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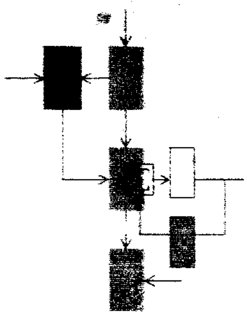

THE DISCRETE MINIMUM PRINCIPLE WITH APPLICATION TO THE LINEAR REGULATOR PROBLEM

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ABSTRACT

The purpose of this report is to summarize in the form of a discrete minimum principle the necessary conditions for optimality which have been derived for the optimal control of a class of discrete-time systems. The discrete minimum principle is used to derive the optimal feedback control law when the discrete dynamical system is linear and the index of performance is quadratic.

CONTENTS

I. INTRODUCTION

In discrete-time systems the time-evolution of the state variables is described by a set of first-order difference equations. The optimal control problem for such systems reduces to the minimization of a cost functional subject to constraints on the control variables and subject to boundary conditions on the state variables.

In this report we modify the results due to Holtzman and Halkin (see References 2, 3, 5, 8, 9) and state a minimum principle for optimality together with the necessary assumptions required for its proof. The minimum principle is then used to derive the results obtained by Kalman and Koepke (see Reference 7) via dynamic programming pertaining to the linear discrete regulator problem.

II. DISCRETE DYNAMICAL SYSTEMS

A. DEFINITIONS

The class of dynamical systems which we shall call discrete are characterized by the following elements:

(1) An ordered subset T of positive integers, called the time set, i.e.,

$$T = \{i\} = \{0, 1, 2, \ldots, N\} \tag{2.1}$$

where N is prespecified.

(2) A set of states $\{\underline{x}\} = X = E_n$ called the state space, where E_n is an n-dimensional Euclidean vector space.

(3) A set of inputs or controls $\{\underline{u}\} = U \subseteq E_r$ called the input space.

(4) A set of outputs $\{\underline{y}\} = Y = E_m$ called the output space.

(5) A difference equation which describes the evolution of the state of the system in time, i.e.,

$$\underline{x}_{i+1} - \underline{x}_i = \underline{f}_i(\underline{x}_i, \underline{u}_i) \quad i = 0, 1, \ldots, N-1 \tag{2.2}$$

where $\underline{x}_i, \underline{u}_i$ are the values of the state vector and the control vector respectively at time i and where $\underline{f}_i(\underline{x}, \underline{u})$ is a vector-valued function which maps X x U into X. The difference Eq. 2.2 is a rule which enables us to compute the state of the system at time i + 1 from knowledge of both the state and the control at time i.

(6) An algebraic equation which relates the output vector $\underline{y}_i$ at time i to the state vector $\underline{x}_i$ and the control vector $\underline{u}_i$, i.e.,

$$\underline{y}_i = \underline{h}_i(\underline{x}_i, \underline{u}_i); \quad i = 0, 1, \ldots N \tag{2.3}$$

where $\underline{h}_i(\underline{x}, \underline{u})$ is a vector-valued function which maps X x U into Y.

A system, Σ, possessing the above properties is called a "discrete dynamical system".

B. ASSUMPTIONS

For every $i = 0, 1, \ldots, N-1$ we shall impose certain constraints upon the vector-valued function $\underline{f}_i(\underline{x}, \underline{u})$ inasmuch as they are central to the proof of the Minimum Principle. These assumptions are:

(1) For every fixed $u \in U$, the function $\underline{f}_i(\underline{x}, \underline{u})$ is twice continuously differentiable with respect to $\underline{x}$.

(2) $\underline{f}_i(\underline{x}, \underline{u})$ and all its first and second partial derivatives with respect to $\underline{x}$ are bounded over $A \times B$ for any bounded sets $A \subset X$, $B \subset U$.

(3) The $n \times n$ matrix

$$\underline{I} + \frac{\partial \underline{f}_i(\underline{x}, \underline{u})}{\partial \underline{x}} \qquad (2.4)$$

is non-singular on $X \times U$.

C. REMARKS

In Section B, assumptions (1) and (2) correspond to the usual "smoothness" conditions also common to the continuous time case. Assumption (3) is, however, of a different nature. In essence, it guarantees that if we know $\underline{x}_{i+1}$ and the control $\underline{u}_i$, then we can uniquely determine $\underline{x}_i$. To see this more clearly let us rewrite Eq. 2.2 in the form

$$\underline{x}_i + \underline{f}_i\,(\underline{x}_i, \underline{u}_i) = \underline{x}_{i+1} \qquad (2.5)$$

Since $\underline{u}_i$ is known, we may think of the left-hand side of Eq. 2.5 as a mapping $\underline{g}_i(\cdot)$ from X into X, so that Eq. 2.5 may be rewritten as

$$\underline{g}_i(\underline{x}_i) = \underline{x}_{i+1} \qquad (2.6)$$

In order to be able to solve Eq. 2.6 for $\underline{x}_i$, the Jacobian matrix

$$\left. \frac{\partial \underline{g}_i(\underline{x})}{\partial \underline{x}} \right|_{\underline{x} = \underline{x}_i} \qquad (2.7)$$

must have an inverse.† But

$$\frac{\partial \underline{g}_i(\underline{x})}{\partial \underline{x}} = \underline{I} + \frac{\partial \underline{f}_i(\underline{x}, \underline{u})}{\partial \underline{x}} \qquad (2.8)$$

and, so, in order to be able to solve for all $\underline{x}_i$, the matrix Eq. 2.8 must have an inverse for all $\underline{x} \in X$ and all $\underline{u} \in U$.

We note, furthermore, that in the case of continuous systems described by a differential equation $\dot{\underline{x}} = \underline{f}(\underline{x}, \underline{u})$, knowledge of the state $\underline{x}(\tau)$ and of the input function $\underline{u}(\cdot)$ is sufficient to uniquely determine $\underline{x}(t)$ for both $t > \tau$ and $t < \tau$. This property is a consequence of the "reversability" with respect to time of the solution of a differential equation. Thus, assumption (3) provides this same property for the class of discrete systems characterized by the difference equation $\underline{x}_{i+1} - \underline{x}_i = \underline{f}_i(\underline{x}_i, \underline{u}_i)$.

As a consequence of the "time reversability" for continuous systems, it may be shown that assumption (3) is always satisfied in the case of a system of difference equations which approximates a system of differential equations.[3]†† However, assumption (3) may well fail to be satisfied for an arbitrary discrete dynamical system.

D. LINEAR DISCRETE SYSTEMS

As in the case of continuous dynamical systems, the linearity of the equations of motion is an extremely strong property which enables us to deduce analytical solutions.

A linear discrete dynamical system is characterized by the difference equation

$$\underline{x}_{i+1} - \underline{x}_i = \underline{A}_i \underline{x}_i + \underline{B}_i \underline{u}_i \qquad (2.9)$$

$$\underline{y}_i = \underline{C}_i \underline{x}_i \qquad (2.10)$$

† This is often called the inverse function theorem. See, for example, Rudin, W., Principles of Mathematical Analysis, Second Edition, McGraw-Hill Book Company, 1964, p. 193.

†† Superscripts refer to numbered items in the References.

where $\underline{A}_i$, $\underline{B}_i$, $\underline{C}_i$ are n x n, n x r and m x n matrices, respectively for all $i = 0, 1, \ldots, N-1$.

Note that assumptions (1) and (2) of Section B are satisfied for the linear system Eq. 2.9. Assumption (3) requires that for all $i = 0, 1, \ldots, N-1$ the n x n matrix $(\underline{I} + \underline{A}_i)$ be invertible.

If the matrices $\underline{A}_i$, $\underline{B}_i$, $\underline{C}_i$ are independent of the time i, then the linear discrete system is time-invariant and is described by the difference equations

$$\underline{x}_{i+1} - \underline{x}_i = \underline{A}\,\underline{x}_i + \underline{B}\,\underline{u}_i \tag{2.11}$$

$$\underline{y}_i = \underline{C}\,\underline{x}_i \tag{2.12}$$

The linearity of Eq. 2.11 enables us to obtain a closed form expression for the state vector $\underline{x}_k$, $0 < k \leq N$ by recursion. Subject to the boundary condition

$$\underline{x}_o = \underline{\xi} \tag{2.13}$$

the solution for $\underline{x}_k$ is given by

$$\underline{x}_k = (\underline{I} + \underline{A})^k \left[\underline{\xi} + \sum_{i=1}^{k} (\underline{I} + \underline{A})^{-i}\, \underline{B}\, \underline{u}_{i-1} \right] \tag{2.14}$$

The derivation of Eq. 2.14 is well-known and may be found elsewhere.†

† See, for example, Zadeh, L. and Desoer, C., Linear System Theory, McGraw-Hill Book Company, 1963.

III. THE OPTIMAL CONTROL PROBLEM AND THE DISCRETE MINIMUM PRINCIPLE

In this chapter we shall state and discuss the discrete version of an optimal control problem and then present a set of necessary conditions for this problem which we shall call the Discrete Minimum Principle. The formulation of the discrete optimization problem will be presented in a framework paralleling that of continuous time optimization problems (see Ref.1, Chapter 5).

A. STATEMENT OF THE PROBLEM

We assume that we are given a discrete dynamical system which is characterized by the difference equation

$$\underline{x}_{i+1} - \underline{x}_i = \underline{f}_i(\underline{x}_i, \underline{u}_i) \qquad i = 0, 1, \ldots, N-1 \tag{3.1}$$

In addition, we are given an initial state $\underline{\xi} \in X$ for $i = 0$, i.e., $\underline{x}_o = \underline{\xi}$ and a specified terminal (or target) set $S \subseteq X$ which is a smooth n-k dimensional manifold of the form

$$S = \{\underline{x} : g_i(\underline{x}) = 0;\ i = 1, 2, \ldots, k \leq n\} \tag{3.2}$$

where the functions $g_i(\underline{x}), \ldots, g_k(\underline{x})$ are given twice continuously differentiable mappings from X into R_1 such that for every $\underline{x} \in X$ the vectors $\frac{\partial}{\partial \underline{x}} g_i(\underline{x})$; $i = 1, 2, \ldots, k$ are linearly independent.

The optimal control problem is to then determine the control sequence

$$\{\underline{u}_i^*,\ i = 0, 1, \ldots, N-1\} \tag{3.3}$$

and the corresponding state trajectory

$$\{\underline{x}_i^*,\ i = 0, 1, \ldots, N\} \tag{3.4}$$

such that

$$\left.\begin{aligned} &\underline{x}_o^* = \underline{\xi} \\ &\underline{x}_{i+1}^* - \underline{x}_i^* = \underline{f}_i(\underline{x}_i^*, \underline{u}_i^*), \quad i = 0, 1, \ldots, N-1 \\ &\underline{u}_i^* \in U, \quad i = 0, 1, \ldots, N-1 \\ &\underline{x}_N^* \in S \end{aligned}\right\} \tag{3.5}$$

and such that amongst all sequences $\{\underline{u}_i\}$ and $\{\underline{x}_i\}$ satisfying the above conditions, the cost functional

$$J(\{\underline{u}_i\}) = \sum_{i=0}^{N-1} L_i(\underline{x}_i, \underline{u}_i) \tag{3.6}$$

attains its minimum value at $\{\underline{u}_i\} = \{\underline{u}_i^*\}$, $\{\underline{x}_i\} = \{\underline{x}_i^*\}$.

Under these conditions, the sequence $\{\underline{u}_i^*\}$ is called the "optimal control" and the sequence $\{\underline{x}_i^*\}$ is called the "optimal trajectory".

We shall assume that for every $i = 0, 1, \ldots, N-1$ the mapping $L_i(\underline{x}, \underline{u})$ from $X \times U$ into R_1 satisfies the same "smoothness" conditions (1) and (2) of Section II B as required of the $\underline{f}_i$.

B. A CONVEXITY ASSUMPTION

Prior to stating a Discrete Minimum Principle expressing necessary conditions for the optimality of $\underline{u}_i^*$ and $\underline{x}_i^*$ we shall require a rather strong assumption which is of central importance in the proof of the Minimum Principle.

> We assume that for all pairs of vectors $\underline{u}$, $\underline{v} \in U$ and for all real numbers $\alpha \in [0, 1]$ there exists a vector $\underline{W}_\alpha \in U$ and a scalar $\beta \geq 0$ such that for all $i = 0, 1, \ldots, N-1$ and for every $\underline{x} \in X$, the $n+1$ dimensional vector
>
> $$\underline{y}_i(\underline{x}, \underline{u}) = \begin{bmatrix} \uparrow \\ \underline{f}_i(\underline{x}, \underline{u}) \\ \downarrow \\ \hline L_i(\underline{x}, \underline{u}) \end{bmatrix} \tag{3.7}$$
>
> satisfies the relation
>
> $$\underline{y}_i(\underline{x}, \underline{W}_\alpha) = \alpha \underline{y}_i(\underline{x}, \underline{v}) + (1-\alpha)\underline{y}_i(\underline{x}, \underline{u}) + \beta \underline{z} \tag{3.8}$$
>
> where $\underline{z}$ is the $n+1$ dimensional unit vector $\text{col}(0, 0, \ldots, 0, 1)$.

This assumption, which is due to J. Holtzman[5] is called "directional convexity". It is a weakening of Halkin's original assumption[3] which required that the set

$$G_i = \{\underline{y}_i(\underline{x}, \underline{u}) : \underline{u} \in U\} \tag{3.9}$$

be convex for every $\underline{x} \epsilon X$. (Note that the requirement that G_i be convex is equivalent to Assumption(4) with $\beta = 0$.)

The convexity requirement, Assumption(4), is the crucial assumption which enables one to derive a Minimum Principle for discrete systems. The proof of the Minimum Principle for both discrete and continuous time systems relies heavily on showing the convexity of certain reachable sets in n+1 dimensions. In the case of control systems which are characterized by differential equations, the time, by its evolution on a continuum, introduces a "convexifying" effect which frees us from the necessity of adding convexity assumptions to the problem statement. On the other hand, for control systems described by difference equations, the time, by its evolution on a finite set, introduces no "convexifying" effect and in order to obtain a Minimum Principle paralleling that of continuous systems, some convexity assumptions must be added to the problem formulation.

A striking example of the "convexifying" effect of time by its evolution on a continuum is most simply afforded by the following theorem.[3]

Theorem: If $f:[0,1] \rightarrow E_n$ is a piecewise continuous function and if A is the set of all subsets of $[0,1]$ which are the union of a finite number of intervals then the set

$$\{ \int_E f(t)\, dt \; : \; E \epsilon A \}$$

is convex.

On the other hand, if $g : \{0, 1, \ldots, k\} \rightarrow E_n$ and if P_k is the set of all subsets of $\{0, 1, \ldots, k\}$ then the set

$$\{ \sum_{i \epsilon S} g(i) \; : \; S \epsilon P_k \}$$

is not convex (unless $g \equiv 0$).

C. SYSTEMS SATISFYING THE CONVEXITY ASSUMPTION

An important and fairly common class of systems for which assumption(4) is satisfied are those for which each $\underline{f}_i(\underline{x}, \underline{u})$ is linear in $\underline{u}$ (or linear in $\underline{x}$ and $\underline{u}$ jointly), the set of controls U is convex and for which the functionals $L_i(\underline{x}, \underline{u})$ are convex functions of $\underline{u}$ for fixed $\underline{x}$ and for all $i = 0, 1, \ldots, N-1$.

The proof of this fact is readily demonstrated since the convexity of $L_i(\underline{x}, \underline{u})$ implies that for all $\underline{u}, \underline{v} \epsilon U$ and for all $\alpha \epsilon [0, 1]$ there exists a $\beta \geq 0$ such that

$$\alpha L_i(\underline{x}, \underline{u}) + (1-\alpha) L_i(\underline{x}, \underline{v}) = \beta + L_i(\underline{x}, \alpha\underline{u} + (1-\alpha)\underline{v}) \qquad (3.10)$$

If we now define

$$\underline{W}_\alpha = \alpha \underline{u} + (1-\alpha)\underline{v} \qquad (3.11)$$

the convexity of U assures us that the vector $\underline{W}_\alpha \epsilon U$. On the other hand, the mapping $\underline{f}_i(\underline{x}, \underline{u})$, being linear on U satisfies

$$\alpha \underline{f}_i(\underline{x}, \underline{u}) + (1-\alpha)\underline{f}_i(\underline{x}, \underline{v}) = \underline{f}_i(\underline{x}, \alpha\underline{u} + (1-\alpha)\underline{v})$$

$$= \underline{f}_i(\underline{x}, \underline{W}_\alpha) \qquad (3.12)$$

Finally, combining Eq. 3.10 and 3.12 we see that

$$\underline{y}_i(\underline{x}, \underline{W}_\alpha) = \alpha \underline{y}_i(\underline{x}, \underline{v}) + (1-\alpha)\underline{y}_i(\underline{x}, \underline{u}) + \beta \underline{z} \qquad (3.13)$$

where $\underline{y}_i(\underline{x}, \underline{u})$ is as defined in Eq. 3.7. Hence, for all $\underline{u}, \underline{v} \epsilon U$ and all $\alpha \epsilon [0, 1]$ we have shown that there exists a $\beta \geq 0$ and a vector $\underline{W}_\alpha \epsilon U$ such that the requirements of assumption (4) are satisfied.

As shown in Reference 3, another important class of discrete optimization problems for which assumption (4) is satisfied are those for which the system of difference Eqs. 3.1 approximates a system of differential equations and the cost functional 3.6 approximates a cost functional of the form

$$J = \int_0^T L(\underline{x}(t), \underline{u}(t))\,dt \qquad (3.14)$$

In other words, we are considering a discrete optimization problem which is an approximation to a continuous time optimization problem. This is quite common in practice, since from a computational viewpoint a discrete system of equations is more adaptable to computer solution than is a system of differential equations.

However, assumption (4) is not necessarily justified in the case of a system of nonlinear difference equations describing a control process which is basically discrete. Considerable care must be taken to assure the validity of this assumption (as well as the preceding ones) prior to applying the Minimum Principle.

D. NECESSARY CONDITIONS FOR OPTIMALITY-- THE MINIMUM PRINCIPLE

Given the optimal control problem formulated in Section III. A and given that assumptions (1) - (4) are satisfied, the following theorem can be proved by geometrical considerations.[3]

Theorem: (Minimum Principle for Discrete Systems)

Let $\{\underline{x}_i^*, i = 0, 1, \ldots, N\}$ be the trajectory of the system 3.1 corresponding to $\{\underline{u}_i^*\}$, where $\underline{u}_i^* \in U$, which originates at $\underline{x}_o^* = \underline{\xi}$ and terminates at $\underline{x}_N^* \in S$, where S is defined by Eq. 3.2. Then in order that $\{\underline{u}_i^*\}$ minimize the cost functional 3.6 it is necessary that there exist a sequence of n-vectors $\{\underline{p}_i^*, i = 0, 1, \ldots, N\}$ called the "costates" such that

1. The scalar function

$$H(\underline{x}_i^*, \underline{p}_{i+1}^*, \underline{u}_i) = L_i(\underline{x}_i^*, u_i) + \langle \underline{p}_{i+1}^*, \underline{f}_i(\underline{x}_i^*, \underline{u}_i) \rangle^{\dagger} \tag{3.15}$$

called the Hamiltonian has an absolute minimum as a function of $\underline{u}_i$ over U at $\underline{u}_i = \underline{u}_i^*$ for every $i = 0, 1, \ldots, N-1$, i.e.,

$$\min_{\underline{u}_i \in U} H(\underline{x}_i^*, \underline{p}_{i+1}^*, \underline{u}_i) = H(\underline{x}_i^*, \underline{p}_{i+1}^*, \underline{u}_i^*) \tag{3.16}$$

or, equivalently

$$H(\underline{x}_i^*, \underline{p}_{i+1}^*, \underline{u}_i^*) \leq H(\underline{x}_i^*, \underline{p}_{i+1}^*, \underline{u}) \quad \text{for all } \underline{u} \in U \tag{3.16a}$$

2. The evolution of $\underline{p}_i^*$ in time is determined by the difference equation

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† $\langle \underline{a}, \underline{b} \rangle$ denotes the inner (or dot product) $\sum_{i=1}^{n} a_i b_i$

$$\underline{p}^*_{i+1} - \underline{p}^*_i = - \frac{\partial H}{\partial \underline{x}} (\underline{x}^*_i , \underline{p}^*_{i+1} , \underline{u}^*_i) \tag{3.17}$$

for all $i = 0, 1, \ldots, N-1$

3. (Transversality conditions)

There exists real numbers $\beta_1, \beta_2, \ldots \beta_k$ such that

$$\underline{p}^*_N = \sum_{i=1}^{k} \beta_i \frac{\partial}{\partial \underline{x}} g_i(\underline{x}^*_N) \tag{3.18}$$

i.e., $\underline{p}^*_N$ is <u>normal</u> to S.

Special cases are (1) If $k = n$ then $S = \underline{\theta}$ is a point in E_n and nothing may be said a priori as to the value of $\underline{p}^*_N$ and (2) If $k = 0$ in which case $S = E_n$ and $\underline{p}^*_N = \underline{0}$.

E. REMARKS ON THE MINIMUM PRINCIPLE

1. In order to be able to handle pathological optimization problems, we should, strictly speaking, include an additional constant p^o in our statement of the Minimum Principle in the same manner as for continuous time optimization problems. In other words, in the statement of the Minimum Principle we should consider the Hamiltonian function

$$H(\underline{x}_i, \underline{p}_{i+1}, \underline{u}_i, p^o) = p^o L_i(\underline{x}_i, \underline{u}_i) + \langle \underline{p}_{i+1}, \underline{f}_i(\underline{x}_i, \underline{u}_i) \rangle \tag{3.19}$$

rather than the function of Eq. 3.15 where p^o is a non-negative constant. The pathological cases occur when $p^o = 0$; when $p^o \neq 0$ we may choose $p^o = 1$ (since the equation for $\underline{p}^*_i$ is linear). It is generally a difficult task to ascertain a priori whether $p^o \neq 0$ (see Ref. 3). However, the linear regulator problem which is considered in the following chapter is not a pathological case.

2. In a manner similar to the Minimum Principle for continuous dynamical systems, the discrete Minimum Principle provides only <u>necessary</u> conditions for a sequence $\{\underline{u}^*_i\}$ to locally minimize† the

† In contradistinction to the continuous case. the concepts of "weak" and "strong" relative minima are equivalent for discrete problems since the control $\underline{u}_i$ exists only on a finite set.

cost functional 3.6. In general, there may be several control sequences, called "extremal" controls, which satisfy all of the necessary conditions and which, therefore, are possible candidates for the optimal control. In order to determine which of these extremals corresponds to the global optimal, one must compute each associated value of $J(\{\underline{u}_i\})$ and simply chose the control which endows $J(\{\underline{u}_i\})$ with the smallest value. This is generally a finite procedure and is analogous to the continuous case (see Ref. 1, Chapter 5).

3. It is assumption(4) which enables us to conclude that if the control sequence $\{\underline{u}_i^*\}$ is to locally minimize $J(\{\underline{u}_i\})$, then $\underline{u}_i^*$ must necessarily absolutely minimize the Hamiltonian 3.15--a fact expressed by Eq. 3.16.

If, on the other hand, we did not require Assumption(4), then as shown by Jordan and Polak,[4] a necessary condition for the sequence $\{\underline{u}_i^*\}$ to be a local minimum of $J(\{\underline{u}_i\})$ is that $\underline{u}_i^*$ locally minimize the Hamiltonian. In other words, in the absence of Assumption(4), we can conclude a "Stationarity" Principle but not a "Minimum" Principle. In fact, it is possible under these circumstances to have a unique control which absolutely minimizes the Hamiltonian but does not locally minimize the cost functional. In cases when Assumption(4) is independently satisfied, the stationarity principle can also be used to determine the optimal control by the procedure outlined in the foregoing remark 2.

4. In the event that the terminal manifold $S = E_n$ and if the cost functional $J(\{\underline{u}_i\})$ is of the form

$$J(\{\underline{u}_i\}) = K(\underline{x}_N) + \sum_{i=0}^{N-1} L_i(\underline{x}_i, \underline{u}_i) \tag{3.20}$$

where $K(\cdot)$ is a twice continuously differentiable functional defined on X, (hence the effect of the final state $\underline{x}_N$ is included in our performance measure) then it can be shown that the transversality condition corresponding to this problem is that

$$\underline{p}_N^* = \frac{\partial}{\partial \underline{x}} K(\underline{x}_N^*) \tag{3.21}$$

This fact will be used in Section IV in the solution of the linear regulator problem.

IV. THE DISCRETE LINEAR REGULATOR PROBLEM

A. THE PROBLEM STATEMENT

In this chapter the necessary conditions of the Minimum Principle will be used to solve a discrete optimization problem. The problem to be considered is the discrete analog of the Linear Regulator problem for continuous time systems (see Reference 1, Chapter 9). To be more specific we suppose that the discrete system is linear on X x U. The difference equation describing the evolution of the state is assumed to be given as

$$\underline{x}_{i+1} - \underline{x}_i = \underline{A}_i \underline{x}_i + \underline{B}_i \underline{u}_i \qquad i = 0, 1, \ldots, N-1 \qquad (4.1)$$

where $\underline{A}_i$ is an $n \times n$ matrix for $i = 0, 1, \ldots, N-1$

$\underline{B}_i$ is an $n \times r$ matrix for $i = 0, 1, \ldots, N-1$

The control space $U = E_r$ (i.e., $\underline{u}_i$ is unconstrained)

The matrix $(\underline{I} + \underline{A}_i)$ is non-singular for $i = 0, 1, \ldots, N-1$

The discrete regulator problem is to determine the control sequence $\{\underline{u}_i^*, \; i = 0, 1, \ldots, N-1\}$ such that the corresponding state sequence $\{\underline{x}_i^*, \; i = 0, 1, \ldots, N\}$ satisfies

$$\underline{x}_o^* = \underline{\xi} = \text{given "initial" state} \qquad (4.2)$$

$$\underline{x}_N^* \in S = E_n, \qquad (4.3)$$

so that $\underline{x}_N^*$ is unconstrained, and such that

$$J(\{\underline{u}_i\}) = \frac{1}{2} \langle \underline{x}_N, \underline{Q}_N \underline{x}_N \rangle + \frac{1}{2} \sum_{i=0}^{N-1} \langle \underline{x}_i, \underline{Q}_i \underline{x}_i \rangle + \langle \underline{u}_i, \underline{R}_i \underline{u}_i \rangle \qquad (4.4)$$

is absolutely minimized, where

$\underline{Q}_i$ is a positive semi-definite $n \times n$ matrix for $i=0, 1, \ldots, N$

$\underline{R}_i$ is a positive definite $r \times r$ matrix for $i=0, 1, \ldots, N-1$

The performance functional Eq. 4.4 is the discrete analog of the quadratic performance functional for continuous problems.

B. NECESSARY CONDITIONS FOR OPTIMALITY

Prior to applying the Minimum Principle to this problem we must insure that assumptions (1) - (4) are satisfied. Assumptions (1) and (2) are clearly satisfied by virtue of the choice of $\underline{f}_i$ and L_i. The $\underline{A}_i$ matrices were assumed to satisfy assumption (3). Finally the convexity assumption (4) is satisfied by virtue of the remarks of Section III C. We are now in position to apply the Minimum Principle.

Let us assume that $\{\underline{u}_i^*\}$ and $\{\underline{x}_i^*\}$ represent an optimal solution to our problem. The first necessary condition of the Minimum Principle requires that the Hamiltonian

$$H(\underline{x}_i^*, \underline{p}_{i+1}^*, \underline{u}) = L_i(\underline{x}_i^*, \underline{u}) + \langle \underline{p}_{i+1}^*, \underline{f}_i(\underline{x}_i^*, \underline{u}) \rangle$$

$$= \frac{1}{2} \langle \underline{x}_i^*, \underline{Q}_i \underline{x}_i^* \rangle + \frac{1}{2} \langle \underline{u}, \underline{R}_i \underline{u} \rangle + \langle \underline{p}_{i+1}^*, \underline{A}_i \underline{x}_i^* \rangle$$

$$+ \langle \underline{p}_{i+1}^*, \underline{B}_i \underline{u} \rangle$$

have an absolute minimum as a function of $\underline{u}$ at $\underline{u} = \underline{u}_i^*$ for every $i=0, 1, \ldots, N-1$. Since $\underline{u}$ is unconstrained and $\frac{\partial H}{\partial \underline{u}}$ exists, the minimum of the Hamiltonian is found by setting $\frac{\partial H}{\partial \underline{u}} = \underline{0}$. At this minimum $\underline{u} = \underline{u}_i^*$. Hence

$$\underline{0} = \frac{\partial H}{\partial \underline{u}} (\underline{x}_i^*, \underline{p}_{i+1}^*, \underline{u}_i^*) = \underline{R}_i \underline{u}_i^* + \underline{B}_i' \underline{p}_{i+1}^*$$

or

$$\underline{u}_i^* = -\underline{R}_i^{-1} \underline{B}_i' \underline{p}_{i+1}^* \qquad (4.5)$$

The required inverse exists since $\underline{R}_i$ is positive definite. Furthermore, the Hamiltonian has a unique minimum at $\underline{u} = \underline{u}_i^*$ by virtue of the fact that $\frac{\partial^2 H}{\partial \underline{u}^2} (\underline{x}_i^*, \underline{p}_{i+1}^*, \underline{u}_i^*) = \underline{R}_i$ is positive definite.

The "costate" vectors $\underline{p}_i^*$, $i = 0, 1, \ldots N-1$ corresponding to $\underline{u}_i^*$ and $\underline{x}_i^*$ satisfy the difference equation

$$\underline{p}_{i+1}^* - \underline{p}_i^* = -\frac{\partial H}{\partial \underline{x}}(\underline{x}_i^*, \underline{p}_{i+1}^*, \underline{u}_i^*)$$

$$= -\underline{Q}_i \underline{x}_i^* - \underline{A}_i' \underline{p}_{i+1}^* \tag{4.6}$$

Application of the transversality condition Eq. 3.21 corresponding to a cost functional of the form Eq. 3.20 with $S = E_n$, yields, since $(\underline{x}_N) = \frac{1}{2} < \underline{x}_N, \underline{Q}_N \underline{x}_N >$, that $\underline{p}_N^* = \underline{Q}_N \underline{x}_N$.

Combining Eqs. 4.1, 4.5 and 4.6 we obtain the $2n \times 2n$ system of "canonical" equations

$$\underline{x}_{i+1}^* - \underline{x}_i^* = \underline{A}_i \underline{x}_i^* - \underline{B}_i \underline{R}_i^{-1} \underline{B}_i' \underline{p}_{i+1}^* \tag{4.7a}$$

$$\underline{p}_{i+1}^* - \underline{p}_i^* = -\underline{Q}_i \underline{x}_i^* - \underline{A}_i' \underline{p}_{i+1}^* \tag{4.7b}$$

with the "split" boundary conditions

$$\underline{x}_o^* = \underline{\xi} \tag{4.8a}$$

$$\underline{p}_N^* = \underline{Q}_N \underline{x}_N^* \tag{4.8b}$$

If we could solve the above two-point boundary value problem for $\underline{p}_{i+1}^*$ we would then obtain (by Eq. 4.5) an expression, in terms of the time i and initial state $\underline{\xi}$, for the optimal control $\underline{u}_i^*$. This is, in general, an extremely difficult problem. As an alternate proceedure, with consistent analogy to the continuous time problem,[1] we shall seek the optimal control law i.e., the optimal control $\underline{u}_i^*$ as a function of the state $\underline{x}_i^*$ and the time i.

$$\underline{u}_i^* = \underline{u}_i^*(\underline{x}_i^*, i) \tag{4.9}$$

To be more specific, we shall assume that $\underline{p}_i^*$ and $\underline{x}_i^*$ are related by a linear transformation for all i, namely

$$\underline{p}_i^* = \underline{K}_i \underline{x}_i^* \tag{4.10}$$

where each $\underline{K}_i$ is an n x n matrix which is to be determined. The optimal control law Eq. 4.9 will then be a linear feedback law,

$$\underline{u}_i^* = -\underline{R}_i^{-1} \underline{B}_i' \underline{K}_{i+1} \underline{x}_{i+1}^* \qquad (4.11)$$

C. THE DISCRETE "RICATTI EQUATION"

In order to determine a recurrence relation for the matrices $\underline{K}_i$, i = 0, 1, . . . , N-1 we shall make use of Eqs. 4.7 a, b. Introducing the assumed expression for $\underline{p}_{i+1}^*$ yields, respectively

$$\underline{x}_{i+1}^* - \underline{x}_i^* = \underline{A}_i \underline{x}_i^* - \underline{B}_i \underline{R}_i^{-1} \underline{B}_i' \underline{K}_{i+1} \underline{x}_{i+1}^* \qquad (4.12a)$$

$$\underline{K}_{i+1} \underline{x}_{i+1}^* - \underline{K}_i \underline{x}_i^* = -\underline{Q}_i \underline{x}_i^* - \underline{A}_i' \underline{K}_{i+1} \underline{x}_{i+1}^* \qquad (4.12b)$$

Rearranging terms we have

$$(\underline{I} + \underline{B}_i \underline{R}_i^{-1} \underline{B}_i' \underline{K}_{i+1}) \underline{x}_{i+1}^* = \underline{\Phi}_i \underline{x}_i^* \qquad (4.13a)$$

$$\underline{\Phi}_i' \underline{K}_{i+1} \underline{x}_{i+1}^* + \underline{Q}_i \underline{x}_i^* - \underline{K}_i \underline{x}_i^* = \underline{0} \qquad (4.13b)$$

where

$$\underline{\Phi}_i \stackrel{\Delta}{=} \underline{I} + \underline{A}_i \qquad (4.14)$$

Solving Eq. 4.13a for $\underline{x}_{i+1}^*$ (assuming that the required inverse exists) we obtain

$$\underline{x}_{i+1}^* = (\underline{I} + \underline{B}_i \underline{R}_i^{-1} \underline{B}_i' \underline{K}_{i+1})^{-1} \underline{\Phi}_i \underline{x}_i^* \qquad (4.15)$$

The inverse term in the above expression can be simplified by making use of the matrix identity[6]

$$(\underline{I}_n + \underline{A}\,\underline{B}')^{-1} = \underline{I}_n - \underline{A}(\underline{I}_r + \underline{B}'\underline{A})^{-1} \underline{B}' \qquad (4.16)$$

where $\underline{A}$, $\underline{B}$ are n x r matrices.

Application of this identity to $(\underline{I} + \underline{B}_i \underline{R}_i^{-1} \underline{B}_i' \underline{K}_{i+1})$, taking $\underline{A} = \underline{B}_i$, $\underline{B}' = \underline{R}_i^{-1} \underline{B}_i \underline{K}_{i+1}$ yields

$$(\underline{I} + \underline{B}_i \underline{R}_i^{-1} \underline{B}_i' \underline{K}_{i+1})^{-1} = \underline{I} - \underline{B}_i(\underline{I}_r + \underline{R}_i^{-1} \underline{B}_i' \underline{K}_{i+1} \underline{B}_i)^{-1} \underline{R}_i^{-1} \underline{B}_i' \underline{K}_{i+1}$$

$$= \underline{I} - \underline{B}_i[\underline{R}_i(\underline{I}_r + \underline{R}_i^{-1}\underline{B}_i'\underline{K}_{i+1}\underline{B}_i]^{-1}\underline{B}_i'\underline{K}_{i+1}$$

$$= \underline{I} - \underline{B}_i(\underline{R}_i + \underline{B}_i'\underline{K}_{i+1}\underline{B}_i)^{-1}\underline{B}_i'\underline{K}_{i+1} \qquad (4.17)$$

Hence Eq. 4.16 becomes

$$\underline{x}_{i+1}^* = [\underline{I} - \underline{B}_i(\underline{R}_i + \underline{B}_i'\underline{K}_{i+1}\underline{B}_i)^{-1}\underline{B}_i'\underline{K}_{i+1}]\underline{\Phi}_i\underline{x}_i^{*\dagger} \qquad (4.18)$$

Substituting this expression into Eq. 4.13b results in

$$\{-\underline{K}_i + \underline{\Phi}_i'[\underline{K}_{i+1} - \underline{K}_{i+1}\underline{B}_i(\underline{R}_i + \underline{B}_i'\underline{K}_{i+1}\underline{B}_i)^{-1}\underline{B}_i'\underline{K}_{i+1}]\underline{\Phi}_i + \underline{Q}_i\}\underline{x}_i^* = \underline{0} \qquad (4.19)$$

for all $i = 0, 1, \ldots, N-1$. Since Eq. 4.19 must hold for any choice of the initial state $\underline{\xi}$ and since the matrix $\underline{K}_i$ does not depend on $\underline{\xi}$, we can conclude that Eq. 4.19 must be satisfied for all $\underline{x}_i^*$. This implies that $\underline{K}_i$ must satisfy the matrix difference equation

$$\underline{K}_i = \underline{\Phi}_i'[\underline{K}_{i+1} - \underline{K}_{i+1}\underline{B}_i(\underline{R}_i + \underline{B}_i'\underline{K}_{i+1}\underline{B}_i)^{-1}\underline{B}_i'\underline{K}_{i+1}]\underline{\Phi}_i + \underline{Q}_i \qquad (4.20)$$

which is the desired result (also obtainable by Dynamic Programming[7]). Equation 4.20 is an explicit rule for obtaining $\underline{K}_i$ from $\underline{K}_{i+1}$ and may be regarded as a discrete analog for the matrix Ricatti (differential) equation.[1]

Turning our attention to the boundary condition for Eq. 4.20 we have, since $\underline{p}_N^* = \underline{Q}_N\underline{x}_N^*$ that

$$\underline{K}_N\underline{x}_N^* = \underline{Q}_N\underline{x}_N^*$$

but since $\underline{x}_N^*$ is unspecified, we conclude that

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† Note that use of the matrix identity Eq. 4.16 has enabled us to replace the problem of inverting an $n \times n$ matrix with one of inverting an $r \times r$ matrix. In many control problems this can truly be advantageous since r is generally $< n$.

$$\underline{K}_N = \underline{Q}_N \tag{4.21}$$

Consequently, the existence and uniqueness of the matrices $\underline{K}_i$, $i = 0, 1, \ldots, N-1$ validates our claim that $\underline{p}_i^* = \underline{K}_i \underline{x}_i^*$, provided, of course, that the inverse term in Eq. 4.15 exists.

In addition, the matrices $\underline{K}_i$, $i = 0, 1, \ldots, N-1$ are symmetric. This is a direct consequence of the fact that $\underline{K}_i$ and $\underline{K}_i'$ both are solutions of the same difference equation with identical boundary conditions--a fact readily verified by taking the transpose of Eq. 4.20, noting that $\underline{R}_i' = \underline{R}_i$ and $\underline{Q}_i' = \underline{Q}_i$. Hence $\underline{K}_i = \underline{K}_i'$ which implies that $\underline{K}_i$ is symmetric.

D. THE OPTIMAL FEEDBACK CONTROL LAW

Having obtained the difference equation for $\underline{K}_i$ we now can obtain an expression for the optimal feedback control law Eq. 4.13. Substituting Eq. 4.18 for $\underline{x}_{i+1}^*$ into Eq. 4.11 yields

$$\begin{aligned} \underline{u}_i^* &= -\underline{R}_i^{-1} \underline{B}_i' \underline{K}_{i+1} [\underline{I} - \underline{B}_i (\underline{R}_i + \underline{B}_i' \underline{K}_{i+1} \underline{B}_i)^{-1} \underline{B}_i' \underline{K}_{i+1}] \underline{\Phi}_i \underline{x}_i^* \\ &= -\underline{R}_i^{-1} [\underline{I} - \underline{B}_i' \underline{K}_{i+1} \underline{B}_i (\underline{R}_i + \underline{B}_i' \underline{K}_{i+1} \underline{B}_i)^{-1}] \underline{B}_i' \underline{K}_{i+1} \underline{\Phi}_i \underline{x}_i^* \\ &= -(\underline{R}_i + \underline{B}_i' \underline{K}_{i+1} \underline{B}_i)^{-1} \underline{B}_i' \underline{K}_{i+1} \underline{\Phi}_i \underline{x}_i^* \end{aligned} \tag{4.22}$$

where the last expression is obtained by use of the matrix identity

$$\underline{I} - \underline{X}(\underline{Y} + \underline{X})^{-1} = Y(Y + \underline{X})^{-1}$$

Eq. 4.22 is the desired expression for the optimal control law at time i. The optimal control requires knowledge of $\underline{K}_{i+1}$ (which can be pre-computed according to Eq. 4.20 and stored) as well as knowledge of the state of the system at time i. A discrete control scheme such as this one is indeed feasible from a practical standpoint--requiring only adjustable feedback gains.

E. AN EXPRESSION FOR THE OPTIMAL COST

We now claim that the matrix $\underline{K}_i$ for $i = 0, 1, \ldots, N$ has the property that

$$J^*(\underline{x}_i^*, j) = \frac{1}{2} < \underline{x}_j^*, \underline{K}_j \underline{x}_j^* > = \frac{1}{2} < \underline{x}_N^*, \underline{Q}_N \underline{x}_N^* > + \frac{1}{2} \sum_{i=j}^{N-1} < \underline{x}_i^*, \underline{Q}_i \underline{x}_i^* >$$

$$+ < \underline{u}_i^*, \underline{R}_i \underline{u}_i^* > \qquad (4.23)$$

= minimum "cost remaining" along the optimal trajectory starting at time j and state $\underline{x}_j^*$

To prove this fact is is only necessary to substitute Eq. 4.22 for $\underline{u}_i^*$ into the above expression. This gives (since $\underline{K}_{i+1}$ is symmetric)

$$J^*(\underline{x}_i^*, j) - \frac{1}{2} < \underline{x}_N^*, \underline{Q}_N \underline{x}_N^* > = \frac{1}{2} \sum_{i=j}^{N-1} < \underline{x}_i^*, \underline{Q}_i + \underline{\Phi}_i' \underline{K}_{i+1} \underline{B}_i (\underline{R}_i + \underline{B}_i' \underline{K}_{i+1} \underline{B}_i)^{-1} \cdot$$

$$\underline{R}_i (\underline{R}_i + \underline{B}_i' \underline{K}_{i+1} \underline{B}_i)^{-1} \underline{B}_i' \underline{K}_{i+1} \underline{\Phi}_i \underline{x}_i^* >$$

$$= \frac{1}{2} \sum_{i=j}^{N-1} < \underline{x}_i^*, \underline{Q}_i + \underline{\Phi}_i' \underline{K}_{i+1} \underline{B}_i [\underline{I} - (\underline{R}_i + \underline{B}_i' \underline{K}_{i+1} \underline{B}_i)^{-1} \cdot$$

$$\underline{B}_i' \underline{K}_{i+1} \underline{B}_i] (\underline{R}_i + \underline{B}_i' \underline{K}_{i+1} \underline{B}_i)^{-1} \underline{B}_i' \underline{K}_{i+1} \underline{\Phi}_i x_i^* >$$

$$= \frac{1}{2} \sum_{i=j}^{N-1} < \underline{x}_i^*, [\underline{Q}_i - \underline{\Phi}_i' \underline{K}_{i+1} \underline{B}_i (\underline{R}_i + \underline{B}_i' \underline{K}_{i+1} \underline{B}_i)^{-1} \underline{B}_i' \underline{K}_{i+1} \underline{\Phi}_i +$$

$$\underline{\Phi}_i' \underline{K}_{i+1} \underline{\Phi}_i] \underline{x}_i^* >$$

$$- < \underline{x}_i^*, \{\underline{\Phi}_i' [I - \underline{K}_{i+1} \underline{B}_i (\underline{R}_i + \underline{B}_i' \underline{K}_{i+1} \underline{B}_i)^{-1} \underline{B}_i'] \cdot$$

$$\underline{K}_{i+1} [\underline{I} - \underline{B}_i (R_i + \underline{B}_i' \underline{K}_{i+1} B_i)^{-1} \underline{B}_i' \underline{K}_{i+1}] \underline{\Phi}_i \} x_i^* >$$

$$= \frac{1}{2} \sum_{i=j}^{N-1} < \underline{x}_i^*, \underline{K}_i \underline{x}_i^* > - < \underline{x}_{i+1}^* \underline{K}_{i+1} \underline{x}_{i+1}^* >$$

$$= \frac{1}{2} < \underline{x}_j^*, \underline{K}_j \underline{x}_j^* > - \frac{1}{2} < \underline{x}_N^*, \underline{K}_N \underline{x}_N^* >$$

Consequently, since $\underline{K}_N = \underline{Q}_N$ we have that

$$J^*(\underline{x}_j^*, j) = \frac{1}{2} < \underline{x}_j^*, \underline{K}_j \underline{x}_j^* > \qquad (4.24)$$

as claimed. In particular, for $j = 0$ we have that

$$\frac{1}{2} < \underline{x}_o, \underline{K}_o \underline{x}_o > \; = \text{ optimal "cost"}$$

$$= \text{ minimum value of } J(\{\underline{u}_i\})$$

It should also be noted that by virtue of the facts (a) $\underline{Q}_i \geq \underline{0}$ and (b) $\underline{R}_i > \underline{0}$ we have that $J^*(\underline{x}_i^*, i) \geq 0$ for all $\{\underline{x}_i^*\}$, $\{\underline{u}_i^*\}$. Thus

$$< \underline{x}_i^*, \underline{K}_i \underline{x}_i^* > \; \geq 0 \qquad \text{for all } \underline{x}_i^*$$

Hence $\underline{K}_i$ is a positive semi-definite matrix for $i = 0, 1, \ldots, N$.

F. CONCLUDING REMARKS

Using the property that $\underline{K}_i$ is positive semi-definite we can now regress and establish the validity of the assumption in Eq. 4.15 that the inverse of $(\underline{I} + \underline{B}_i \underline{R}_i^{-1} \underline{B}_i' \underline{K}_{i+1})$exists. Since $\underline{K}_{i+1}$ is $\geq \underline{0}$, the mr x mr matrix $\underline{B}_i' \underline{K}_{i+1} \underline{B}_i$ is also positive semi-definite. Furthermore, since the sum of a positive definite matrix and a positive semi-definite matrix is itself positive definite, we conclude that $(\underline{R}_i + \underline{B}_i' \underline{K}_{i+1} \underline{B}_i) > \underline{0}$. Hence the claim that $(\underline{R}_i + \underline{B}_i' \underline{K}_{i+1} \underline{B}_i)^{-1}$ exists is validated. Finally, by the matrix identity 4.16 we establish that $(\underline{I} + \underline{B}_i \underline{R}_i^{-1} \underline{B}_i' \underline{K}_{i+1})^{-1}$ indeed exists.

Consequently, we have shown, by use of the Minimum Principle, that if the optimal control exists, then it is uniquely determined by Eqs. 4.20 and 4.22. By independent arguments (such as those used in the continuous time case) it is possible to conclude that the optimal control indeed exists. Consequently, for the discrete optimum linear regulator problem the minimum principle supplies us with both a <u>necessary</u> and <u>sufficient</u> condition for optimality. This completes our investigation of the above optimization problem.

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