SID 66-46-2

A STUDY OF LONGITUDINAL OSCILLATIONS OF PROPELLANT TANKS AND WAVE PROPAGATIONS IN FEED LINES

Part II—Wave Propagation in Elastic Pipe Filled With Incompressible Viscous Fluid

21 January 1966 NAS8-11490

Prepared by

Michael M. H. Loh, Staff Investigator
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Approved by

F. C. Hung
Assistant Director, Structures and Dynamics

NORTH AMERICAN AVIATION, INC.
SPACE and INFORMATION SYSTEMS DIVISION

Accession No.

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FOREWORD

This report was prepared by the Space and Information Systems Division of North American Aviation, Inc., Downey, California, for the George C. Marshall Space Flight Center, National Aeronautics and Space Administration, Huntsville, Alabama, under Contract No. NAS8-11490, "Study of Longitudinal Oscillations of Propellant Tanks and Wave Propagations in Feed Lines," dated Januay 6, 1965. Dr. George F. McDonough (Principal) and Mr. Robert S. Ryan (Alternate) of Aero-Astrodynamics Laboratory, MSFC are Contracting Officer Representatives. The work is being published in five separate parts:

Part I - One-Dimensional Wave Propagation in a Feed Line

Part II - Wave Propagation in an Elastic Pipe Filled With Incompressible Viscous Fluid

Part III - Wave Propagation in an Elastic Pipe Filled With Incompressible Viscous Streaming Fluid

Part IV - Longitudinal Oscillation of a Propellant-Filled Flexible Hemispherical Tank

Part V - Longitudinal Oscillation of a Propellant-Filled Flexible Oblate Spheroidal Tank

The project was carried out by the Launch Vehicle Dynamics Group, Structures and Dynamics Department of Research and Engineering Division, S&ID. Dr. F.C. Hung was the Program Manager for North American Aviation, Inc. The study was conducted by Dr. Clement L. Tai (Principal Investigator), Dr. Michael M.H. Loh, Mr. Henry Wing, Dr. Sui A. Fung, and Dr. Shoichi Uchiyama. Dr. James Sheng, who started the investigation of Part IV, left in the middle of the program to teach at the University of Wisconsin. The computer program was developed by Mr. R.A. Pollock, Mr. F.W. Egeling, and Mr. S. Miyashiro.

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CONTENTS

ILLUSTRATIONS

NOMENCLATURE

A, B	Constants
c	Phase velocity
E	Young's modulus of material
F	Force
h	Thickness of pipe wall
i	$\sqrt{-1}$
J_o	Bessel function of zero order
J_1	Bessel function of first order
k	Complex propagation constant, $k = k_1 + i\, k_2$
k_1	Phase factor
k_2	Attenuation factor
p	Pressure
P	A constant
Q	Rate of mean volume flow
r	Coordinate in radial direction
R	Radius of pipe
t	Time
u	Longitudinal displacement of pipe wall
U	A constant
v	Fluid velocity component

V	A function of r
w	Radial displacement of pipe wall
W	A constant
x	Coordinate in longitudinal direction
y	A dimensionless parameter, $y = \pm \frac{k}{\omega}\left(\frac{Eh}{2\rho R_o}\right)^{1/2}$
z	Viscosity parameter, $z = R_o\left(\frac{\omega\rho}{\mu}\right)^{1/2}$
α	A dimensionless parameter, $\alpha = \left(\frac{i\ \omega\ \rho}{\mu}\right)^{1/2}$
σ	Poisson's ratio
ω	Circular frequency of disturbance
ρ	Mass density of fluid
ρ_o	Mass density of pipe wall material
μ	Viscosity
λ	Proportionment of wave length, $2\pi\ \lambda$ = wave length
η	Radial inertia parameter, $\eta = \frac{\rho_o \omega^2 R_o^2}{E}$
ξ	Longitudinal inertia parameter, $\xi = \frac{\rho_o h}{\rho R_o}$
$\tau_1, \tau_2, \tau_3, \tau_4$	Constants
$\underset{\approx}{O}$	Indicates the order of magnitude

Subscripts

r	Radial direction
x	Longitudinal direction
θ	Circumferential direction
0	Unstressed

INTRODUCTION

The problems of wave propagation of fluid in a pipe have attracted the attention of many investigators. Water flowing through the penstock of a hydro-power plant, blood circulation in a human body, pressure transmission in a pneumatic system—all depend on the dynamics of wave propagation of fluid in the conduit. Since the introduction of liquid rockets, the wave propagation of propellant feed systems becomes more interesting than ever.

The purpose of this study is to investigate the theoretical aspect of the problem of longitudinal wave propagation in an elastic pipe filled with an incompressible and viscous fluid. Special attention is directed toward the influence of inertia of pipe wall on wave propagation and its correlation with fluid viscosity. The investigation is divided into two parts. The first part, which is documented in this volume, is devoted to the analysis of pressure waves being propagated through a system at rest. The second part, which extends the investigation to a more general situation when the pressure waves travel through a system filled with streaming fluid, is published in a separate volume as Part III of the report.

The reflection of pressure waves in blood circulation has been studied by Karreman (Reference 1), following Witzig's approach. His results indicated that the influence of the viscosity on the velocity propagation is very slight and were considered by Morgan and Kiely (Reference 2) to be incorrect. Uchida (Reference 3) investigated the viscous flow in a circular pipe by superposing a pulsating flow on the steady motion. He linearized the fundamental equations of motion by introducing the assumption of axially parallel flow. In addition to the restraint of parallel flow, the elasticity of the pipe was ignored in the analysis.

An attempt to take account of the steady stream in the investigation of wave propagation of viscous fluid has also been made by Jacobs (Reference 4). In his work, the average stream velocity was used instead of the actual non-uniform velocity profile and the tube wall was assumed to move in the radial direction only. However, the fluid velocity cannot be uniform as it has to satisfy the condition of no slip at the wall no matter how small the viscosity. The longitudinal inertia force, as it will be seen later in the discussion, has greater influence on the wave propagation than the radial inertia force.

The present investigation follows the classical approach in formulating the problem with specific importance being accredited to the work of G. W. Morgan and J. P. Kiely (Reference 2). The classical Navier-Stokes equations of two dimensional flow with axial symmetric motion are first introduced. The elastic equilibrium equations of the pipe are established on the basis of a thin shell. These equations are linearized by assuming small amplitude oscillation and large wave length. Further simplification is made by omitting terms of small order of magnitude. The terms contributed by the pipe wall inertia forces, which were neglected by Morgan and Kiely, are retained in the analysis. From the fluid and elastic equations with appropriate boundary conditions, a characteristic equation is obtained. After a lengthy mathematical manipulation, the characteristic equation is reduced to a quadratic form of the complex propagation constants. This characteristic equation is being solved with the aid of a digital computer from which two sets of phase velocities and attenuation factors are obtained for various viscosity and inertia parameters. This phenomenon, the existence of two phase velocities, was not recognized by previous investigators. The phase velocities and attenuation factors are obtained in terms of three dimensionless parameters, longitudinal and radial inertia parameters which represent the effect of pipe wall inertia, and viscosity parameter which exhibits the influence of viscosity. These relations are delineated in the form of graphs.

ASSUMPTIONS

In formulating the mathematical model, the following assumptions are made.

(1) Materials of pipe follow Hooke's Law.

(2) Thickness of pipe, h, is small in comparison with radius R, thus $h/R \ll 1$.

(3) Slope of pipe wall disturbances, $\frac{dR}{dx}$, is small; therefore, shear and bending stresses in pipe are neglected.

(4) Forced disturbances are harmonic in time.

(5) Damping is small.

(6) Wave length is large compared with radius, more specifically
$$\left(\frac{R_o}{\lambda}\right)^2 \left(\frac{\mu}{\omega \rho R_o^2}\right) \ll 1$$

(7) The disturbance of pipe is small. The boundary conditions at the pipe wall may be linearized.

(8) The fluid is incompressible.

BASIC EQUATIONS

EQUATIONS OF MOTION OF FLUID

In the problem considered here, all motions are assumed to be axially symmetric; therefore, the circumferential velocity component and its derivatives are ignored. Cylindrical coordinates are adopted with x being the distance along the pipe axis and r the coordinate in the radial direction. By restricting the problem to small disturbances so that the nonlinear terms are negligible, the Navier-Stokes' equations are simplified in the following form:

$$\rho \frac{\partial v_r}{\partial t} = - \frac{\partial p}{\partial r} + \mu \left(\frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r} \frac{\partial v_r}{\partial r} + \frac{\partial^2 v_r}{\partial x^2} - \frac{v_r}{r^2} \right) \tag{1}$$

$$\rho \frac{\partial v_x}{\partial t} = - \frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 v_x}{\partial r^2} + \frac{1}{r} \frac{\partial v_x}{\partial r} + \frac{\partial^2 v_x}{\partial x^2} \right) \tag{2}$$

where v_r and v_x are the velocity components in the radial and axial directions, p is the pressure, ρ is the mass density, μ is the viscosity and t is the time.

The equation of conservation of mass is

$$\frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{\partial v_x}{\partial x} = 0 \tag{3}$$

EQUATIONS OF EQUILIBRIUM OF ELASTIC PIPE

When the thickness h of an elastic pipe is small compared with the radius, then the hoop tension and the tensile force in the axial direction are respectively given by

$$T_\theta = \frac{Eh}{1 - \sigma^2} \left(\frac{w}{R_o} + \sigma \frac{\partial u}{\partial x} \right) \tag{4}$$

$$T_x = \frac{Eh}{1 - \sigma^2} \left(\frac{\partial u}{\partial x} + \sigma \frac{w}{R_o} \right) \tag{5}$$

where w and u are components of displacement of the pipe wall in radial and axial directions, E is Young's modulus, σ is Poisson's ratio, h is the thickness of pipe wall and R_o is the unstressed radius of the pipe.

The normal force of fluid acting on an elementary area normal to the radius is

$$F_r = p - 2\mu \frac{\partial v_r}{\partial r} \tag{6}$$

and the shear force of fluid acting on an elementary area normal to the radius is

$$F_{rx} = \mu \left[\frac{\partial v_x}{\partial r} + \frac{\partial v_r}{\partial x} \right] \tag{7}$$

From assumption (3), it is stipulated that shear and bending stresses in the pipe are neglected. The equations of equilibrium of the pipe may be written as

$$\rho_o h \frac{\partial^2 w}{\partial t^2} = F_r(R, x, t) - \frac{T_\theta}{R_o} \tag{8}$$

$$\rho_o h \frac{\partial^2 u}{\partial t^2} = - F_{rx}(R, x, t) + \frac{\partial T_x}{\partial x} \tag{9}$$

where ρ_o = mass density of the pipe wall material.

Substituting the stresses and fluid forces as given in Eq. (4), (5), (6), and (7) and linearizing the boundary conditions by evaluating the forces at R_o instead of at variable radius R, the equations of equilibrium become

$$\rho_o h \frac{\partial^2 w}{\partial t^2} = \left[p - 2\mu \frac{\partial v_r}{\partial r} \right]_{r=R_o} - \frac{Eh}{1 - \sigma^2} \left[\frac{w}{R_o^2} + \frac{\sigma}{R_o} \frac{\partial u}{\partial x} \right] \tag{10}$$

$$\rho_o h \frac{\partial^2 u}{\partial t^2} = - \mu \left[\frac{\partial v_x}{\partial r} + \frac{\partial v_r}{\partial x} \right]_{r=R_o} + \frac{Eh}{1 - \sigma^2} \left[\frac{\partial^2 u}{\partial x^2} + \frac{\sigma}{R_o} \frac{\partial w}{\partial x} \right] \tag{11}$$

SOLUTION OF EQUATIONS

As we limit ourselves to the investigation of pressure waves which are harmonic in time, we may assume v_r, v_x, w, u and p all vary as $e^{i(kx-\omega t)}$ where

ω = circular frequency of the forced disturbance

$k = k_1 + ik_2$ = complex propagation constant

k_1 = a phase factor which represents a phase shift

k_2 = an attenuation factor, which represents a measure of decay of the disturbance as the wave travels along the pipe.

Therefore we set

$$\begin{aligned} v_r &= V_r (r)\, e^{i(kx-\omega t)} \\ v_x &= V_x (r)\, e^{i(kx-\omega t)} \\ w &= W\, e^{i(kx-\omega t)} \\ u &= U\, e^{i(kx-\omega t)} \end{aligned} \tag{12}$$

where V_r and V_x are functions of r, and W and U are constants. By introducing the expressions in Equation (12) into the continuity Equation (3), we have the order of magnitude of v_r/v_x

$$\frac{v_r}{v_x} = \frac{V_r}{V_x} \underset{\approx}{O} Rk$$

$\underset{\approx}{O}$ indicates the order of magnitude. For small damping, $k = k_1 = \frac{1}{\lambda}$ where $2\pi\lambda$ is the wave length. If we restrict ourselves to the problem of long waves as stipulated in assumption(6), $\frac{v_r}{v_x} \underset{\approx}{O} Rk \underset{\approx}{O} \frac{R}{\lambda} \ll 1$. Since v_r and its derivatives are expected to be very small, neglecting those terms in Equation (1) implies that the pressure is constant across the cross-section of the conduit and becomes a function of x and t only. Thus, we may write

$$p = P e^{i(kx - \omega t)} \tag{13}$$

where P is a constant.

Substituting the expressions in (12) and (13) into Equation (2)

$$- i \omega \rho V_x = - i k P + \mu \left(\frac{d^2 V_x}{dr^2} + \frac{1}{r} \frac{d V_x}{dr} - k^2 V_x \right) \tag{14}$$

Compare the relative magnitude of the coefficients of the two terms containing V_x in the above equation

$$\frac{k^2}{\frac{\omega \rho}{\mu}} = \frac{k^2 R_o^2}{\frac{\omega \rho}{\mu} R_o^2} \underset{\approx}{O} \left(\frac{R_o}{\lambda} \right)^2 \left(\frac{\mu}{\omega \rho R_o^2} \right)$$

This is a very small value, in accordance with assumption (6). The term $k^2 V_x$ may be neglected. Equation (14) becomes

$$\frac{d^2 V_x}{d r^2} + \frac{1}{r} \frac{d V_x}{dr} + \frac{i \omega \rho}{\mu} V_x = \frac{i k P}{\mu} \tag{15}$$

This is a Bessel equation. Its solution is found to be

$$V_x (r) = A J_o (\alpha r) + \frac{k}{\omega \rho} p \tag{16}$$

where $\alpha = \left(\frac{i \omega \rho}{\mu} \right)^{1/2}$, J_o is the Bessel function of zero order and A is an arbitrary constant; and

$$\frac{d V_x}{dr} = - \alpha A J_1 (\alpha r) \tag{17}$$

J_1 is the Bessel function of first order. By substituting Equation (12) into Equation (3), we have

$$\frac{1}{r} \frac{d(r V_r)}{dr} + i k V_x = 0$$

Because of the condition of symmetry, the radial velocity vanishes at the axis of the pipe. Therefore, $V_r(r)$ may be obtained by integrating the above equation:

$$V_r \ (r) = -\frac{ik}{r}\int_0^r r \ V_x \ (r) \ dr = -\frac{i \ k \ A}{\alpha} J_1 \ (\alpha r) - \frac{i \ k^2 \ r}{2 \ \omega\rho} P \qquad (18)$$

and

$$\frac{d \ V_r}{dr} = - i \ k \ A \left[J_o \ (\alpha r) - \frac{J_1 \ (\alpha r)}{\alpha r} \right] - \frac{i \ k^2}{2 \ \omega\rho} P \qquad (19)$$

Substituting the forms of (12) and (13) into elastic pipe Equations (10) and (11)

$$- \rho_o \ h \ \omega^2 \ W = P - 2\mu \left. \frac{d \ V_r}{dr} \right|_{r=R_o} - \frac{Eh}{1 - \sigma^2} \left[\frac{W}{R_o^2} + \frac{\sigma}{R_o} i \ k \ U \right] \qquad (20)$$

$$- \rho_o \ h \ \omega^2 \ U = - \mu \left[\frac{d \ V_x}{dr} + i \ k \ V_r \right]_{r=R_o} + \frac{Eh}{1 - \sigma^2} \left[- k^2 \ U + \frac{i \ \sigma \ k}{R_o} W \right] \qquad (21)$$

Introducing the values in Equations (17), (18) and (19), and evaluating at the boundary $r = R_o$, Equations (20) and (21) become

$$\left[- \rho_o \ h \ \omega^2 + \frac{Eh}{1 - \sigma^2} \frac{1}{R_o^2} \right] W + \frac{Eh}{1 - \sigma^2} \frac{i \ \sigma \ k}{R_o} U - P$$

$$+ 2\mu \left\{ - i \ k \ A \left[J_o \ (\alpha R_o) - \frac{J_1 \ (\alpha R_o)}{\alpha R_o} \right] - \frac{i \ k^2}{2 \ \omega\rho} P \right\} = 0 \qquad (22)$$

$$\left[- \rho_o \ h \ \omega^2 + \frac{Eh}{1 - \sigma^2} k^2 \right] U - \frac{Eh}{1 - \sigma^2} \frac{i \ \sigma \ k}{R_o} W$$

$$+ \mu \left\{ - \alpha \ A \ J_1 \ (\alpha R_o) + \frac{k^2 \ A}{\alpha} J_1 \ (\alpha R_o) + \frac{k^3 \ R_o}{2 \ \omega\rho} P \right\} = 0 \qquad (23)$$

Now, comparing the order of magnitude of the terms containing P in Equation (22),

$$\frac{\mu k^2}{\omega\rho} \Big/ 1 \underset{\approx}{O} \frac{\mu}{\omega\rho R_o^2} k^2 R_o^2 \underset{\approx}{O} \frac{\mu}{\omega\rho R_o^2} \left(\frac{R_o}{\lambda}\right)^2$$

Also, comparing the terms containing A in Equation (23)

$$\frac{\frac{k^2}{\alpha}}{\alpha} = \frac{k^2}{\alpha^2} \underset{\approx}{O} \frac{k^2 \mu}{\omega\rho} \underset{\approx}{O} \frac{\mu}{\omega\rho R_o^2} k^2 R_o^2 \underset{\approx}{O} \frac{\mu}{\omega\rho R_o^2} \left(\frac{R_o}{\lambda}\right)^2$$

From assumption (6), $\frac{\mu}{\omega\rho R_o^2}\left(\frac{R_o}{\lambda}\right)^2$ is a very small number; therefore, we may neglect the last term in Equation (22) and the term $\mu \frac{k^2 A}{\alpha} J_1 (\alpha R_o)$ in Equation (23). Rearranging, Equations (22) and (23) become

$$\frac{Eh}{1-\sigma^2} \frac{i\sigma k}{R_o} U + \left[-\rho_o h \omega^2 + \frac{Eh}{1-\sigma^2} \frac{1}{R_o^2}\right] W - P$$

$$- 2i \mu k \left[J_o (\alpha R_o) - \frac{J_1 (\alpha R_o)}{\alpha R_o}\right] A = 0 \qquad (24)$$

$$\left[-\rho h \omega^2 + \frac{Eh}{1-\sigma^2} k^2\right] U - \frac{Eh}{1-\sigma^2} \frac{i\sigma k}{R_o} W + \mu \frac{k^3 R_o}{2\omega\rho} P$$

$$- \mu \alpha J_1 (\alpha R_o) A = 0 \qquad (25)$$

Let us now examine the coefficients of W in Equation (24).

$$\frac{\rho_o h \omega^2}{\frac{Eh}{1-\sigma^2} \frac{1}{R_o^2}} \underset{\approx}{O} \frac{\frac{\omega^2}{k^2} k^2 R_o^2}{\frac{E}{\rho_o \left(1-\sigma^2\right)}}$$

For small damping, $\frac{\omega}{k} \simeq \frac{\omega}{k_1}$ and represents the velocity of a wave which is of the order $\left(\frac{Eh}{2\rho R_o}\right)^{1/2}$ if the viscosity is neglected. Thus the above ratio becomes

$$\frac{\frac{Eh}{2\rho R_o} k^2 R_o^2}{\frac{E}{\rho_o\left(1-\sigma^2\right)}} \underset{\approx}{O} \frac{1-\sigma^2}{2}\left(\frac{R_o}{\lambda}\right)^2 \frac{\rho_o h}{\rho R_o}$$

Examine also the coefficients of U in Equation (25).

$$\frac{\rho_o h \omega^2}{\frac{Eh}{1-\sigma^2} k^2} \underset{\approx}{O} \frac{\rho_o\left(1-\sigma^2\right)}{E} \frac{Eh}{2\rho R_o} \underset{\approx}{O} \frac{1-\sigma^2}{2} \frac{\rho_o h}{\rho R_o}$$

The terms $\rho_o h \omega^2 W$ and $\rho_o h \omega^2 U$ in Equations (24) and (25) are the inertia forces of the pipe wall. For propellant lines made of metallic tubes filled with light fluid such as liquid hydrogen, the ratio $\frac{\rho_o}{\rho} = \frac{500}{4.4} > 100$. The pipe wall inertia terms are no longer small forces and will be retained in the analysis.

BOUNDARY CONDITIONS

Assuming no slip of the fluid at the pipe wall, we have

$$\frac{\partial u}{\partial t} = -i \omega U e^{i(kx-\omega t)} = V_x (R_o) e^{i(kx-\omega t)}$$

and

$$\frac{\partial w}{\partial t} = -i \omega W e^{i(kx-\omega t)} = V_r (R_o) e^{i(kx-\omega t)}$$

Substituting values of Equations (16) and (18), we get

$$i\,\omega\,U + \frac{k}{\omega\rho}P + J_o\,(\alpha R_o)\,A = 0 \tag{26}$$

$$-i\,\omega\,W + \frac{i\,k^2\,R_o}{2\,\omega\rho}\,P + \frac{i\,k\,J_1\,(\alpha R_o)}{\alpha}\,A = 0 \tag{27}$$

Equations (24), (25), (26) and (27) form a set of homogeneous equations of U, W, P and A. The characteristic equation is obtained by setting the determinant of the coefficients of the above equations equal to zero. Thus

$$\begin{array}{cccc} U & W & P & A \end{array}$$

$$\begin{vmatrix} \frac{Eh}{1-\sigma^2}\frac{i\,\sigma\,k}{R_o} & \left[-\rho_o\,h\,\omega^2 + \frac{Eh}{1-\sigma^2}\frac{1}{R_o^2}\right] & -1 & -2i\,\mu\,k\left[J_o(\alpha R_o) - \frac{J_1(\alpha R_o)}{\alpha R_o}\right] \\ \left[-\rho_o\,h\,\omega^2 + \frac{Eh}{1-\sigma^2}k^2\right] & -\frac{Eh}{1-\sigma^2}\frac{i\,\sigma\,k}{R_o} & \mu\frac{k^3 R_o}{2\,\omega\rho} & -\mu\,\alpha\,J_1(\alpha R_o) \\ i\omega & 0 & \frac{k}{\omega\rho} & J_o(\alpha R_o) \\ 0 & -\omega & \frac{k^2\,R_o}{2\,\omega\rho} & \frac{k\,J_1(\alpha R_o)}{\alpha} \end{vmatrix} = 0 \tag{28}$$

After a lengthy and laborious determinant manipulation, and observing that $\left(\frac{R_o}{\lambda}\right)^2\left(\frac{\mu}{\omega\rho\,R_o^2}\right) << 1$, Equation (28) reduces to the following form

$$\left[2\,J_o(\alpha R_o) - 4\frac{J_1(\alpha R_o)}{\alpha R_o}\right](1-\eta)\,y^4 - \left[\left\{2 + \xi - (1-\sigma^2)\,\eta\xi\right\}J_o(\alpha R_o) + \left\{(1-4\sigma) - 2\xi - (1-\sigma^2)\,\eta\,(1-2\xi)\right\}\frac{J_1(\alpha R_o)}{\alpha R_o}\right]y^2 + (1-\sigma^2)\left[\xi\,J_o\,(\alpha R_o) + \frac{J_1(\alpha R_o)}{\alpha R_o}\right] = 0 \tag{29}$$

where

$$y^2 = \left(\frac{k}{\omega}\right)^2 \left(\frac{Eh}{2\rho R_o}\right), \quad \eta = \frac{\rho_o \omega^2 R_o^2}{E}, \text{ and } \xi = \frac{\rho_o h}{\rho R_o}$$

η and ξ are two dimensionless parameters derived from the radial and longitudinal inertia force terms of the pipe wall and are named radial and longitudinal inertia parameters respectively. Equation (29) is a quadratic equation from which two sets of complex solutions can be found, and then $\frac{k}{\omega} = \frac{k_1 + ik_2}{\omega}$ can be computed. The phase velocity, $c = \frac{\omega}{k_1}$, and the damping factor, k_2, may be determined by the real and imaginary parts of the solution of y. When η and ξ both are zero, Equation (29) reduces to the form given by Morgan and Kiely.

VELOCITY DISTRIBUTION

The longitudinal velocity at any cross section of the pipe is given by Equation (16) as

$$V_x (r) = A J_o (\alpha r) + \frac{k}{\omega\rho} P \tag{16}$$

If we define Q as the rate of mean volume flow in the pipe, then

$$Q = \int_o^{R_o} 2\pi r V_x \, dr = 2\pi A \frac{R_o}{\alpha} J_1 (\alpha R_o) + \pi P \frac{k R_o^2}{\omega\rho} \tag{30}$$

Denote the mean average velocity over a cross section by $\underline{V_x}$,

$$\underline{V_x} = \frac{Q}{\pi R_o^2} = \frac{2A}{\alpha R_o} J_1 (\alpha R_o) + \frac{k}{\omega\rho} P \tag{31}$$

Eliminating U and W from Equations (24), (26) and (27), we find the relations between A and P as follows

$$\frac{A}{P} = \frac{\frac{2(1-\sigma^2)}{k^2 R_o^2}\frac{\eta}{\xi} + (1-\sigma^2)\eta - (1-2\sigma)}{\left[1-(1-\sigma^2)\eta\right]\frac{J_1(\alpha R_o)}{\alpha R_o} - \sigma J_o(\alpha R_o)} \frac{k}{2\omega\rho} = B\frac{k}{2\omega\rho} \tag{32}$$

where k depends on η, ξ, σ and αR_o; therefore, B is a function of η, ξ, σ and αR_o.

Using the relation (32), we get the ratio of the longitudinal velocity and the mean average velocity to be

$$\frac{V_x(r)}{\underline{V_x}} = \frac{1+\frac{B}{2}J_o(\alpha r)}{1+B\frac{J_1(\alpha R_o)}{\alpha R_o}} \tag{33}$$

and

$$B = \frac{\frac{2(1-\sigma^2)}{k^2 R_o^2}\frac{\eta}{\xi} + (1-\sigma^2)\eta - (1-2\sigma)}{\left[1-(1-\sigma^2)\eta\right]\frac{J_1(\alpha R_o)}{\alpha R_o} - \sigma J_o(\alpha R_o)}$$

However, in some respects it is more interesting and meaningful to express the velocity profile in terms of ratio of the fluid velocity and pipe wall velocity. From Equations (12) and (26), the longitudinal pipe wall velocity is

$$V_x(R_o) = -i\omega U = A J_o(\alpha R_o) + \frac{k}{\omega\rho}P$$

Using again the relation (32), the longitudinal velocity ratio between fluid and pipe wall becomes

$$\frac{V_x(r)}{V_x(R_o)} = \frac{1+\frac{B}{2}J_o(\alpha r)}{1+\frac{B}{2}J_o(\alpha R_o)} \tag{34}$$

Similarly, the radial velocity ratio between fluid and pipe wall is

$$\frac{V_r\ (r)}{V_r\ (R_o)} = \frac{\dfrac{r}{R_o} + B\,\dfrac{J_1\ (\alpha r)}{\alpha R_o}}{1 + B\,\dfrac{J_1\ (\alpha R_o)}{\alpha R_o}} \tag{35}$$

DISCUSSION OF RESULTS

Introducing the notation $z = R_o \left(\frac{\omega\rho}{\mu}\right)^{1/2}$, thus $\alpha R_o = R_o \left(\frac{i\omega\rho}{\mu}\right)^{1/2} = \sqrt{i}\, z$, it is recognized that the solution of Equation (29) is characterized by the dimensionless quantity, z, which for the convenience of discussion is named viscosity parameter. The characteristic equation is composed of Bessel functions with complex arguments. A general solution of the closed form is difficult to achieve. Therefore, a numerical method is used with the IBM 7094 digital computer. For simplicity, Poisson's ratio σ is assumed to be 0.3 throughout the calculations. Equation (29) is an equation of the quadratic form. For each set of parameter values η, ξ, and z, we get two sets of complex solutions, namely

$$y_1 = \pm \frac{k_1 + ik_2}{\omega}\left(\frac{Eh}{2\rho\ R_o}\right)^{1/2} = \tau_1 + i\,\tau_2 \tag{36}$$

$$y_2 = \pm \frac{k_1 + ik_2}{\omega}\left(\frac{Eh}{2\rho\ R_o}\right)^{1/2} = \tau_3 + i\,\tau_4 \tag{37}$$

which lead to two sets of phase velocity, c, and damping factor k_2.

Equation (36) gives

$$c = \frac{\omega}{k_1} = \pm \frac{1}{\tau_1}\left(\frac{Eh}{2\rho\ R_o}\right)^{1/2}$$

$$k_2 = \pm\, \tau_2\, \omega \left(\frac{2\rho\ R_o}{Eh}\right)^{1/2} \tag{38}$$

and Equation (37) gives

$$c = \frac{\omega}{k_1} = \pm \frac{1}{\tau_3}\left(\frac{Eh}{2\rho\ R_o}\right)^{1/2}$$

$$k_2 = \pm\, \tau_4\, \omega \left(\frac{2\rho\ R_o}{Eh}\right)^{1/2} \tag{39}$$

The relations between phase velocities and viscosity parameters are shown in Figures 1 and 2. Figure 1 gives the velocity-viscosity-parameter relationship when the radial inertia parameter is very small or $\eta = 0$, and Figure 2 gives the relationship when the longitudinal inertia parameter is very small or $\xi = 0$. Figures 3a through 3e, 4a, and 4b are the longitudinal velocity profiles. In each of these figures, there are two sets of curves corresponding to the two sets of results given by Equations (38) and (39). One set of the results, where the pipe wall velocity is relatively small as compared with the fluid velocity, is here named wave propagation of the first kind. The other set of results, where the pipe wall has relatively large longitudinal velocity, is identified as wave propagation of the second kind.

For wave propagation of the first kind the motion of the pipe wall is very small (see Figures 3a through 3e, 4a and 4b); therefore, the phase velocities are more or less independent from the pipe inertia parameters but are greatly influenced by the viscosity parameter of the fluid, as shown in Figures 1 and 2. When the viscosity parameter is small, corresponding to large viscosity and small disturbing frequency, the velocity profile, as shown in Figures 3a through 3e, 4a and 4b, is nearly independent from the inertia parameters and follows the well-known pattern of parabolic distribution. When the viscosity parameter increases, the influence of inertia parameters on velocity profiles increases also. When the viscosity parameter is very large, very sharp velocity gradients are developed near the pipe wall and the velocity profiles exhibit the boundary layer flow.

For wave propagation of the second kind, the phase velocity of the fluid is nearly equal to $\sqrt{\frac{E}{\rho_o}}$, the elastic wave velocity of the pipe wall, and is nearly independent from the viscosity parameters as shown in Figure 1. The pipe wall has large motion in the longitudinal direction for the wave propagation of the second kind, hence the velocity profiles are greatly influenced by the longitudinal inertia. When the longitudinal inertia parameter is small (see Figures 3a and 3b), the velocity profiles follow the similar pattern of the first kind and the wall velocity is generally out of phase with the fluid velocity except for fluid of large viscosity (corresponding to small z-value). As the longitudinal inertia parameter increases, the wall velocity surges forward and finally the velocity distributions reverse the pattern of those of the first kind, as depicted in Figures 3d and 3e.

CONCLUSIONS

From the results, it may be concluded that the longitudinal inertia parameters have much greater influence on the wave propagations than the radial inertia parameters. The radial inertia parameters, which may be expressed as

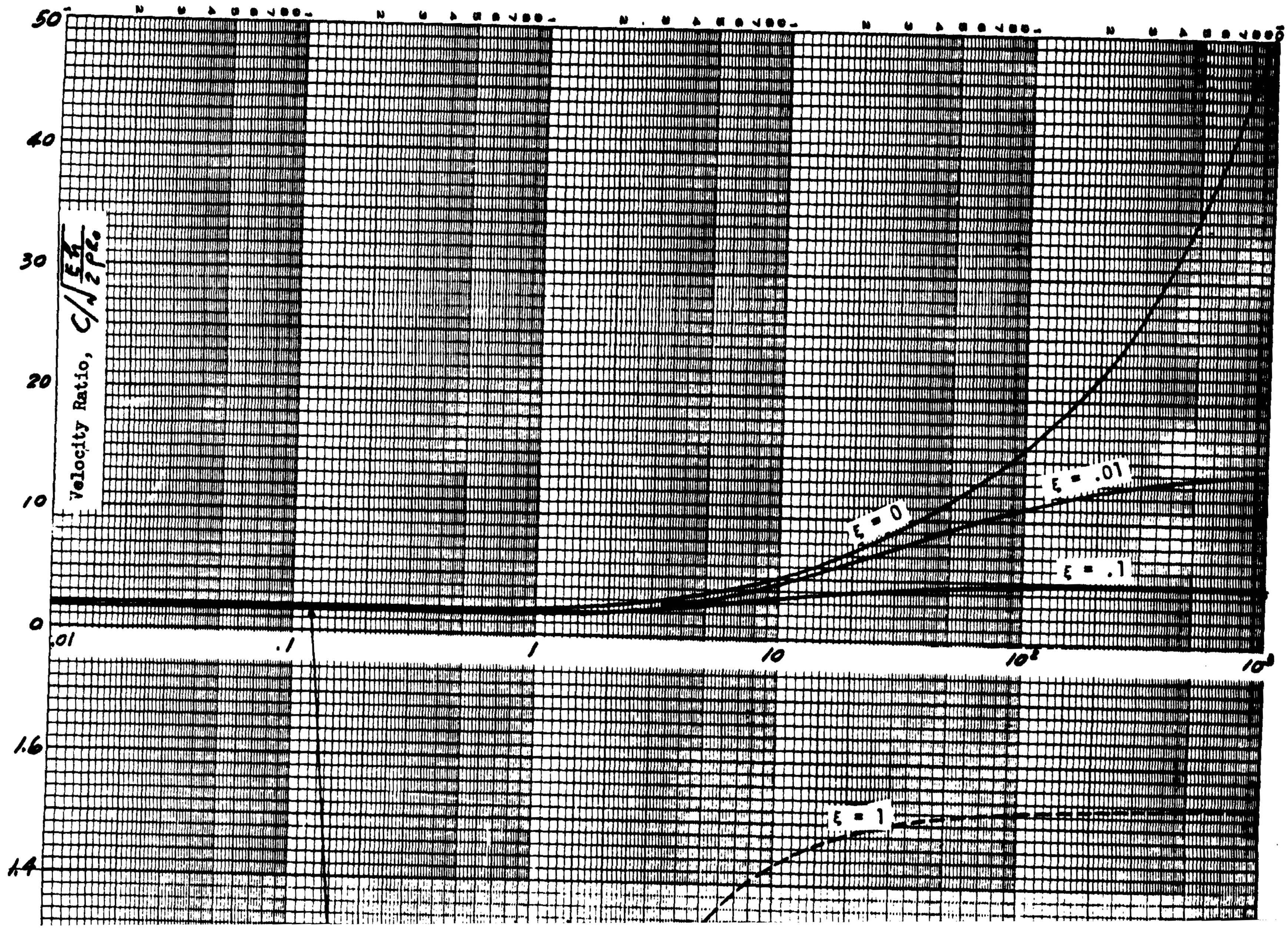
$$\eta = \frac{\rho_o \omega^2 R_o^2}{E} = \frac{\omega^2 R_o^2}{\frac{E}{\rho_o}} = \left(\frac{R_o}{\lambda_p}\right)^2 \ll 1$$

(where $2\pi\lambda_p$ is the wave length of the wave traveling in the elastic material of the pipe) are very small values. The phase velocities and velocity profiles shown in Figures 2, 4a, and 4b are practically identical with those shown in Figures 1 and 3a for $\xi = 0$. Thus, the effects of radial inertia parameters on wave propagation are negligible. In the case where the wave length of the elastic material of the pipe is very small and approaching the order of the radius of the pipe, the influence of radial inertia parameters on wave propagation may become prominent.

The radial velocity patterns of the fluid for various parameters are shown in Figures 7a through 7c and 8a through 8c in the form of ratio to wall radial velocity.

The decay of the disturbing waves depends on the Poisson's ratio and the viscosity. As we fixed the Poisson's ratio in our investigation, the damping factor varies with the viscosity parameter only, and it is almost independent from the pipe wall inertia parameters. When the viscosity parameter is small, which corresponds to large viscosity and small disturbing frequency, the wave propagation of the first kind possesses large values of attenuation factor while the waves of the second kind have very small damping effects. When the viscosity parameter is small, waves of the second kind move as a rigid body (refer to Figures 3a through 3e with $z = 1$); the viscosity of the fluid plays a very small role in creating the shear forces which are the source of damping. When the viscosity parameter is large, the attenuation factors become very small in both cases. Figure 9 gives the plots of attenuation factor vs. viscosity parameter of wave propagation of both kinds. These curves are practically the same for all values of pipe wall inertia parameters.

The work reported here deals with wave propagation in an elastic pipe filled with incompressible viscous fluid at rest. The extension of this investigation to a more general situation where the pressure waves travel through a system filled with streaming fluid is discussed in Part III of this report.



Velocity Ratio, $C/\sqrt{\frac{Eh}{2\rho c_0}}$
50
40
30
20
10
0
1.6
1.4
.01
.1
1
10
10^2
10^3
ξ = 0
ξ = .01
ξ = .1
ξ = 1

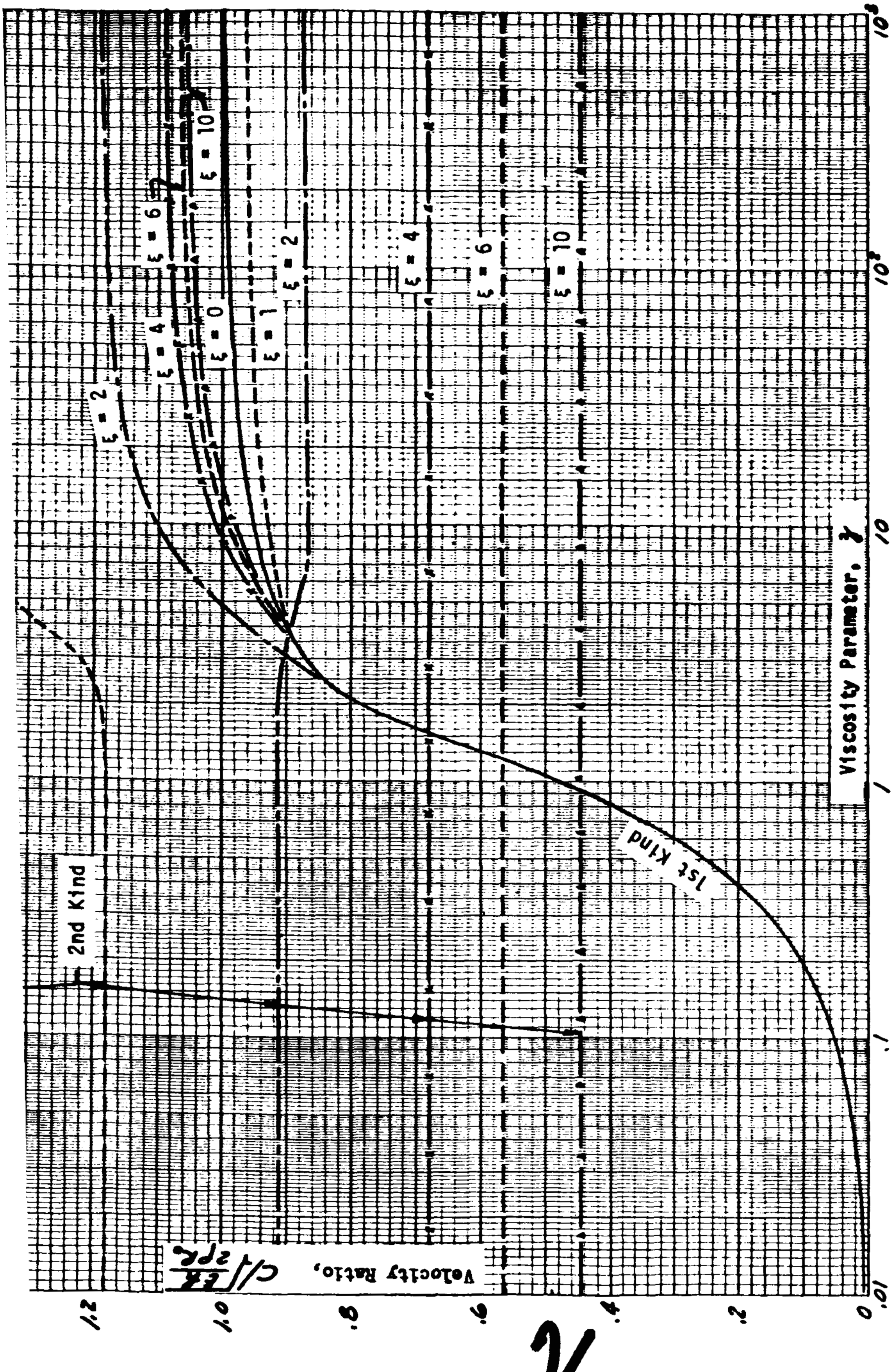


Figure 1. Phase Velocity vs Viscosity Parameter for Very Small Radial Inertia Parameter (η = 0)

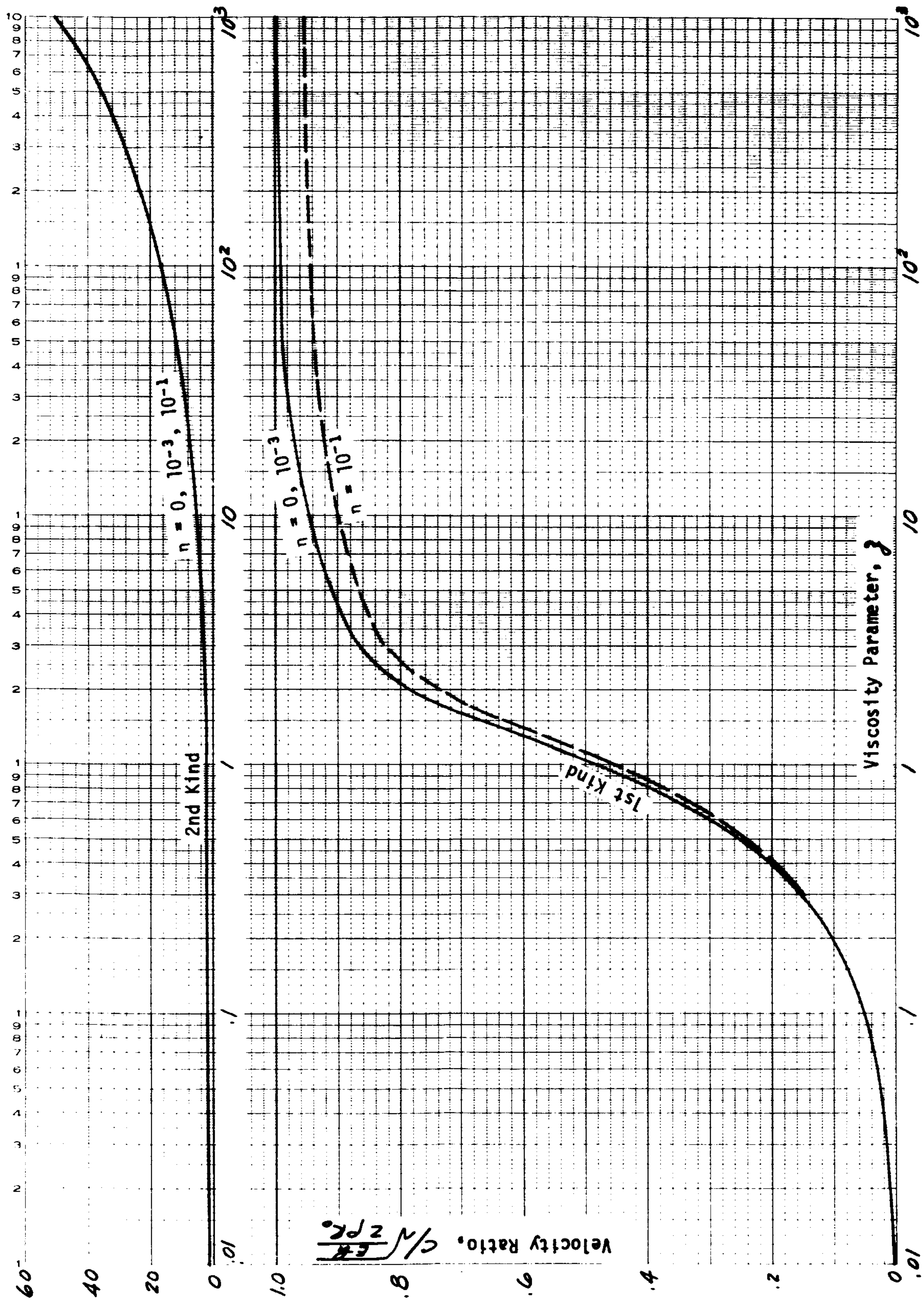


Figure 2. Phase Velocity vs Viscosity Parameter for Very Small Longitudinal Inertia Parameter ($\xi = 0$)

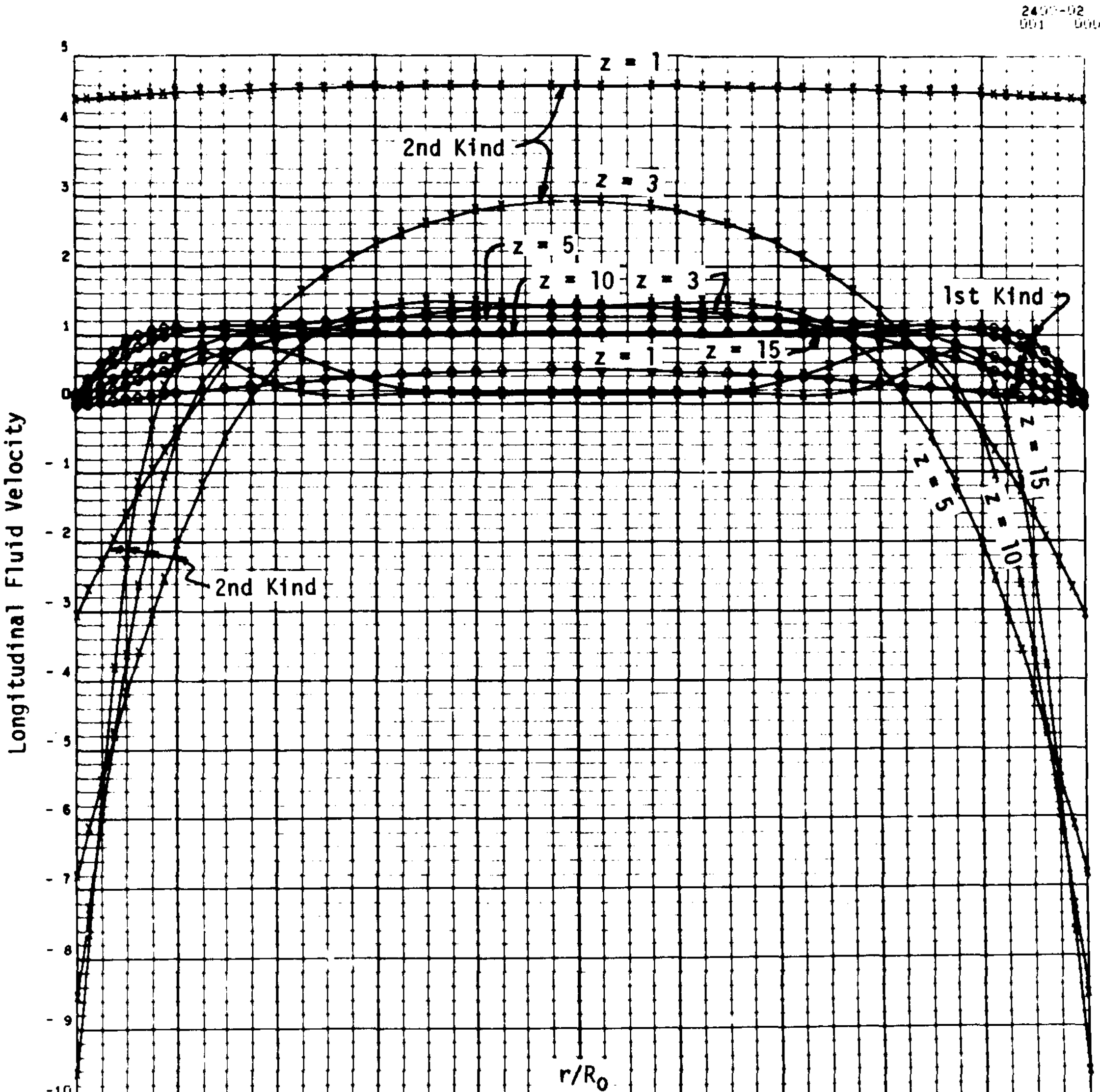


Figure 3a. Longitudinal Velocity Profile for $\eta = 0$, $\xi = 0$

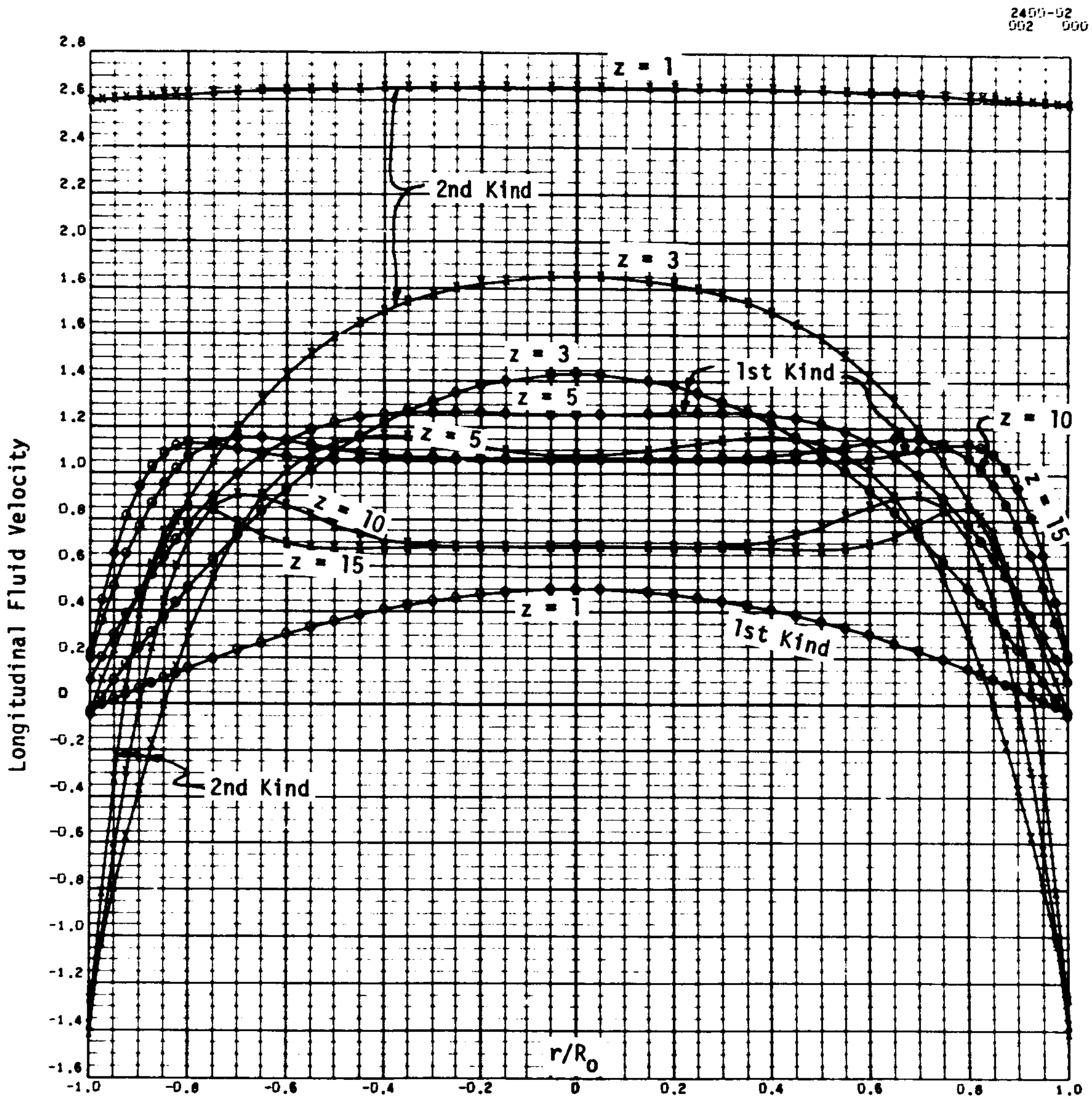


Figure 3b. Longitudinal Velocity Profile for $\eta = 0$, $\xi = 1$

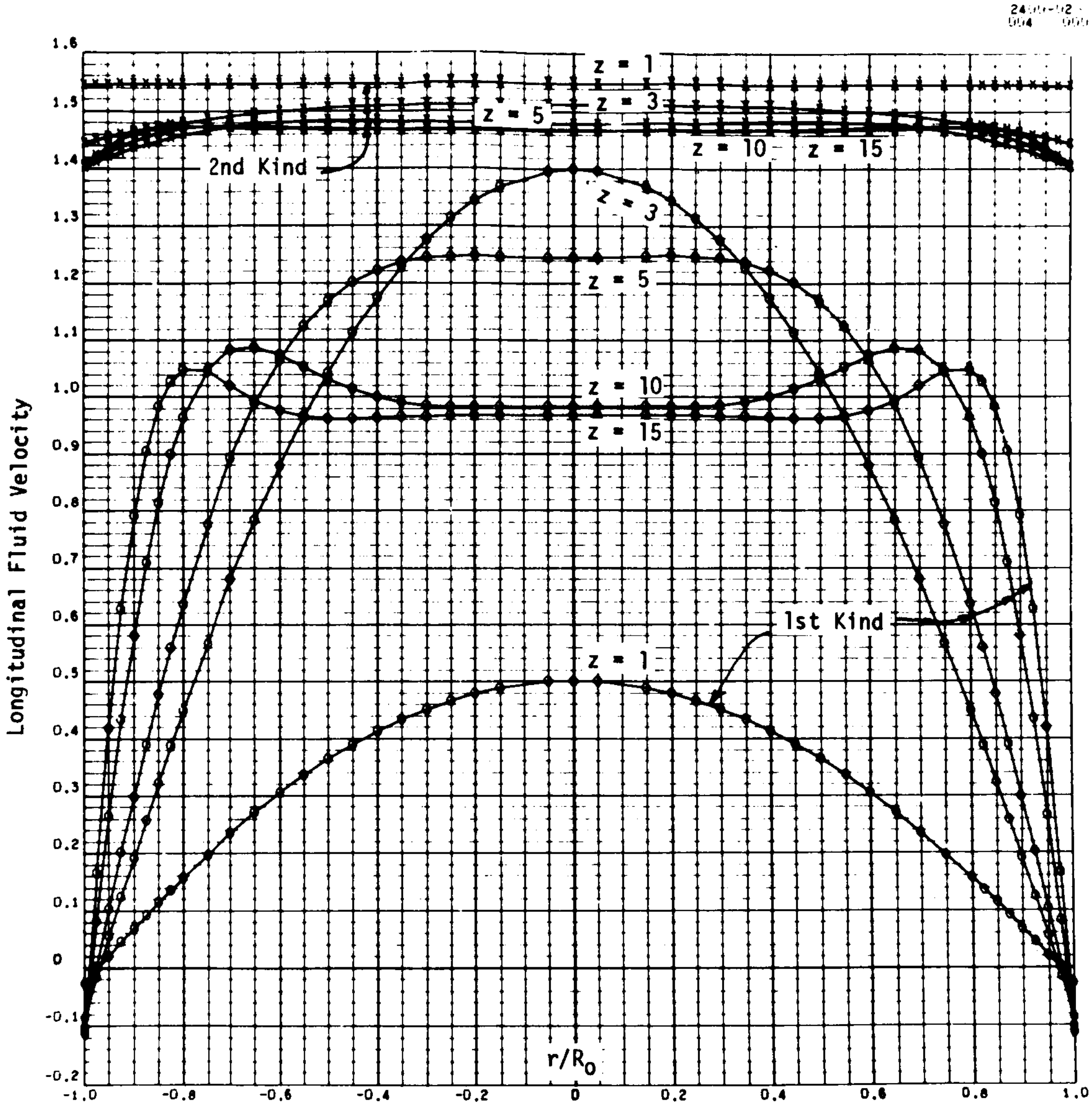


Figure 3c. Longitudinal Velocity Profile for $\eta = 0$, $\xi = 4$

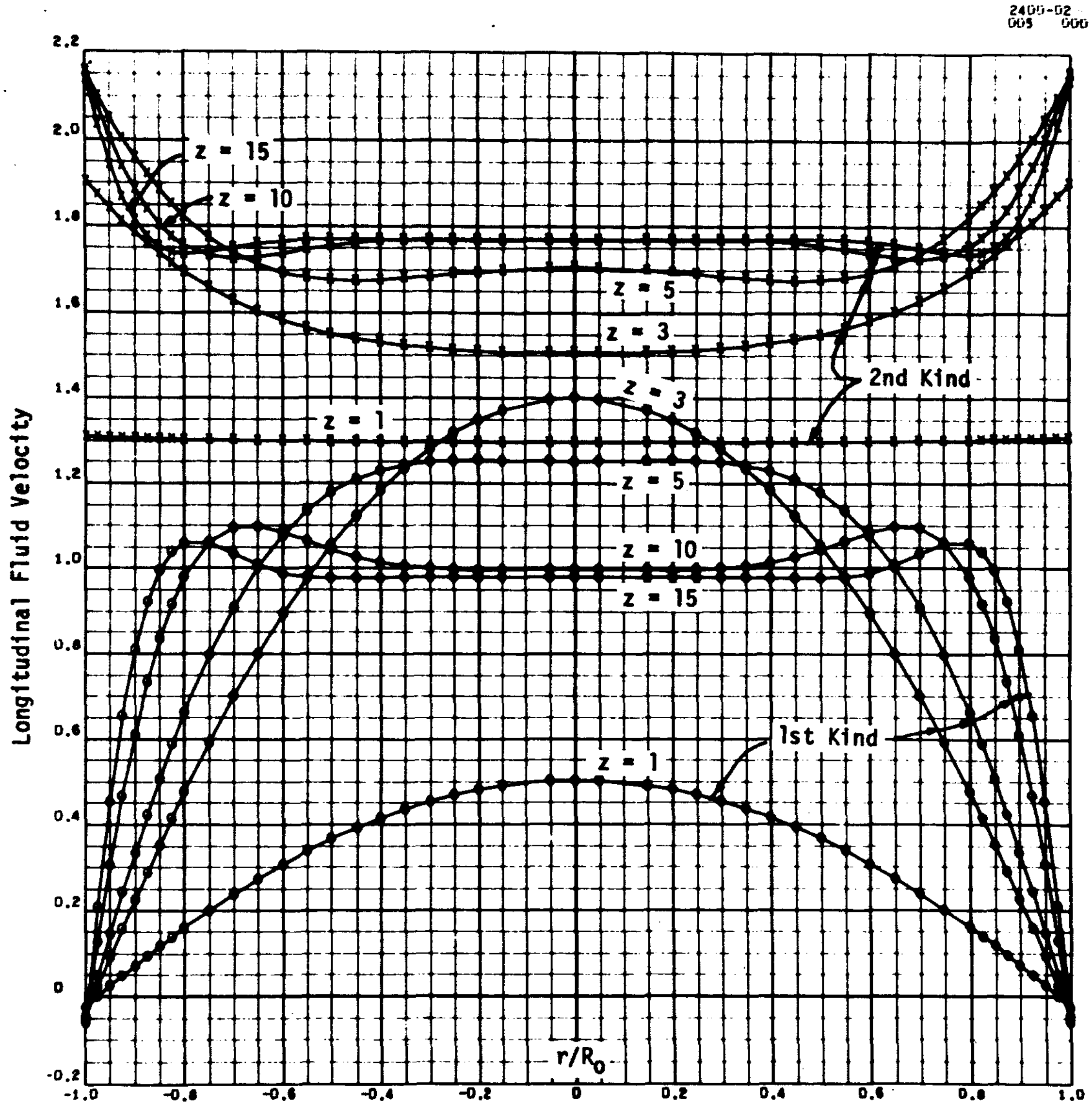


Figure 3d. Longitudinal Velocity Profile for $\eta = 0$, $\xi = 6$

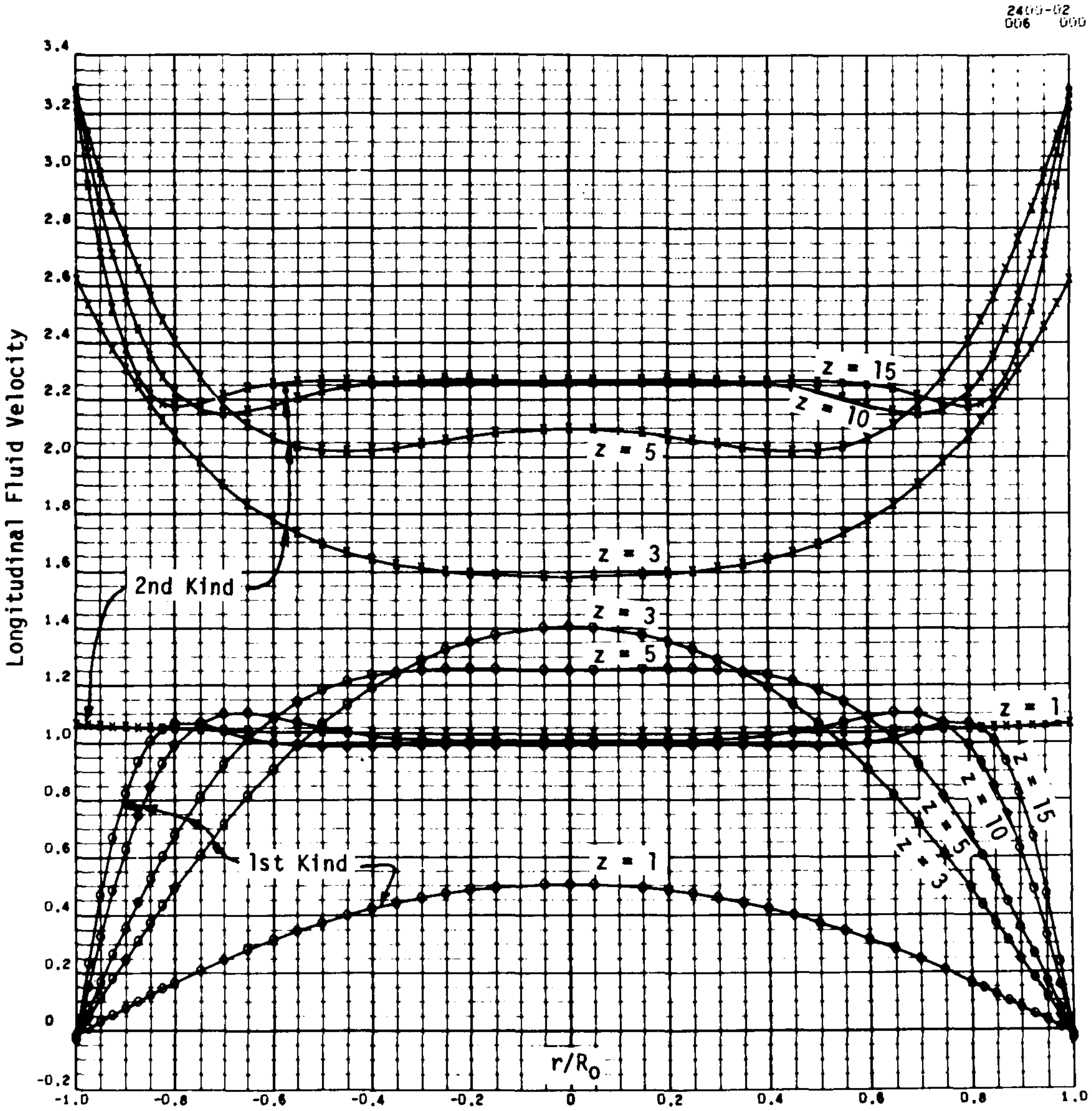


Figure 3e. Longitudinal Velocity Profile for $\eta = 0$, $\xi = 10$

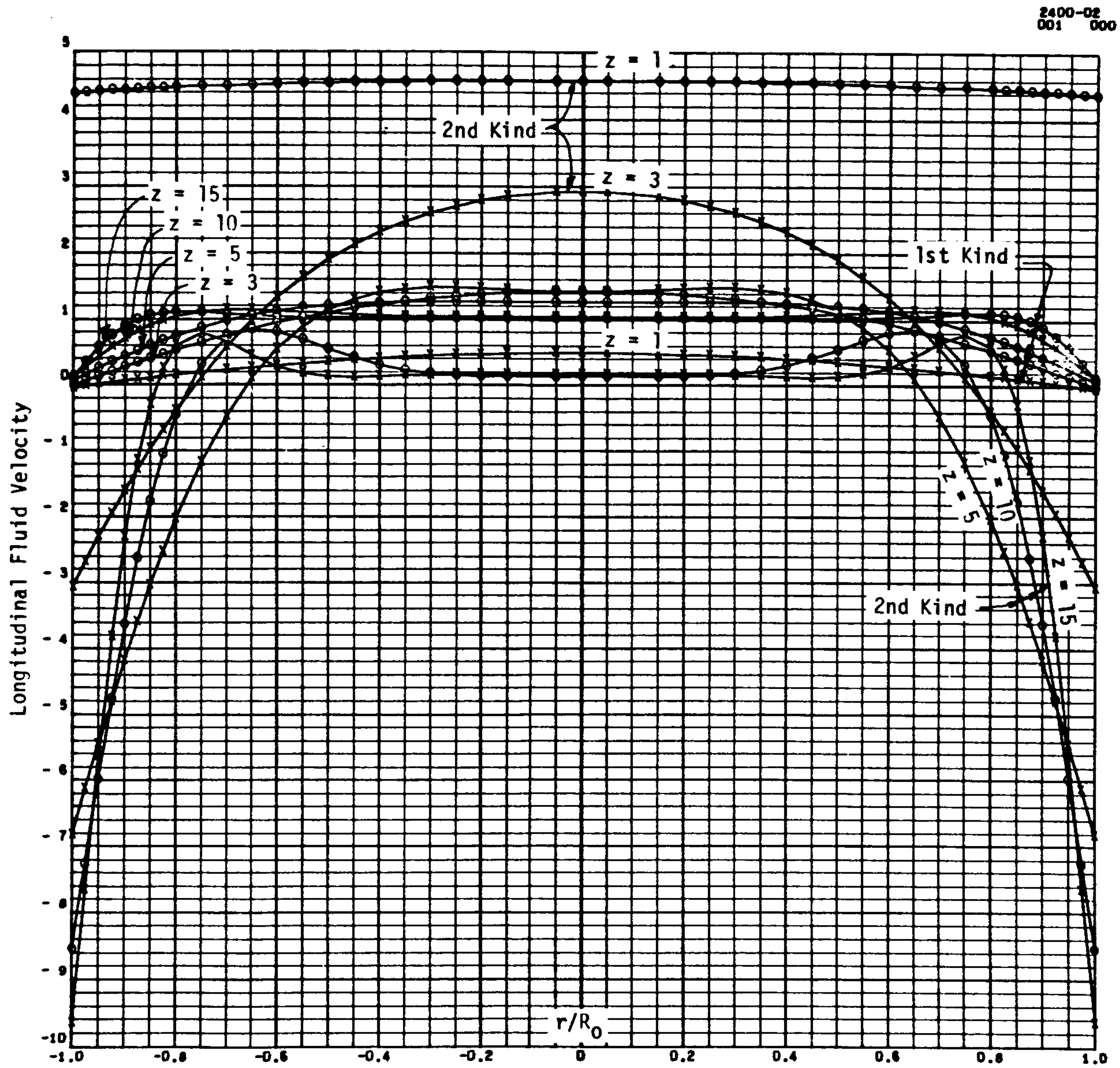


Figure 4a. Longitudinal Velocity Profile for $\xi = 0$, $\eta = 10^{-6}$

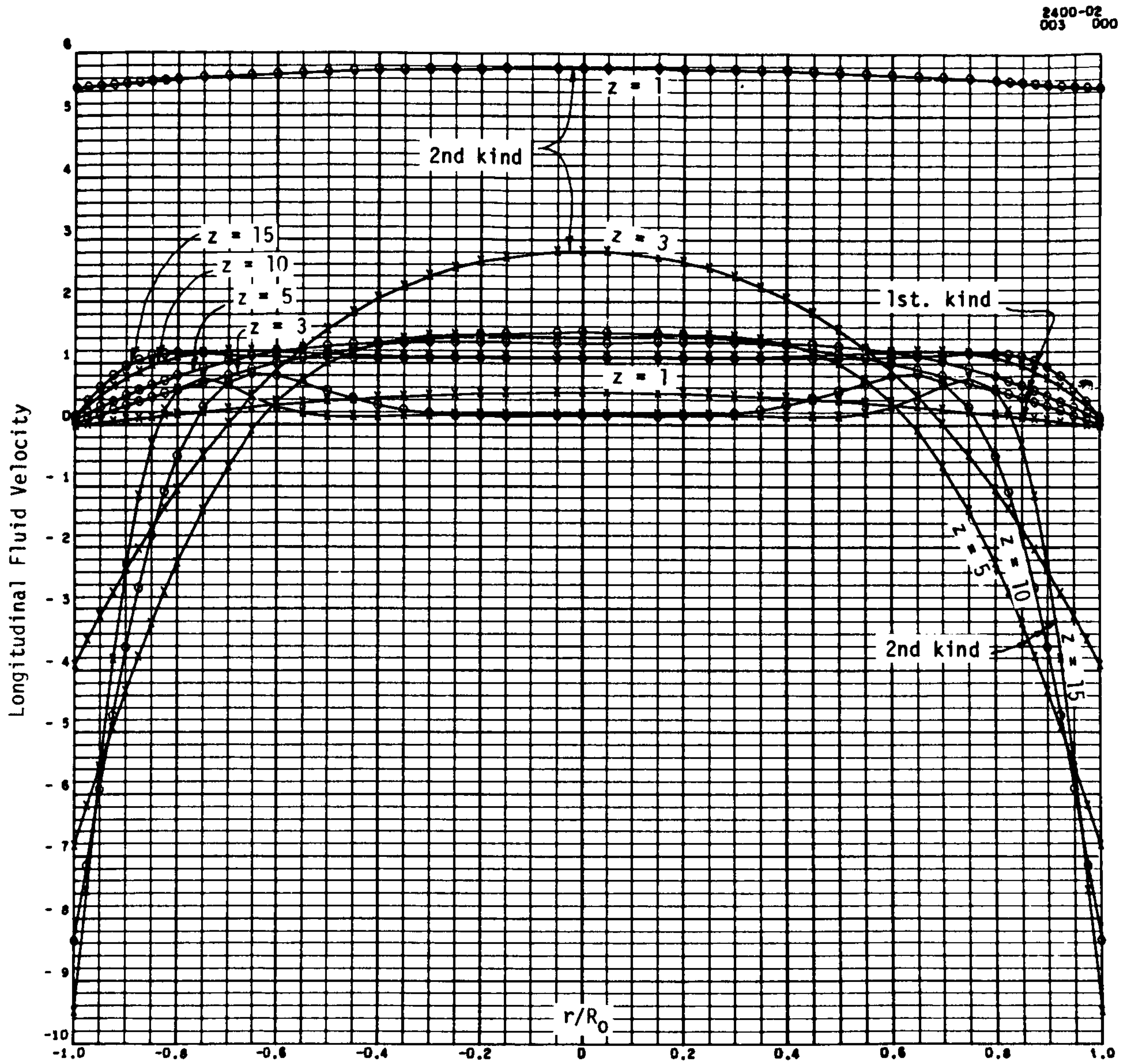


Figure 4b. Longitudinal Velocity Profile for $\xi = 0$, $\eta = 10^{-1}$

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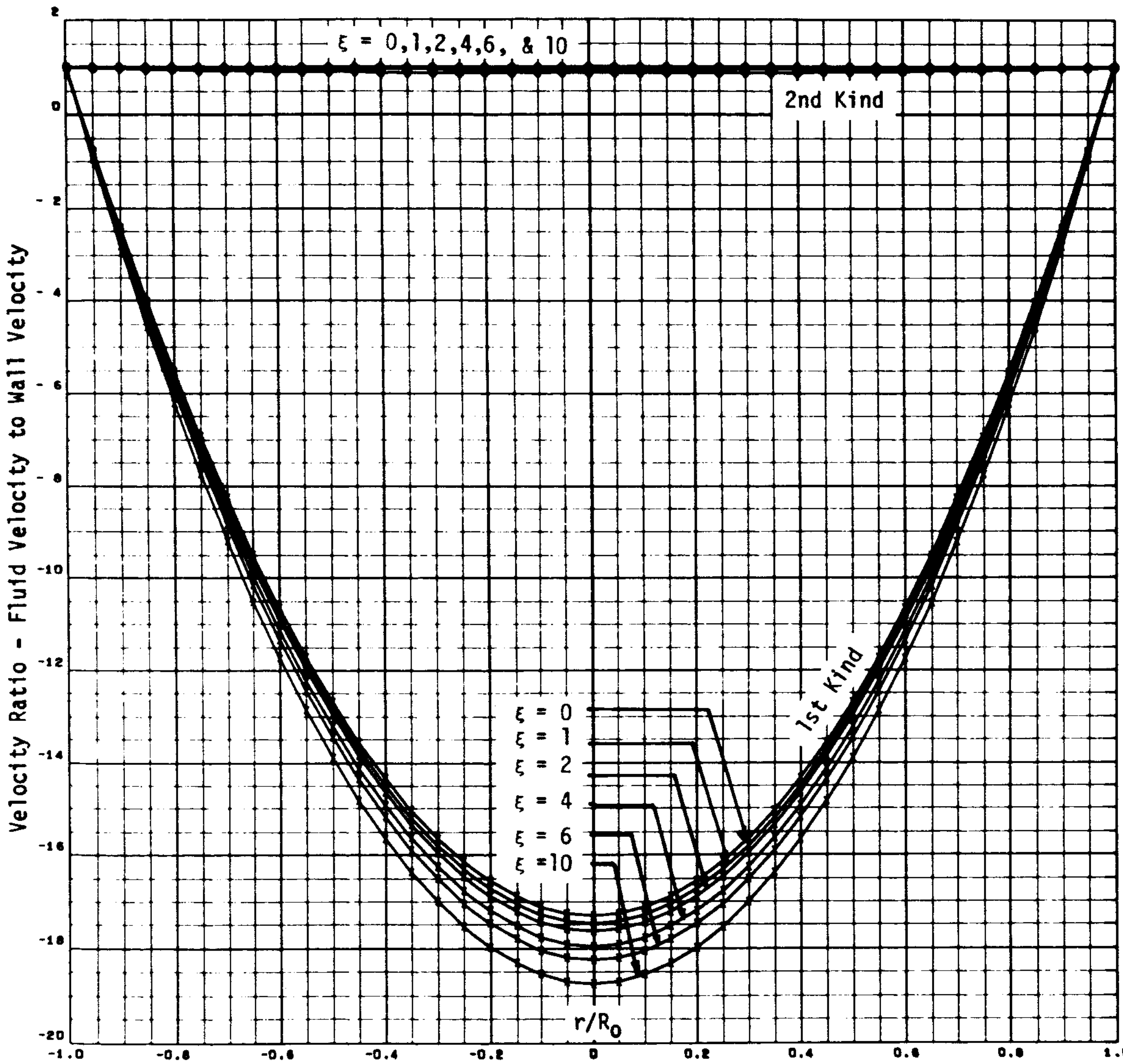

Figure 5a. Longitudinal Velocity Ratio—Fluid Velocity to Wall Velocity for $\eta = 0$, $z = 1$

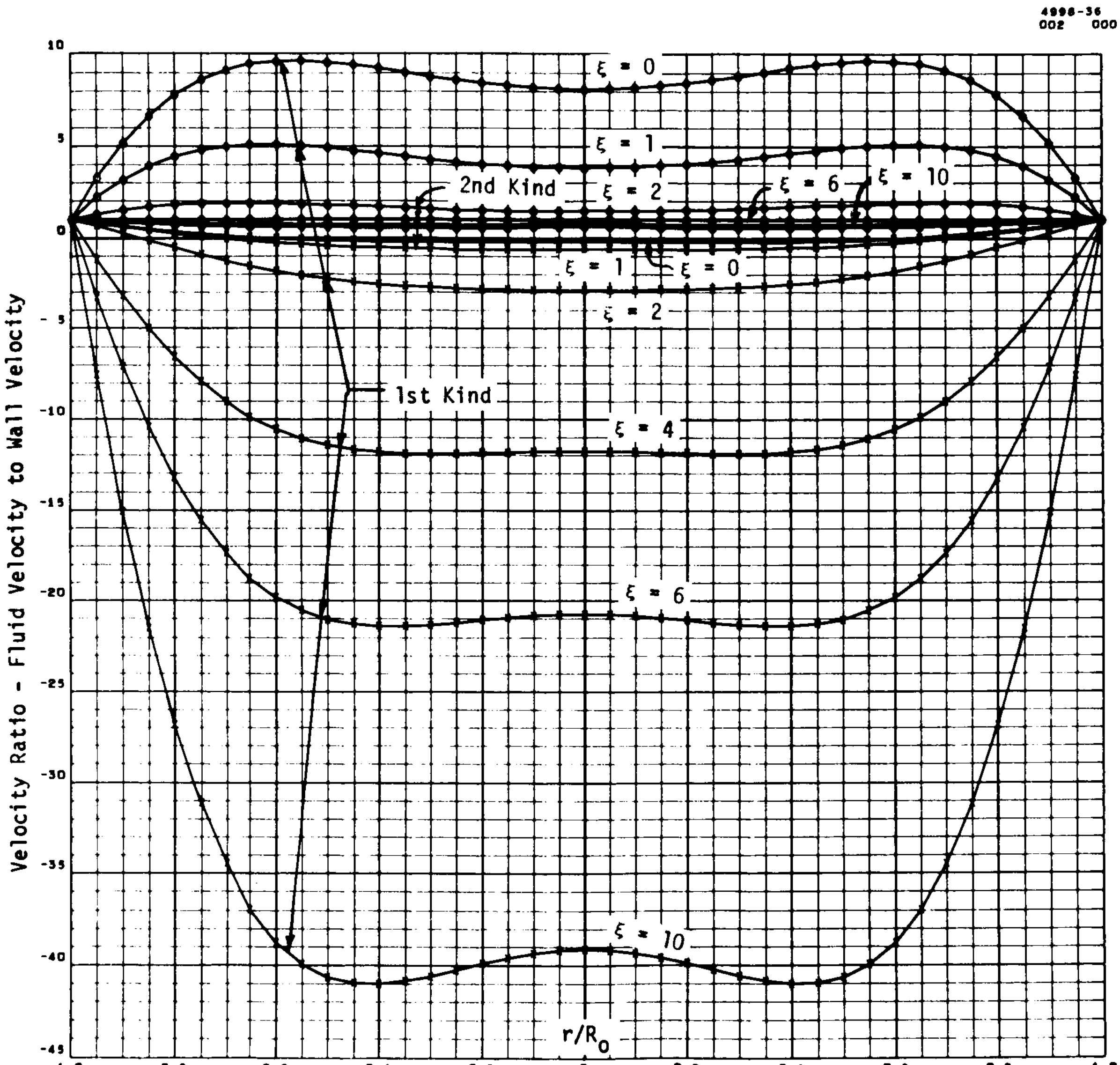

Figure 5b. Longitudinal Velocity Ratio—Fluid Velocity to Wall Velocity for $\eta = 0$, $z = 5$

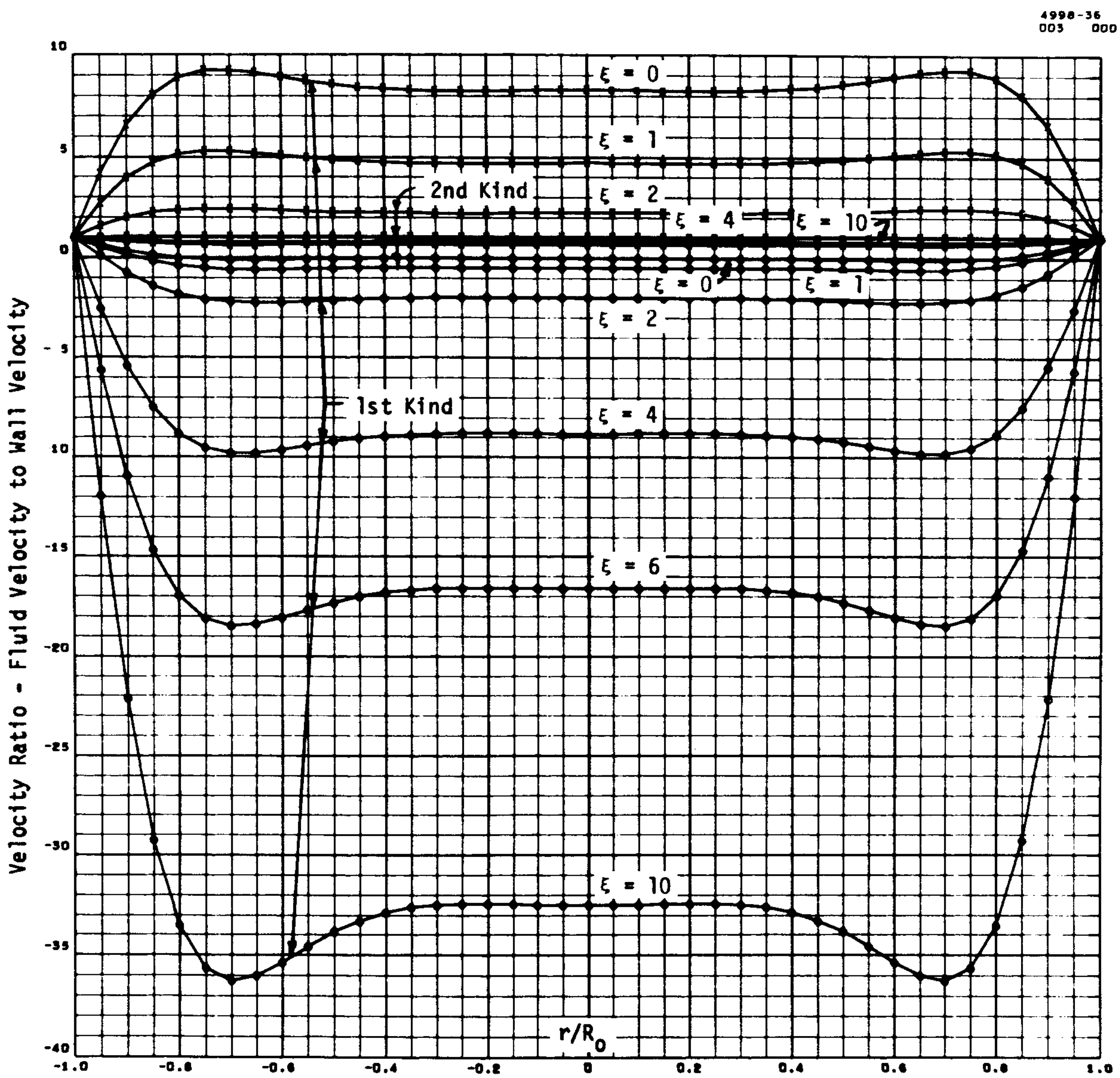

Figure 5c. Longitudinal Velocity Ratio—Fluid Velocity to Wall Velocity for $\eta = 0$, $z = 10$

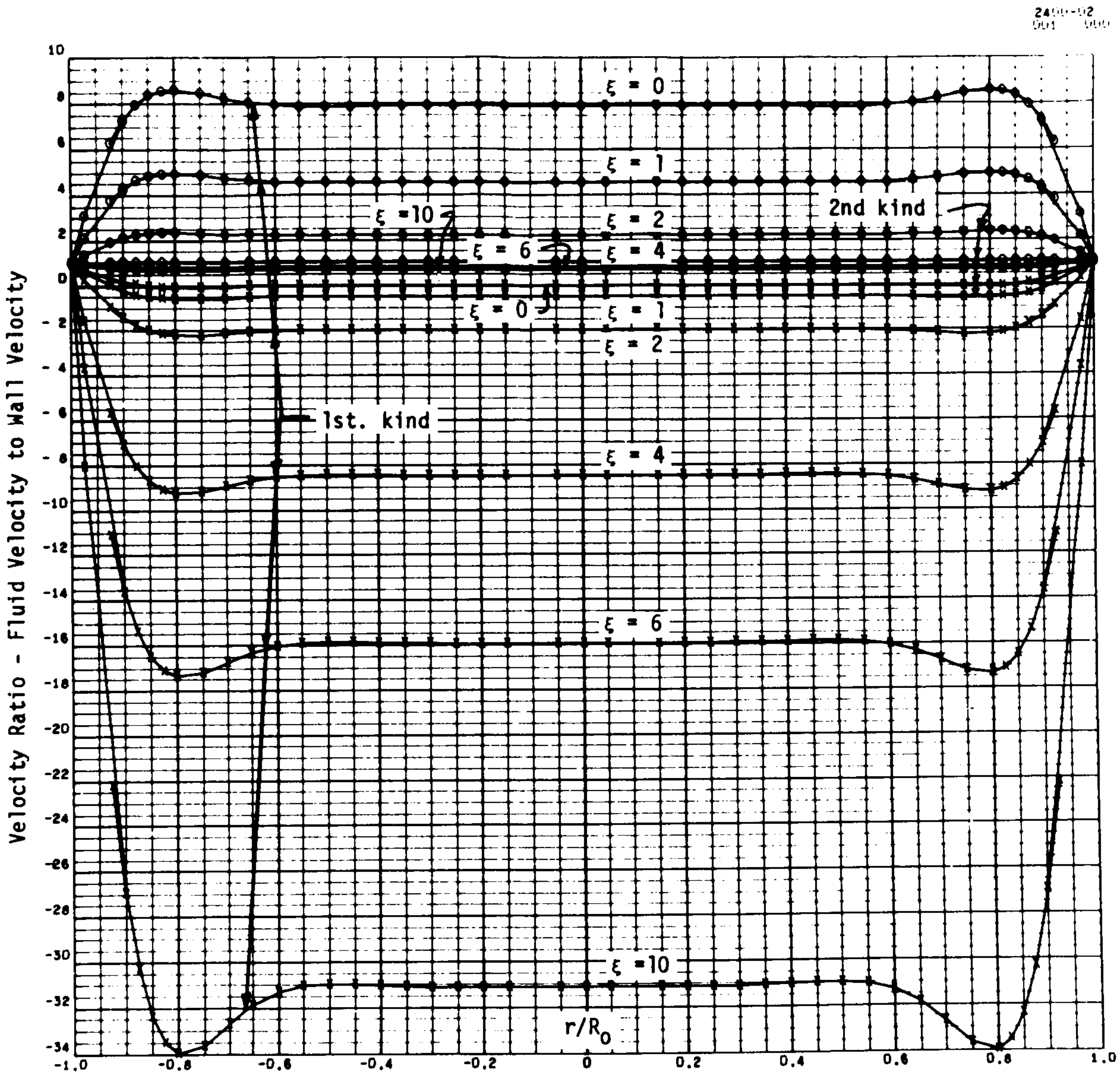

Figure 5d. Longitudinal Velocity Ratio—Fluid Velocity to Wall Velocity for $\eta = 0$, $z = 15$

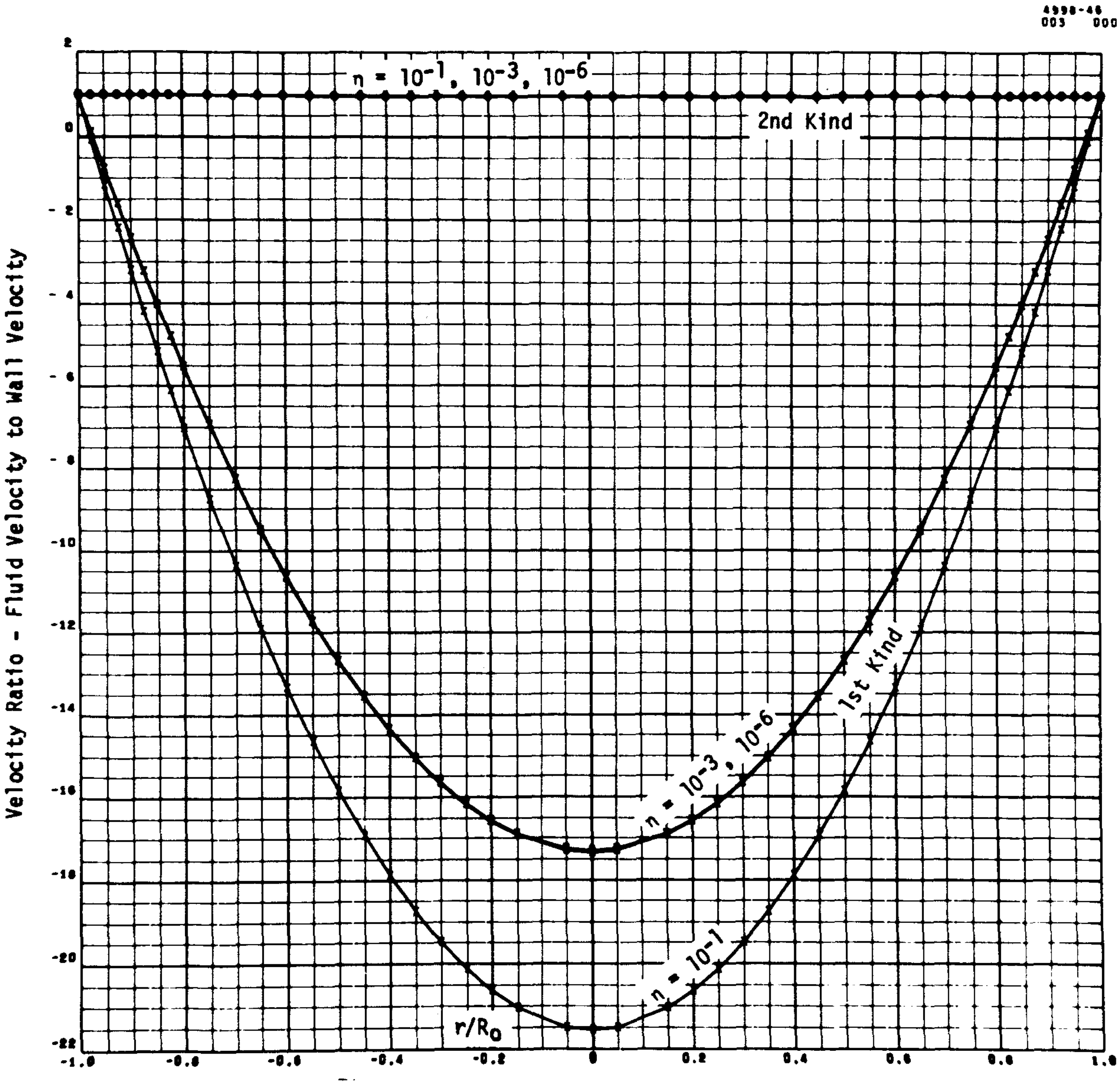

Figure 6a. Longitudinal Velocity Ratio—Fluid Velocity to Wall Velocity for $\xi = 0$, $z = 1$

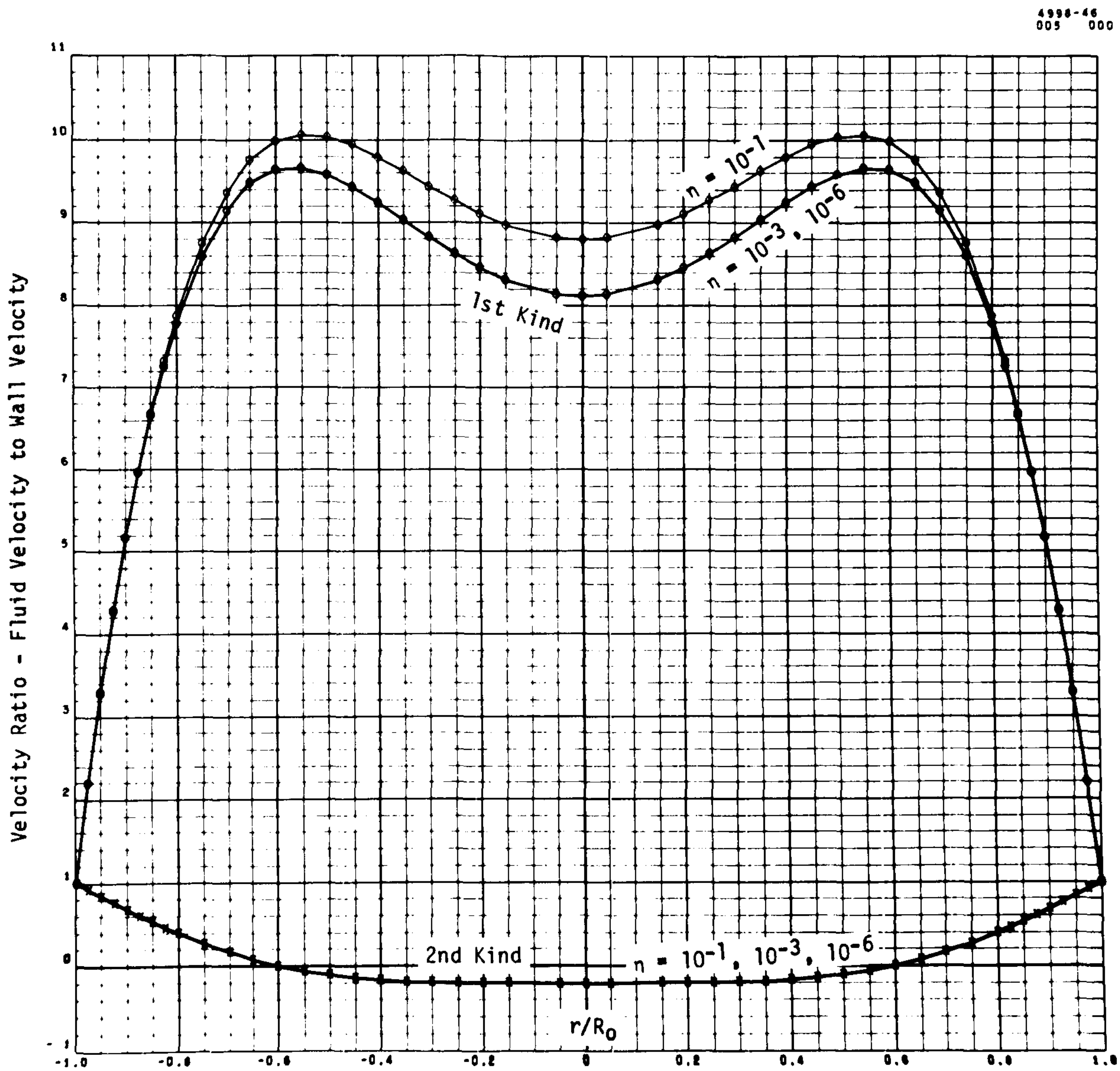

Figure 6b. Longitudinal Velocity Ratio—Fluid Velocity to Wall Velocity for $\xi = 0$, $z = 5$

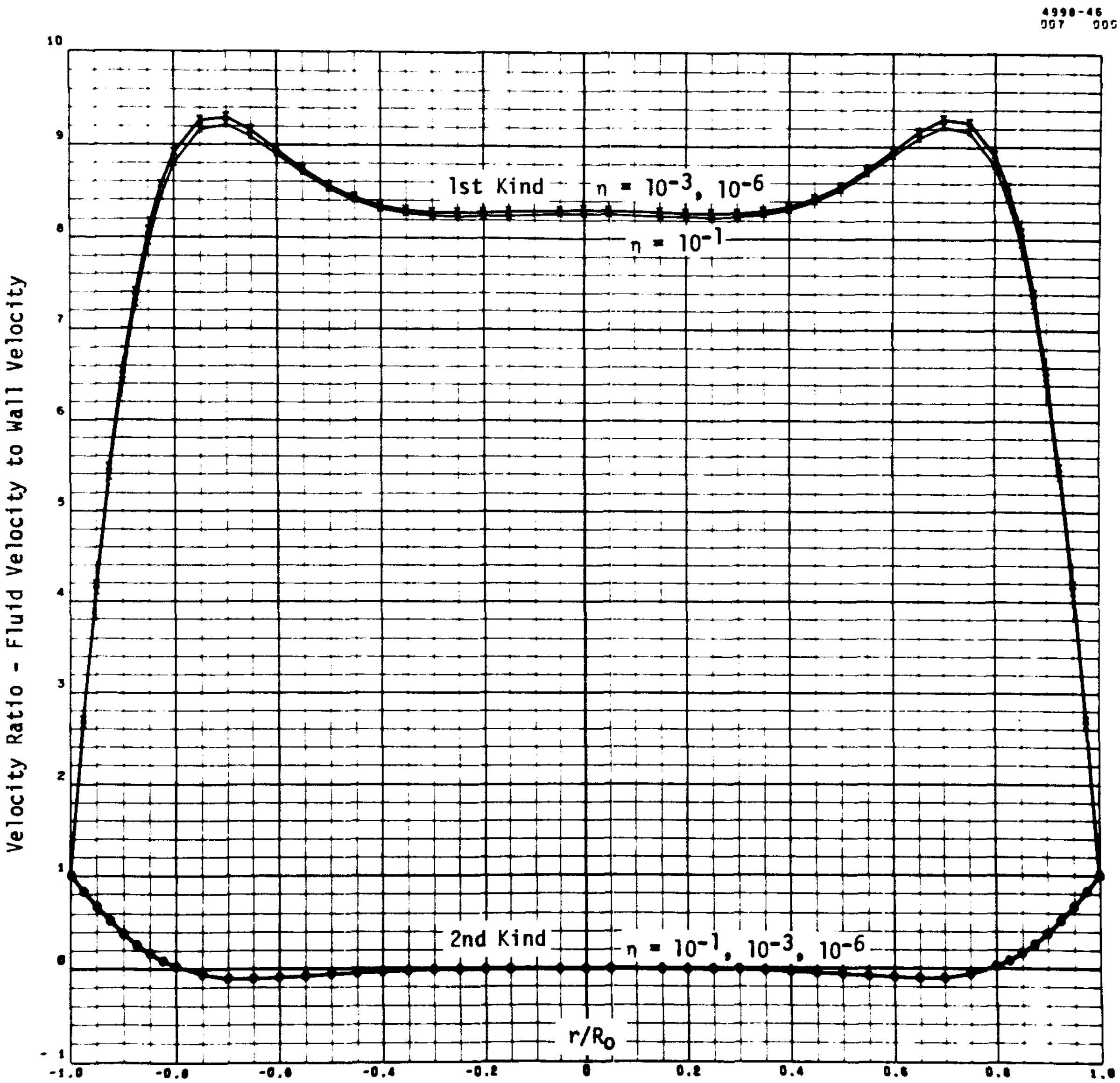

Figure 6c. Longitudinal Velocity Ratio— Fluid Velocity to Wall Velocity for $\xi = 0$, $z = 10$

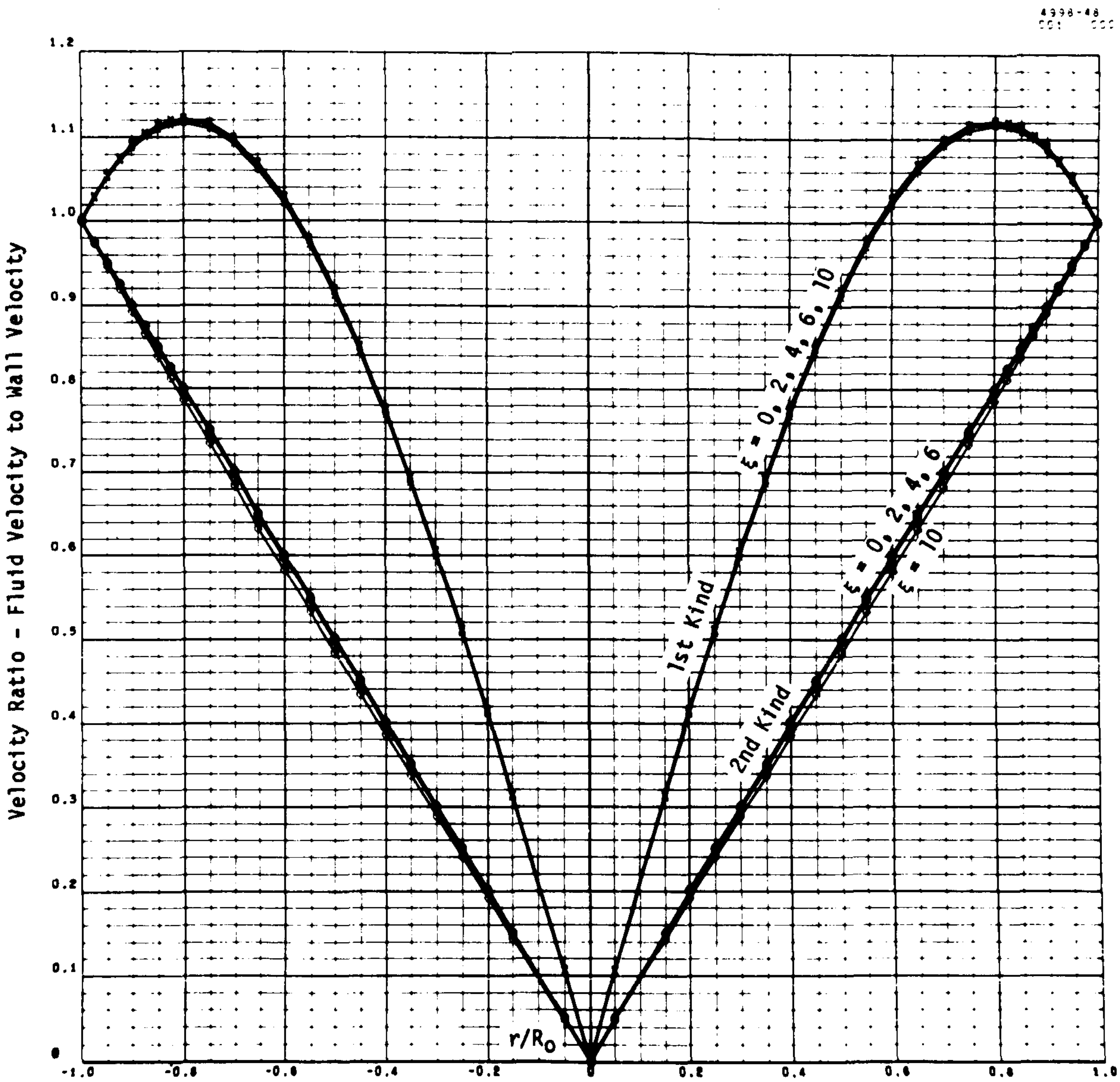

Figure 7a. Radial Velocity Ratio—Fluid Velocity to Wall Velocity for $\eta = 0$, $z = 1$

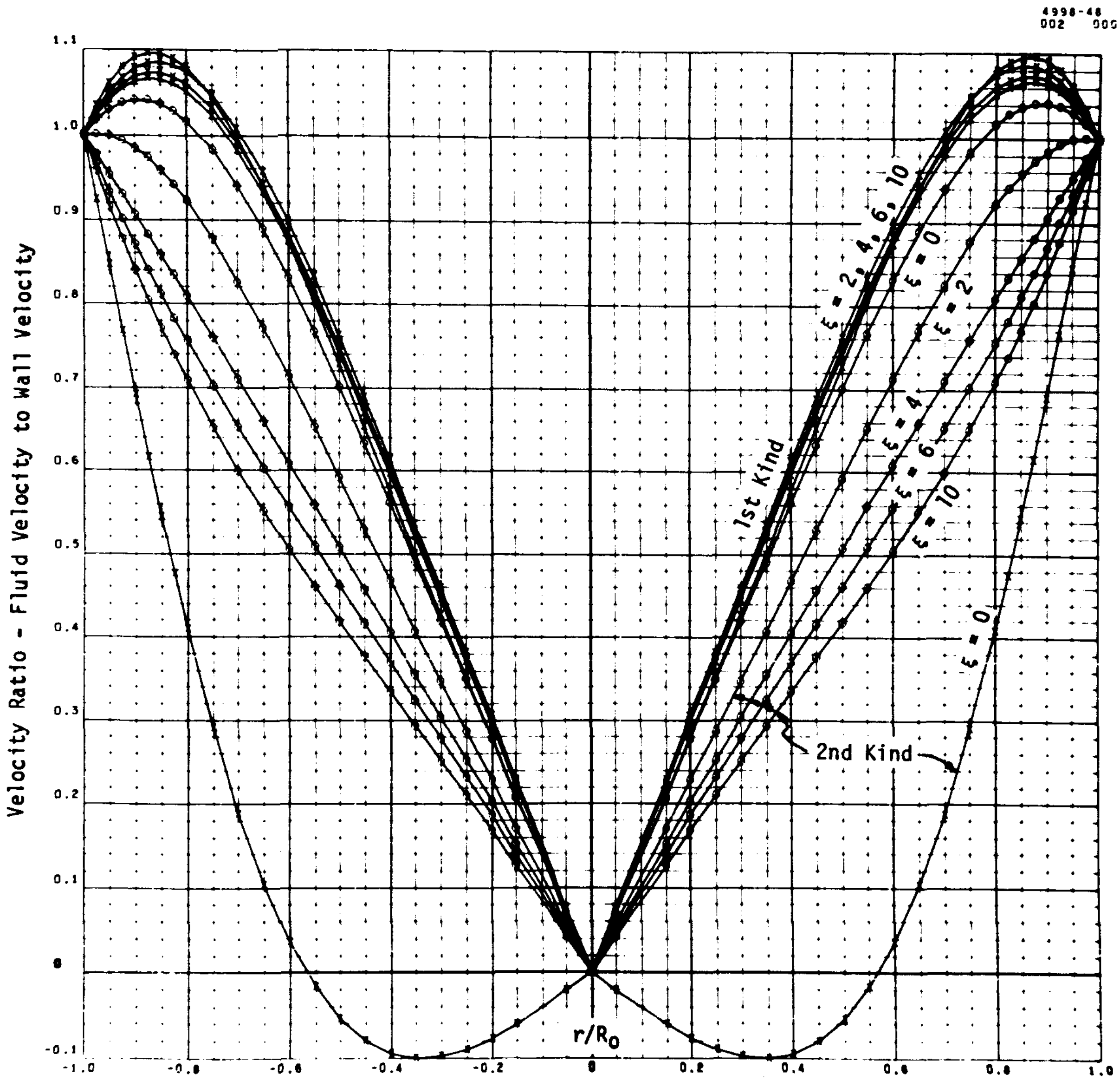

Figure 7b. Radial Velocity Ratio—Fluid Velocity to Wall Velocity for $\eta = 0$, $z = 5$

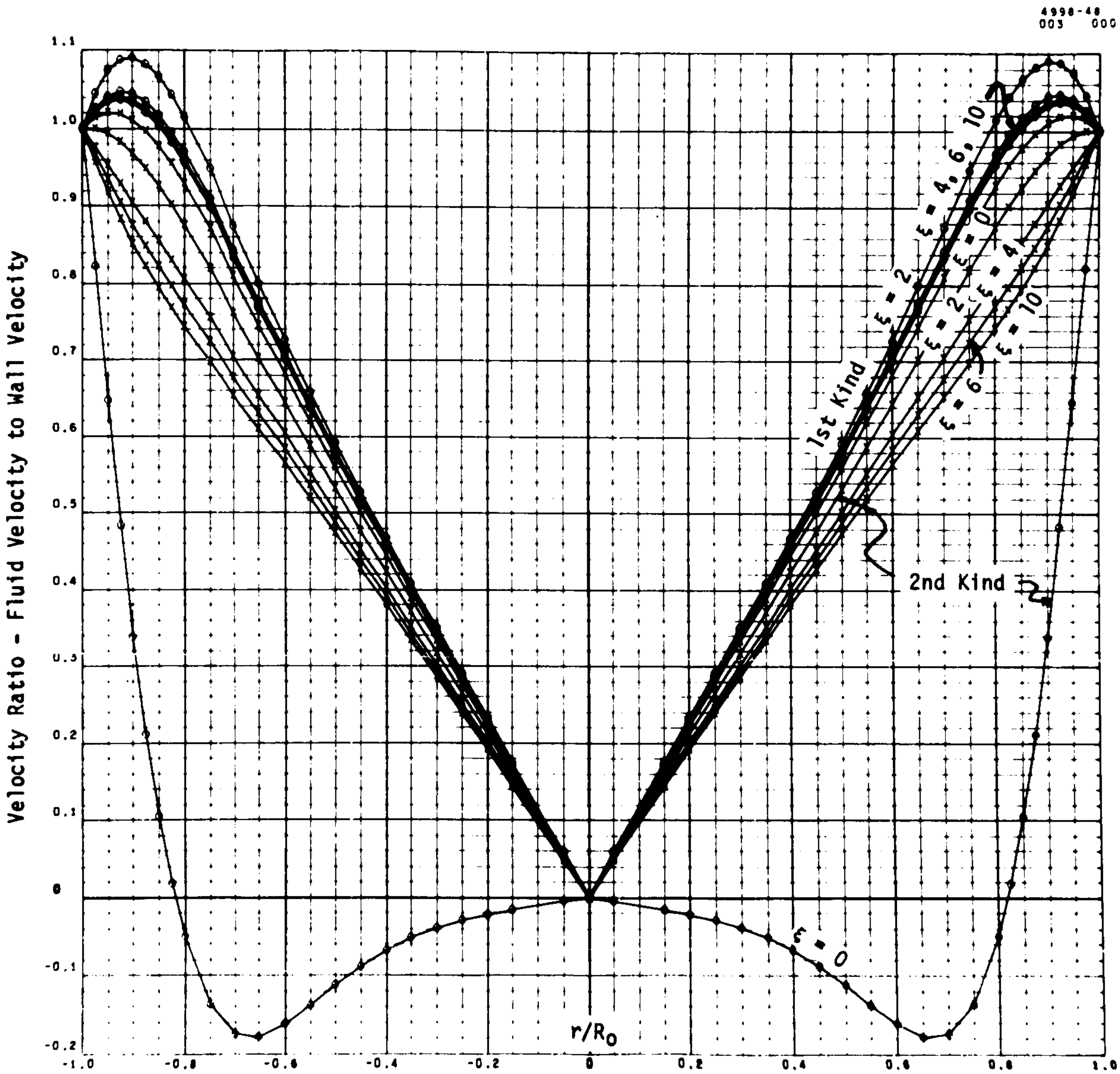

Figure 7c. Radial Velocity Ratio— Fluid Velocity to Wall Velocity for $\eta = 0, \quad z = 10$

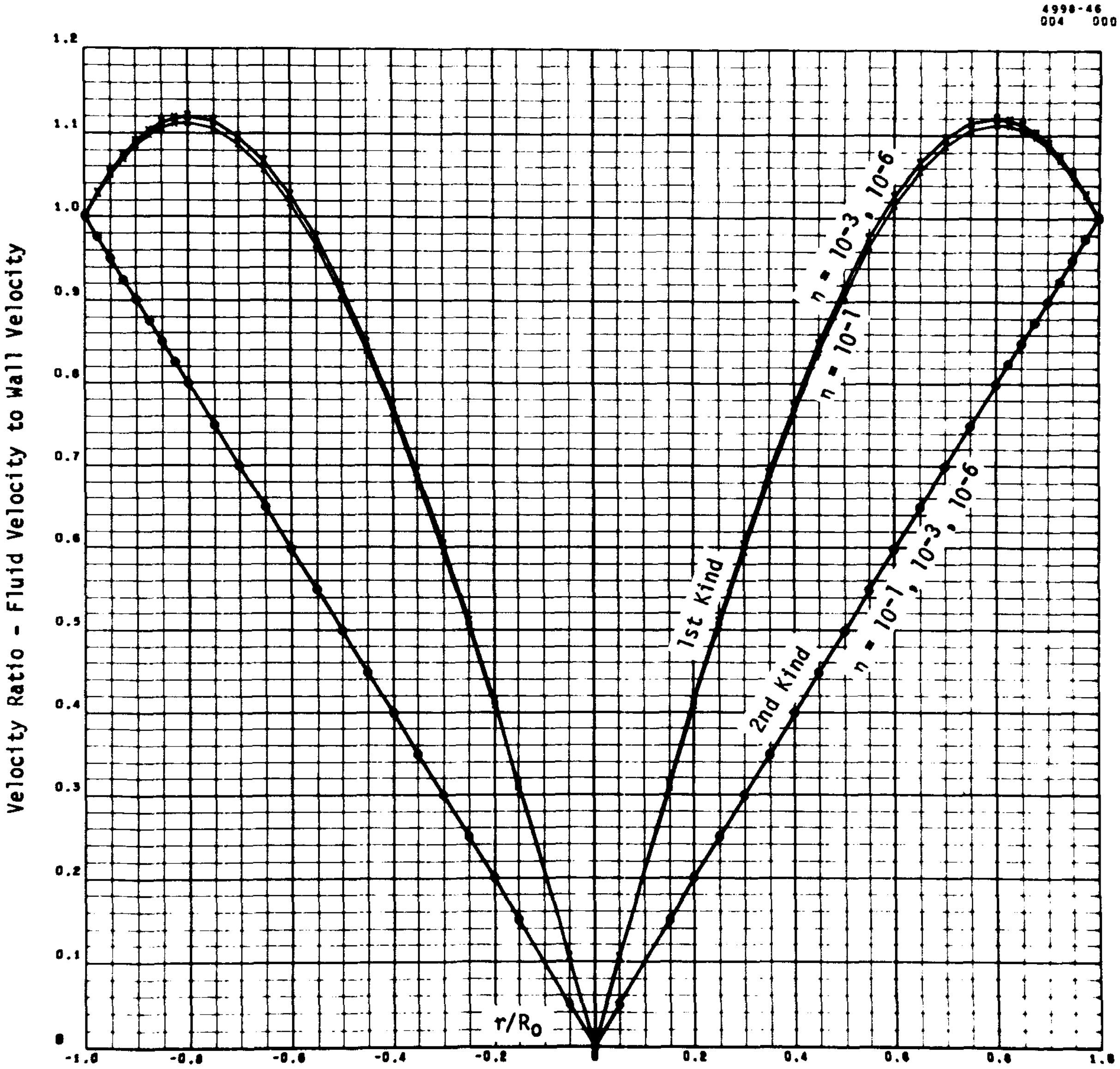

Figure 8a. Radial Velocity Ratio—Fluid Velocity to Wall Velocity for $\xi = 0$, $z = 1$

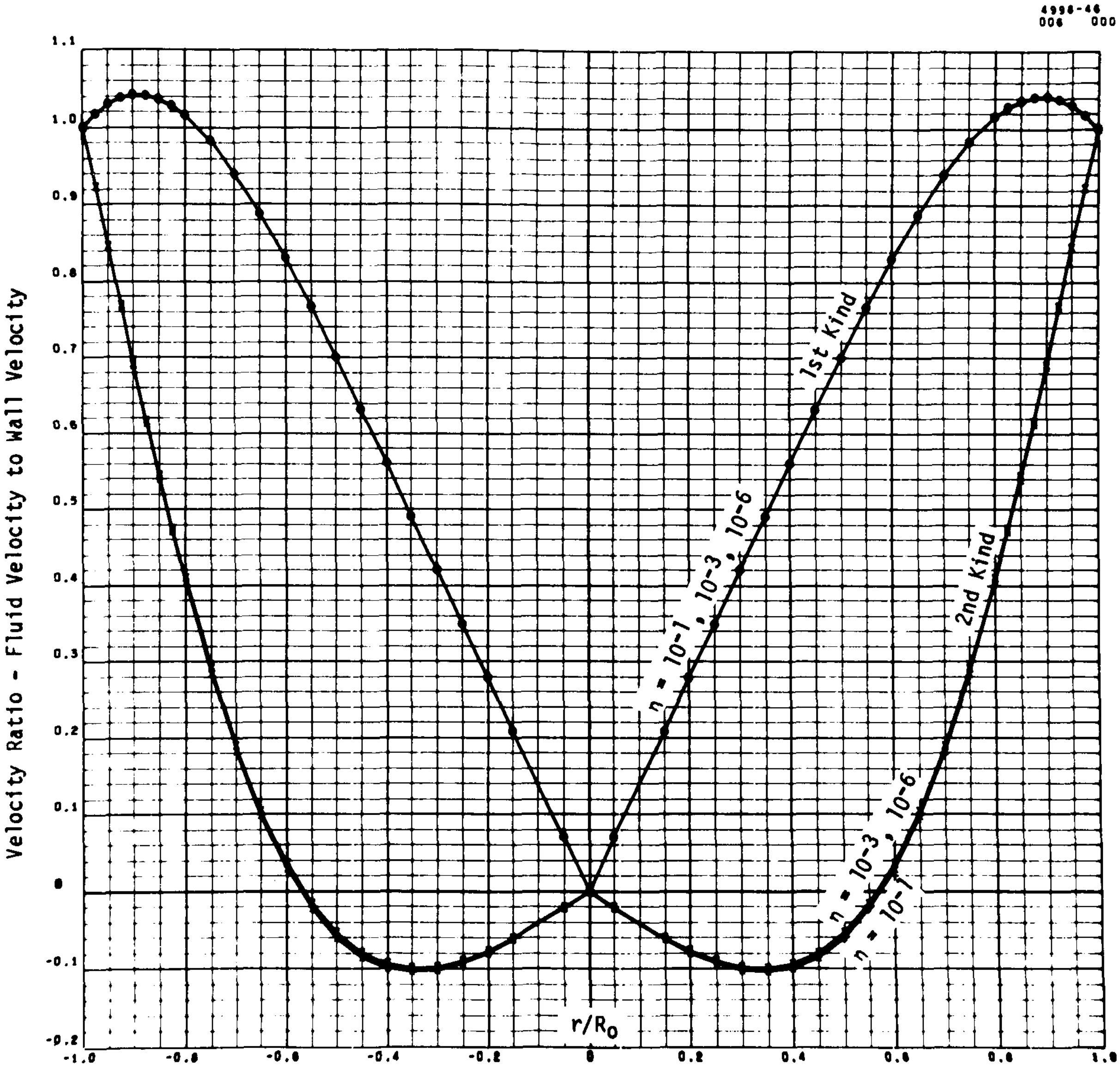

Figure 8b. Radial Velocity Ratio—Fluid Velocity to Wall Velocity for $\xi = 0$, $z = 5$

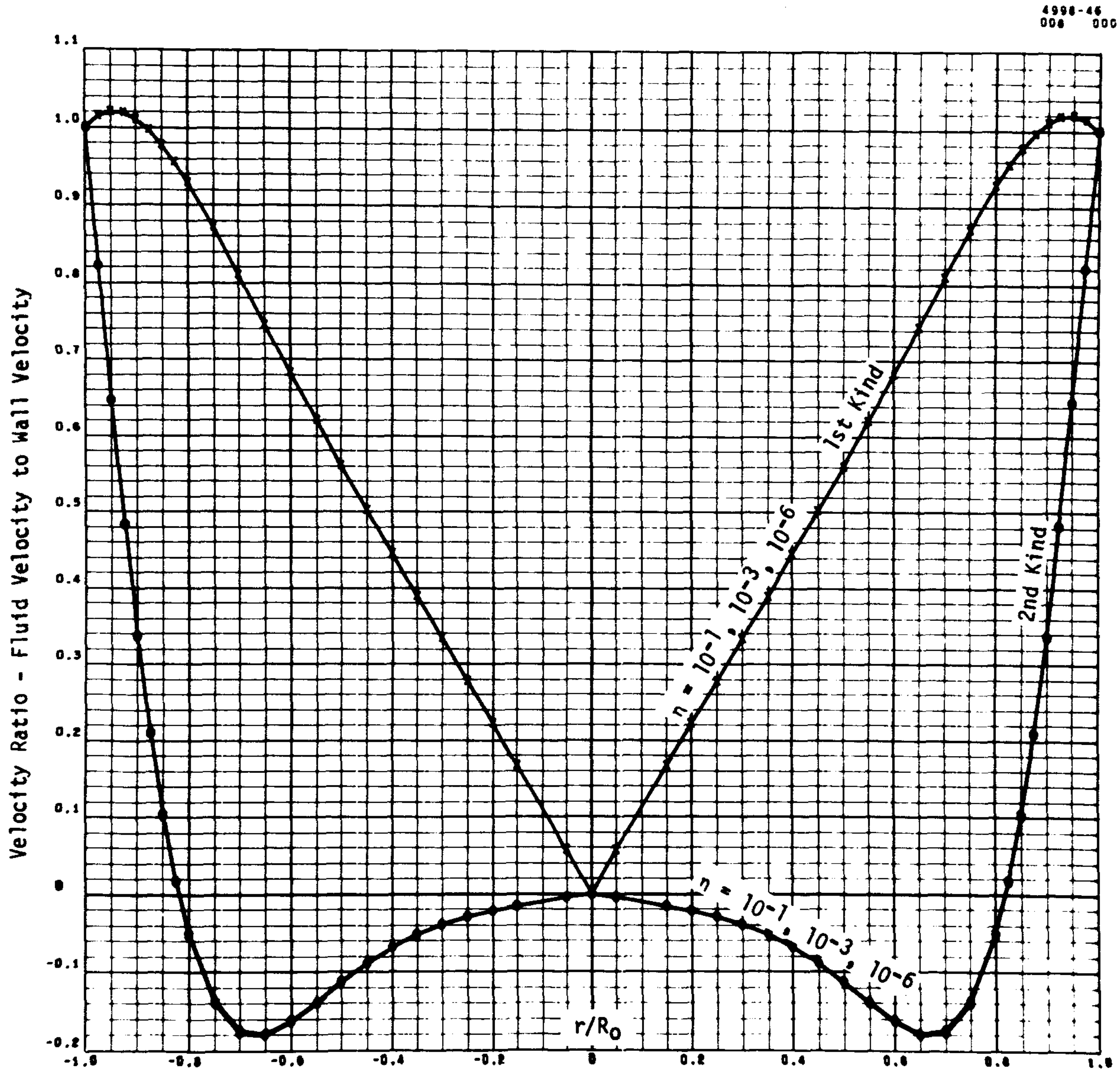

Figure 8c. Radial Velocity Ratio—Fluid Velocity to Wall Velocity for $\xi = 0$, $z = 10$

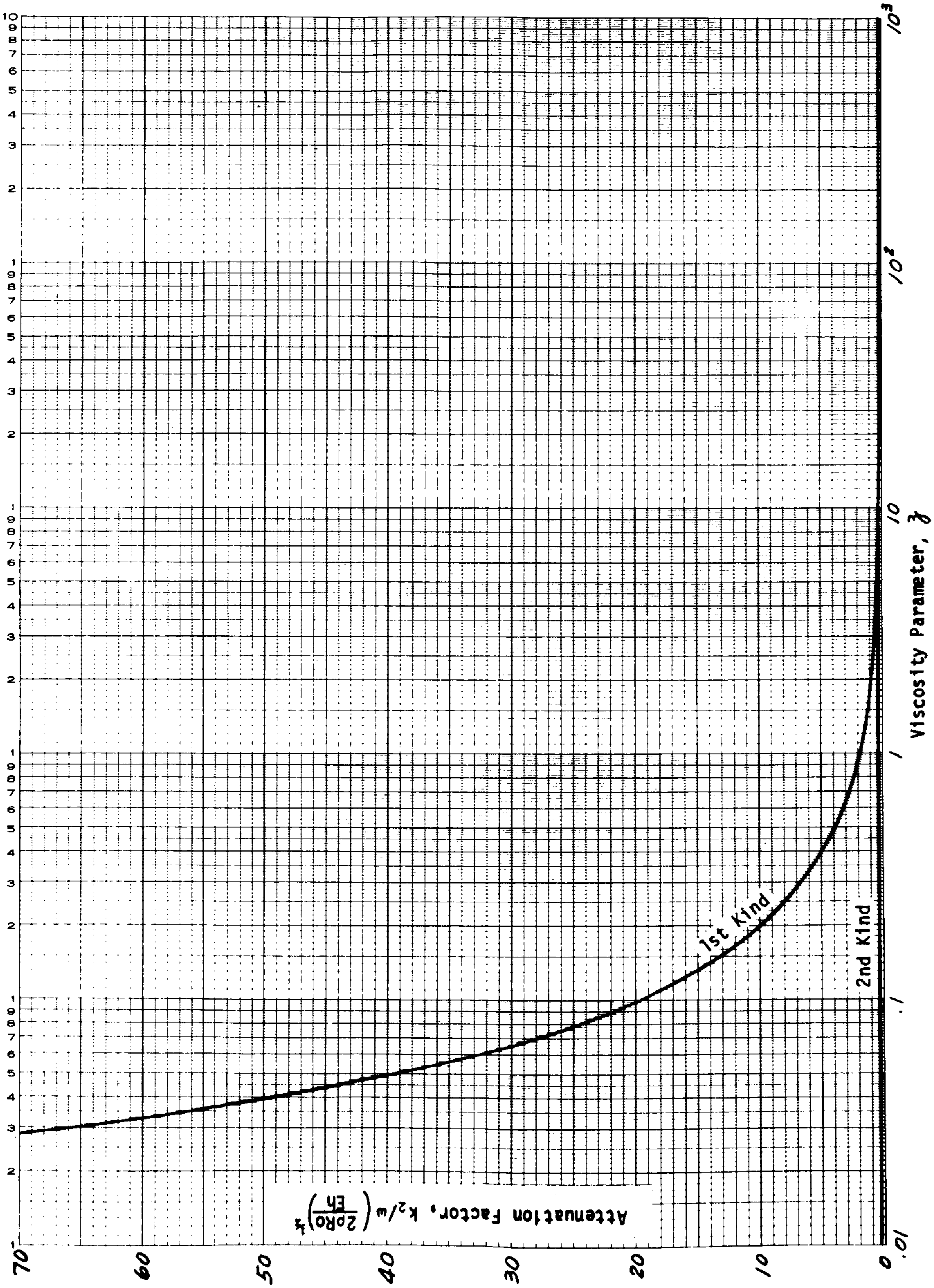

Figure 9. Attenuation Factor vs Viscosity Parameter

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