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LIST OF SYMBOLS

A	wing area of aircraft
A_p	area of propeller
B	constant defined on page 6
C_L	lift coefficient
C_D	drag coefficient
D	drag
E	Weierstrass function defined on page 7
F	functional form defined on page 6
g	acceleration due to gravity, 32.2 ft/sec^2
G	quantity to be optimized
H	Hamiltonian function defined on page 7
K	thrust/weight
L	lift
m	mass
n	lift/weight
P	power input to air
r	(thrust minus drag)/weight
R	thrust minus drag
t	time
T	thrust
u	dimensionless velocity
v	velocity
v_r	reference velocity
W	weight of aircraft
x	range coordinate
y	altitude coordinate
α	real variable defined on page 4
β	real variable defined on page 4 constant defined on page 11
γ	flight path angle
Γ	real variable defined on page 4
η	dimensionless altitude
λ	Lagrange multiplier
ξ	dimensionless range
ρ_o	sea level density of atmosphere
σ	ratio of atmospheric density to sea level density
τ	dimensionless time
Δu_o	change in velocity of airstream across propeller

Subscripts

1	initial conditions
2	final conditions
no	non-optimal
o	optimal

Notation

A prime denotes differentiation with respect to non-dimensional time.
A dot denotes differentiation with respect to time.

ABSTRACT

General equations which describe optimum flight trajectories in a plane for an assumed model aircraft are obtained in this paper by the use of the methods of calculus of variations. These general equations are presented in a form applicable for analog computer solution. A few typical two point minimum time trajectories are obtained using the analog computer for thrust only, drag only, and thrust and drag cases. Two interesting features are illustrated by these results.

1. Optimal trajectories do not exist to all physically obtainable endpoints in the plane.

2. For the cases which include drag, if the endpoint is a sufficient distance from the origin, then a portion of the flight trajectory is flown in a quasi-steady manner.

The sensitivity of the solutions to certain unknown constants as a function of endpoint location is illustrated and discussed. Also some methods of making approximations to an optimal trajectory are outlined.

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INTRODUCTION

When a person is in control of a vehicle, he is usually able to travel along one of an infinite number of paths and bring himself and his vehicle to one of any number of final states. He must, however, operate the vehicle within certain bounds which restrict his maneuvers. Briefly, he is limited to operate within the laws of nature, government, and self-preservation.

It is quite natural to ask if it is possible to operate a given vehicle in an optimum fashion and still obey all of the above mentioned laws. Since the formal procedure for obtaining an answer based on analysis is usually quite formidable, answers to questions of this type are often obtained in an intuitive and/or semi-analytical way with practical experience being used, if possible, as a guide. However, an analytical optimal control solution, if only to a much simplified case of the original problem, is of considerable value since this program should give considerable insight into the nature of the optimization and will give an upper or lower bound to which intuitive or other sub-optimal control programs may be compared. In this way the significance of an optimization or the gains to be made by an optimization are apparent.

An aircraft capable of control may be flown in an optimal fashion with considerable advantage. The determination of the flight trajectory and the corresponding control program which results in optimum performance for a simplified model aircraft is the subject of this paper. It is well known that the calculus of variations may be used for the analysis of this problem. Among the first individuals to set up problems in aircraft performance using the calculus of variations were Hestenes[1] and Garfinkel[2] in 1951. Later Cicala[3] and Miele[4] set up aircraft performance problems using a particular case of the general mathematical problem from the calculus of variations known as the problem of Mayer. The Mayer formulation has been highly popular ever since because of its generality and adaption to the concepts of state and control variables. The Mayer formulation will be used in this paper.

There is no particular difficulty in setting up a problem in optimal flight mechanics and displaying the optimizing conditions which must be met along an optimal trajectory. The essential steps used in order to arrive at this state of analysis are to describe mathematically the quantity to be optimized and the applicable constraint conditions, define an augmented function in terms of these conditions, and then utilize the results of the theory of the calculus of variations to write down the Euler-Lagrange equations and associated optimizing conditions. There are numerous papers which illustrate the procedures mentioned above[5]. However, due to the very complex nature of the optimizing conditions, there are very few papers on aircraft performance in which the solution to the optimizing equations is obtained.

In order to make a problem in flight mechanics tractable analytically, using the indirect methods of the calculus of variations, certain approximations and assumptions will usually have to be made. For example the thrust of a turbojet engine or the drag coefficient of an aircraft may have to be assumed constant throughout the flight. Under these assumptions, an analysis is made by operating on the principle that the form of an optimal trajectory is not microscopically dependent on relatively small changes that would actually take place in these parameters. In comparison, the direct methods of dynamic programming or steepest ascent when applied to problems in flight mechanics can and usually do include the details of engine data, etc. The intent of this paper is to present analytical solutions to simplified yet realistic aircraft models so that an understanding of the significance or physical meaning associated with an optimal solution will be made apparent. Rather than investigate the details of a particular optimal flight, general results will be obtained in terms of performance parameters such as wing loading, thrust to weight ratio, etc. If general principles governing optimal flight are to be obtained, they most naturally arise from an analytical indirect approach.

Only plane motion of an aircraft operating within the atmosphere will be discussed. Furthermore, the earth will be assumed to be flat with a constant gravitational force. The aircraft will be considered to be a particle of constant mass operating under gravitational, thrust, lift, and drag forces.

The quantity to be optimized will be assumed as expressible in the form $G(x,y,v,\gamma,t)$ where

x = coordinate in horizontal direction
y = coordinate in vertical direction
v = velocity of vehicle
γ = flight path angle
t = time

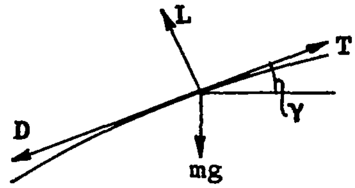
The equations of constraint consist of Newton's second law of motion and related kinematical conditions plus certain inequality constraints which result from design limitations of the aircraft and limitations to the possible flight regime as dictated by the laws mentioned on page 1.

Flight trajectories which satisfy the necessary optimizing conditions as set forth in the next section will be referred to as optimal trajectories.

DEVELOPMENT OF OPTIMIZING CONDITIONS

Equations of Constraint

The analysis of optimal aircraft performance will be confined to an aircraft assumed to operate in a plane under the forces shown on page 3.



Where:

T = thrust. It will be assumed that the thrust for the various types of aircraft can be expressed as a function of velocity and height only, with no control possible.

D = drag. In general the drag force is a function of height, velocity and lift. In this analysis the dependence of drag on lift will be omitted.

L = lift. Lift is in general a function of height, velocity and lift coefficient.

mg = weight of the aircraft. The mass of the aircraft will be assumed constant.

For a high speed aircraft the lift, drag, and thrust would be more properly expressed in terms of Mach number. However, for simplification in order to make the comparison between high speed and low speed aircraft more direct, the atmosphere will be assumed to be isothermal. In this case lift, drag, and thrust become velocity dependent for both high and low speed flight.

The quantity to be optimized as a result of flight will be assumed to be as expressible in the form

$$G(x,y,v,\gamma,t)\Big|_2 \tag{2.1}$$

The initial values of x,y,v,γ, and t will all be assumed as fixed. The subscript 2 refers to final values. For example a maximum range problem is given by $G = x_2$.

Let $T(v,y) - D(v,y) = R(v,y)$, then the equations of motion in the tangential and normal directions may be written as follows:

$$m\dot{v} = R(y,v) - mg\sin\gamma\ , \tag{2.2}$$

$$mv\dot{\gamma} = L(y,v,C_L) - mg\cos\gamma\ , \tag{2.3}$$

with the following kinematical relations between the variables:

$$\dot{x} = v\cos\gamma\ , \tag{2.4}$$

$$\dot{y} = v\sin\gamma\ . \tag{2.5}$$

These four equations are written using the assumptions already listed, and they represent dynamical bounds on the possible motion of the aircraft as required by the laws of physics. They contain five dependent variables x, y, v, γ, and C_L so that one degree of freedom remains for control. In this case the control variable is C_L, and the four state variables are x, y, v, and γ.

If the aircraft is to be flown in an allowable fashion, additional inequality conditions must be included in the statement of the problem.

In order to determine the effect of realistic constraints of this type on an optimal trajectory, the constraints on the state variables will be given by the following inequalities

$$y \geq 0 \ , \tag{2.6}$$

$$v_{stall} \leq v \leq v_{max} \ . \tag{2.7}$$

The amount of lift which can be generated is limited by the inequality constraint on the control variable C_L,

$$C_{Lmin} \leq C_L \leq C_{Lmax} \ . \tag{2.8}$$

For the purposes of analysis, the three inequalities given by Equations (2.6), (2.7), and (2.8) may be expressed in terms of equalities by the method suggested by Hancock[6]. New real variables, $\bar{\alpha}$, $\bar{\beta}$, and Γ are introduced as follows:

$$y = \bar{\alpha}^2 \ , \tag{2.9}$$

$$(v_{max} - v)(v - v_{stall}) = \bar{\beta}^2 \ , \tag{2.10}$$

$$(C_{Lmax} - C_L)(C_L - C_{Lmin}) = \Gamma^2 \ . \tag{2.11}$$

Since $\bar{\alpha}^2$, $\bar{\beta}^2$, and Γ^2 are always positive, Equations (2.9), (2.10), and (2.11) express the same inequalities as Equations (2.6), (2.7), and (2.8).

A general problem in optimal aircraft performance for the assumed mathematical model aircraft may be formulated as follows:
Extremize the function

$$G(x, y, v, \gamma, t)\Big|_2 \ , \tag{2.12}$$

subject to constraints

$$\dot{v} = R/m - g \sin\gamma \ , \tag{2.13}$$

$$\dot{\gamma} = L/mv - g \cos\gamma / v \ , \tag{2.14}$$

$$\dot{x} = v \cos\gamma \ , \tag{2.15}$$

$$\dot{y} = v \sin\gamma \ , \tag{2.16}$$

$$y = \bar{\alpha}^2 \ , \tag{2.17}$$

$$(v_{max} - v)(v - v_{stall}) = \bar{\beta}^2 , \tag{2.18}$$

$$(C_{L_{max}} - C_L)(C_L - C_{L_{min}}) = \Gamma^2 . \tag{2.19}$$

In order to compare optimal performance between various aircraft, the above equations will be put into dimensionless form. If a reference velocity is defined by the velocity of an aircraft in steady level flight at $C_L = 1$ and given by

$$v_r^2 = \frac{2\ W_1}{\rho_o A} \tag{2.20}$$

where W_1 = initial weight of the aircraft,
A = wing area of the aircraft,
ρ_o = sea level density,

then the following non-dimensional parameters may be defined:

Non-dimensional	velocity	$u = v/v_r$,	(2.21)
"	range	$\xi = gx/v_r^2$,	(2.22)
"	altitude	$\eta = g(y-y_1)/v_r^2$,	(2.23)
"	time	$\tau = gt/v_r$,	(2.24)
"	lift	$n = L/W_1$,	(2.25)
"	T-D	$r = R/W_1$.	(2.26)

In terms of these dimensionaless variables, Equations (2.12) through (2.19) may be written as follows:

$$G(\xi,\eta,u,\gamma,\tau)\Big|_2 , \tag{2.27}$$

$$u' = r - \sin\gamma , \tag{2.28}$$

$$\gamma' = n/u - \cos\gamma/u , \tag{2.29}$$

$$\xi' = u \cos\gamma , \tag{2.30}$$

$$\eta' = u \sin\gamma , \tag{2.31}$$

$$\alpha^2 = \eta + gy_1/{v_r}^2 , \tag{2.32}$$

$$\beta^2 = (u_{max} - u)(u - u_{min}) , \tag{2.33}$$

$$\Gamma^2 = (C_{L_{max}} - C_L)(C_L - C_{L_{min}}) . \tag{2.34}$$

The four state variables ξ,η,u and γ and the four control variables C_L,α,β and Γ, are subject to seven equations of constraint, leaving one degree of freedom for optimal control.

Necessary Optimal Conditions

If the following function is defined:

$$F = \lambda_\xi(u\cos\gamma - \xi') + \lambda_\eta(u\sin\gamma - \eta') + \lambda_u(r - \sin\gamma - u') + \lambda_\gamma\left(\frac{n}{u} - \frac{\cos\gamma}{u} - \gamma'\right)$$

$$+ \lambda_\alpha\left(\alpha^2 - \frac{y_1 g}{v_r^2} - \eta\right) + \lambda_\beta\left[\beta^2 - f(u)\right] + \lambda\ \left[\Gamma^2 - g(C_L)\right], \qquad (2.35)$$

where $f(u) = (u_{max} - u)(u - u_{min})$,

$g(C_L) = (C_{L_{max}} - C_L)(C_L - C_{L_{min}})$,

then the necessary Euler-Lagrange optimizing conditions are given by

$$\frac{\partial F}{\partial y_i} - \frac{d}{d\tau}\frac{\partial F}{\partial y_i'} = 0 . \qquad i = 1,2,\ldots,8 \qquad (2.36)$$

Written out, Equations (2.36) are

$$\xi : \quad \lambda_\xi' = 0 , \qquad (2.37)$$

$$\eta : \quad \lambda_u \frac{\partial r}{\partial \eta} + \lambda_\gamma \frac{\partial n}{\partial \eta}\frac{1}{u} - \lambda_\alpha + \lambda_\eta' = 0 , \qquad (2.38)$$

$$u : \quad \lambda_\xi \cos\gamma + \lambda_\eta \sin\gamma + \lambda_u \frac{\partial r}{\partial u} - \lambda_\gamma\left(\frac{n}{n^2} - \frac{\cos\gamma}{u^2} - \frac{\lambda_\xi}{u}\frac{\partial n}{\partial u}\right) - \lambda_\beta \frac{\partial f}{\partial u} + \lambda_u' = 0, \qquad (2.39)$$

$$\gamma : \quad -\lambda_\xi u \sin\gamma + \lambda_\eta u \cos\gamma - \lambda_u \cos\gamma + \lambda_\gamma \frac{\sin\gamma}{u} + \lambda_\gamma' = 0 , \qquad (2.40)$$

$$C_L: \quad \lambda_\gamma \frac{\partial n}{\partial C_L}\ \frac{1}{u} - \lambda\,\Gamma\,\frac{\partial g}{\partial C_L} = 0 , \qquad (2.41)$$

$$\alpha : \quad 2\alpha\,\lambda_\alpha = 0 , \qquad (2.42)$$

$$\beta : \quad 2\beta\lambda_\beta = 0 , \qquad (2.43)$$

$$\Gamma : \quad 2\Gamma\,\lambda_\Gamma = 0 . \qquad (2.44)$$

The above set of equations has the following first integral given by

$$F - \frac{\partial F}{\partial y_i'}\,y_i' = B , \qquad i = 1,2,\ldots,8 \qquad (2.45)$$

or $$\lambda_\xi \xi' + \lambda_\eta \eta' + \lambda_u u' + \lambda_\gamma \gamma' = B \ , \tag{2.46}$$

which may be written as

$$\lambda_\xi u \cos\gamma + \lambda_\eta u \sin\gamma + \lambda_u (r - \sin\gamma) + \lambda_\gamma \left(\frac{n}{u} - \frac{\cos\gamma}{u}\right) = B \ . \tag{2.46a}$$

The end values of the state variables on the minimizing arc must satisfy the transversality condition

$$\left[Bd\tau + dG + \frac{\partial F}{\partial y_i'} dy_i \right]^2 = 0 \qquad i = 1,2,\ldots,8 \tag{2.47}$$

and for the problem as formulated this condition takes the form

trans: $$\left[Bd\tau + dG - \lambda_\xi d\xi - \lambda_\eta d\eta - \lambda_u du - \lambda_\gamma d\gamma \right]^2 = 0 \ . \tag{2.47a}$$

Often an extremal curve is composed of more than one arc, forming a cusp, and/or is partially composed of boundary curves resulting from constraints on the state and control variables. The following corner condition must be satisfied at boundary points or at points of discontinuity:

$$\left[Bd\tau + dG + \frac{\partial F}{\partial y_i'} dy_i \right]_-^+ = 0 \ , \qquad i = 1,2,\ldots,8 \tag{2.48}$$

or $$\left[Bd\tau + dG - \lambda_\xi d\xi - \lambda_\eta d\eta - \lambda_u du - \lambda_\gamma d\gamma \right]_-^+ = 0 \ . \tag{2.49}$$

Along a minimizing arc, the following Weierstrass function must be everywhere greater than or equal to zero.

$$E = F_{no} - F_o - (y_{ino}' - y_{io}') \left. \frac{\partial F}{\partial y_i'} \right|_o \geq 0 \ , \ i = 1,2,\ldots,8 \tag{2.50}$$

where "no" refers to the function evaluated for non-optimal but permissible control. The subscript o refers to optimal control. For the problem as formulated F = 0 so that the above condition takes the form

$$\lambda_\xi \xi_{no}' + \lambda_\eta \eta'_{no} + \lambda_u u_{no}' + \lambda_\gamma \gamma_{no}' \geq \lambda_\xi \xi_o' + \lambda_\eta \eta_o' + \lambda_u u_o' + \lambda_\gamma \gamma_o' \ , \tag{2.51}$$

which is equivalent to requiring that the function

$$H = \lambda_\xi \xi' + \lambda_\eta \eta' + \lambda_u u' + \lambda_\gamma \gamma' \ , \tag{2.52}$$

take on a minimum with respect to optimal control.

The Γ Equation (2.44) can be satisfied by either $\Gamma = 0$ or $\lambda_\Gamma = 0$. If $\Gamma = 0$ then it follows from Equation (2.34) that the lift program is given by either $C_L = C_{L_{max}}$ or $C_L = C_{L_{min}}$. If $\lambda_\Gamma = 0$ then the flight program consists of intermediate lift coefficients. Note that in this case the C_L Equation (2.41) reduces to

$$\lambda_\gamma \frac{\partial n}{\partial C_L} \frac{1}{u} = 0 \ , \tag{2.53}$$

which implies that $\lambda_\gamma = 0$. If $\lambda_\gamma = 0$ then the normal equation of motion (2.29) becomes uncoupled from the other equations of constraint, and therefore the flight path angle, instead of the lift coefficient may be considered as the control variable.

The β Equation (2.43) can be satisfied by either $\beta = 0$ or $\lambda_\beta = 0$. If $\beta = 0$ it follows from Equation (2.33) that the velocity program is given by either $u = u_{max}$ or $u = u_{min}$. If $\lambda_\beta = 0$ then the flight program consists of intermediate velocities.

Similarly the altitude program consists of intermediate altitudes or is given by

$$\eta = - y_1 g/v_r^2 \ .$$

An optimal trajectory may be comprised of six types of arcs:

(i) arcs of intermediate velocity, altitude, and lift coefficients
(ii) maximum lift coefficient arcs
(iii) minimum lift coefficient arcs
(iv) maximum velocity arcs
(v) minimum velocity arcs
(vi) minimum altitude arcs

The order in which these arcs are joined to form an optimal trajectory is determined by using the Weierstrass condition and the corner conditions.

Unbounded Solutions

In general an optimum trajectory is composed of arcs where the variables take on values inside their region of definition (i.e., arcs i) plus arcs along the boundaries of the regions of definition (arcs ii - vi). However, for comparison purposes it is of interest to investigate first optimal maneuvers which are unbounded. The unbounded trajectory will provide an upper or lower limit to the solution of bounded optimal trajectories, and the unbounded solution may also give considerable insight into the form of a bounded solution.
By setting

$$\lambda_\alpha = \lambda_\beta = \lambda_\Gamma = \lambda_\gamma = 0 \ , \tag{2.54}$$

the optimizing equations for unbounded flight which minimize $G(\xi,\eta,u,\tau)\Big|_2$ are obtained and are summarized below.

$$\xi \ : \ \lambda_\xi = \text{constant} \ , \tag{2.55}$$

$$\eta \ : \ \lambda_u \, \partial r/\partial \eta + \lambda_\eta' = 0 \ , \tag{2.56}$$

$$u \ : \ \lambda_\xi \cos\gamma + \lambda_\eta \sin\gamma + \lambda_u \, \partial r/\partial u + \lambda_u' = 0 \ , \tag{2.57}$$

$$\gamma \quad : -\lambda_\xi u \sin\gamma + \lambda_\eta u \cos\gamma - \lambda_u \cos\gamma = 0 , \tag{2.58}$$

$$\text{1st} : \lambda_\xi u \cos\gamma + \lambda_\eta u \sin\gamma + \lambda_u (r-\sin\gamma) = B , \tag{2.59}$$

$$\text{Trans}: (Bd\tau + dG - \lambda_\xi d\xi - \lambda_\eta d\eta - \lambda_u du)|^2 = 0 , \tag{2.60}$$

$$\text{Corner}: (Bd\tau + dG - \lambda_\xi d\xi \,\sigma\, \lambda_\eta d\eta - \lambda_u du)|^{\pm} = 0 , \tag{2.61}$$

$$\text{Weierstrass}: \lambda_\xi \xi'_{no} + \lambda_\eta \eta'_{no} + \lambda_u u'_{no} \geq \lambda_\xi \xi'_o + \lambda_\eta \eta'_o + \lambda_u u'_o . \tag{2.62}$$

Equation (2.62) is equivalent to requiring that the function

$$H = \lambda_\xi u \cos\gamma + \lambda_\eta u \sin\gamma + \lambda_u (r-\sin\gamma) \tag{2.63}$$

be a minimum with respect to the control variable γ . Thus it is necessary that

$$\partial H/\partial\gamma = 0 , \tag{2.64}$$

and

$$\partial^2 H/\partial\gamma^2 \geq 0 . \tag{2.65}$$

The first condition is identically satisfied along an optimal trajectory by Equation (2.58).

The second condition may be written as

$$-\lambda_\xi u \cos\gamma - \lambda_\eta u \sin\gamma + \lambda_u \sin\gamma \geq 0 . \tag{2.66}$$

If the γ Equation (2.58) is solved for λ_η to give

$$\lambda_\eta = \lambda_u/u + \lambda_\xi \tan\gamma , \tag{2.67}$$

and substituted into Equation (2.66), it reduces to

$$-\lambda_\xi u \sec\gamma \geq 0 . \tag{2.68}$$

Thus $\sec\gamma$ must maintain a sign opposite to the constant λ_ξ throughout the trajectory.

If equality can occur in Equation (2.68) then the minimizing arc may have a corner at such a point. Thus if $\lambda_\xi \neq 0$ the solution will have no corners. Solving for the control variable from the γ Equation (2.58) gives

$$\lambda_\xi \tan\gamma = \lambda\eta - \lambda_u/u . \tag{2.69}$$

Differentiating with respect to τ yields

$$\lambda_\xi \sec^2\gamma \, \gamma' = \lambda'_\eta - (u\lambda'_u - \lambda_u u')/u^2 , \tag{2.70}$$

then substituting for λ'_η , λ'_u , and u' from Equations (2.56), (2.57) and (2.28) and introducing the 1st integral Equation (2.59) results in the following expression:

$$\lambda_\xi \sec^2\gamma \ \gamma' = \lambda_u \ (1/u \ \frac{\partial r}{\partial u} - \frac{\partial r}{\partial \eta}) + B/u^2 \quad . \tag{2.71}$$

Substituting Equation (2.67) into the 1st integral Equation (2.59) gives

$$r\lambda_u = (B - \lambda_\xi u \ \sec\gamma) \ . \tag{2.72}$$

If $r = 0$ then $u \ \sec\gamma = B/\lambda_\xi$, a constant which represents an optimal condition for an aircraft if thrust equals drag, or when $T = D = 0$. This latter case corresponds to the well-known brachistochrone problem.

If $\lambda_\xi = 0$ and $r \neq 0$ then Equations (2.71) and (2.72) may be combined to give

$$u \ \frac{\partial(ru)}{\partial\eta} = \frac{\partial(ru)}{\partial u} \quad . \tag{2.73}$$

The condition $\lambda_\xi = 0$ is required by the transversality condition if no final restriction is imposed on ξ . With $\lambda_\xi = 0$ the additional solution $\cos\gamma = 0$ is obtained from Equation (2.58). Since in this case a corner may exist in the solution both $\cos\gamma = 0$ and Equation (2.73) are usually needed for a solution[6].

If $r \neq 0$ and $\lambda_\xi \neq 0$ then Equation (2.72) may be solved for λ_u and substituted into Equation (2.71) and rearranged to give

$$\gamma' = \frac{\cos\gamma}{u} \left\{ \frac{1}{r}(u \ \frac{\partial r}{\partial u} - u^2 \ \frac{\partial r}{\partial\eta})(\frac{B \ \cos\gamma}{\lambda_\xi u} - 1) + \frac{B \ \cos\gamma}{\lambda_\xi u} \right\} \quad . \tag{2.74}$$

If the function G does not explicitly contain time and if the final time is unrestricted then from the transversality condition Equation (2.60), $B = 0$ and Equation (2.74) reduces to

$$\gamma' = - \frac{\cos\gamma}{u} \left\{ \frac{1}{r} \ (u \ \frac{\partial r}{\partial u} - u^2 \ \frac{\partial r}{\partial\eta}) \right\} . \tag{2.75}$$

The solution of the optimizing Equation (2.75) in conjunction with the constraint Equations (2.28), (2.30), and (2.31) form a one parameter family of curves with γ_1 as the parameter. The value of γ_1 used for a particular problem will depend upon both G and the end conditions.

If the function $G = \tau$ then from the transversality condition Equation (2.60), $B = -1$ and Equation (2.74) may be written as

$$\gamma' = - \frac{\cos\gamma}{u} \left\{ \frac{1}{r}(u \ \frac{\partial r}{\partial u} - u^2 \ \frac{\partial r}{\partial\eta})(\frac{\cos\gamma}{\lambda_\xi u} + 1) + \frac{\cos\gamma}{\lambda_\xi u} \right\} \quad . \tag{2.76}$$

Since $\cos\gamma$ must maintain a constant sign throughout the trajectory, if endpoints in the ξ,η plane are chosen to the right of the initial point then it follows from Equations (2.68) that λ_ξ is negative.

The solution of the optimizing Equation (2.76) in conjunction with the constraint Equations (2.28), (2.30), and (2.31) form a two parameter family of curves with γ_1 and λ_ξ as the parameters. The choice of γ_1, and λ_ξ depend upon the restrictions imposed on the end conditions.

Since λ_ξ is a constant, $\cos\gamma$ is required by Equation (2.68) to remain positive throughout an optimaltrajectory.

Performance Characteristics

The optimizing Equation (2.76) is a function of the performance characteristics of a particular aircraft introduced through the quantity

$$r = \frac{T-D}{W} \ .$$

Thus r must be expressed functionally before a solution can be obtained.

The drag to weight ratio may be expressed in terms of a drag coefficient as

$$\frac{D}{W} = \frac{\rho v^2 C_D A}{2W} \ , \tag{2.77}$$

where in general C_D is a function of either the Reynolds number or Mach number, depending on the flight regime. Density variations up to 50,000 ft. can be accurately approximated by the expression $\rho = \rho_o e^{-\beta y}$ where $\beta = 1/30,100$. In terms of non-dimensional parameters Equation (2.77) reduces to

$$D/W = \sigma\, C_D\, u^2 \tag{2.78}$$

where

$$\sigma = \frac{\rho}{\rho_o} = e^{-\beta y} = e^{-\beta y_1}\, e^{\frac{-Bv_r{}^2\eta}{g}} \ . \tag{2.79}$$

The relationship for the thrust to initial weight ratio for an aircraft depends upon the power plant. An approximate expression for the thrust to weight ratio for various aircraft can be obtained from simple momentum consideration and are summarized in the following table:

TABLE 1

Power Plant Relations

Power Plant	T/W_1	Assumptions and Comments
Rocket	$T/W_1 = K$	Exit pressure is assumed to be ambient and exit velocity is assumed constant.
Ramjet	$T/W_1 = K\sigma u^2$	Exhaust velocity is assumed to be proportional to the free stream velocity.
Turboject	$T/W_1 = K\sigma$	The change in velocity through the engine is assumed to decrease as the free stream velocity increases.
Piston engine or turboprop	$T/W_1 = \frac{1}{v_r}\left(\frac{P/W_1}{u+\Delta u_o/2}\right)$	The velocity change across the propellor is assumed to be zero. For high speed flight Δu_o may be set equal to zero.

$$\text{where } (2u + \Delta u_o)^2 \Delta u_o = \frac{4P}{\rho_o A_p \sigma v_r{}^3}$$

where K = Constant

P = power input to the air (P = power available x propeller efficiency)
Δu_o = change in velocity upstream and downstream of the propeller
A_p = area of the propeller

It is apparent from the relations of Table 1 and Equation (2.77) that, except for a piston engine aircraft at low speeds, the expressions for thrust and drag are quite well-behavedfunctions of altitude and velocity (assuming that the drag coefficient is constant). However, even these simplified equations when substituted into the optimizing Equation (2.76) yield a complicated non-linear equation to be solved in conjunction with the non-linear equations of constraint. Since general shapes and trends associated with an optimization problem are of interest, further approximations and assumptions may be made to simplify the analysis but still retain the nature of the effects of aerodynamic, thrust, and gravity forces on the trajectory.

The thrust and drag variations with altitude are small if the optimum trajectory lies within a small height variation, and in this case the density may be considered constant by setting $\sigma = 1$. This assumption will be used even for large height variations in order to simplify the analysis and obtain a first approximation to an optimal solution. Note that with $\sigma = 1$ the thrust expressions for the rocket and turbojet aircraft are identical and the expression for r becomes

$$r = K - C_D u^2 \quad . \tag{2.80}$$

The difference between high and low performance aircraft may be approximated by adjusting the thrust to initial weight ratios between high and low values.

Minimum Time Solutions

If the final range is specified, then the optimizing condition for a minimum time solution is given by Equation (2.76). If the end conditions on η and u are either left free or specified then using the transversality condition and Equations (2.67) and (2.72) the Lagrange multiplied λ_ξ can be expressed in terms of end conditions as listed below:

TABLE 2

Endpoint Conditions and Corresponding λ_ξ

Range, ξ	Altitude, η	Velocity, u	λ_ξ
1. Fixed	Free	Fixed	$\lambda_\xi = \frac{\cos\gamma_2}{u_2}\left(\frac{1}{r_2\sin\gamma_2 - 1}\right)$
2. Fixed	Free	Free	λ_ξ = constant such that $\tan\gamma_2 = 0$
3. Fixed	Fixed	Free	$\lambda_\xi = -\frac{\cos\gamma_2}{u_2}$
4. Fixed	Fixed	Fixed	λ_ξ = constant such that $u_2 = u_{fixed}$

The form of a minimum time solution is independent of whichever case is chosen and only case 3 will be considered in detail in this paper. The optimizing condition for a minimum time, fixed coordinate endpoint trajectory with the final velocity free, operating in a uniform gravitational field with tangential forces given by Equation (2.80) is determined from Equation (2.76) which reduces to

$$\gamma' = + \frac{\cos\gamma}{u}\left\{\frac{2C_D u^2}{K - C_D u^2}\left(1 + \frac{\cos\gamma}{\lambda_\xi u}\right) - \frac{\cos\gamma}{\lambda_\xi u}\right\} \quad , \tag{2.81}$$

where

$$\lambda_\xi = -\frac{\cos\gamma_2}{u_2} \quad . \tag{2.82}$$

Equation (2.81) is to be solved in conjunction with the equations of constraint.

$$u' = K - C_D u^2 - \sin\gamma \, , \tag{2.83}$$

$$\xi' = u \cos\gamma \, , \tag{2.84}$$

$$\eta' = u \sin\gamma \, . \tag{2.85}$$

METHOD OF SOLUTION AND RESULTS

The Brachistochrone Problem

Equations (2.81), (2.82), (2.83), and (2.84) are four first order non-linear differential equations in the four variables γ, u, ξ, and η. Four constants of integration and a value for λ_ξ are needed in order to completely determine a solution. For a performance type of problem, three of the constants of integration are the known initial conditions on the state variables ξ, η, and u. The fourth constant of integration γ_1 is unknown and must be determined by using a trial and error procedure. In order to solve the two point minimum time problem, it is necessary to not only guess the initial value for the flight path angle γ_1, but to guess a value for λ_ξ as well and adjust them until a trajectory passes through the desired endpoint with the proper conditions associated with the value of λ_ξ for that trajectory. In order to obtain some insight as to the effect of γ_1 and λ_ξ on a solution, the above problem may be reduced to the familiar Brachistochrone problem by setting thrust and drag equal to zero. With $r = 0$ the optimizing condition is obtained from Equations (2.71) and (2.72) which may be combined to give

$$\gamma' = \frac{\cos\gamma}{u} \tag{3.1}$$

Equations (2.83), (2.84), and (2.85) become

$$u' = -\sin\gamma \, , \tag{3.2}$$

$$\xi' = u \cos\gamma \, , \tag{3.3}$$

$$\eta' = u \sin\gamma \, . \tag{3.4}$$

Although λ_ξ has been eliminated from the equations, γ_1 is still a function of the endpoint location, and Equations (3.1), (3.2), (3.3), and (3.4) are still to be solved by trial and error methods. The initial flight path angle γ_1 may be adjusted by trial and error, and the equations integrated until the optimal trajectory curve goes through the specified endpoint. This procedure is readily adaptable to analog computer methods. The difficulties which arise by using trial and error techniques to solve boundary value problems are easily visualized from the results of analog computer solutions.

Figure 1, for example, illustrates how the solution to the brachistochrone problem from Equations (3.1), (3.2), (3.3), and (3.4) is affected by varying

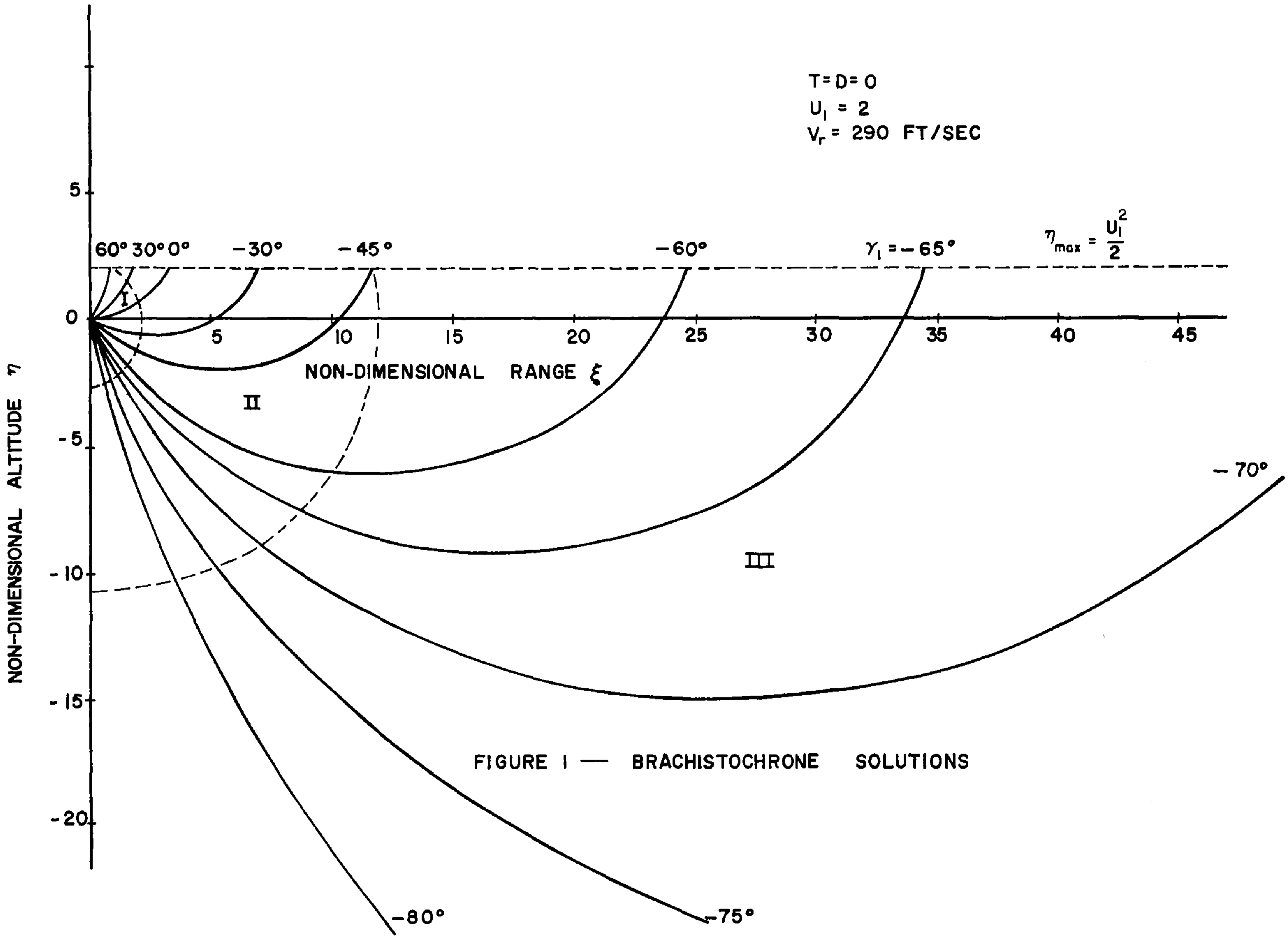


FIGURE 1 — BRACHISTOCHRONE SOLUTIONS

initial flight path angle γ_1. Although the curves in Figure 1 represent the optimum trajectories for each initial flight path angle, they also have the unique property of brachistochronic curves in that they represent the loci of optimal endpoints. In other words, for a given initial angle, the optimal trajectory represents the locus of all possible endpoints that can be reached in an optimal fashion starting with that initial angle.

Because of energy considerations, it is clear that no trajectory and hence no optimal endpoint can lie above the line

$$\eta_{max} = \frac{u_1^2}{2} \quad .$$

The region above this line is therefore called an unattainable or "forbidden" region. All points below this line may be reached in an optimal fashion where the trajectories are a function of the initial flight path angle.

For the purpose of discussion, three regions are indicated in Figure 1. In the first region, it is necessary to vary the initial flight path angle, γ_1, throughout its entire range from -90° to + 90° in order to cover the entire area with optimal trajectories. In region II, however, the entire area can be covered by varying γ_1 between -90° and +60°. Finally, in region III, which includes the entire remaining portion of the plane, initial flight path angles need only be varied from -90° to -45°. Since fewer initial angles are needed to cover greater areas, as indicated above, the solution becomes more sensitive to this initial condition as the distance from the origin increases. As a result, in order to reach an endpoint that is at a considerable distance from the origin, the initial values of the flight path angle, γ_1, must be specified to a large number of significant figures. If an endpoint is to be reached within certain prescribed tolerances, then endpoints at relatively large distances will require greater accuracy than an analog computer or even a digital computer can supply.

Thus for trajectories of short length, an analog computer solution to the fixed endpoint brachistochrone problem can be obtained by making trial and error adjustments to the initial flight path angle until a trajectory passes through the selected endpoint. Although this iterative procedure cannot be successfully used for problems with long trajectories, the form of the solution can still be obtained by observing the results of varying γ_1, through a finite number of values of γ_1 as is shown in Figure 1. For any given endpoint, an approximate value for γ_1 can be determined, and the shape of the actual trajectory can be approximated by visual interpolation.

With the brachistochrone problem as a model, it is easier to consider the more general fixed endpoint minimum time problem which includes the effects of thrust and drag. In this case, the optimizing equation is given by Equation (2.81), and the equations of constraint are given by Equations (2.83), (2.84), and (2.85). It is evident that in addition to the difficulties encountered with choosing γ_1, as mentioned in the case of the brachistochrone problem, there will be further difficulties associated with the choice of the constant λ_ξ. The process of iterating with γ_1 and λ_ξ in order to determine an optimal trajectory which goes through a given endpoint can become a nearly impossible task for trajectories of considerable length. As a result, instead of solving the two point boundary value problem as such, the optimizing equations are integrated in a manner similar to that described for the brachistochrone problem. For each initial flight path angle, a range of values for the constant λ_ξ are chosen, and a set of trajectory curves generated. The value of $\frac{\cos\gamma}{u}$ is monitored (Equation 2.82), and when it reaches the assigned value of $-\lambda_\xi$, the trajectory is terminated. In this manner, the complete region of space in which solutions are possible can be determined, and a manifold of optimal trajectories is generated.

Time Optimal Trajectories with Constant Thrust

The brachistochrone problem represents an aircraft performing under the influence of gravity alone. If thrust is included, the problem becomes considerably more difficult due to the fact that the constant λ_ξ enters into the problem. The optimizing equation for an aircraft operating under thrust and gravity forces only, can be determined from Equation (2.81) by setting $C_D = 0$. The optimizing equation becomes

$$\gamma' = - \frac{\cos\gamma}{u} \left\{ \frac{\cos\gamma}{\lambda_\xi u} \right\} \tag{3.5}$$

and the velocity equation becomes

$$u' = K - \sin\gamma \ . \tag{3.6}$$

The kinematic relations, Equations (3.3) and (3.4), remain the same.

Using the method outlined above for the general case, optimal trajectories generated for the thrust-only case are shown in Figures 2 and 3. For each initial angle, a range of λ_ξ's are selected, and the corresponding trajectories are plotted. At the same time a second set of plots of a "cut off" parameter,

$$- \frac{\cos\gamma}{\lambda_\xi u} \ ,$$

is made (Figure 3). When the "cut off" parameter equals unity, the condition

$$- \lambda_\xi = \frac{\cos\gamma_2}{u_2}$$

is satisfied and the integration is terminated. If $-\lambda_\xi$ is picked such that

$$- \lambda_\xi > \frac{\cos\gamma_1}{u_1} \ ,$$

then the "cut off" parameter never approaches unity as shown in Figure 3. Thus $-\lambda_\xi$ is limited to values within a certain restricted range. Specifically, $-\lambda_\xi$ must vary between $\frac{\cos\gamma_1}{u_1}$ and zero.

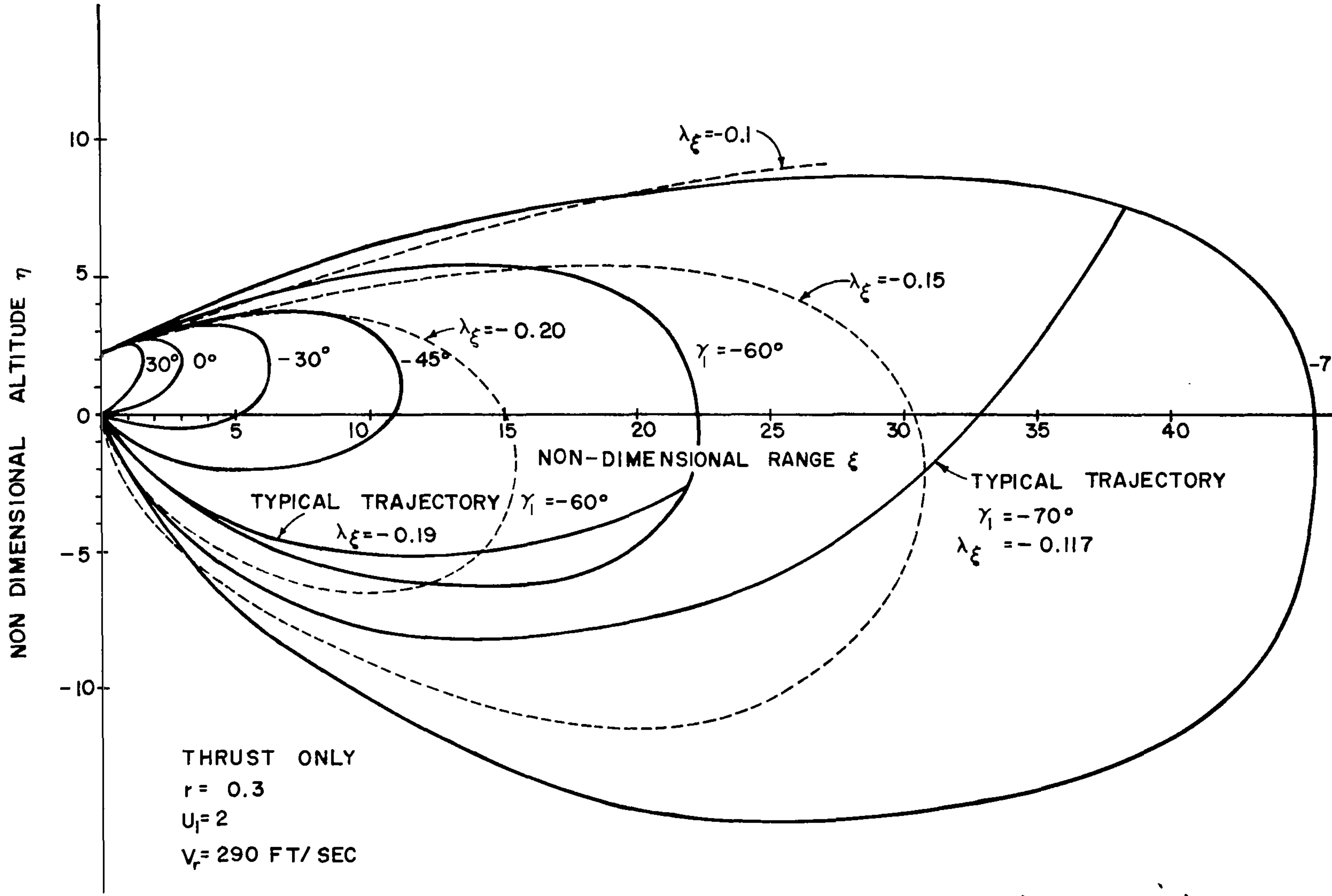


FIGURE 2— LOCUS OF OPTIMAL ENDPOINTS (THRUST ONLY)

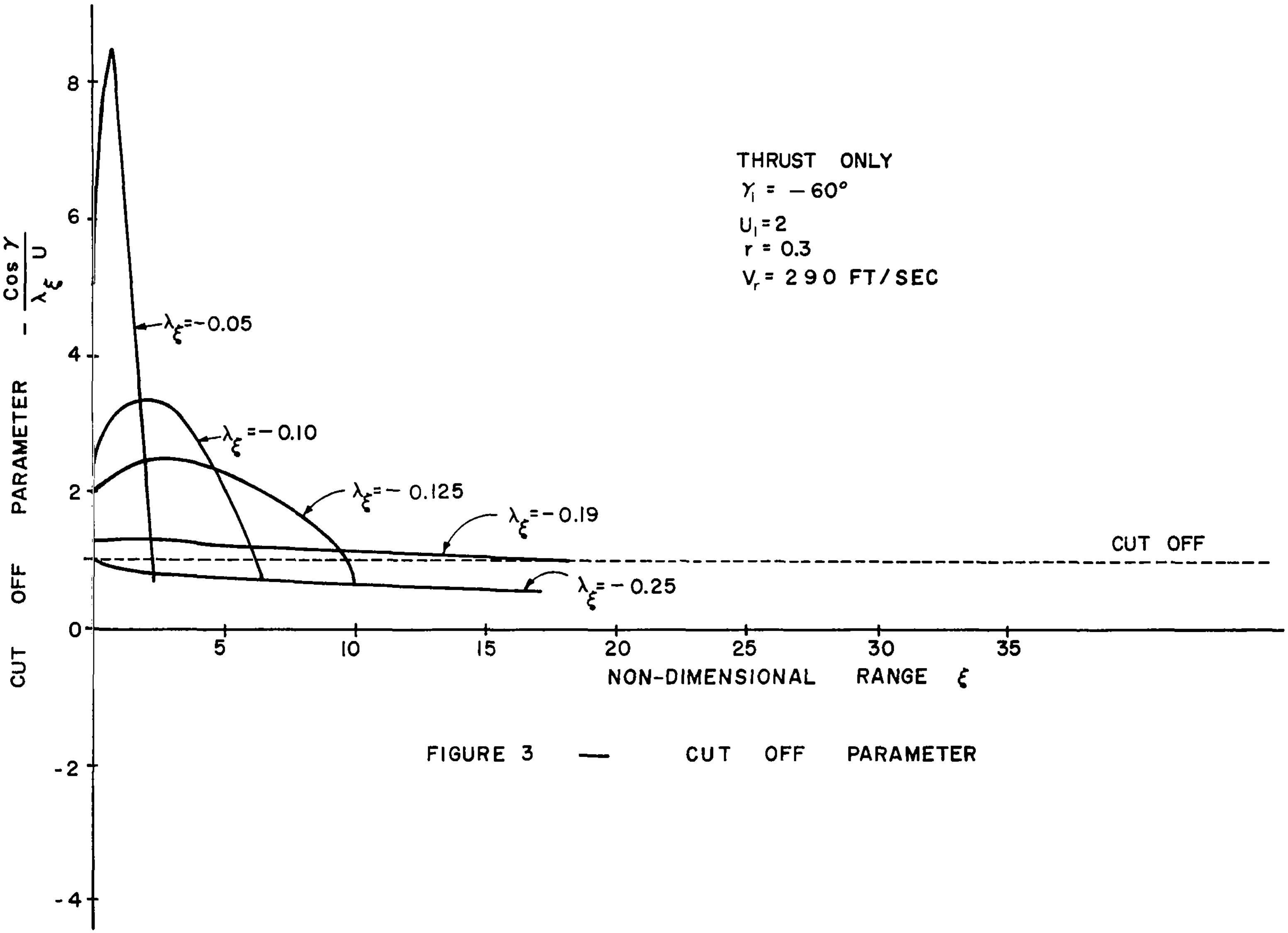


FIGURE 3 — CUT OFF PARAMETER

The locus of optimal endpoints that can be reached by varying λ_ξ throughout its permissible range for a given initial flight path angle is shown in Figure 2. Unlike the brachistochrone problem, the locus of optimal endpoints does not coincide with the optimal trajectories. Also, the locus of optimal endpoints for each initial angle forms a closed loop. In addition all the loops are very nearly tangent to each other at the upper point where they meet the η axis forming bounding region of space, in which optimal solution exist. This situation is similar to the brachistochrone problem, but unlike the brachistochrone problem, the aircraft simulated in this thrust-only case can reach points outside the bounded region.

The endpoints at a considerable distance from the origin are not only increasingly sensitive to the initial flight path angle γ_1, but also to the constant λ_ξ. This is easily seen by comparing the areas between the loops at various distances from the origin, recalling the range limitations on λ_ξ, and noting the lengths of the loops in Figure 2. For example, the area in which endpoints are obtained starting with γ_1 between -60° and -70° is more than twice as large as the area in which endpoints are obtained starting with γ_1 between $+90^\circ$ and -60°. All the points on the relatively short 0° locus can be reached by varying $-\lambda_\xi$ between 0 and 0.5. Whereas all the points on the much longer 70° locus are reached by varying $-\lambda_\xi$ between a smaller range of 0 to 0.171.

Since the sensitivity of any given trajectory endpoint to the two constants γ_1 and λ_ξ greatly increases with the distance from the origin, the two point boundary value problem, as such, is virtually impossible to solve for long trajectories. However in a manner analogous to the brachistochrone problem an approximation of the shape of the trajectory as well as initial flight path angle and λ_ξ can be made from Figure 2.

The lift coefficients (C_L) required to fly a couple of typical trajectories for the thrust-only case are shown in Figure 4. It can be seen that the range of lift coefficient requirements for this case lies within the capabilities of most aircraft.

Time Optimal Trajectories with Drag

The optimizing equation for a glider type of aircraft can be obtained from Equation (2.81) by setting $K = 0$. The optimizing equation in this case becomes

$$\gamma' = -\frac{\cos\gamma}{u}\left\{2 + \frac{3\cos\gamma}{\lambda_\xi u}\right\} , \tag{3.7}$$

with the velocity equation given by

$$u' = -C_D u^2 - \sin\gamma . \tag{3.8}$$

The kinematic relations, Equations (3.3) and (3.4), remain the same.

Solutions to the drag-only case from Equations (3.7), (3.8), (3.3), and (3.4) are obtained in the same manner as for the thrust-only case and typical

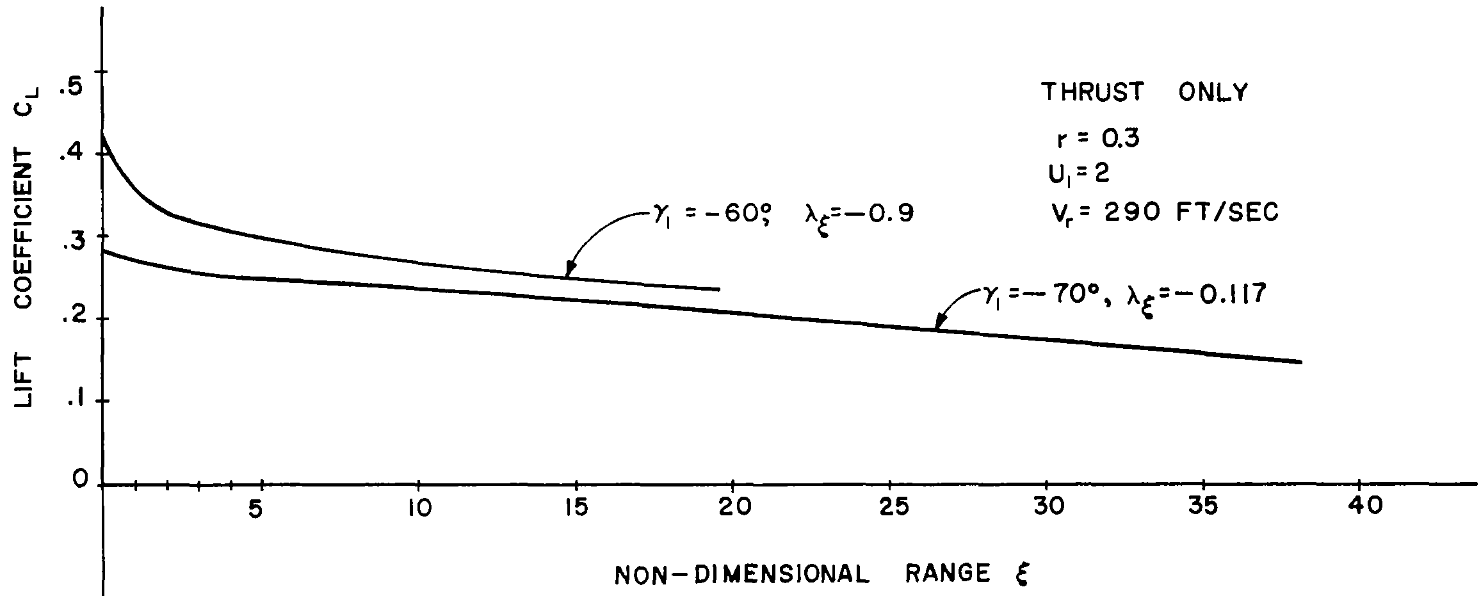


FIGURE 4 — LIFT COEFFICIENT

trajectories and their associated cut off parameters are shown in Figures 5 and 6. The loci of optimal endpoints for each initial flight path angle obtained by varying the constant λ_ξ are shown in Figure 5.

As in the brachistochrone problem and for the same energy consideration, there is a region of unattainable points and hence a "forbidden" region in which there are no possible endpoints. Unlike the thrust-only case, the loci are not closed loops but a manifold of similarly shaped curves which extend to infinity. The further an endpoint is from the origin along any given locus of endpoints, the smaller will be the value of λ_ξ for the corresponding optimum trajectory. In the limit the value of λ_ξ asymptotically approaches a fixed value which is dependent upon the initial conditions. If λ_ξ is selected below this value, the cut-off parameter will not approach unity (Figure 6). The trajectory in this case is a diving one. Furthermore, the value of λ_ξ cannot be chosen greater than

$$-\frac{\cos\gamma_1}{u_1},$$

and have the cut-1ff criteria satisfied. Figure 7 indicates the approximate range through which λ_ξ is allowed to vary for a given initial angle. It is evident from Figure 7 that the sensitivity of a solution to λ_ξ is increased for the lower initial flight path angles. At considerable distances from the origin all solutions are extremely sensitive to λ_ξ due to the asymptotic nature of λ_ξ.

Again the sensitivity of the solution to both γ_1 and λ_ξ virtually prevents the two point boundary value problem from being solved by iteration. However, approximate trajectories can be obtained from Figure 5.

All of the loci in Figure 5 become straight lines at some distance from the origin. Long flight trajectories also have a linear portion to them at approximately the same slope as their corresponding loci. Along the flattened portion of the trajectory, the flight path angle and velocity are nearly constant. Hence the aircraft may be considered to be flying in a quasi-steady manner. As the length of the trajectory increases, the quasi-steady portion increases, and the trajectory becomes a short dive at the beginning, a short climb at the end with the middle portion flown at a constant flight path angle. A plot of the final locus angle to the initial flight path angle is given in Figure 13.

The lift coefficient C_L necessary to fly a few typical trajectories for the drag only case are shown in Figure 8. It can be seen that during the early part of the trajectory, the range of required lift coefficients is within the capabilities of most aircraft. During the short climb at the end of the optimal trajectories, the lift coefficient has a tendency to increase appreciably due to the rapid loss of velocity.

Time Optimal Trajectories with Thrust and Drag

The optimization equation for the general case of an aircraft with thrust and drag is given by Equation (2.81) with the equation of motion and kinematical

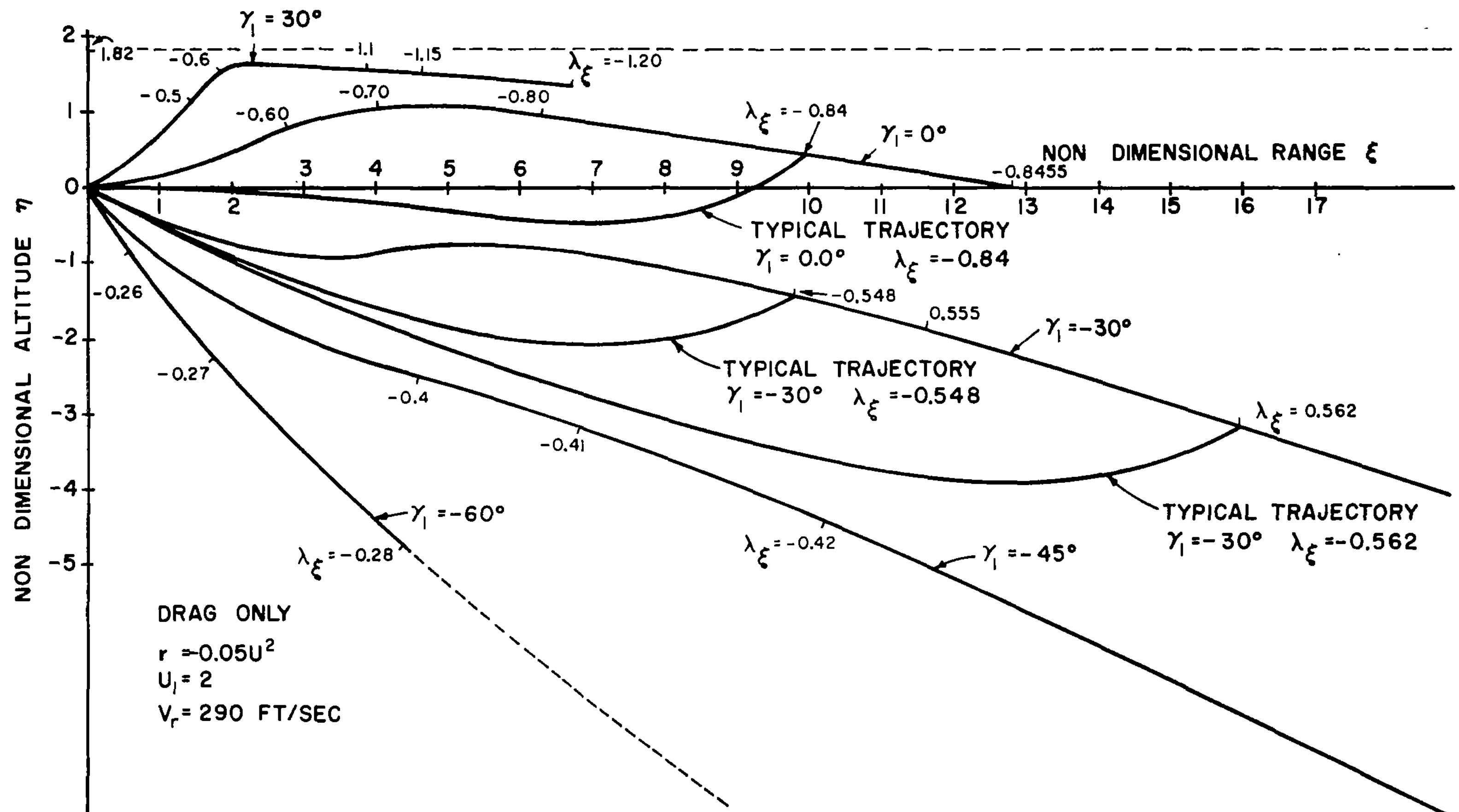


FIGURE 5 — LOCUS OF OPTIMAL END POINTS (DRAG ONLY)

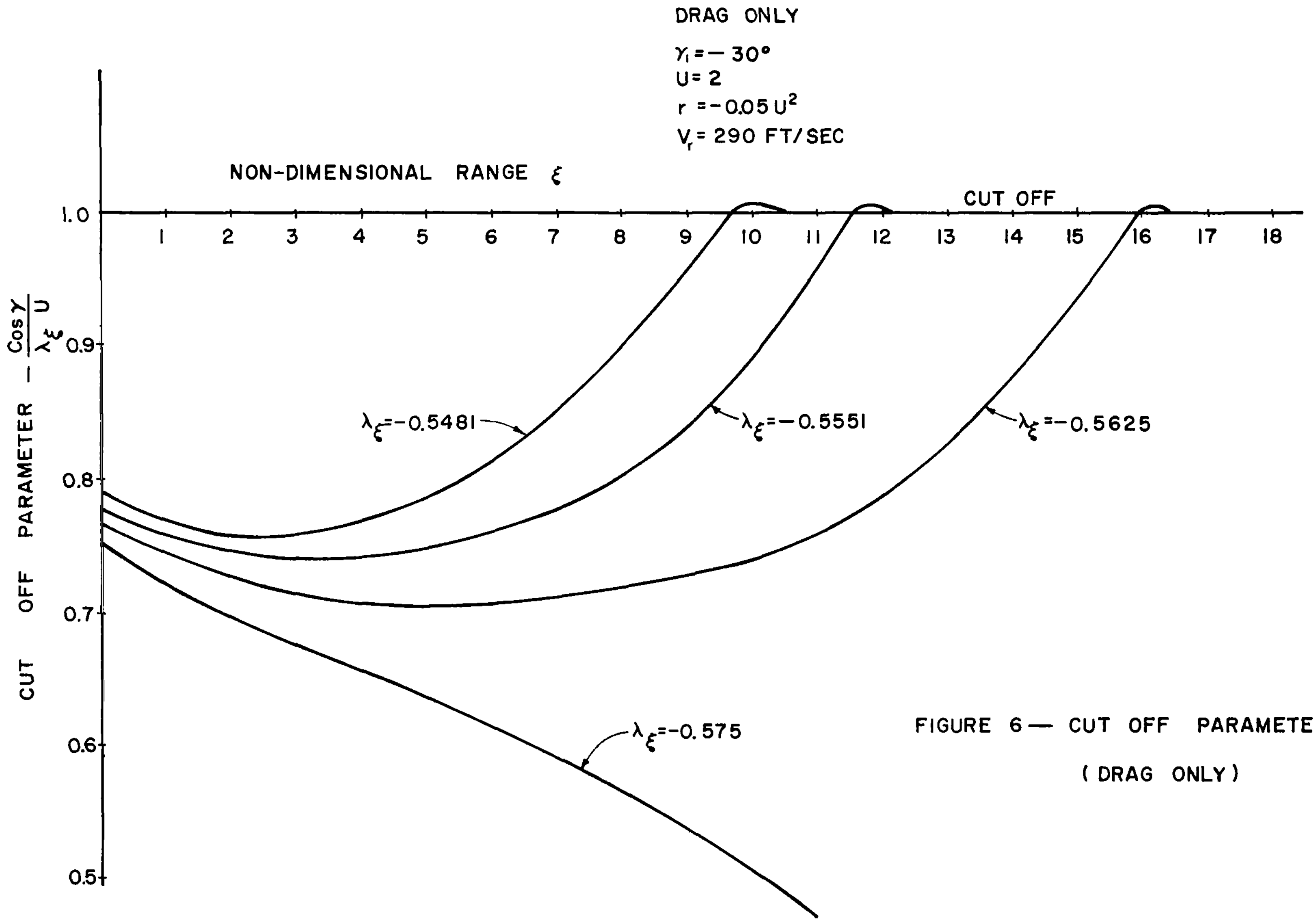


FIGURE 6 — CUT OFF PARAMETER

(DRAG ONLY)

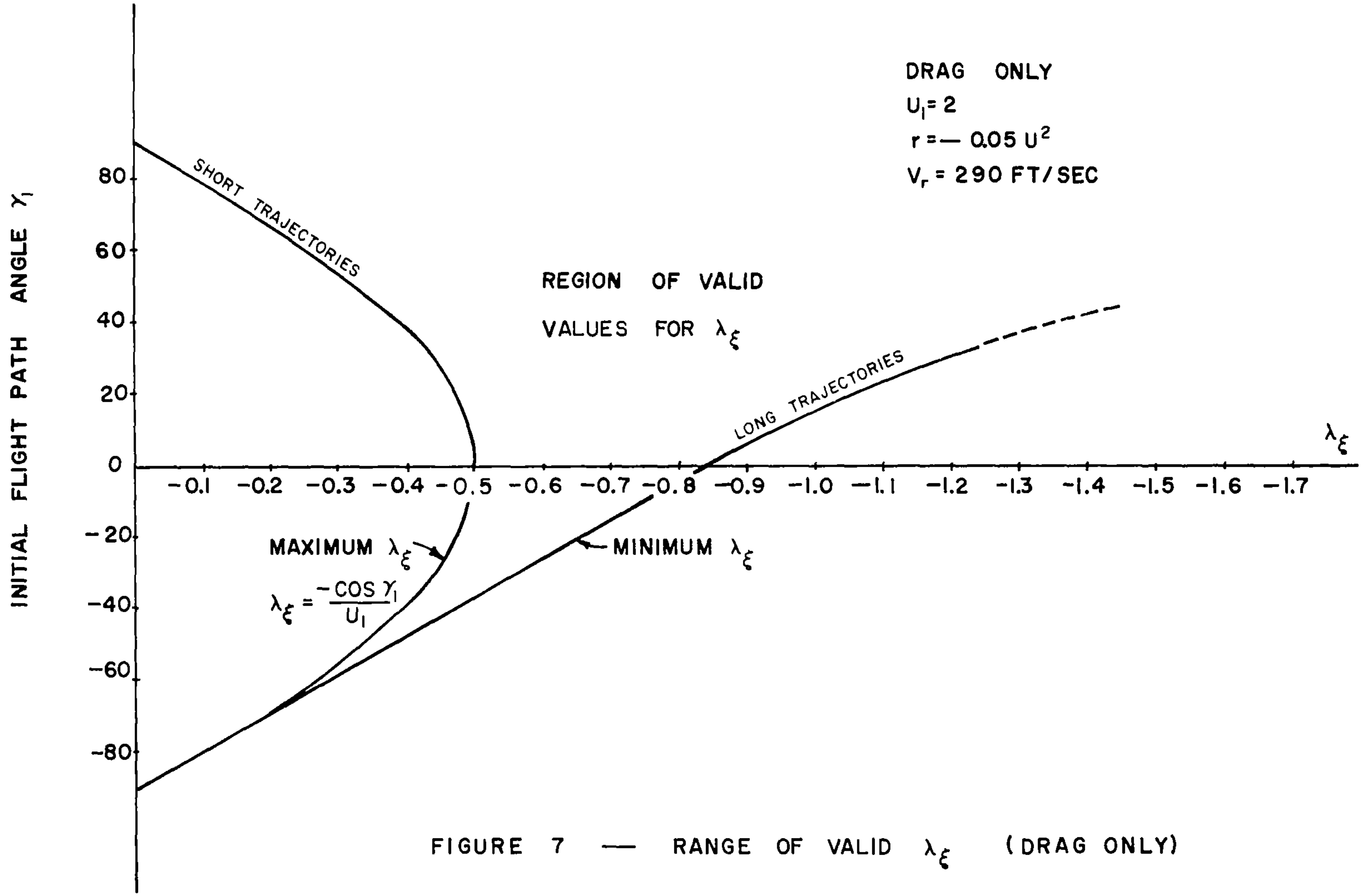


FIGURE 7 — RANGE OF VALID λ_ξ (DRAG ONLY)

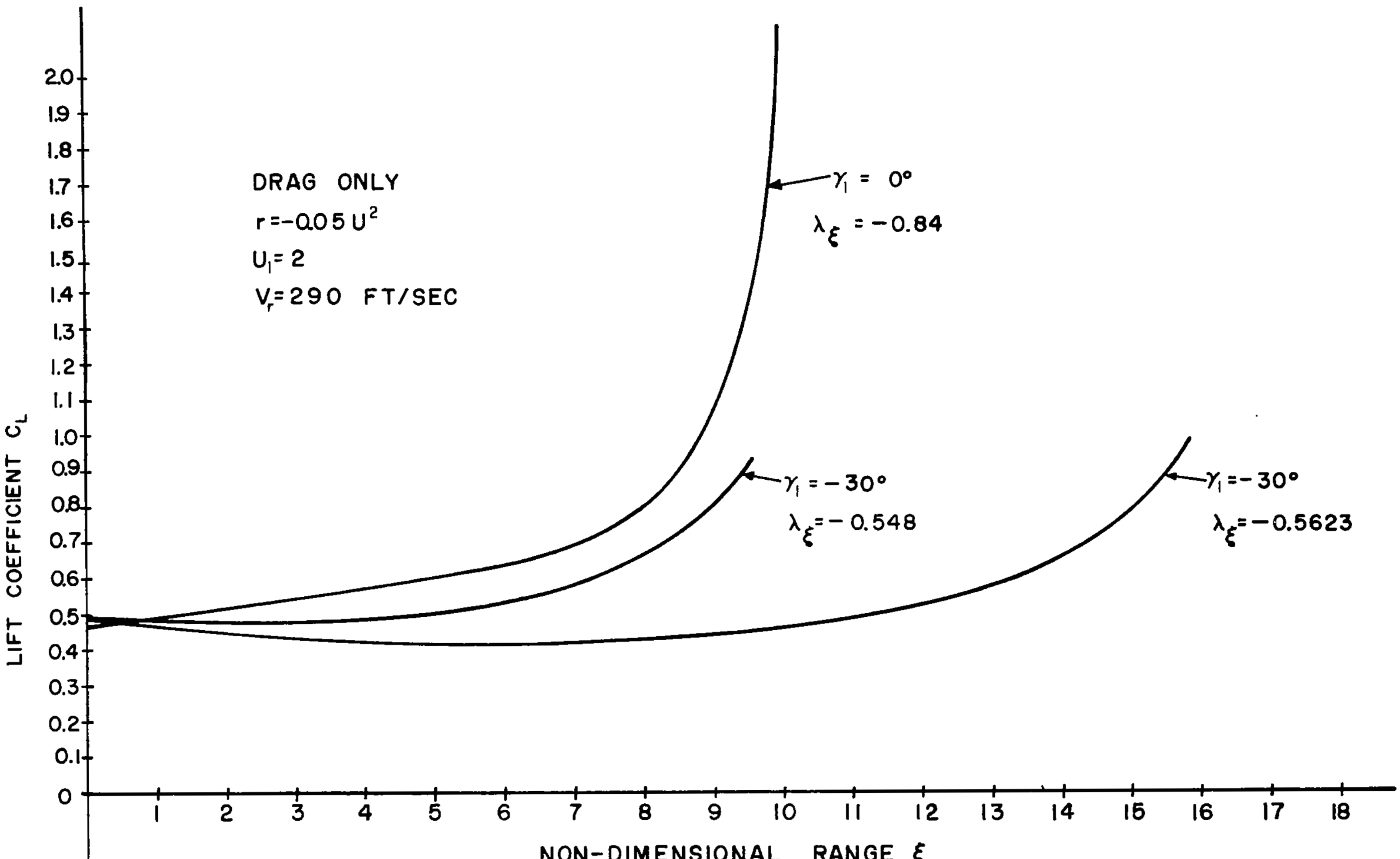


FIGURE 8 — LIFT COEFFICIENT (DRAG ONLY)

constraint equations given by Equations (2.83), (2.84), and (2.85). Solutions to these equations are obtained in a manner similar to that for the drag-only case as shown in Figures 9 and 10. However, the solution procedure is somewhat complicated by the fact that at certain points along some of the trajectories thrust and drag are equal. At these points

$$-\frac{\cos\gamma}{\lambda_\xi u} = 1 \, .$$

This condition had previously been used as the criterion for cut off. In this case, however, the transversality condition requirement $\lambda_{u2} = 0$ is not satisfied. When either $\lambda_{u2} = 0$ or $r = 0$ is substituted into Equation (2.72)

$$r\lambda_u = (B - \lambda_\xi u \sec\gamma) \qquad (3.9)$$

along with the time optimal condition $B = -1$, the condition

$$\lambda_\xi = -\frac{\cos\gamma}{u} \qquad (3.10)$$

is obtained. Thus the proper cut-off condition occurs when $-\frac{\cos\gamma}{\lambda_\xi u}$ is equal to unity and $r \neq 0$.

Some computational difficulties are encountered with the optimizing equation

$$\gamma' = \frac{\cos\gamma}{u}\left\{\frac{2C_D u^2}{r}\left(1 + \frac{\cos\gamma}{\lambda_\xi u}\right) - \frac{\cos\gamma}{\lambda_\xi u}\right\} \, . \qquad (3.11)$$

as r approaches zero. However, γ' does not become infinite as r approaches zero as can be seen if λ_ξ is eliminated from Equation (3.11) by using Equation (3.9) with $B = -1$,

$$\gamma' = \frac{\cos\gamma}{u}\left\{\frac{2C_D u^2\lambda_u + 1}{1 + r\lambda_u}\right\} \, , \qquad (3.12)$$

and taking the limit as $r \longrightarrow 0$ which gives

$$\gamma' = \frac{\cos\gamma}{u}\left\{2C_D u^2 \lambda_u + 1\right\} . \qquad (3.13)$$

Both analog and digital computer results using Equation (3.11) are unreliable if r approaches or passes through zero. This difficulty can often be avoided on the digital computer if the time increments in the vicinity of $r = 0$ are taken large.

The loci of optimal endpoints for various initial flight path angles are shown in Figure 9. The curves are of similar shape to the drag only case and extend to infinity. At some distance from the origin, they all tend to approach straight lines, each at a different slope. This slope is plotted to the initial

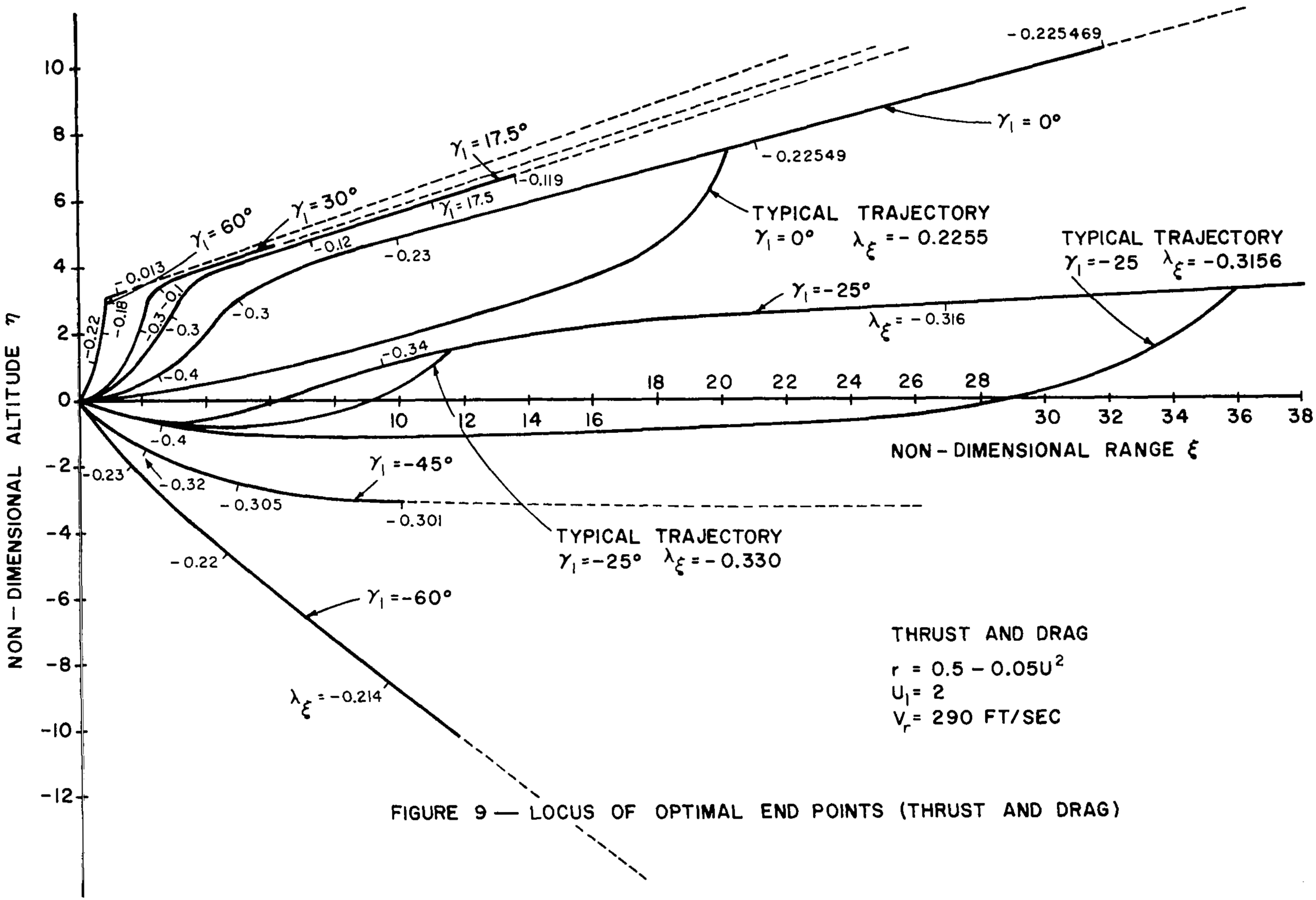


FIGURE 9 — LOCUS OF OPTIMAL END POINTS (THRUST AND DRAG)

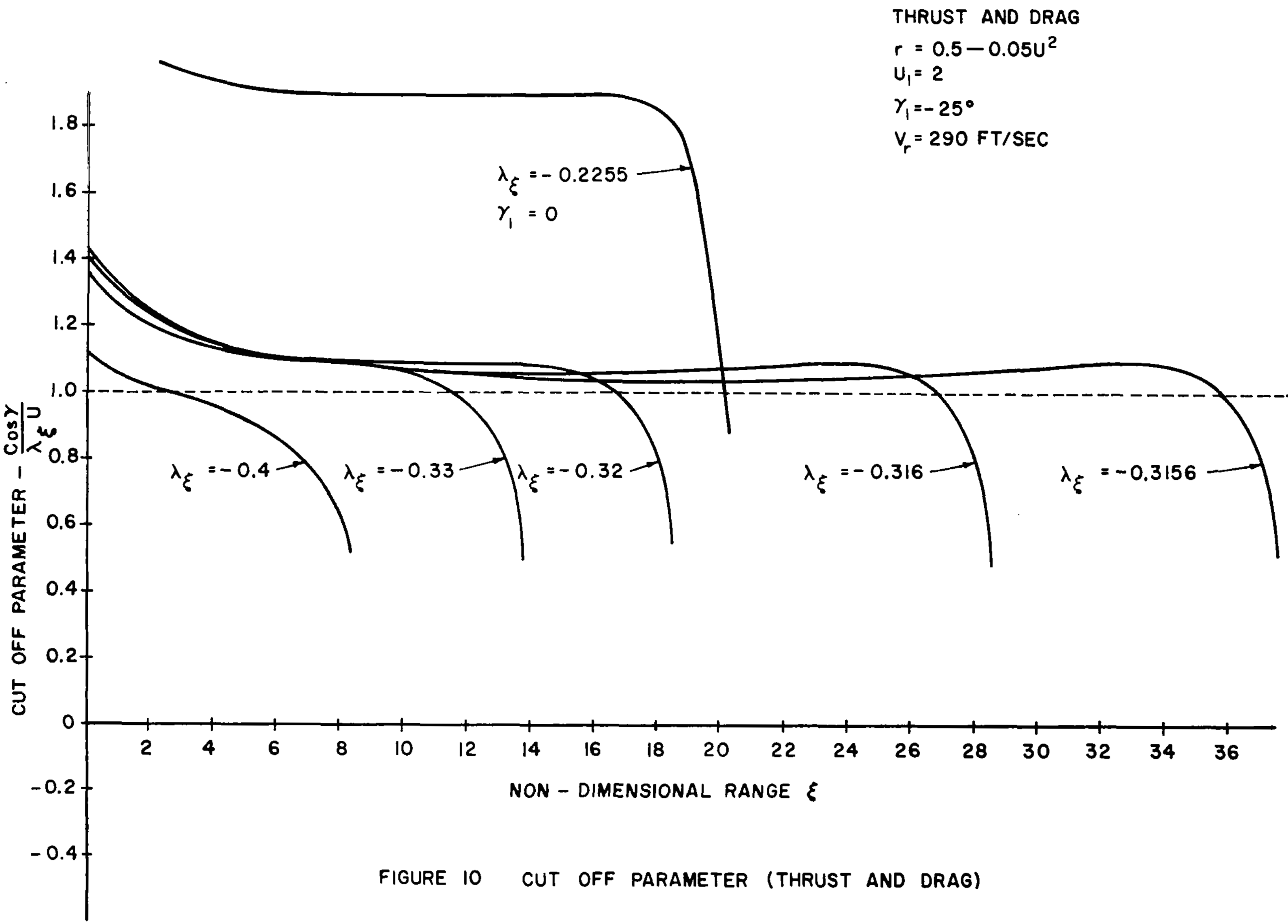


FIGURE 10 CUT OFF PARAMETER (THRUST AND DRAG)

flight path angle as shown in Figure 13. Trajectories shown in Figure 9 are similar to the drag trajectories in that each has a quasi-steady portion.

In a manner similar to the drag-only case the constant λ_ξ along any given locus of endpoints asymptotically approaches a fixed value. Contrary to the drag-only case, this value is a maximum instead of a minimum. If λ_ξ is picked above this maximum value the trajectory dives. The approximate region of valid values for λ_ξ is shown in Figure 11. Again it is easy to see that as in the drag case, the solution is more sensitive to λ_ξ at lower initial flight path angles than higher ones although due to the asymptotic property of λ_ξ, all solutions are sensitive to λ_ξ at large distances from the origin.

Figure 12 shows the lift coefficients C_L necessary to fly some typical trajectories for the thrust and drag case. The range of the required lift coefficient is well within the capabilities of most aircraft.

DISCUSSION AND CONCLUSIONS

The aircraft in this paper was assumed to operate under thrust, drag, lift, and gravity forces. For the cases in which thrust was set equal to zero, the maximum energy of the aircraft is given by its initial kinetic plus potential energy. The magnitude of the initial total energy determines a boundary of endpoints outside of which it is physically impossible to have a trajectory. For the cases which include thrust, every point in the plane is a physically obtainable point, even though a zigzag path may be necessary for an aircraft with a thrust to weight ratio less than one to reach them. However, there is a boundary outside of which optimum solutions are not obtained. The boundary of optimal solutions for the thrust-only case is given by the locus of optimal endpoints for $\gamma_1 = -90^\circ$ and the boundary of optimal solutions for the thrust and drag case is given by the locus of optimal endpoints for $\gamma_1 = +90^\circ$. Due to computational complexities, these boundaries were not evaluated for Figures 2 and 9. However, the locus of optimal endpoints for the -70° case in Figure 2 and the $+60^\circ$ case in Figure 9 very closely approximates the boundary of optimal solutions.

It is evident from Figures 1 and 2 that when drag is not included in a problem, there is no tendency for a portion of the flight trajectory to be linear. This quasi-steady effect appears only when velocity dependent drag is introduced into the problem and is illustrated in Figures 5 and 9. A feature of the flight trajectories which are partially quasi-steady is that the steady part of the trajectory is preceded by a diving type of maneuver and followed by a climbing type of maneuver. During the steady portion of the flight the flight path slope and the slope of the corresponding locus of optimum endpoints are nearly equal. The trajectory slope is slightly greater than the slope of the locus for endpoints located a finite distance from the origin but approaches the same slope as the endpoint is moved to infinity.

The parameter λ_ξ has a great influence on the length and shape of an optimal trajectory. Values of λ_ξ given by the equation

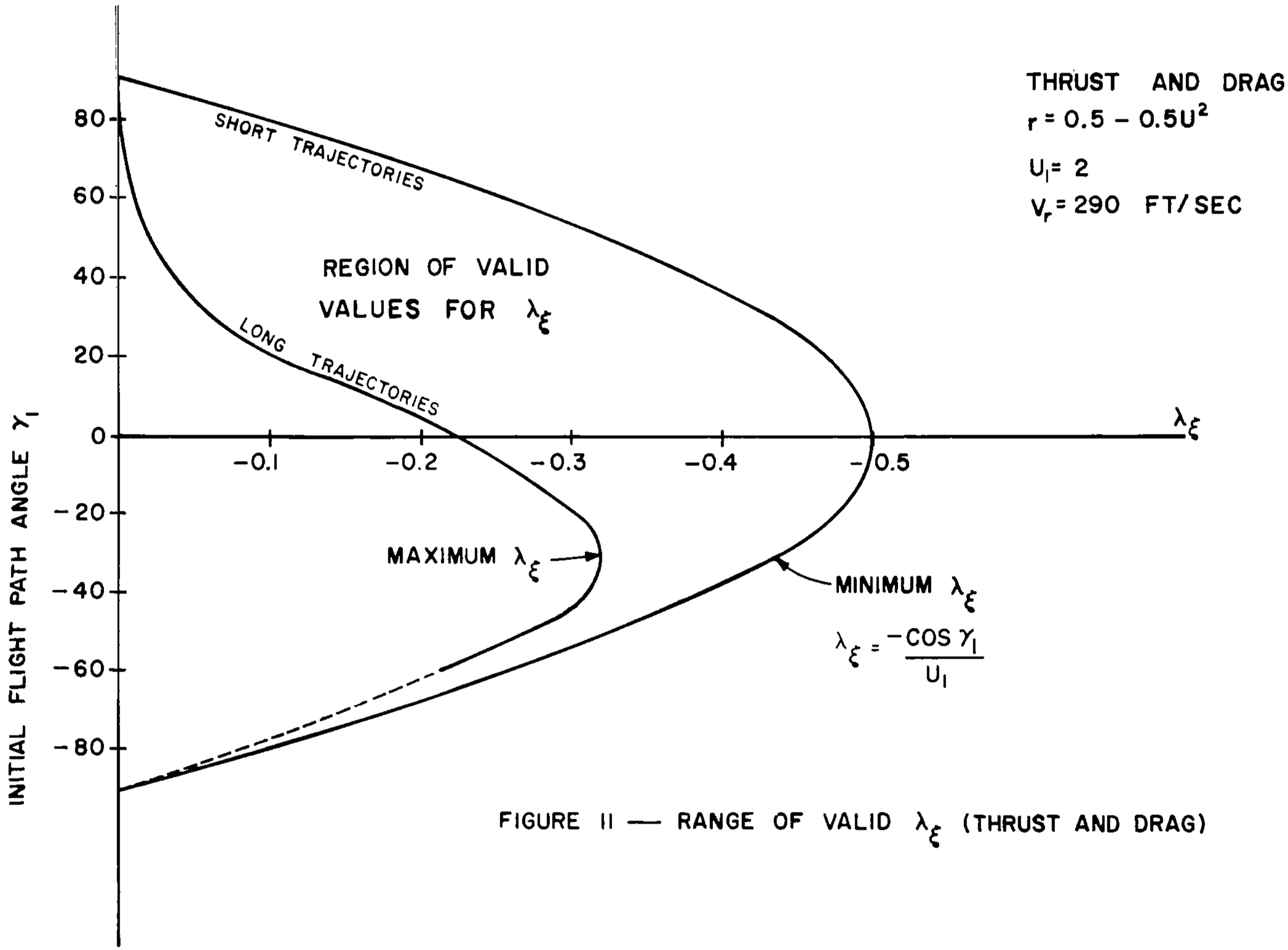


FIGURE 11 — RANGE OF VALID λ_ξ (THRUST AND DRAG)

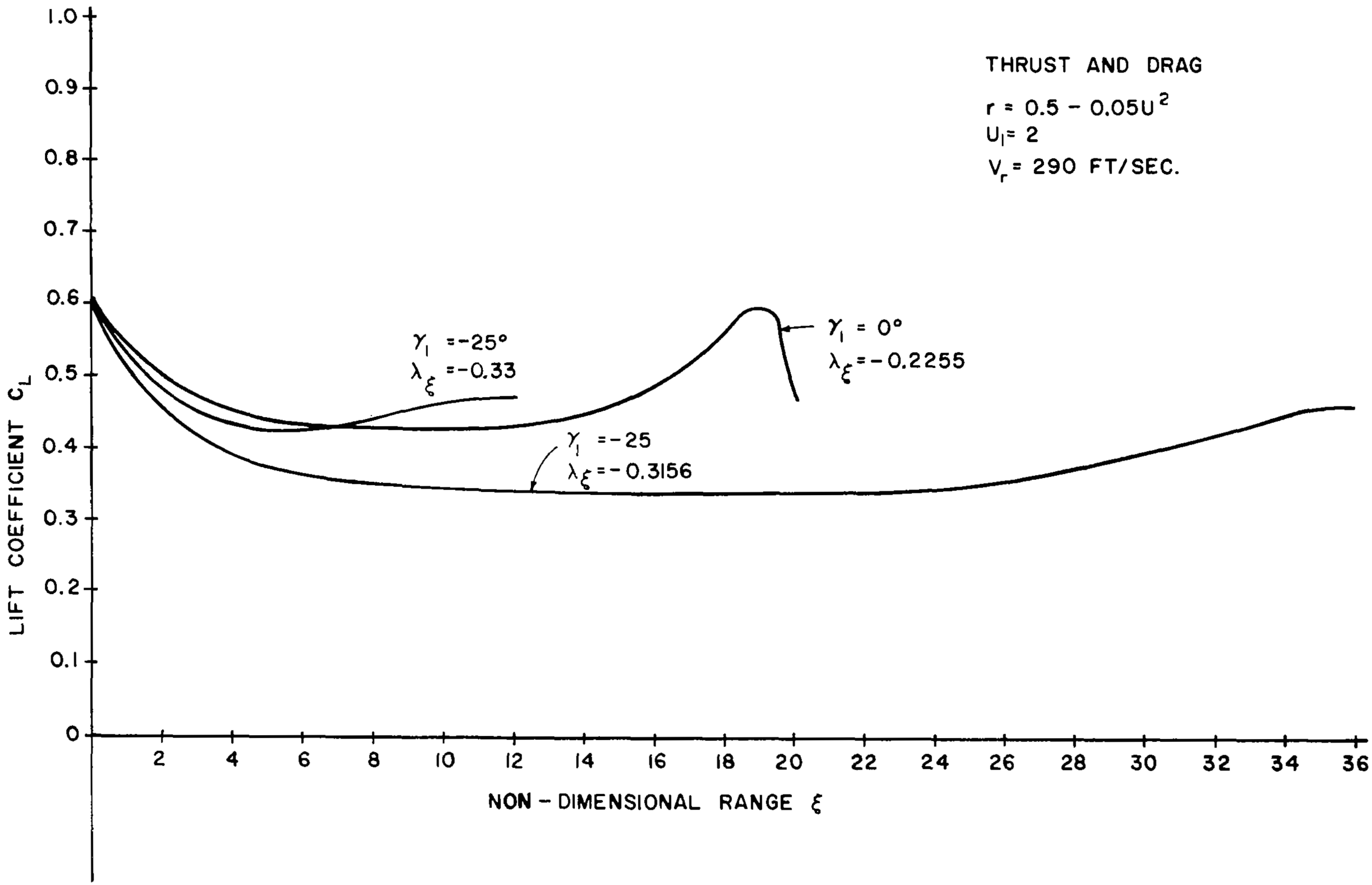


FIGURE 12 — LIFT COEFFICIENT (THRUST AND DRAG)

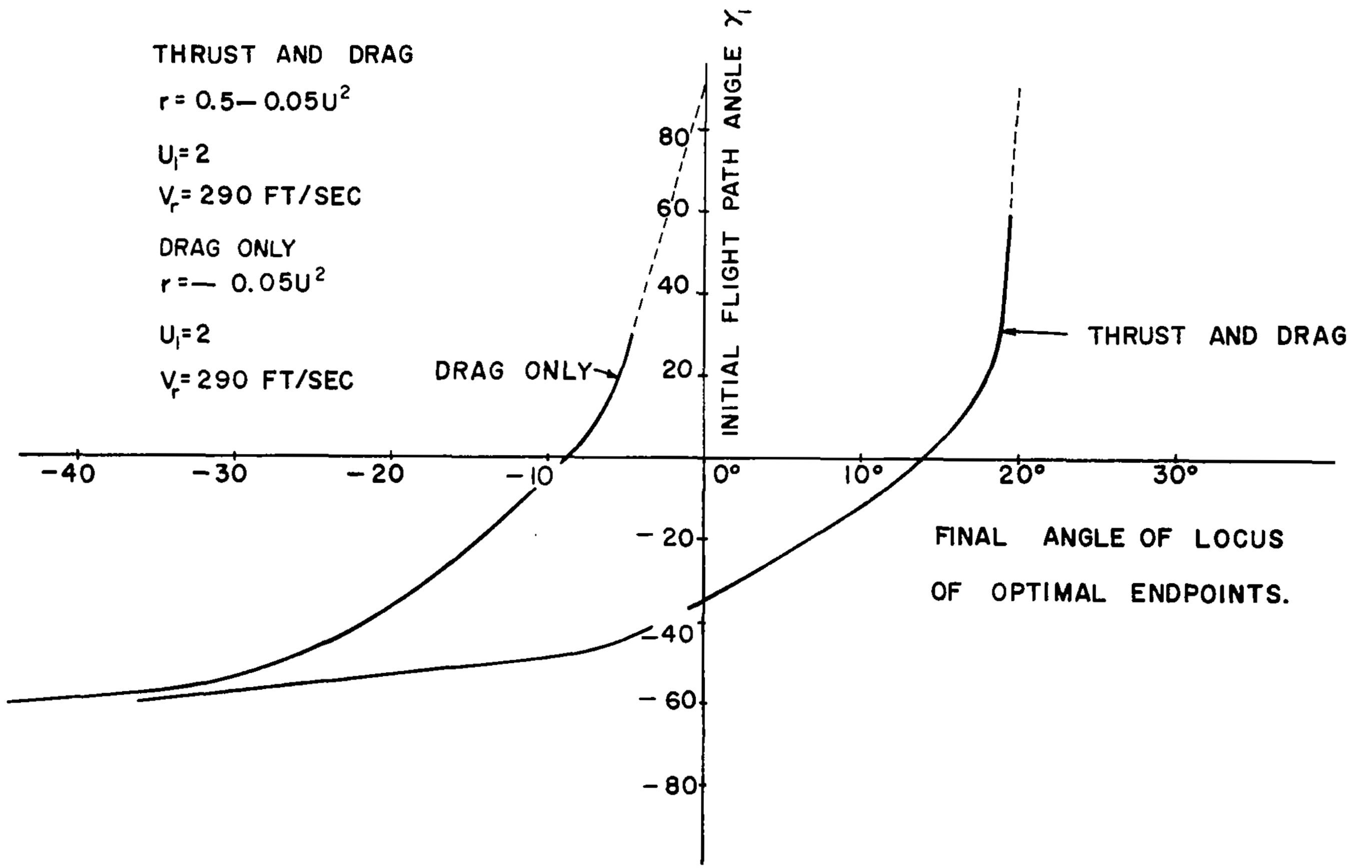


FIGURE 13 — FINAL LOCUS ANGLE AND INITIAL γ RELATION

$$\lambda_\xi = -\frac{\cos\gamma_1}{u_1} \quad , \tag{4.1}$$

represent the minimum value of λ_ξ for the thrust and drag case (with $r_1 > 0$) and the maximum value of λ_ξ for the drag case (with $r_1 < 0$). The number specified by Equation (4.1) is the only value for the ratio $\frac{\cos\gamma}{u}$ for the brachistochrone problem (with $r = 0$). In this respect the brachistochrone problem represents the border line case between trajectories flown with thrust greater than drag and vice versa. From Figures 5 and 9 or Figures 6 and 10, it is seen that for trajectories of short length, λ_ξ is very close in value to that given by Equation (4.1). A consequence of this situation is that trajectories of short length exhibit the characteristics of a brachistochrone solution in that the locus of optimal endpoints nearly coincides with the trajectory itself. As a result, a brachistochronic approximation can be used for generating short trajectories. This approximation is obtained by setting

$$\frac{\cos\gamma}{\lambda_\xi u} = -1 \tag{4.2}$$

in Equation (2.81) to give

$$\gamma' = \frac{\cos\gamma}{u} \tag{4.3}$$

which is independent of λ_ξ and is precisely the optimizing condition for the brachistochrone problem. When Equation (4.3) is used as the optimizing condition along with the equations of constraint (2.83), (2.84), and (2.85), a trajectory solution which passes through the desired endpoint sufficiently "close" to the origin will be a good approximation to the true optimal trajectory.

For a given initial flight path angle, longer flight trajectories are obtained for the drag-only case by decreasing the value of λ_ξ from

$$-\frac{\cos\gamma_1}{u_1}$$

and for the thrust and drag case by increasing the value of λ_ξ from

$$-\frac{\cos\gamma_1}{u_1} \quad .$$

In both cases the trajectory lengthens at an ever increasing rate as λ_ξ approaches some fixed number. For example, in Figure 9, in order to lengthen the $\gamma_1 = 0$ trajectory from the point $\xi = 21.0$, $\eta = 7.7$, to the point $\xi = 31.9$, $\eta = 10.5$, the multiplier λ_ξ must be changed only the small amount of 0.000021 (digital computer data). Beyond this point, the number of significant figures needed to specify λ_ξ increase at a fantastic rate. As a result it is simply not possible to solve for an optimum trajectory with an endpoint a great distance from the origin by directly integrating the optimizing equations using trial and error to determine γ_1 and λ_ξ. However, the fact that a good portion of an optimum trajectory to a distant point will be quasi-steady, can be used to construct an approximate trajectory to any distant point within the boundary of optimal solutions. This may be done by using Figure 9 to estimate the value of γ_1 and λ_ξ needed to reach the selected endpoint. The optimizing equation can then be integrated until the solution becomes quasi-steady. The steady portion of the solution can then be extended until a zoom using the brachistochronic approximation will extend the trajectory through the selected endpoint. The basis for using the brachistochronic approximation in the final zoom is that at the endpoint

$$\frac{\cos\gamma}{\lambda_\xi u} = -1 \quad .$$

By modifying the brachistochronic approximation through the introduction of an appropriate constant multiplying factor into Equation (3.1), trajectories of considerable length can be approximated. Although at the present time this method has not been used extensively, it appears that it may be of considerable aid when generating trajectories including constraints.

REFERENCES

1. Hestenes, M. R., "A General Problem in the Calculus of Variations with Applications to Paths of Least Time," Rand RM-100, March, 1950.

2. Garfinkel, Boras, "Minimal Problems in Airplane Performance," Quarterly of Applied Mathematics, Volume 9, No. 2, 1951.

3. Cicala, P., "An Engineering Approach to the Calculus of Variations," Levrotto and Bella, Torino, Italy, 1957.

4. Miele, Angelo, "Optimum Climbing Technique for a Rocket Powered Air-Craft," Jet Propulsion, Volume 25, pp. 385-391, August, 1955.

5. Leitmann, George, Optimization Techniques, New York, Academic Press, 1962. (See references, Chapter IV.)

6. Hancock, Harris, Theory of Maxima and Minima, New York, Dover, 1960.

7. Heermann, Hugo and Kretsinger, Phil, "The Minimum Time Problem," Journal of the Astronautical Sciences, Winter, 1964.

NASA-Langley, 1966 CR-499