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APPLICATIONS OF RATE DIAGRAMS TO THE ANALYSIS AND DESIGN OF A CLASS OF ON-OFF CONTROL SYSTEMS

by Ralph C. Lake

Prepared by
UNIVERSITY OF TENNESSEE
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for

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CHAPTER I

INTRODUCTION

The design and analysis of on-off control systems can be very laborious if the system configuration is complex. This is because no general technique exists which would provide accurate information about the system performance for all classes of on-off systems.

Patapoff presented a method[1*] in which the performance of a class of on-off control systems may be analyzed. Patapoff's method, called the "rate diagram", is a plot of the output rate of a controlled element at "control removal" (removal of plant input) versus the rate at "control application" (application of plant input).

Patapoff's method used a Laplace Transformation of the error signal. Such an approach constrains the error signal filter to be linear. In this research the rate diagram idea is formulated by utilizing the state variable representation. This approach removes the constraint of the linear filter, thereby making the method applicable to a wide class of on-off systems.

It is the purpose of this paper to apply the rate diagram technique to some configurations of on-off control stystems. It is hoped that the illustration of specific applications will encourage

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*The superscript numbers represent similarly numbered references in the "List of References."

further study of this technique and its possible extensions to other classes of on-off systems.

CHAPTER II

THE RATE DIAGRAM METHOD

Consider the block diagram of a system shown in Figure 1. The controlled element is a second order pure inertia plant whose output position and output rate are represented by x_1 and x_2, respectively. It is assumed that the switch has dead space such that the loop transient always dies out before the application of control effort, λ , to plant input. Very often the dead space is deliberately introduced to avoid erratic switching caused by random noise. The switch may also possess hysteresis.

In a physical system there is a time delay, τ_R, between switch-on and the application of control effort. Similarly, a time delay, τ_F, exists between switch-off and cut-off of control effort. This phenomena is represented by the "delay" block in the figure. Notice that τ_R and τ_F are, in general, not equal. The "filter" block is inserted into the system to obtain the desired switching characteristic and to reduce noise effect. The filter can be either linear or nonlinear.

Systems of this type may appear as the stability subsystem or reaction subsystem of a spacecraft command module. The pure inertia plant may represent a spacecraft traveling outside the earth atmosphere.

The rate diagram is a plot of the system rate at control removal (x_{2f}) versus the rate at control application (x_{2i}). Since the loop transient dies out before control application, the output rate at

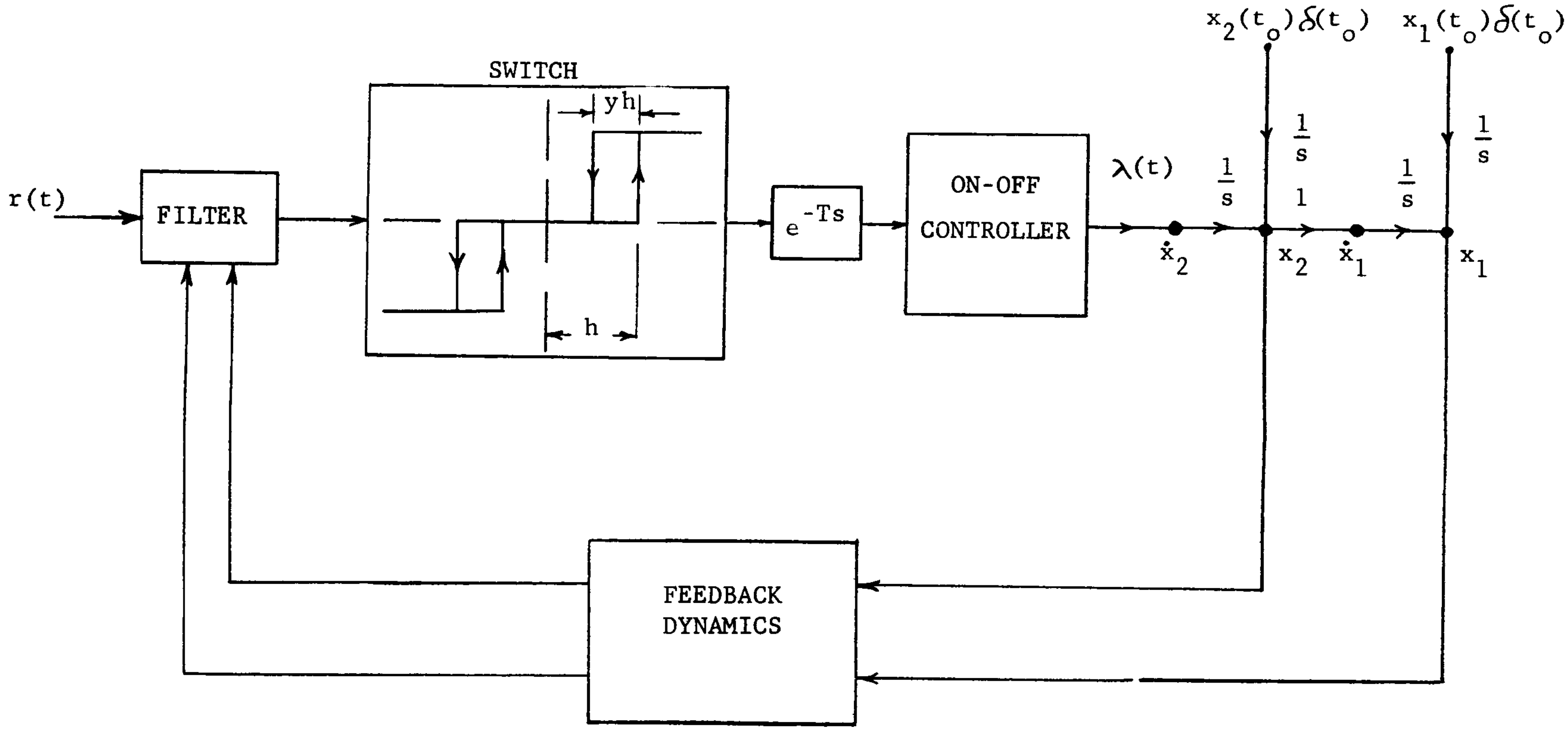
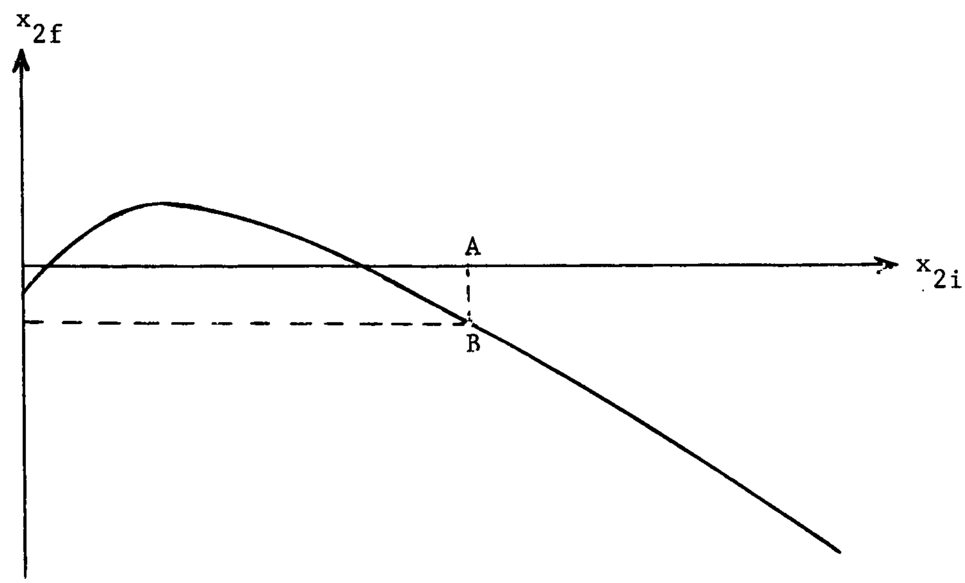


Fig. 1. An on-off control system with a pure inertia plant.

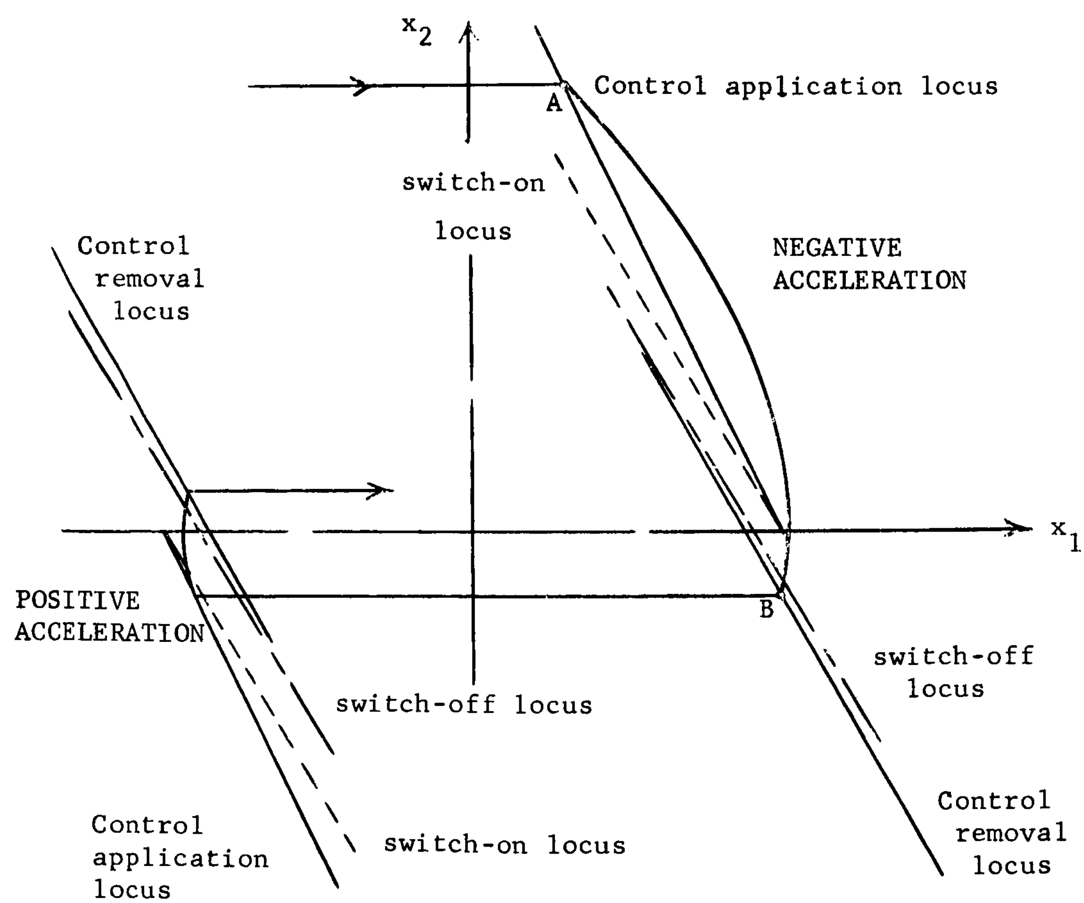
control removal can be expressed as a function of control application. If output amplitude characteristics of the switch and of the controller have odd symmetry with respect to their inputs, the rate diagram is also symmetrical with respect to the origin. Therefore, it is sufficient to consider only positive x_{2i} and determine the values of x_{2f} relative to x_{2i}.

The system stability, transient response, and limit cycle behavior can be analyzed via rate diagrams. Phase plane trajectories can readily be constructed from the rate diagram, and vice versa. Figure 2 illustrates how to construct the phase plane trajectory from the rate diagram. Fig. 2a represents a rate diagram of a typical system and Fig. 2b is the phase plane. Draw the control application line on phase plane, which is a function of x_1 and x_2 only. Before control application the system output coasts at a constant rate until it reaches the control application line at A. Starting from A the system follows a constant acceleration trajectory until it reaches B. Point B is the point where the control is removed from the plant input. This point is given directly from the corresponding B point on the rate diagram. Starting from B the system again coasts at constant rate until it reaches the controller application line on the opposite side.

The behavior of the on-off system can be studied with the aid of the rate diagram. When the rate curve lies in the first quadrant of the rate plane, a stepping action occurs (Fig. 3a). That is, the phase trajectory, which alternates between free coasting and constant acceler-



(a)



(b)

Fig. 2. Constructing phase trajectory from rate diagram.

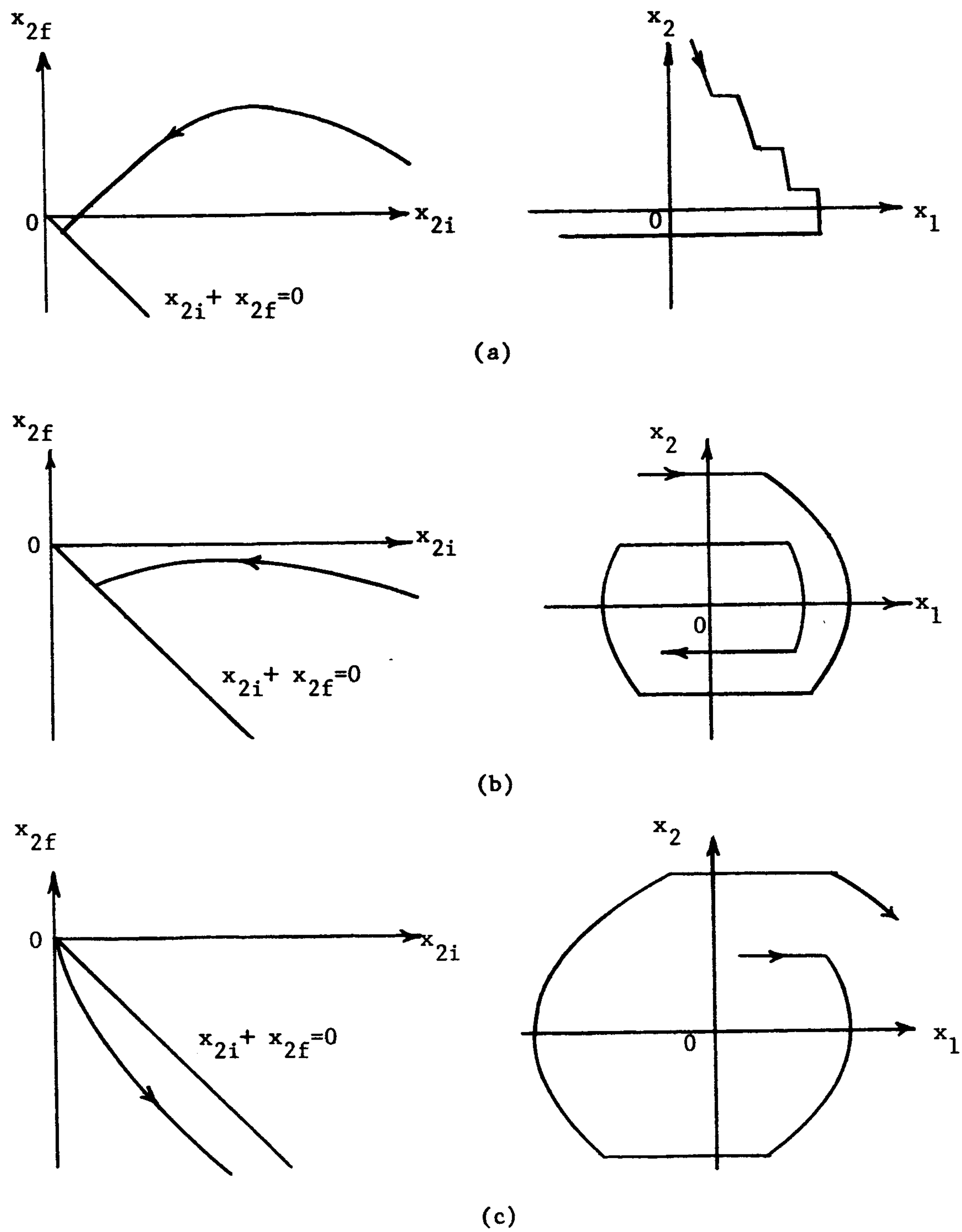


Fig. 3. Rate diagrams and corresponding phase trajectories.

ation, has a step pattern on phase plane. Under this condition the system eventually converges to a stable limit cycle. When the rate curve lies between the x_{2i}-axis and the line $x_{2i} + x_{2f} = 0$, the system output is oscillatory but converges toward a stable limit cycle (Fig. 3b, page 7). If the rate curve lies between the x_{2f}-axis and the line $x_{2i} + x_{2f} = 0$, the system rate is divergent. And, unless the rate curve recrosses the $x_{2i} + x_{2f} = 0$ line as x_{2i} increases, the system would be unstable.

For most cases, the rate curve will intersect the line: $x_{2i} + x_{2f} = 0$. An intersection indicates the existance of a limit cycle, since $x_{2f} = -x_{2i}$ at such a point.

The vector-state differential equations for the state variables of the controlled element are:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ \lambda(t) \end{bmatrix} . \qquad (1)$$

In state-vector notation, the above equation may be rewritten:

$$\underline{\dot{x}} = A\,\underline{x} + C\,\underline{m} \qquad (2)$$

where:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad (3)$$

and

$$C = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \qquad (4)$$

The vector state differential equation may be solved by any of several methods.[3,4,5] For the case where the input of the controlled element is of the form: $\lambda(t) = \lambda$ = a constant, the solution is:

$$\begin{bmatrix} x_1\ (t - t_o) \\ \\ x_2\ (t - t_o) \end{bmatrix} = \begin{bmatrix} x_1(t_o) + (t - t_o)\ x_2(t_o) - \frac{\lambda}{2}(t - t_o)^2 \\ \\ x_2\ (t_o) - (t - t_o) \end{bmatrix} . \quad (5)$$

For the derivation of the rate diagram equations, consider r(t) = 0. This defines an equilibrium state: $\underline{x} = 0$. Thus, the response from any initial state $\underline{x}(t_o)$ will be defined.

From the preceding discussion, it is clearly seen that much information is contained in a rate diagram. The advantages of the rate diagram over phase plane methods are obvious: (1) the rate diagram curve considers all non-zero values of initial system rate on one plot. This would be impractical for phase plane plots. (2) The effects of changes in the values of the system parameters are shown directly. This can often be difficult to surmise using phase plane techniques. (3) The rate curves of different system configurations can be shown on one plot in order to compare the characteristics of the different configurations. This would be virtually impossible to do using phase plane methods without a resulting mesh of trajectories, thus making it difficult to obtain meaningful information.

CHAPTER III

RATE DIAGRAM ANALYSIS OF ON-OFF SYSTEM 1

DERIVATION OF EQUATIONS

Consider the system shown in Figure 4, with an initial positive rate and a displacement such that the error signal lies within the deadband of the switch. By inspection of Figure 4, the error signal for the above configuration may be written as:

$$\epsilon = - G_1 G_3 x_1 - G_2 x_2 \tag{6}$$

where G_1, G_2, and G_3 are constants.

The system will switch on when $\epsilon = - h$. Denote x_{1s} and x_{2s} as the values of x_1 and x_2 at the instant of switch-on, and note that x_2 will be constant prior to control application. By substituting Equation (5) into Equation (6), x_{1s} is found to be:

$$x_{1s} = \frac{h}{G_1 G_3} - \frac{G_2 x_{2s}}{G_1 G_3} . \tag{7}$$

If t_o is the instant of switch-on, τ_R seconds later control will be applied to the plant. During this time interval $\lambda = 0$, we have from Equation (5)

$$x_1(t_o + \tau_R) = x_1(t_o) + \tau_R x_2(t_o) \tag{8}$$

and

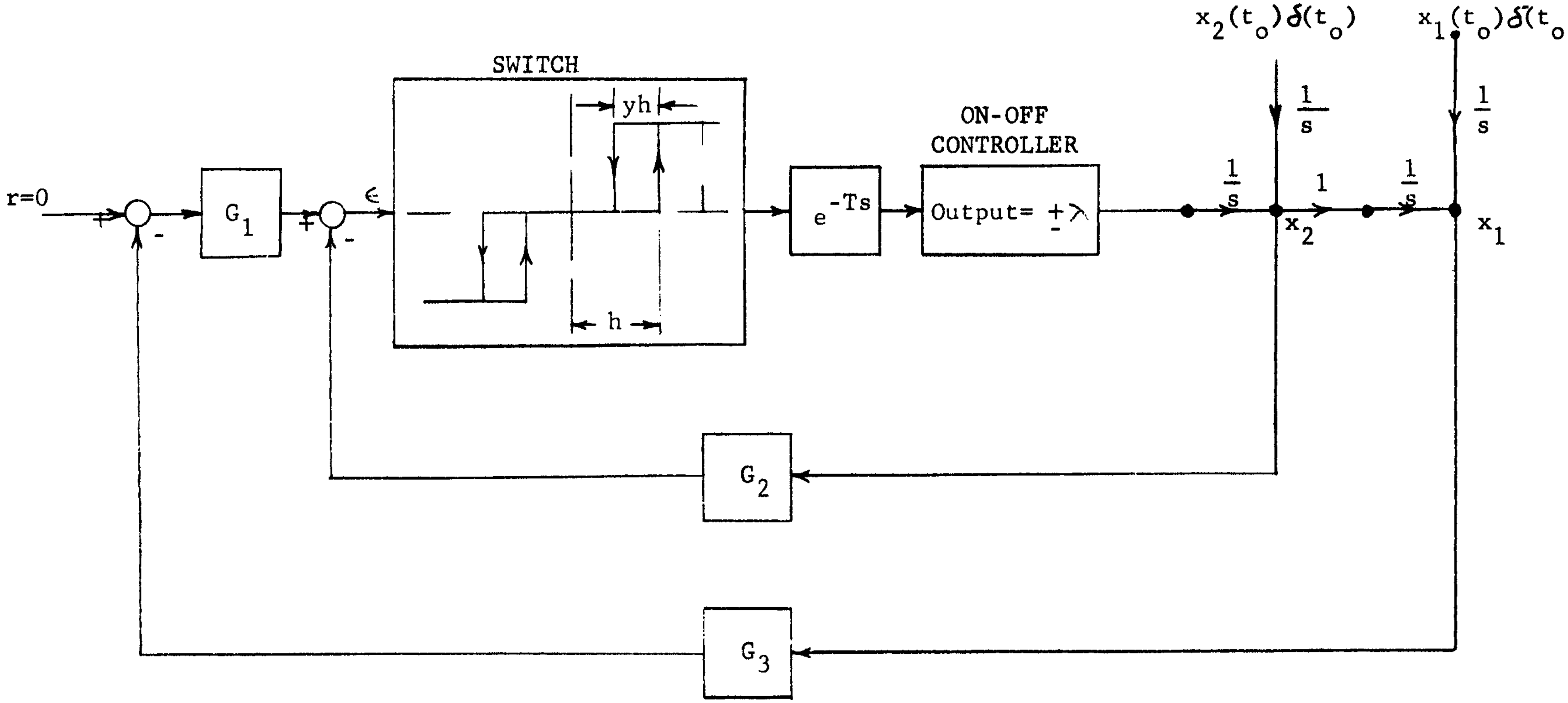


Fig. 4. Block diagram of on-off system 1.

$$\begin{bmatrix} x_{1i} \\ \\ x_{2i} \end{bmatrix} = \begin{bmatrix} x_1(t_o + \tau_R) \\ \\ x_2(t_o + \tau_R) \end{bmatrix} = \begin{bmatrix} \frac{h}{G_1G_2} - \frac{G_2x_{2s}}{G_1G_3} + x_{2s}\tau_R \\ \\ x_{2s} \end{bmatrix} . \quad (9)$$

If t = 0 is chosen as the time instant of control application, then by substituting Equation (5) into (6) the error may be expressed as

$$\epsilon(t) = -G_1G_3x_{1i} - G_1G_3x_{2i}t + G_1G_3 \frac{\lambda t^2}{2}$$

$$- G_2x_{2i} + G_2 \lambda t . \quad (10)$$

The system will switch-off when $\epsilon = -h + yh$. Define t_1 as the interval between the beginning of control application and switch-off. Using this condition in Equation (10), t_1 is found to be

$$t_1 = \frac{-B_1 + \sqrt{B_1^2 - 4A_1C_1}}{2A_1} \quad (11)$$

where

$$\left. \begin{aligned} A_1 &= \frac{G_1G_3\lambda}{2} \\ B_1 &= G_2\lambda - G_1G_3x_{2i} \\ C_1 &= h - G_1G_3x_{1i} - G_2x_{2i} - yh \end{aligned} \right\} . \quad (12)$$

τ_F seconds after switch-off, control is removed from the plant. Denote x_{1f} and x_{2f} as the values of x_1 and x_2 at control removal. Using Equation (5)

$$\begin{bmatrix} x_{1f} \\ x_{2f} \end{bmatrix} = \begin{bmatrix} x_{1i} + x_{2i}(t_1 + \tau_F) - \frac{\lambda}{2}(t_1 + \tau_F)^2 \\ x_{2i} - \lambda(t_1 + \tau_F) \end{bmatrix} . \qquad (13)$$

The rate diagram for this system may now be constructed by computing and plotting values of x_{2f} for arbitrary values of x_{2i} using Equations (11), (12) and the second one of (13).

THE RATE DIAGRAMS

The nominal values of system parameters used for constructing the diagrams are listed in Table I. Figures 5 to 11 are rate diagrams for various system parameters.

REMARKS

The rate diagrams for System 1 show that the system is stable for the values of system parameters considered. In addition, they indicate that an "over-shooting" response results from large values of initial rate, and a "stepping" response results from small values of initial rate.

The limit cycle rate is affected as follows:

1. The limit cycle rate increases as G_1 increases.
2. The limit cycle rate decreases as G_2 increases.
3. The limit cycle rate increases as h_1 increases.
4. The limit cycle rate increases as y increases.
5. The limit cycle rate increases as λ increases.

TABLE I

NOMINAL VALUES OF SYSTEM PARAMETERS FOR SYSTEM 1

Parameter	Nominal Value	Units
G_1	2.0	ND
G_2	1.0	Degree-sec./degree
G_3	1.0	ND
h	2.0	Degrees
y	0.05	ND
λ	6.0	Degrees/sec.^{2}
τ_R	0.02	Seconds
τ_F	0.02	Seconds

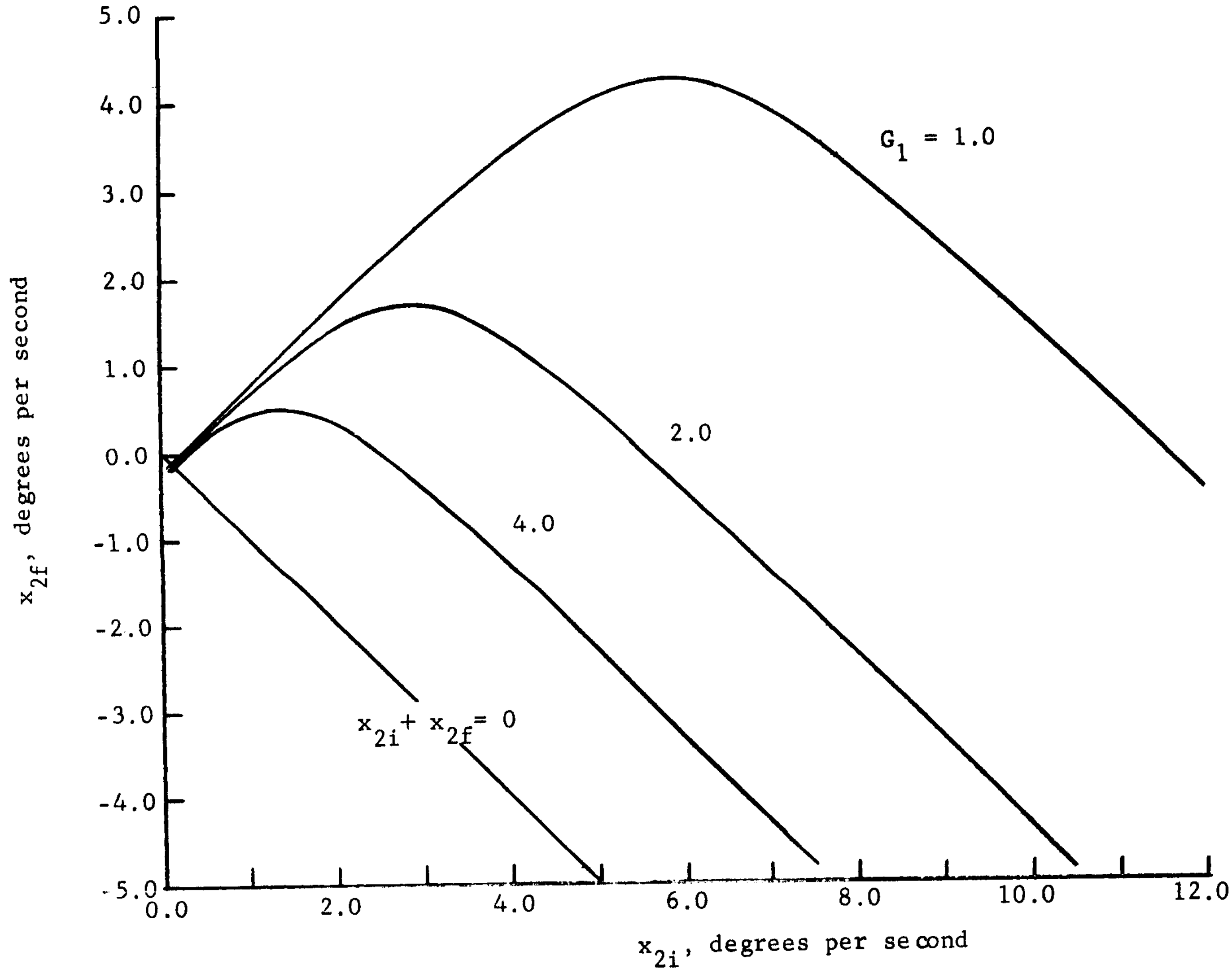


Fig. 5. Rate diagram of system 1 for variations of G_1.

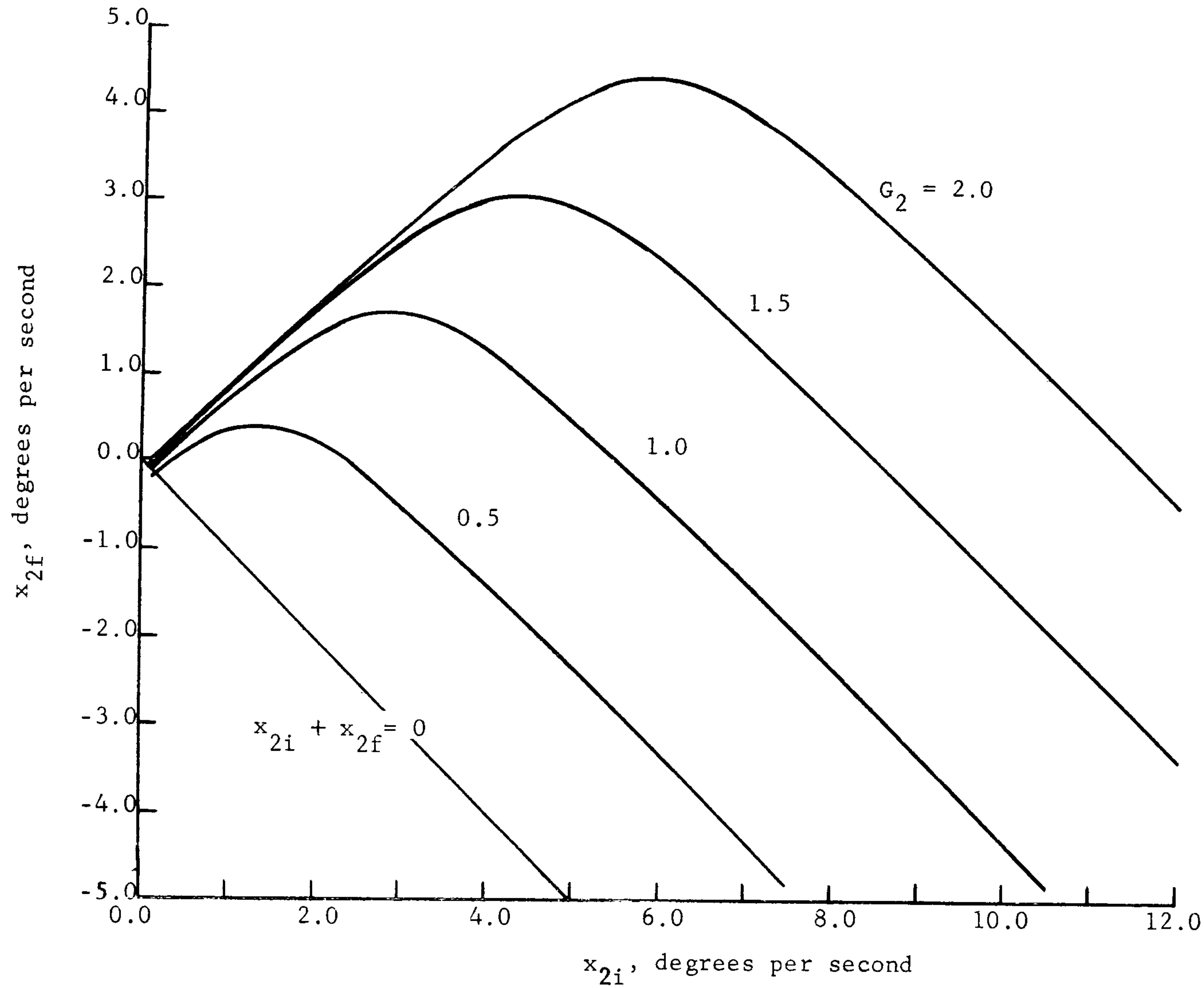


Fig. 6. Rate diagram of system 1 for variations of G_2.

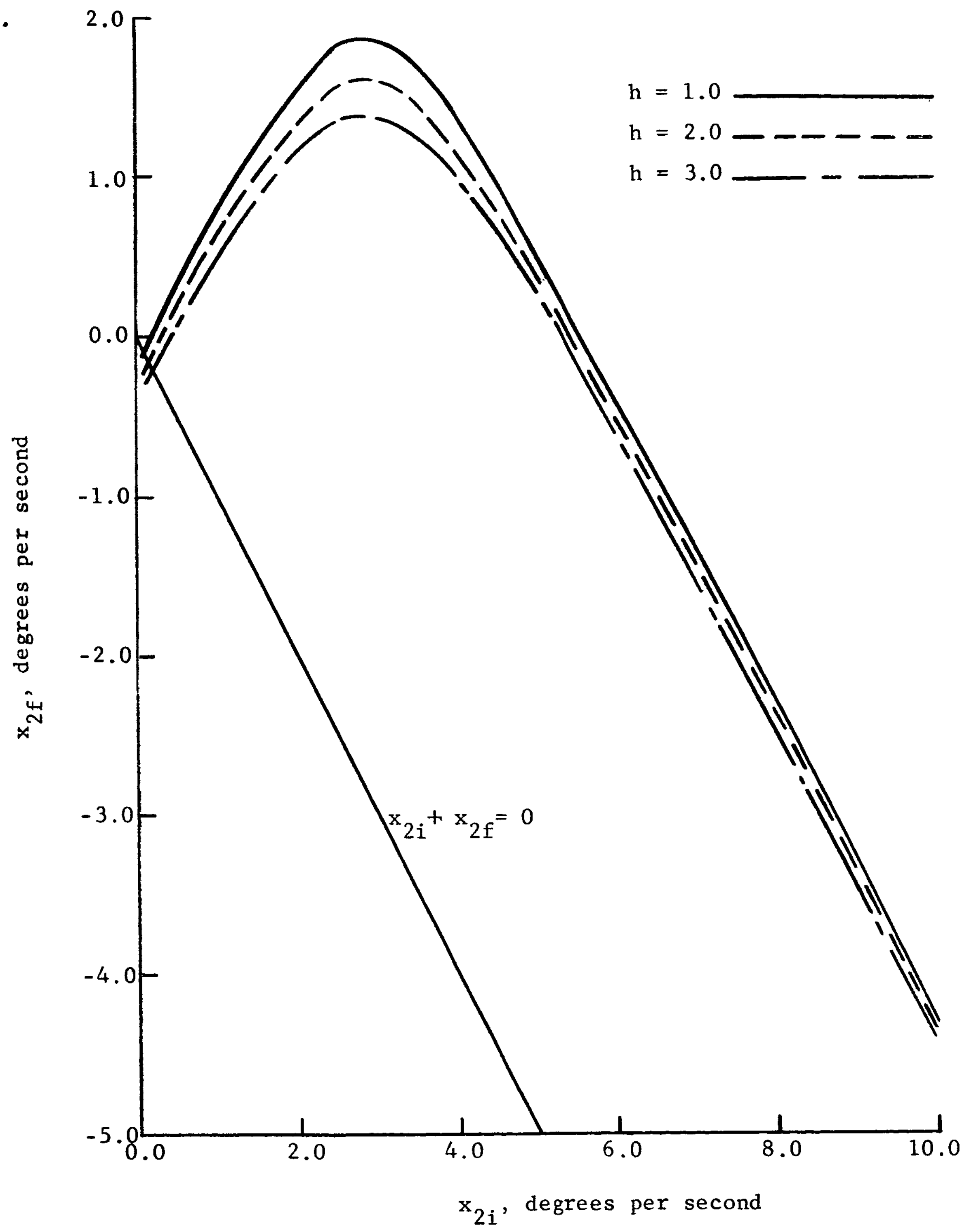


Fig. 7. Rate diagram of system 1 for variations of h.

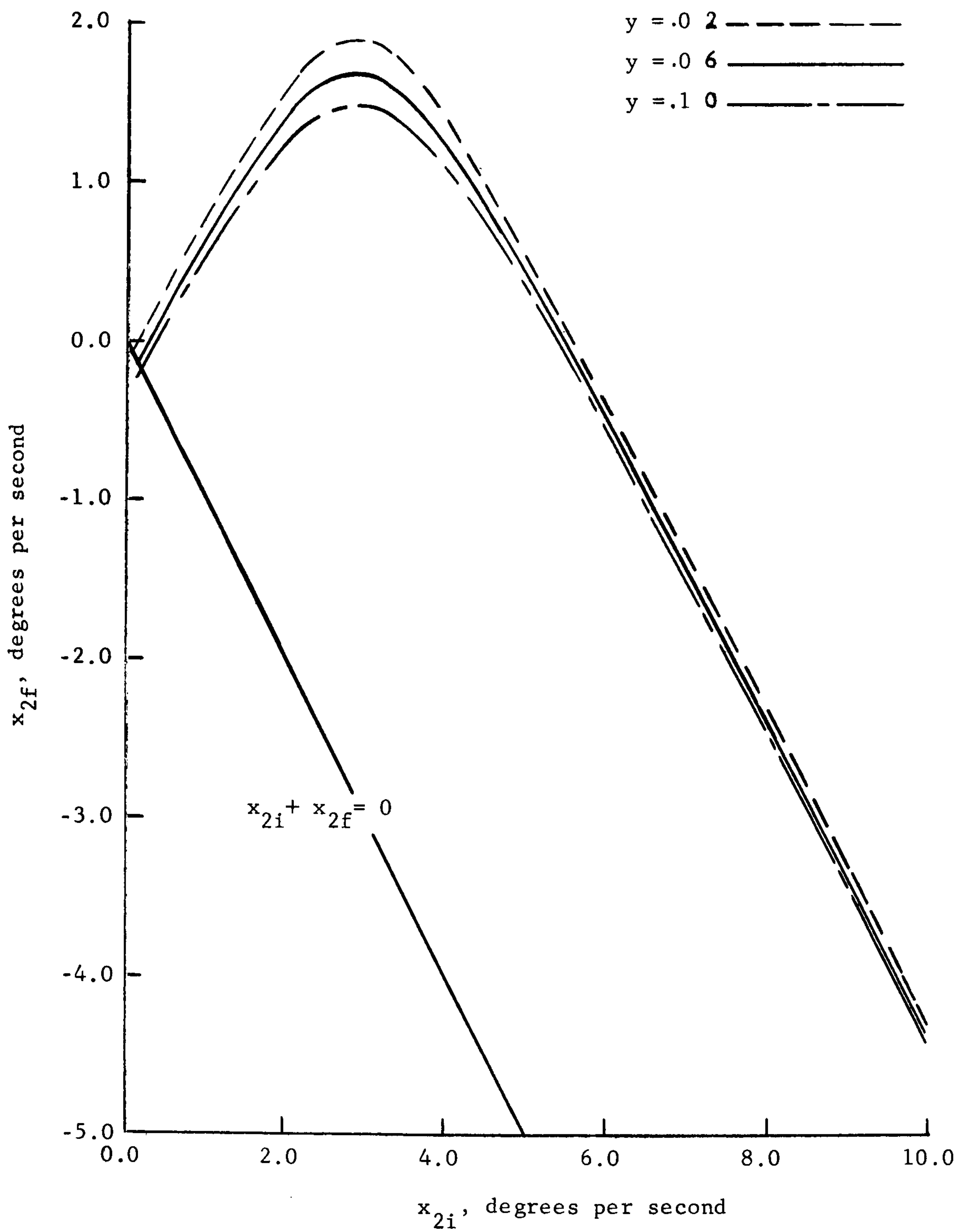


Fig. 8. Rate Diagram of System 1 for variations of y.

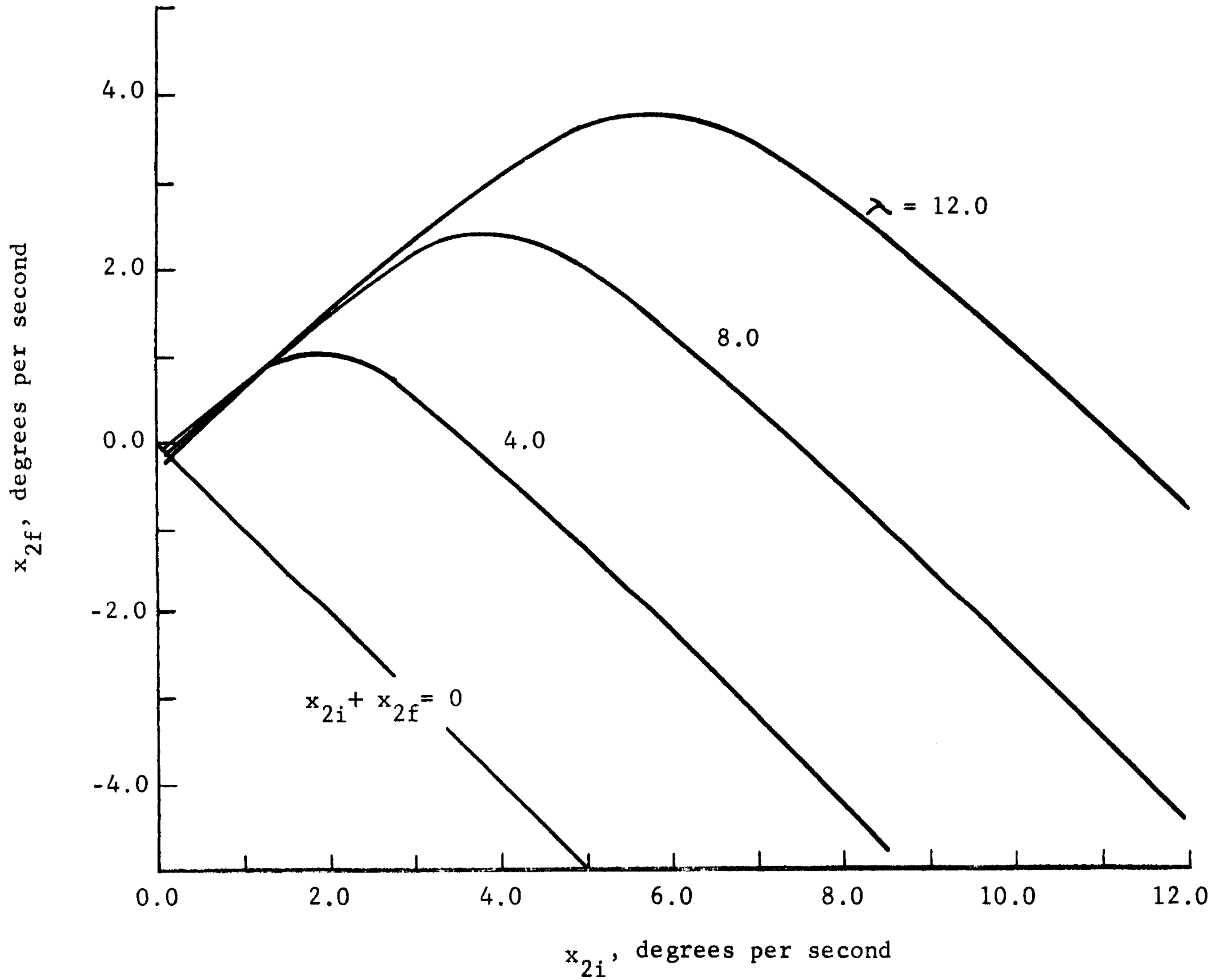


Fig. 9. Rate diagram of system 1 for variations of λ.

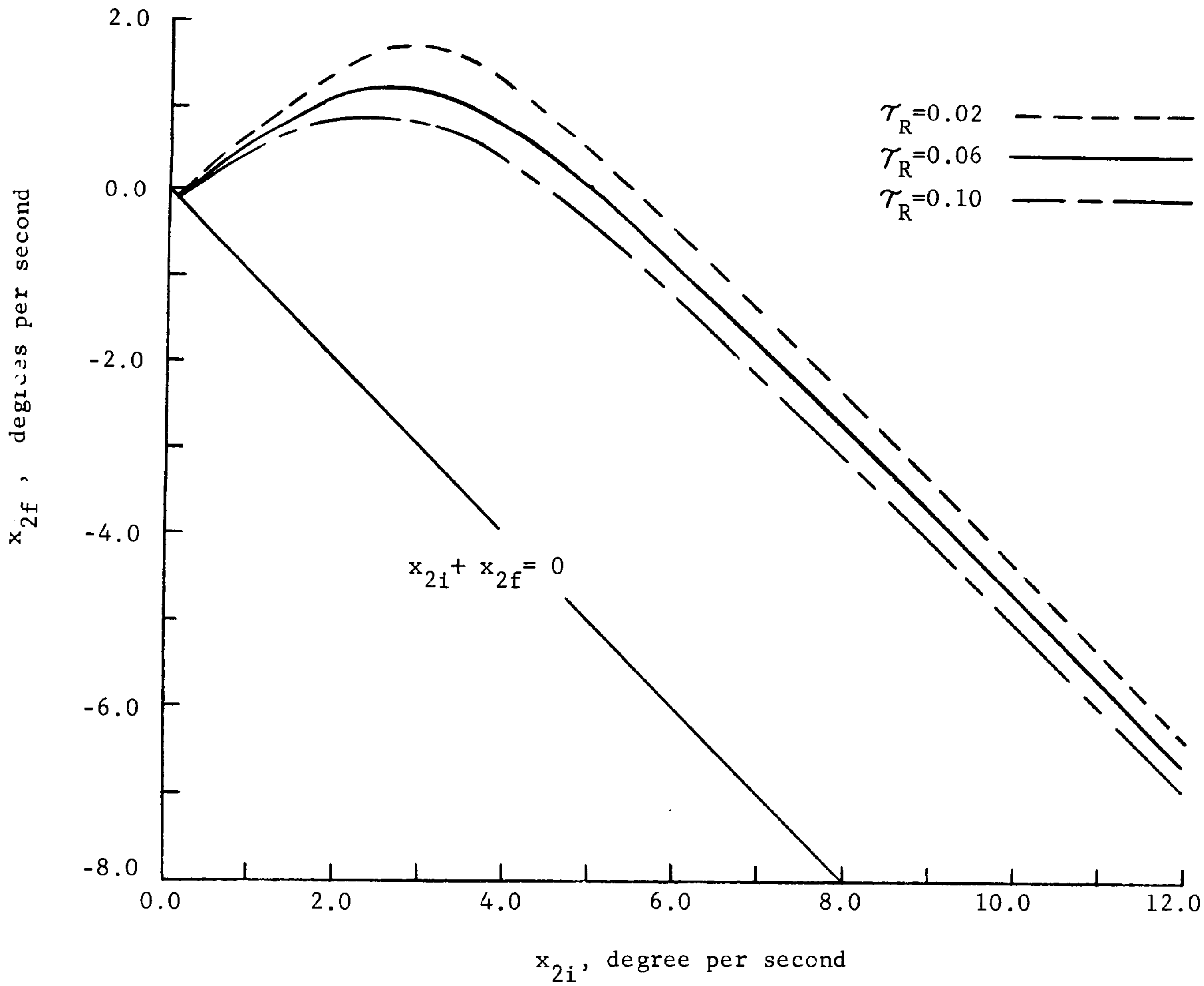


Fig. 10. Rate diagram of system 1 for variations of τ_R, the transport lag of thrust build-up.

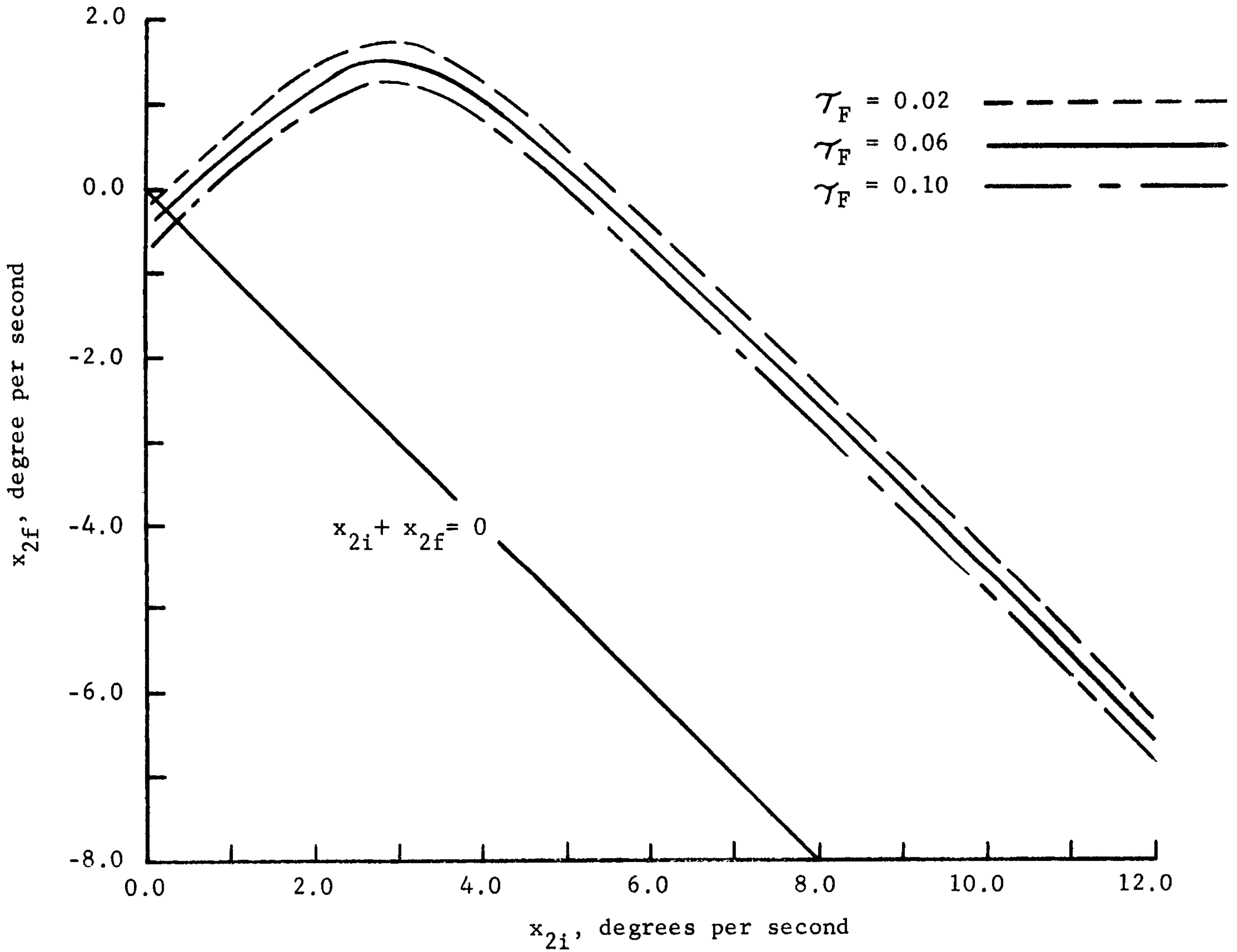


Fig. 11. Rate diagram of system 1 for variations of τ_F, the transport lag of thrust decay.

6. The limit cycle rate is independent of τ_R.

7. The limit cycle rate increases as τ_F increases.

CHAPTER IV

RATE DIAGRAM ANALYSIS OF ON-OFF SYSTEM 2

DERIVATION OF EQUATIONS

Consider the system shown in Figure 12 with an initial positive rate and a displacement such that the error signal lies within the deadband of the switch. It will be necessary to classify the magnitude of x_{2i} in order to determine the correct error signal mode.

I. $\left| G_2 \, x_{2i} \right| \leq h$

By inspection of Figure 12, the error signal mode for this magnitude of x_{2i} is given by:

$$\epsilon = - G_2 \, x_2 + G_1(-G_3 \, x_1 + d) \tag{14}$$

The system will switch on when $\epsilon = - h$.

Denoting the value of x_1 and x_2 at switch-on as $x_1(t_o)$ and $x_2(t_o)$ respectively, the expression for $x_1(t_o)$ may be obtained by substituting the above boundary condition into Equation (14) to yield:

$$x_1(t_o) = \frac{- G_2 \, x_2(t_o) + h \; + G_1 d}{G_1 G_3} \quad . \tag{15}$$

Control will be applied to the system τ_R seconds after switch-on occurs. Denoting the values of x_1 and x_2 at the beginning of control application as x_{1i} and x_{2i}, and by utilizing Equation (5), the following expressions are obtained:

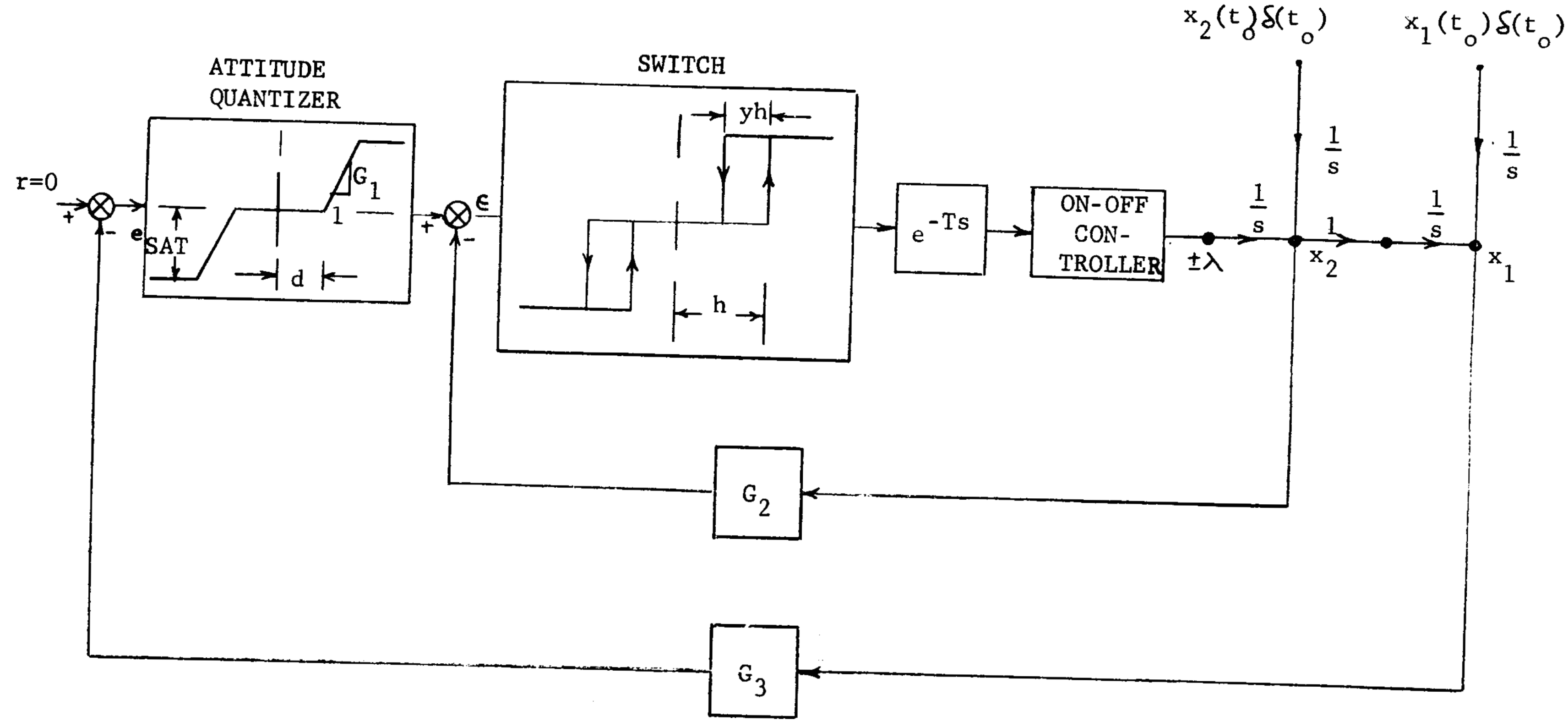


Fig. 12. Block diagram for on-off system 2.

$$\begin{bmatrix} x_{1i} \\ x_{2i} \end{bmatrix} = \begin{bmatrix} x_1(t_o) + x_2(t_o)\ \mathcal{T}_R \\ x_2(t_o) \end{bmatrix} \tag{16}$$

Determine the value of time at which $G_1(-G_3\ x_1 + d) = -\ e_{SAT}$. Define this value of time as t_{ESAT}. When control is applied to the system, x_1, is given by the expression below.

$$x_1(t) = x_{1i} + x_{2i}t - \frac{\lambda}{2}\ t^2 \tag{17}$$

Applying the given boundary conditions, t_{ESAT} is found to be:

$$t_{ESAT} = \frac{-B_{ESAT} \pm \sqrt{B_{ESAT}^{\ 2} - 4\ A_{ESAT}\ C_{ESAT}}}{2\ A_{ESAT}} \tag{18}$$

where

$$\left.\begin{aligned} A_{ESAT} &= \frac{G_1 G_3\ \lambda}{2} \\ B_{ESAT} &= -G_1 G_3\ x_{2i} \\ C_{ESAT} &= -(G_1 G_3\ x_{1i} - G_1\ d - e_{SAT}) \end{aligned}\right\} \tag{19}$$

It should be noted that the choice of sign for the radical is determined by using the sign which results in the smallest positive real value of t_{ESAT}. It should also be noted that if the solution has a complex value, then clearly, the stated boundary conditions will not be attained by the system for the given initial conditions. This procedure will be observed in the remainder of this analysis whenever there exists a possibility that the error signal does not saturate the attitude quantizer.

Define: $\psi = G_1(-G_3\, x_1 + d)$ (20)

Expanding the expression for ψ yields:

$$\psi = -G_1G_3\, x_{1i} - G_1G_3\, x_{2i}t + \frac{G_1G_3 \quad t^2}{2} + G_1\, d \quad . \tag{21}$$

Differentiating the above expression and setting the result equal to zero yields:

$$t_{MAX} = \frac{x_{2i}}{\lambda} \tag{22}$$

where t_{MAX} is defined as the interval between the beginning of control application and the instant at which the magnitude of ψ is a maximum.

If $t_{MAX} \leq t_{ESAT}$, then saturation does not occur.

Define t_1 as the interval between the beginning of control application and switch-off. At switch-off,

$$= -h \;\; + yh \quad . \tag{23}$$

Thus:

$$-h + yh = -G_1G_3\, x_{1i} - G_1G_3\, x_{2i}\, t_1 + \frac{G_1G_3\lambda}{2}\, t_1^2 + G_1\, d - G_2\, x_{2i} + G_2\lambda t_1 \quad . \tag{24}$$

Solving for t_1 ,

$$t_1 = \frac{-B_1 \pm \sqrt{B_1^2 - 4\, A_1C_1}}{2\, A_1} \tag{25}$$

where:

$$\left.\begin{aligned} A_1 &= \frac{G_1 G_3 \lambda}{2} \\ B_1 &= G_2 \lambda - G_1 G_3 x_{2i} \\ C_1 &= -(G_1 G_3 x_{1i} + G_2 x_{2i} + yh - h - G_1 d) \end{aligned}\right\} \quad . \tag{26}$$

T_F seconds after switch-off, control is removed.

If $t_{MAX} > t_{ESAT}$, saturation would occur. For this case, the analysis is continued as follows:

Evaluate x_1 and x_2 at time $t = t_{ESAT}$, as shown below:

$$\begin{bmatrix} x_1 (t_{ESAT}) \\ x_2 (t_{ESAT}) \end{bmatrix} = \begin{bmatrix} x_{1i} + x_{2i} t_{ESAT} - \frac{\lambda}{2} t_{ESAT}^2 \\ x_{2i} - \lambda t_{ESAT} \end{bmatrix} \quad . \tag{27}$$

Define t_2 as the interval between t_{ESAT} and the instant when switch-off occurs. During this interval,

$$\epsilon = - G_1 x_2 - e_{SAT} \quad . \tag{28}$$

Switch-off occurs when $\epsilon = - h + yh$. From the above information, t_2 is found to be:

$$t_2 = \frac{e_{SAT} + G_2 x_2(t_{ESAT}) + yh - h}{G_2 \lambda} \quad . \tag{29}$$

To investigate the possibility that switch-off does not occur while the filter signal is saturated, the following procedure may be used.

Define t_5 as the interval between the beginning of saturation and the end of saturation. The following equation is now applicable:

$$- e_{SAT} = - G_1 G_3 \, x_1(t_{ESAT}) - G_1 G_3 \, x_2(t_{ESAT}) \, t_5 + \frac{G_1 G_3 \lambda t_5^2}{2} + G_1 d \quad . \tag{30}$$

Solving for t_5,

$$t_5 = \frac{-B_5 \pm \sqrt{B_5^2 - 4 A_5 C_5}}{2 A_5} \tag{31}$$

where

$$\left. \begin{aligned} A_5 &= \frac{G_1 G_3 \lambda}{2} \\ B_5 &= - G_1 G_3 \, x_2(t_{ESAT}) \\ C_5 &= - \left[G_1 G_3 \, x_1 (t_{ESAT}) - G_1 d - e_{SAT} \right] \end{aligned} \right\} \quad . \tag{32}$$

If the expression for t_5 has a negative or complex value, switch-off would occur at the end of the t_2 interval.

If t_5 has a positive value, the following procedure must be used.

If $t_5 \geq t_2$, the system would switch off at the end of the t_2 interval. Control would be removed τ_F seconds later. Thus, the total "on" time for application of control effort to the system is given by:

$$t_{on} = t_{ESAT} + t_2 + t_F \quad . \tag{33}$$

x_2 may be evaluated at this time with respect to x_{2i} using Eqaution (5), and the rate diagram may be constructed.

If $t_5 < t_2$, the filter signal would operate in the linear mode again, and the approach would be identical to that used in Equation (24) with x_{1i} and x_{2i} being replaced by the values of x_1 and x_2 at the end of of the t_5 interval.

II. $h + e_{SAT} > |G_2 x_{2i}| > h$

In order for the switch to be off, the polarity and magnitude of x_1 must be such that the output of the attitude quantizer summed with the rate feedback signal lies within the switching dead-band.

Consider the case of positive x_{2i}. From inspection of Figure 6, page 16, the error signal mode for this magnitude of x_{2i} is given by

$$\epsilon = - G_2 x_2 + G_1(- G_3 x_1 - d) \quad . \tag{34}$$

The system will switch on when $\epsilon = -h$.

Denoting the value of x_1 and x_2 at switch-on as $x_1(t_o)$ and $x_2(t_o)$ respectively, the expression for $x_1(t_o)$ may be obtained by substituting the above boundary condition into Equation (34) to yield:

$$x_1(t_o) = \frac{h - G_2 x_2(t_o) - G_1 d}{G_1 G_3} \quad . \tag{35}$$

Control will be applied to the system τ_R seconds after switch-on occurs. Denoting the values of x_1 and x_2 at the beginning of control application as x_{1i} and x_{2i}, the following expressions are obtained:

$$\begin{bmatrix} x_{1i} \\ \\ x_{2i} \end{bmatrix} = \begin{bmatrix} x_1(t_o) + x_2(t_o) \cdot \tau_R \\ \\ x_2(t_o) \end{bmatrix} \quad . \tag{36}$$

Assume that the system will switch off while the error mode is in the positive linear portion of the quantizer. In this mode, the error signal is given by:

$$\epsilon = - G_2 x_2 + G_1(- G_3 x_1 - d) \tag{37}$$

$$= - G_2 x_{2i} + G_2 \lambda t - G_1G_3 x_{1i} - G_1G_3 x_{2i}t + \frac{G_1G_3\lambda}{2}t^2 - G_1d \tag{38}$$

The system will switch off when $\epsilon = -h + yh$. Define t_1 as the interval between the beginning of control application and switch-off. Substituting the above boundary conditions into Equation (38) yields:

$$t_1 = \frac{- B \pm \sqrt{B^2 - 4AC}}{2A} \tag{39}$$

where:

$$\left.\begin{aligned} A &= \frac{G_1G_3 \lambda}{2} \\ B &= G_2 \lambda - G_1G_3 x_{2i} \\ C &= -(G_1G_3 x_{1i} + G_2 x_{2i} + G_1 d + yh - h) \end{aligned}\right\} . \tag{40}$$

Denote x_{1i} and x_{2i} as $x_1(t_o)$ and $x_2(t_o)$.

Define t_{dp} as the interval between t_o and the instant when $G_1(-G_3 x_1 - d) = d$.

$$-G_1G_3 x_1(t_o) - G_1G_3 x_2(t_o) t_{dp} + \frac{G_1G_3 \lambda}{2} t_{dp}^2 - G_1 d = d \tag{41}$$

Thus:

$$t_{dp} = \frac{- B_p \pm \sqrt{B_p^2 - 4A_pC_p}}{2A_p} \tag{43}$$

where:

$$\left.\begin{aligned} A_p &= \frac{G_1 G_3 \lambda}{2} \\ B_p &= -G_1 G_3\, x_2(t_o) \\ C_p &= -(G_1 G_3\, x_1(t_o) + G_1\, d + d) \end{aligned}\right\} \quad . \tag{43}$$

If $t_1 \leq t_{dp}$, the assumption was correct. τ_F seconds later, control is removed from the system.

However, if $t_1 > t_{dp}$, the assumption was incorrect. In this case, evaluate x_1 and x_2 at the time $t = t_{dp}$, using Equation (5)

$$\begin{bmatrix} x_1(t_{dp}) \\ \\ x_2(t_{dp}) \end{bmatrix} = \begin{bmatrix} x_{1i} + x_{2i}\, t_{dp} - \frac{\lambda}{2}\, t_{dp}^2 \\ \\ x_{2i} - \lambda\, t_{dp} \end{bmatrix} \quad . \tag{44}$$

Define t_{dn} as the interval between t_{dp} and the instant when $-G_3\, x_1 = -d$.

$$-G_3\, x_1(t_{dp}) - G_3\, x_2(t_{dp})\, t_{dn} + \frac{G_3 \lambda}{2}\, t_{dn}^2 = -d \tag{45}$$

Thus:

$$t_{dn} = \frac{-B_n \pm \sqrt{B_n^2 - 4\, A_n C_n}}{2A_n} \tag{46}$$

where:

$$A_n = \frac{G_3\, \lambda}{2} \tag{47}$$

$$\left.\begin{aligned} B_n &= -G_3\, x_2(t_{dp}) \\ C_n &= -(G_3\, x_1\,(t_{dp}) - d) \end{aligned}\right\} \qquad (48)$$

Assume switch-off occurs during this interval. Note that during this interval $\epsilon = -G_2\, x_2.$ (49)

Switch-off would occur when $\epsilon = -h + yh$. Define t_1'' as the interval between t_{dp} and the instant of switch-off.

Substituting the given boundary conditions into Equation (49), t_1'' is found to be:

$$t_1'' = \frac{G_2\, x_2(t_{dp}) + yh - h}{G_2 \lambda} \quad . \qquad (50)$$

If $t_1'' \leq t_{dn}$, then switch-off occurs during this interval. In this event,

$$t_1 = t_{dp} + t_1'' \qquad (51)$$

where t_1 is the interval between the beginning of control application and switch-off. Control is removed from the system τ_F seconds later.

If $t_1'' > t_{dn}$, switch-off would not have occurred during the interval. For this case, the analysis may be continued as follows:

Evaluate x_1 and x_2 at time t_{dn}, using Equation (5).

$$\begin{bmatrix} x_1(t_{dn}) \\ x_2(t_{dn}) \end{bmatrix} = \begin{bmatrix} x_1(t_{dp}) + x_2(t_{dp})\, t_{dn} - \frac{\lambda}{2}\, t_{dn}^2 \\ x_2(t_{dp}) - \lambda\, t_{dn} \end{bmatrix} \quad . \qquad (52)$$

Define t_{SATN} as the interval between t_{dn} and the instant when $G_1(-G_3\, x_1 + d) = -\, e_{SAT}$.

$$-e_{SAT} = -G_1G_3\, x_1(t_{dn}) - G_1G_3\, x_2(t_{dn})\, t_{SATN} + \frac{G_1G_3\,\lambda}{2}\, t_{SATN}^2 + G_1 d \tag{53}$$

t_{SATN} may be solved from the following expression:

$$t_{SATN} = \frac{-\,B_x \pm \sqrt{B_x^2 - 4A_xC_x}}{2A_x} \tag{54}$$

where:

$$\left.\begin{aligned} A_x &= \frac{G_1G_3\,\lambda}{2} \\ B_x &= -G_1G_3\, x_2(t_{dn}) \\ C_x &= -\left[G_1G_3\, x_1(t_{dn}) - G_1 d - e_{SAT} \right] \end{aligned}\right\} . \tag{55}$$

During this interval, the error signal is given by the following expression:

$$\epsilon = -\, G_2\, x_2 + G_1(-G_3\, x_1 + d)\ . \tag{56}$$

If switch-off occurs during this interval,

$$-h + yh = -G_2\, x_2(t_{dn}) + G_2\,\lambda\, t_3 - G_1G_3\, x_1(t_{dn}) - G_1G_3\, x_2(t_{dn})t_3 + \frac{G_1G_3\,\lambda}{2}\, t_3^{\ 2} + G_1 d \tag{57}$$

where t_3 is defined as the interval between t_{dn} and the instant when switch-off occurs. By solving the above equation, t_3 is found to be:

$$t_3 = \frac{-B_3 + \sqrt{B_3^2 - 4 A_3 C_3}}{2 A_3} \tag{58}$$

where:

$$\left.\begin{aligned} A_3 &= \frac{G_1 G_3 \lambda}{2} \\ B_3 &= G_2 \lambda - G_1 G_3 x_2(t_{dn}) \\ C_3 &= -\left[G_1 G_3 x_1(t_{dn}) + G_2 x_2(t_{dn}) + yh - h G_1 d\right] \end{aligned}\right\} \tag{59}$$

Define $\chi = G_1(-G_3 x_1 + d)$

Expanding the expression for χ yields:

$$\chi = -G_1 G_3 x_1(t_{dn}) - G_1 G_3 x_2(t_{dn}) t + \frac{G_1 G_3 \lambda t^2}{2} + G_1 d \quad . \tag{60}$$

Differentiating the above expression and setting the result equal to zero yields:

$$t_{MAX} = \frac{x_2(t_{dn})}{\lambda} \tag{61}$$

where t_{MAX} is defined as the interval between t_{dn} and the instant at which the magnitude of x is a maximum.

If $t_{MAX} \leq t_{SATN}$, saturation does not occur. For this case,

$$t_1 = t_{dp} + t_{dn} + t_3 \quad . \tag{62}$$

Control would be removed from the system τ_F seconds later.

If $t_{MAX} > t_{SATN}$, saturation would occur. The analysis may be continued as follows:

Evaluate x_1 and x_2 at time $t = t_{SATN}$, as shown below:

$$\begin{bmatrix} x_1(t_{SATN}) \\ \\ x_2(t_{SATN}) \end{bmatrix} = \begin{bmatrix} x_1(t_{dn}) + x_2(t_{dn})\, t_{SATN} - \frac{\lambda}{2}\, t^2_{SATN} \\ \\ x_2(t_{dn}) - \lambda\, t_{SATN} \end{bmatrix} . \quad (63)$$

Define t_4 as the interval between t_{SATN} and the instant when switch-off occurs. During this interval,

$$\epsilon = -G_2\, x_2 - e_{SAT} \quad . \qquad (64)$$

Switch-off occurs when $\epsilon = -h + yh$. From the above information t_4 is found to be:

$$t_4 = \frac{e_{SAT} + G_2\, x_2(t_{SATN}) + yh - h}{G_2} \qquad (65)$$

For the above case,

$$t_1 = t_{dp} + t_{dn} + t_{SATN} + t_4 \quad . \qquad (66)$$

Note that in all cases, control is removed from the system T_F seconds after switch-off. Evaluate x_{1f} and x_{2f} as the values of $x_1(t_1 + \tau_F)$ and $x_2(t_1 + \tau_F)$ with respect to x_{1i} and x_{2i}.

III. $|G_2\, x_{2i}| > h + e_{SAT}$

In this mode, the system will be switched on independently of the value of x_{1i}. This may be easily verified as follows:

Note that when the attitude quantizer is operating in the saturation mode, the error signal is given by:

$$\epsilon = - G_2\, x_2 + e_{SAT} \quad . \qquad (67)$$

The system will be switched on when $\epsilon = -h$

$$-h = -G_2 x_2 + e_{SAT} \quad . \tag{68}$$

Define x_{2SAT} as the minimum positive value of x_{2i} which results in the system switching on independently of the initial value of x_1. Thus, from Equation (68),

$$x_{2SAT} = \frac{h + e_{SAT}}{G_2} \quad . \tag{69}$$

Q.E.D.

τ_R seconds after switch-on, control is applied to the system.

$$\begin{bmatrix} x_{1i} \\ \\ x_{2i} \end{bmatrix} = \begin{bmatrix} x_1(t_o) + x_2(t_o) \tau_R \\ \\ x_2(t_o) \end{bmatrix} \tag{70}$$

Define t_{SAT} as the interval between the beginning of control application and the instant when the attitude quantizer signal becomes unsaturated. At this instant,

$$G_1(-G_3 x_1 - d) = e_{SAT} \tag{71}$$

Thus:

$$e_{SAT} = -G_1G_3 x_{1i} - G_1G_3 x_{2i} t_{SAT} + G_1G_3 \frac{\lambda}{2} t_{SAT}^2 - G_1 d \quad . \tag{72}$$

Therefore,

$$t_{SAT} = \frac{-B_{SAT} \pm \sqrt{B_{SAT}^2 - 4 A_{SAT} C_{SAT}}}{2 A_{SAT}} \tag{73}$$

where:

$$\left.\begin{aligned} A_{SAT} &= G_1 G_3 \frac{\lambda}{2} \\ B_{SAT} &= -G_1 G_3 x_{2i} \\ C_{SAT} &= -(G_1 G_3 x_{1i} + G_1 d + e_{SAT}) \end{aligned}\right\} \quad . \tag{74}$$

Evaluate x_1 and x_2 at $t = t_{SAT}$. Denote these as $x_1(t_{SAT})$ and $x_2(t_{SAT})$ respectively.

If $\left| -G_2 \, x_2(t_{SAT}) + e_{SAT} \right| \geq h + yh$, the system will still be "on". In this case, denote $x_1(t_{SAT})$ and $x_2(t_{SAT})$ as x_{1i} and x_{2i} respectively, and follow the procedure outlined in Section II, beginning with Equation (41), and adding t_{SAT} to the expressions for controller "on" time.

If $\left| -G_2 \, x_2(t_{SAT}) + e_{SAT} \right| < h + yh$, the system has switched off before the quantizer signal becomes unsaturated. Although this is unlikely, this condition is investigated as follows:

$$\epsilon = -G_2 \, x_2 + e_{SAT} \tag{75}$$

$$-h + yh = -G_2 \, x_{2i} + G_2 \lambda \, t_1 + e_{SAT} \tag{76}$$

therefore,

$$t_1 = \frac{G_2 \, x_{2i} + yh - h - e_{SAT}}{G_2 \lambda} \quad . \tag{77}$$

τ_F seconds later, control is removed from the system.

THE RATE DIAGRAMS

The nominal values of system parameters used for constructing the rate diagrams are listed In Table II. Figures 13-17 are the rate

TABLE II

NOMINAL VALUES OF SYSTEM PARAMETERS FOR SYSTEM 2

Parameter	Nominal Value	Units
G_1	2.0	ND
d	2.5	Degrees
e_{SAT}	15.0	Degrees
G_2	1.0	Degree-sec./degree
G_3	1.0	ND
h	2.0	Degrees
y	0.05	ND
λ	6.0	Degrees/sec.^{2}
τ_R	0.02	Seconds
τ_F	0.02	Seconds

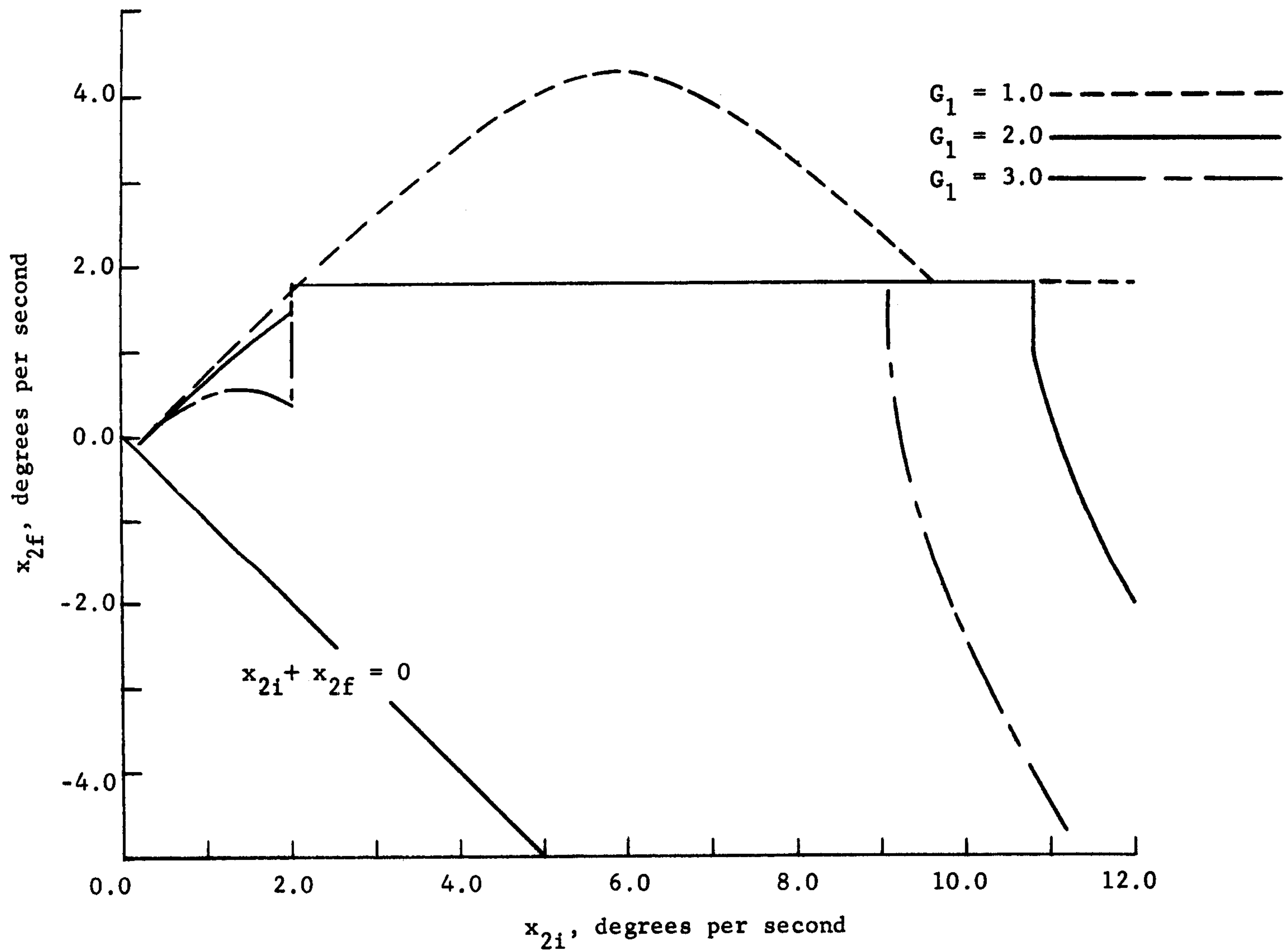


Fig. 13. Rate diagram of system 2 for variations of G_1.

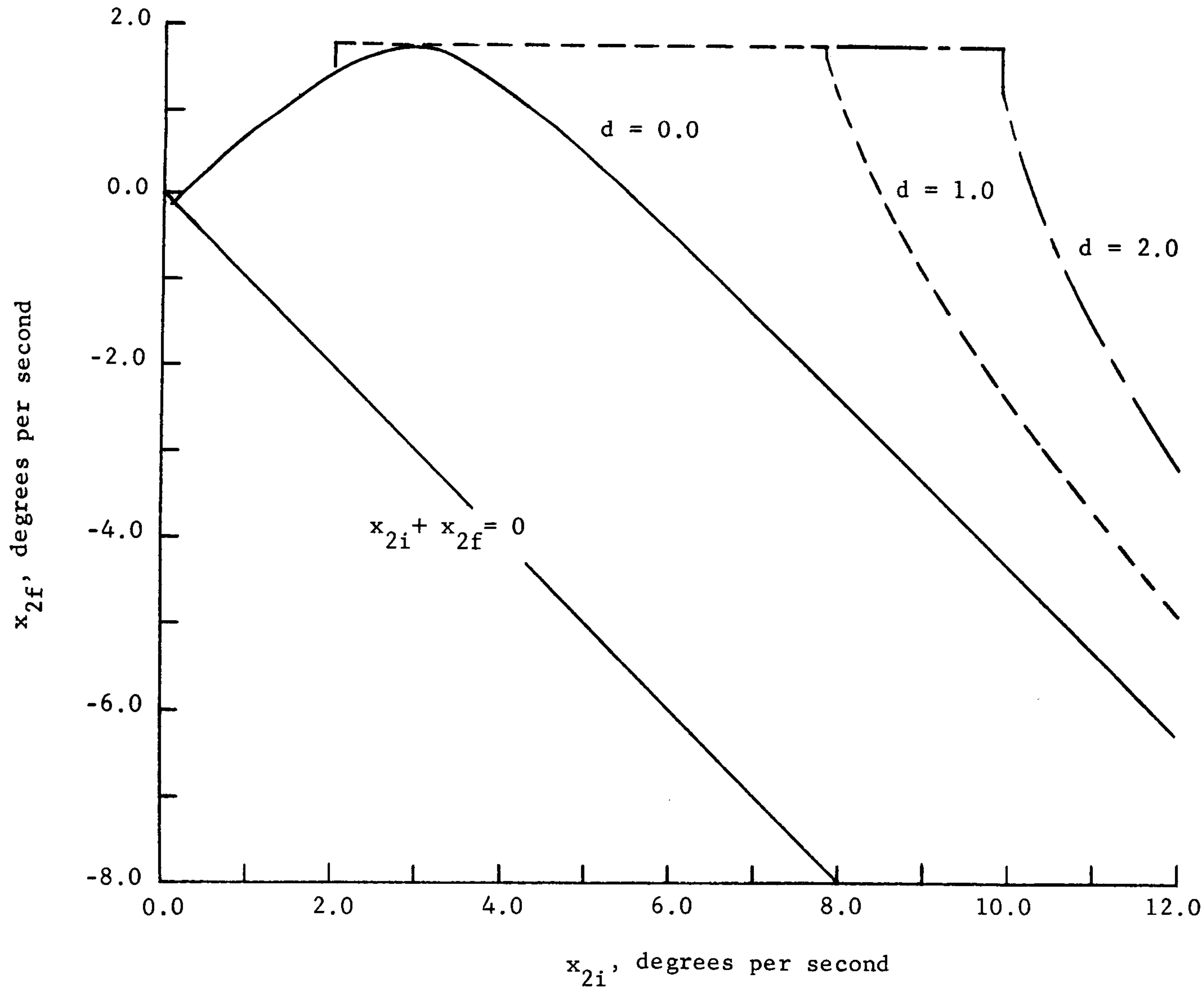


Fig. 14. Rate diagram of system 2 for variations of d.

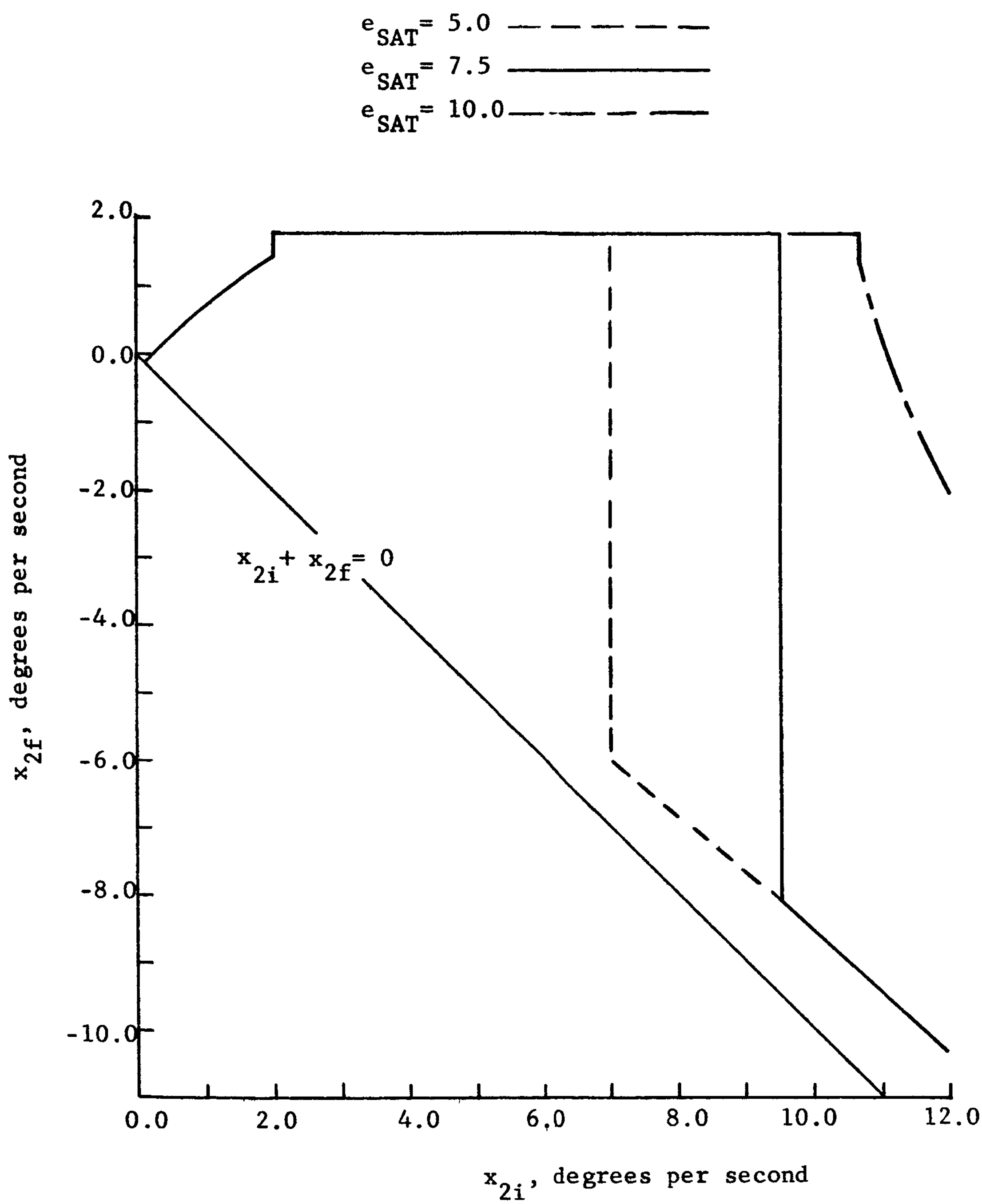


Fig. 15. Rate diagram of system 2 for variations of e_{SAT}.

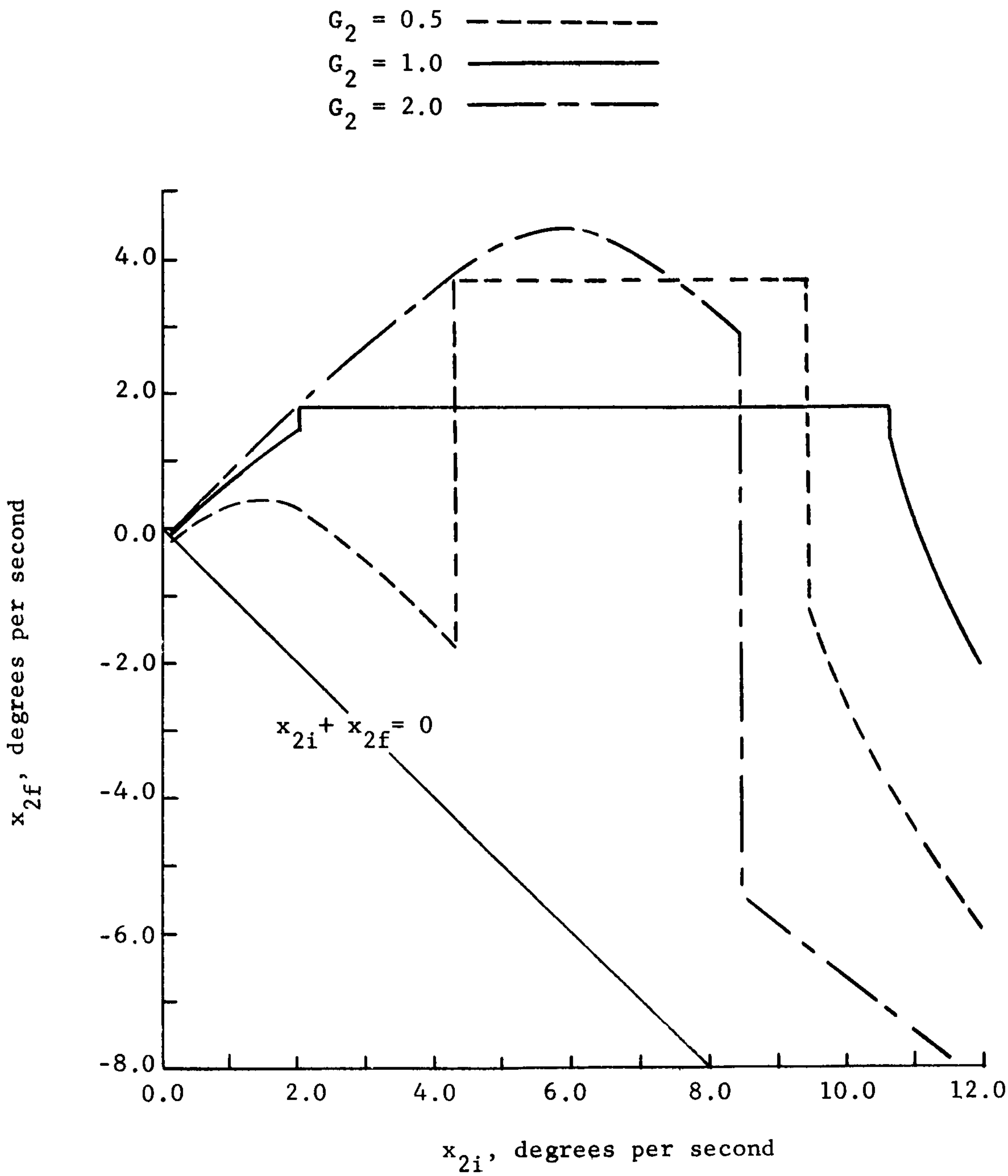


Fig. 16. Rate diagram of system 2 for variation of G_2.

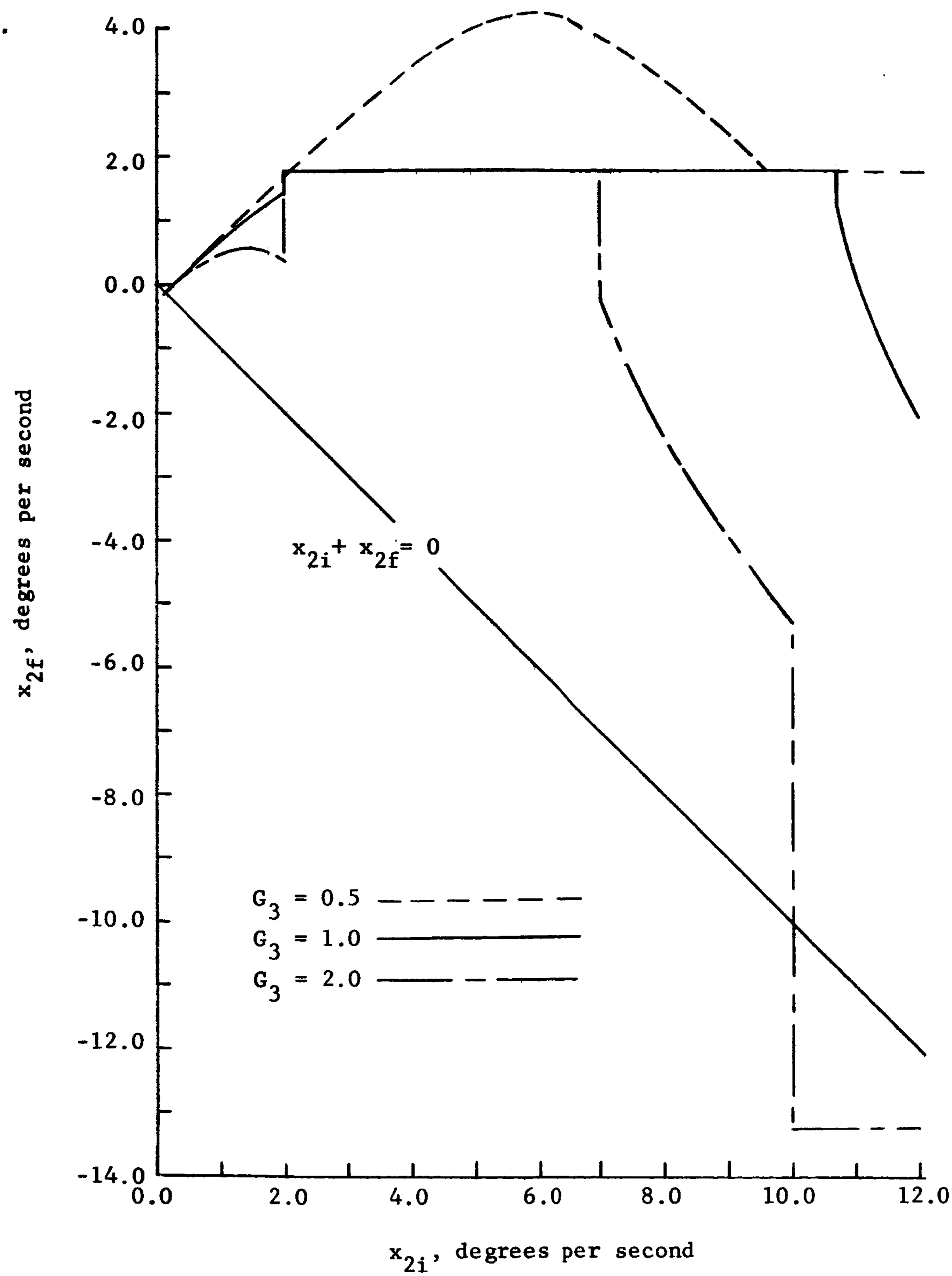


Fig. 17. Rate diagram of system 2 for variations of G_3.

diagrams for various system parameters.

REMARKS

The rate diagrams for System 2 indicate that a "stepping" response results from a large range of values of initial rate. This type of response is much faster than the "over-shooting" response. This illustrates the improvements which may be obtained using a non-linear filter for the error signal.

CONCLUSION

The use of rate diagrams for the design and evaluation of a class of on-off control systems has been illustrated. An important example of this class of control systems would be an attitude control system for a space craft operating beyond an atmosphere. For such systems, the use of an ideal step function is a very good approximation to the control torque. It should be noted however, that the rate diagram is not restricted to systems using an ideal step function as the control action. For other forms of control action the equations become more complicated and it may be necessary to resort to numerical or graphical methods to solve the equations.

A suggested topic for further research would be to investigate possible modifications of the rate diagram concept to include other classes of on-off control systems.

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APPENDIX I

A PROGRAM TO DETERMINE THE TRANSIENT RESPONSE OF ON-OFF CONTROL SYSTEM 1

```
C       THE TRANSFER FUNCTION OF THE CONTROLLED ELEMENT IS
C       OF THE FORM 1/S**2.
C          *          *          *          *          *          *
C       VARIABLE             DESCRIPTION
C       G1                   FORWARD ATTITUDE GAIN
C       G2                   RATE FEEDBACK GAIN
C       G3                   ATTITUDE FEEDBACK GAIN
C       SWON                 CONTROLLER SWITCH-ON THRESHOLD
C       PUHYS                PER UNIT HYSTERESIS OF SWITCH
C       ACCEL                ANGULAR ACCELERATION LEVEL
C       TR                   TRANSPORT LAG FOR THRUST BUILD-UP
C       TF                   TRANSPORT LAG FOR THRUST DECAY
C       X1                   ANGULAR POSITION
C       X2                   ANGULAR RATE
C          *          *          *          *          *          *
C       THE FOLLOWING VALUES OF SYSTEM PARAMETERS WILL BE
C       CONSIDERED NOMINAL. EACH PARAMETER WILL BE VARIED
C       SEPARATELY, WHILE THE OTHERS ARE HELD CONSTANT.
C          *          *          *          *          *          *
           G1 = 2.0
           G2 = 1.0
           G3 = 1.0
           SWON = 2.0
           PUHYS = 0.05
           ACCEL = 6.0
           TR = 0.02
           TF = 0.02
C          *          *          *          *          *          *
           PRINT 101
  101      FORMAT(34H G1 WILL BE VARIED FROM 0.5 TO 4.0)
           DO 1 K1 = 50,400,25
           FK1 = K1
           G1 = FK1/100.
           DO 2 IX = 20,1200,20
    3      FLOATX = IX
           X2IN = FLOATX/ 100.
           CALL SUB1 (G1, G2, G3, SWON, PUHYS, ACCEL, TR, TF,
     1     X1IN,X1FIN,X2IN,X2FIN, TEEONE, TIME)
           PRINT 100,G1,G2,G3,SWON,PUHYS,ACCEL,TR,TF,X1IN,
     1     X1FIN,X2IN,X2FIN,TEEONE,TIME
    2      CONTINUE
    1      CONTINUE
           G1 = 2.0
```

```
      PRINT 102
  102 FORMAT(34H G2 WILL BE VARIED FROM 0.5 TO 2.0)
      DO 5 K2 = 50,200,25
      FK2 = K2
      G2 = FK2/ 100.
      DO 6  IX = 20,1200,20
    7 FLOATX = IX
      X2IN = FLOATX/ 100.
      CALL SUB1 (G1, G2, G3, SWON, PUHYS, ACCEL, TR, TF,
     1X1IN,X1FIN,X2IN,X2FIN, TEEONE, TIME)
      PRINT 100,G1,G2,G3,SWON,PUHYS,ACCEL,TR,TF,X1IN,
     1X1FIN,X2IN,X2FIN,TEEONE,TIME
    6 CONTINUE
    5 CONTINUE
      G2 = 1.0
C     *        *        *        *        *        *
C     *        *        *        *        *        *
      PRINT 103
  103 FORMAT(36H SWON WILL BE VARIED FROM 1.0 TO 5.0)
      DO 10 KSWON = 1,5
      SWON = KSWON
      DO 20  IX = 20,1200,20
   30 FLOATX = IX
      X2IN = FLOATX / 100.
      CALL SUB1 (G1, G2, G3, SWON, PUHYS, ACCEL, TR, TF,
     1X1IN,X1FIN,X2IN,X2FIN, TEEONE, TIME)
      PRINT 100,G1,G2,G3,SWON,PUHYS,ACCEL,TR,TF,X1IN,
     1X1FIN,X2IN,X2FIN,TEEONE,TIME
   20 CONTINUE
   10 CONTINUE
      SWON = 2.0
C     *        *        *        *        *        *
C     *        *        *        *        *        *
      PRINT 104
  104 FORMAT(38H PUHYS WILL BE VARIED FROM 0.01 TO 0.1)
      DO 50 KPUHYS=1,10
      FKPHYS=KPUHYS
      PUHYS=FKPHYS/100.
      DO 60  IX = 20,1200,20
   70 FLOATX=IX
      X2IN=FLOATX/100.
      CALL SUB1 (G1, G2, G3, SWON, PUHYS, ACCEL, TR, TF,
     1X1IN,X1FIN,X2IN,X2FIN, TEEONE, TIME)
      PRINT 100,G1,G2,G3,SWON,PUHYS,ACCEL,TR,TF,X1IN,
     1X1FIN,X2IN,X2FIN,TEEONE,TIME
   60 CONTINUE
   50 CONTINUE
      PUHYS = 0.05
```

```
      PRINT 105
105   FORMAT(38H ACCEL WILL BE VARIED FROM 4.0 TO 12.0)
      DO 150 KACCEL=4,12,2
      ACCEL=KACCEL
      DO 200  IX = 20,1200,20
300   FLOATX=IX
      X2IN=FLOATX/100.
      CALL SUB1 (G1, G2, G3, SWON, PUHYS, ACCEL, TR, TF,
     1 X1IN,X1FIN,X2IN,X2FIN, TEEONE, TIME)
      PRINT 100,G1,G2,G3,SWON,PUHYS,ACCEL,TR,TF,X1IN,
     1 X1FIN,X2IN,X2FIN,TEEONE,TIME
200   CONTINUE
150   CONTINUE
      ACCEL = 6.0
C     *        *        *        *        *        *
C     *        *        *        *        *        *
      PRINT 106
106   FORMAT(35H TR WILL BE VARIED FROM 0.01 TO 0.1)
      DO 500 KTR=1,10
      FTR=KTR
      TR=FTR/100.
      DO 600  IX = 20,1200,20
700   FLOATX=IX
      X2IN=FLOATX/100.
      CALL SUB1 (G1, G2, G3, SWON, PUHYS, ACCEL, TR, TF,
     1 X1IN,X1FIN,X2IN,X2FIN, TEEONE, TIME)
      PRINT 100,G1,G2,G3,SWON,PUHYS,ACCEL,TR,TF,X1IN,
     1 X1FIN,X2IN,X2FIN,TEEONE,TIME
600   CONTINUE
500   CONTINUE
      TR = 0.02
C     *        *        *        *        *        *
C     *        *        *        *        *        *
      PRINT 107
107   FORMAT(35H TF WILL BE VARIED FROM 0.01 TO 0.1)
      DO 15 KTF=1,10
      FTF=KTF
      TF=FTF/100.
      DO 25  IX = 20,1200,20
 35   FLOATX=IX
      X2IN=FLOATX/100.
      CALL SUB1 (G1, G2, G3, SWON, PUHYS, ACCEL, TR, TF,
     1 X1IN,X1FIN,X2IN,X2FIN, TEEONE, TIME)
      PRINT 100,G1,G2,G3,SWON,PUHYS,ACCEL,TR,TF,X1IN,
     1 X1FIN,X2IN,X2FIN,TEEONE,TIME
 25   CONTINUE
 15   CONTINUE
      TF = 0.02
```

```
C          *          *          *          *          *          *.
  100      FORMAT(7F11.6/7F11.6//)
           END
C          *          *          *          *          *          *
$IBFTC SUB1
C          *          *          *          *          *          *
           SUBROUTINE SUB1(G1,G2,G3,SWON,PUHYS,ACCEL,TR,TF,
     1     X1IN,X1FIN,X2IN,X2FIN, TEEONE, TIME )
C          *          *          *          *          *          *
           X1IN=SWON/(G1*G3)-(G2*X2IN)/(G1*G3)+X2IN*TR
           A=G1*G3*ACCEL/2.0
           B=G2*ACCEL-G1*G3*X2IN
           C=-(G2*X2IN+G1*G3*X1IN+PUHYS*SWON-SWON)
C          TEEONE IS THE INTERVAL FROM THE BEGINNING OF
C          CONTROL APPLICATION UNTIL SWITCH-OFF OCCURS.
           TEEONE=(-B+SQRT (B**2-4.*A*C))/(2.*A)
           TIME=TEEONE+TF
           X1FIN=X1IN+TIME*X2IN-TIME**2*ACCEL/2.0
           X2FIN=X2IN-ACCEL*TIME
C          THE RATE DIAGRAM IS CONSTRUCTED BY PLOTTING X2FIN
C          VERSUS THE GIVEN VALUE OF X2IN.
           RETURN
           END
$ENTRY
```

APPENDIX II

A PROGRAM TO DETERMINE THE TRANSIENT RESPONSE OF ON-OFF CONTROL SYSTEM 2

```
C      THE TRANSFER FUNCTION OF THE CONTROLLED ELEMENT IS
C      OF THE FORM 1/S**2.
C         *          *          *          *          *          *
C      VARIABLE              DESCRIPTION
C      G1                    GAIN OF ATTITUDE QUANTIZER
C      DEE                   DEADBAND OF ATTITUDE QUANTIZER
C      ESAT                  SATURATION LEVEL OF QUANTIZER
C      G2                    RATE FEEDBACK GAIN
C      G3                    ATTITUDE FEEDBACK GAIN
C      SWON                  CONTROLLER SWITCH-ON THRESHOLD
C      PUHYS                 PER UNIT HYSTERESIS OF SWITCH
C      ACCEL                 ANGULAR ACCELERATION LEVEL
C      TR1                   TRANSPORT LAG FOR THRUST BUILD-UP
C      TF1                   TRANSPORT LAG FOR THRUST DECAY
C      X1                    ANGULAR POSITION
C      X2                    ANGULAR RATE
C         *          *          *          *          *          *
C      THE FOLLOWING VALUES OF SYSTEM PARAMETERS WILL BE
C      CONSIDERED NOMINAL. EACH PARAMETER WILL BE VARIED
C      SEPARATELY, WHILE THE OTHERS ARE HELD CONSTANT.
C         *          *          *          *          *          *
       G1=2.0
       DEE=2.5
       ESAT=15.0
       G2=1.0
       G3=1.0
       SWON=2.0
       PUHYS=0.05
       ACCEL=6.0
       TR1=0.02
       TF1=0.02
C         *          *          *          *          *          *
       PRINT 101
 101   FORMAT(34H G1 WILL BE VARIED FROM 0.5 TO 5.0)
       DO 1    K1 = 50,500,50
       FK1=K1
       G1=FK1/100.
       DO 2  IX = 2,120,2
 3     FLOATX=IX
       X2IN = FLOATX/10.
       IF(G2*X2IN-SWON)5,5,6
 5     CALL SUBH1(G1,DEE,ESAT,G2,G3,SWON,PUHYS,ACCEL,TR1,
      1     TF1,X1IN,X2IN,X1FIN,X2FIN,ONTIME)
       GO TO 7
```

```
 6     CALL SUBGH1(G1,DEE,ESAT,G2,G3,SWON,PUHYS,ACCEL,TR1,
     1    TF1,X1IN,X2IN,X1FIN,X2FIN,ONTIME)
 7     PRINT 100,G1,DEE,ESAT,G2,G3,SWON,PUHYS,ACCEL,TR1,
     1    TF1,X1IN,X2IN,X1FIN,X2FIN,ONTIME
 100   FORMAT(8F11.6/7F11.6//)
 2     CONTINUE
 1     CONTINUE
       G1=2.0
C         *          *          *          *          *          *
       PRINT 102
 102   FORMAT(35H DEE WILL BE VARIED FROM 0.0 TO 5.0)
       DO 201   KDEE = 0,500,100
       FKDEE=KDEE
       DEE=FKDEE/100.
       DO 202  IX = 20,1200,20
 203   FLOATX=IX
       X2IN=FLOATX/100.
       IF(G2*X2IN-SWON)205,205,206
 205   CALL SUBH1(G1,DEE,ESAT,G2,G3,SWON,PUHYS,ACCEL,TR1,
     1    TF1,X1IN,X2IN,X1FIN,X2FIN,ONTIME)
       GO TO 207
 206   CALL SUBGH1(G1,DEE,ESAT,G2,G3,SWON,PUHYS,ACCEL,TR1,
     1    TF1,X1IN,X2IN,X1FIN,X2FIN,ONTIME)
 207   PRINT 200,G1,DEE,ESAT,G2,G3,SWON,PUHYS,ACCEL,TR1,
     1    TF1,X1IN,X2IN,X1FIN,X2FIN,ONTIME
 200   FORMAT(8F11.6/7F11.6//)
 202   CONTINUE
 201   CONTINUE
       DEE=2.5
C         *          *          *          *          *          *
       PRINT 103
 103   FORMAT(37H ESAT WILL BE VARIED FROM 5.0 TO 15.0)
       DO 301 KESAT=10,30,5
       FESAT=KESAT
       ESAT=FESAT/2.
       DO 302  IX = 20,1200,20
 303   FLOATX=IX
       X2IN=FLOATX/100.
       IF(G2*X2IN-SWON)305,305,306
 305   CALL SUBH1(G1,DEE,ESAT,G2,G3,SWON,PUHYS,ACCEL,TR1,
     1    TF1,X1IN,X2IN,X1FIN,X2FIN,ONTIME)
       GO TO 307
 306   CALL SUBGH1(G1,DEE,ESAT,G2,G3,SWON,PUHYS,ACCEL,TR1,
     1    TF1,X1IN,X2IN,X1FIN,X2FIN,ONTIME)
 307   PRINT 300, G1,DEE,ESAT,G2,G3,SWON,PUHYS,ACCEL,TR1,
     1    TF1,X1IN,X2IN,X1FIN,X2FIN,ONTIME
 300   FORMAT(8F11.6/7F11.6//)
 302   CONTINUE
 301   CONTINUE
```

```
      ESAT=15.0
C         *          *          *          *          *          *
      PRINT 104
  104 FORMAT(34H G2 WILL BE VARIED FROM 0.5 TO 4.0)
      DO 401    K2 = 50,400,50
      FK2=K2
      G2=FK2/100.
      DO 402  IX = 20,1200,20
  403 FLOATX=IX
      X2IN=FLOATX/100.
      IF(G2*X2IN-SWON)405,405,406
  405 CALL SUBHI(G1,DEE,ESAT,G2,G3,SWON,PUHYS,ACCEL,TR1,
     1    TF1,X1IN,X2IN,X1FIN,X2FIN,ONTIME)
      GO TO 407
  406 CALL SUBGHI(G1,DEE,ESAT,G2,G3,SWON,PUHYS,ACCEL,TR1,
     1    TF1,X1IN,X2IN,X1FIN,X2FIN,ONTIME)
  407 PRINT 400, G1,DEE,ESAT,G2,G3,SWON,PUHYS,ACCEL,TR1,
     1    TF1,X1IN,X2IN,X1FIN,X2FIN,ONTIME
  400 FORMAT(8F11.6/7F11.6//)
  402 CONTINUE
  401 CONTINUE
      G2=1.0
C         *          *          *          *          *          *
      PRINT 105
  105 FORMAT(34H G3 WILL BE VARIED FROM 0.5 TO 4.0)
      DO 501     K3 = 50,400,50
      FK3=K3
      G3=FK3/100.
      DO 502  IX = 20,1200,20
  503 FLOATX=IX
      X2IN=FLOATX/100.
      IF(G2*X2IN - SWON) 505,505,506
  505 CALL SUBHI(G1,DEE,ESAT,G2,G3,SWON,PUHYS,ACCEL,TR1,
     1    TF1,X1IN,X2IN,X1FIN,X2FIN,ONTIME)
      GO TO 507
  506 CALL SUBGHI(G1,DEE,ESAT,G2,G3,SWON,PUHYS,ACCEL,TR1,
     1    TF1,X1IN,X2IN,X1FIN,X2FIN,ONTIME)
  507 PRINT 500, G1,DEE,ESAT,G2,G3,SWON,PUHYS,ACCEL,TR1,
     1    TF1,X1IN,X2IN,X1FIN,X2FIN,ONTIME
  500 FORMAT(8F11.6/7F11.6//)
  502 CONTINUE
  501 CONTINUE
      G3=1.0
      END
C         *          *          *          *          *          *
$IBFTC SUBHI
C         *          *          *          *          *          *
      SUBROUTINE SUBHI(G1,DEE,ESAT,G2,G3,SWON,PUHYS,ACCEL
     1,TR1,TF1,X1IN,X2IN,X1FIN,X2FIN,ONTIME)
```

```
C          *           *           *           *           *           *
C     X1TO WILL BE CONSIDERED AS THE VALUE OF X1 AT THE
C     INSTANT OF SWITCH-ON.
      X1TO=(-G2*X2IN+SWON+G1*DEE)/(G1*G3)
C     X1IN IS DEFINED AS THE VALUE OF X1 AT THE BEGINNING
C     OF CONTROL APPLICATION .
      X1IN=X1TO+X2IN*TR1
C     DEFINE T1 AS THE INTERVAL BETWEEN THE BEGINNING OF
C     CONTROL APPLICATION AND SWITCH-OFF.
      A1=G1*G3*ACCEL/2.
      B1=G2*ACCEL-G1*G3*X2IN
      C1=-(G1*G3*X1IN+G2*X2IN+PUHYS*SWON-G1*DEE - SWON)
      T1=(-B1+SQRT(B1**2-4.*A1*C1))/(2.*A1)
      AESAT = G1*G3*ACCEL/2.
      BESAT = -G1*G3*X2IN
      CESAT = -(G1*G3*X1IN - ESAT - G1*DEE)
      RAD = BESAT**2 - 4.*AESAT*CESAT
      IF (RAD) 50,11,11
   11 TESAT  = (-BESAT-SQRT(RAD))/(2.*AESAT)
      IF (TESAT ) 12,13,13
   12 TESAT = (-BESAT+SQRT(RAD))/(2.*AESAT)
      IF (TESAT) 50,13,13
   13 TMAX = X2IN/ACCEL
      IF (TMAX - TESAT) 50,50,15
   15 X1ESAT = X1IN+X2IN*TESAT-TESAT**2*ACCEL/2.
      X2ESAT = X2IN - TESAT*ACCEL
      T2 = (-SWON+SWON*PUHYS+ESAT+G2*X2ESAT)/(G2*ACCEL)
      IF (T2) 17,16,16
   17 PRINT 111
  111 FORMAT(35H ANALYSIS MUST BE CONTINUED FURTHER)
      ONTIME = 0.0
      GO TO 51
   16 T1 = TESAT + T2
   50 ONTIME =T1+TF1
   51 X1FIN=X1IN+X2IN*ONTIME-ONTIME**2*ACCEL/2.
      X2FIN=X2IN-ACCEL*ONTIME
      RETURN
      END
C          *           *           *           *           *           *

$IBFTC SUBGH1

C          *           *           *           *           *           *
      SUBROUTINE SUBGH1(G1,DEE,ESAT,G2,G3,SWON,PUHYS,
     1ACCEL,TR1,TF1,X1IN,X2IN,X1FIN,X2FIN,ONTIME)
C          *           *           *           *           *           *
C     DEFINE X2SAT AS THE MINIMUM POSITIVE VALUE OF X2IN
C     WHICH RESULTS IN THE SYSTEM BEING SWITCHED ON
C     INDEPENDENTLY OF THE INITIAL VALUE OF X1.
```

```
      X2SAT=(SWON+ESAT)/G2
      IF(X2IN-X2SAT)1,1,2
    2 X1TO = 0.0
C     NOTE THAT FOR THIS EXAMPLE, X1TO WAS CHOSEN
C     ARBITRARILY.
      X1IN = X1TO + X2IN*TR1
C     DEFINE TSAT AS THE INTERVAL BETWEEN THE BEGINNING
C     OF CONTROL APPLICATION AND THE INSTANT WHEN THE
C     ATTITUDE QUANTIZER SIGNAL BECOMES UNSATURATED.
      ASAT = G1*G3*ACCEL/2.
      BSAT = -G1*G3*X2IN
      CSAT = -(G1*G3*X1IN+G1*DEE+ESAT)
      RAD = BSAT**2 - 4.*ASAT*CSAT
      IF (RAD) 56,57,57
   56 PRINT 222
  222 FORMAT (18H TSAT IS UNDEFINED)
      ONTIME = 0.0
      GO TO 51
   57 TSAT = (-BSAT-SQRT(RAD))/(2.*ASAT)
      IF (TSAT)58,59,59
   58 TSAT = (-BSAT+SQRT(RAD))/(2.*ASAT)
   59 X1TSAT = X1IN+X2IN*TSAT-TSAT**2*ACCEL/2.
      X2TSAT=X2IN-ACCEL*TSAT
      SWOFF=SWON-SWON*PUHYS
    3 ERTSAT=-G2*X2TSAT+ESAT
      IF (ABS(ERTSAT) - SWOFF) 5,6,6
    5 T1 = (-SWON-ESAT+SWON*PUHYS+G2*X2IN)/(G2*ACCEL)
C     T1 IS THE TIME FROM THE BEGINNING OF CONTROL
C     APPLICATION UNTIL SWITCH OFF.
      IF (T1) 6,60,60
   60 IF (T1 - TSAT) 50,50,6
    6 X1TO = X1TSAT
      X2TO = X2TSAT
      GO TO 11
    1 X1IN=(-G2*X2IN+SWON-G1*DEE)/(G1*G3)+X2IN*TR1
      X1TO=X1IN
      X2TO=X2IN
      TSAT=0.0
C     DEFINE T1P AS THE INTERVAL BETWEEN TO AND SWITCH-OFF.
   11 A1P = G1*G3*ACCEL/2.
      B1P=G2*ACCEL-G1*G3*X2TO
      C1P=-(G1*G3*X1TO+G2*X2TO+G1*DEE+SWON*PUHYS
     1         - SWON)
      RAD = B1P**2 - 4.*A1P*C1P
      T1P = (-B1P - SQRT(RAD))/(2.*A1P)
      IF (T1P) 76,75,75
   76 T1P = (-B1P + SQRT(RAD))/(2.*A1P)
```

```
   75 ADP = G1*G3*ACCEL/2.
      BDP = - G1*G3*X2TO
      CDP = -(G1*G3*X1TO+G2*X2TO+G1*DEE+DEE)
C     TDP IS DEFINED AS THE INTERVAL BETWEEN TO AND THE
C     INSTANT WHEN G1*(-G3*X1 - DEE) = DEE.
      RAD = BDP**2 - 4.*ADP*CDP
      IF (RAD) 18,19,19
   18 T1 = TSAT + T1P
      GO TO 50
   19 TDP = (-BDP-SQRT(RAD))/(2.*ADP)
      IF (TDP) 20,21,21
   20 TDP = (-BDP + SQRT(RAD))/(2.*ADP)
      IF (TDP) 18,21,21
   21 IF (T1P  - TDP) 12,12,13
   12 T1 = TSAT + T1P
      GO TO 50
   13 X1TDP=X1TO+X2TO*TDP-TDP**2*ACCEL/2.
      X2TDP=X2TO-ACCEL*TDP
   61 ADN=G3*ACCEL/2.
      BDN=-G3*X2TDP
      CDN=-(G3*X1TDP-DEE)
      RAD = BDN**2 - 4.*ADN*CDN
C     DEFINE TDN AS THE INTERVAL BETWEEN TDP AND THE
C     INSTANT WHEN (-G3*X1=-DEE).
      IF (RAD) 122,23,23
  122 T1PP = (G2*X2TDP+SWON*PUHYS-SWON)/(G2*ACCEL)
C     DEFINE T1PP AS THE INTERVAL BETWEEN TDP AND THE
C     INSTANT WHEN SWITCH-OFF OCCURS.
      IF (T1PP) 66,22,22
   66 T1 = TSAT + TDP
      GO TO 50
   23 TDN = (-BDN-SQRT(RAD))/(2.*ADN)
      IF(TDN) 24,25,25
   24 TDN = (-BDN+SQRT(RAD))/(2.*ADN)
   25 T1PP = (G2*X2TDP+SWON*PUHYS-SWON)/(G2*ACCEL)
      IF (T1PP) 15,123,123
  123 IF (T1PP - TDN)22,22,15
   22 T1 = TSAT + TDP + T1PP
      GO TO 50
   15 X1TDN=X1TDP+X2TDP*TDN-TDN**2*ACCEL/2.
      X2TDN=X2TDP-ACCEL*TDN
C     DEFINE T3 AS THE INTERVAL BETWEEN TDN AND THE
C     INSTANT WHEN SWITCH-OFF OCCURS.
      A3=G1*G3*ACCEL/2.
      B3=G2*ACCEL-G1*G3*X2TDN
      C3=-(G1*G3*X1TDN+G2*X2TDN+SWON*PUHYS-SWON-G1*DEE)
      RAD = B3**2 - 4.*A3*C3
   28 T3 = (-B3 - SQRT(RAD))/(2.*A3)
      IF (T3) 29,30,30
```

```
   29 T3 = (-B3 + SQRT(RAD))/(2.*A3)
      GO TO 30
   30 AN=G1*G3*ACCEL/2.
      BN=-G1*G3*X2TDN
      CN=-(G1*G3*X1TDN-G1*DEE-ESAT)
      RAD = BN**2 - 4.*AN*CN
      IF (RAD) 31,32,32
   31 T1 = TSAT + TDP + TDN + T3
      GO TO 50
   32 TSATN = (-BN - SQRT(RAD))/(2.*AN)
      IF (TSATN) 33,34,34
   33 TSATN = (-BN + SQRT(RAD))/(2.*AN)
      IF (TSATN) 31,34,34
   34 TMAX = X2TDN/ACCEL
      IF (TMAX - TSATN) 31,31,17
   17 X2SATN=X2TDN-ACCEL*TSATN
      X1SATN = X1TDN+X2TDN*TSATN-TSATN**2*ACCEL/2.
C     DEFINE T4 AS THE INTERVAL BETWEEN TSATN AND THE
C     INSTANT OF SWITCH OFF.
      T4=(ESAT+G2*X2SATN+SWON*PUHYS-SWON)/(G2*ACCEL)
      A5 = G1*G3*ACCEL/2.
      B5 = -G1*G3*X2SATN
      C5 = -(G1*G3*X1SATN-G1*DEE-ESAT)
      RAD = B5**2 - 4.*A5*C5
      IF (RAD) 41,37,37
   37 T5 = (-B5-SQRT(RAD))/(2.*A5)
      IF (T5) 38,39,39
   38 T5 = (-B5+SQRT(RAD))/(2.*A5)
   39 IF (T4 - T5) 41,41,42
   42 X1T5 = X1SATN +X2SATN*T5-T5**2*ACCEL/2.
      X2T5 = X2SATN - T5*ACCEL
      A6 = G1*G3*ACCEL/2.
      B6 = G2*ACCEL - G1*G3*X2T5
      C6 = -(G1*G3*X1T5+G2*X2T5+SWON*PUHYS-SWON-G1*DEE)
      RAD = B6**2 - 4.*A6*C6
      T6 = (-B6-SQRT(RAD))/(2.*A6)
      IF (T6) 43,44,44
   43 T6 = (-B6+SQRT(RAD))/(2.*A6)
   44 ATDN = G1*G3*ACCEL/2.
      BTDN = -G1*G3*X2T5
      CTDN = -(G1*G3*X1T5-G1*DEE-DEE)
      RAD = BTDN**2 - 4.*ATDN*CTDN
      IF (RAD) 48,45,45
   45 TDN2 = (-BTDN-SQRT(RAD))/(2.*ATDN)
      IF (TDN2) 46,47,47
   46 TDN2 = (-BTDN+SQRT(RAD))/(2.*ATDN)
      47 IF (TDN2 - T6) 40,48,48
   48 T1 = TSAT+TDP+TDN+TSATN +T5+T6
      GO TO 50
```

```
   40 PRINT 111
  111 FORMAT(35H ANALYSIS MUST BE CONTINUED FURTHER)
      ONTIME = 0.0
      GO TO 51
   41 T1=TSAT+TDP+TDN+TSATN+T4
   50 ONTIME =T1+TF1
   51 X1FIN=X1IN+X2IN*ONTIME-ONTIME**2*ACCEL/2.
      X2FIN=X2IN-ACCEL*ONTIME
      RETURN
      END
$ENTRY
$IBSYS
```


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