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A STUDY OF LONGITUDINAL OSCILLATIONS OF PROPELLANT TANKS AND WAVE PROPAGATIONS IN FEED LINES

Part IV—Longitudinal Oscillation of a Propellant-Filled Flexible Hemispherical Tank

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FOREWORD

This report was prepared by the Space and Information Systems Division of North American Aviation, Inc., Downey, California, for the George C. Marshall Space Flight Center, National Aeronautics and Space Administration, Huntsville, Alabama, under Contract No. NAS8-11490, "Study of Longitudinal Oscillations of Propellant Tanks and Wave Propagations in Feed Lines," dated January 6, 1965. Dr. George F. McDonough (Principal) and Mr. Robert S. Ryan (Alternate) of Aero-Astrodynamics Laboratory, MSFC, are Contracting Officer Representatives. The work is published in five separate parts:

Part I - One-Dimensional Wave Propagation in a Feed Line

Part II - Wave Propagation in an Elastic Pipe Filled With Incompressible Viscous Fluid

Part III - Wave Propagation in an Elastic Pipe Filled With Incompressible Viscous Streaming Fluid

Part IV - Longitudinal Oscillation of a Propellant-Filled Flexible Hemispherical Tank

Part V - Longitudinal Oscillation of a Propellant-Filled Flexible Oblate Spheroidal Tank

The project was carried out by the Launch Vehicle Dynamics Group, Structures and Dynamics Department of Research and Engineering Division, S&ID. Dr. F.C. Hung was the Program Manager for North American Aviation, Inc. The study was conducted by Dr. Clement L. Tai (Principal Investigator), Dr. Michael M.H. Loh, Mr. Henry Wing, Dr. Sui-An Fung, and Dr. Shoichi Uchiyama. Dr. James Sheng, who started the investigation of Part IV, left in the middle of the program to teach at the University of Wisconsin. The computer program was developed by Mr. R.A. Pollock, Mr. F.W. Egeling, and Mr. S. Miyashiro.

ABSTRACT

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A procedure has been formulated to determine the natural frequencies of an elastic liquid-filled hemispherical shell subjected to axisymmetric vibrations. It is assumed that the fluid is inviscid and incompressible, and its motion is assumed irrotational. Under these assumptions, a velocity potential is obtained from the solution of Laplace's equation in spherical coordinates. This velocity potential, together with Bernoulli's equation, permits the evaluation of the fluctuating fluid pressure at the interface. Treating the interface pressure as a forcing function in the shell equations, the shell displacement components are then determined analytically. The free surface boundary condition and the interface condition for the radial velocities can only be satisfied approximately. An eigenvalue problem is formulated by minimizing the integrated squared error for the interface condition subject to the constraints that the integrated error for the free surface condition also be a minimum, and the prescribed radial deflection along the edge of the shell be satisfied.

CONTENTS

ILLUSTRATIONS

NOMENCLATURE

$A, B, b_o, b_1, \tilde{C}, \tilde{D}$	Arbitrary constants
a	Radius of shell
C_n	Undetermined constants
D	$Eh/1-\nu^2$
E	Young's modulus
F	Variable defined in Equation (26)
$F_1(z)$	Hypergeometric function
f	Function defined by Equation (36)
h	Shell thickness
N_θ, N_ϕ	Meridional and circumferential stress resultant, respectively
$P_n(\mu)$	Legendre polynomials
P	Pressure
R(r)	Separated dependent variable of Laplace's equation
r	Radial coordinate
t	Time
u(z)	Integration factor defined by Equation (33)
v, w	Meridional and radial displacement component, respectively
$\bar{v}, \bar{w}$	Nondimensional meridional and radial displacement component, respectively

z	$\frac{1}{2}(1-\mu)$
α, β	Hypergeometric function parameters defined by Equations (30b) and (30c), respectively
ϵ	Function defined by Equation (48)
ϵ_1	Integrated error of η_1
ϵ_2	Integrated squared error of η_2
ρ_{ij}	Elements in frequency matrix obtained from ϵ_2
η_1, η_2	Functional error defined by Equations (44) and (46), respectively
$\Theta(\theta)$	Separated dependent variable of Laplace's equation
θ	Meridional spherical coordinate
κ	$\frac{\rho a^2 h}{D}$
λ_1, λ_2	Lagrangian multipliers
μ	Cos θ
ρ	Density of shell material
ρ_o	Fluid density
ν	Poisson's ratio
Φ	Velocity potential
ϕ	Velocity potential for steady flow
χ	Parameter defined by Equation (24b)
ψ	Parameter defined by Equation (24c)
ω	Natural frequency

INTRODUCTION

Analysis of the axisymmetric vibrations of a liquid-filled oblate spheroidal shell is known to be a difficult problem. The magnitude of the difficulty is indicated by the absence of any published paper on this problem, despite the fact that fuel-sloshing problems alone have attracted considerable attention during the past.

No doubt, the difficulty stems from the horrendous and lengthy mathematical expressions required in the treatment of the oblate spheroidal shell and its associated boundary conditions. Due to the complexity involved, the present state of the art for analyzing such an interaction problem would dictate the use of an energy approach. If this method were to be pursued, the investigator would soon be lost in a maze of algebra. To avoid the obstacles that would be encountered in attacking the oblate spheroidal shell-liquid interaction problem without some definite approach other than the energy method, the most logical first step would be to analyze a liquid-filled, hemispherical shell. This particular configuration not only contains the same features as the liquid-filled, oblate spheroidal shell, but its principal attraction is in its simpler geometry. For this reason, the present study was undertaken in order to develop an analytical approach and numerical procedure that, with suitable modification, will be used for solving the problem of the axisymmetric oscillation of a propellant in a flexible, oblate spheroidal tank. In addition, the solution for the spherical case will serve as a check for the oblate spheroidal solution when it degenerates into a hemisphere in the limit.

SECTION 1. MATHEMATICAL MODEL

In this report, a thin elastic hemispherical shell of uniform thickness and completely filled with a liquid propellant that is assumed to be inviscid and incompressible under irrotational motion is considered. Because the shell is treated as a membrane, bending effects are ignored. For a thin hemispherical shell whose thickness-to-radius ratio is very small, such as in the present case, this assumption is justified. Also attention was confined to small-amplitude, longitudinal, or axisymmetric, motion only.

The shell coordinate system used in this report is shown in Figure 1.

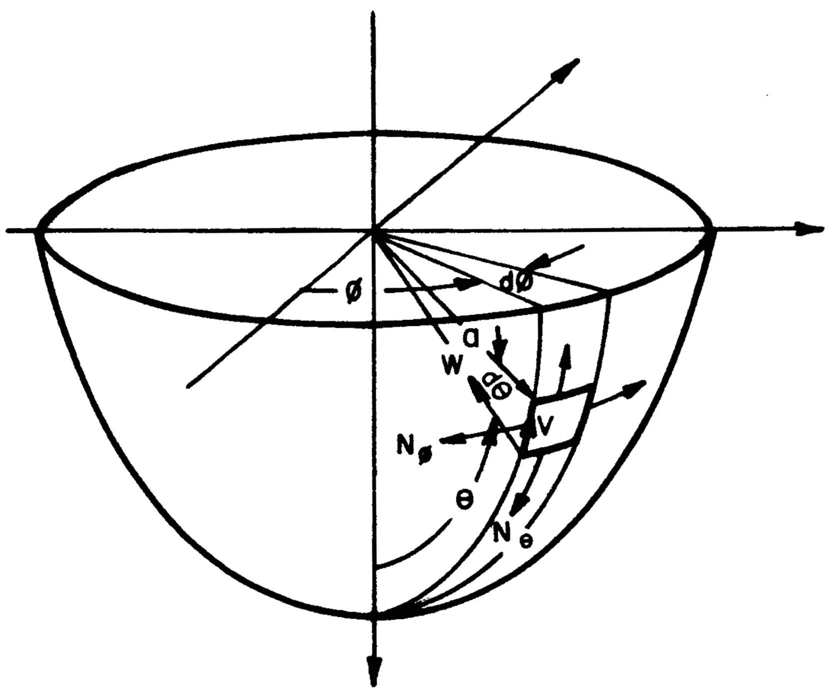

Figure 1. Hemispherical Shell Coordinate System

SECTION 2. METHOD OF SOLUTION

The behavior of the vibrating hemisphere of fluid was first determined from the velocity potential that satisfied Laplace's equation. This permitted the evaluation of the dynamic fluid pressure at the interface, which was then taken as the external forcing function acting on the hemispherical shell. The solution of the shell equations then gave both meridional and radial displacement components in terms of the velocity potential.

An eigenvalue problem is formulated from the boundary condition of the liquid at the free surface, the compatibility of radial velocities of the shell and fluid at the interface, and the boundary condition imposed on the radial displacement at the edge of the shell. Since none of these three equations can be satisfied exactly, the integrated squared error method is employed. Finally, the eigenvalue problem is reduced to matrix form by requiring the integrated squared error of the interface condition to be a minimum subject to the constraints that (1) the integrated error for the free surface condition also be a minimum and (2) the boundary condition for the radial displacement at the edge be satisfied within the limitation of using only a finite series expansion in Legendre polynomials for the fluid pressure.

SECTION 3. LAPLACE'S EQUATION

In addition to assuming that the fluid is inviscid and incompressible, it is also assumed that the motion of the fluid is irrotational. This latter assumption implies the existence of a velocity potential that satisfies Laplace's equation. Under the above assumptions, the problem of small-amplitude axisymmetric vibrations of a hemisphere of fluid is reduced to the solution of Laplace's equation, subject to appropriate boundary conditions.

In spherical coordinates, for the case of axial symmetry, Laplace's equation takes the form

$$\sin\theta \frac{\partial}{\partial r}\left(r^2 \frac{\partial \Phi}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \Phi}{\partial \theta}\right) = 0 \tag{1}$$

where Φ is the velocity potential, and r, θ are coordinates that denote the distance measured from the origin and the "cone angle" measured from the vertical axis, respectively.

BOUNDARY CONDITIONS

The solution of the problem requires specification of the boundary conditions at the free surface of the liquid and at the interface. These conditions require the application of Bernoulli's equation, which is derived from the integration of Euler's equation of motion.

It is assumed that the pressure at the free surface is a constant. This is justified, provided that we consider motion of only infinitely small amplitudes. Under this assumption, the velocity vector at the surface is small, and so the velocity squared term in Bernoulli's equation is a small term of higher order and, hence, may be neglected. Another implication of this same restriction is that the normal to the free surface will make only a very small angle with the vertical. It then follows that the normal component of the fluid velocity at the free surface is essentially equal to the normal velocity of the surface itself. In terms of the linearized Bernoulli's equation, this condition can be expressed as

$$-\frac{g}{r}\frac{\partial \Phi}{\partial \theta} = \frac{\partial^2 \Phi}{\partial t^2} \quad \text{at } \theta = \frac{\pi}{2} \tag{2}$$

This is the linearized boundary condition that is to be satisfied at the free surface, which represents one of the conditions required in the formulation of the eigenvalue problem.

INTERFACE CONDITION

The interface condition is the one that couples the vibrating system represented by the hemisphere of fluid and the hemispherical shell. Due to our original assumption that the fluid is inviscid, this condition is concerned with the normal velocity of the fluid and of the shell only. For compatibility, both of these velocities must be equal. Thus,

$$\frac{\partial w}{\theta t}\bigg|_{\text{shell}} = \frac{\partial w}{\partial t}\bigg|_{\text{fluid}} \qquad \text{at } r = a \tag{3}$$

This is the second condition required in formulating the eigenvalue problem.

SOLUTION OF LAPLACE'S EQUATION

In the case of simple harmonic motion, the velocity potential may be taken in the form

$$\Phi = \phi\,(r, \theta) \cos \omega t \tag{4}$$

where $\phi\,(r, \theta)$ is the velocity potential for steady flow, ω is the natural frequency of the system, and t is the time. The velocity potential is related to the meridional and radial velocity components of the fluid, respectively, by the relations

$$\frac{\partial v}{\partial t} = \frac{1}{r}\,\frac{\partial \Phi}{\partial \theta} \tag{5a}$$

$$\frac{\partial w}{\partial t} = \frac{\partial \Phi}{\partial r} \tag{5b}$$

On substituting Equation (4) into Equation (1), a differential equation that is identical in form to Equation (1) is obtained for $\phi\,(r, \theta)$. This differential equation for ϕ can be solved by employing the standard technique of separation of variables. Therefore, we can assume that

$$\phi\,(r, \theta) = R(r)\,\Theta(\theta) \tag{6}$$

Upon substituting into Laplace's equation, two ordinary differential equations for determining $R(r)$ and $\Theta(\theta)$ are obtained. They are

$$r^2 \frac{d^2R}{dr^2} + 2r \frac{dR}{dr} - n(n+1)R = 0 \tag{7}$$

$$\frac{d^2\Theta}{d\theta^2} + \cot\theta \frac{d\Theta}{d\theta} + n(n+1)\Theta = 0 \tag{8}$$

where n is a non-negative integer.

Equation (7) has as its solution

$$R(r) = Ar^n + Br^{-(n+1)} \tag{9}$$

where A and B are arbitrary constants. In order to avoid the singularity at $r = 0$ so that the velocity potential will remain finite, the constant B is assumed zero.

Equation (8) may be transformed into Legendre's equation by introducing a new variable, μ, defined by

$$\mu = \cos\theta \tag{10}$$

The result is

$$\frac{d}{d\mu}\left[(1 - \mu^2)\frac{d\Theta}{d\mu}\right] + n(n+1)\Theta = 0 \tag{11}$$

which has the solution

$$\Theta(\theta) = \tilde{C}\, P_n(\mu) + \tilde{D}\, Q_n(\mu) \tag{12}$$

where $\tilde{C}$ and $\tilde{D}$ are arbitrary constants, and $P_n(\mu)$ and $Q_n(\mu)$ are Legendre polynomials of the first and second kind, respectively, of degree n. In a similar fashion, with regard to the constant B in Equation (9), the constant $\tilde{D}$ is assumed zero because $Q_n(\mu)$ has logarithmic singularities at $\mu = 1$ corresponding to $\theta = 0°$.

With both R(r) and $\Theta(\theta)$ determined, the steady-state velocity potential can be derived immediately from Equation (6). Since each term of degree n is a solution, the most general expression for the velocity potential is given by the sum of all such solutions plus an additional arbitrary constant. For convenience, the radius r is non-dimensionalized by dividing it by the radius "a" of the hemisphere. In addition, the constants are redefined so that ϕ finally takes the form

$$\phi = a^2\omega\left[C_o + \sum_{n=1}^{\infty} \frac{C_n}{n}\left(\frac{r}{a}\right)^n P_n(\mu)\right] \tag{13}$$

where C_o, C_1, . . . , C_n are nondimensional arbitrary constants. Thus, substituting this expression back into Equation (4), the velocity potential is

$$\Phi = a^2\omega\left[C_o + \sum_{n=1}^{\infty} \frac{C_n}{n}\left(\frac{r}{a}\right)^n P_n(\mu)\right]\cos\omega t \tag{14}$$

FREE SURFACE BOUNDARY CONDITION

On substituting Equation (14) into Equation (2), the following relation for the free surface boundary condition is obtained:

$$\frac{a\omega^2}{g} C_o + \sum_{n=1}^{\infty} \frac{C_n}{n}\left(\frac{r}{a}\right)^{n-1}\left[\sqrt{1-\mu^2}\,\frac{d}{d\mu}P_n(\mu) + \frac{a\omega^2}{g}\left(\frac{r}{a}\right)P_n(\mu)\right]_{\mu=\cos\frac{\pi}{2}=0} = 0 \tag{15}$$

It is seen that this linearized boundary condition cannot be satisfied for each n^{th} - degree term of the legendre functions for C_n's different from zero. Hence, the relation will be averaged over the entire series by integrating it over the free surface of the liquid. It is this resulting expression that will be used later in formulating the eigenvalue problem to determine the natural frequencies of the system.

PRESSURE AT INTERFACE

The dynamic fluid pressure can be obtained from the velocity potential through the linearized Bernoulli equation. This relation is given by

$$p = -\rho_o \frac{\partial\Phi}{\partial t} \tag{16}$$

where ρ_o is the fluid density. Inserting the expression for Φ from Equation (14), we have

$$p = a^2 \omega^2 \rho_o \left[C_o + \sum_{n=1}^{\infty} \frac{C_n}{n} \left(\frac{r}{a}\right)^n P_n(\mu) \right] \sin \omega t \tag{17}$$

At the interface where r = a, the pressure is

$$p(a, \theta) = a^2 \omega^2 \rho_o \left[C_o + \sum_{n=1}^{\infty} \frac{C_n}{n} P_n(\mu) \right] \sin \omega t \tag{18}$$

which represents the fluctuating pressure at the interface between the fluid and the hemispherical shell.

SECTION 4. EQUATIONS OF MOTION FOR HEMISPHERICAL SHELL

The equations of equilibrium for a shell of revolution under a symmetrical loading are given in Timoshenko and Woinowsky-Krieger (Reference 5). On specializing these equations to the case of a spherical or hemispherical shell under inertial loading and fluid pressure, they take the form

$$\frac{\partial N_\theta}{\partial \theta} + (N_\theta - N_\phi) \cot \theta = \rho h a \frac{\partial^2 v}{\partial t^2} \tag{19a}$$

$$N_\theta + N_\phi = \rho h a \frac{\partial^2 w}{\partial t^2} + a p (a, \theta) \tag{19b}$$

where N_θ and N_ϕ are the stress resultants in the direction of the meridional and circumferential coordinates, respectively; ρ is the density of the shell material; and h is the shell thickness. The meridional displacement v is taken positively in the direction of increasing θ, where θ is measured from the axis of symmetry, and the radial displacement w is positive in the direction of the inward normal. Since the expression for the pressure $p(a,\theta)$ given by Equation (18) is in terms of the constants C_n's, there is no loss of generality by assuming that it acts in the direction of the outward normal, as was done in Equation (19b). The solution for the constants will automatically adjust themselves to conform to the true physical situation as long as a consistent sign convention is maintained.

Equations (19) may be expressed in terms of the displacements by utilizing the following relations for the stress resultants given in Reference 5.

$$N_\theta = \frac{D}{a} \left[\frac{\partial v}{\partial \theta} + \nu v \cot \theta - (1 + \nu) w \right] \tag{20a}$$

$$N_\phi = \frac{D}{a} \left[\nu \frac{\partial v}{\partial \theta} + v \cot \theta - (1 + \nu) w \right] \tag{20b}$$

where

$$D = \frac{Eh}{1 - \nu^2}$$

On substituting Equations (20) into Equations (19) and introducing the non-dimensionalized displacement components $\overline{v}$ and $\overline{w}$ defined by

$$v = \frac{\rho a^4 \omega^2}{D} \overline{v} \sin \omega t \tag{21a}$$

$$w = \frac{\rho a^4 \omega^2}{D} \overline{w} \sin \omega t \tag{21b}$$

and the pressure given by Equation (18), we obtain

$$\frac{d^2\overline{v}}{d\theta^2} + \cot\theta \frac{d\overline{v}}{d\theta} - (\nu + \cot^2\theta - \kappa\omega^2)\overline{v} - (1+\nu)\frac{d\overline{w}}{d\theta} = 0 \tag{22}$$

$$(1+\nu)\left(\frac{d\overline{v}}{d\theta} + \overline{v}\cot\theta - 2\overline{w}\right) + \kappa\omega^2\overline{w}$$

$$= \frac{\rho_o}{\rho}\left[C_o + \sum_{n=1}^{\infty} \frac{C_n}{n} P_n(\mu)\right] \tag{23}$$

where

$$\kappa = \frac{\rho a^2 h}{D}$$

The displacement components in nondimensional form will simply be referred to as displacements or displacement components in the remainder of the report.

UNCOUPLED DIFFERENTIAL EQUATION FOR MERIDIONAL DISPLACEMENT

Equations (22) and (23) may be decoupled to yield a differential equation expressed in terms of the displacement component $\overline{v}$ alone. If the expression for $\overline{w}$ from Equation (23) is substituted into Equation (22), we obtain

$$\frac{d^2\overline{v}}{d\theta^2} + \cot\theta\frac{d\overline{v}}{d\theta} + (\chi - \csc^2\theta)\overline{v} + \psi\left(\frac{\rho_o}{\rho}\right)\sum_{n=1}^{\infty}\frac{C_n}{n}\frac{d}{d\theta}P_n(\mu) = 0 \tag{24a}$$

where

$$\chi = \frac{2(1-\nu^2) + (1+3\nu)\kappa\omega^2 - \kappa^2\omega^4}{1-\nu^2-\kappa\omega^2} \tag{24b}$$

$$\psi = \frac{1+\nu}{(1-\nu^2)-\kappa\omega^2} \tag{24c}$$

This equation may, in turn, be expressed in terms of independent variable μ according to the relation given in Equation (10). Thus,

$$\frac{d}{d\mu}\left[(1-\mu^2)\frac{d\bar{v}}{d\mu}\right] + (\chi - \frac{1}{1-\mu^2})\bar{v}$$

$$- \psi\,(\frac{\rho_o}{\rho})\sum_{n=1}^{\infty}\frac{C_n}{n}(1-\mu^2)^{1/2}\frac{d}{d\mu}P_n(\mu) = 0 \tag{25}$$

<u>Transposition to Hypergeometric Equation</u>

The first two terms of Equation (25) may be transposed into a hypergeometric equation. To do this, first let

$$\bar{v} = (1-\mu^2)^{1/2}F \tag{26}$$

Then Equation (25) becomes

$$(1-\mu^2)\frac{d^2F}{d\mu^2} - 4\mu\frac{dF}{d\mu} - (2-\chi)F$$

$$- \psi\,(\frac{\rho_o}{\rho})\sum_{n=1}^{\infty}\frac{C_n}{n}\frac{d}{d\mu}P_n(\mu) = 0 \tag{27}$$

The introduction of a new variable z, defined as

$$z = \frac{1}{2}(1-\mu) \tag{28}$$

into this equation leads to

$$z(1-z)\frac{d^2F}{dz^2} + 2(1 - 2z)\frac{dF}{dz} - (2 - \chi)F$$

$$= -\frac{\psi}{2}\left(\frac{\rho_o}{\rho}\right)\sum_{n=1}^{\infty}\frac{C_n}{n}\frac{d}{dz}P_n(1 - 2z) \tag{29}$$

The homogeneous part of Equation (29) is recognized as a hypergeometric differential equation. Since the constant coefficient dF/dz is an integer, only one solution exists (Reference 6), and this is given by the hypergeometric function

$$F_1 = F_1(\alpha, \beta; 2; z) \tag{30a}$$

where

$$\alpha = \frac{1}{2}\left[3 + (1 + 4\chi)^{1/2}\right] \tag{30b}$$

$$\beta = \frac{1}{2}\left[3 - (1 + 4\chi)^{1/2}\right] \tag{30c}$$

Reduction of Order of Differential Equation

Having one homogeneous solution of Equation (29), a new linear differential equation of one order lower can be obtained. Generally speaking, this is analogous to reducing the degree of an algebraic equation when one solution is known. From the theory of ordinary differential equations, it may be recalled that this homogeneous solution will enable the determination of the complete solution of the original second order differential equation by quadratures (Reference 7). The method of solution is based on the variation of parameters.

Let

$$F = f(z) \cdot F_1(\alpha, \beta; 2; z) \tag{31}$$

Substituting Equation (31) into Equation (29), and bearing in mind that $F_1(\alpha, \beta; 2; z)$ is a homogeneous solution of the original equation, a new differential equation of second order is obtained for determining f(z). However, this equation is of the first order in the variable df/dz. Hence, it can be written as

$$\frac{d}{dz}\left(\frac{df}{dz}\right) + 2\left(\frac{1}{F_1}\frac{dF_1}{dz} + \frac{1-2z}{z-z^2}\right)\frac{df}{dz}$$

$$= -\frac{\psi}{2}\frac{\rho_o}{\rho}\sum_{n=1}^{\infty}\frac{C_n}{n}\cdot\frac{1}{z(1-z)F_1}\frac{d}{dz}P_n(1-2z) \tag{32}$$

Equation (32) has an integrating factor u(z) of the form

$$u(z) = \exp 2\int\left(\frac{1}{F_1}\frac{dF_1}{dz} + \frac{1-2z}{z-z^2}\right)dz = z^2(1-z)^2F_1^{\,2} \tag{33}$$

Multiplying Equation (32) through by this integration factor, the result is

$$u\frac{d}{dz}\left(\frac{df}{dz}\right) + \frac{du}{dz}\frac{df}{dz} = -\frac{\psi}{2}\left(\frac{\rho_o}{\rho}\right)\sum_{n=1}^{\infty}\frac{C_n}{n}\,z(1-z)\,F_1\frac{d}{dz}P_n(1-2z) \tag{34}$$

or

$$\frac{d}{dz}\left[z^2(1-z)^2F_1^{\,2}\frac{df}{dz}\right] = -\frac{\psi}{2}\left(\frac{\rho_o}{\rho}\right)\sum_{n=1}^{\infty}\frac{C_n}{n}z(1-z)\,F_1\frac{d}{dz}P_n(1-2z) \tag{34}$$

Integrating this equation once and reintroducing μ as the independent variable results in

$$(1-\mu^2)^2F_1^{\,2}\frac{df}{d\mu} = b_1 + \psi\left(\frac{\rho_o}{\rho}\right)\int_1^{\mu}\sum_{n=1}^{\infty}\frac{C_n}{n}(1-\mu^2)\,F_1\frac{d}{d\mu}P_n(\mu)d\mu \tag{35}$$

where b_1 is an arbitrary constant. Integrating once again, the equation

$$f = b_o + b_1 \int_1^{\mu} \frac{d\mu}{(1 - \mu^2)^2 F_1^{\,2}}$$

$$+ \psi\left(\frac{\rho_o}{\rho}\right) \int_1^{\mu} \frac{1}{(1 - \mu^2)^2 F_1^{\,2}} \int_1^{\mu} \sum_{n=1}^{\infty} \frac{C_n}{n} (1 - \mu^2)\, F_1 \frac{d}{d\mu} P_n(\mu) d\mu d\mu \tag{36}$$

is obtained, where b_o is another constant of integration. By retaining the two arbitrary constants introduced in each of the integrations, the solution for the meridional displacement $\overline{v}$, obtained by using f(z), will represent the complete solution for the differential Equation (25).

SOLUTION FOR MERIDIONAL DISPLACEMENT

Upon integration, the coefficient of b_1 in Equation (36) leads to terms that contain a singularity at $\mu = 1$ ($\theta = 0°$). Consequently, for a finite solution to exist at that point, b_1 must be taken as zero. In view of Equations (26), (31), and (36), then, the following expression is obtained for the meridional displacement:

$$\overline{v} = b_o (1 - \mu^2)^{1/2} F_1$$

$$+ \psi\left(\frac{\rho_o}{\rho}\right)(1 - \mu^2)^{1/2} F_1 \int_1^{\mu} \frac{1}{(1 - \mu^2)^2 F_1^{\,2}} \int_1^{\mu} \sum_{n=1}^{\infty} \frac{C_n}{n} (1 - \mu^2) F_1 \frac{d}{d\mu} P_n(\mu) d\mu d\mu \tag{37}$$

Boundary Conditions

Because of axial symmetry, the displacement $\overline{v}$ must be zero at $\theta = 0°$; this condition is automatically satisfied by Equation (37). The constant b_o can be determined from the boundary condition at $\theta = \pi/2$. In the present case, the axial motion is restricted so that $\overline{v}$ is zero at this point. On applying this boundary condition, b_o is found to be

$$b_o = \psi\left(\frac{\rho_o}{\rho}\right) \int_1^{o} \frac{1}{(1 - \mu^2)^2 F_1^{\,2}} \int_1^{o} \sum_{n=1}^{\infty} \frac{C_n}{n} (1 - \mu^2) F_1 \frac{d}{d\mu} P_n(\mu) d\mu d\mu \tag{38}$$

Substitution of this expression for b_o into Equation (37) gives for $\overline{v}$

$$\overline{v} = -\psi\left(\frac{\rho_o}{\rho}\right)(1-\mu^2)^{\frac{1}{2}}F_1\int_\mu^o \frac{1}{(1-\mu^2)^2 F_1^{\,2}}\int_\mu^o \sum_{n=1}^{\infty}\frac{C_n}{n}(1-\mu^2)F_1\frac{d}{d\mu}P_n(\mu)\,d\mu d\mu \tag{39}$$

SOLUTION FOR RADIAL DISPLACEMENT

Since $\overline{v}$ is now known, the radial displacement for the hemispherical shell may be obtained directly from Equation (23). Thus, on substitution for $\overline{v}$ from Equation (39), Equation (23) gives

$$\overline{w} = -\frac{\left(\frac{\rho_o}{\rho}\right)}{2(1+\nu)-\kappa\omega^2}\Bigg\{\psi(1+\nu)\left[2\mu F_1 - (1-\mu^2)\frac{dF_1}{d\mu}\right]$$

$$\cdot\int_\mu^o \frac{1}{(1-\mu^2)^2F_1^{\,2}}\int_\mu^o \sum_{n=1}^{\infty}\frac{C_n}{n}(1-\mu^2)F_1\frac{d}{d\mu}P_n(\mu)d\mu d\mu$$

$$+\frac{\psi\left(\frac{\rho_o}{\rho}\right)}{(1-\mu^2)F_1}\int_1^\mu \sum_{n=1}^{\infty}\frac{C_n}{n}(1-\mu^2)F_1\frac{d}{d\mu}P_n(\mu)d\mu$$

$$+ C_o + \sum_{n=1}^{\infty}\frac{C_n}{n}P_n(\mu)\Bigg\} \tag{40}$$

Since this expression for $\overline{w}$ was obtained without performing any integration, there are no arbitrary constants. Hence, no boundary conditions can be applied to $\overline{w}$. This is the consequence of using the membrane shell theory in which bending stiffness is neglegted. Because of the original stipulation, $\overline{w}$ will satisfy the condition of axial symmetry; by implication, it will not contain any singularity at $\theta = 0°$, since both $\overline{v}$ and $p(a, \theta)$ are regular at this point.

If it is assumed that the hemispherical shell is provided with a rigid ring at the equatorial plane, thus giving the edge simple support, then

$$\overline{w}\left(\frac{\pi}{2}\right) = 0 \tag{41}$$

However, the absence of an arbitrary constant precludes the satisfaction of this condition. Therefore, this or whatever other boundary conditions may be imposed on $\overline{w}$ must be treated as a constraint.

IMPOSITION OF INTERFACE CONDITION

The interface condition, already discussed in Section 3, represents the compatibility condition that must exist between the fluid and the shell at their common boundary and that will be applied to couple the system.

In view of Equation (5b), the condition stipulated by Equation (3) for compatibility of velocities may be rewritten.

$$\frac{\partial \dot{w}}{\partial t} = - \left. \frac{\partial \Phi}{\partial r} \right|_{r = a} \tag{42}$$

The minus sign must be present since the positive radial direction for the shell and the hemisphere of fluid are opposite each other. Substituting from Equations (14) and (21b), the foregoing relation becomes

$$\sum_{n=1}^{\infty} C_n P_n(\mu) + \rho \frac{a^3 \omega^2}{D} \overline{w} = 0 \tag{43}$$

SECTION 5. FORMULATION OF EIGENVALUE PROBLEM

The eigenvalue problem for determining the natural frequencies and corresponding mode shapes of the coupled fluid-shell system is formulated using three equations that were obtained as a result of imposing the necessary boundary conditions of the problem. These three equations are the liquid free-surface boundary condition given in Equation (15), the boundary condition of zero-radial displacement at the equatorial plane given in Equation (41), and the condition of compatible velocities at the interface given by Equation (43). The solution of these three equations taken simultaneously will yield the natural frequencies. Other than the trivial case in which all the C_n's are zero, it is apparent that the series in these equations cannot be satisfied term by term. This, together with the complexity of the expressions involved, excludes all possibility of an analytical solution. Hence, some approximate numerical method of solution must be used. The approximate method used herein to formulate the eigenvalue problem is based on the least squared error technique.

LEAST SQUARED ERROR FORMULATION

Since, from a practical point of view, the series representation for the velocity potential must be truncated, the boundary conditions given by Equations (15), (41), and (43) will be satisfied only approximately. However, the intention is to treat the boundary condition for $\overline{w}$ as a constraint so that it will be satisfied exactly within the limitation of using only a finite series representation for the velocity potential. Similarly, within the same limitations, the equation obtained for the integrated error resulting from the approximate satisfaction of the free surface boundary condition for the fluid is also treated as a constraint.

The difference between the exact and approximate satisfaction of the free surface boundary condition is denoted by the functional error η_1. Thus,

$$\eta_1 = \frac{a\omega^2}{g} C_o + \sum_{n=0}^{N} \frac{C_n}{n} \left(\frac{r}{a}\right)^{n-1} \left[\frac{d}{d\mu} P_n(0) + \frac{a\omega^2}{g}\left(\frac{r}{a}\right) P_n(0)\right] \qquad (44)$$

If this expression is integrated over the free surface of the fluid, the integrated error ϵ_1 is obtained. Accordingly,

$$\epsilon_1 = 2\pi a^2 \int_0^1 \eta_1 \left(\frac{r}{a}\right) d\left(\frac{r}{a}\right) \tag{45}$$

From Equation (43), the error η_2 along a meridian, resulting from the truncation of the series expansion in Legendre polynomials, is

$$\eta_2 = \sum_{n=1}^{N} C_n P_n(\mu) + \rho \frac{a^3 \omega^2}{D} \bar{w} \tag{46}$$

Since it is the interface condition that will have the most influence on the frequency for the interaction problem, it is desirable to make η_2 as small as possible. For this reason, we will consider it from a least squared error standpoint instead of averaging the error out over the series as was done with η_1.

To obtain the integrated squared error for this case, η_2 is first squared and then integrated over the surface of the hemisphere so that

$$\epsilon_2 = 2\pi a^2 \int_0^{\frac{\pi}{2}} \eta_2^{\,2} \sin\theta d\theta \tag{47}$$

Equation (47) contains a total of N + 2 unknowns, as does Equation (45). There are N + 1 unknown C_n's plus the unknown frequency ω. In order for the finite series representation of the velocity potential to provide the best possible fit for the satisfaction of the boundary conditions, these N + 1 unknown C_n's are determined in such a manner as to render ϵ_2 a minimum subject to Equations (41) and (45) treated as constraints. This is equivalent to minimizing the function

$$\epsilon = \epsilon_2 + \lambda_1 \epsilon_1 + \lambda_2 \bar{w}\left(\frac{\pi}{2}\right) \tag{48}$$

where λ_1 and λ_2 are Lagrangian multipliers. Hence,

$$\frac{\partial \epsilon}{\partial C_n} = \left[\begin{array}{c|c|c} \zeta_{ij} & \dfrac{\epsilon_1}{C_j} & \dfrac{\bar{w}\,(\frac{\pi}{2}}{C_j} \\ \hline \epsilon_1 / C_j & 0 & 0 \\ \hline \bar{w}\,(\frac{\pi}{2}) / C_j & 0 & 0 \end{array}\right] \left\{\begin{array}{c} C_j \\ \hline \lambda_1 \\ \hline \lambda_2 \end{array}\right\} = 0$$

where

$$i, j = 0, 1, \ldots, N$$

$$\zeta_{ij} = \frac{1}{C_j} \frac{\partial \epsilon_2}{\partial C_i} \tag{49}$$

This linear algebraic system of N + 2 homogeneous simultaneous equations has a nontrivial solution only if the determinant of the above equation is zero.

The solution of the determinantal equation will yield N + 1 natural frequencies. Only the first few frequencies will have any physical significance, however, since it takes more and more terms in the series expansion for the fluid pressure in order to be able to represent it accurately as the frequency increases.

SECTION 6. SOLUTION FOR FREQUENCIES

The eigenvalue problem as formulated cannot be solved unless a digital computer is used. The method of solution as programmed is essentially a systematic trial-and-correction process which searches for an ω until the frequency determinant vanishes. A modal frequency is assumed that permits the calculation of the hypergeometric function F_1 (α, β; 2, z) and the displacements $\overline{v}$ and $\overline{w}$ corresponding to this estimated frequency. The integrals that appear in the expressions for $\overline{v}$ and $\overline{w}$ are evaluated numerically by using the trapezoidal rule. Since the constants C_n's are unknown, only their coefficients are actually calculated at each discrete point used in the numerical process. This calculation requires the arranging of these values in a matrix to facilitate the bookkeeping involved.

With the determination of the displacement $\overline{w}$, numerical values for all the elements appearing in matrix Equation (49) are calculated. The determinant of the matrix is then evaluated. This process is repeated a second time with the frequency incremented to initiate the Newton-Raphson method of root determination, which continues until a zero value for the determinant is obtained within the specified degree of accuracy.

CONCLUDING REMARKS

In this section of the report, it was hoped that some numerical results could have been obtained and conclusions drawn. However, this was not possible due to numerous difficulties encountered during programming and also in the formulation of the eigenvalue problem.

The ground work for the numerical procedure for solving this problem was started approximately six months ago. Besides the usual problems that are encountered during the development of a new computer program, an additional obstacle presented itself in the formulation of the eigenvalue problem from Equations (15), (41) and (43) which would give numerical results having physical meaning.

In the first attempt, the integrated squared error for the approximate satisfaction of both the free surface boundary condition and the interface condition for compatible velocities was summed to yield the total integrated squared error for the system. This expression was then minimized, subject to the constraint that $\overline{w}(\pi/2) = 0$. When the resulting determinant of this system of linear algebraic homogeneous equations was evaluated, it was always positive for all frequencies, with the possible exception of $\omega = 0$. Later on, it occurred that the integrated squared error for the two boundaries should be given different weights instead of the same weight, since one boundary condition must have a greater influence on the frequency than the other. Since at that time it was not apparent as to how this weighing procedure should be carried out, it was decided to formulate the eigenvalue problem using the three pertinent equations by the collocation method. The value of the frequency determinant resulting from this formulation was always negative.

Both of the above formulations of the eigenvalue problem are in conflict with physical reality and, hence, were discarded. The present formulation discussed in Section 5 appears to be most promising. Although no substantiating evidence is available, very preliminary computing results seem to support this observation, as indicated in Figures 2 and 3. Therefore, it is recommended that additional funds be allocated for continuation of this study in order to bring it to a successful conclusion.

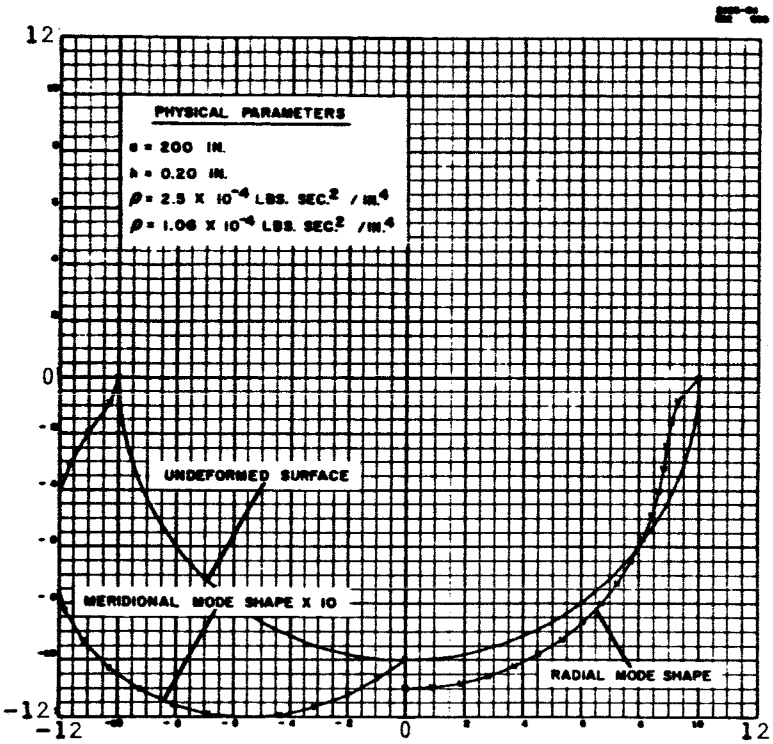

Figure 2. Mode Shape for Hemispherical Shell Filled with Liquid Oxygen, f = 0.495 cps

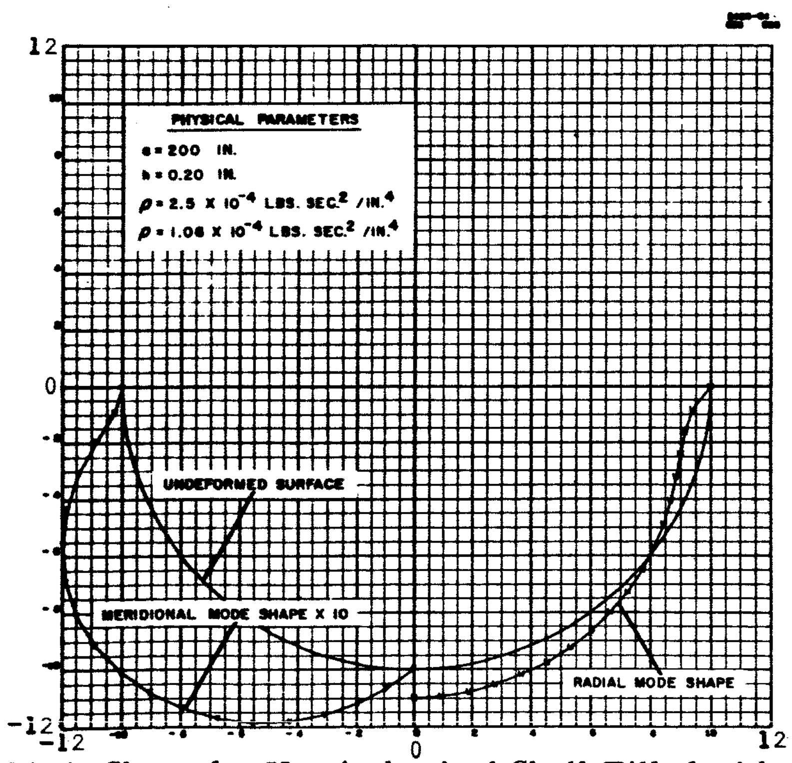

Figure 3. Mode Shape for Hemispherical Shell Filled with Liquid Oxygen, f = 0.699 cps

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