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A STUDY OF LONGITUDINAL OSCILLATIONS
OF PROPELLANT TANKS AND
WAVE PROPAGATIONS IN FEED LINES

Part V - Longitudinal Oscillation of a Propellant-
Filled Flexible Oblate Spheroidal Tank

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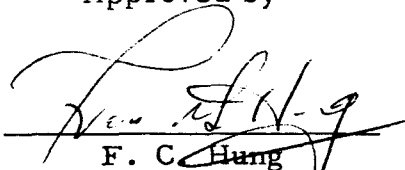
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FOREWORD

This report was prepared by the Space and Information Systems Division of North American Aviation, Inc., Downey, California, for the George C. Marshall Space Flight Center, National Aeronautics and Space Administration, Huntsville, Alabama, under Contract No. NAS8-11490, "Study of Longitudinal Oscillations of Propellant Tanks and Wave Propagations in Feed Lines," dated January 6, 1965. Dr. George F. McDonough (Principal) and Mr. Robert S. Ryan (Alternate) of Aero-Astrodynamics Laboratory, MSFC, are Contracting Officer Representatives. The work is published in five separate parts:

- Part I - One-Dimensional Wave Propagation in a Feed Line
- Part II - Wave Propagation in an Elastic Pipe Filled With Incompressible Viscous Fluid
- Part III - Wave Propagation in an Elastic Pipe Filled With Incompressible Viscous Streaming Fluid
- Part IV - Longitudinal Oscillation of a Propellant-Filled Flexible Hemispherical Tank
- Part V - Longitudinal Oscillation of a Propellant-Filled Flexible Oblate Spheroidal Tank

The project was carried out by the Launch Vehicle Dynamics Group, Structures and Dynamics Department of Research and Engineering Division, S&ID, Dr. F.C. Hung was the Program Manager for North American Aviation, Inc. The study was conducted by Dr. Clement L. Tai (Principal Investigator), Dr. Michael M. H. Loh, Mr. Henry Wing, Dr. Sui-An Fung, and Dr. Shoichi Uchiyama. Dr. James Sheng, who started the investigation of Part IV, left in the middle of the program to teach at the University of Wisconsin. The computer program was developed by Mr. R. A. Pollock, Mr. F. W. Egeling, and Mr. S. Miyashiro.



ABSTRACT

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The present study describes an analytical method for determining the axisymmetric longitudinal mode shapes and frequencies of an incompressible and inviscid fluid contained in a pressurized, flexible oblate spheroidal propellant tank. Series expansions for the fluid velocity potential and the tank wall deflections are combined through the boundary conditions and shell equations of motion to obtain an eigenvalue problem whose solutions are the system frequencies and the coefficients of the series. In the analysis, the effect of the ullage gas pressure is included. The matrix eigenvalue problem for the case of a hemispherical tank was previously programmed for the computer solutions by Tai and Wing (Reference 1). This program will be directly applied to the present eigenvalue problem for the numerical solutions.



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NOMENCLATURE

a	Semi-major axis of oblate spheroid
\vec{a}_n	Unit vector
b	Semi-minor axis of oblate spheroid
$B_0, B_1 \cdots B_n$	Nondimensional arbitrary constants, Equation (10)
C	Half the distance between foci of the ellipse and equal to $(b-a)$
D	Extensional modulus, Equation (52) and differential operator, Equation (78)
D_1, D_2, D_3	Arbitrary constants, Equations (6) and (8)
$e_{\eta\eta}, e_{\phi\phi}$	Strains in the middle surface of the shell
E	Young's modulus
$E_1, E_2 \cdots E_m$	Nondimensional arbitrary constants, Equation (17)
$f_{\xi}, f_{\eta}, f_{\phi}$	Inertia forces
$g_{\xi 1}, g_{\xi 2}, g_{\xi 3}$	Functional coefficients, Equation (65)
h	Thickness of the shell
$h_1, h_2 \cdots h_n$	Scale factors, Equation (29)
$h_{\eta 1}, h_{\eta 2} \cdots h_{\eta 5}$	Functional coefficients, Equation (60)
$h_{\xi 1}, h_{\xi 2}, h_{\xi 3}$	Functional coefficients, Equation (60)
i	Imaginary number
$k_{\eta 1}, k_{\eta 2}$	Functional coefficients, Equations (67) and (68)
$L\bar{w}$	Difference terms of the first order, Equation (84)



$M_\eta, M_\phi, M_{\eta\phi}, M_{\phi\eta}$	Stress couple resultants
$N_\eta, N_\phi, N_{\eta\phi}, N_{\phi\eta}$	Stress resultants
P_n	Legendre polynomials
P_u	Ullage pressure
P_{us}	Static ullage pressure
P_ξ, P_η, P_ϕ	External loads in the shell equations of equilibrium and P_ξ is the liquid pressure in the present analysis
P_ξ	Nondimensional liquid pressure
q	Resultant velocity of liquid
Q_n	Legendre polynomials of the second kind
Q_η, Q_ϕ	Transverse shearing stress resultants
r, θ, ϕ	Cylindrical coordinates, Figure 1
R	Function of independent variable η , Equation (66)
R_η, R_ϕ	Principal radii of curvature
$S\bar{w}$	Difference terms of order higher than the second, Equation (85)
S_I	Integrated squared error over the interface, Equation (108)
S_s	Integrated squared error over the liquid surface, Equation (107)
S_T	Total integrated squared error, Equation (106)
t	Time
u, v, w	Hoop, meridian and normal displacements
u, v	Homogeneous solutions of the linear differential equation of second order, Equation (72)



\bar{w}	Particular solution of the linear differential equation of second order, Equation (72)
\bar{V}, \bar{W}	Nondimensional meridian and normal displacements
V_{gm}	Change in the ullage volume due to a unit deflection of the m^{th} normal deflection, Equation (17)
V_o	Displacement of liquid surface, Equation (15)
V_g	Change in the ullage volume due to the shell motion, Equation (16)
V_u	Ullage volume
V_{us}	Static ullage volume
w_u	Ullage normal displacement
X, Y	Separated dependent variables, Equation (3)
x, y, z	Cartesian coordinates, Figure 1
α_η, α_ϕ	Functions of independent variable η , Equation (42)
γ	Isentropic exponent, Equation (13)
$\delta^2, \delta^4, \dots, \delta^{2m}$	Symbols of difference defined by Equation (80)
$\mu\delta, \mu\delta^3, \dots, \mu\delta^{2m+1}$	Symbols of difference defined by Equation (82)
λ	Lagrange multiplier
ν	Poisson's ratio
ξ, η, ϕ	Oblate spheroidal coordinates, Figure 1
ξ_o	= b/c
ξ_η, ξ_ϕ	Oblate spheroidal coordinates in meridian and hoop-directions, respectively
ρ	Mass density of the shell
ρ_f	Mass density of the liquid



ϕ	Velocity potential of the liquid for steady flow
Φ	Velocity potential of the liquid
ω	Natural frequency
Ω	$= \omega^2$



INTRODUCTION

Knowledge of the dynamic behavior of thin-walled fuel tanks is of great importance in launch vehicle and spacecraft design. Thin elastic shells are used as structural elements of such fuel tanks. With respect to the dynamic behavior, of prime importance are the natural modes and corresponding natural frequencies of vibrations. Analysis of such response must account for deformational properties of both the liquid and the shell and for their mutual constraints. To develop an understanding of the action of a liquid and shell combination, the longitudinal oscillation of liquid in a pressurized thin-walled oblate spheroidal tank is considered.

The small motion of an ideal liquid in a fixed or moving rigid container of simple geometrical shape, such as a circular cylindrical tank and a spherical tank, is well known; and the methods of analysis for such problems are available for obtaining approximate solutions. Budiansky (Reference 2), Hwang (Reference 3), and Chu (Reference 4) have analyzed the natural oscillations of liquid in rigid tanks by the source-sink approach, leading to integral equation statements of the eigenvalue problem. A number of studies have also been made for a thin-shell container of cylindrical and spherical shapes. Coale and Nagano (Reference 5) have dealt with the axisymmetric dynamic behavior of a cylindrical tank with a hemicylindrical bottom by the method of minimization of the integrated squared error. Gossard (Reference 6) has introduced the energy method for the axisymmetric dynamic response calculation of liquid-filled, hemispherical elastic membrane shells and has demonstrated the numerical calculations by free and forced response calculations for sloshing liquid. Palmer and Asher (Reference 7) have discussed the axisymmetric longitudinal oscillation of thin elastic shells of revolution partially filled with an incompressible liquid and containing a pressurized gas in the remaining volume by the direct stiffness method. The very recent work of Tai and Wing (Reference 1) is concerned with the axisymmetric oscillation of a propellant—filled flexible hemispherical tank and the method of approach is basically the same as in the present report.

The method of analysis reported herein is based on a mathematical model of a membrane shell filled with an incompressible and inviscid fluid in axisymmetric harmonic motion. It followed that the velocity potential satisfying the Laplace's equation is first established, and then the equations of motion of the shell elements, including the forcing functions obtained from the velocity potential, are solved for the shell displacement components



with the boundary conditions. Finally, the natural modes and corresponding natural frequencies of the liquid-filled oblate spheroidal tank are determined by the method of minimization.



METHOD OF ANALYSIS

The tank model upon which the present analysis is based is a flexible, liquid-filled, oblate spheroidal shell. In Figure 1, the shell configuration and the oblate spheroidal coordinate system (ξ, η, ϕ) are shown, together with the rectangular (x, y, z) and the spherical (r, θ, ϕ) coordinate systems. The displacement components (u, v, w) of the shell are also shown. The liquid in the shell is assumed to be both incompressible and inviscid. The volume above the liquid's free surface is filled with a gas that is assumed to have no significant dynamics of its own in the frequency range under consideration. The pressure P_u due to such gas acts uniformly on the free surface of the liquid. Only an axisymmetric motion is considered. The inertia forces of the shell in both the radial and meridian directions are included.

1. VELOCITY POTENTIAL OF LIQUID

It is shown in hydrodynamic theory (Reference 8) that the continuity requirement for an incompressible, irrotational liquid is stated by Laplace's

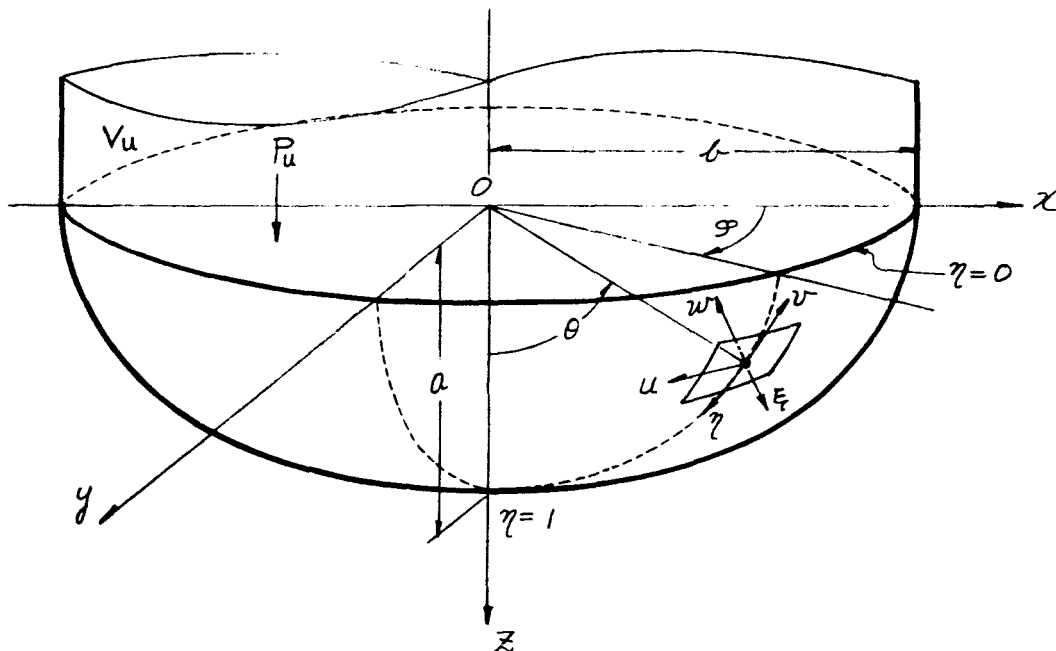


Figure 1. Oblate Spheroid Shell Geometry and Shell Displacement



equation. The velocity potential that satisfies the Laplace's equation may be taken to be

$$\Phi = \phi \cos \omega t \quad (1)$$

for harmonic motion where ω is the natural frequency, t is the time, and ϕ represents the velocity potential for steady flow. The Laplace's equation for the case of the axisymmetric flow of a propellant contained in an oblate spheroidal shell is written as

$$\nabla^2 \phi = \frac{\partial}{\partial \xi} \left[(1 + \xi^2) \frac{\partial \phi}{\partial \xi} \right] + \frac{\partial}{\partial \eta} \left[(1 - \eta^2) \frac{\partial \phi}{\partial \eta} \right] = 0 \quad (2)$$

By the method of separation,

$$\phi = X(\xi) Y(\eta) \quad (3)$$

Then, two separate ordinary differential equations are

$$\frac{d}{d\xi} \left[(1 + \xi^2) \frac{dX}{d\xi} \right] - n(n+1) X = 0 \quad (4)$$

$$\frac{d}{d\eta} \left[(1 - \eta^2) \frac{dY}{d\eta} \right] + n(n+1) Y = 0 \quad (5)$$

where n is a non-negative integer.

Solving Equation (4) for X gives

$$X = D_1 P_n(i\xi) + D_2 Q_n(i\xi) \quad (6)$$



where D_1 and D_2 are arbitrary constants and P_n and Q_n are Legendre functions. In order to avoid singularities at $\xi = 0$, D_2 is assumed zero. For this case,

$$X = D_1 P_n (i\xi) \quad (7)$$

Solving Equation (5) for Y gives

$$Y = D_3 P_n (\eta) \quad (8)$$

where D_3 is an arbitrary constant.

From the solutions given by Equations (7) and (8), it is reasonable to assume that

$$\phi = C^2 \xi_o^2 \omega \left[B_0 + \sum_{n=1}^{\infty} B_n P_n (i\xi) P_n (\eta) \right] \quad (9)$$

where

C = half the distance between foci of the ellipse and equal to $b - a$

$$\xi_o = \frac{b}{C}$$

B_0, B_n = nondimensional arbitrary constants to be determined

$$P_n (i\xi) = \frac{(2n)!}{2^n (n!)^2} \left\{ (i\xi)^n - \frac{n(n-1)}{2(2n-1)} (i\xi)^{n-2} + \right. \\ \left. \frac{n(n-1)(n-2)(n-3)}{2 \cdot 4 (2n-1)(2n-3)} (i\xi)^{n-4} - \dots \right\}$$



$$P_n(\eta) = \frac{(2n)!}{2^n (n!)^2} \left\{ \eta^n - \frac{n(n-1)}{2(2n-1)} \eta^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2 \cdot 4 (2n-1)(2n-3)} \eta^{n-4} - \dots \right\}$$

The substitution of Equation (9) into Equation (1) yields the velocity potential:

$$\Phi = C^2 \xi_o^2 \omega \left[B_o + \sum_{n=1}^{\infty} B_n P_n(i\xi) P_n(\eta) \right] \cos \omega t \quad (10)$$

that is capable of describing any admissible axisymmetric velocity pattern of liquid within the oblate spheroidal shell.

2. PRESSURE ON THE SHELL WALL

The fluid pressure on the shell wall, P_ξ , can be related to the velocity potential, Φ , through Kelvin's equation (Reference 7)

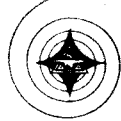
$$P_\xi + \rho_f \frac{\partial \Phi}{\partial t} + \frac{1}{2} \rho_f q^2 = P_u \quad (11)$$

where ρ_f is the density of liquid, q denotes the resultant velocity of the fluid, and P_u is the ullage gas pressure. The term $\frac{1}{2} \rho_f q^2$ is small compared with the other terms and so it will be neglected here. It is assumed in the present study that the ullage gas dynamics is neglected. Hence, the liquid surface is subjected to uniform pressure, P_u .

Substituting Equation (10) for the velocity potential, Φ , in Equation (11), and neglecting the term $\frac{1}{2} \rho_f q^2$ gives

$$P_\xi = C^2 \xi_o^2 \rho_f \Omega \sin \omega t \left[B_o + \sum_{n=1}^{\infty} B_n P_n(i\xi) P_n(\eta) \right] + P_u \quad (12)$$

where $\Omega = \omega^2$.



The ullage pressure variations can be related to variations in the ullage volume through the linearized form of the adiabatic, isentropic, perfect gas relation, which is derived from

$$P_{us} V_{us}^{\gamma} = (P_{us} + P_u) (V_{us} + V_u)^{\gamma} \quad (13)$$

where P_{us} is the static ullage pressure, V_{us} is the static ullage volume, V_u is the fluctuating ullage volume and γ is the isentropic exponent. Thus, the binomial expansion of the right-hand side of Equation (13), after linearization, gives

$$P_u = - \frac{\gamma P_{us}}{V_{us}} V_u \quad (14)$$

The displacement of the liquid, v_{l0} at $\eta = 0$, as shown in Figure 2, is obtained from the expression:

$$\frac{\partial v_{l0}(\xi, 0, t)}{\partial t} = \frac{1}{C\xi} \frac{\partial \Phi}{\partial \eta} \Big|_{\eta=0} \quad (15)$$

The fluctuating ullage volume, V_u can be related to the shell and liquid motions by the equation

$$dV_u = 2\pi C\xi v_{l0} d(C\xi) + dV_{lg} \quad (16)$$

where the first term on the right-hand side gives the effect of the liquid surface, and the second term is due to the shell motions. This term, V_{lg} , is determined by the expression

$$dV_{lg} = \sin \omega t \sum_{m=0}^{\infty} E_m dV_{gm} \quad (17)$$

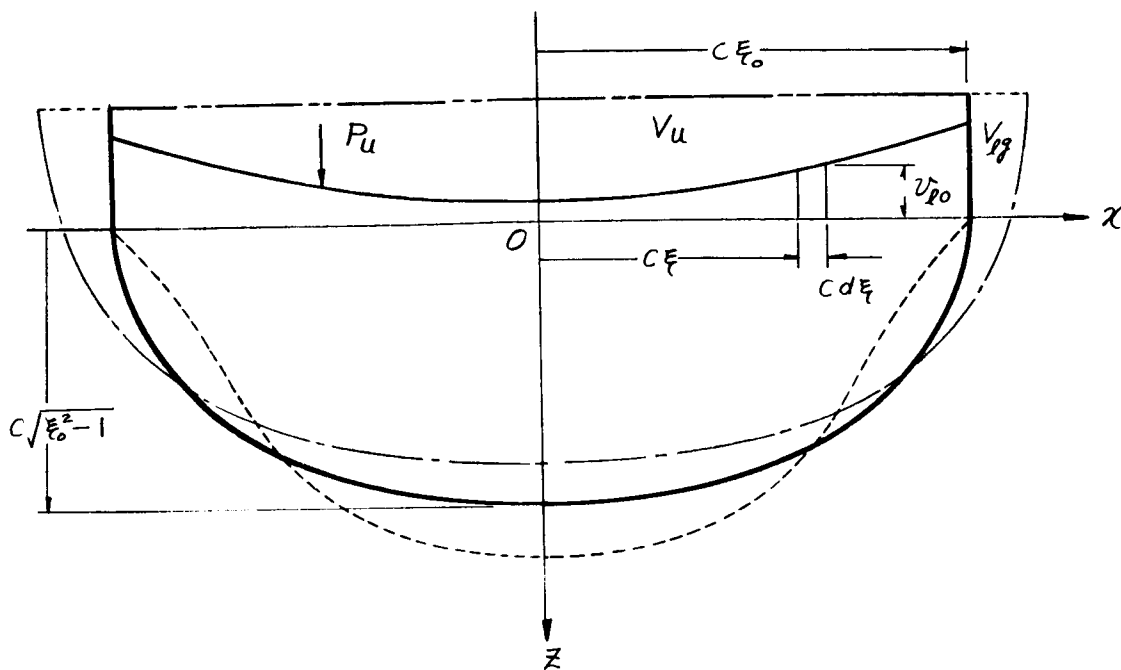


Figure 2. Ullage Space and Liquid Displacement

where E_0, E_1, E_2, \dots are nondimensional arbitrary constants and should be determined as part of the solution. V_{lg} is the change in the ullage volume due to a unit deflection of the m^{th} assumed normal deflection, w_u for the ullage space as defined by

$$w_u = \sin \omega t \sum_{m=0}^{\infty} E_m \tag{18}$$

Integrating Equation (16) from $\xi = 0$ to $\xi = \xi_0$ gives

$$V_u = 2\pi C^2 \int_0^{\xi_0} \xi v_{l_0} d\xi + V_{lg} \tag{19}$$

Integrating Equation (17) gives

$$V_{lg} = \sin \omega t \sum_{m=0}^{\infty} E_m V_{gm} \tag{20}$$



Substituting Equation (1) for Φ in Equation (15), and integrating the result from $t = 0$ to $t = t$ gives

$$v_{l_0} = \int_0^t \frac{1}{C\xi} \left. \frac{\partial \Phi}{\partial \eta} \right|_{\eta=0} dt = \frac{1}{C\xi} \left. \frac{\partial \Phi}{\partial \eta} \right|_{\eta=0} \frac{\sin \omega t}{\omega} \quad (21)$$

Differentiating Equation (9) with respect to η gives

$$\left. \frac{\partial \Phi}{\partial \eta} \right|_{\eta=0} = C^2 \xi_0^2 \omega \sum_{n=1}^{\infty} n B_n P_{n-1}(0) P_n(i\xi) \quad (22)$$

Thus, the displacement of the liquid, v_{l_0} , at $\eta = 0$ is obtained as

$$v_{l_0} = C \xi_0^2 \sin \omega t \sum_{n=1}^{\infty} n B_n P_{n-1}(0) \frac{P_n(i\xi)}{\xi} \quad (23)$$

The fluctuating ullage volume, V_u , is then determined by substituting Equations (20) and (23) into Equation (19) as

$$V_u = 2\pi C^3 \xi_0^2 \sin \omega t \int_0^{\xi_0} \sum_{n=1}^{\infty} n B_n P_{n-1}(0) P_n(i\xi) d\xi \\ + \sin \omega t \sum_{m=0}^{\infty} E_m V_{gm} \quad (24)$$



where the integration of $P_n(i\xi)$ is

$$\int_0^{\xi_0} P_n(i\xi) d\xi = \frac{1}{n+1} \left[\xi_0 P_n(i\xi_0) + i P_{n-1}(i\xi_0) - i P_{n-1}(0) \right]$$

Substituting Equation (24) into Equation (14) gives

$$P_u = - \frac{2\pi C^3 \gamma P_{us} \xi_0^2 \sin \omega t}{V_{us}} \sum_{n=1}^{\infty} \frac{n B_n P_{n-1}(0)}{n+1} \left[\xi_0 P_n(i\xi_0) + i P_{n-1}(i\xi_0) - i P_{n-1}(0) \right]$$

$$- \frac{\gamma P_{us} \sin \omega t}{V_{us}} \sum_{m=0}^{\infty} E_m V_{gm} \quad (25)$$

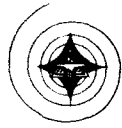
Substituting Equations (10) and (25) into Equation (11) gives the fluid pressure acting at the oblate spheroidal shell as

$$P_{\xi} \Big|_{\xi=\xi_0} = C^2 \xi_0^2 \rho_f \Omega \sin \omega t \left[B_0 + \sum_{n=1}^{\infty} B_n P_n(i\xi_0) P_n(\eta) \right]$$

$$- \frac{2\pi C^3 \gamma P_{us} \xi_0^2 \sin \omega t}{V_{us}} \sum_{n=1}^{\infty} \frac{n}{n+1} B_n P_{n-1}(0) \cdot$$

$$\left[\xi_0 P_n(i\xi_0) + i P_{n-1}(i\xi_0) - i P_{n-1}(0) \right]$$

$$- \frac{\gamma P_{us}}{V_{us}} \sin \omega t \sum_{m=0}^{\infty} E_m V_{gm} \quad (26)$$



3. BOUNDARY CONDITIONS

According to Lamb's hydrodynamics, if the Cartesian coordinate system is used, the normal liquid velocity at the free surface is expressed as

$$\frac{\omega^2}{g} \Phi = \frac{\partial \Phi}{\partial z} \quad \text{at } z = 0 \quad (27)$$

where g is acceleration of gravity.

For the general case, the gradient of the velocity potential, $\nabla \Phi$ is expressed as (Reference 9)

$$\nabla \Phi = \sum_{n=1}^3 \frac{1}{h_n} \frac{\partial \Phi}{\partial \xi_n} \vec{a}_n \quad (28)$$

where ξ_1 , ξ_2 , and ξ_3 are orthogonal curvilinear coordinates, \vec{a}_1 , \vec{a}_2 , and \vec{a}_3 are unit vectors, and h_1 , h_2 , and h_3 are scale factors and expressed as

$$h_n^2 = \left(\frac{\partial x}{\partial \xi_n} \right)^2 + \left(\frac{\partial y}{\partial \xi_n} \right)^2 + \left(\frac{\partial z}{\partial \xi_n} \right)^2 \quad (29)$$

$$n = 1, 2, \text{ and } 3$$

The Cartesian coordinates are related to the oblate spheroidal coordinates by

$$\left. \begin{aligned} x &= C \sqrt{(1 + \xi^2)(1 - \eta^2)} \cos \phi \\ y &= C \sqrt{(1 + \xi^2)(1 - \eta^2)} \sin \phi \\ z &= C \xi \eta \end{aligned} \right\} \quad (30)$$



For the case of oblate spheroidal coordinates, ξ_n , \vec{a}_n , and h_n are written as (Reference 10):

$$\left. \begin{aligned} \xi_1 &= C \xi, \quad \xi_2 = \eta, \quad \xi_3 = \cos \phi \\ \vec{a}_1 &= \vec{a}_\xi, \quad \vec{a}_2 = \vec{a}_\eta, \quad \vec{a}_3 = \vec{a}_\phi \\ h_1 &= \sqrt{\frac{\xi^2 + \eta^2}{\xi^2 + 1}}, \quad h_2 = C \sqrt{\frac{\xi^2 + \eta^2}{1 - \eta^2}}, \quad h_3 = C \frac{\sqrt{(\xi^2 + 1)(1 - \eta^2)}}{\sin \phi} \end{aligned} \right\} \quad (31)$$

Thus, the scalar component of $\nabla\Phi$ in the η -direction is obtained from Equations (28) and (31),

$$\frac{1}{h_2} \frac{\partial \Phi}{\partial \xi_2} = \frac{1}{C} \sqrt{\frac{1 - \eta^2}{\xi^2 + \eta^2}} \frac{\partial \Phi}{\partial \eta} \quad (32)$$

Hence, the boundary condition at the free surface of the liquid contained in the oblate spheroidal tank is obtained from Equations (27) and (32) as

$$\frac{\omega^2}{g} \Phi = \frac{1}{C \xi} \frac{\partial \Phi}{\partial \eta} \quad \text{at } \eta = 0 \quad (33)$$

At the interface, the normal velocity of the liquid and the shell must be the same. Such condition is expressed in terms of spherical coordinates as

$$\frac{\partial \Phi}{\partial r} = - \frac{\partial w}{\partial t} \quad \text{at } r = a \quad (34)$$

where a is the radius of the spherical shell and the normal displacement of the shell, w , will be determined in the shell analysis. The scalar component



of $\nabla\Phi$ in the ξ -direction is obtained from Equations (28) and (31),

$$\frac{1}{h_1} \frac{\partial\Phi}{\partial\xi_1} = \frac{1}{C} \sqrt{\frac{\xi^2 + 1}{\xi^2 + \eta^2}} \frac{\partial\Phi}{\partial\xi} \quad (35)$$

Hence, the boundary condition at the interface between the liquid and the oblate spheroidal tank is obtained from Equations (34) and (35) as

$$\frac{1}{C} \sqrt{\frac{\xi_0^2 + 1}{\xi_0^2 + \eta^2}} \frac{\partial\Phi}{\partial\xi} = -\frac{\partial w}{\partial t} \quad \text{at } \xi = \xi_0 \quad (36)$$

4. EQUATIONS OF MOTION FOR MEMBRANE SHELL

The equations of motion in terms of oblate spheroidal coordinates can be obtained from the general expression introduced in Reference 11 as follows:

$$\frac{\partial a_\phi N_\eta}{\partial\xi_\eta} + \frac{\partial a_\eta N_{\phi\eta}}{\partial\xi_\phi} + N_{\eta\phi} \frac{\partial a_\eta}{\partial\xi_\phi} - N_\phi \frac{\partial a_\phi}{\partial\xi_\eta} - Q_\eta \frac{a_\eta a_\phi}{R_\eta} + a_\eta a_\phi (P_\eta + f_\eta) = 0 \quad (37)$$

$$\frac{\partial a_\eta N_\phi}{\partial\xi_\phi} + \frac{\partial a_\phi N_{\eta\phi}}{\partial\xi_\eta} + N_{\phi\eta} \frac{\partial a_\phi}{\partial\xi_\eta} - N_\eta \frac{\partial a_\eta}{\partial\xi_\phi} - Q_\phi \frac{a_\eta a_\phi}{R_\phi} + a_\eta a_\phi (P_\phi + f_\phi) = 0 \quad (38)$$

$$\frac{\partial a_\phi Q_\eta}{\partial\xi_\eta} + \frac{\partial a_\eta Q_\phi}{\partial\xi_\phi} + N_\eta \frac{a_\eta a_\phi}{R_\eta} + N_\phi \frac{a_\eta a_\phi}{R_\phi} + a_\eta a_\phi (P_\xi + f_\xi) = 0 \quad (39)$$

$$\frac{\partial a_\phi M_{\eta\phi}}{\partial\xi_\eta} - \frac{\partial a_\eta M_\phi}{\partial\xi_\phi} + M_\eta \frac{\partial a_\eta}{\partial\xi_\phi} + M_{\phi\eta} \frac{\partial a_\phi}{\partial\xi_\eta} + Q_\phi a_\eta a_\phi = 0 \quad (40)$$



$$\frac{\partial a_\eta}{\partial \xi_\phi} M_{\phi\eta} - \frac{\partial a_\phi}{\partial \xi_\eta} M_{\eta\phi} + M_\phi \frac{\partial a_\phi}{\partial \xi_\eta} + M_{\eta\phi} \frac{\partial a_\eta}{\partial \xi_\phi} + Q_\eta a_\eta a_\phi = 0 \quad (41)$$

where N_η , N_ϕ , $N_{\eta\phi}$, and $N_{\phi\eta}$ are stress resultants; M_η , M_ϕ , $M_{\eta\phi}$, and $M_{\phi\eta}$ are stress couple resultants; Q_η and Q_ϕ are transverse shearing stress resultants; P_ξ , P_η , and P_ϕ are lateral loads; f_ξ , f_η , and f_ϕ are inertia forces; R_η and R_ϕ are principal radii of curvature; and it is found from curvilinear coordinates that

$$\left. \begin{aligned} a_\eta &= C \sqrt{\frac{\xi_o^2 + \eta^2}{1 - \eta^2}}, \quad a_\phi = C \frac{\sqrt{(\xi_o^2 + 1)(1 - \eta^2)}}{\sin \phi} \\ \xi_\eta &= \eta, \quad \xi_\phi = \cos \phi \\ R_\eta &= C \frac{\sqrt{(\xi_o^2 + 1)(\xi_o^2 + \eta^2)}}{\xi_o}, \quad R_\phi = C \frac{(\xi_o^2 + \eta^2)^{3/2}}{\xi_o \sqrt{\xi_o^2 + 1}} \end{aligned} \right\} \quad (42)$$

If the components u , v and w of displacement are defined as shown in Figure 1, the expressions to be substituted for f_ξ , f_η , and f_ϕ , respectively are

$$f_\xi = \rho h \frac{\partial^2 w}{\partial t^2}, \quad f_\eta = -\rho h \frac{\partial^2 v}{\partial t^2} \quad \text{and} \quad f_\phi = -\rho h \frac{\partial^2 u}{\partial t^2} \quad (43)$$

where ρ is the mass density of the shell and h is the thickness of the shell.

In the present study, a thin oblate spheroidal membrane shell is assumed. If the bending stresses in the shell are neglected, it follows that

$$Q_\eta = Q_\phi = 0 \quad (44)$$



It is also assumed that the oblate spheroidal tank is subjected to an axisymmetric motion. Thus, if the tank oscillates symmetrically with respect to the z-axis, then

$$\left. \begin{aligned} P_\phi = P_\eta = 0, \frac{\partial}{\partial \xi_\phi} = 0 \\ N_{\eta\phi} = N_{\phi\eta} = 0 \end{aligned} \right\} \quad (45)$$

Under the conditions of Equations (44) and (45), the equations of motion, Equations (37) through (39), are simplified to

$$a_\phi \frac{\partial N_\eta}{\partial \xi_\eta} + (N_\eta - N_\phi) \frac{\partial a_\phi}{\partial \xi_\eta} = a_\eta a_\phi \rho h \frac{\partial^2 v}{\partial t^2} \quad (46)$$

$$0 = a_\eta a_\phi \rho h \frac{\partial^2 u}{\partial t^2} \quad (47)$$

$$\frac{N_\eta}{R_\eta} + \frac{N_\phi}{R_\phi} + P_\xi = \rho h \frac{\partial^2 w}{\partial t^2} \quad (48)$$

The substitution of Equations (42) into Equations (46), (47), and (48), respectively, yields

$$\frac{\partial N_\eta}{\partial \eta} - (N_\eta - N_\phi) \frac{\eta}{1 - \eta^2} = -C \sqrt{\frac{\xi_0^2 + \eta^2}{1 - \eta^2}} \rho h \frac{\partial^2 v}{\partial t^2} \quad (49)$$

$$0 = C^2 \frac{\sqrt{(\xi_0^2 + \eta^2)(\xi_0^2 + 1)}}{\sin \phi} \rho h \frac{\partial^2 u}{\partial t^2} \quad (50)$$



$$\frac{\xi_0}{C\sqrt{(\xi_0^2 + 1)(\xi_0^2 + \eta^2)}} N_\eta + \frac{\xi_0\sqrt{\xi_0^2 + 1}}{C(\xi_0^2 + \eta^2)^{3/2}} N_\phi + P_\xi = -\rho h \frac{\partial^2 w}{\partial t^2} \quad (51)$$

Equation (50) indicates that either the velocity in ϕ -direction is constant or the corresponding displacement is zero. Hence, the equations of motion, Equations (49) and (51), are used simultaneously to solve for the displacements v and w . Under the usual assumption for thin shells, that the stress components normal to the middle surface are small compared with the other stress components and may be neglected in the stress-strain relations, the stress resultants, N_η and N_ϕ , are obtained as

$$\left. \begin{aligned} N_\eta &= D (e_{\eta\eta} + \nu e_{\phi\phi}) \\ N_\phi &= D (e_{\phi\phi} + \nu e_{\eta\eta}) \end{aligned} \right\} \quad (52)$$

Where D is extensional modulus and equal to $Eh/(1 - \nu^2)$, and $e_{\eta\eta}$ and $e_{\phi\phi}$ are the strains in the middle surface of the shell.

The strains in the middle surface of the shell, in terms of oblate spheroidal coordinates, can be obtained from the general expression introduced in Reference 11 as follows:

$$\left. \begin{aligned} e_{\eta\eta} &= \frac{1}{a_\eta} \frac{v}{\partial \xi_\eta} + \frac{u}{a_\eta a_\phi} \frac{\partial a_\eta}{\partial \xi_\phi} - \frac{w}{R_\eta} \\ e_{\phi\phi} &= \frac{1}{a_\phi} \frac{\partial u}{\partial \xi_\phi} + \frac{v}{a_\eta a_\phi} \frac{\partial a_\phi}{\partial \xi_\eta} - \frac{w}{R_\phi} \end{aligned} \right\} \quad (53)$$



The substitution of Equations (42) into Equations (53) yields

$$\left. \begin{aligned}
 e_{\eta\eta} &= -\frac{1}{C} \sqrt{\frac{1-\eta^2}{\xi_o^2 + \eta^2}} \cdot \frac{\partial v}{\partial \eta} + \frac{u}{C^2 \sqrt{(\xi_o^2 + \eta^2)(\xi_o^2 + 1)}} \frac{\partial}{\partial \phi} \left(C \sqrt{\frac{\xi_o^2 + \eta^2}{1-\eta^2}} \right) \\
 &\quad - \frac{\xi_o w}{C \sqrt{(\xi_o^2 + 1)(\xi_o^2 + \eta^2)}} \\
 e_{\phi\phi} &= \frac{1}{C \sqrt{(\xi_o^2 + 1)(1-\eta^2)}} \frac{\partial u}{\partial \phi} - \frac{v}{C^2 \sqrt{(\xi_o^2 + \eta^2)(\xi_o^2 + 1)}} \times \\
 &\quad \frac{\partial}{\partial \eta} \left[C \sqrt{(\xi_o^2 + 1)(1-\eta^2)} \right] - \frac{\xi_o \sqrt{\xi_o^2 + 1}}{C (\xi_o^2 + \eta^2)^{3/2}} w
 \end{aligned} \right\} (54)$$



For an axisymmetric motion, Equations (54) are simplified to

$$\left. \begin{aligned} e_{\eta\eta} &= \frac{1}{C} \sqrt{\frac{1-\eta^2}{\xi_o^2 + \eta^2}} \frac{\partial v}{\partial \eta} - \frac{\xi_o w}{C \sqrt{(\xi_o^2 + 1)(\xi_o^2 + \eta^2)}} \\ e_{\phi\phi} &= \frac{\eta v}{C \sqrt{(\xi_o^2 + \eta^2)(1-\eta^2)}} - \frac{\xi_o \sqrt{\xi_o^2 + 1} w}{C (\xi_o^2 + \eta^2)^{3/2}} \end{aligned} \right\} \quad (55)$$

The substitution of Equations (55) into Equations (52) yields the stress resultants in terms of the displacements:

$$\begin{aligned} N_{\eta} = -D \left[\frac{1}{C} \sqrt{\frac{1-\eta^2}{\xi_o^2 + \eta^2}} \frac{\partial v}{\partial \eta} + \frac{\xi_o w}{C \sqrt{(\xi_o^2 + 1)(\xi_o^2 + \eta^2)}} - \frac{\nu \eta v}{C \sqrt{(\xi_o^2 + \eta^2)(1-\eta^2)}} \right. \\ \left. + \frac{\nu \xi_o \sqrt{\xi_o^2 + 1} w}{C (\xi_o^2 + \eta^2)^{3/2}} \right] \end{aligned} \quad (56)$$

$$\begin{aligned} N_{\phi} = D \left[\frac{\eta v}{C \sqrt{(\xi_o^2 + \eta^2)(1-\eta^2)}} - \frac{\xi_o \sqrt{\xi_o^2 + 1} w}{C (\xi_o^2 + \eta^2)^{3/2}} - \frac{\nu \sqrt{1-\eta^2}}{C \sqrt{(\xi_o^2 + \eta^2)}} \frac{\partial v}{\partial \eta} \right. \\ \left. - \frac{\nu \xi_o w}{C \sqrt{(\xi_o^2 + 1)(\xi_o^2 + \eta^2)}} \right] \end{aligned}$$

Substituting Equations (56) for N_{η} and N_{ϕ} in Equations (49) and (51), and performing the differentiation, the equations of motion in terms of the displacements are obtained as



$$\begin{aligned}
 & \frac{1}{C} \sqrt{\frac{1-\eta^2}{\xi_o^2 + \eta^2}} \frac{\partial^2 v}{\partial \eta^2} - \left[\frac{\eta}{C \sqrt{(\xi_o^2 + \eta^2)(1-\eta^2)}} + \frac{\eta \sqrt{1-\eta^2}}{C (\xi_o^2 + \eta^2)^{3/2}} + \frac{\eta}{C \sqrt{(\xi_o^2 + \eta^2)(1-\eta^2)}} \right] \frac{\partial v}{\partial \eta} \\
 & - \left[\frac{\nu}{C \sqrt{(\xi_o^2 + \eta^2)(1-\eta^2)}} - \frac{\nu \eta^2}{C (\xi_o^2 + \eta^2)^{3/2} \sqrt{1-\eta^2}} + \frac{\eta^2}{C \sqrt{\xi_o^2 + \eta^2} (1-\eta^2)^{3/2}} \right] v \\
 & + \left[\frac{\xi_o}{C \sqrt{(\xi_o^2 + 1)(\xi_o^2 + \eta^2)}} + \frac{\nu \xi_o \sqrt{\xi_o^2 + 1}}{C (\xi_o^2 + \eta^2)^{3/2}} \right] \frac{\partial w}{\partial \eta} - \left[\frac{\xi_o \eta}{C \sqrt{\xi_o^2 + 1} (\xi_o^2 + \eta^2)^{3/2}} + \frac{3 \nu \xi_o \eta \sqrt{\xi_o^2 + 1}}{C (\xi_o^2 + \eta^2)^{5/2}} \right. \\
 & \left. + \frac{(1-\nu) \xi_o \eta}{C (1-\eta^2) \sqrt{(\xi_o^2 + 1)(\xi_o^2 + \eta^2)}} - \frac{(1-\nu) \eta \xi_o \sqrt{\xi_o^2 + 1}}{C (1-\eta^2) (\xi_o^2 + \eta^2)^{3/2}} \right] w = \frac{c \rho h}{D} \sqrt{\frac{\xi_o^2 + \eta^2}{1-\eta^2}} \frac{\partial^2 v}{\partial t^2} \quad (57)
 \end{aligned}$$

$$\begin{aligned}
 & \left[\frac{\xi_o \sqrt{1-\eta^2}}{C^2 \sqrt{\xi_o^2 + 1} (\xi_o^2 + \eta^2)} + \frac{\nu \xi_o \sqrt{(\xi_o^2 + 1)(1-\eta^2)}}{C^2 (\xi_o^2 + \eta^2)^2} \right] \frac{\partial v}{\partial \eta} - \left[\frac{\nu \xi_o \eta}{C^2 \sqrt{(\xi_o^2 + 1)(1-\eta^2)} (\xi_o^2 + \eta^2)} \right. \\
 & \left. + \frac{\xi_o \eta \sqrt{\xi_o^2 + 1}}{C^2 \sqrt{1-\eta^2} (\xi_o^2 + \eta^2)^2} \right] v + \left[\frac{\xi_o^2}{C^2 (\xi_o^2 + 1) (\xi_o^2 + \eta^2)} + \frac{\xi_o^2 (\xi_o^2 + 1)}{C^2 (\xi_o^2 + \eta^2)^3} + \frac{2 \nu \xi_o^2}{C^2 (\xi_o^2 + \eta^2)^2} \right] w \\
 & - \frac{P_\xi}{D} = - \frac{\rho h}{D} \frac{\partial^2 w}{\partial t^2} \quad (58)
 \end{aligned}$$



To simplify the mathematical manipulation and nondimensionalize the notations, the following substitutions are made in Equations (57) and (58):

$$\left. \begin{aligned}
 v &= C \xi_o \bar{V} \sin \omega t \\
 w &= C \xi_o \bar{W} \sin \omega t \\
 P_{\xi} &= \frac{Eh}{(1-\nu^2) C \xi_o} \bar{P}_{\xi} \\
 \frac{(1-\nu^2) \rho C^2 \xi_o^2}{E} &= K \\
 \omega^2 &= \Omega
 \end{aligned} \right\} \quad (59)$$

Furthermore, the following functional coefficients are introduced:

$$h_{\eta 1}(\eta) = -\left(\frac{2\eta}{1-\eta^2} - \frac{\eta}{\xi_o^2 + \eta^2} \right)$$

$$h_{\eta 2}(\eta) = -\frac{\nu}{1-\eta^2} + \frac{\nu \eta^2}{(\xi_o^2 + \eta^2)(1-\eta^2)} - \frac{\eta^2}{(1-\eta^2)^2} + \frac{\xi_o^2 + \eta^2}{\xi_o^2 (1-\eta^2)} K \Omega$$

$$h_{\eta 3}(\eta) = \frac{\xi_o}{\sqrt{(\xi_o^2 + 1)(1-\eta^2)}} + \frac{\nu \xi_o \sqrt{\xi_o^2 + 1}}{(\xi_o^2 + \eta^2) \sqrt{1-\eta^2}}$$

$$h_{\eta 4}(\eta) = -\frac{\xi_o \eta}{\sqrt{(\xi_o^2 + 1)(1-\eta^2)}(\xi_o^2 + \eta^2)} - \frac{3\nu \xi_o \eta \sqrt{\xi_o^2 + 1}}{\sqrt{1-\eta^2}(\xi_o^2 + \eta^2)^2} - \frac{(1-\nu)\xi_o \eta}{(1-\eta^2)^{3/2} \sqrt{\xi_o^2 + 1}}$$



$$\begin{aligned}
 & + \frac{(1-\nu)\eta \xi_o \sqrt{\xi_o^2 + 1}}{(1-\eta^2)^{3/2} (\xi_o^2 + \eta^2)} \\
 h_{\eta 5}(\eta) &= \frac{\xi_o^3 \sqrt{1-\eta^2} \left[\xi_o^2 + \eta^2 + \nu (\xi_o^2 + 1) \right]}{\sqrt{\xi_o^2 + 1} (\xi_o^2 + \eta^2)^2} \\
 h_{\xi 1}(\eta) &= - \frac{(1+\nu)\eta \xi_o^2 + \eta(1+\nu\eta^2)}{(1-\eta^2) \left[\xi_o^2 + \eta^2 + \nu (\xi_o^2 + 1) \right]} \\
 h_{\xi 2}(\eta) &= \frac{\xi_o K \Omega}{\left[\xi_o^2 + \eta^2 + \nu (\xi_o^2 + 1) \right]} \left[\frac{\xi_o^2 + \eta^2}{\sqrt{(\xi_o^2 + 1)(1-\eta^2)}} + \frac{(1+\xi_o^2)^{3/2}}{\sqrt{1-\eta^2} (\xi_o^2 + \eta^2)} + \frac{2\nu \sqrt{\xi_o^2 + 1}}{\sqrt{1-\eta^2}} \right. \\
 & \quad \left. - \frac{\sqrt{\xi_o^2 + 1} (\xi_o^2 + \eta^2)^2}{\xi_o^4 \sqrt{1-\eta^2}} \right] \\
 h_{\xi 3}(\eta) &= - \frac{\sqrt{\xi_o^2 + 1} (\xi_o^2 + \eta^2)^2}{\xi_o^3 \sqrt{1-\eta^2} \left[\xi_o^2 + \eta^2 + \nu (\xi_o^2 + 1) \right]}
 \end{aligned} \tag{60}$$

The equations of motion, Equations (57) and (58), are then written in the forms

$$\frac{d^2 \bar{V}}{d\eta^2} + h_{\eta 1}(\eta) \frac{d\bar{V}}{d\eta} + h_{\eta 2}(\eta) \bar{V} + h_{\eta 3}(\eta) \frac{d\bar{W}}{d\eta} + h_{\eta 4}(\eta) \bar{W} = 0 \tag{61}$$

$$\frac{d\bar{V}}{d\eta} + h_{\xi 1}(\eta) \bar{V} + h_{\xi 2}(\eta) \bar{W} + h_{\xi 3}(\eta) \bar{P}_\xi = 0 \tag{62}$$



It has been verified that the equations of motion for spherical shells are obtained by substituting the following expressions into Equations (61) and (62):

$$\left. \begin{aligned} \xi_o &= \frac{a}{C} \\ C &= 0 \\ \eta &= \cos \theta \end{aligned} \right\} \quad (63)$$

5. SOLUTIONS OF EQUATIONS OF MOTION

Meridional Displacement \bar{V}

Equations (61) and (62) constitute two simultaneous linear differential equations for two unknowns, \bar{V} and \bar{W} , which will be solved numerically here as a function of \bar{P}_ξ for given boundary conditions. From Equation (62), \bar{W} is written in the form

$$\bar{W} = g_{\xi 1}(\eta) \frac{d\bar{V}}{d\eta} + g_{\xi 2}(\eta) \bar{V} + g_{\xi 3}(\eta) \bar{P}_\xi \quad (64)$$

where

$$\left. \begin{aligned} g_{\xi 1}(\eta) &= \frac{\xi_o^3 \sqrt{(1+\xi_o^2)(1-\eta^2)} (\xi_o^2 + \eta^2) [\xi_o^2 + \eta^2 + \nu(1+\xi_o^2)]}{(1+\xi_o^2)(\xi_o^2 + \eta^2)^3 - \xi_o^4(\xi_o^2 + \eta^2)^2 - \xi_o^4(1+\xi_o^2)^2 - 2\nu\xi_o^4(1+\xi_o^2)(\xi_o^2 + \eta^2)} \\ g_{\xi 2}(\eta) &= - \frac{\xi_o^3 \sqrt{1+\xi_o^2} (\xi_o^2 + \eta^2) [\xi_o^2 + \eta^2 + \nu(1+\xi_o^2)]}{\sqrt{1-\eta^2} [(1+\xi_o^2)(\xi_o^2 + \eta^2)^3 - \xi_o^4(\xi_o^2 + \eta^2)^2 - \xi_o^4(1+\xi_o^2)^2 - 2\nu\xi_o^4(1+\xi_o^2)(\xi_o^2 + \eta^2)]} \\ g_{\xi 3}(\eta) &= - \frac{(1+\xi_o^2)(\xi_o^2 + \eta^2)^3}{(1+\xi_o^2)(\xi_o^2 + \eta^2)^3 - \xi_o^4(\xi_o^2 + \eta^2)^2 - \xi_o^4(1+\xi_o^2)^2 - 2\nu\xi_o^4(1+\xi_o^2)(\xi_o^2 + \eta^2)} \end{aligned} \right\} \quad (65)$$



Substituting Equation (64) for \bar{W} in Equation (61) gives

$$\frac{d^2 \bar{V}}{d\eta^2} + k_{\eta 1}(\eta) \frac{d\bar{V}}{d\eta} + k_{\eta 2}(\eta) \bar{V} = R(\eta) \quad (66)$$

where

$$k_{\eta 1}(\eta) = \frac{H(\xi_o, \eta)}{H(\xi_o, \eta) + \xi_o^4 \left[\xi_o^2 + \eta^2 + \nu(1 + \xi_o^2) \right]^2} \times \left[- \left(\frac{2\eta}{1-\eta^2} + \frac{\eta}{\xi_o^2 + \eta^2} \right) + \right.$$

$$\frac{\xi_o^3 \eta \sqrt{1 + \xi_o^2}}{H(\xi_o, \eta) \sqrt{1 - \eta^2}} \left\{ (2 - \xi_o^2 - 3\eta^2) \left[\frac{\xi_o}{\sqrt{(1 + \xi_o^2)(1 - \eta^2)}} + \frac{\nu \xi_o \sqrt{1 + \xi_o^2}}{(\xi_o^2 + \eta^2) \sqrt{1 - \eta^2}} \right] \left[\xi_o^2 + \eta^2 + \nu(1 + \xi_o^2) \right] \right.$$

$$\left. + 2(1 - \eta^2)(\xi_o^2 + \eta^2) \right\} - \frac{2\xi_o^3 \eta \sqrt{(1 + \xi_o^2)(1 - \eta^2)}}{H(\xi_o, \eta)^2} (\xi_o^2 + \eta^2) \times$$

$$\left[\frac{\xi_o}{\sqrt{(1 + \xi_o^2)(1 - \eta^2)}} + \frac{\nu \xi_o \sqrt{1 + \xi_o^2}}{(\xi_o^2 + \eta^2) \sqrt{1 - \eta^2}} \right] \left[\xi_o^2 + \eta^2 + \nu(1 + \xi_o^2) \right] \times$$

$$\left[3(1 + \xi_o^2)(\xi_o^2 + \eta^2)^2 - 2\xi_o^4(\xi_o^2 + \eta^2) - 2\nu\xi_o^2(1 + \xi_o^2) \right] - \frac{\xi_o^3 \eta \sqrt{1 + \xi_o^2}}{H(\xi_o, \eta) \sqrt{1 - \eta^2}} (\xi_o^2 + \eta^2) \times$$



$$\begin{aligned}
 & \left[\frac{\xi_0}{\sqrt{(1+\xi_0^2)(1-\eta^2)}} + \frac{\nu \xi_0 \sqrt{1+\xi_0^2}}{(\xi_0^2 + \eta^2) \sqrt{1-\eta^2}} \right] \left[(1+\nu) \xi_0^2 + (1+\nu \eta^2) \right] - \\
 & - \left[\frac{\xi_0 \eta}{\sqrt{(1+\xi_0^2)(1-\eta^2)} (\xi_0^2 + \eta^2)} + \frac{3\nu \xi_0 \eta \sqrt{1+\xi_0^2}}{\sqrt{1-\eta^2} (\xi_0^2 + \eta^2)^2} + \frac{(1-\nu) \xi_0 \eta}{(1-\eta^2)^{3/2} \sqrt{1+\xi_0^2}} \right. \\
 & \left. - \frac{(1-\nu) \eta \xi_0 \sqrt{1+\xi_0^2}}{(1-\eta^2)^{3/2} (\xi_0^2 + \eta^2)} \right] \times \\
 & \frac{\xi_0^3 \sqrt{(1+\xi_0^2)(1-\eta^2)}}{H(\xi_0, \eta)} \times (\xi_0^2 + \eta^2) \left[\xi_0^2 + \eta^2 + \nu (1+\xi_0^2) \right] \quad (67)
 \end{aligned}$$

$$\begin{aligned}
 k_{\eta^2}(\eta) = & \frac{H(\xi_0, \eta)}{H(\xi_0, \eta) + \xi_0^4 \left[\xi_0^2 + \eta^2 + \nu (1+\xi_0^2) \right]^2} \times \left[-\frac{\nu}{1-\eta^2} + \frac{\nu \eta^2}{(\xi_0^2 + \eta^2)(1-\eta^2)} - \frac{\eta^2}{(1-\eta^2)^2} \right] \\
 & + \frac{\xi_0^2 + \eta^2}{\xi_0^2 (1-\eta^2)} K \Omega - \frac{\xi_0^3 \sqrt{1+\xi_0^2}}{H(\xi_0, \eta) (1-\eta^2)^{3/2}} \left\{ (2 + \xi_0^2 - \eta^2) \eta^2 \left[(1+\nu) \xi_0^2 + (1+\nu \eta^2) \right] \right\} \times
 \end{aligned}$$



$$\left[\frac{\xi_0}{\sqrt{(1+\xi_0^2)(1-\eta^2)}} + \frac{\nu \xi_0 \sqrt{1+\xi_0^2}}{(\xi_0^2 + \eta^2) \sqrt{1-\eta^2}} \right] + (\xi_0^2 + \eta^2)(1-\eta^2) \left[(1+\nu)\xi_0^2 + 1 + 3\nu\eta^2 \right] \Bigg\} +$$

$$\frac{2\xi_0^3 \eta^2 \sqrt{1+\xi_0^2}}{H(\xi_0, \eta)^2 \sqrt{1-\eta^2}} (\xi_0^2 + \eta^2) \left[(1+\nu)\xi_0^2 + (1+\nu\eta^2) \right] \times$$

$$\left[\frac{\xi_0}{\sqrt{(1+\xi_0^2)(1-\eta^2)}} + \frac{\nu \xi_0 \sqrt{1+\xi_0^2}}{(\xi_0^2 + \eta^2) \sqrt{1-\eta^2}} \right] \left[3(1+\xi_0^2)(\xi_0^2 + \eta^2)^2 - 2\xi_0^4(\xi_0^2 + \eta^2) \right.$$

$$\left. - 2\nu\xi_0^4(1+\xi_0^2) \right] + \left[\frac{\xi_0 \eta}{\sqrt{(1+\xi_0^2)(1-\eta^2)}(\xi_0^2 + \eta^2)} + \frac{3\nu \xi_0 \eta \sqrt{1+\xi_0^2}}{(\xi_0^2 + \eta^2)^2 \sqrt{1-\eta^2}} \right.$$

$$\left. + \frac{(1-\nu)\xi_0 \eta}{(1-\eta^2)^{3/2} \sqrt{1+\xi_0^2}} - \frac{(1-\nu)\eta \xi_0 \sqrt{1+\xi_0^2}}{(1-\eta^2)^{3/2} \sqrt{(\xi_0^2 + \eta^2)}} \right] \times$$

$$\frac{\xi_0^3 \eta \sqrt{1+\xi_0^2}}{H(\xi_0, \eta) \sqrt{1-\eta^2}} (\xi_0^2 + \eta^2) \left[(1+\nu)\xi_0^2 + (1+\nu\eta^2) \right] \Bigg] \quad (68)$$



$$R(\eta) = \frac{-H(\xi_o, \eta)}{H(\xi_o, \eta) + \xi_o^4 [\xi_o^2 + \eta^2 + \nu(1 + \xi_o^2)]^2} \times \left[\frac{(1 + \xi_o^2)(\xi_o^2 + \eta^2)^3}{H(\xi_o, \eta)} C^2 \xi_o^2 \rho_f \Omega \sin \omega t \times \right.$$

$$\left. \left[\frac{\xi_o}{\sqrt{(1 + \xi_o^2)(1 - \eta^2)}} + \frac{\nu \xi_o \sqrt{1 + \xi_o^2}}{(\xi_o^2 + \eta^2) \sqrt{1 - \eta^2}} \right] \times \sum_{n=1}^{\infty} B_n \frac{1}{1 - \eta^2} \times \left[n P_n(i \xi_o) P_{n-1}(\eta) - \right.$$

$$\left. n \eta P_n(i \xi_o) P_n(\eta) \right] + \left\{ \frac{6 \eta (1 + \xi_o^2)(\xi_o^2 + \eta^2)^2}{H(\xi_o, \eta)} \left[\frac{\xi_o}{\sqrt{(1 + \xi_o^2)(1 - \eta^2)}} + \frac{\nu \xi_o \sqrt{1 + \xi_o^2}}{(\xi_o^2 + \eta^2) \sqrt{1 - \eta^2}} \right] - \right.$$

$$\left. \frac{2 \eta (1 + \xi_o^2)(\xi_o^2 + \eta^2)^3}{H(\xi_o, \eta)^2} \left[3(1 + \xi_o^2)(\xi_o^2 + \eta^2)^2 - 2 \xi_o^4 (\xi_o^2 + \eta^2) - 2 \nu \xi_o^4 (1 + \xi_o^2) \right] \times \right.$$

$$\left. \left[\frac{\xi_o}{\sqrt{1 + \xi_o^2} (1 - \eta^2)} + \frac{\nu \xi_o \sqrt{1 + \xi_o^2}}{(\xi_o^2 + \eta^2) \sqrt{1 - \eta^2}} \right] - \frac{(1 + \xi_o^2)(\xi_o^2 + \eta^2)^3}{H(\xi_o, \eta)} \left[\frac{\xi_o \eta}{\sqrt{(1 + \xi_o^2)(1 - \eta^2)} (\xi_o^2 + \eta^2)} + \right.$$

$$\left. \frac{3 \nu \xi_o \eta \sqrt{1 + \xi_o^2}}{\sqrt{1 + \eta^2} (\xi_o^2 + \eta^2)^2} + \frac{(1 - \nu) \xi_o \eta}{(1 - \eta^2)^{3/2} \sqrt{1 + \xi_o^2}} - \frac{(1 - \nu) \xi_o \eta \sqrt{1 + \xi_o^2}}{(1 - \eta^2)^{3/2} (\xi_o^2 + \eta^2)} \right] \times \left\{ C^2 \xi_o^2 \rho_f \Omega \sin \omega t \times \right.$$

$$\left. \left[B_o + \sum_{n=1}^{\infty} B_n P_n(i \xi_o) P_n(\eta) \right] + \frac{2 \pi \gamma C^3 \xi_o^2 P_{us} \sin \omega t}{V_{us}} \times \right.$$



$$\sum_{n=1}^{\infty} \frac{n}{n+1} B_n P_{n-1}(0) P_n(i\xi_0) \left[\xi_0 P_n(i\xi_0) + i P_{n-1}(i\xi_0) - i P_{n-1}(0) \right] -$$

$$\left. \frac{\gamma P_{us}}{V_{us}} \sin \omega t \sum_{m=0}^{\infty} E_m V_{gm} \right\} \quad (69)$$

$$H(\xi_0, \eta) = (1 + \xi_0^2)(\xi_0^2 + \eta^2)^3 - \xi_0^4(\xi_0^2 + \eta^2)^2 - \xi_0^4(1 + \xi_0^2)^2 - 2\nu\xi_0^4(1 + \xi_0^2)(\xi_0^2 + \eta^2) \quad (70)$$

One of the most obvious ways to solve a linear differential equation numerically is to replace the derivatives in the equation by their formal expressions in terms of differences and solve the resulting difference equation. The present method of solution follows the one introduced in Reference 12.

To solve Equation (66), two boundary conditions are required. Thus, the boundary conditions for the displacement, \bar{V} , are

$$\left. \begin{aligned} \bar{V} &= 0 & \text{at } \eta &= 0 \\ \bar{V} &= 0 & \text{at } \eta &= 1 \end{aligned} \right\} \quad (71)$$

The general solution of Equation (66) has the form

$$\bar{V} = C_1 \bar{u} + C_2 \bar{v} + \bar{w} \quad (72)$$

where C_1 and C_2 are arbitrary constants, \bar{u} and \bar{v} are independent solutions of the equation obtained by setting $R(\eta) = 0$, and \bar{w} is a particular solution of the original equation. One condition is available at the starting point, sufficient to reduce the number of arbitrary constants by one, so that the solution is now effectively of the form

$$\bar{V} = C_1 \bar{u} + \bar{w} \quad (73)$$



Let \bar{w} be taken as a solution of Equation (66) so that

$$\bar{w}(0) = 0 \quad (74)$$

and \bar{u} as a solution of the homogeneous equation

$$\frac{d^2 \bar{V}}{d\eta^2} + k_{\eta 1}(\eta) \frac{d\bar{V}}{d\eta} + k_{\eta 2}(\eta) \bar{V} = 0 \quad (75)$$

such that

$$\bar{u}(0) = 0 \quad (76)$$

The other boundary condition that $\bar{V} = 0$ when $\eta = 1$ provides the equation

$$0 = C_1 \bar{u}(1) + \bar{w}(1) \quad (77)$$

for the determination of C_1 ; $\bar{u}(1)$ and $\bar{w}(1)$ are obtained in the solution $\bar{u}(\eta)$ and $\bar{w}(\eta)$, respectively. With C_1 , \bar{u} and \bar{w} the solution \bar{V} satisfying the boundary conditions, Equations (71) will be found.

A Particular Solution (\bar{w})

By using a customary differential operator, D , Equation (66) may be written in terms of \bar{w} as

$$[D^2 + k_{\eta 1}(\eta)D + k_{\eta 2}(\eta)] \bar{w} = R(\eta) \quad (78)$$

In terms of central differences in Sheppard's notation (Reference 12), the differential operator D is given formally by

$$D = \frac{1}{h} \left(\mu\delta - \frac{\mu\delta^3}{3!} + \frac{2^2 \mu\delta^5}{5!} - \frac{2^2 \cdot 3^2 \mu\delta^7}{7!} + \frac{2^2 \cdot 3^2 \cdot 4^2 \mu\delta^9}{9!} - \dots \right) \quad (79)$$



where $h = \eta_{n+1} - \eta_n$, and the symbols $\mu\delta, \mu\delta^3, \dots$ are defined as

$$\left. \begin{aligned} \mu\delta \bar{w}_n &= \frac{1}{2} (\bar{w}_{n+1} - \bar{w}_{n-1}) \\ \mu\delta^3 \bar{w}_n &= \frac{1}{2} (\delta^2 \bar{w}_{n+1} - \delta^2 \bar{w}_{n-1}) \\ &\dots \\ \mu\delta^{2m+1} \bar{w}_n &= \frac{1}{2} (\delta^{2m} \bar{w}_{n+1} - \delta^{2m} \bar{w}_{n-1}) \end{aligned} \right\} n=0, 1, 2, \dots \quad (80)$$

From Equations (79) and (80) and the symbolic identity $\mu^2 = 1 + \delta^2/4$, the formulas for D^2, D^3, \dots are obtained. In particular,

$$D^2 = \frac{1}{h^2} \left(\delta^2 - \frac{2\delta^4}{4!} + \frac{2 \cdot 2^2 \delta^6}{6!} - \frac{2 \cdot 2^2 \cdot 3^2 \delta^8}{8!} + \frac{2 \cdot 2^2 \cdot 3^2 \cdot 4^2 \delta^{10}}{10!} \dots \right) \quad (81)$$

where the symbols $\delta^2, \delta^4, \dots$ are defined as

$$\left. \begin{aligned} \delta^2 \bar{w}_n &= \bar{w}_{n+1} - 2\bar{w}_n + \bar{w}_{n-1} \\ \delta^4 \bar{w}_n &= \bar{w}_{n+2} - 4\bar{w}_{n+1} + 6\bar{w}_n - 4\bar{w}_{n-1} + \bar{w}_{n-2} \\ &\dots \\ \delta^{2m} \bar{w}_n &= \sum_{k=0}^{2m} (-1)^k \binom{2m}{k} \bar{w}_{n+m-k} \end{aligned} \right\} \quad (82)$$

Substituting for D and D^2 in Equation (78) the values given by Equations (79) and (81), and collecting together in a group on the right all differences of order higher than the second, it follows that

$$L\bar{w} = h^2 R(\eta) + S\bar{w} \quad (83)$$



where

$$L\bar{w} = \delta^2 w + hk_{\eta_1}(\eta) \mu \delta \bar{w} + h^2 k_{\eta_2}(\eta) \bar{w} \quad (84)$$

$$S\bar{w} = \left[hk_{\eta_1}(\eta) \left(\frac{\mu \delta^3}{6} - \frac{\mu \delta^5}{30} + \dots \right) + \left(\frac{\delta^4}{12} - \frac{\delta^6}{90} + \dots \right) \right] \bar{w} \quad (85)$$

To obtain a first approximation, \bar{w}_0 , for the solution of Equation (83), the second term, $S\bar{w}$, is ignored. Hence, \bar{w}_0 is obtained by solving the simple second-order differential equation

$$L \bar{w}_0 = h^2 R(\eta) \quad (86)$$

From the values of \bar{w}_0 obtained above and Equation (85), the value of $S\bar{w}_0$ is obtained. The first correction, \bar{w}_1 , is the solution of the simple difference equation

$$L \bar{w}_1 = S\bar{w}_0; \quad (87)$$

the second correction satisfies

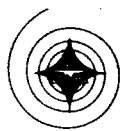
$$L \bar{w}_2 = S\bar{w}_1 \quad (88)$$

and so on, the desired value of \bar{w} being given finally by

$$\bar{w} = \bar{w}_0 + \bar{w}_1 + \bar{w}_2 + \dots \quad (89)$$

By the aid of the definition of $\mu \delta$ and δ^2 , the value \bar{w}_n of the first order is obtained from Equation (84) as

$$L \bar{w}_n = \bar{w}_{n+1} - 2\bar{w}_n + \bar{w}_{n-1} + \frac{1}{2} hk_{\eta_1}(\eta_n) (\bar{w}_{n+1} - \bar{w}_{n-1}) + h^2 k_{\eta_2}(\eta_n) \bar{w}_n \quad (90)$$



Hence, Equation (83) is written as

$$\left[1 + \frac{1}{2} h k_{\eta 1}(\eta_n)\right] \bar{w}_{n+1} - \left[2 - h^2 k_{\eta 2}(\eta_n)\right] \bar{w}_n + \left[1 - \frac{1}{2} h k_{\eta 1}(\eta_n)\right] \bar{w}_{n-1} = h^2 R(\eta_n) + S \bar{w}_n \quad (91)$$

Similarly, ignoring all differences of order higher than the third in Equation (85), $S \bar{w}_n$ may be obtained as

$$S \bar{w}_n = \frac{1}{12} \left\{ \left[1 + h k_{\eta 1}(\eta_n)\right] \bar{w}_{n+2} - 2 \left[2 + h k_{\eta 1}(\eta_n)\right] \bar{w}_{n+1} + 6 \bar{w}_n - 2 \left[2 - h k_{\eta 1}(\eta_n)\right] \bar{w}_{n-1} + \left[1 - h k_{\eta 1}(\eta_n)\right] \bar{w}_{n-2} \right\} \quad (92)$$

Thus, with initially given values:

$$\left. \begin{array}{l} \bar{w} = 0 \text{ at } \eta = 0 \\ \bar{w} = 0 \text{ at } \eta = h \end{array} \right\} \quad (93)$$

Equation (91) gives successively the particular solution \bar{w}_n at η_n as

$$\bar{w}_n = \bar{w}_{n0} + \bar{w}_{n1} + \bar{w}_{n2} + \dots \quad (94)$$

A Homogeneous Solution (\bar{u})

From Equation (75), the homogeneous equation for \bar{u} is written as

$$\left[D^2 + k_{\eta 1}(\eta) D + k_{\eta 2}(\eta) \right] \bar{u} = 0 \quad (95)$$



The value of \bar{u} is found by the same method in computing \bar{w} above. Thus, with initially given values:

$$\left. \begin{aligned} \bar{u} &= 0 \text{ at } \eta = 0 \\ \bar{u} &= 0 \text{ at } \eta = h \end{aligned} \right\}, \quad (96)$$

the homogeneous solution \bar{u} is obtained from the following difference equation:

$$\begin{aligned} \left[1 + \frac{1}{2} h k_{\eta 1}(\eta_n) \right] \bar{u}_{n+1} - \left[2 - h^2 k_{\eta 2}(\eta_n) \right] \bar{u}_n + \\ \left[1 - \frac{1}{2} h k_{\eta 1}(\eta_n) \right] \bar{u}_{n-1} = S \bar{u}_n \end{aligned} \quad (97)$$

where

$$\begin{aligned} S \bar{u}_n = \frac{1}{12} \left\{ \left[1 + h k_{\eta 1}(\eta_n) \right] \bar{u}_{n+2} - 2 \left[2 + h k_{\eta 1}(\eta_n) \right] \bar{u}_{n+1} \right. \\ \left. + 6 \bar{u}_n - 2 \left[2 - h k_{\eta 1}(\eta_n) \right] \bar{u}_{n-1} + \left[1 - h k_{\eta 1}(\eta_n) \right] \bar{u}_{n-2} \right\} \end{aligned} \quad (98)$$

Equation (97) gives successively the homogeneous solution \bar{u}_n at η_n as

$$\bar{u}_n = \bar{u}_{n0} + \bar{u}_{n1} + \bar{u}_{n2} + \dots \quad (99)$$

A General Solution (\bar{V})

From Equations (91) and (97), the end values $\bar{u}(1)$ and $\bar{w}(1)$ are determined; hence, the constant C_1 can be obtained from Equation (77). Finally, the general solution of Equation (66) at η_n is obtained from Equation (73) as:

$$\bar{V}_n = - \frac{\bar{w}(1)}{\bar{u}(1)} \bar{u}_n + \bar{w}_n \quad (100)$$



Normal Displacement, \bar{w}

The derivative of \bar{V} with respect to η may be approximately obtained in the difference form from Taylor's series:

$$\bar{V}_{n+1} = \bar{V}_{n-1} + 2h \left. \frac{d\bar{V}}{d\eta} \right|_{\eta=\eta_n} + \frac{h^3}{3} \left. \frac{d^3\bar{V}}{d\eta^3} \right|_{\eta=\eta_n} + (\text{higher powers of } h) \quad (101)$$

Thus, by ignoring all derivatives of order higher than the third of the series above,

$$\left. \frac{d\bar{V}}{d\eta} \right|_{\eta=\eta_n} \cong \frac{\bar{V}_{n+1} - \bar{V}_{n-1}}{2h} \quad (102)$$

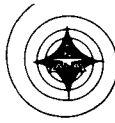
The normal displacement, \bar{w}_n , is then obtained by substituting Equation (102) for $d\bar{V}/d\eta$ in Equation (64) as

$$\bar{w}_n = \frac{1}{2h} g_{\xi 1}(\eta_n) (\bar{V}_{n+1} - \bar{V}_{n-1}) + g_{\xi 2}(\eta_n) \bar{V}_n + g_{\xi 3}(\eta_n) \bar{P}_{\xi}(\eta_n) \quad (103)$$

6. MATRIX EIGENVALUE PROBLEM

Since the form of modal solutions is complex, an exact solution can not be obtained. Although the series representation for the velocity potential, Equation (10), would represent an exact solution if an infinite number of terms were used, as a practical matter, these series must be truncated. Since the velocity potential, Equation (3), satisfies the differential equation, Equation (2), and the noninterface boundary condition exactly term-by-term, it is the liquid surface condition, Equation (33), and the interface condition, Equation (36), that suffer. These conditions can be satisfied only approximately. The functional errors can then be defined as:

$$\epsilon_s = \frac{1}{C_{\xi}} \left. \frac{\partial \Phi}{\partial \eta} \right|_{\eta=0} - \frac{\omega^2}{g} \left. \Phi \right|_{\eta=0} \quad (104)$$



$$\epsilon_I = \frac{1}{C} \sqrt{\frac{\xi_o^2 + 1}{\xi_o^2 + \eta^2}} \left. \frac{\partial \Phi}{\partial \xi} \right|_{\xi = \xi_o} - \frac{\partial w}{\partial t} \quad (105)$$

With these expressions, the total integrated squared error over the boundaries involved can be expressed as

$$S_T = S_S + S_I \quad (106)$$

where

$$S_S = 2\pi C^2 \int_0^{\xi_o} \epsilon_s^2 (B_n, E_m, \xi) \xi d\xi \quad (107)$$

$$S_I = 2\pi C^2 \int_0^1 \epsilon_I^2 (B_n, E_m, \eta) \sqrt{\xi_o^2 + \eta^2} d\eta \quad (108)$$

$$n = 0, 1, 2, \dots; m = 0, 1, 2, \dots$$

In addition to the boundary conditions, Equations (33) and (36), the normal displacement, w , must satisfy one boundary condition since the differential equation, Equation (61), is of the first order with respect to w ; hence, one boundary condition is required for such displacement to solve. In the present analysis, the following boundary condition is considered:

$$w = 0 \text{ at } \eta = 0 \quad (109)$$

The frequency, ω , and constants, B_n and E_m , are then determined by minimizing the total integrated squared error, subject to a constraint, $w = 0$. The conditions for this minimum are

$$\frac{\partial S_T}{\partial B_n} + \lambda \frac{\partial w}{\partial B_n} = 0 \quad (110)$$



$$\frac{\partial S_T}{\partial E_m} + \lambda \frac{\partial w}{\partial E_m} = 0 \quad (111)$$

where λ is a Lagrange multiplier to be determined in the solution.

Equations (110) and (111) are two sets of homogeneous algebraic equations for the constants B_n , E_m , and a Lagrange multiplier λ . These equations can be put in the form of a matrix eigenvalue problem.

7. SOLUTION OF EIGENVALUE PROBLEM

The matrix eigenvalue problem formulated above can be solved only with the aid of a IBM computer. The method of solution consists of searching for values of ω that make the determinant of the eigenvalue matrix vanish. An estimated value of the modal frequency ω is chosen. Then numerical values of the coefficients and the Lagrange multiplier at the matrix is evaluated. This process is repeated for successive values of ω until a zero value of the determinant is found to a desired degree of accuracy.

The numerical calculation will be performed by using the existing computer program which has been generated by Tai and Wing (Reference 1) for the case of longitudinal oscillations of a propellant-filled, flexible hemispherical tank.



CONCLUDING REMARKS

The present method of analysis seems not to have been used for the analysis of oscillations of liquid-carrying elastic tanks. Instead, all the methods that have been published to date are based on the energy method, requiring extremely elaborate numerical calculations. The present method is straightforward and does not require such elaborate numerical calculations as the energy method if the motion is axisymmetric.

Since the method of solution reported herein can be applied to any boundary conditions and expressed in a general form, the analysis for any other shapes of the shell and the numerical procedure for different boundary conditions are not complicated. Hence, the following problem areas are suggested for future work:

1. The axisymmetric dynamic behavior of membrane shells of arbitrary contour;
2. The axisymmetric dynamic behavior of a cylindrical tank with a flexible, inverted oblate spheroidal bulkhead;
3. The axisymmetric dynamic behavior of a cylindrical tank with a flexible, inverted arbitrary-shaped bulkhead;
4. Combined membrane and bending theory for the axisymmetric dynamic behavior of a cylindrical tank with a flexible, inverted oblate spheroidal bulkhead.



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