means that the weighting function $W_3 = H_{\overline{uu}}^*$ has been modified and has become

$$W_3 = H_{\overline{uu}}^* \quad \text{for} \quad |\det(H_{\overline{uu}}^*)| > \varepsilon, \text{ and}$$

$$W_3 = I \quad \text{for} \quad |\det(H_{\overline{uu}}^*)| \leq \varepsilon \tag{4.39}$$

where $|\det(\ )|$ means the absolute value of the determinant of ( ),

I is the m x m identity matrix, and

ε is an arbitrary small positive number.

No stipulations are made as to the continuity of the elements of W_3. The redefined W_3 has no singular points but still has intervals in which it is negative definite.

The following development is entirely heuristic and cannot be backed by more than intuition. It is presented here only as an indication of a possible area of future investigation.

Assume that although the reference trajectory is only nearly optimal, the approximation for changes in the performance index given in Equation (4.29) can still be used. Further, let us assume that the redefined W_3 of Equation (4.39) is positive definite for $t_o \leq t \leq t_1$ and is negative definite for $t_1 < t < t_f$. The integral of Equation (4.29) can be broken into

$$\Delta P_1 = \int_{t_o}^{t_1} \delta\bar{u}^T W_3 \; \delta\bar{u} \, dt > 0$$

$$\Delta P_2 = \int_{t_1}^{t_f} \delta\bar{u}^T W_3 \; \delta\bar{u} \, dt < 0 \tag{4.40}$$

where $\Delta P_1 + \Delta P_2 = (V - V^*)$.

Three cases are possible, i.e.,

$$|\Delta P_1| > |\Delta P_2| ,$$

$$|\Delta P_1| = |\Delta P_2| , \text{ and}$$

$$|\Delta P_1| < |\Delta P_2| \tag{4.41}$$

In the first and second cases, the disturbed trajectory is no better than the reference trajectory, but under the assumption of this section, the possibility of obtaining the third case implies that it might be possible to obtain a disturbed trajectory for which $V < V^*$ and thus improve the trajectory using the EC scheme.

CHAPTER 5

NUMERICAL PERFORMANCE EVALUATION

Numerical results used in the evaluation of the comparative guidance capabilities of three members of the first order control family (Section 4.5) were obtained using a Control Data Corporation 1604 digital computer. The reference trajectory along which the control schemes were compared is the approximately optimal three-dimensional low-thrust Earth-Mars trajectory described in Chapter 3.

5.1 Numerical Implementation

The reference trajectory was obtained using the steepest descent optimization procedure described in Appendix B. Next, the first order control schemes were programmed in such a manner that any member of the first order control family could be used by selecting the weighting function $W(t)$.

The accuracy of the computations involved in obtaining the reference trajectory and applying the control schemes is highly dependent on the processes of numerical matrix inversion and numerical integration. The accuracy of the routines which carry out these numerical operations was checked separately using problems having known solutions. After the routines were inserted into the program and actual problem data was being used, the matrix inversion was again checked by multiplying the matrices to be inverted by the inverse. The matrix inversions were carried out in full double precision. Also, the dependent variables in the integrations were carried in double precision in order to control round off error.

The numerical integrations were carried out using a fifth order Runge-Kutta numerical integration scheme[17]. The integration interval size was chosen in the following manner for both the steepest descent process and for the control scheme simulation programs. First, an arbitrary interval size, h, was chosen and a trial integration was made. Then a second trial integration using an interval size of h/2 was made. The data for the two trials was then compared. This data consisted of tabulated values of the integrated functions. The following criterion was used to determine the suitability of a particular interval size, h.

Each entry in each column of data obtained using the interval size h was compared with the corresponding entry obtained using h/2. In each column (consisting of the values of a single variable) the entry having the largest absolute value was found and the decimal place defined by the fourth significant digit in this number was chosen as the critical decimal place for that column. A difference of more than one unit in this decimal place between any two corresponding numbers (comparing data for interval sizes of h and h/2) for any variable was sufficient to disqualify h as an acceptable interval size. If h was disqualified, then h/2 became the new interval size and was checked against h/4. This process was repeated until an acceptable interval size was found.

The interval chosen was h = 0.25 days. The range of integration was 184 days. The accuracy of the integrals obtained using this interval size compared very well with check data obtained using both fifth order and ninth order Adams numerical integration schemes[18].

The first process which was undertaken after obtaining the reference trajectory was the determination of the adjoint solutions. These were obtained as a part of the control scheme program.

5.2 Adjoint Solutions

Six of the seven sets of adjoint solutions obtained when Equations (4.3) are integrated backward in time from t_f to t_o with starting conditions given by Equations (4.14) are shown in Figures 13 through 18. Each quantity plotted, $\lambda_{ij}(t)$, is a time dependent sensitivity coefficient which indicates the sensitivity of the terminal value of the i^{th} state variable to an uncorrected error in the j^{th} state variable at time. t, i.e.,

$$\lambda_{ij} = \frac{\partial x_i(t_f)}{\partial x_j(t)} \tag{5.1}$$

A complete discussion of the adjoint solutions is outside the scope of this study. However, several facts about the adjoint solutions will be noted. For additional information concerning the properties of adjoint solutions, the reader is referred to References 2 and 19.

The maximum absolute value of $\lambda_{ij}(t)$ represents the maximum sensitivity of $x_i(t_f)$ to a unit error in x_j. The time t at which such a maximum for λ_{ij} occurs denotes that time during the mission at which a unit error in x_j would most affect $x_i(t_f)$. For example, the maximum absolute value of λ_{15} (See Figure 13) occurs approximately 47 days after the beginning of the mission. Thus, x_1. is most sensitive to a disturbance in x_5 around 47 days after the start of the mission.

The sensitivities of the first six state variables (velocity and position components) to an error in the seventh state variable show abrupt changes in slope beginning approximately 80 days after the start of the mission. It is at this time that the vehicle begins to turn around in order to start deceleration so that rendezvous with Mars can be achieved. Large changes in the sensitivities of the kinematic state variables (velocity and position

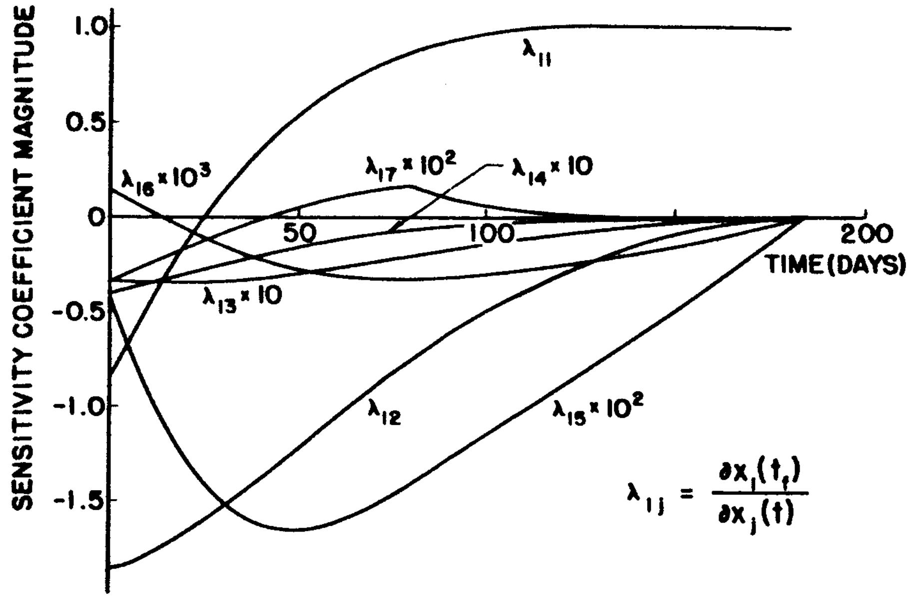

FIGURE 13. SENSITIVITY OF $x_1(t_f)$ TO STATE PERTURBATIONS

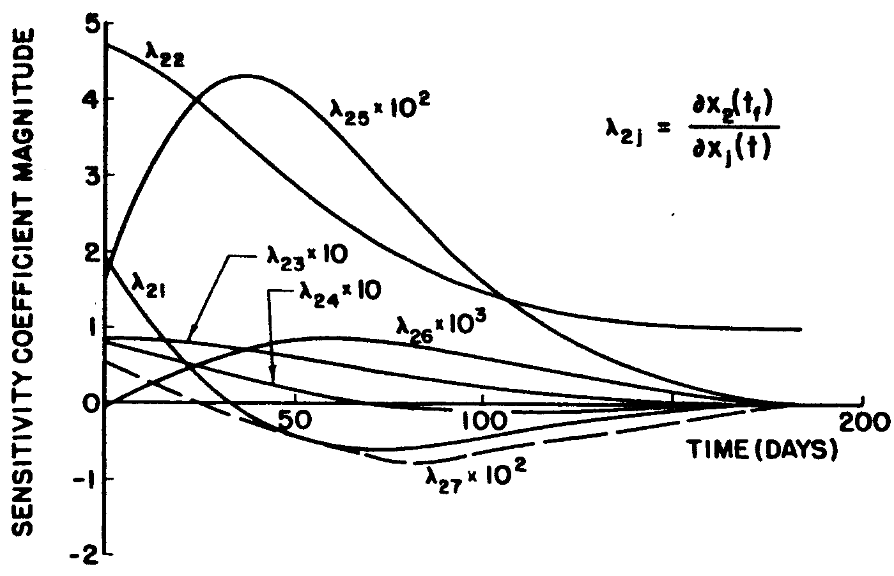

FIGURE 14. SENSITIVITY OF $x_2(t_f)$ TO STATE PERTURBATIONS

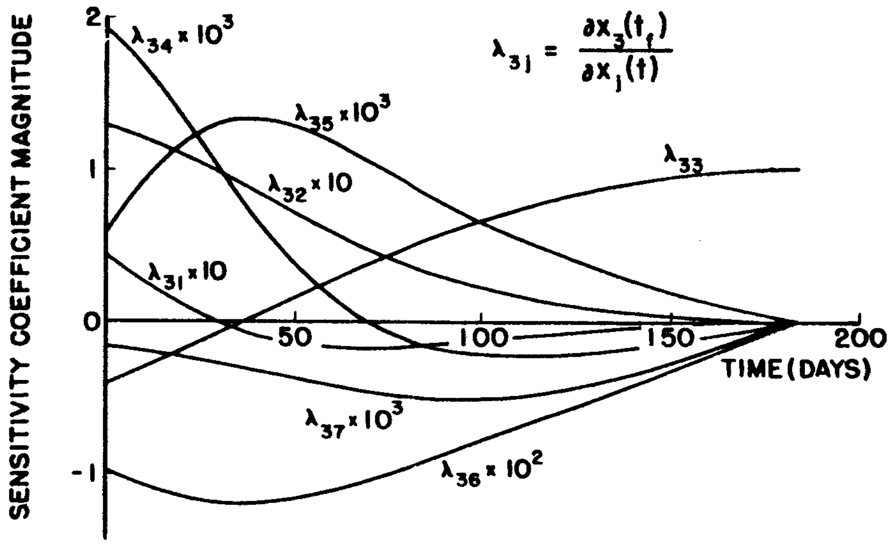

FIGURE 15. SENSITIVITY OF $x_3(t_f)$ TO STATE PERTURBATIONS

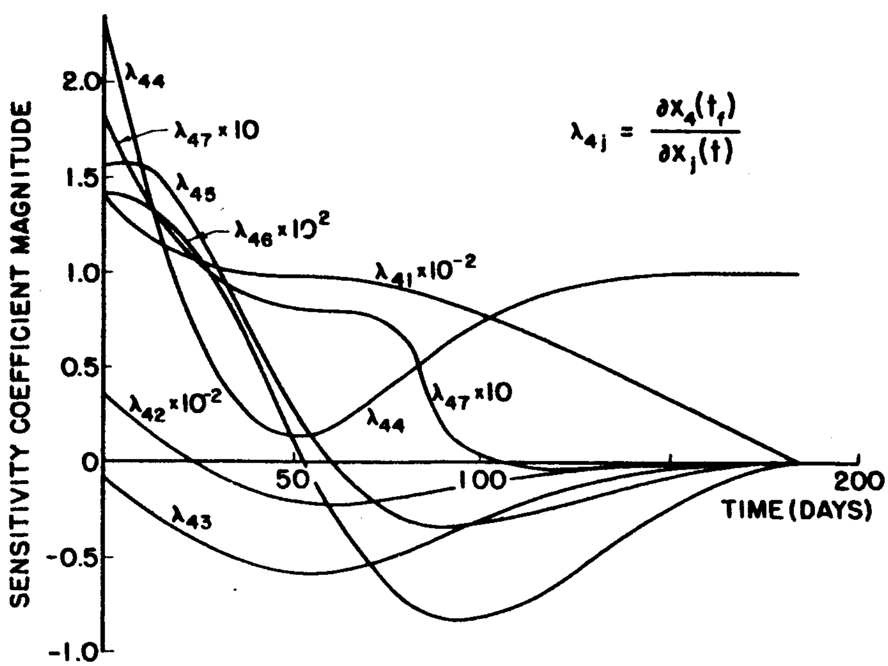

FIGURE 16. SENSITIVITY OF $x_4(t_f)$ TO STATE PERTURBATIONS

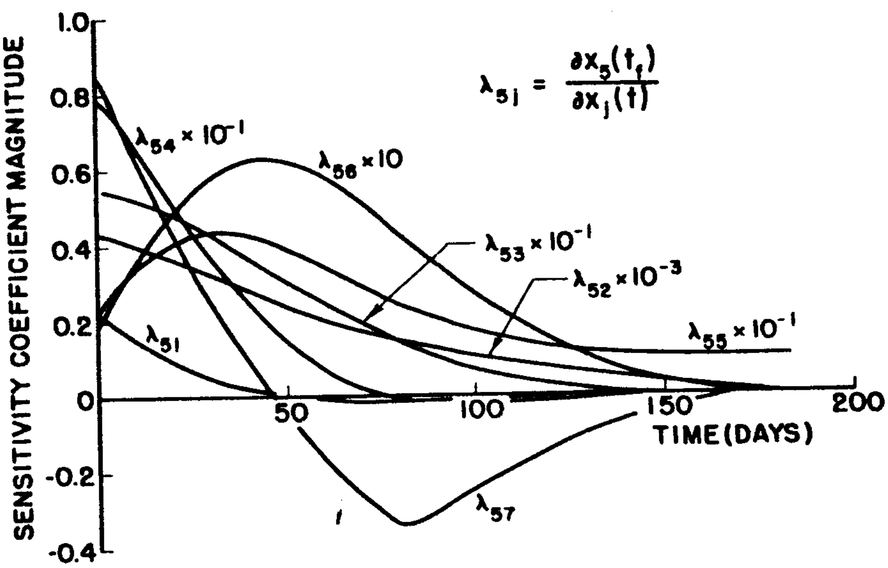

FIGURE 17. SENSITIVITY OF $x_5(t_f)$ TO STATE PERTURBATIONS

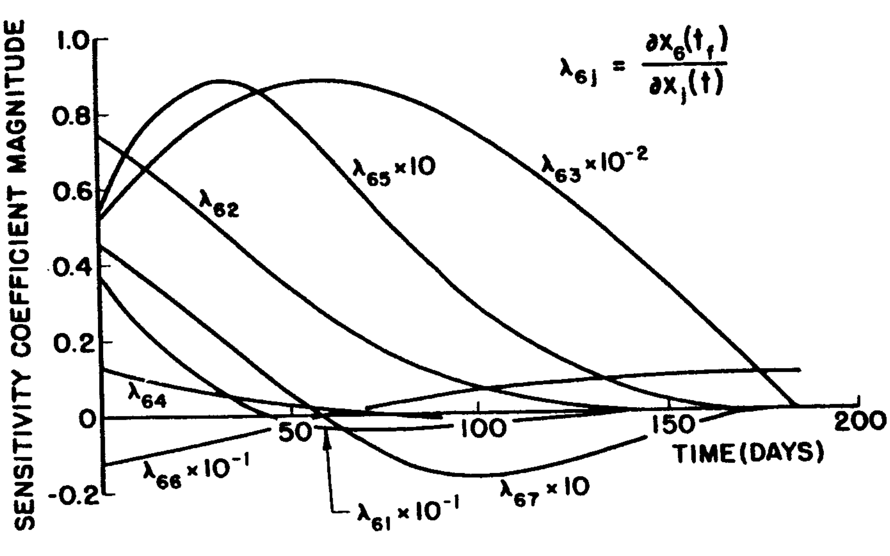

FIGURE 18. SENSITIVITY OF $x_6(t_f)$ TO STATE PERTURBATIONS

components) during the thrust reversal period are to be expected since changes in thrust angle are accompanied by corresponding changes in thrust acceleration direction. These thrust acceleration direction changes in turn affect the velocity and position variables.

The seventh set of adjoint solutions, i.e., those obtained by setting

$$\lambda_{7i}(t_f) = 0 \qquad (i = 1, \ldots, 6)$$

$$\lambda_{77}(t_f) = 1 \tag{5.2}$$

are not plotted because this set of solutions is constant. For a constant mass ejection rate, u_1, the mass cannot be disturbed by errors in position or velocity. In addition, for fixed final time, t_f, an error in the mass itself cannot be changed in a system characterized by a constant mass ejection rate.

5.3 Numerical Comparisons

As discussed in Chapter 4, three members of the first order control scheme family were chosen for comparison. Two members are characterized by arbitrarily chosen constant weighting functions while the third has its weighting function given uniquely as is described in the section on E-Function control (Section 4.3). The three weighting functions used are

$$W_1(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{5.3}$$

$$W_2(t) = \begin{bmatrix} 10 & 0 \\ & \\ 0 & 1 \end{bmatrix} \tag{5.4}$$

and

$$W_3(t) = H_{\overline{uu}}^{\;*}(t) = \begin{bmatrix} \left(\frac{\partial^2 H}{\partial u_3^{\,2}}\right)^* & \left(\frac{\partial^2 H}{\partial u_3 \partial u_4}\right)^* \\ & \\ \left(\frac{\partial^2 H}{\partial u_3 \partial u_4}\right)^* & \left(\frac{\partial^2 H}{\partial u_4^{\,2}}\right)^* \end{bmatrix} \tag{5.5}$$

Weighting function W_2 was chosen after the data for W_1 had been obtained. It was noted that when using W_1 to correct errors in x_1, x_2, x_4, and x_5 (the velocity and position components in the x-y plane), the resulting deviations in x_3 and x_6 (the velocity and position in the z-direction) were disproportionately large. For example, the maximum deviation in x_6 was larger than the maximum deviation in x_4 even when x_4 was the disturbed variable. It was apparent that with the equal weighting of W_1, the control variable u_3 was exerting too much influence in correcting errors in the x-y plane. The weighting function W_2 was chosen in order that the influence of u_3 be decreased.

For W_2, however, it was noted that the control u_4 had too great an effect when x_3 or x_6 was found to be in error. Thus, W_1 proved to be better than the W_2 weighting function when errors were in x_1, x_2, x_4, or x_5, but W_2 was better when the errors were in x_3 and x_6.

Note that since the reference trajectory is approximately optimal, the weighting function W_3 should give a more desirable control than W_1

or W_2. It should be noted at this point, however, that the weighting function $W_3 = H^*_{\overline{uu}}$ as defined in Section (4.3) is not positive definite along the entire reference trajectory. This results from the fact that the reference trajectory is only approximately optimal and was obtained using a gradient technique. It was found that the determinant of W_3 changed signs three times along the reference trajectory.

The matrix W_3 was used despite the lack of positive definiteness in order to see what type of control would be produced. Inversion of the matrix W_3 was not attempted near the points where the determinant changed signs.

In the following discussion the three control schemes are denoted by their respective weighting matrices W_1, W_2, and W_3. The three control schemes are compared by numerically simulating the performance of each scheme in correcting typical state disturbances. In the simulated control maneuvers shown here, it is assumed that the errors occurred at the start of the mission.

Data are presented for the case where the mass ejection rate, u_1, and the relative exhaust speed, u_2, are held constant. The only actual control quantities are the angles u_3 and u_4.

Each control scheme is used to correct seven disturbances, i.e., an error in the initial value of each of the seven state variables. The magnitudes of the errors are 10^{-6} AU day for the velocity variables, 10^{-6} AU for the position variables, and 10^{-6} times the initial vehicle mass for the seventh state variable. Only one variable was disturbed at a time. Hence, the values of the resulting curves at any time, t, can be interpreted as elements of a state transition matrix $\phi(t, t_1)$, i.e.,

$$\delta\bar{x}(t) = \phi(t, t_1)\, \delta\bar{x}(t_1) \tag{5.6}$$

State variable deviation histories and the control deviation programs which produced them are given in Figures 19 through 25. Each figure consists of eight parts which display three curves each. The transpose of the initial state deviation vector for each of Figures 19 through 25 is shown below.

For Figure 19,

$$\delta\bar{x}^T = [(10^{-6}\ \text{AU/Day}) \quad 0 \quad 0 \quad 0 \quad 0 \quad 0] \tag{5.7}$$

For Figure 20,

$$\delta\bar{x}^T = [\, 0 \quad (10^{-6}\ \text{AU/Day}) \quad 0 \quad 0 \quad 0 \quad 0] \tag{5.8}$$

For Figure 21,

$$\delta\bar{x}^T = [\, 0 \quad 0 \quad (10^{-6}\text{AU/Day}) \quad 0 \quad 0 \quad 0] \tag{5.9}$$

For Figure 22,

$$\delta\bar{x}^T = [\, 0 \quad 0 \quad 0 \quad (10^{-6}\ \text{AU}) \quad 0 \quad 0 \quad 0] \tag{5.10}$$

For Figure 23,

$$\delta \bar{x}^T = [\, 0 \quad 0 \quad 0 \quad 0 \quad (10^{-6}\ \mathrm{AU}) \quad 0 \quad 0 \,] \qquad (5.11)$$

For Figure 24,

$$\delta \bar{x}^T = [\, 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad (10^{-6}\ \mathrm{AU}) \quad 0 \,] \qquad (5.12)$$

And, for Figure 25,

$$\delta \bar{x}^T = [\, 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad (10^{-6}\ m_o) \,] \qquad (5.13)$$

where m_o is the value of the mass of the vehicle at the start of the mission.

The first six parts of Figures 19 through 25, parts (a) through (f), are the time-dependent responses of the state variables x_1 through x_6 to an initial error and the subsequent control deviation. The seventh and eighth part of each figure, parts (g) and (h), are the control deviation programs δu_3 and δu_4. Each of the eight parts of a figure contains three curves. These three curves show the responses given to the state disturbance by the three control schemes W_1, W_2, and W_3.

For example, Figure 21 shows the response of each of the control schemes to the state disturbance given in Equation (5.9). Parts (g) and (h) show the control deviation programs $\delta u_3(t)$ and $\delta u_4(t)$. Part (a) shows δx_1, the deviation in the variable x_1, resulting from the initial state error defined by Equation (5.9) and the subsequent control deviation programs shown in Figures 21 (g) and 21 (h). Part (f) exhibits the behavior of δx_6 for the same initial error and resulting control deviations.

Table I below summarizes the data contained in parts (a) through (h) of any of Figures 19 through 25

Table I

Summary of Data Displayed for a Typical Figure from Figure 19 through Figure 25

Part	Plots Contained
(a)	δx_1 (t) for W_1, W_2, and W_3
(b)	δx_2 (t) for W_1, W_2, and W_3
(c)	δx_3 (t) for W_1, W_2, and W_3
(d)	δx_4 (t) for W_1, W_2, and W_3
(e)	δx_5 (t) for W_1, W_2, and W_3
(f)	δx_6 (t) for W_1, W_2, and W_3
(g)	δu_3 (t) for W_1, W_2, and W_3
(h)	δu_4 (t) for W_1, W_2, and W_3

With u_1 held constant, an error in the seventh state variable, x_7, leads to a constant state deviation history for δx_7. Also, an error in any of the first six state variables cannot cause a state deviation in x_7. Note, however, that an error in x_7 will lead to deviations in all other state variables since then the reference control program will not lead to satis-

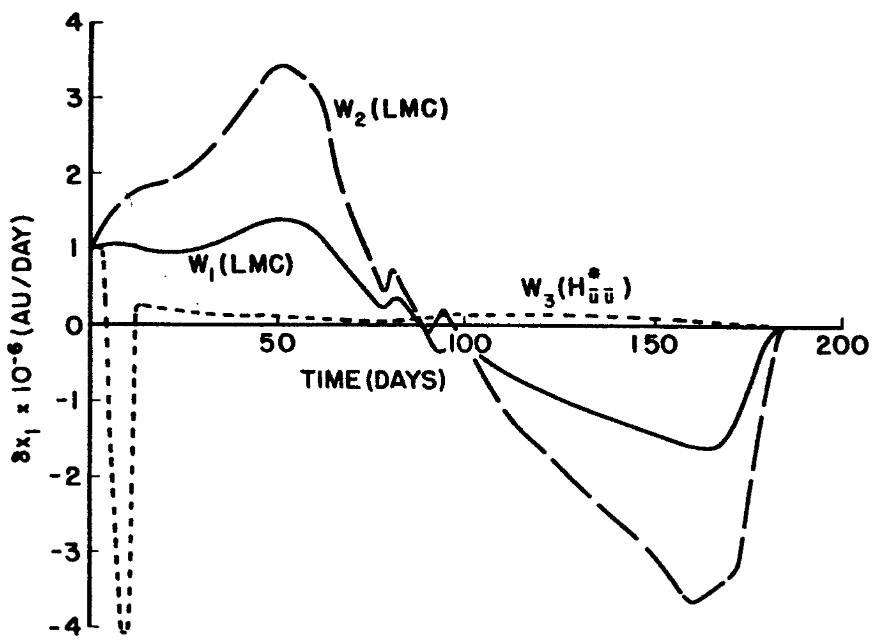

FIGURE 19(a). STATE DEVIATIONS FOR $\delta x_1(0) = 10^{-6}$ AU/DAY

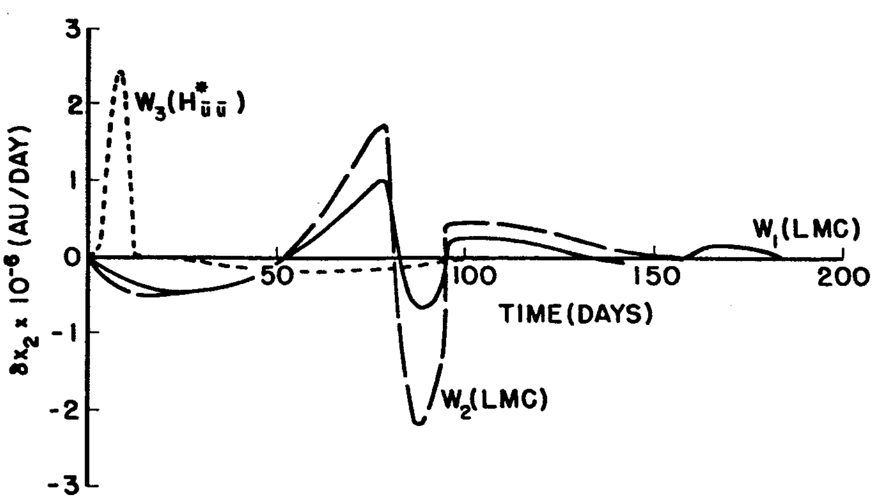

FIGURE 19(b). STATE DEVIATIONS FOR $\delta x_1(0) = 10^{-6}$ AU/DAY

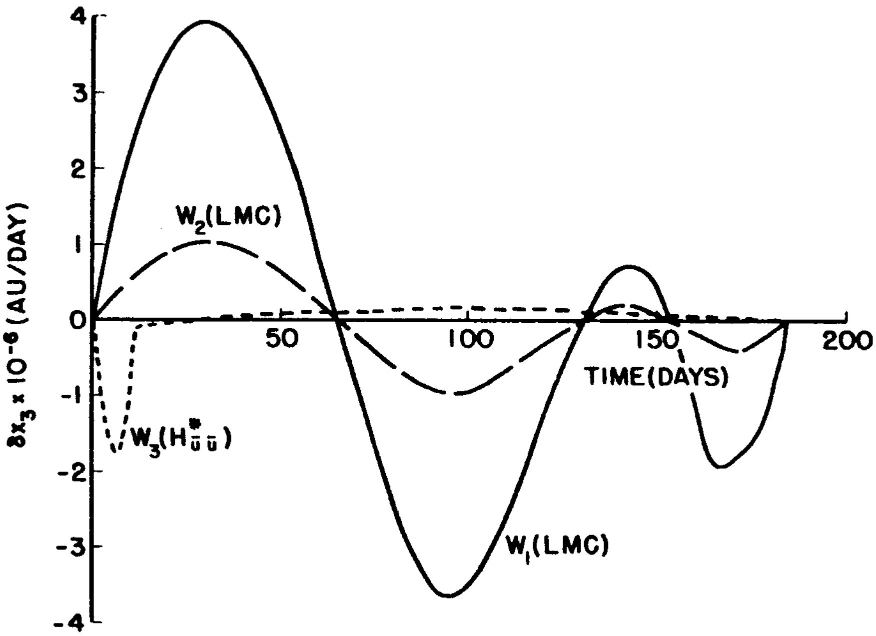

FIGURE 19(c). STATE DEVIATIONS FOR $\delta x_1(0) = 10^{-6}$ AU DAY

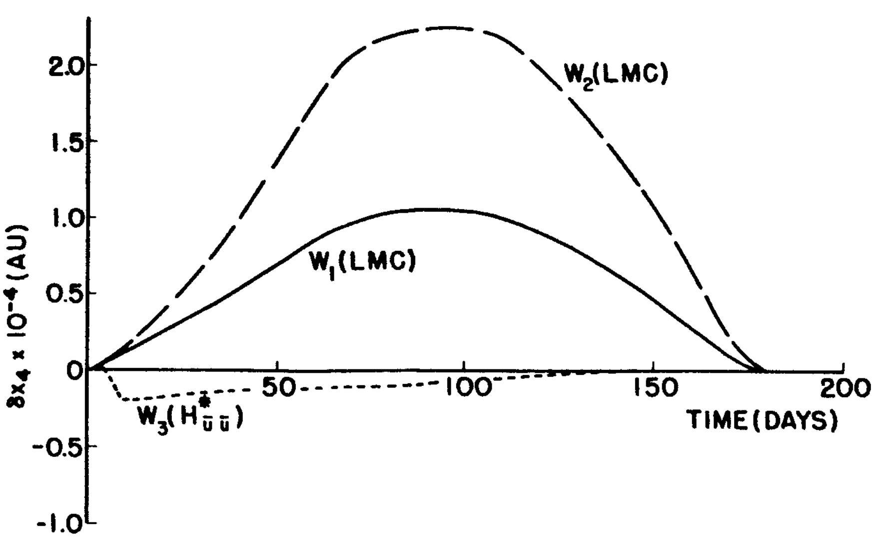

FIGURE 19(d). STATE DEVIATIONS FOR $\delta x_1(0) = 10^{-6}$ AU/DAY

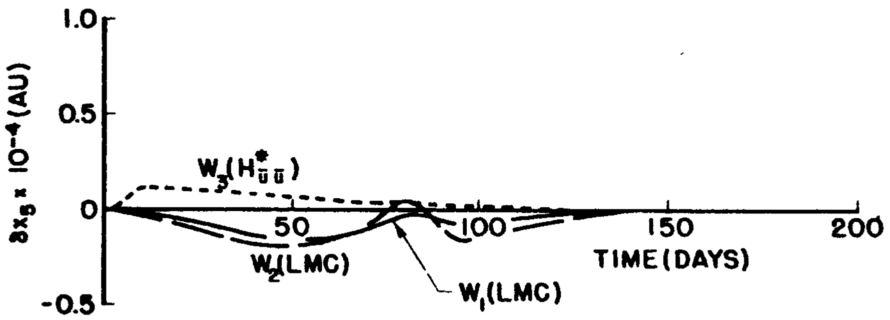

FIGURE 19(e). STATE DEVIATIONS FOR $\delta x_1(0) = 10^{-6}$ AU/DAY

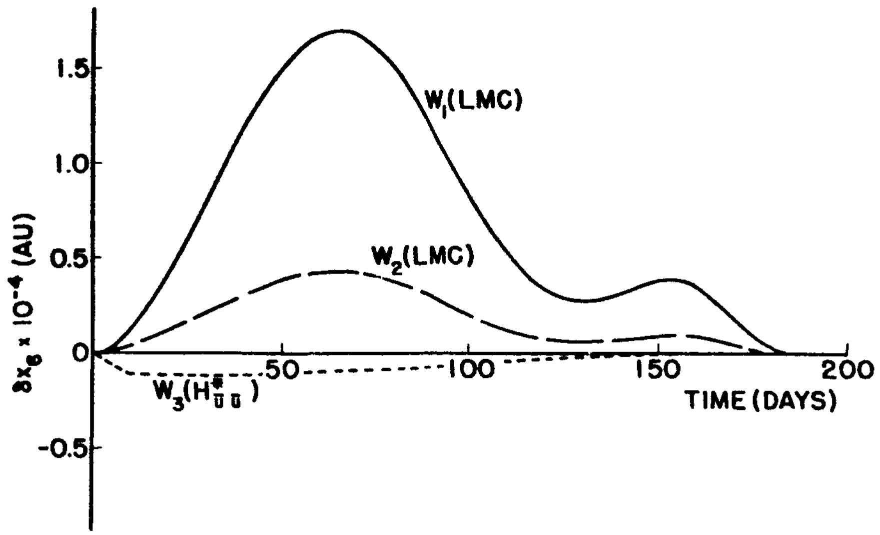

FIGURE 19(f). STATE DEVIATIONS FOR $\delta x_1(0) = 10^{-6}$ AU/DAY

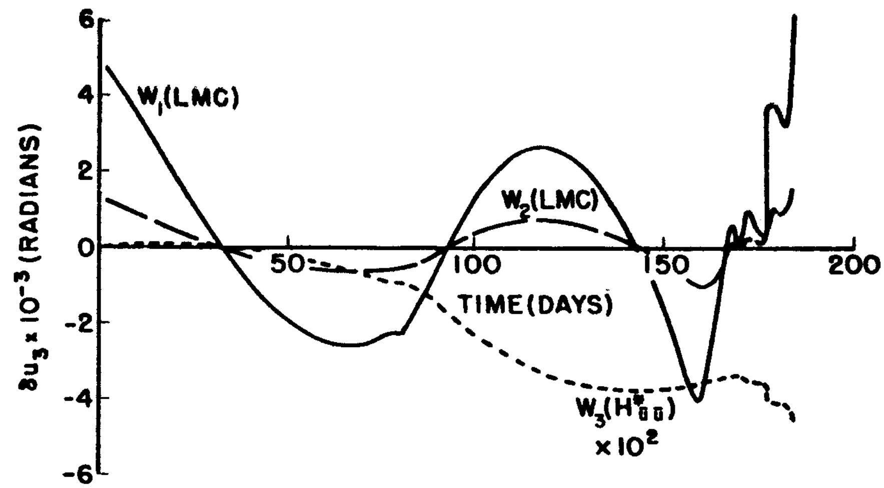

FIGURE 19(g). CONTROL DEVIATIONS FOR $\delta x_1(0) = 10^{-6}$AU/DAY

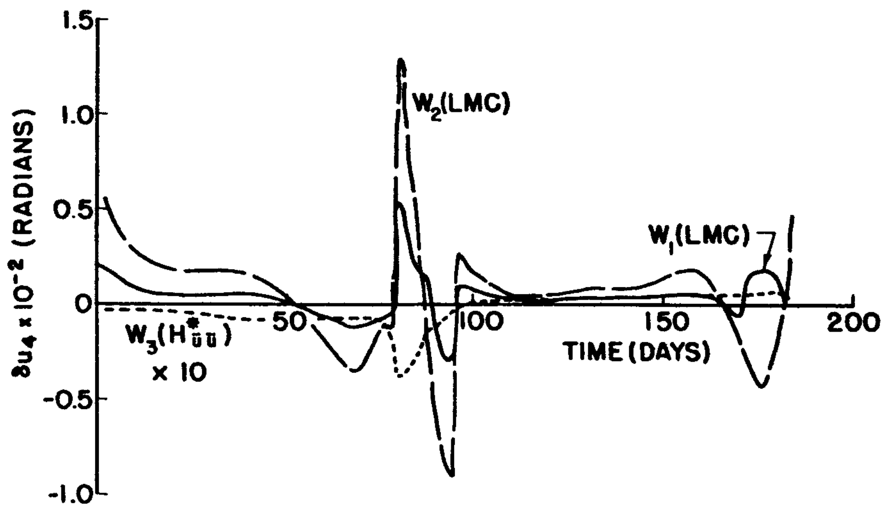

FIGURE 19(h). CONTROL DEVIATIONS FOR $\delta x_1(0) = 10^{-6}$AU/DAY

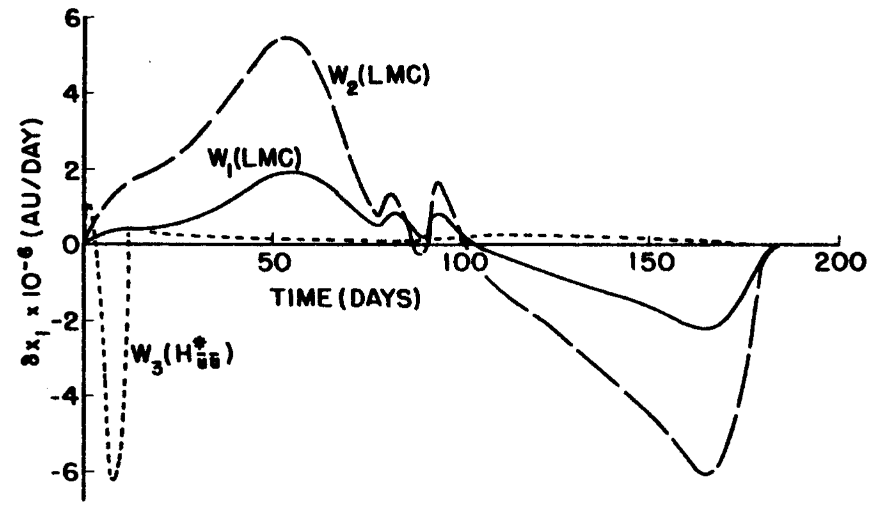

FIGURE 20(a). STATE DEVIATIONS FOR $\delta x_2(0) = 10^{-6}$ AU/DAY

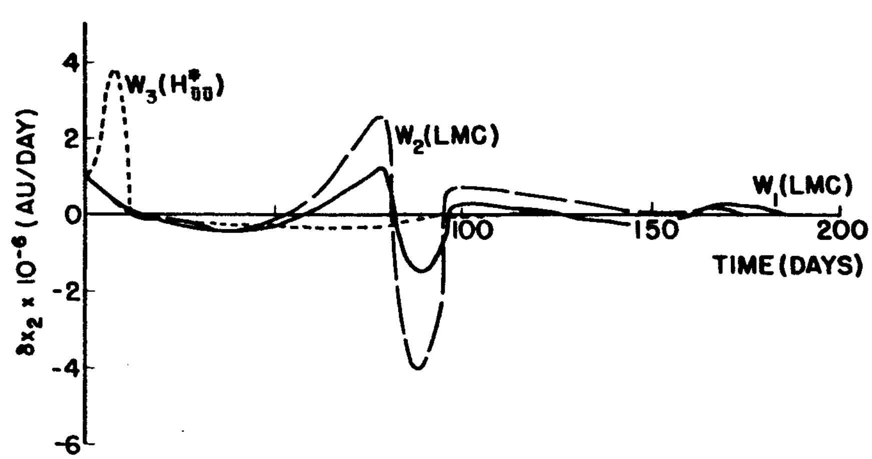

FIGURE 20(b). STATE DEVIATIONS FOR $\delta x_2(0) = 10^{-6}$ AU/DAY

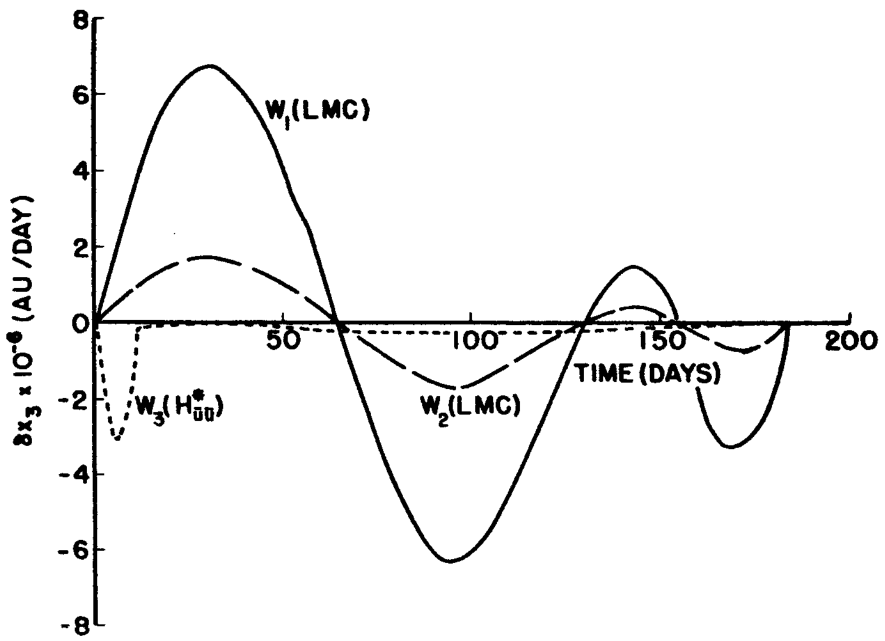

FIGURE 20(c). STATE DEVIATIONS FOR $\delta x_2(0) = 10^{-6}$ AU/DAY

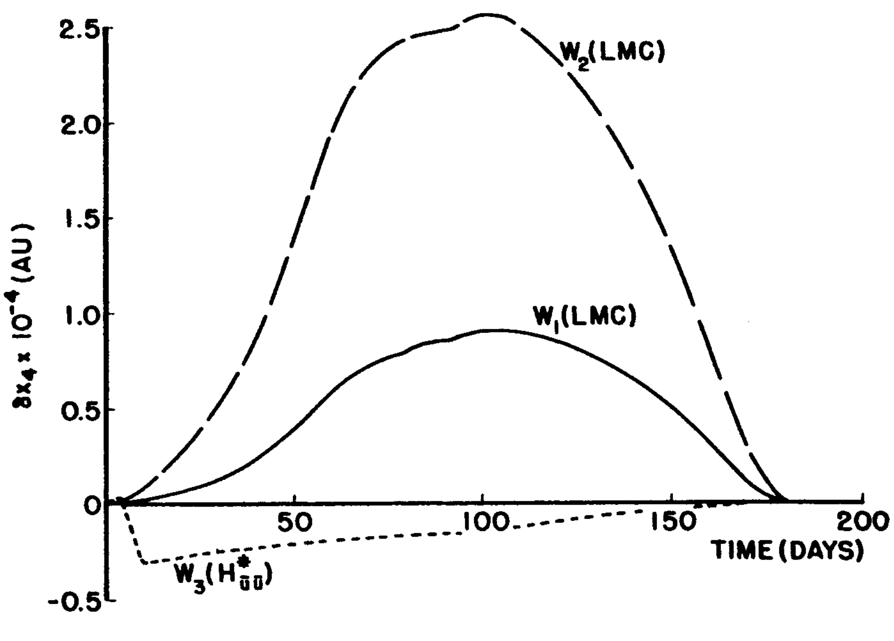

FIGURE 20(d). STATE DEVIATIONS FOR $\delta x_2(0) = 10^{-6}$ AU/DAY

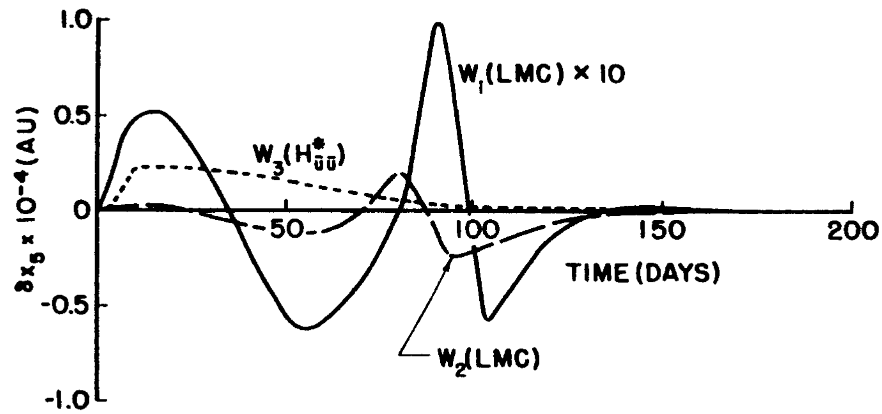

FIGURE 20(e). STATE DEVIATIONS FOR $\delta x_2(0) = 10^{-6}$AU/DAY

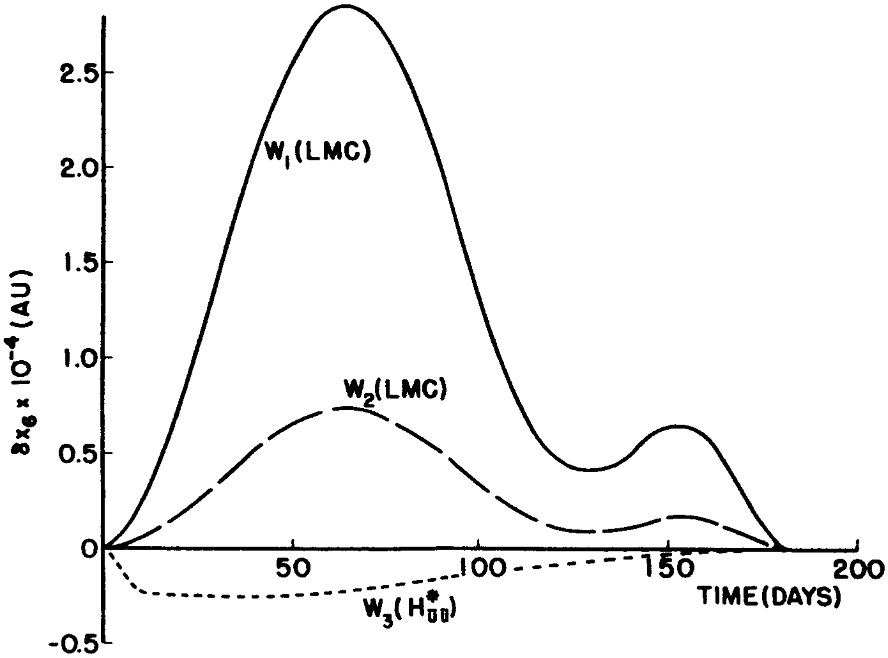

FIGURE 20(f). STATE DEVIATIONS FOR $\delta x_2(0) = 10^{-6}$ AU/DAY

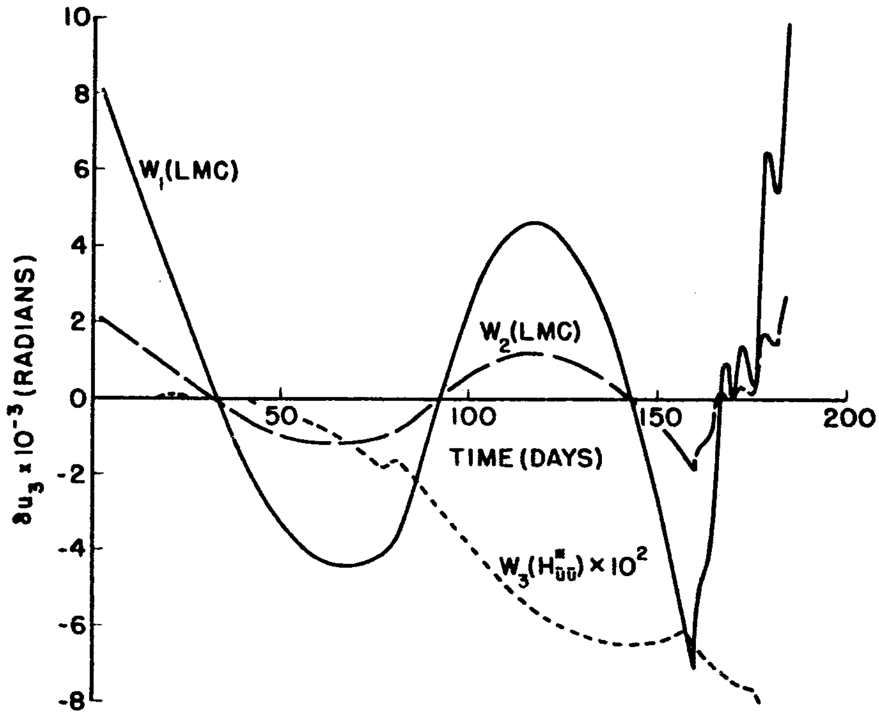

FIGURE 20(g). CONTROL DEVIATIONS FOR $\delta x_2(0) = 10^{-6}$AU

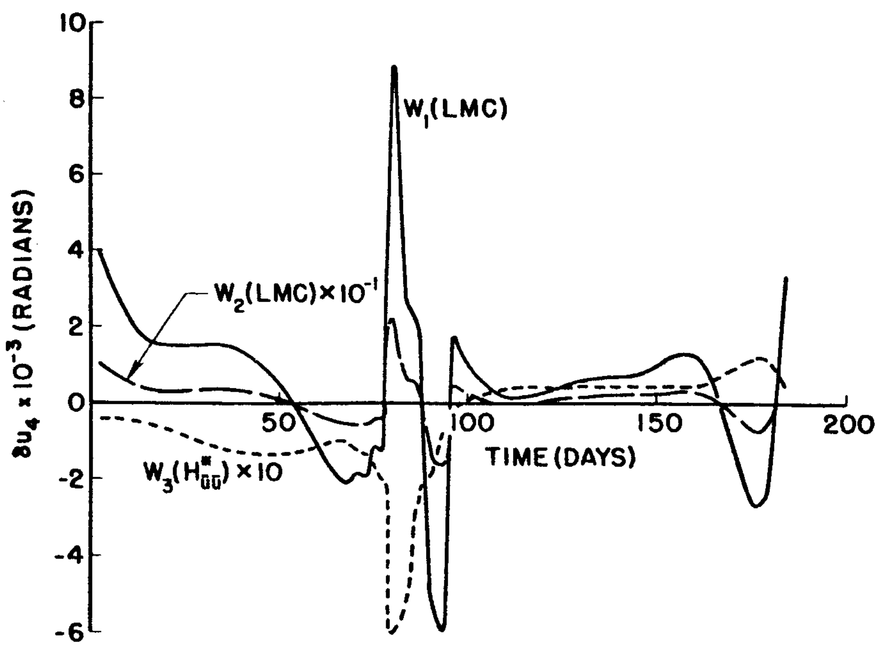

FIGURE 20(h). CONTROL DEVIATIONS FOR $\delta x_2(0) = 10^{-6}$AU

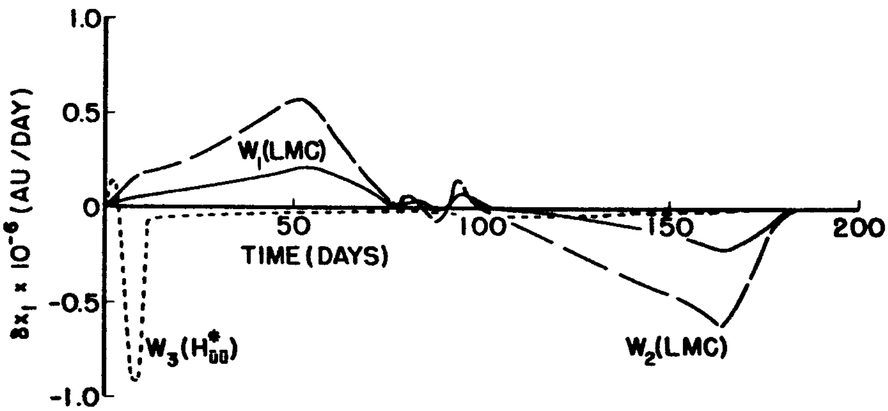

FIGURE 21(a). STATE DEVIATIONS FOR $\delta x_3(0) = 10^{-6}$AU/DAY

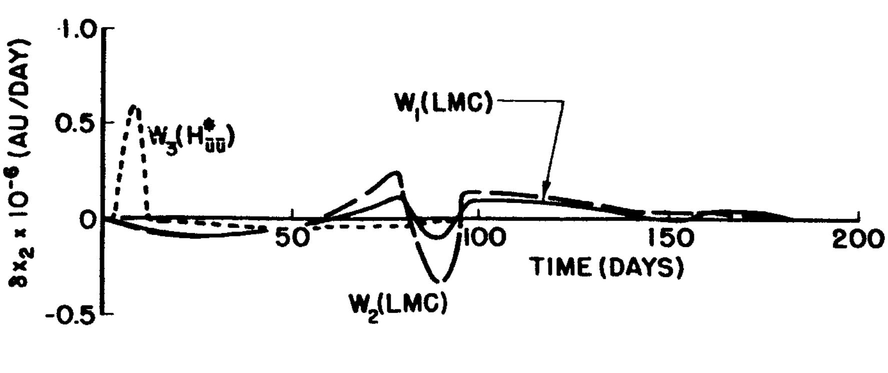

FIGURE 21 (b). STATE DEVIATIONS FOR $\delta x_3(0) = 10^{-6}$ AU/DAY

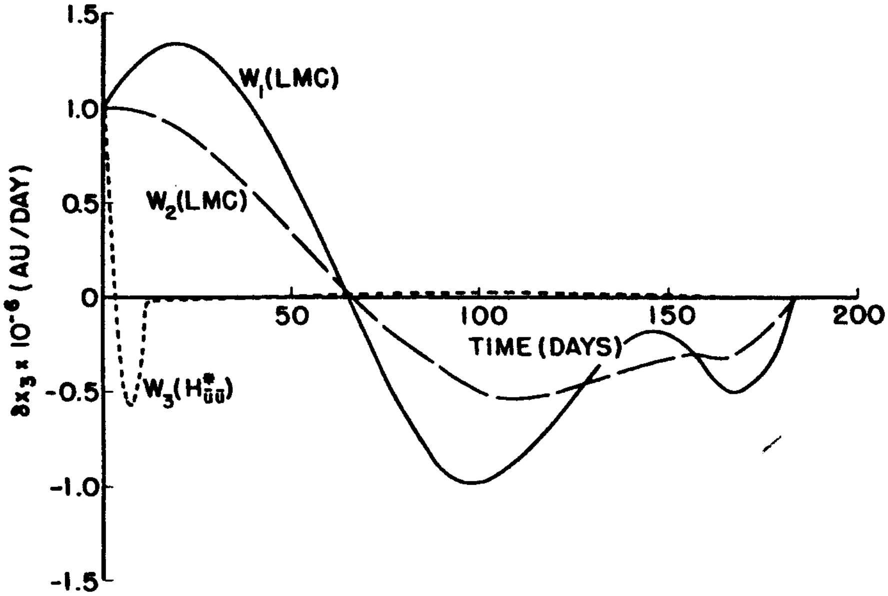

FIGURE 21(c). STATE DEVIATIONS FOR $\delta x_3(0) = 10^{-6}$ AU/DAY

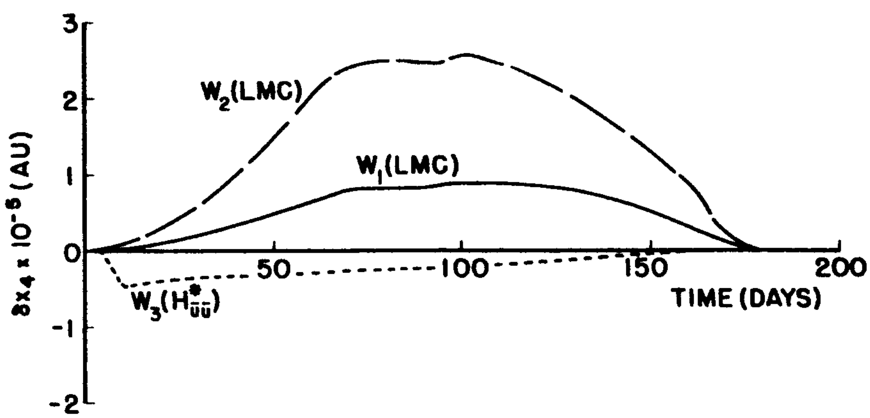

FIGURE 21(d). STATE DEVIATIONS FOR $\delta x_3(0) = 10^{-6}$ AU/DAY

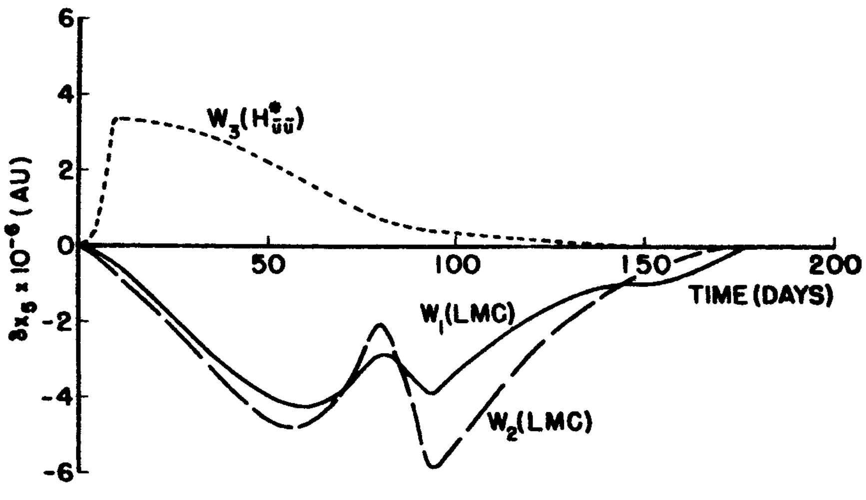

FIGURE 21(e). STATE DEVIATIONS FOR $\delta x_3(0) = 10^{-6}$ AU/DAY

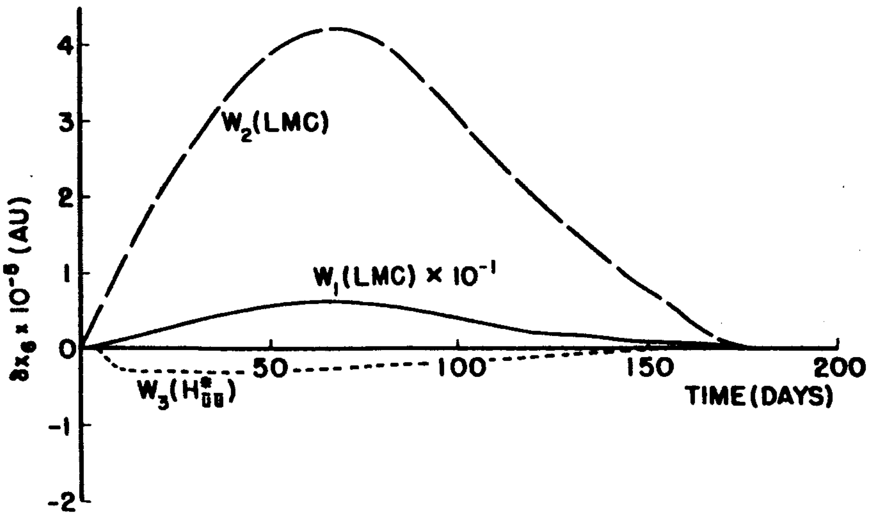

FIGURE 21(f). STATE DEVIATIONS FOR $\delta x_3(0) = 10^{-6}$ AU/DAY

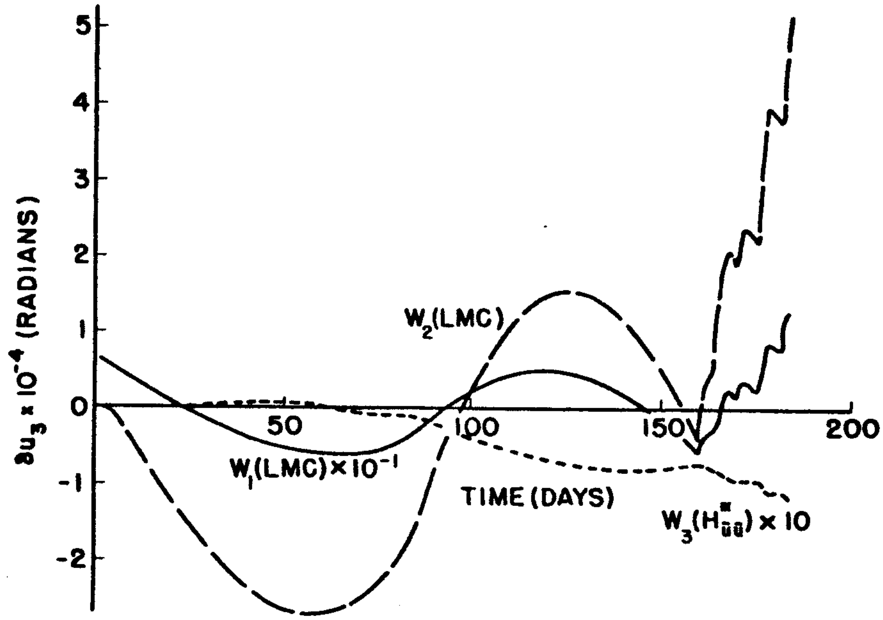

FIGURE 21(g). CONTROL DEVIATIONS FOR $\delta x_3(0) = 10^{-6}$AU

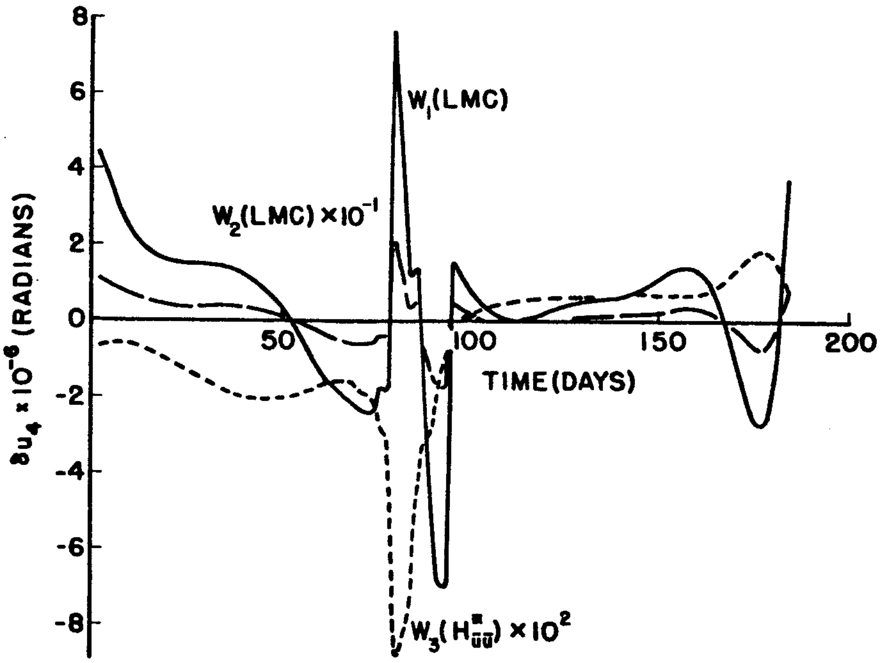

FIGURE 21(h). CONTROL DEVIATIONS FOR $\delta x_3(0) = 10^{-6}$AU

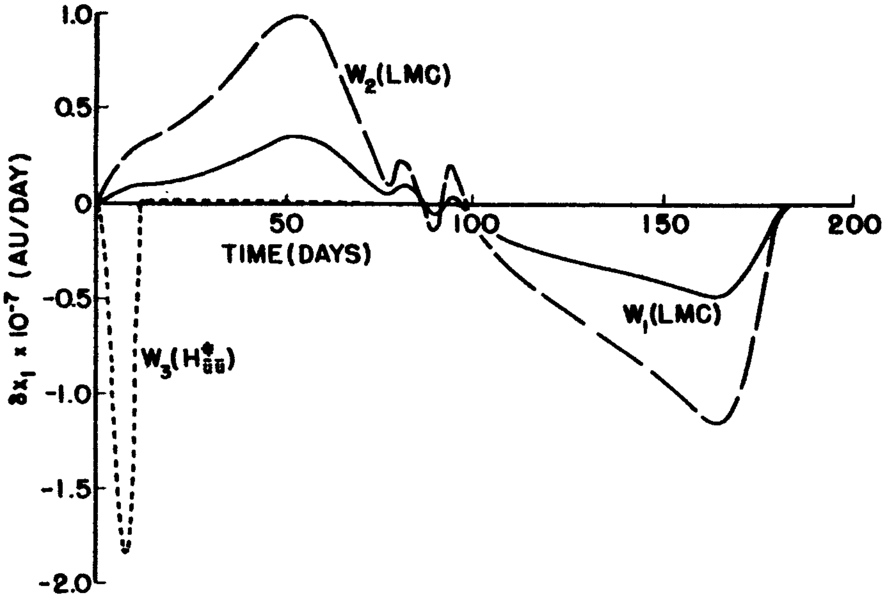

FIGURE 22(a). STATE DEVIATIONS FOR $\delta x_4(0) = 10^{-6}$AU

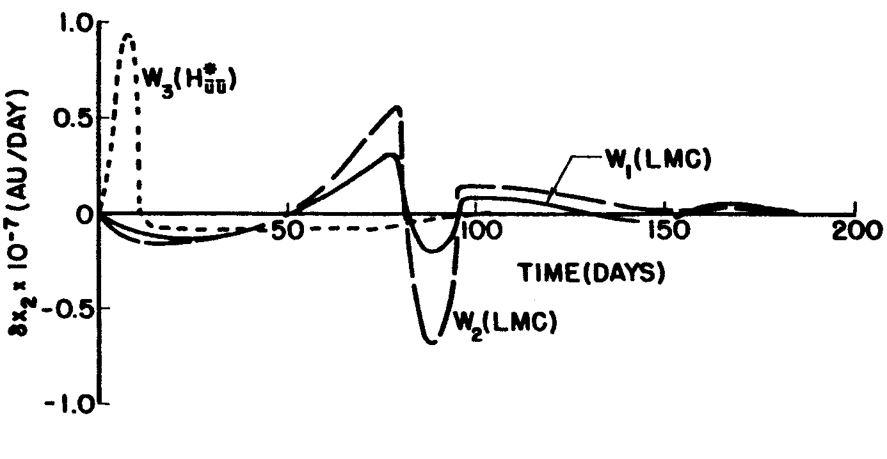

FIGURE 22(b). STATE DEVIATIONS FOR $\delta x_4(0) = 10^{-6}$AU

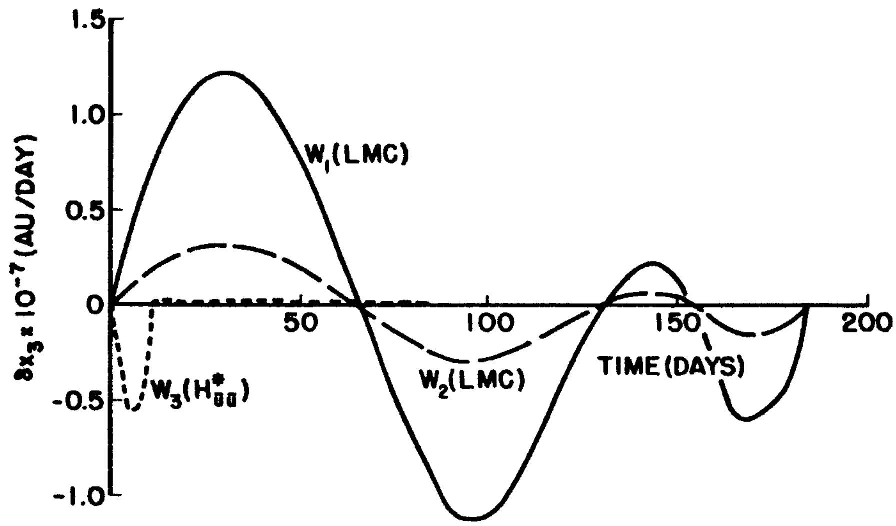

FIGURE 22(c). STATE DEVIATIONS FOR $\delta x_4(0) = 10^{-6}$ AU

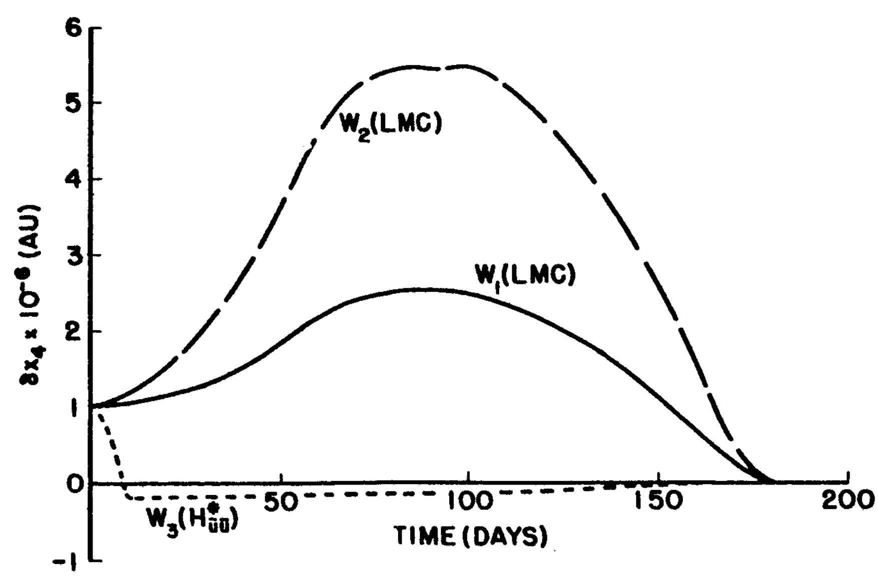

FIGURE 22(d). STATE DEVIATIONS FOR $\delta x_4(0) = 10^{-6}$AU

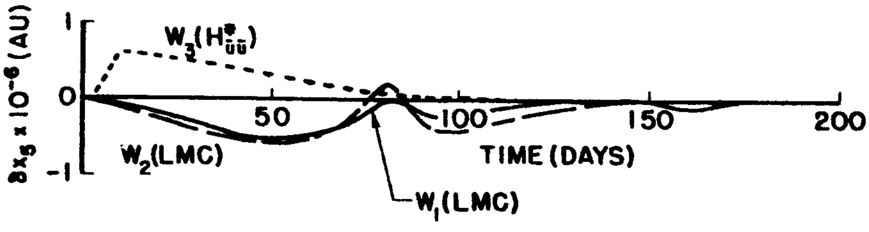

FIGURE 22(e). STATE DEVIATIONS FOR $\delta x_4(0) = 10^{-6}$ AU

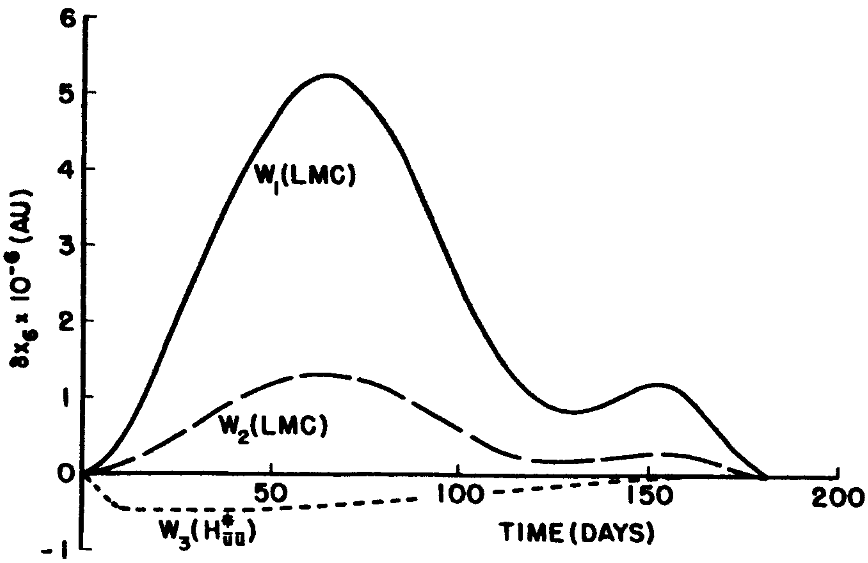

FIGURE 22(f). STATE DEVIATIONS FOR $\delta x_4(0) = 10^{-6}$AU

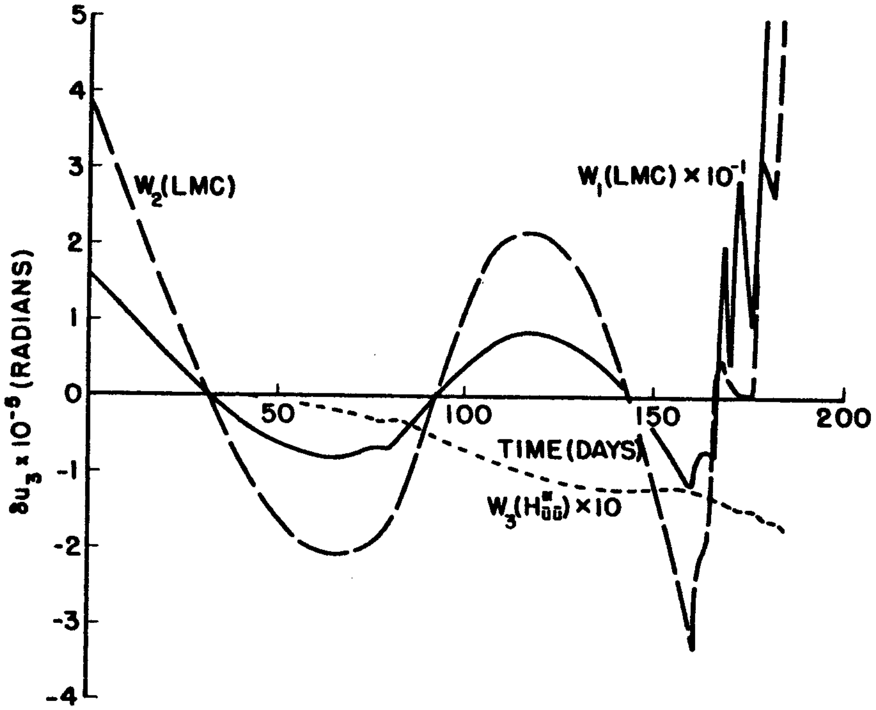

FIGURE 22(g). CONTROL DEVIATIONS FOR $\delta x_4(0) = 10^{-6}$AU

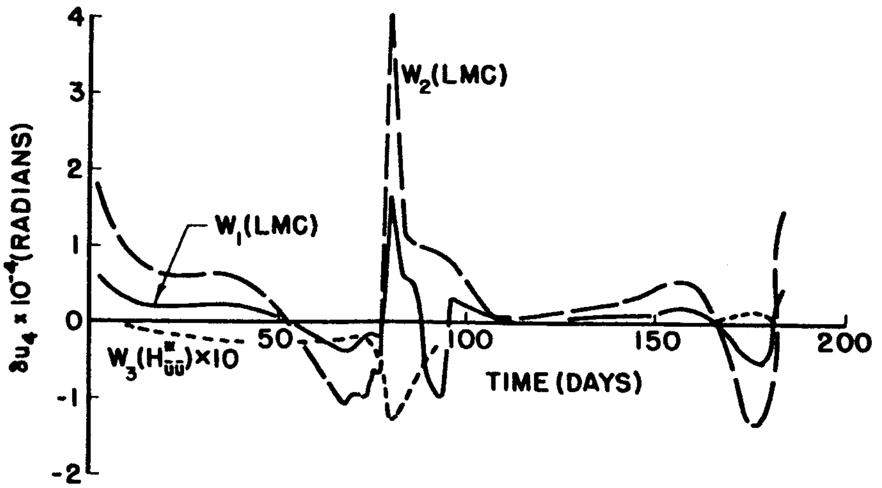

FIGURE 22(h). CONTROL DEVIATIONS FOR $\delta x_4(0) = 10^{-6}$AU

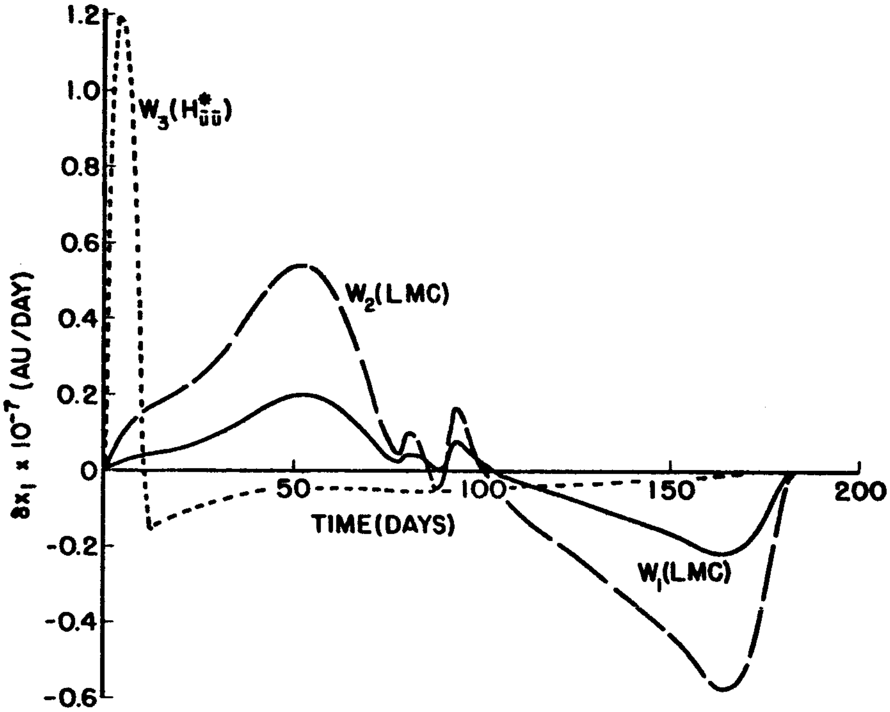

FIGURE 23(a). STATE DEVIATIONS FOR $\delta x_5(0) = 10^{-6}$AU

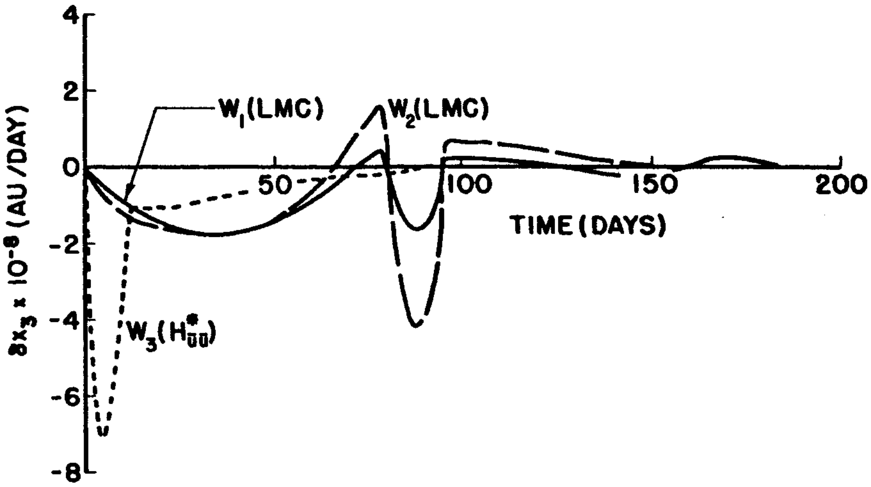

FIGURE 23(b). STATE DEVIATIONS FOR $\delta x_5(0) = 10^{-6}$ AU

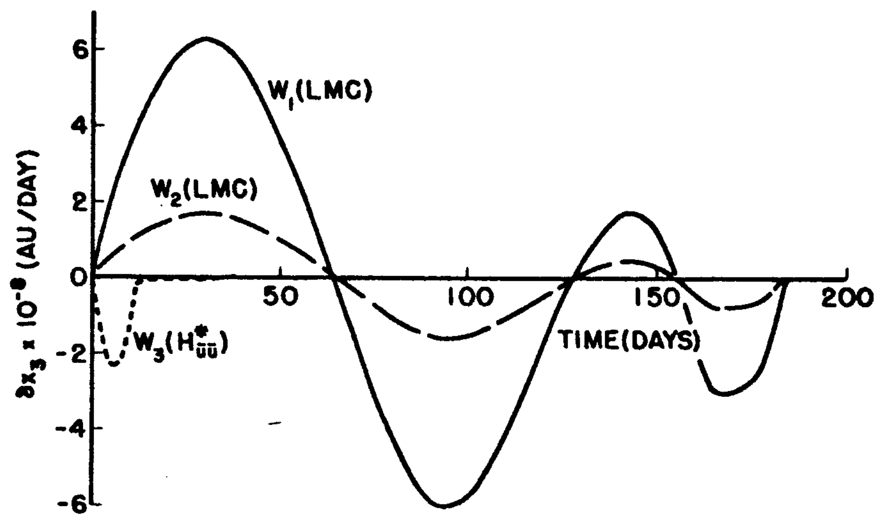

FIGURE 23(c). STATE DEVIATIONS FOR $\delta x_5(0) = 10^{-6}$AU

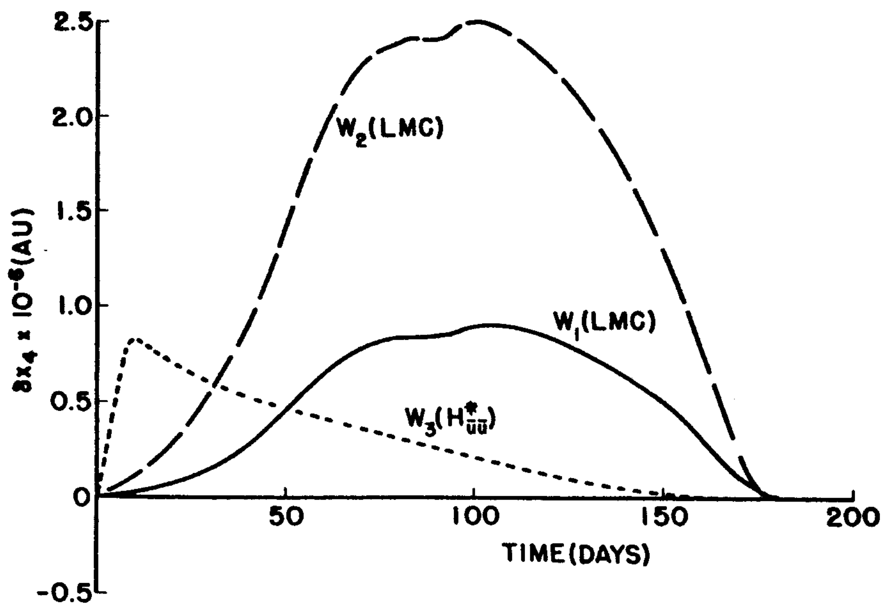

FIGURE 23(d). STATE DEVIATIONS FOR $\delta x_5(0) = 10^{-6}$ AU

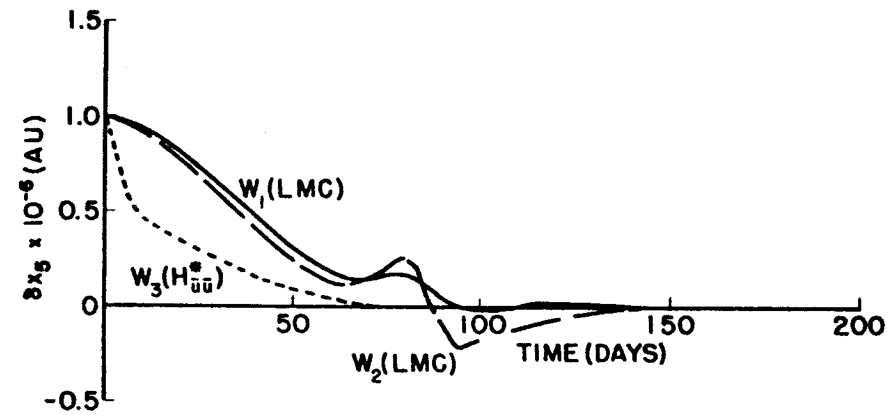

FIGURE 23(e). STATE DEVIATIONS FOR $\delta x_5(0) = 10^{-6}$AU

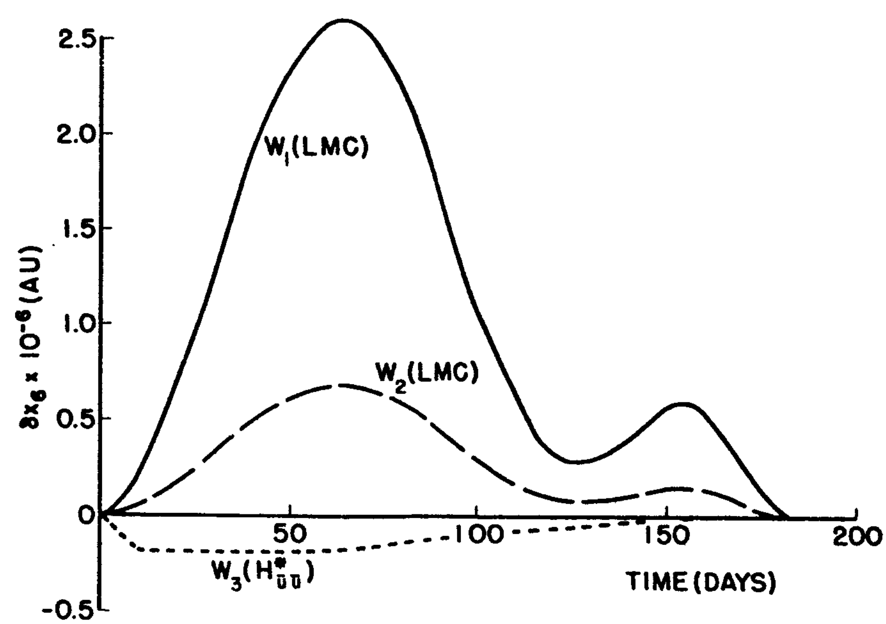

FIGURE 23(f). STATE DEVIATIONS FOR $\delta x_5(0) = 10^{-6}$AU

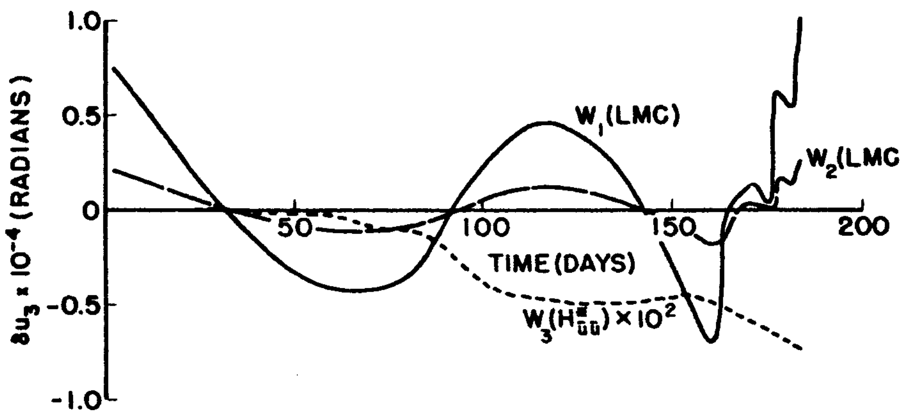

FIGURE 23(g). CONTROL DEVIATIONS FOR $\delta x_5(0) = 10^{-6}$AU

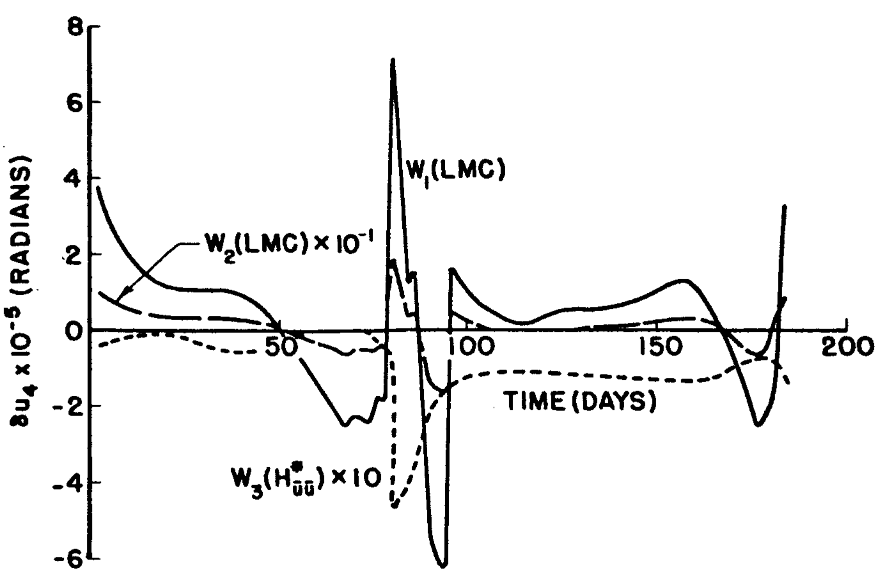

FIGURE 23(h). CONTROL DEVIATIONS FOR $\delta x_5(0) = 10^{-6}$AU

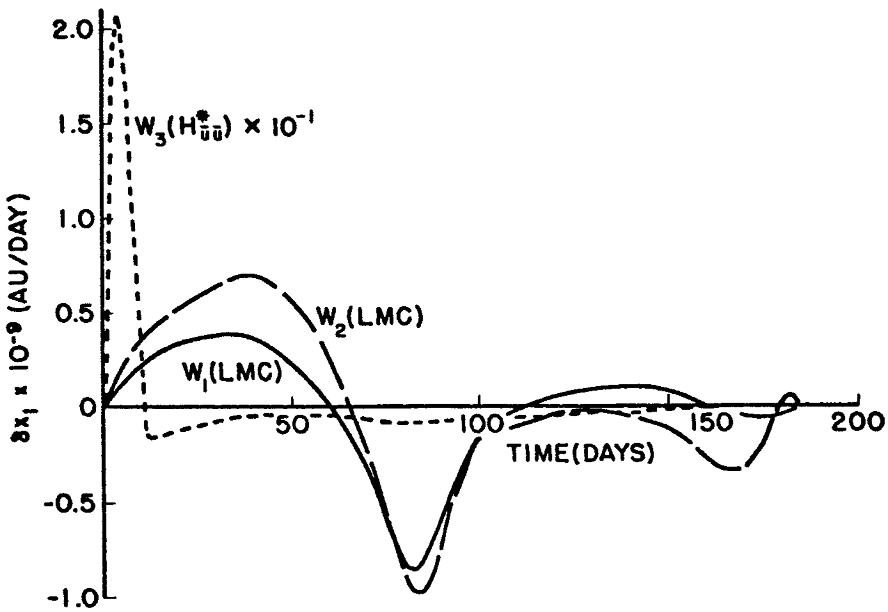

FIGURE 24(a). STATE DEVIATIONS FOR $\delta x_6(0) = 10^{-6}$AU

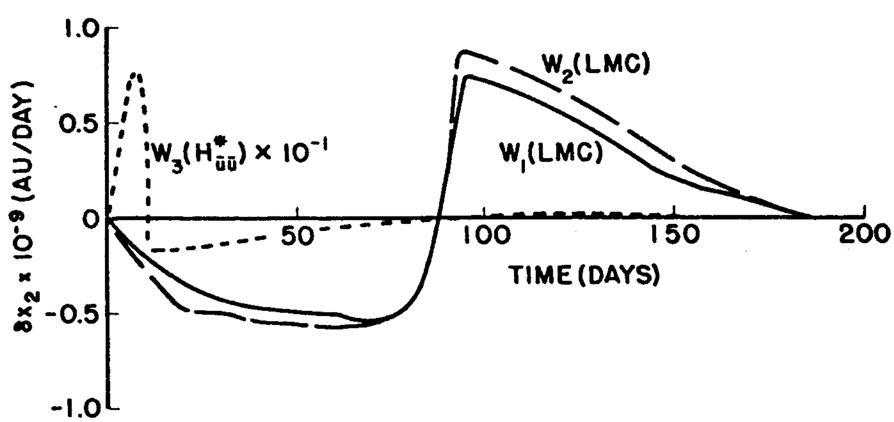

FIGURE 24(b). STATE DEVIATIONS FOR $\delta x_6(0) = 10^{-6}$AU

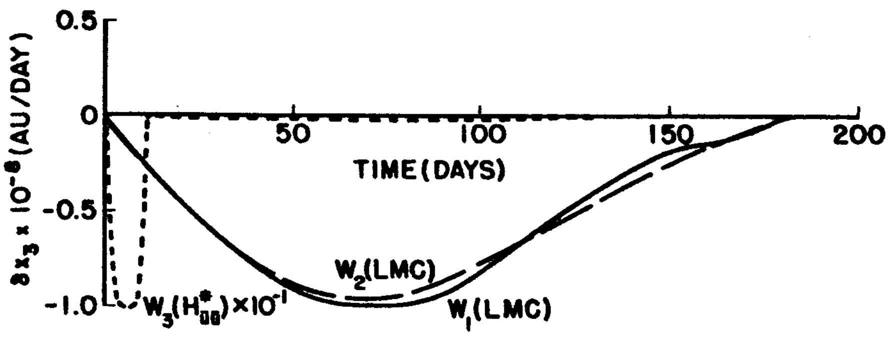

FIGURE 24(c). STATE DEVIATIONS FOR $\delta x_6(0) = 10^{-6}$AU

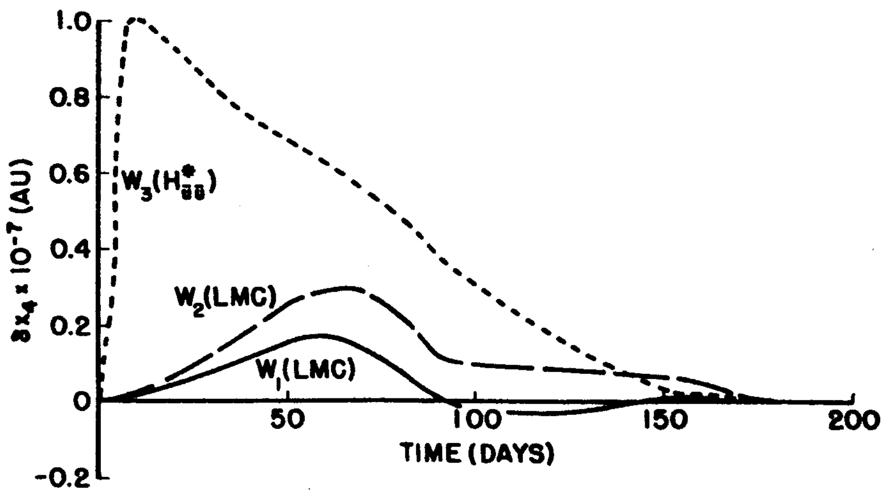

FIGURE 24(d). STATE DEVIATIONS FOR $\delta x_6(0) = 10^{-6}$AU

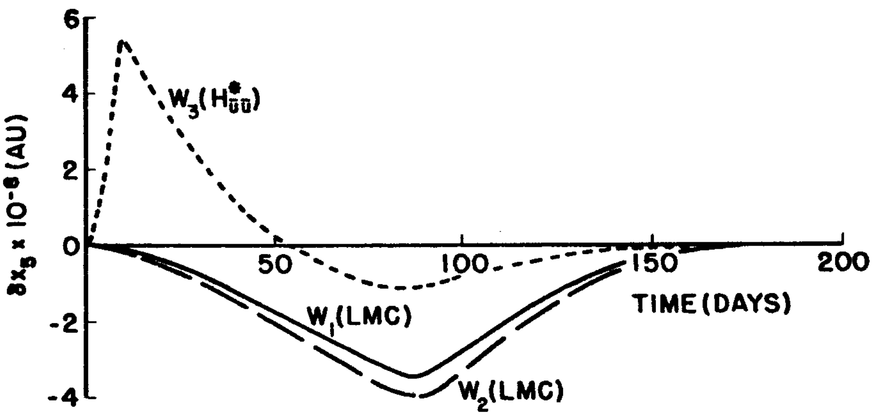

FIGURE 24(e). STATE DEVIATIONS FOR $\delta x_6(0) = 10^{-6}$ AU

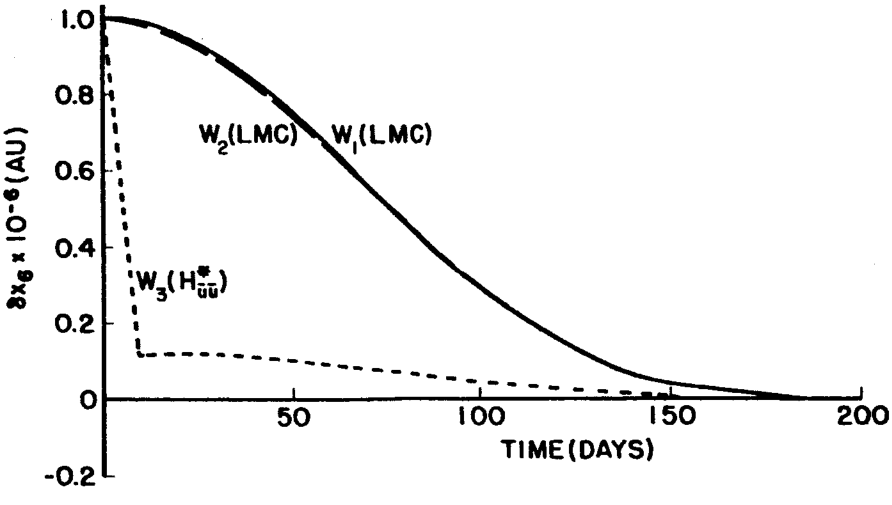

FIGURE 24(f). STATE DEVIATIONS FOR $\delta x_6(0) = 10^{-6}$ AU

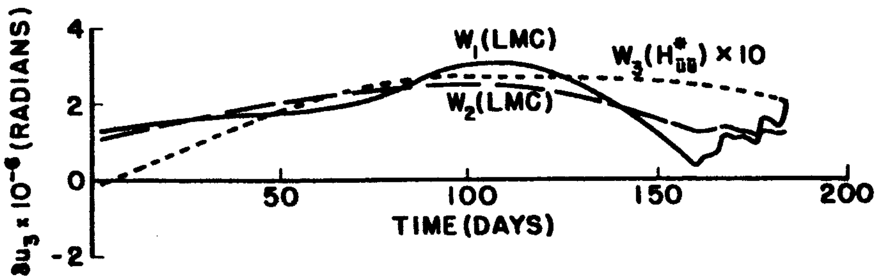

FIGURE 24(g). CONTROL DEVIATIONS FOR $\delta x_6(0) = 10^{-6}$AU

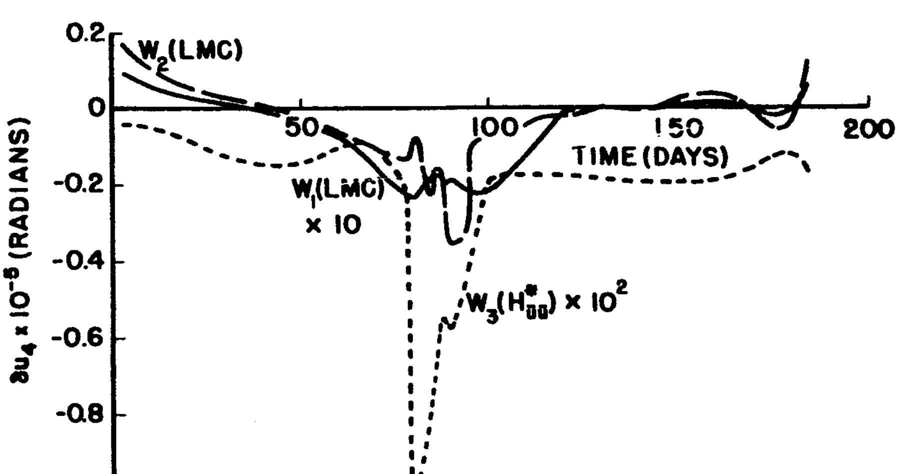

FIGURE 24(h). CONTROL DEVIATIONS FOR $\delta x_6(0) = 10^{-6}$AU

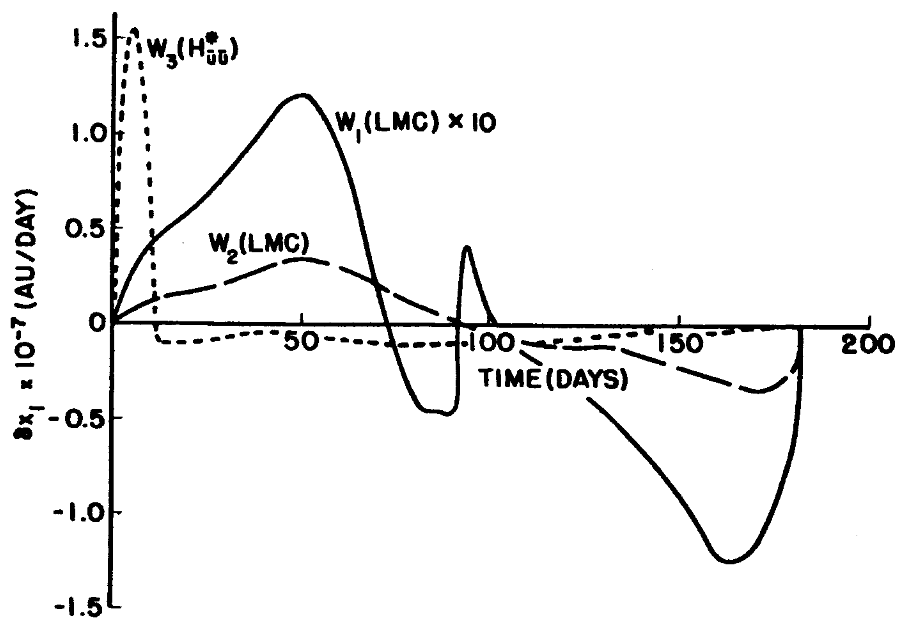

FIGURE 25(a). STATE DEVIATIONS FOR $\delta x_7(0) = 10^{-6} \times m_0$

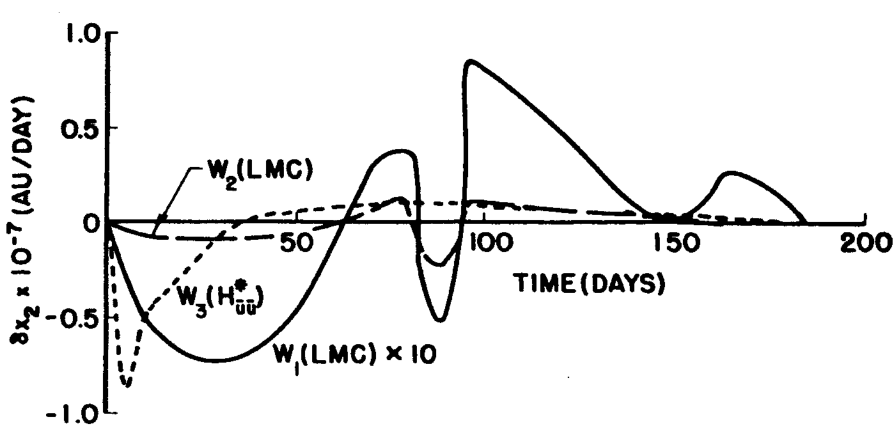

FIGURE 25(b). STATE DEVIATIONS FOR $\delta x_7(0) = 10^{-6} \times m_0$

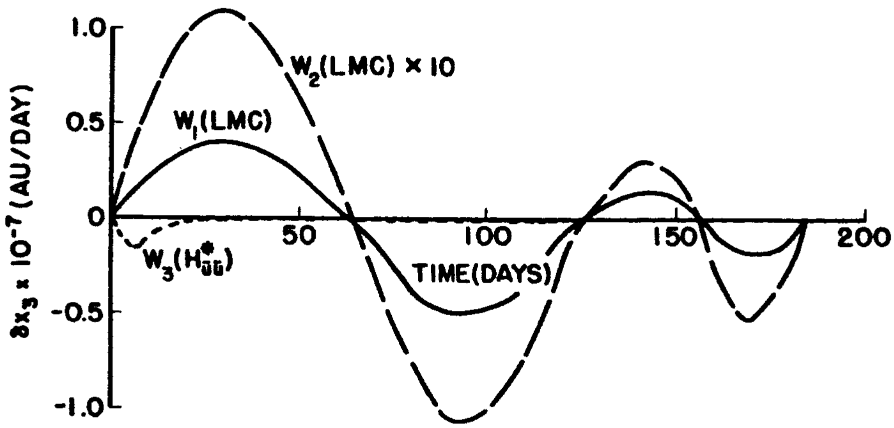

FIGURE 25(c). STATE DEVIATIONS FOR $\delta x_7(0) = 10^{-6} \times m_0$

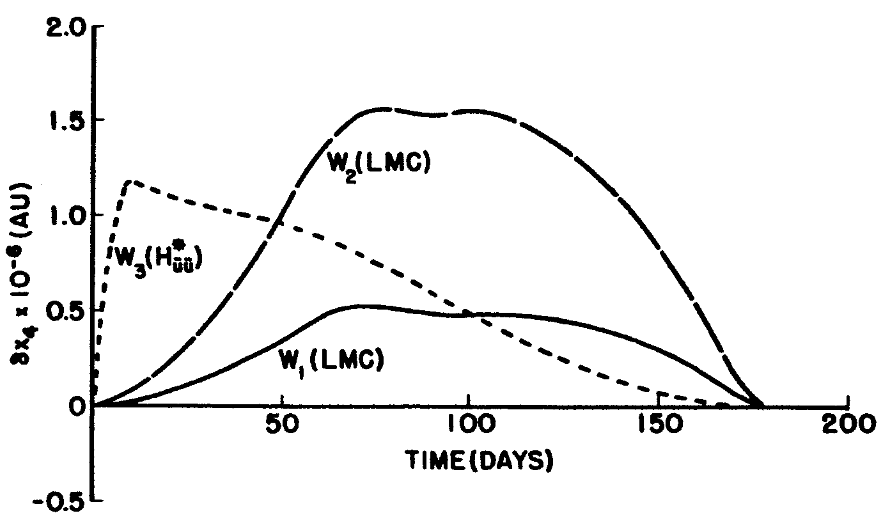

FIGURE 25(d). STATE DEVIATIONS FOR $\delta x_7(0) = 10^{-6} \times m_0$

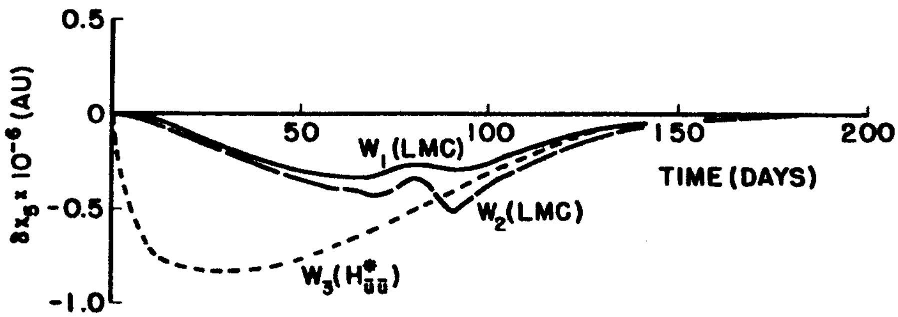

FIGURE 25(e). STATE DEVIATIONS FOR $\delta x_7(0) = 10^{-6} \times m_0$

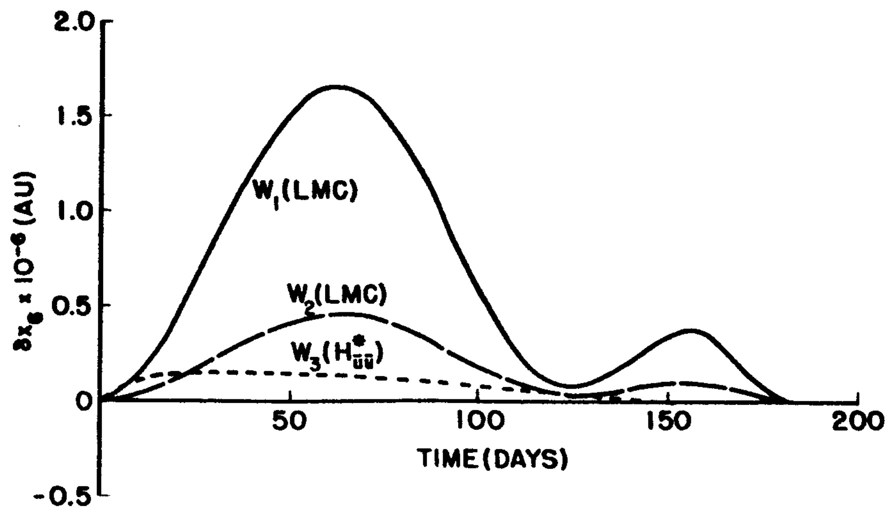

FIGURE 25(f). STATE DEVIATIONS FOR $\delta x_7(0) = 10^{-6} \times m_0$

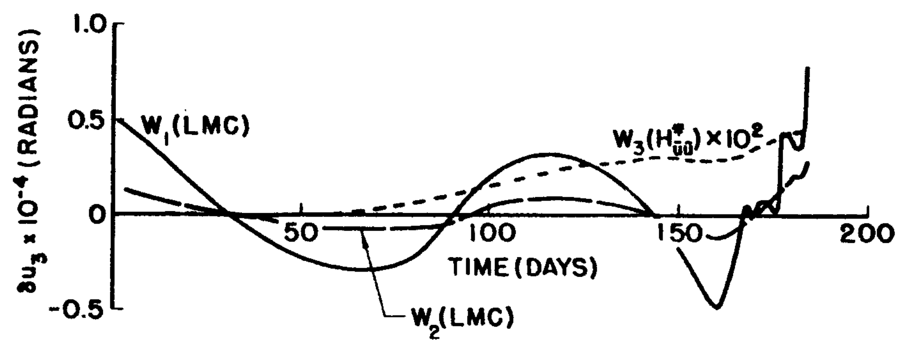

FIGURE 25(g). CONTROL DEVIATIONS FOR $\delta x_7(0) = 10^{-6} \times m_0$

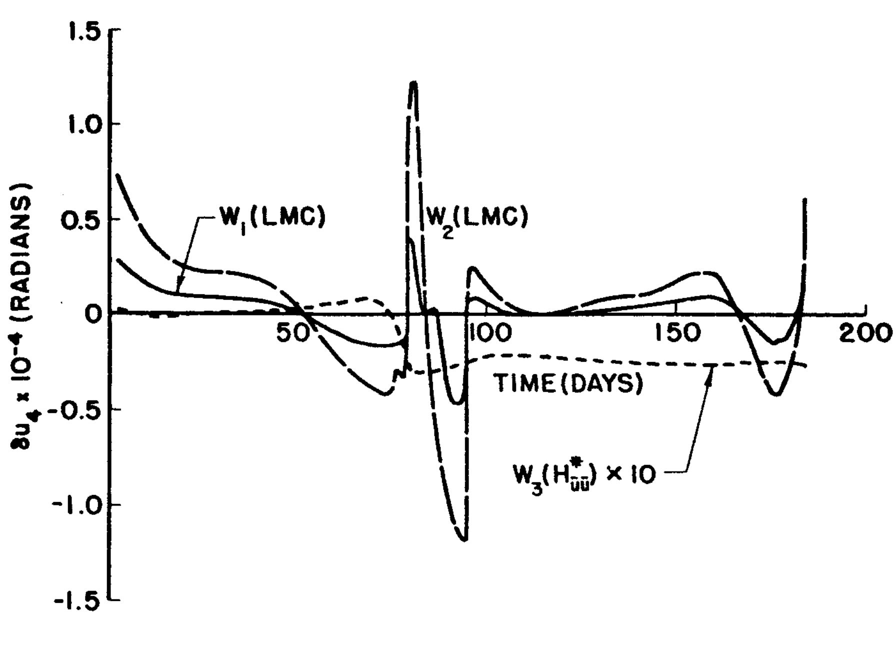

FIGURE 25(h). CONTROL DEVIATIONS FOR $\delta x_7(0) = 10^{-6} \times m_0$

faction of the terminal conditions. The reason for this is that a constant thrust magnitude applied to a body whose mass differs from the reference value will not produce the reference acceleration.

The state deviation histories for the variable x_7 are constant and are not plotted. However, if u_1 were variable, then the state deviation histories δx_7 (t) would no longer be constant and δx_7 (t) would have to be considered.

In the comparison presented here, the state variables x_1 through x_6 are terminally constrained while x_7 is terminally free. Thus, the state deviations produced by an error in x_7 are expected to differ from those produced by an error in any other state variable. When an error in x_7 occurs, the control schemes do not attempt to correct it. Instead, the other state variables are adjusted in order to insure rendezvous for the vehicle with its altered mass.

In contrast, state errors in variables x_1 through x_6 are corrected by the control schemes. An error in any one of the first six (terminally constrained) state variables initiates a correction maneuver while an error in the seventh (terminally free) state variable initates a control maneuver which insures rendezvous but does not correct the original state error. Thus, the control maneuver for a deviation in a terminally constrained state variable should differ from the control maneuver for an error in a terminally free state variable.

5.4 Discussion of Results

For an error in any of the variables, the δx_1 and δx_2 curves [i. e., parts (a) and (b) of Figures 19 through 25] for W_1 and W_2 exhibit erratic behavior between 75 and 105 days. During this time, (the thrust reversal period) the control variable u_4 swings through an angle of almost 180 degrees (See Figure 12). During reversal the δu_4 curves [i.e., parts (h) of Figures 19 through 25] for W_1 and W_2 show rapid changes also. The curves for W_3, however, show δx_1, δx_2, and δu_4 as smooth functions during reversal. This behavior is not unexpected since the elements of W_3 are related to the sensitivity of the system to changes in the control variables. On the other hand, the elements of W_1 and W_2 are constant. The smoothness of the state deviation curves for W_3 during this period indicates that the use of W_3 leads to an adjustment in the control deviations to account for the changes in the system's sensitivity to the control deviation. The constant weighting functions W_1 and W_2 allow no such adjustment.

Comparisons of the data for W_1 and W_2 show that W_2 produces state deviations of smaller amplitudes for δx_1, δx_2, δx_4, and δx_5 than does W_1 when the error occurs in one of the variables x_1, x_2, x_4, or x_5 (Figures 19, 20, 22, 23) but the δx_3 and δx_6 curves for W_2 have larger amplitudes than for W_1. When x_3 or x_6 is the variable in which the error occurs (Figures 21 and 24), the δx_3 and δx_6 are smaller in amplitude for W_1 than for W_2. This implies that W_2 is a better weighting function than W_1 when the error is in x_1, x_2, x_4, or x_5, but if x_3 or x_6 is disturbed, then W_1 is better than W_2.

In general, for W_1, W_2, and W_3, errors in velocity result in much larger state deviations than do disturbances in position and mass. This effect is illustrated in Figures 13 through 18 and it has been observed in previous control studies of this type [2, 3, 8].

The control deviations characteristic of W_1 and W_2 are larger in every case than those of W_3. The state deviation curves indicate that

W_1 and W_2 overcorrect and produce oscillatory state deviations while W_3 produces much smaller but more effective corrections. If W_1 or W_2 is used, it appears that the control scheme corrects the current state error without compensating for the effect of this correction on the other state variables.

The W_3 control scheme corrects velocity errors in the following manner. Rapid velocity changes are introduced which quickly bring the vehicle's velocity to a value very near the characteristic value for the reference trajectory. Meanwhile, the position variables have been allowed to drift slightly away from the reference values. The small velocity deviation which remains is then used to slowly correct the small position error which occurred during the correction of the major portion of the velocity error.

A position error is controlled by the W_3 scheme as follows. Rapid velocity changes occur which bring the value of the disturbed position variable closer to its reference value. The velocity variables then return to values very near the reference values and again slowly correct the remaining position variable errors.

Thus, the use of W_3 as the weighting function gives a state deviation history very different from that given by W_1 and W_2. The W_3 state deviation history seems to be the low-thrust equivalent of the impulsive corrections characteristic of optimal high-thrust ballistic guidance.

Comparison of Figure 25 with Figures 19 through 24 show that the control maneuver executed in the case of an error in a terminally free state variable (x_7) does not differ significantly in character from the control maneuvers for erros in terminally constrained state variables. This result is surprising if one notes that in the case of Figure 25, the control schemes are physically unable to correct the error in x_7 while in Figures 19 through 24, it is shown that the main action of the control scheme is to correct the original state errors.

CHAPTER 6

CONCLUSIONS AND RECOMMENDATIONS

6.1 Summary

In the investigation presented here, a new control scheme is presented. The scheme, referred to as the E-Function Control (EC) is similar in form to the Lambda Matrix Control scheme (LMC). The EC scheme is a first order scheme as is the LMC scheme, but it is derived by using a performance index similar to that used with the second order Extremal Field Control scheme. The EC scheme replaces the arbitrary weighting function characteristic of the LMC scheme with a uniquely determined matrix. Furthermore, a general state prediction procedure is presented which is able to handle terminally free state variables as well as state variables which are terminally constrained.

The EC scheme is compared numerically with two LMC schemes (two different arbitrary weighting functions) on a three-dimensional Earth-Mars low-thrust transfer trajectory. The numerical results show that where applicable, the EC scheme is superior to the LMC scheme characterized by an arbitrary weighting function.

The limitations of the EC scheme are discussed and suggestions for extensions of the study are made. The applicability of the EC scheme to control about a near optimal reference trajectory is discussed also.

6.2 Conclusions

1. The user of the LMC scheme, given no criterion on which to base his choice of weighting function, W (t), would generally choose a weighting function which is convenient and easy to use.

Thus, since the choice of weighting function determines the control law, the usual LMC scheme leads to a control deviation program which is a function of the user's idea of ease and convenience. It would seem that any control-sensitive, non-arbitrary weighting function, however approximate, would yield a more desirable control law than an arbitrary weighting function. The results obtained using W_3 tend to substantiate this idea. Even though for the non-optimal reference trajectory, W_3 fails to meet one of the requirements placed on the weighting function, i.e., that it be positive definite over the entire trajectory, the control deviations produced by W_3 tend to null the state deviations much more quickly than do the control deviations produced by W_1 and W_2.

2. A constant weighting function, such as W_1 or W_2, which does not respond to the changing sensitivities of the system to the controls leads to a control deviation program which tends to over-correct state errors and produce oscillatory state deviation histories. The weighting function W_3, derived using the EC scheme, does provide for control-scheme response to changing sensitivities of the system to the controls and produces state deviation histories which do not oscillate after a short correction period which occurs immediately after the discovery of a state error.

3. Despite the fact that W_3 is not positive definite over the entire trajectory, use of the W_3 matrix as the weighting function seems to produce a low-thrust equivalent of the impulsive corrections characteristic of optimal high-thrust guidance. It is not claimed that $W_3 = H_{\underline{uu}}^*$ yields the ultimate in first order control, but it seems clear that this choice of the weighting function for the model

used here gives definite improvement over the entirely arbitrary constant weighting functions W_1 and W_2. It is probable that $W_3 = H_{\overline{uu}}^{*}$ is a better weighting function than any constant weighting function.

4. As has been found in all previous control studies of this type, errors in velocity are more serious than errors in position. The resulting state deviations in all variables are larger.

6.3 Recommendations for Further Study

It is recommended that the study be extended to the following areas:

1. First, a truly optimal reference trajectory should be obtained and the control method developed here should be retested. The optimal reference trajectory should be obtained by using a method such as that proposed by Jazwinski[20] in order that $H_{\overline{u}} = 0$ and $H_{\overline{uu}}$ be positive definite along the reference trajectory (necessary conditions for a true optimum). Use of the Jazwinski scheme requires that $H_{\overline{uu}}$ be positive definite and requires that $H_{\overline{u}} = 0$ along each attempt to obtain the optimal trajectory. An iterative process is used to obtain the initial values of the Lagrange multipliers required to satisfy the terminal constraint relations.

2. It is suggested that a second order scheme, such as the Extremal Field Control scheme be compared with the EC scheme presented here.

3. It is also suggested that the possibility of using the E-Function Control scheme presented here to improve a nonoptimal reference trajectory be investigated. This might be accomplished by requiring the control scheme to guide the vehicle to a mid-course

point slightly away from the reference path and then requiring the scheme to correct the resulting error.

4. The possiblity of developing the ideas of Section 4.6 should be investigated. Here, it is suggested that given an approximately optimum reference trajectory, the EC scheme might be used to generate a more nearly optimal trajectory.

APPENDICES

APPENDIX A

Definitions of Terms

Control - the process of regulating the state of the vehicle (system) by varying the parameters of the propulsion system.

Control deviation program - a sequence of control changes consisting of a control change vector at each point on the trajectory following the initiation of a change in the reference control program.

Control program - a unique specification of the values for all of the control variables at each point in time over the span of the mission.

Control scheme (guidance control scheme) - as used in this report, a method for calculating changes in the control variables to correct for deviations from the reference trajectory.

Control variables - quantities which regulate the effect of the propulsion system on the vehicle state (the control variables appear as undifferentiated time-dependent parameters in the differential equations of state).

Reference control program - the control program which, along with the reference initial state, will produce the reference trajectory.

Reference state - the values of the state variables at any point in time which result from the use of the reference initial state and the reference control program.

Reference trajectory - the values of position and velocity at each point in time along the planned mission trajectory (the kinematic portion of the reference state at each point).

State deviation - the difference between the actual value of a state variable and its reference value at a point in time.

State deviation prediction - refers to the process of predicting the state deviations which will occur due to the use of a control deviation program.

State equations - the seven differential equations which describe the behavior of the state variables.

State perturbation - a disturbance in state or difference between the actual and reference states.

State variables - the seven quantities which represent the physical condition of the vehicle.

Terminally constrained state variable - a state variable which has its value at the end of the mission explicitly or implicitly specified by mission requirements.

Terminally free state variable - a state variable which has its value unrestricted at the end of the mission.

APPENDIX B

The Steepest Descent Procedure

The steepest descent (or ascent) procedure in the calculus of variations is a numerical procedure for finding solutions of optimum programming problems for a system governed by a set of ordinary differential equations. Certain independent variables, called the control variables, must be programmed in such a way that certain terminal conditions are satisfied and a quantity called the performance index is extremized. The steepest descent procedure determines an approximation to the optimum control variable program. The accuracy of the approximation depends on the number of iterations over which the process is applied, the closeness of the original guess of the control program, and the degree of nonlinearity of the differential equations which govern the system.

The problems to which the steepest descent procedure is applied are, in the classical calculus of variation formulation, two-point boundary value problems for a set of ordinary nonlinear differential equations. The steepest descent procedure provides a systematic iterative procedure for the solution of such problems which takes advantage of the capabilities of the modern high-speed computer.

The problem is basically the one formulated in Chapter 2. Given a system governed by a set of n simultaneous first order nonlinear differential equations

$$\dot{\bar{x}} = \bar{f}(\bar{u}, \bar{x}, t) \tag{B.1}$$

with initial conditions

$$\bar{x}(t_o) = \bar{x}_o \tag{B.2}$$

subject to p terminal constraint relations

$$\overline{M}(\bar{x}(t_f), t_f) = 0 \tag{B.3}$$

find the control program $\bar{u}(t)$ such that a performance index

$$V = G(\bar{x}_f, t_f) + \int_{t_o}^{t_f} Q(\bar{u}, \bar{x}, t)\, dt \tag{B.4}$$

is extremized. Following Chapter 2, the Euler-Lagrange equations

$$\dot{\bar{\lambda}}^T = - H_{\bar{x}} \tag{B.5}$$

must be satisfied along the extremal path. The end conditions for (B. 5) are

$$\bar{\lambda}^T(t_f) = P_{\bar{x}}\Big|_{t_f} \tag{B.6}$$

where $P = G + \bar{\nu}^T \overline{M}$.

Equations (B. 1) and (B. 5) with conditions (B. 2) and (B. 6) form a two-point boundary value problem for the 2n differential equations in $\bar{x}$ and $\bar{\lambda}$.

In general, the p vector of constant multipliers is unknown. Also, if the final time is not constant, some method of eliminating it must be used or it must be added to the list of unknown quantities.

Choose any of the p terminal constraint relations as a condition to determine the time at which the integration will be stopped, i.e., the final time t_f. Let this relation be called K. Thus the stopping condition

$$K(\bar{x}_f, t_f) = 0 \tag{B.7}$$

determines the final time t_f and is satisfied for all nominal trajectories. The remaining p - 1 terminal constraint relations and their corresponding ν's take the place of the p terminal constraint relations in (B.6) since the ν corresponding to K is indeterminate because $K(t_f) = 0$.

In order to avoid having to know the $(p - 1)\lambda$'s at t_f, let

$$\bar{\lambda}_G^T(t_f) = G_{\bar{x}}\Big|_{t_f} \tag{B.8}$$

$$\bar{\lambda}_{\overline{M}}^T = M_{\bar{x}}\Big|_{t_f} \tag{B.9}$$

$$\bar{\lambda}_K^T = K_{\bar{x}}\Big|_{t_f} \tag{B.10}$$

where $\bar{\lambda}_G$ is an n vector, $\bar{\lambda}_{\overline{M}}$ is an n x (p - 1) matrix of (p - 1) n vectors and $\bar{\lambda}_K$ is an n vector. The quantities $\bar{\lambda}_G$, $\bar{\lambda}_{\overline{M}}$, and $\bar{\lambda}_K$ are vectors of influence functions or sensitivity coefficient functions which satisfy the adjoint equations given in Equation (B.5). Then, to first order, at t_f,

$$\delta G = \bar{\lambda}_G^T\, \delta\bar{x} \tag{B.11}$$

$$\delta\overline{M} = \bar{\lambda}_{\overline{M}}^T\, \delta\bar{x} \tag{B.12}$$

$$\delta K = \bar{\lambda}_K^T \, \delta\bar{x} \tag{B. 13}$$

The numerical procedure for obtaining the control program which produces an optimal trajectory is as follows: Choose a nominal control program $\bar{u}^*(t)$ which will lead to eventual satisfaction of the stopping condition (B. 7). Integrate the Equation (B. 1) forward in time until (B. 7) is satisfied. Since the λ's are not known at this point, Equation (B. 5) cannot be integrated back in time from t_f to t_o with the end conditions (B. 6). The Fundamental Guidance Equation (4. 6) states that for small deviations away from the nominal trajectory,

$$(\bar{\lambda}_G^T \, \delta\bar{x})_{t_f} = (\bar{\lambda}_G^T \, \delta\bar{x})_{t_o} + \int_{t_o}^{t_f} \bar{\lambda}_G^T \, B \delta\bar{u} \, dt \tag{B. 14}$$

$$(\bar{\lambda}_{\overline{M}}^T \, \delta\bar{x})_{t_f} = (\bar{\lambda}_{\overline{M}}^T \, \delta\bar{x})_{t_o} + \int_{t_o}^{t_f} \bar{\lambda}_{\overline{M}}^T \, B \delta\bar{u} \, dt \tag{B. 15}$$

$$(\bar{\lambda}_K^T \, \delta\bar{x})_{t_f} = (\bar{\lambda}_K^T \, \delta\bar{x})_{t_o} + \int_{t_o}^{t_f} \bar{\lambda}_K^T \, B \delta\bar{u} \, dt \tag{B. 16}$$

The total changes in G, $\overline{M}$, and K at the final time t_f are given by

$$\Delta G = \delta G + \dot{G} \, \Delta t \tag{B. 17}$$

$$\Delta \overline{M} = \delta \overline{M} + \dot{\overline{M}} \, \Delta t \tag{B. 18}$$

$$\Delta K = \delta K + \dot{K} \, \Delta t \tag{B. 19}$$

Substituting (B. 11) through (B. 13) into (B. 14) through (B. 16) and then eliminating δG, $\delta\overline{M}$, and δK with Equation (B. 17) through (B. 19), we obtain

$$\Delta G = \int_{t_o}^{t_f} \bar{\lambda}_G^T B\delta\bar{u}dt + (\dot{G}\Delta t)_{t_f} + (\bar{\lambda}_G^T \delta\bar{x})_{t_o} \qquad (B.20)$$

$$\Delta\bar{M} = \int_{t_o}^{t_f} \bar{\lambda}_{\bar{M}}^T B\delta\bar{u}dt + (\dot{\bar{M}}\Delta t)_{t_f} + (\bar{\lambda}_{\bar{M}}^T \delta\bar{x})_{t_o} \qquad (B.21)$$

$$\Delta K = \int_{t_o}^{t_f} \bar{\lambda}_K^T B\delta\bar{u}dt + (\dot{K}\Delta t)_{t_f} + (\bar{\lambda}_K^T \delta\bar{x})_{t_o} \qquad (B.22)$$

where $\dot{G} = (\frac{\partial G}{\partial t}) + (\frac{\partial G}{\partial \bar{x}} \bar{f})$, etc.

Since $K = 0$ determines t_f, $\Delta K = 0$ on each approximation to the optimal trajectory. Solving (B.22) for Δt and eliminating Δt from (B.20) and (B.21) we get

$$\Delta G = \int_{t_o}^{t_f} \bar{\lambda}_{GK}^T B\delta\bar{u}dt + (\bar{\lambda}_{GK}^T \delta\bar{x})_{t_o} \qquad (B.23)$$

$$\Delta\bar{M} = \int_{t_o}^{t_f} \bar{\lambda}_{MK}^T B\delta\bar{u}dt + (\bar{\lambda}_{MK}^T \delta\bar{x})_{t_o} \qquad (B.24)$$

where

$$\bar{\lambda}_{GK} = \bar{\lambda}_G - \frac{\dot{G}}{\dot{K}} \bar{\lambda}_K, \text{ and}$$

$$\bar{\lambda}_{MK} = \bar{\lambda}_{\bar{M}} - \frac{\dot{\bar{M}}}{\dot{K}} \bar{\lambda}_K$$

The $\bar{\lambda}$'s are influence functions as was noted in the discussion of the fundamental guidance equation. The steepest ascent computation scheme seeks to find the control deviation program which maximizes ΔG for specified values of $\Delta\bar{M}$, ΔK, and ΔR^2, where ΔR^2 is defined by the following integral

$$\Delta R^2 = \int_{t_o}^{t_f} \delta\bar{u}^T (t)\, W (t)\, \delta\bar{u} (t)\, dt \tag{B.25}$$

where $W(t)$ is an arbitrary symmetric $m \times m$ weighting matrix. In general, Equation (B.3) is not satisfied along a nominal path, so values $\Delta\bar{M}_c$ for $\Delta\bar{M}$ (usually some percentage of the needed $\Delta\bar{M}$) are chosen to bring the nominal solution closer to satisfying the terminal constraints.

Choosing ΔG as the performance index, adjoining (B.21) and (B.25) to it with Lagrange multipliers $\bar{\nu}$ and μ respectively, the procedure of Chapter 2 can be followed to obtain

$$\bar{\nu} = -2\mu\, I_{MM}^{-1}\, \Delta\bar{\beta}\, I_{MM}^{-1}\, I_{MG} \tag{B.26}$$

$$2\mu = \pm \sqrt{\frac{I_{GG} - I_{MG}^T\, I_{MM}^{-1}\, I_{MG}}{\Delta R^2 - \Delta\bar{\beta}^T\, I_{MM}^{-1}\, \Delta\bar{\beta}}} \tag{B.27}$$

where

$$\Delta\bar{\beta} = \Delta\bar{M}_c - \bar{\lambda}_{MK}^T (t_o)\, \delta\bar{x} (t_o)$$

$$I_{ij} = \int_{t_o}^{t_f} \bar{\lambda}_{iK}^T\, B W^{-1} B^T \bar{\lambda}_{jK}\, dt$$

The quantities i and j are dummy indices which take on values M and G. The proper values for the $\delta\bar{u}(t)$ can be shown to be

$$\delta\bar{u}(t) = \pm \frac{1}{2\mu} W^{-1} B^T [\bar{\lambda}_{GK} - \bar{\lambda}_{MK} I_{MM}^{-1} I_{MG}]$$

$$+ W^{-1} B^T \bar{\lambda}_{MK} I_{MM}^{-1} \Delta\bar{\beta} \tag{B.28}$$

where all functions on the right are evaluated at time t and the + (-) sign is to be used if ΔG is to be increased (decreased).

Note that if the $\Delta\overline{M}_c$ asked for is too large, then the denominator in (B. 27) can become negative. If this happens, then $\Delta\overline{M}_c$ must be scaled down until the denominator becomes positive.

The $\delta\bar{u}(t)$ at each point is added to the $\bar{u}(t)$ at each point to generate a new nominal control program. This new nominal control program is then fed back into the scheme to generate a new nominal trajectory, etc. The process is repeated until no further significant improvement in the nominal trajectory can be economically obtained.

A more complete discussion of the method can be found in References 12 and 13.

APPENDIX C

Partial Derivatives of the Variational Hamiltonian

Since in the problem considered here $Q = 0$, the variatonal Hamiltonian, $H = Q + \bar{\lambda}^T \bar{f}$ is given by

$$H = \lambda_1 \left(-\frac{\mu x_4}{r^3} + \frac{u_1 u_2}{x_7} \cos u_3 \cos u_4 \right)$$

$$+ \lambda_2 \left(-\frac{\mu x_5}{r^3} + \frac{u_1 u_2}{x_7} \cos u_3 \sin u_4 \right)$$

$$+ \lambda_3 \left(-\frac{\mu x_6}{r^3} + \frac{u_1 u_2}{x_7} \sin u_3 \right) + \lambda_4 x_1 + \lambda_5 x_2$$

$$+ \lambda_6 x_3 - \lambda_7 u_1 \tag{C.1}$$

where $\bar{f}$ is defined in Equations (1.5) and (1.7).

The partial derivatives of H with respect to the control vector, $\bar{u}$, $H_{\bar{u}}$, are given by

$$H_{\bar{u}} = \left[\frac{\partial H}{\partial u_1} \quad \frac{\partial H}{\partial u_2} \quad \frac{\partial H}{\partial u_3} \quad \frac{\partial H}{\partial u_4} \right] \tag{C.2}$$

where

$$\frac{\partial H}{\partial u_1} = \frac{u_2}{x_7} (\lambda_1 \cos u_3 \cos u_4 + \lambda_2 \cos u_3 \sin u_4 + \lambda_3 \sin u_3) - \lambda_7$$

$$\frac{\partial H}{\partial u_2} = \frac{u_1}{x_7} (\lambda_1 \cos u_3 \cos u_4 + \lambda_2 \cos u_3 \sin u_4 + \lambda_3 \sin u_3)$$

$$\frac{\partial H}{\partial u_3} = \frac{u_1 u_2}{x_7} \left(-\lambda_1 \sin u_3 \cos u_4 - \lambda_2 \sin u_3 \sin u_4 + \lambda_3 \cos u_3 \right)$$

$$\frac{\partial H}{\partial u_4} = \frac{u_1 u_2}{x_7} \left(-\lambda_1 \cos u_3 \sin u_4 + \lambda_2 \cos u_3 \cos u_4 \right)$$

The second partial derivatives of H^* with respect to the control vector form a symmetric 4 x 4 matrix and are given by

$$H_{\overline{uu}} = \begin{bmatrix} 0 & \frac{\partial^2 H}{\partial u_1 \partial u_2} & \frac{\partial^2 H}{\partial u_1 \partial u_3} & \frac{\partial^2 H}{\partial u_1 \partial u_4} \\ & 0 & \frac{\partial^2 H}{\partial u_1 \partial u_3} & \frac{\partial^2 H}{\partial u_2 \partial u_4} \\ \text{symmetric} & & \frac{\partial^2 H}{\partial u_3^2} & \frac{\partial^2 H}{\partial u_3 \partial u_4} \\ & & & \frac{\partial^2 H}{\partial u_4^2} \end{bmatrix} \qquad (C.3)$$

where

$$\frac{\partial^2 H}{\partial u_1 \partial u_2} = \left(\frac{\partial H}{\partial u_2}\right) / u_1$$

$$\frac{\partial^2 H}{\partial u_1 \partial u_3} = \left(\frac{\partial H}{\partial u_3}\right) / u_1$$

$$\frac{\partial^2 H}{\partial u_1 \partial u_4} = \left(\frac{\partial H}{\partial u_4}\right) / u_1$$

---

* $H_{\overline{\alpha\beta}} = \frac{\partial}{\partial} \left(\frac{\partial H}{\partial \bar{\alpha}}\right)^T$

$$\frac{\partial^2 H}{\partial u_2 \partial u_3} = \left(\frac{\partial H}{\partial u_3}\right)/u_1$$

$$\frac{\partial^2 H}{\partial u_2 \partial u_4} = \left(\frac{\partial H}{\partial u_4}\right)/u_2$$

$$\frac{\partial^2 H}{\partial u_3^{\ 2}} = -\left(\frac{\partial H}{\partial u_2}\right)/u_2$$

$$\frac{\partial^2 H}{\partial u_3 \partial u_4} = \frac{u_1 u_2}{x_7}(\lambda_1 \sin u_3 \sin u_4 - \lambda_2 \sin u_3 \cos u_4)$$

$$\frac{\partial^2 H}{\partial u_4^{\ 2}} = -\frac{u_1 u_2}{x_7}(\lambda_1 \cos u_3 \cos u_4 + \lambda_2 \cos u_3 \sin u_4)$$

The partial derivatives of H with respect to the state variables, $H_{\bar{x}}$, are given by

$$H_{\bar{x}} = \left[\frac{\partial H}{\partial x_1} \quad . \quad . \quad . \quad \frac{\partial H}{\partial x_7}\right] \tag{C.4}$$

where

$$\frac{\partial H}{\partial x_1} = \lambda_4$$

$$\frac{\partial H}{\partial x_2} = \lambda_5$$

$$\frac{\partial H}{\partial x_3} = \lambda_6$$

$$\frac{\partial H}{\partial x_4} = \mu r^{-5}\left[\lambda_1 (3x_4^{\ 2} - r^2) + 3\lambda_2 x_4 x_5 + 3\lambda_3 x_4 x_6\right]$$

$$\frac{\partial H}{\partial x_5} = \mu r^{-5}\,[3\lambda_1 x_5 x_4 + \lambda_2\,(3x_5{}^2 - r^2) + 3\lambda_3 x_5 x_6]$$

$$\frac{\partial H}{\partial x_6} = \mu r^{-5}\,[3\lambda_1 x_6 x_4 + 3\lambda_2 x_6 x_5 + \lambda_3\,(3x_6{}^2 - r^2)]$$

$$\frac{\partial H}{\partial x_7} = -\left(\frac{\partial H}{\partial u_2}\right)\frac{u_2}{x_7}$$

The second partial derivatives of H with respect to the state variables, $H_{\overline{xx}}$, form a symmetric 7×7 matrix and are given by

$$H_{\overline{xx}} = \left[\begin{array}{ccc|ccc|c}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & \frac{\partial^2 H}{\partial x_4{}^2} & \frac{\partial^2 H}{\partial x_4 \partial x_5} & \frac{\partial^2 H}{\partial x_4 \partial x_6} & 0 \\
0 & 0 & 0 & & \frac{\partial^2 H}{\partial x_5{}^2} & \frac{\partial^2 H}{\partial x_5 \partial x_6} & 0 \\
0 & 0 & 0 & \text{symmetric} & & \frac{\partial^2 H}{\partial x_6{}^2} & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & \frac{\partial^2 H}{\partial x_7{}^2}
\end{array}\right] \tag{C.5}$$

where, letting $S = \lambda_1 x_4 + \lambda_2 x_5 + \lambda_3 x_6$,

$$\frac{\partial^2 H}{\partial x_4^2} = \frac{\mu}{r^7} [3r^2 (2\lambda_1 x_4 + S) - 15x_4^2 S]$$

$$\frac{\partial^2 H}{\partial x_4 x_5} = \frac{\mu}{r^7} [3r^2 (\lambda_1 x_5 + \lambda_2 x_4) - 15x_4 x_5 S]$$

$$\frac{\partial^2 H}{\partial x_4 x_6} = \frac{\mu}{r^7} [3r^2 (\lambda_1 x_6 + \lambda_3 x_4) - 15x_4 x_6 S]$$

$$\frac{\partial^2 H}{\partial x_5^2} = \frac{\mu}{r^7} [3r^2 (2\lambda_2 x_5 + S) - 15x_5^2 S]$$

$$\frac{\partial^2 H}{\partial x_5 x_6} = \frac{\mu}{r^7} [3 r^2 (\lambda_2 x_6 + \lambda_3 x_5) - 15x_5 x_6 S]$$

$$\frac{\partial^2 H}{\partial x_6^2} = \frac{\mu}{r^7} [3r^2 (2\lambda_3 x_6 + S) - 15x_6^2 S]$$

$$\frac{\partial^2 H}{\partial x_7^2} = - (\frac{\partial H}{\partial x_7})/x_7 = (\frac{\partial H}{\partial u_2}) \frac{u_2}{x_7^2}$$

The mixed second partial derivatives of H with respect to $\overline{u}$ and $\overline{x}$, form a 4 x 7 matrix given by

$$H_{\overline{ux}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \frac{\partial^2 H}{\partial x_7 \partial u_1} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{\partial^2 H}{\partial x_7 \partial u_2} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{\partial^2 H}{\partial x_7 \partial u_3} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{\partial^2 H}{\partial x_7 \partial u_4} \end{bmatrix} \tag{C. 6}$$

where

$$\frac{\partial^2 H}{\partial x_7 \partial u_i} = -\left(\frac{\partial H}{\partial u_i}\right)/x_7 \qquad (i = 1, \ldots, 4)$$

The partial derivatives of H with respect to $\overline{\lambda}$, $H_{\overline{\lambda}}$, is a 7 vector given

$$H_{\overline{\lambda}} = [f_1 \ f_2 \ f_3 \ f_4 \ f_5 \ f_6 \ f_7]$$

where the $\overline{f}_i = \dot{\overline{x}}_i$ and are given by Equations (2. 5) and (2. 7).

The mixed second partial derivatives of H with respect to $\overline{u}$ and $\overline{\lambda}$, $H_{\overline{u\lambda}}$, forms a 4 x 7 matrix given by

$$H_{\overline{u\lambda}} = \begin{bmatrix} \frac{\partial^2 H}{\partial\lambda_1 \partial u_1} & \frac{\partial^2 H}{\partial\lambda_2 \partial u_1} & \frac{\partial^2 H}{\partial\lambda_3 \partial u_1} & 0 & 0 & 0 & -1 \\ \frac{\partial^2 H}{\partial\lambda_1 \partial u_2} & \frac{\partial^2 H}{\partial\lambda_2 \partial u_2} & \frac{\partial^2 H}{\partial\lambda_3 \partial u_2} & 0 & 0 & 0 & 0 \\ \frac{\partial^2 H}{\partial\lambda_1 \partial u_3} & \frac{\partial^2 H}{\partial\lambda_2 \partial u_3} & \frac{\partial^2 H}{\partial\lambda_3 \partial u_3} & 0 & 0 & 0 & 0 \\ \frac{\partial^2 H}{\partial\lambda_1 \partial u_4} & \frac{\partial^2 H}{\partial\lambda_2 \partial u_4} & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{C. 7}$$

where

$$\frac{\partial^2 H}{\partial\lambda_1 \partial u_1} = \frac{u_2}{x_7} \cos u_3 \cos u_4$$

$$\frac{\partial^2 H}{\partial\lambda_2 \partial u_1} = \frac{u_2}{x_7} \cos u_3 \sin u_4$$

$$\frac{\partial^2 H}{\partial\lambda_3 \partial u_1} = \frac{u_2}{x_7} \sin u_3$$

$$\frac{\partial^2 H}{\partial\lambda_i \partial u_2} = \left(\frac{\partial^2 H}{\partial\lambda_i \partial u_1}\right) \frac{u_1}{u_2} \quad (i = 1, 2, 3)$$

$$\frac{\partial^2 H}{\partial \lambda_1 \partial u_3} = -\frac{u_1 u_2}{x_7} \sin u_3 \cos u_4$$

$$\frac{\partial^2 H}{\partial \lambda_2 \partial u_3} = -\frac{u_1 u_2}{x_7} \sin u_3 \sin u_4$$

$$\frac{\partial^2 H}{\partial \lambda_3 \partial u_3} = \frac{u_1 u_2}{x_7} \cos u_3$$

$$\frac{\partial^2 H}{\partial \lambda_1 \partial u_4} = -\frac{u_1 u_2}{x_7} \cos u_3 \sin u_4$$

$$\frac{\partial^2 H}{\partial \lambda_2 \partial u_4} = \frac{u_1 u_2}{x_7} \cos u_3 \cos u_4$$

The mixed second partial derivatives of H with respect to $\overline{x}$ and $\overline{\lambda}$, $H_{\overline{x\lambda}}$, form a 7 x 7 matrix given by

$$H_{\overline{\lambda x}} = \left[\begin{array}{ccc|ccc|c}
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
\hline
\frac{\partial^2 H}{\partial\lambda_1 \partial x_4} & \frac{\partial^2 H}{\partial\lambda_2 \partial x_4} & \frac{\partial^2 H}{\partial\lambda_3 \partial x_4} & 0 & 0 & 0 & 0 \\
\diagdown & \frac{\partial^2 H}{\partial\lambda_2 \partial x_5} & \frac{\partial^2 H}{\partial\lambda_3 \partial x_5} & 0 & 0 & 0 & 0 \\
\text{symmetric} & \diagdown & \frac{\partial^2 H}{\partial\lambda_3 \partial x_6} & 0 & 0 & 0 & 0 \\
\hline
\frac{\partial^2 H}{\partial\lambda_1 \partial x_7} & \frac{\partial^2 H}{\partial\lambda_2 \partial x_7} & \frac{\partial^2 H}{\partial\lambda_3 \partial x_7} & 0 & 0 & 0 & 0
\end{array}\right] \qquad (C.8)$$

where

$$\frac{\partial^2 H}{\partial\lambda_i \partial x_i} = \frac{\mu}{r^5} (3x_i^2 - r^2) \qquad (i = 1, 2, 3)$$

$$\frac{\partial^2 H}{\partial\lambda_i \partial x_j} = \frac{3\mu}{r^5} (x_i + 3)(x_j)$$ for (i, j) taking on the values $(2, 4)$, $(3, 4)$ and $(3, 5)$.

$$\frac{\partial^2 H}{\partial\lambda_i \partial x_7} = -\frac{u_2}{x_7} \left(\frac{\partial^2 H}{\partial\lambda_i \partial u_2}\right) \qquad (i = 1, 2, 3)$$

If the control variables u_1 and u_2 are held constant, the partials of H with respect to the control, state, and adjoint variables are given by the following equations:

$$H_{\bar{u}} = \begin{bmatrix} \frac{\partial H}{\partial u_3} & \frac{\partial H}{\partial u_4} \end{bmatrix} \tag{C.9}$$

where the elements are defined in Equations (C. 2),

$$H_{\overline{uu}} = \begin{bmatrix} \frac{\partial^2 H}{\partial u_3^{\,2}} & \frac{\partial^2 H}{\partial u_3 \partial u_4} \\ \frac{\partial^2 H}{\partial u_3 \partial u_4} & \frac{\partial^2 H}{\partial u_4^{\,2}} \end{bmatrix} \tag{C.10}$$

where the elements are defined in Equations (C. 3),

$$H_{\overline{ux}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \frac{\partial^2 H}{\partial x_7 \partial u_3} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{\partial^2 H}{\partial x_7 \partial u_4} \end{bmatrix} \tag{C.11}$$

where the elements are defined in Equations (C. 6)

$$H_{\overline{u\lambda}} = \begin{bmatrix} \frac{\partial^2 H}{\partial \lambda_1 \partial u_3} & \frac{\partial^2 H}{\partial \lambda_2 \partial u_3} & \frac{\partial^2 H}{\partial \lambda_3 \partial u_3} & 0 & 0 & 0 & 0 \\ \frac{\partial^2 H}{\partial \lambda_1 \partial u_4} & \frac{\partial^2 H}{\partial \lambda_2 \partial u_4} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(C. 12)

where the elements are defined in Equations (C. 7).

The matrices $H_{\bar{x}}$, $H_{\bar{\lambda}}$, $H_{\overline{xx}}$, and $H_{\overline{x\lambda}}$ are not changed.

APPENDIX D

Equation Summary

1. Equations of state $[\,\dot{\bar{x}} = \bar{f}\,(\bar{u}, \bar{x}, t)\,]$

$$\dot{x}_1 = \frac{-\mu x_4}{r^3} + \frac{u_1 u_2}{x_7} \cos u_3 \cos u_4$$

$$\dot{x}_2 = \frac{-\mu x_5}{r^3} + \frac{u_1 u_2}{x_7} \cos u_3 \sin u_4$$

$$\dot{x}_3 = \frac{-\mu x_6}{r^3} + \frac{u_1 u_2}{x_7} \sin u_3$$

$$\dot{x}_4 = x_1$$

$$\dot{x}_5 = x_2$$

$$\dot{x}_6 = x_3$$

$$\dot{x}_7 = -u_1$$

where $r = (x_4^{\,2} + x_5^{\,2} + x_6^{\,2})^{1/2}$

2. Performance Index for reference trajectory.

$$G = -\,x_7\,(t_f) \text{ - to be minimized}$$

3. Adjoint equations ($\dot{\bar{\lambda}} = - H_{\bar{x}}$)

$$\dot{\lambda}_1 = \lambda_4$$

$$\dot{\lambda}_2 = \lambda_5$$

$$\dot{\lambda}_3 = \lambda_6$$

$$\dot{\lambda}_4 = \frac{\mu}{r^5} \left[\lambda_1 (3x_4^2 - r^2) + 3\lambda_2 x_5 x_4 + 3\lambda_3 x_6 x_4\right]$$

$$\dot{\lambda}_5 = \frac{\mu}{r^5} \left[3\lambda_1 x_4 x_5 + \lambda_2 (3x_5^2 - r^2) + 3\lambda_3 x_6 x_5\right]$$

$$\dot{\lambda}_6 = \frac{\mu}{r^5} \left[3\lambda_1 x_4 x_6 + 3\lambda_2 x_5 x_6 + \lambda_3 (3x_6^2 - r^2)\right]$$

$$\dot{\lambda}_7 = \frac{u_1 u_2}{x_7^2} \left[\lambda_1 \cos u_3 \cos u_4 + \lambda_2 \cos u_3 \sin u_4 + \lambda_3 \sin u_3\right]$$

4. Terminal Constraint Relations for Control Schemes ($\overline{M} = 0$)

$$M_1 = x_1(t_f) - u_m(t_f) = 0$$

$$M_2 = x_2(t_f) - v_m(t_f) = 0$$

$$M_3 = x_3(t_f) - w_m(t_f) = 0$$

$$M_4 = x_4(t_f) - x_m(t_f) = 0$$

$$M_5 \quad = \quad x_5\,(t_f) = y_m\,(t_f) \quad = \quad 0$$

$$M_6 \quad = \quad x_6\,(t_f) - z_m\,(t_f) \quad = \quad 0$$

where (u_m, v_m, w_m) and (x_m, y_m, z_m) are the components of Mars' velocity vector and position vector at t_f, respectively.

APPENDIX E

CONSTANTS

1. Conversion factors

   a. Distance 1 AU $= 9.283 \times 10^{7}$ miles

   $= 1.494 \times 10^{11}$ meters

   $= 4.902 \times 10^{11}$ feet

   b. Velocity 1 AU/Day $= 3.868 \times 10^{6}$ miles/hr.

   $= 5.673 \times 10^{6}$ ft/sec

2. Mars Orbital Data

a.	Semimajor axis for orbit	a_m	1.523691 AU
b.	Eccentricity	e	0.093393
c.	Argument of ascending node		0 (for coordinates used)
d.	Angle of incliniation	i_m	0.032289 rad.
e.	Argument of perihelion	ω_m	5.8541335 rad.
f.	Eccentric anomaly (t_o)	F_m	4.250885 rad.

3. Earth Position and Velocity at 12:00 noon, May 9, 1971, (t_o)

Position

a.	x	$= x_4$	$= -0.99980$	AU
b.	y	$= x_5$	$= 2.009 \times 10^{-2}$	AU
c.	z	$= x_6$	$= 0.0$	AU

Velocity

a.	u	$= x_1$	$= -3.455906 \times 10^{-4}$	AU/day
b.	v	$= x_2$	$= -1.719868 \times 10^{-2}$	AU/day

c. $w = x_3 = 0.0$ AU/day

4. Vehicle thrust constants

   a. Mass flow rate — 0.00108 vehicle mass/day

   b. Exhaust Velocity — 0.045365 AU/day

5. Solar Gravitational Constant $\mu = 2.96007536 \times 10^{-4} \frac{AU^3}{DAY^2}$

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VITA

Wallace Thomas Fowler was born in Greenville, Texas, on August 27, 1938, the first son of Winton Thomas and Bessie Routh Fowler. He attended public schools in Greenville, Texas, and was graduated from Greenville High School in June 1956. He attended Rice Institute in Houston, Texas, from September 1956 to June 1957. In September 1957, he enrolled in The University of Texas. In June 1960, he received a Bachelor of Arts degree in Mathematics and entered graduate school at The University of Texas. In August 1961, he received a Master of Science degree in Engineering Mechanics, at which time he began work toward the doctorate in this field.

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