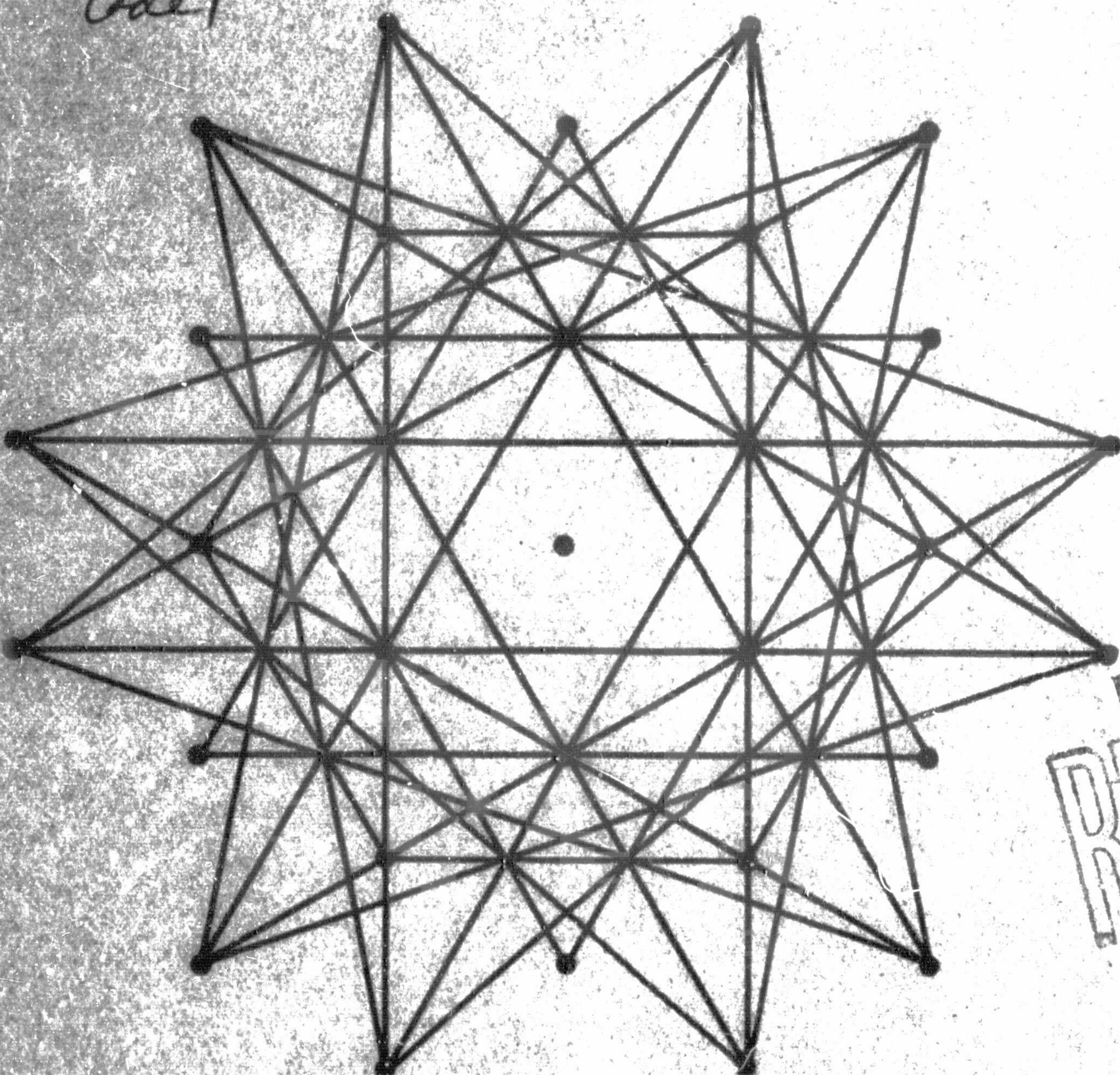
A PROBABILITY LAW OBTAINED BY COMPOUNDING THE POISSON AND HALF-NORMAL PROBABILITY LAWS

Michael F. Dacey



Geography Branch
Office of Naval Research
Task No. 389-142
Contract Nonr 1228(35)

Technical Report Number 4
Geographic Information Systems
Department of Geography
Northwestern University

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TECHNICAL REPORT NO. 4

of

ONR Task No. 389-142
Contract Nonr 1228(35)

OFFICE OF NAVAL RESEARCH

GEOGRAPHY BRANCH

Deparcment of Geography
Northwestern University
Evanston, Illinois

December 1965

FUNDING PROVIDED BY
Remote Sensing Program
Office of Space Sciences and Applications
National Aeronautics and Space Administration

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ABSTRACT

The discrete probability law considered in this report has the probability mass function

$$\pi(x;\ \sigma) = \sigma^{x}\sqrt{(2/\pi)}\,e^{\sigma^2/4}D_{-(x\,+\,1)}(\sigma) \qquad \begin{array}{l}\sigma > 0\\ x = 0,\ 1,\ 2,\ \ldots\end{array}$$

$$= \frac{2\sigma^{x}}{x!\sqrt{\pi}}\,(-1)^{x}\,\frac{d^{x}}{d\sigma^{x}}\,[e^{\sigma^2/2}\mathrm{Erfc}(\sigma/\sqrt{2})].$$

where $D_{\nu}(\cdot)$ is the parabolic cylinder function and $\mathrm{Erfc}(\cdot)$ is the complementary error function. This probability law is obtained by assuming that the parameter of a Poisson variable is distributed according to the half-normal probability law. A recursive relation for the individual probability terms and the rth factorial moment are obtained. A small table of probability terms is included.

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The discrete probability law considered in this report has the probability mass function

$$\pi(x; \sigma) = \sigma^x \surd(2/\pi) e^{\sigma^2/4} D_{-(x+1)}(\sigma), \qquad \sigma > 0, \; x = 0, 1, 2, \ldots$$

$$= 0, \qquad \text{elsewhere}$$

where

$$D_\nu(z) = \frac{e^{-z^2/4}}{\Gamma(-\nu)} \int_0^\infty e^{-zt - t^2/2} t^{-\nu - 1} dt \qquad \nu < 0$$

is the parabolic cylinder function. This probability law has the recursive property

$$\pi(x; \sigma) = \frac{\sigma^2}{x} [\pi(x - 2; \sigma) - \pi(x - 1; \sigma)]. \qquad x = 2, 3, \ldots$$

Expressions for the moments are obtained.

Derivation of the Probability Law

A large number of probability laws are obtained by compounding two well-known probability laws. An early derivation of this kind is Greenwood and Yules's compound Poisson-gamma distribution, which is a special case of the negative binomial distribution. In discussing this probability law Williamson and Bretherton [1963, p. 9] note that the choice of the gamma distribution function is arbitrary but state that "a positively skew distribution is preferred to the normal distribution." This comment provoked my curiosity about the compound Poisson-normal distribution, but I could not find this type of probability law mentioned in standard sources such as Haight [1961]. One interpretation of a compound Poisson-normal probability law is considered. The derivation is a rather obvious extension of the Poisson-gamma distribution and presents no mathematical complexity, so it is difficult to believe that the probability law has not been previously published.

A random variable X is assumed to follow the Poisson probability law with parameter m. This parameter is in turn treated as a random variable. It is assumed that $m = |y|$ where y has the normal distribution with mean zero and standard deviation σ or, what is the same thing, that m is a positive random variable having the half-normal distribution with parameter σ. In symbols

$$p(x;\ m) = e^{-m} m^{x}/x!, \qquad \begin{array}{l} m > 0 \\ x = 0,\ 1,\ 2,\ \ldots \end{array}$$

and

$$f(m;\ \sigma) = \frac{\sqrt{2}}{\sigma\sqrt{\pi}}\, e^{-m^2/2\sigma^2}. \qquad m,\ \sigma > 0$$

The resulting probability law is obtained by integrating the product of $p(x;\ m)$ and $f(m;\ \sigma)$ with respect to m, and the result is

$$\pi(x;\ \sigma) = \int_0^\infty p(x;\ m) f(m;\ \sigma)\,dm$$

$$= \frac{\sqrt{2}}{x!\,\sigma\sqrt{\pi}} \int_0^\infty e^{-m - m^2/2\sigma^2} m^{x}\,dm$$

$$= \sigma^x \sqrt{(2/\pi)}\, e^{\sigma^2/4} D_{-(x+1)}(\sigma).$$

All properties of the parabolic cylinder function required for this analysis are given in Volume II, Chapter 8 of the Bateman Manuscript Project [1953]. This function is tabulated by Miller [1964]; the $D_{-(x+1)}(z)$ is represented by $U(x + 1/2, z)$.

From the recursive formula

$$D_{\nu+1}(z) - zD_{\nu}(z) + \nu D_{\nu-1}(z) = 0,$$

it is possible to obtain

$$\pi(x;\sigma) = \frac{\sigma^2}{x}\,[\pi(x-2,\sigma) - \pi(x-1;\sigma)]. \qquad x = 2, 3, \ldots$$

The parabolic cylinder function of negative integer orders may be expressed in terms of the error function

$$D_{-(m+1)}(z) = \sqrt{2}\,\frac{(-1)^m}{m!}\, e^{-z^2/4}\,\frac{d^m}{dz^m}\,[e^{z^2/2}\mathrm{Erfc}(z/\sqrt{2})];$$

so, the probability law may also be expressed in terms of the error function

$$\pi(x;\sigma) = \frac{2\sigma^x}{x!\sqrt{\pi}}(-1)^x\,\frac{d^x}{d\sigma^x}\,[e^{\sigma^2/2}\mathrm{Erfc}(\sigma/\sqrt{2})].$$

Because the analysis of expressions containing $D_{\nu}(z)$ is relatively simple, the parabolic cylinder form is used.

Moments

It is convenient to obtain the factorial moments. The _r_th factorial moment is

$$\mu'_{[r]}(x;\,\sigma) = \sum_{x=0}^{\infty} x^{[r]}\pi(x;\,\sigma)$$

$$= \sqrt{(2/\pi)}e^{\sigma^2/4} \sum_{x=0}^{\infty} x^{[r]}\sigma^x D_{-(x+1)}(\sigma)$$

$$= \sqrt{(2/\pi)}e^{\sigma^2/4} \sum_{x=0}^{\infty} x^{[r]}\sigma^x \int_0^{\infty} \frac{e^{-\sigma^2/4}}{x!} e^{-\sigma t - t^2/2} t^x dt$$

$$= \sqrt{(2/\pi)} \int_0^{\infty} e^{-\sigma t - t^2/2} \left[\sum_{x=0}^{\infty} x^{[r]} \frac{(\sigma t)^x}{x!} \right] dt$$

$$= \sqrt{(2/\pi)} \int_0^{\infty} e^{-t^2/2 - \sigma t} (\sigma t)^r \left[\sum_{x=0}^{\infty} \frac{(\sigma t)^{x-r}}{(x-r)!} \right] dt$$

$$= \sqrt{(2/\pi)}\sigma^r \int_0^{\infty} e^{-t^2/2} t^r dt$$

$$= (2\sigma^2)^{r/2}\Gamma\left(\frac{r+1}{2}\right) \Big/ \Gamma(1/2).$$

Hence,

$$\mu'_{[1]}(x;\,\sigma) = \sigma\sqrt{(2/\pi)}, \qquad \mu'_{[2]}(x;\,\sigma) = \sigma^2,$$

$$\mu'_{[3]}(x;\,\sigma) = 2\sigma^3\sqrt{(2/\pi)}, \qquad \mu'_{[4]}(x;\,\sigma) = 3\sigma^4.$$

The mean and variance are

$$E(x) = \sigma\sqrt{(2/\pi)}, \quad V(x) = \sigma[\sqrt{(2/\pi)} + \sigma - 2\sigma/\pi],$$

respectively. The coefficient of variation is

$$\text{c.v.} = \frac{V(x)}{E(x)} = 1 + \sigma\left(\frac{\pi - 2}{\sqrt{(2\pi)}}\right) \doteq 1 + \sigma(.45542958).$$

Probability Terms

Individual probability terms are given in Table 1 for $\sigma = .1(.1)1(.5)2.5$. The $\pi(0; \sigma)$ and $\pi(1; \sigma)$ were computed from values tabulated by Miller; the $\pi(x; \sigma)$ for $x = 2, 3, \ldots,$ were computed from the recursive formula.

TABLE 1

INDIVIDUAL PROBABILITY TERMS OF $\pi(x;\ \sigma)$

	σ												
x	.1	.2	.3	.4	.5	.6	.7	.8	.9	1.0	1.5	2.0	2.5
0	9250	8584	7993	7466	6992	6567	6183	5835	5519	5232	4115	3362	2827
1	0705	1252	1674	1997	2241	2423	2556	2649	2710	2747	2708	2510	2281
2	0042	0147	0284	0437	0594	0746	0889	1020	1138	1242	1583	1705	1706
3	0002	0015	0042	0083	0137	0201	0272	0347	0425	0502	0844	1073	1197
4		0001	0005	0014	0029	0049	0075	0107	0144	0185	0416	0632	0795
5			0001	0002	0005	0011	0019	0031	0045	0063	0192	0352	0503
6					0001	0002	0004	0008	0013	0020	0084	0187	0304
7							0001	0002	0004	0006	0035	0095	0177
8									0001	0002	0014	0046	0099
9											0005	0021	0054
10											0002	0010	0029
11											0001	0004	0014
12												0002	0007
13													0003
14													0002
15													0001

Decimal points are left out.

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Haight, F. A. 1961. Index to the Distributions of Mathematical Statistics, Journal of Research, B, 65, 23-60.

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Williamson, E. and Bretherton, M. H. 1963. Tables of the Negative Binomial Distribution, New York, Wiley.

Unclassified
Security Classification

DOCUMENT CONTROL DATA - R&D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) Department of Geography, Northwestern University, Evanston, Illinois 60621	2a. REPORT SECURITY CLASSIFICATION Unclassified 2b. GROUP

3. REPORT TITLE

A PROBABILITY LAW OBTAINED BY COMPOUNDING THE POISSON AND HALF-NORMAL PROBABILITY LAWS

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)
Technical report

5. AUTHOR(S) (Last name, first name, initial)

Dacey, Michael F.

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
December 1965	7	4

8a. CONTRACT OR GRANT NO. Nonr 1228(35) b. PROJECT NO. c. Task No. 389-142 d.	9a. ORIGINATOR'S REPORT NUMBER(S) Technical Report No. 4 9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)

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