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TOLERANCE AND CONFIDENCE LIMITS FOR CLASSES OF DISTRIBUTIONS BASED ON FAILURE RATE

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PREFACE

This Memorandum stems from RAND's continuing interest in the assessment of reliability. It reports the results of research on the mathematical theory of tolerance and confidence limits.

The body of the Memorandum is addressed primarily to mathematical statisticians. The Summary is longer and more detailed than usual in RAND Memoranda. In addition to summarizing the contents of this Memorandum, an effort is made to relate the results to relevant earlier studies [3],[4]. The Summary is written for the user of statistical procedures who may not be a mathematician.

The authors, who have been consultants to The RAND Corporation, performed some of the research for this Memorandum during the course of the reliability assessment study that RAND is conducting for the Apollo Reliability and Quality Office, Hq NASA, under Contract NASr-21(11). Another part was done under the auspices of the Boeing Scientific Research Laboratories. Therefore, the results described herein are also being disseminated in essentially the same form, save for this Preface and for the Summary, as Mathematical Note No. 446 (DI-82-0503) by the Boeing Scientific Research Laboratories, Seattle, Washington.

SUMMARY

This Memorandum extends the validity of exponential tolerance and confidence limits, under certain restrictions, to classes of distributions based on failure rate. In particular, the usual exponential lower tolerance limit is shown to be conservative for the increasing failure rate class of distributions in the range of population coverages and confidence coefficients of practical interest. Conservative confidence limits are also obtained on the mean.

The rest of this Summary will be devoted to putting the results of this Memorandum in perspective for the user of statistical procedures who may not be a mathematician. This involves separating the material (definitions, theorems, corollaries) that pertains directly to tolerance and confidence limits from the techniques used in obtaining these results. Also, comparisons with "standard" procedures will be made. Certain preliminaries are needed.

Let $\underline{X} = (X_1, X_2, \ldots, X_n)$ denote an ordered (i.e., $0 \le X_1 \le X_2 \le \ldots \le X_n$) sample of times to failure of an item with distribution F. If L is a function such that

$$P_F\{1 - F[L(\underline{X})] \ge 1-q\} = 1-\alpha,$$

we say that $[L(\underline{X}), \infty)$ is a lower tolerance interval (or that $L(\underline{X})$ is a lower tolerance limit) for the population, with coverage 1-q and confidence coefficient $1-\alpha$. That is, the probability equals $1-\alpha$ that the interval $[L(\underline{X}), \infty)$ covers at least a fraction 1-q of the population of lifetimes. Similarly, if U is a function such that

$$P_F\{F[U(\underline{X})] \geq q\} = 1-\alpha,$$

we say that $[0,U(\underline{X})]$ is an upper tolerance interval (or that $U(\underline{X})$ is an upper tolerance limit) for the population, with coverage q and confidence coefficient $1-\alpha$.

The q^{th} percentile, ξ_q, of a continuous distribution F is defined by

$$F(\xi_q) = q.$$

If a function S_1 satisfies

$$P_F\{\xi_q \geq S_1(\underline{X})\} = 1-\alpha,$$

we say that $[S_1(\underline{X}),\infty)$ is a $100(1-\alpha)$ percent lower confidence interval (or that $S_1(\underline{X})$ is a lower confidence limit) for ξ_q. Similarly, if a function S_2 satisfies

$$P_F\{\xi_q \leq S_2(\underline{X})\} = 1-\alpha,$$

we say that $[0,S_2(\underline{X})]$ is a $100(1-\alpha)$ percent upper confidence interval (or that $S_2(\underline{X})$ is an upper confidence limit) for ξ_q.

If, in the above definitions for tolerance and confidence limits, the probabilities on the left-hand side are greater than or equal to ($\geq$) instead of equal to ($=$) $1-\alpha$, then the adjective "conservative" is added to the term being defined. Thus one speaks of a conservative lower tolerance interval with coverage 1-q and confidence coefficient $1-\alpha$, of a conservative $100(1-\alpha)$ percent lower confidence limit for the q^{th} percentile, etc.

A lower tolerance limit with coverage 1-q and confidence coefficient $1-\alpha$ is a $100(1-\alpha)$ percent lower confidence limit for the percentile ξ_q. Similarly, an upper tolerance limit with coverage q and confidence coefficient $1-\alpha$ is a $100(1-\alpha)$ percent upper confidence limit for the percentile ξ_q.

A censored sampling plan is one in which a fixed number of items, say n, are placed on life test and the testing is terminated when a fixed number of them, r $(1 \le r \le n)$, have failed. If the life distribution is exponential with mean θ, then the maximum likelihood estimate of θ based on the censored sample is

$$\hat{\theta}_{r,n}(\underline{X}) = \frac{1}{r}\left[\sum_{1}^{r} X_i + (n-r)X_r\right].$$

This can also be written, as on p. 6 of this Memorandum, with $X_0 = 0$, as

$$\hat{\theta}_{r,n}(\underline{X}) = \sum_{1}^{r} \frac{n-i+1}{r} (X_i - X_{i-1}).$$

For the exponential distribution with mean θ, based on a sampling plan censored at r out of n observations, the following facts hold:*

(i) A lower tolerance limit with coverage 1-q and confidence coefficient $1-\alpha$ is given by

$$\frac{-2r \log (1-q)\ \hat{\theta}_{r,n}(\underline{X})}{\chi^2_{1-\alpha}(2r)}.$$

*In what follows, $\chi^2_\beta(m)$ denotes the β^{th} percentile of the chi-square distribution with m degrees of freedom.

This is also a 100(1-α) percent lower confidence limit for ξ_q, the q^{th} percentile.

(ii) An upper tolerance limit with coverage q and confidence coefficient 1-α is given by

$$\frac{-2r \log (1-q)\ \hat{\theta}_{r,n}(\underline{X})}{\chi^2_{\alpha}(2r)}$$

This is also a 100(1-α) percent upper confidence limit for ξ_q, the q^{th} percentile.

(iii) A lower 100(1-α) percent confidence limit for the mean, θ, is given by

$$\frac{2r\ \hat{\theta}_{r,n}(\underline{X})}{\chi^2_{1-\alpha}(2r)} .$$

(iv) An upper 100(1-α) percent confidence limit for the mean, θ, is given by

$$\frac{2r\ \hat{\theta}_{r,n}(\underline{X})}{\chi^2_{\alpha}(2r)} .$$

Definitions of increasing failure rate (IFR) distributions, decreasing failure rate (DFR) distributions, distributions whose failure rate increases on the average (IFRA), and distributions whose failure rate decreases on the average (DFRA), are given on pages 3 and 4. If a distribution is IFR (DFR), it is also IFRA (DFRA), but the converse is not, in general, true. An example is given on p. 4 showing an IFRA distribution that is not an IFR distribution.

The exponential distribution belongs to each of the four classes of distributions -- its failure rate being constant.

The remainder of this Summary is devoted to relating the results of this Memorandum to the "standard" results listed in (i) through (iv), above.

Section 2 gives lower tolerance limits, and lower confidence limits for percentiles and for the mean for IFR or IFRA distributions. Theorem 2.3, p. 7, states that if, for IFR distributions, $\chi^2_{1-\alpha}(2r) \geq -2n \log (1-q)$, the exponential lower tolerance limit in (i) above is a conservative lower tolerance limit; if $\chi^2_{1-\alpha}(2r) \leq -2n \log (1-q)$, then $\frac{r}{n} \hat{\theta}_{r,n}(\underline{X})$ provides the conservative lower tolerance limit. Similarly for the lower confidence limit for a percentile. Corollary 2.4, p. 9, states that if, for an IFR distribution, $1-\alpha \geq 1-e^{-1}$ and $1-q \geq e^{-r/n}$, then the results in (i) provide conservative lower tolerance and confidence limits. Corollary 2.7, p. 10, states that the results of Theorem 2.4 hold even for IFRA distributions, provided r = 1; that is, only the earliest failure time is used. Theorem 2.8, p. 10, provides a conservative lower confidence limit for the mean of an IFR distribution. It differs from the result stated in (iii) by a factor

$$1 - \exp \frac{-\chi^2_{1-\alpha}(2r)}{2n} .$$

This factor being less than unity, the lower limit of Theorem 2.8 is less than that of (iii).

Section 3 gives upper tolerance limits, and upper confidence limits for percentiles and for the mean of IFR or IFRA distributions.

Theorem 3.3, p. 14, states that if for IFRA distributions, $\chi^2_\alpha(2r) \le -2(n-r+1) \log (1-q)$, then the exponential upper tolerance limit of (ii) above is a conservative upper tolerance limit; if $\chi^2_\alpha(2r) \ge -2(n-r+1) \log (1-q)$, then $\frac{r}{n-r+1} \hat{\theta}_{r,n}(\underline{X})$ provides the conservative upper tolerance limit. Similarly for the upper confidence limit for a percentile. Corollary 3.4, p. 16, states that if for an IFRA distribution, $1-\alpha \ge 1-e^{-1}$ and $q \ge 1 - \exp\{-\frac{r}{n-r+1}\}$, then the results in (ii) provide conservative upper tolerance and confidence limits. Theorem 3.3 and Corollary 3.4, holding for IFRA distributions, hold, _a fortiori_, for IFR distributions. Theorem 3.5, p. 16, states that for IFR distributions, the upper confidence limit for the mean given in (iv) is conservative if $\chi^2_\alpha(2r) \le 2(n-r+1)$; if $\chi^2_\alpha(2r) \ge 2(n-r+1)$, then $\frac{r}{n-r+1} \hat{\theta}_{r,n}(\underline{X})$ provides the upper confidence limit. Corollary 3.6, p. 19, states that for IFR distributions, if $1-\alpha > 1-e^{-1}$ and $r \le \frac{n+1}{2}$, then the upper confidence limit for the mean given in (iv) is conservative.

Section 4 gives upper and lower tolerance limits, and a lower confidence limit for the mean for DFR or DFRA distributions. Theorem 4.1, p. 20, states that if, for DFRA distributions, $\chi^2_{1-\alpha}(2r) \le -2(n-r+1) \log (1-q)$, the exponential lower tolerance limit in (i) above is conservative; if $\chi^2_{1-\alpha}(2r) \ge -2(n-r+1) \log (1-q)$, then $\frac{r}{n-r+1} \hat{\theta}_{r,n}(\underline{X})$ provides the conservative lower tolerance limit. Theorem 4.2, p. 21, states that if, for DFR distributions, $\chi^2_\alpha(2r) \ge -2n \log (1-q)$, then the exponential upper tolerance limit in (ii) is conservative; if $\chi^2_\alpha(2r) \le -2n \log (1-q)$, then $\frac{r}{n} \hat{\theta}_{r,n}(\underline{X})$ provides the conservative upper tolerance limit. Finally, Theorem 4.3, p. 21, states for DFR distributions, if $\chi^2_{1-\alpha}(2r) \leq 2(n-r+1)$, then the exponential lower confidence limit for the mean given in (iii) is conservative; if $\chi^2_{1-\alpha}(2r) \geq 2(n-r+1)$, then a conservative lower confidence limit for the mean is given by

$$\frac{r}{n-r+1} \exp\left[1 - \frac{\chi^2_{1-\alpha}(2r)}{2(n-r+1)}\right] \hat{\theta}_{r,n}(\underline{X}) .$$

CONTENTS

1. INTRODUCTION

A fundamental problem in statistical reliability theory and life testing is to obtain lower tolerance limits as a function of sample data, say $\underline{X} = (X_1, X_2, \ldots, X_n)$. That is, if X denotes the time to failure of an item with distribution F, then we seek a function $L(\underline{X})$ such that

$$P_F\{1 - F[L(\underline{X})] \geq 1 - q\} \geq 1 - \alpha.$$

We call $1 - q$ the population coverage for the interval $[L(\underline{X}), \infty)$, and $1 - \alpha$ the confidence coefficient. Also, we want $U(\underline{X})$ such that $P_F\{F[U(\underline{X})] \geq q\} \geq 1 - \alpha$. Related problems are those of obtaining confidence limits on moments and percentiles.

Parametric tolerance limits based on the normal and exponential distributions are well known.[1][2][3] Goodman and Madansky[4] examine various criteria for goodness of tolerance intervals and certain optimum properties of the usual exponential tolerance limits are demonstrated. Recently, a great deal of effort has been devoted to obtaining various confidence limits for the Weibull distribution. Dubey[5] obtains asymptotic confidence limits on $1 - F(T)$ and the failure rate for the class of Weibull distributions with non-decreasing failure rate. He also studies the properties of various estimators for Weibull parameters.[6] Johns and Lieberman[7] present a method for obtaining exact lower confidence limits for $1 - F(T)$ when F is the Weibull distribution with both scale and shape parameters unknown. Unlike Dubey, they do not require that the Weibull distribution in question have a non-decreasing failure rate. These confidence limits are obtained both for the censored and non-censored cases and are asymptotically efficient.

There exist distribution-free tolerance limits[8] based on say, the k^{th} order statistic X_k for certain values of q, α, k and sample size N. They have one unfortunate disadvantage, however. For given α, q, k there is a minimum sample size $N(\alpha, q, k)$ such that

$$P_F\{1 - F(X_k) \geq 1 - q\} \geq 1 - \alpha$$

is true only if $N \geq N(\alpha, q, k)$. Hanson and Koopmans[9] obtain upper tolerance limits for the class of distributions with increasing hazard rate, and lower tolerance limits for the class of distributions with PF_2 density, f (i.e., log f(x) is concave where finite). They do not assume non-negative random variables as we do. In Ref. 10, sharper results for distributions with monotone failure rate are obtained. This Memorandum extends and generalizes the results of Ref. 10 and in the process provides more elegant proofs.

Assuming that the sample data arise from a distribution with monotone failure rate (either non-decreasing or non-increasing and $F(0^-) = 0$) or with monotone failure rate average, we obtain conservative confidence limits for most reliability parameters of interest.*
These confidence limits are, in part, derived as in the case of the exponential distribution. In many instances these are optimum confidence limits when the failure distribution is actually exponential

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*See Chapter 2 and Appendix 2 of Ref. 11 for a discussion of such distributions and a test for monotone failure rate.

(Goodman and Madansky[4]). They also have the advantage of being convenient to compute and are not based on a strong, non-verifiable, parametric assumption.

PRELIMINARIES

Throughout this Memorandum we use the following notation and assumptions. Let $0 \le X_1 \le X_2 \le \ldots \le X_n$ $(0 \le Y_1 \le Y_2 \le \ldots \le Y_n)$ denote an ordered sample from a distribution F (G), and define $X_0 = Y_0 = 0$. We assume that F is continuous, $F(0) = G(0) = 0$, and let $G(x) = 1 - e^{-x}$ for $x \ge 0$. We say that a distribution F with density f is an increasing failure rate (IFR) distribution if its failure rate $r(t) = f(t)/[1 - F(t)]$ is increasing. It is easy to verify that if F is IFR, $G^{-1}F(t) = -\log[1 - F(t)]$ is convex where finite. This motivates the more general definition: We say that F is IFR if $-\log[1 - F(t)]$ is convex where finite. Similarly, F is a decreasing failure rate (DFR) distribution if $G^{-1}F(t)$ is concave on $[0,\infty)$. Barlow and Proschan[12] obtain inequalities for expected values of statistics based on the exponential assumption when, in fact, the true distribution has a monotone failure rate.

We will also be interested in a considerably weaker restriction on F. If F has a density f and failure rate r(x) such that

$$\frac{1}{t}\int_0^t r(x)\ dx$$

is increasing (decreasing) in t, we say that F has an increasing (decreasing) failure rate average. We write F is IFRA (DFRA).

More generally, F is IFRA (DFRA) if and only if

$$\frac{G^{-1}F(x)}{x} \equiv \frac{-\log[1-F(x)]}{x}$$

is increasing where finite (decreasing on $[0,\infty)$). See Ref. 13 for additional properties of this class. If F is IFR (DFR) and $F(0) = 0$, then it follows that F is IFRA (DFRA).

Perhaps a simple example will motivate the IFRA class of distributions. Let

$$F(x) = \begin{cases} 0 & , x < 0 \\ (1-e^{-x})(1-e^{-kx}), & x \geq 0 \end{cases}$$

where $k > 1$. Then it is easy to check that F is IFRA but <u>not</u> IFR. This is the life distribution of a structure composed of two substructures in parallel, the first having k components in series, the second consisting of a single component, with component life lengths independently distributed according to the unit exponential distribution. Any "reasonable" structure built from components having exponential or IFR failure distributions will have an IFRA failure distribution (cf. Ref. 13).*

We use the symbol $\overset{st}{\leq}$ for stochastic inequality and $\overset{st}{=}$ for stochastic equivalence.

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*The preceding example also shows that a structure built from DFR components will not, in general, be DFRA.

2. LOWER CONFIDENCE LIMITS

To obtain lower confidence limits we need the following lemma which is proved in Ref. 15.

Lemma 2.1:

$$\phi\left[\sum_{1}^{n} a_i x_i\right] \leq \sum_{1}^{n} a_i \phi(x_i)$$

for all convex ϕ such that $\phi(0) \leq 0$, and for all $0 \leq x_1 \leq \ldots \leq x_n$ if and only if

$$0 \leq \sum_{j=i}^{n} a_j \leq 1$$

for $i = 1, 2, \ldots, n$. Furthermore, if

$$\phi\left[\sum_{1}^{n} a_i x_i\right] \leq \sum_{1}^{n} a_i \phi(x_i)$$

whenever $0 \leq x_1 \leq \ldots \leq x_n$, and

$$0 \leq \sum_{j=i}^{n} a_j \leq 1$$

for $i = 1, 2, \ldots, n$, then ϕ is convex and $\phi(0) \leq 0$.

The following theorem, an immediate consequence of Lemma 2.1, is the key tool used in obtaining lower confidence limits.

<u>THEOREM 2.2</u>: If $G^{-1}F$ is convex on the support of F, F(0) = 0 = G(0) and

$$0 \leq \sum_{j=i}^{n} a_j \leq 1$$

for i = 1, 2, ..., n, then

$$F\left[\sum_1^n a_i X_i\right] \overset{st}{\leq} G\left[\sum_1^n a_i Y_i\right].$$

<u>Proof</u>: By the previous lemma

$$G^{-1}F\left[\sum_1^n a_i X_i\right] \leq \sum_1^n a_i G^{-1}F(X_i) \equiv \sum_1^n a_i Y_i',$$

where Y_1', ..., Y_n' are jointly distributed as the order statistics from G. By applying a lemma in Lehman[16, p. 73] we have the result.||

Let

$$\hat{\theta}_{r,n}(\underline{X}) = \sum_1^r \frac{(n-i+1)}{r} (X_i - X_{i-1}),$$

and let $\chi^2_{1-\alpha}(2r)$ denote the $(1-\alpha)100$ percent point of the chi-square distribution with 2r degrees of freedom. If

$$L(\underline{X}) = \frac{-2r \log (1-q)}{\chi^2_{1-\alpha}(2r)} \hat{\theta}_{r,n}(\underline{X}),$$

then

$$P_G\{1 - G[L(\underline{Y})] \geq 1 - q\} = 1 - \alpha.$$

Also define

$$C_{1-\alpha,q}(r) = \begin{cases} \dfrac{-2r \log (1-q)}{\chi^2_{1-\alpha}(2r)} & \text{if } \chi^2_{1-\alpha}(2r) \geq -2n \log (1-q) \\ \dfrac{r}{n} & \text{if } \chi^2_{1-\alpha}(2r) \leq -2n \log (1-q) . \end{cases}$$

<u>THEOREM 2.3</u>: If F is IFR, $F(0) = 0$, $F(\xi_q) = q$, then

(2.1) $$P_F\{1 - F[C_{1-\alpha,q}(r)\hat{\theta}_{r,n}] \geq 1 - q\} \geq 1 - \alpha,$$

or equivalently,

(2.2) $$P_F\{\xi_q \geq C_{1-\alpha,q}(r)\hat{\theta}_{r,n}\} \geq 1 - \alpha.$$

<u>Proof</u>: Since (2.1) and (2.2) are equivalent, we need only show (2.1). Note that

$$\sum_1^r a_i X_i = \sum_1^r A_i (X_i - X_{i-1})$$

where

$$A_i = \sum_{j=i}^r a_j .$$

By Theorem 2.2,

$$F\left[\sum_1^r A_i (X_i - X_{i-1})\right] \overset{st}{\leq} G\left[\sum_1^r A_i (Y_i - Y_{i-1})\right]$$

when $0 \le A_i \le 1$ for $i = 1, 2, \ldots, r$. Choosing

$$A_i = \frac{-2 \log (1-q)(n-i+1)}{\chi^2_{1-\alpha}(2r)},$$

we have

$$F\left[\frac{-2 \log (1-q)}{\chi^2_{1-\alpha}(2r)} \sum_1^r (n-i+1)(X_i - X_{i-1})\right] \overset{st}{\le} G\left[\frac{-2 \log (1-q)}{\chi^2_{1-\alpha}(2r)} \sum_1^r (n-i+1)(Y_i - Y_{i-1})\right]$$

when

$$\frac{-2n \log (1-q)}{\chi^2_{1-\alpha}(2r)} \le 1.$$

It follows that, in this case,

$$P_F\left\{1 - F\left[\frac{-2r \log (1-q)}{\chi^2_{1-\alpha}(2r)} \hat{\theta}_{r,n}\right] \ge 1 - q\right\} \ge 1 - \alpha.$$

If

$$\frac{-2n \log (1-q)}{\chi^2_{1-\alpha}(2r)} > 1,$$

then let $A_i = \frac{(n-i+1)}{n}$, so that

$$F\left[\frac{1}{n} \sum_1^r (n-i+1)(X_i - X_{i-1})\right] \overset{st}{\le} G\left[\frac{1}{n} \sum_1^r (n-i+1)(Y_i - Y_{i-1})\right].$$

Also,

$$P_G\left\{1 - G\left[\frac{1}{n}\sum_1^r (n-i+1)(Y_i - Y_{i-1})\right] \geq 1 - q\right\} \geq P_G\left\{1 - G[L(\underline{Y})] \geq 1 - q\right\} = 1 - \alpha$$

so that (2.1) follows.||

Corollary 2.4: If F is IFR, $1-\alpha \geq 1 - e^{-1}$ and $1 - q \geq e^{-r/n}$, then

$$P_F\left\{1 - F\left[\frac{-2r \log (1-q)}{\chi^2_{1-\alpha}(2r)}\hat{\theta}_{r,n}\right] \geq 1 - q\right\} \geq 1 - \alpha.$$

Proof: By Theorem 2.3 we need only show

$$\frac{\chi^2_{1-\alpha}(2r)}{2r} \geq -\frac{n}{r}\log (1-q).$$

Let H denote the chi-square distribution with 2r degrees of freedom. Since H is IFR, $H(2r) \leq 1 - e^{-1}$. This implies $\chi^2_{1-\alpha}(2r) \geq 2r$, i.e., $\chi^2_{1-\alpha}(2r)/2r \geq 1$ when $1 - \alpha > 1 - e^{-1}$. Since $1 - q \geq e^{-r/n}$, then $-\frac{n}{r}\log (1-q) \leq 1$. The result follows.||

Theorem 2.3 can be partially extended to IFRA distributions by using the following lemma proved in Ref. 15.

Lemma 2.5:

$$\phi\left[\sum_1^n a_i x_i\right] \leq \sum_1^n a_i\, \phi(x_i)$$

for all $0 \leq x_1 \leq x_2 \leq \ldots \leq x_n$ and for all ϕ such that $\phi(x)/x$ is increasing in x if and only if for some k $(1 \leq k \leq n)$, $0 \leq A_1 \leq A_2 \leq \ldots \leq A_k \leq 1$ and $A_{k+1} = \ldots = A_n = 0$, with A_i defined on p. 7.

The following theorem and its corollary are an immediate consequence of Lemma 2.5.

THEOREM 2.6: If $[G^{-1}F(x)]/x$ is increasing on the support of F, and $0 \le A_1 \le A_2 \le \ldots \le A_k \le 1$; $A_{k+1} = \ldots = A_n = 0$ for some k $(1 \le k \le n)$, then

$$F\left[\sum_1^n a_i X_i\right] \overset{st}{\le} G\left[\sum_1^n a_i Y_i\right].$$

Corollary 2.7: If F is IFRA, $F(0) = 0$ and $F(\xi_q) = q$, then

$$P_F\left\{1 - F\left[C_{1-\alpha,q}(1)X_1\right] \ge 1 - q\right\} \ge 1 - \alpha,$$

or equivalently,

$$P_F\left\{\xi_q \ge C_{1-\alpha,q}(1)X_1\right\} \ge 1 - \alpha.$$

THEOREM 2.8: If F is IFR and $\theta = \int_0^\infty x \, dF(x)$, then

$$P_F\left\{\theta \ge \left[1 - \exp\left(\frac{-\chi^2_{1-\alpha}(2r)}{2n}\right)\right]\frac{2r}{\chi^2_{1-\alpha}(2r)}\,\hat{\theta}_{r,n}\right\} \ge 1 - \alpha.$$

Proof: We use the bound

$$F(t;\theta) \ge b(t;\theta) = \begin{cases} 0 & , \ t < \theta \\ 1 - e^{-wt}, & t \ge \theta \end{cases}$$

where w depends on t and satisfies $\int_0^t e^{-wx}\,dx = \theta$; see Ref. 11, p. 28.

By Theorem 2.2,

$$G\left[\sum_1^r A_i\,(Y_i-Y_{i-1})\right] \overset{st}{\geq} F\left[\sum_1^r A_i\,(X_i-X_{i-1});\theta\right] \overset{st}{\geq} b\left[\sum_1^r A_i\,(X_i-X_{i-1});\theta\right]$$

if $0 \leq A_i \leq 1$ for $i = 1, 2, \ldots, r$. Choose $k_{1-\alpha}$ so that

$$\text{(2.3)} \qquad P_G\left\{G\left[\sum_1^r A_i\,(Y_i-Y_{i-1})\right] \leq k_{1-\alpha}\right\} = 1-\alpha .$$

Then

$$P_F\left\{b\left[\sum_1^r A_i\,(X_i-X_{i-1});\theta\right] \leq k_{1-\alpha}\right\} \geq 1-\alpha .$$

Thus, since for $t < \theta$, $b(t;\theta) = 1-e^{-wt}$ (where $w(\theta)$ satisfies $\frac{1-e^{-wt}}{w} = \theta$),

$$P_F\left\{w(\theta) \leq \frac{-\log\,(1-k_{1-\alpha})}{\sum_1^r A_i\,(X_i-X_{i-1})}\right\} \geq 1-\alpha .$$

Since $w(\theta)$ is decreasing in θ, using the condition just above governing $w(\theta)$, we find

$$P_F\left\{\theta \geq \frac{k_{1-\alpha}\sum_1^r A_i\,(X_i-X_{i-1})}{-\log\,(1-k_{1-\alpha})}\right\} \geq 1-\alpha .$$

Now choose $A_i = c(n-i+1)$ where $0 \leq c \leq 1/n$. Hence (2.3) becomes

$$P_G\left\{1-\exp\left[-c\sum_1^r (n-i+1)\,(Y_i-Y_{i-1})\right] \leq k_{1-\alpha}\right\} = 1-\alpha,$$

implying

$$\frac{-\ 2\ \log\ (1-k_{1-\alpha})}{c} = \chi^2_{1-\alpha}(2r)\,,$$

or

$$k_{1-\alpha} = 1 - \exp\left[\frac{-\ c\chi^2_{1-\alpha}(2r)}{2}\right]\,.$$

Therefore,

$$P_F\left\{\theta \geq \left[1 - \exp\left(\frac{-\ c\chi^2_{1-\alpha}(2r)}{2}\right)\right]\frac{2r}{\chi^2_{1-\alpha}(2r)}\,\hat{\theta}_{r,n}\right\} \geq 1-\alpha\,.$$

To maximize the bound subject to $c \leq 1/n$, set $c = 1/n$.||

3. UPPER CONFIDENCE LIMITS

To obtain upper confidence limits, we need the following lemma which is proved in Ref. 15.

Lemma 3.1: Assume $\phi(x)/x$ is increasing in $x \in [0,b]$. Then

$$\phi\left[\sum_1^n a_i x_i\right] \geq \sum_1^n a_i \phi_i(x_i),$$

for all $0 \equiv x_0 \leq x_1 \leq \ldots \leq x_n \leq b$ and

$$0 \leq \sum_1^n a_i x_i \leq b,$$

if and only if for some k $(0 \leq k \leq n)$, $a_1 \geq 0$, $a_2 \geq 0$, ..., $a_{k-1} \geq 0$, $a_k \geq 1$, $a_{k+1} = a_{k+2} = \ldots = a_n = 0$.

The following theorem is the key tool used in obtaining upper confidence limits:

THEOREM 3.2: If $G^{-1}F(x)/x$ is increasing on the support of F and $a_n \geq 1$, $a_i \geq 0$, for i = 1, 2, ..., n, then

$$F\left[\sum_1^n a_i X_i\right] \overset{st}{\geq} G\left[\sum_1^n a_i Y_i\right].$$

Proof: By the previous lemma

$$G^{-1}F\left[\sum_1^n a_i X_i\right] \geq \sum_1^n a_i G^{-1}F(X_i) \equiv \sum_1^n a_i Y_i',$$

where $Y_1', \ldots, Y_n'$ are jointly distributed as the order statistics from G. By applying a lemma in Lehman[16, p. 73] we have the result.||

Note that F is IFRA if and only if $G^{-1}F(x)/x$ is increasing on the support of F when $F(0) = 0$ and

$$G(x) = \begin{cases} 1 - e^{-x}, & x \geq 0 \\ 0 & , x < 0. \end{cases}$$

It will be convenient to let

$$C^*_{\alpha,q}(r) = \begin{cases} \dfrac{-2r \log(1-q)}{\chi^2_\alpha(2r)}, & \text{if } \chi^2_\alpha(2r) \leq -2(n-r+1) \log(1-q) \\ \dfrac{r}{n-r+1}, & \text{if } \chi^2_\alpha(2r) \geq -2(n-r+1) \log(1-q). \end{cases}$$

<u>THEOREM 3.3</u>: If F is IFRA, $F(0) = 0$ and $F(\xi_q) = q$, then

$$P_F\left\{F\left[C^*_{\alpha,q}(r)\hat{\theta}_{r,n}\right] \geq q\right\} \geq 1-\alpha, \tag{3.1}$$

or equivalently,

$$P_F\left\{\xi_q \leq C^*_{\alpha,q}(r)\hat{\theta}_{r,n}\right\} \geq 1-\alpha. \tag{3.2}$$

<u>Proof:</u> Since (3.1) and (3.2) are equivalent, we need only show (3.1). Let

$$A_i = \sum_{j=i}^{r} a_j$$

as before. By Theorem 3.2,

$$F\left[\sum_{1}^{r} A_i (X_i - X_{i-1})\right] \overset{st}{\geq} G\left[\sum_{1}^{r} A_i (Y_i - Y_{i-1})\right]$$

when $a_i \geq 0$ and $A_i \geq 1$ for $i = 1, 2, \ldots, r$. Hence

$$F\left[\frac{-2 \log (1-q)}{\chi_\alpha^2(2r)} \sum_{1}^{r} (n-i+1)(X_i - X_{i-1})\right] \overset{st}{\geq} G\left[\frac{-2 \log (1-q)}{\chi_\alpha^2(2r)} \sum_{1}^{r} (n-i+1)(Y_i - Y_{i-1})\right]$$

when

$$\frac{-2 \log (1-q)(n-r+1)}{\chi_\alpha^2(2r)} \geq 1.$$

It follows that in this case,

$$P_F\left\{F\left[\frac{-2r \log (1-q)}{\chi_\alpha^2(2r)} \hat{\theta}_{r,n}\right] \geq q\right\} \geq 1-\alpha.$$

If

$$\frac{-2(n-r+1) \log (1-q)}{\chi_\alpha^2(2r)} < 1,$$

then

$$P_G\left\{G\left[\frac{1}{(n-r+1)} \sum_{1}^{r} (n-i+1)(Y_i - Y_{i-1})\right] \geq q\right\}$$

$$\geq P_G\left\{G\left[\frac{-2 \log (1-q)}{\chi_\alpha^2(2r)} \sum_{1}^{r} (n-i+1)(Y_i - Y_{i-1})\right] \geq q\right\} = 1-\alpha,$$

and (3.1) follows.||

Corollary 3.4: If F is IFRA, $1-\alpha \geq 1-e^{-1}$ and $q \geq 1 - \exp\left\{-\frac{r}{n-r+1}\right\}$, then

$$P_F\left\{F\left[\frac{-2r \log(1-q)}{\chi^2_\alpha(2r)}\hat{\theta}_{r,n}\right] \geq q\right\} \geq 1-\alpha.$$

Proof: By Theorem 3.3 we need only show

$$\frac{\chi^2_\alpha(2r)}{2r} \leq \frac{-(n-r+1)}{r}\log(1-q).$$

Let H denote the chi-square distribution with 2r degrees of freedom. Since log H(x) is concave, $H(2r) \geq e^{-1}$ by Jensen's inequality, which implies $\chi^2_\alpha(2r) \leq 2r$, or $\chi^2_\alpha(2r)/2r \leq 1$, when $1-\alpha > 1-e^{-1}$. Since $1 \leq \frac{-(n-r+1)}{r}\log(1-q)$ by hypothesis, the result follows.||

It will be convenient to let

$$c_{\alpha,r} = \begin{cases} \frac{r}{n-r+1}, & \text{if } \chi^2_\alpha(2r) \geq 2(n-r+1) \\ \frac{2r}{\chi^2_\alpha(2r)}, & \text{if } \chi^2_\alpha(2r) \leq 2(n-r+1). \end{cases}$$

THEOREM 3.5: If F is IFR and $\theta = \int_0^\infty x\,dF(x)$, then

$$P_F\left\{\theta \leq c_{\alpha,r}\hat{\theta}_{r,n}\right\} \geq 1-\alpha.$$

Proof: We use the bound

$$F(t;\theta) \le B(t;\theta) = \begin{cases} 1 - e^{-t/\theta}, & t < \theta \\ 1, & t \ge \theta. \end{cases}$$

(See Ref. 11, p. 27.) By Theorem 3.2,

$$G\left[\sum_{1}^{r} A_i (Y_i - Y_{i-1})\right] \overset{st}{\le} F\left[\sum_{1}^{r} A_i (X_i - X_{i-1});\theta\right] \overset{st}{\le} B\left[\sum_{1}^{r} A_i (X_i - X_{i-1});\theta\right]$$

if $a_i \ge 0$ and $A_i \ge 1$ for $i = 1, 2, \ldots, r$, where

$$A_i = \sum_{j=i}^{r} a_j .$$

Choose k_α so that

$$P_G\left\{G\left[\sum_{1}^{r} A_i (Y_i - Y_{i-1})\right] \ge k_\alpha\right\} = 1-\alpha .$$

Now let $A_i = c(n-i+1)$ for $i = 1, 2, \ldots, r$, where $c \ge \frac{1}{n-r+1}$. Hence as in the proof of Theorem 2.8,

$$-\log(1-k_\alpha) = \frac{c\chi_\alpha^2(2r)}{2}$$

and

$$1-k_\alpha = \exp\left[-\frac{c\chi_\alpha^2(2r)}{2}\right].$$

<u>Case 1</u>. $k_\alpha < 1 - e^{-1}$ or $\frac{c\chi_\alpha^2(2r)}{2} < 1$. Now

$$P_F\left\{B\left[c\sum_1^r (n-i+1)(X_i - X_{i-1});\theta\right] \geq k_\alpha\right\} \geq 1-\alpha ,$$

implying

$$P_F\left\{\theta \leq \frac{c}{-\log(1-k_\alpha)}\sum_1^r (n-i+1)(X_i - X_{i-1})\right\} \geq 1-\alpha,$$

or

$$P_F\left\{\theta \leq \frac{2r}{\chi_\alpha^2(2r)}\hat{\theta}_{r,n}\right\} \geq 1-\alpha.$$

We now choose c as small as possible subject to $c \geq \frac{1}{n-r+1}$; i.e., choose $c = \frac{1}{n-r+1}$. We do this so the exponential upper confidence bound will be valid for as many combinations of α and r as possible.

<u>Case 2</u>. $k_\alpha > 1 - e^{-1}$ or $\chi_\alpha^2(2r) \geq 2(n-r+1)$. Now

$$P_F\left\{B\left[\frac{r}{n-r+1}\hat{\theta}_{r,n};\theta\right] \geq k_\alpha\right\} \geq 1-\alpha,$$

which implies

$$P_F\left\{\theta \leq \frac{r}{n-r+1}\hat{\theta}_{r,n}\right\} \geq 1-\alpha. \|$$

Confidence bounds on θ assuming F IFRA can be similarly derived using the probability bounds in Ref. 14.

Corollary 3.6: If F is IFR, $\theta = \int_0^\infty x\,dF(x)$, $1-\alpha > 1-e^{-1}$, and $r \le \frac{n+1}{2}$, then

$$P_F\left\{\theta \le \frac{2r}{\chi^2_\alpha(2r)}\,\hat{\theta}_{r,n}\right\} \ge 1-\alpha .$$

Proof: By Theorem 3.5 we need only show $\chi^2_\alpha(2r) \le 2(n-r+1)$. As in the proof of Corollary 3.4, $\chi^2_\alpha(2r)/2r \le 1$ when $1-\alpha > 1-e^{-1}$. Since $\frac{n-r+1}{r} \ge 1$ when $r \le \frac{n+1}{2}$, the result follows.||

4. CONFIDENCE LIMITS FOR DFR DISTRIBUTIONS

Confidence limits for DFR and DFRA distributions can also be obtained using the techniques of the previous sections.

Let

$$C^{**}_{1-\alpha,q}(r) = \begin{cases} \dfrac{-2r \log(1-q)}{\chi^2_{1-\alpha}(2r)}, & \text{if } \chi^2_{1-\alpha}(2r) \le -2(n-r+1)\log(1-q) \\[2ex] \dfrac{r}{n-r+1}, & \text{if } \chi^2_{1-\alpha}(2r) > -2(n-r+1)\log(1-q). \end{cases}$$

THEOREM 4.1: If F is DFRA, then

$$P_F\left\{1-F\left[C^{**}_{1-\alpha,q}(r)\,\hat{\theta}_{r,n}\right] \ge 1-q\right\} \ge 1-\alpha.$$

Proof: The proof is similar to the proof of Theorem 3.3 where now $F^{-1}G(x)/x$ is increasing in $x \ge 0$. Hence

$$G\left[\sum_1^r A_i\,(Y_i-Y_{i-1})\right] \overset{st}{\ge} F\left[\sum_1^r A_i\,(X_i-X_{i-1})\right]$$

when $A_i \ge 1$ for $i = 1, 2, \ldots, r$, by Theorem 3.2. Letting

$$A_i = \frac{-2(n-i+1)}{\chi^2_{1-\alpha}(2r)} \log(1-q),$$

we see that

$$P_F\left\{1-F\left[\frac{-2\log(1-q)}{\chi^2_{1-\alpha}(2r)}\sum_1^r (n-i+1)(X_i-X_{i-1})\right] \ge 1-q\right\} \ge 1-\alpha,$$

when

$$\frac{-2(n-r+1)\log(1-q)}{\chi^2_{1-\alpha}(2r)} \geq 1.$$

The remainder of the proof is obvious.||

The upper tolerance limits for DFR distributions are not as useful. Let

$$c^{***}_{\alpha,q}(r) = \begin{cases} \dfrac{-2r\log(1-q)}{\chi^2_{\alpha}(2r)}, & \text{if } \chi^2_{\alpha}(2r) > -2n\log(1-q) \\ \dfrac{r}{n}, & \text{if } \chi^2_{\alpha}(2r) \leq -2n\log(1-q). \end{cases}$$

THEOREM 4.2: If F is DFR, then

$$P_F\left\{F\left[c^{***}_{\alpha,q}(r)\hat{\theta}_{r,n}\right] \geq q\right\} \geq 1-\alpha.$$

We omit the proof since it is similar to previous proofs.

Let

$$c^{*}_{\alpha,r} = \begin{cases} \dfrac{2r}{\chi^2_{1-\alpha}(2r)}, & \text{if } \chi^2_{1-\alpha}(2r) \leq 2(n-r+1) \\ \dfrac{r}{n-r+1}\exp\left[1-\dfrac{\chi^2_{1-\alpha}(2r)}{2(n-r+1)}\right], & \text{if } \chi^2_{1-\alpha}(2r) \geq 2(n-r+1). \end{cases}$$

THEOREM 4.3: If F is DFR and $\theta = \int_0^\infty x\,dF(x) < \infty$, then

$$P_F\left\{\theta \geq c^{*}_{\alpha,r}\hat{\theta}_{r,n}\right\} \geq 1-\alpha.$$

Proof: We use the bound[11, p.31]

$$F(t;\theta) \geq b(t;\theta) = \begin{cases} 1 - e^{-t/\theta}, & t \leq \theta \\ 1 - \dfrac{\theta e^{-1}}{t}, & t > \theta. \end{cases}$$

By Theorem 3.2 with G and F interchanged,

$$G\left[\sum_1^r A_i\,(Y_i - Y_{i-1})\right] \overset{st}{\geq} F\left[\sum_1^r A_i\,(X_i - X_{i-1});\theta\right] \overset{st}{\geq} b\left[\sum_1^r A_i\,(X_i - X_{i-1});\theta\right]$$

when $A_i \geq 1$ for $i = 1, 2, \ldots, r$.

Choose $k_{1-\alpha}$ so that

$$P_G\left\{G\left[\sum_1^r A_i\,(Y_i - Y_{i-1})\right] \leq k_{1-\alpha}\right\} = 1-\alpha.$$

Let $A_i = c(n-i+1)$ for $i = 1, 2, \ldots, r$ so that $c \geq \frac{1}{n-r+1}$. Then it follows that

$$-2 \log\,(1-k_{1-\alpha}) = c\chi^2_{1-\alpha}(2r),$$

as in the proof of Theorem 2.8.

Case 1. $k_{1-\alpha} < 1 - e^{-1}$ or $\chi^2_{1-\alpha}(2r) \leq \frac{2}{c}$. Now

$$P_F\left\{1 - \exp\left[-\frac{c}{\theta}\sum_1^r (n-i+1)\,(X_i - X_{i-1})\right] \leq k_{1-\alpha}\right\} \geq 1-\alpha,$$

implying

$$P_F\left\{\theta \geq \frac{2r\hat{\theta}_{r,n}}{\chi^2_{1-\alpha}(2r)}\right\} \geq 1-\alpha.$$

We want to choose c as small as possible subject to $c \geq \frac{1}{n-r+1}$. Hence, let $c = \frac{1}{n-r+1}$.

<u>Case 2</u>. $k_{1-\alpha} \geq 1 - e^{-1}$ or $\chi^2_{1-\alpha}(2r) \geq 2(n-r+1)$. In this case

$$P_F\left\{1 - \frac{(n-r+1)e^{-1}\theta}{\sum_1^r (n-i+1)(X_i - X_{i-1})} \leq k_{1-\alpha}\right\} \geq 1-\alpha,$$

implying

$$P_F\left\{\theta \geq (1-k_{1-\alpha})\frac{er}{(n-r+1)}\,\hat{\theta}_{r,n}\right\} \geq 1-\alpha.$$

The bound is obtained by substituting for $1-k_{1-\alpha}$. ||

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