REPORT 1691-18

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ABSTRACT

The accuracies of two approximate formulations which are used in edge diffraction theory are checked. The formulations apply for the near-field diffraction of a cylindrical wave by a wedge. The approximate formulations are compared with the exact solution which can be evaluated in the region where the approximations need to be checked. The approximate formulations are found to be accurate, with errors which are generally less than a few per cent.

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ACCURACY OF APPROXIMATE FORMULATIONS FOR NEAR-FIELD WEDGE DIFFRACTION OF A LINE SOURCE

I. INTRODUCTION

A number of antenna diffraction problems have been treated in terms of diffraction by a perfectly conducting wedge.[1] These problems are treated by taking advantage of the property of wedge diffraction that the diffracted wave behaves as a cylindrical wave from the edge of the wedge. Thus a problem is formulated by superimposing the waves from each edge involved in the structural geometry.

In general either the diffraction by a plane wave or the far-field diffraction by a cylindrical wave is used in these problems. (The term "far field" is meant to imply that the distance to the observation point from the edge is large compared to that of the line source.) However, in certain problems, such as the coupling between waveguides,[2] the near-field diffraction by a line source is employed. An exact solution for the diffraction of a cylindrical wave by a wedge is available in the form of an eigenfunction expansion. However, the practical evaluation of this expansion is prohibitive for either the source or the observation distance larger than a wavelength or so. Two approximate formulations have been used for this solution. The purpose of this report is to show the accuracies of these approximate solutions.

II. CYLINDRICAL WAVE DIFFRACTION BY A WEDGE

The diffraction of a cylindrical wave by a wedge is illustrated in Fig. 1. The solution to the problem may be expressed in terms of a scalar function which represents the component of the electromagnetic field normal to the plane of study. The geometrical optics fields depicted in Fig. 2 are given by

$$(1) \qquad U_o = \frac{e^{-jkR}}{\sqrt{R}} = \frac{e^{-jk(r^2 + r_o^2 - 2r\, r_o \cos(\psi - \psi_o))^{\frac{1}{2}}}}{(r^2 + r_o^2 - 2r\, r_o \cos(\psi - \psi_o))^{\frac{1}{4}}}$$

incident region;

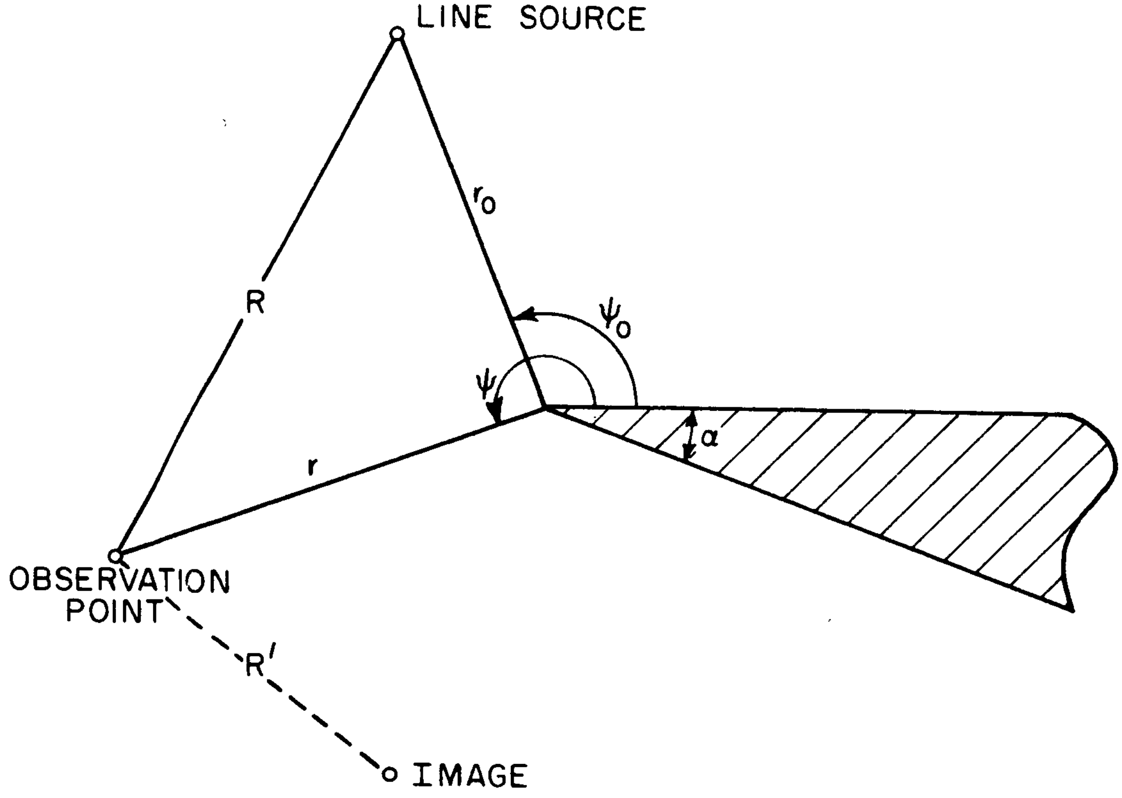

Fig. 1. Geometry of wedge and line source near field diffraction.

$$(2) \qquad U_o = \frac{e^{-jkR}}{\sqrt{R}} \pm \frac{e^{-jkR'}}{\sqrt{R'}}$$

$$= \frac{e^{-jk(r^2 + r_o^2 - 2r\, r_o \cos(\psi - \psi_o))^{\frac{1}{2}}}}{(r_o^2 + r_o^2 - 2r\, r_o \cos(\psi - \psi_o))^{\frac{1}{4}}}$$

$$\pm \frac{e^{-jk(r^2 + r_o^2 - 2r\, r_o \cos(\psi + \psi_o))^{\frac{1}{2}}}}{(r^2 + r_o^2 - 2r\, r_o \cos(\psi + \psi_o))^{\frac{1}{4}}}$$

incident and reflected region; and

$$(3) \qquad U_o = 0, \text{ shadow region.}$$

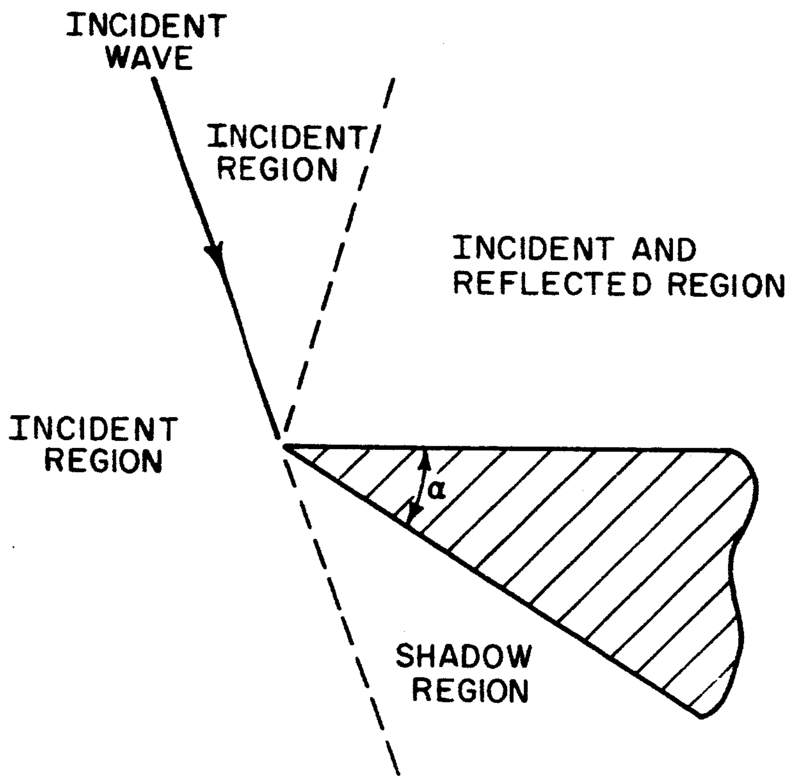

Fig. 2. Geometrical optics region.

The minus sign in Eq. (2) applies for the electric field polarization parallel to the edge and the plus sign applies for perpendicular polarization. R and R' are the distances of the line source and its image, respectively, to the observation point. The total field is given by

$$(4) \qquad U = U_o + U_d ,$$

where U_d is the diffracted field.

The nature of the diffracted wave U_d is that of a cylindrical wave radiating from the edge. One diffracted field formulation is obtained by modifying the solution given by Obha[3] for the diffraction of a half-plane illuminated by a dipole source. This solution has been reduced to the two-dimensional form and extended to wedge diffraction. The diffracted wave of this formulation is thus given by

$$(5) \qquad U_d(r, r_o, \psi, \psi_o) = \frac{e^{-jk(r+r_o)}}{\sqrt{r + r_o}} \; e^{jk \frac{r r_o}{r + r_o}}$$

$$\cdot \left[V_B\left(\frac{r r_o}{r + r_o}, \psi - \psi_o\right) \mp V_B\left(\frac{r r_o}{r + r_o}, \psi + \psi_o\right) \right] .$$

The same convention is used for the choice of signs in the diffracted field equations as in Eq. (2). The diffraction function, for plane wave diffraction introduced by Pauli[4] is given by

$$(6) \qquad V_B(r, \phi) = \frac{1}{\sqrt{\pi}} \; \frac{\sin \frac{\pi}{n} \, e^{j\frac{\pi}{4}}}{n} \; \frac{2\left|\cos \frac{\phi}{2}\right|}{\cos \frac{\pi}{n} - \cos \frac{\phi}{n}}$$

$$\times \, e^{jkr \cos \phi} \int_{(akr)^{\frac{1}{2}}}^{\infty} e^{-j\tau^2} \, d\tau$$

$$+ \left[\text{higher-order terms} \right] ,$$

where $a = 1 + \cos\phi$.

The other diffracted field formulation is obtained from Born and Wolf[5] for the diffraction of a cylindrical wave by a half-plane. This formulation is extended to wedge diffraction to give the diffracted wave as

$$(7) \qquad U_d(r, r_o, \psi, \psi_o) = \frac{e^{-jk\left[R + \frac{2r\,r_o}{R_1+R}\cos(\psi-\psi_o)\right]}}{\sqrt{\frac{R_1+R}{2}}} \times V_B\left(\frac{2r\,r_o}{R_1+R}, \psi-\psi_o\right)$$

$$\pm \frac{e^{-jk\left[R' + \frac{2r\,r_o}{R_1+R'}\cos(\psi+\psi_o)\right]}}{\sqrt{\frac{R_1+R'}{2}}} V_B\left(\frac{2r\,r_o}{R_1+R'}, \psi+\psi_o\right),$$

where $R_1 = r + r_o$.

Both formulations have been extended for non-zero wedge angles to agree with Pauli's formulation for plane wave diffraction ($r_o \to \infty$). It should be noted that both approximations are increasingly accurate for large distances to either the source or observation points.

The exact solution for diffraction of a cylindrical wave by a wedge as given by Harrington[6] is used to check the degree of approximation of the above formulations. The diffracted field, expressed as an eigenfunction solution is given by

$$(8) \qquad U_d(r, r_o, \psi, \psi_o) = \frac{\pi}{n} \frac{e^{-j\frac{\pi}{4}}}{\sqrt{\lambda}} \sum_{m=0}^{\infty} \epsilon_m H^{(2)}_{\frac{m}{n}}(kr_o)$$

$$\times J_{\frac{m}{n}}(kr)\left[\cos\left\{\frac{m}{n}(\psi - \psi_o)\right\} \pm \cos\left\{\frac{m}{n}(\psi + \psi_o)\right\}\right]$$

$$- \begin{Bmatrix} \frac{\pi}{\sqrt{\lambda}} e^{-j\frac{\pi}{4}} H_o^{(2)}(kr), & \phi > \pi \\ 0 \qquad\qquad , & \phi < \pi \end{Bmatrix},$$

for $r < r_o$, and where $\epsilon_\nu = 2$ for $\nu \neq 0$
$\qquad\qquad\qquad\qquad\qquad\qquad 1$ for $\nu = 0$.

The bracketed term corresponds to the geometrical optics term in Eqs. (1), (2), and (3).

The exact series expansion of Eq. (8) is generally impractical to compute. However, it can be computed for small arguments (r and r_o), which is the region in which the approximate formulation needs to be checked.

III. RESULTS

To check the accuracies of the approximate formulations, only one of the diffracted-field components needs to be computed as can be seen from Eqs. (5) and (7). Computations for Eqs. (5) and (7) are similar to those in previous publications. A Scatran subprogram for computing V_B may be found in Reference 7. The computation of Eq. (8) involves using known identities of Hankel and Bessel functions in a computer subprogram for calculating Bessel functions of arbitrary order (see Appendix).

Several representative values of r, r_o, and wedge angles were chosen. The results of the exact formulation, Eq. (8), are shown in Figs. 3-10. The accuracies of the approximate formulation, Eqs. (5) and (7) are tabulated in Tables 1 through 8 as deviations of the magnitude and phase from that of the exact formulation.

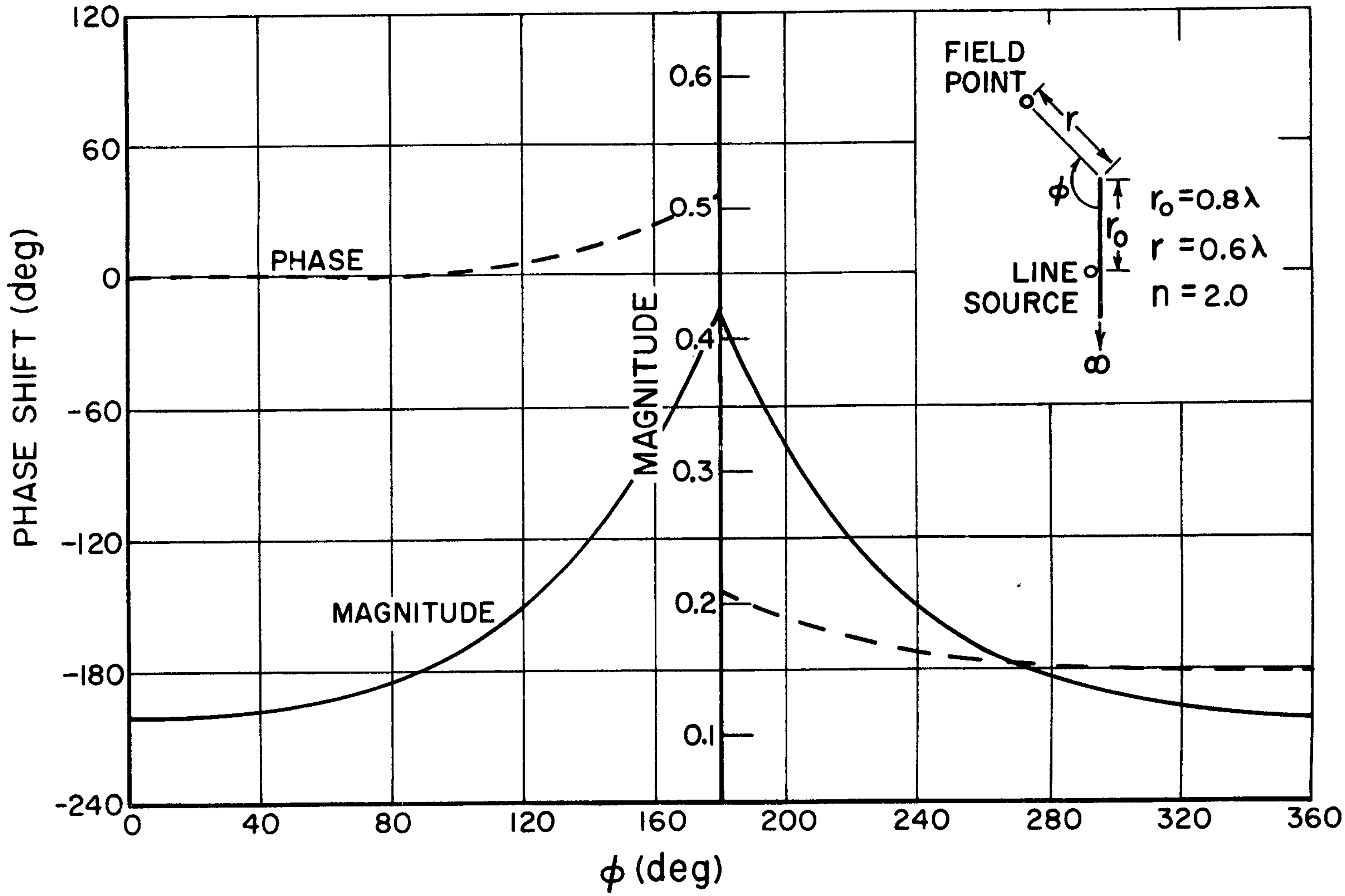

Fig. 3. Diffracted field U_d (exact formulation).

TABLE 1

$r_o = 0.8\lambda$ $\quad r = 0.6\lambda$ $\quad n = 2.0$

	U_d from Eq. (5)		U_d from Eq. (7)	
ϕ	Magnitude % Error	Phase Diff. in Degrees	Magnitude % Error	Phase Diff. in Degrees
0	-2.768	-1.635	-1.228	-3.961
20	-1.398	-0.787	0.103	-2.923
40	0.148	0.147	1.601	-1.677
60	-0.458	-0.300	0.943	-1.814
80	-1.271	-0.881	0.082	-2.085
100	-0.743	-0.762	0.537	-1.641
120	-0.193	-0.748	0.901	-1.292
140	-0.264	-1.026	0.470	-1.264
160	-0.052	-1.029	0.217	-1.072
180	-0.021	-0.817	-0.021	-0.817
200	0.023	-1.011	0.292	-1.055
220	-0.007	-0.936	0.729	-1.174
240	-0.496	-0.883	0.594	-1.427
260	-0.930	-0.858	0.347	-1.738
280	-0.633	-0.521	0.730	-1.725
300	-0.569	-0.366	0.831	-1.880
320	-1.015	-0.575	0.421	-2.399
340	-0.826	-0.428	0.684	-2.564

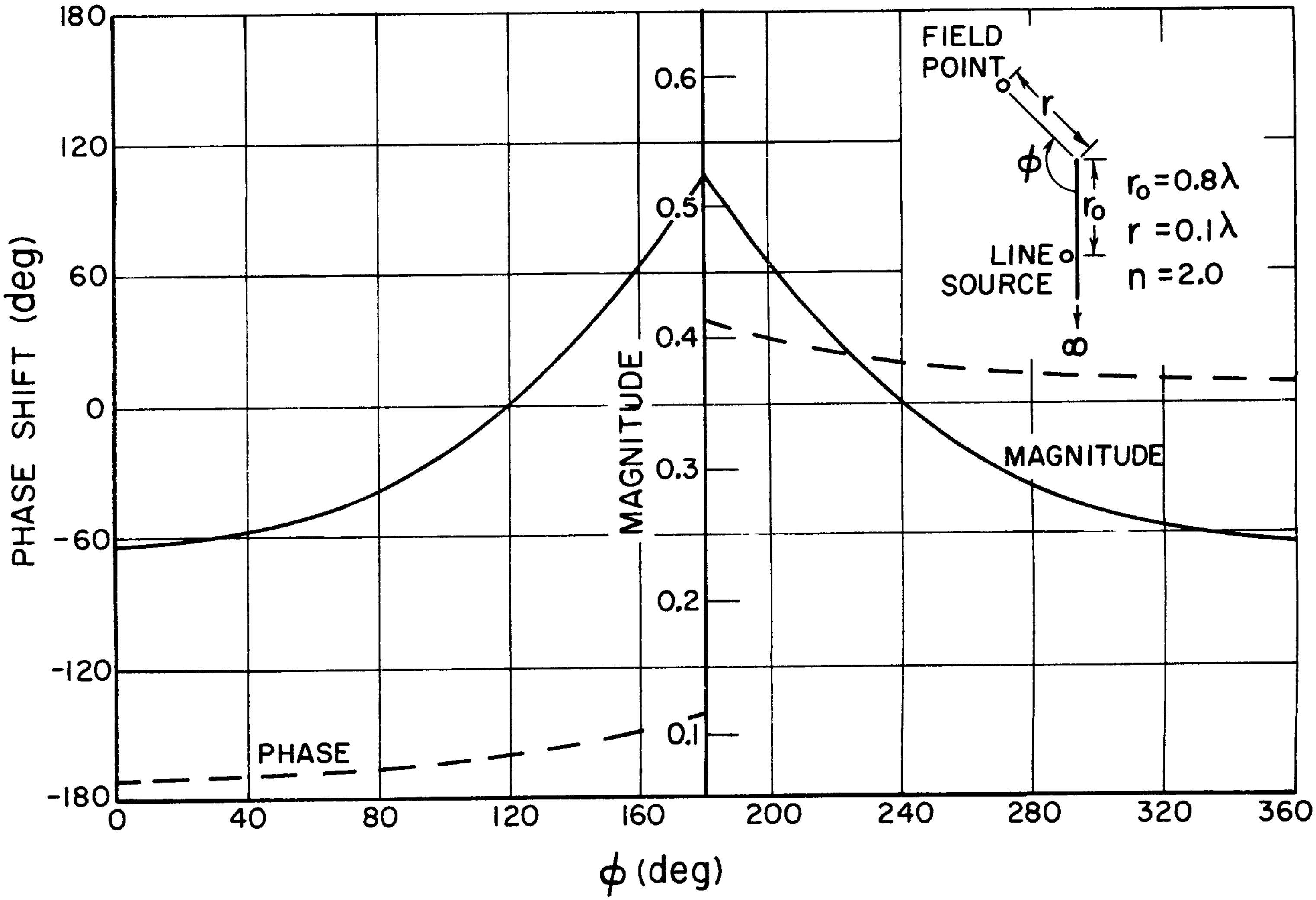

Fig. 4. Diffracted field U_d (exact formulation).

TABLE 2

$r_o = 0.8\lambda$ $r = 0.1\lambda$ $n = 2.0$

ϕ	U_d from Eq. (5) Magnitude % Error	U_d from Eq. (5) Phase Diff. in Degrees	U_d from Eq. (7) Magnitude % Error	U_d from Eq. (7) Phase Diff. in Degrees
0	-0.728	-1.378	1.091	-2.317
20	-0.708	-1.390	1.085	-2.298
40	-0.646	-1.422	1.068	-2.242
60	-0.538	-1.469	1.040	-2.154
80	-0.380	-1.521	0.996	-2.040
100	-0.174	-1.562	0.932	-1.905
120	0.059	-1.570	0.836	-1.755
140	0.267	-1.522	0.692	-1.591
160	0.354	-1.409	0.483	-1.420
180	0.187	-1.248	0.187	-1.248
200	0.354	-1.409	0.483	-1.420
220	0.267	-1.522	0.692	-1.591
240	0.059	-1.570	0.836	-1.755
260	-0.174	-1.562	0.932	-1.905
280	-0.380	-1.521	0.996	-2.040
300	-0.538	-1.469	1.040	-2.154
320	-0.646	-1.422	1.068	-2.242
340	-0.708	-1.390	1.085	-2.298

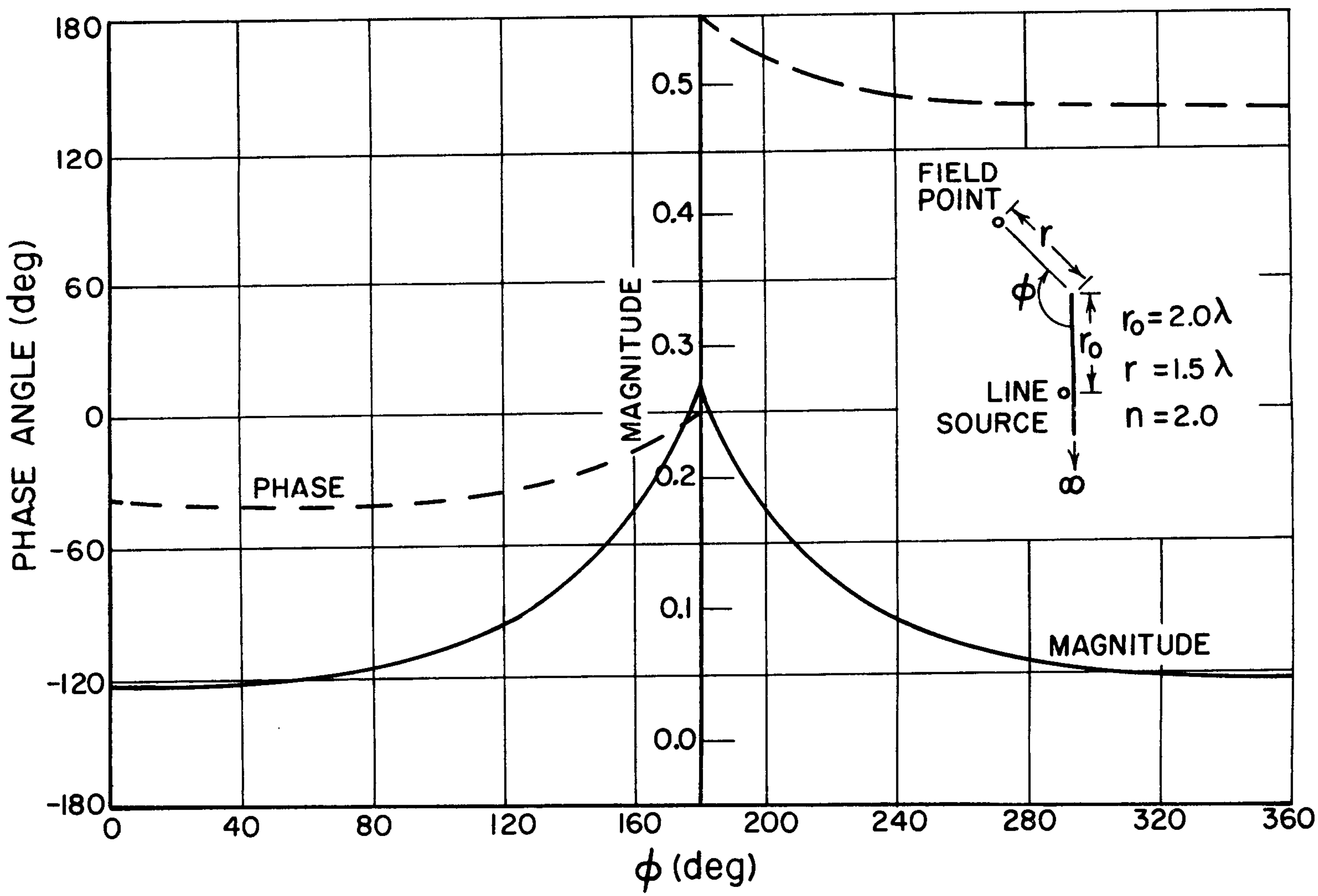

Fig. 5. Diffracted field U_d (exact formulation).

TABLE 3

$r_o = 2.0\lambda$ $r = 1.5\lambda$ $n = 2.0$

ϕ	U_d from Eq. (5) Magnitude % Error	U_d from Eq. (5) Phase Diff. in Degrees	U_d from Eq. (7) Magnitude % Error	U_d from Eq. (7) Phase Diff. in Degrees
0	-0.454	-2.354	-0.122	-3.444
20	-0.136	0.714	0.190	-0.299
40	-0.121	1.048	0.206	0.153
60	-0.256	-0.334	0.089	-1.120
80	-0.345	-0.684	0.039	-1.368
100	-0.311	-0.198	0.127	-0.769
120	-0.250	0.037	0.231	-0.390
140	-0.199	-0.218	0.238	-0.455
160	-0.089	-0.458	0.126	-0.513
180	0.083	-0.286	0.083	-0.286
200	-0.044	-0.403	0.172	-0.458
220	-0.286	-0.411	0.151	-0.647
240	-0.330	-0.262	0.151	-0.689
260	-0.285	-0.056	0.153	-0.627
280	-0.251	-0.021	0.133	-0.704
300	-0.236	-0.156	0.109	-0.942
320	-0.215	-0.180	0.112	-1.075
340	-0.185	0.009	0.140	-1.004

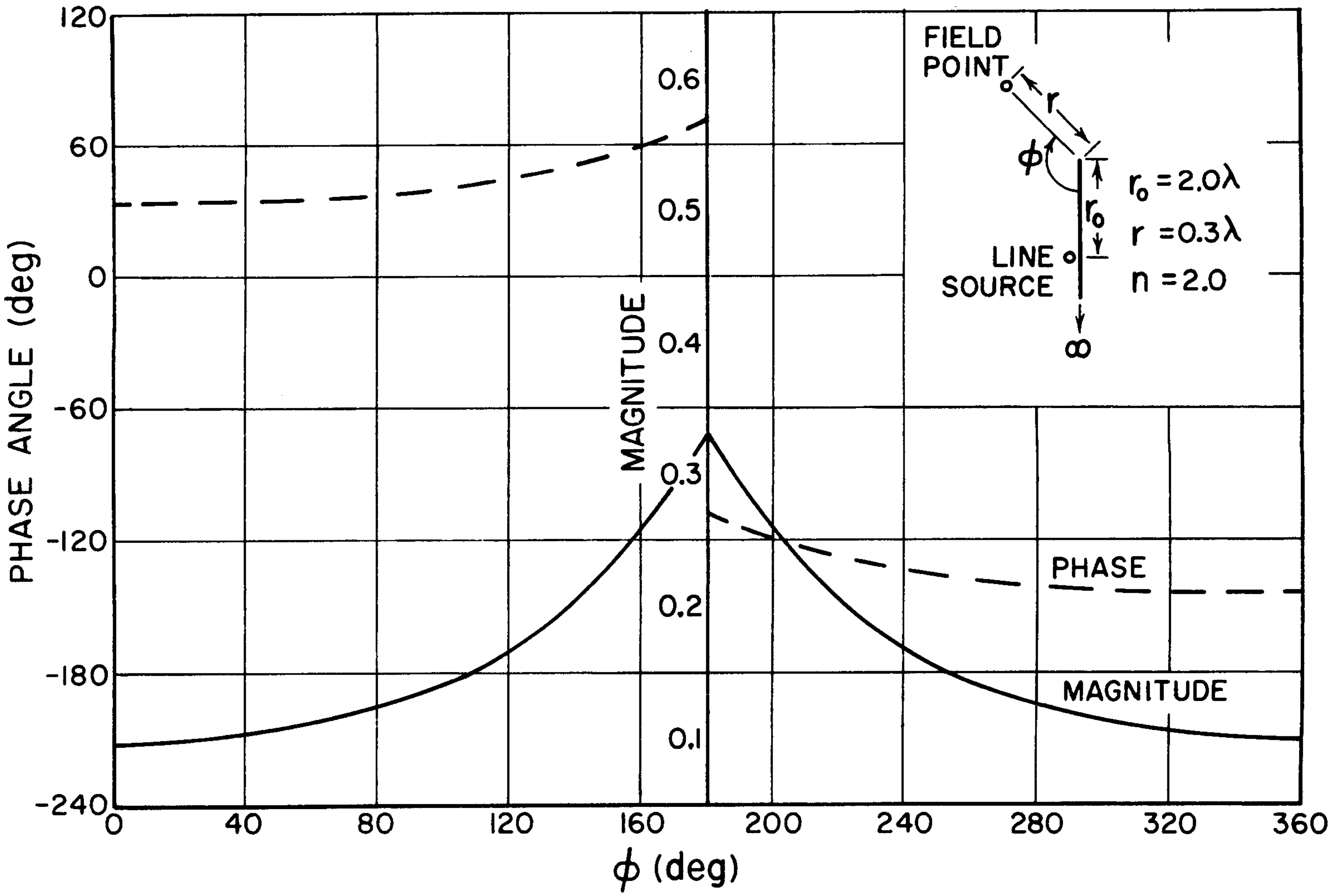

Fig. 6. Diffracted field U_d (exact formulation).

TABLE 4

$r_o = 2.0\lambda$ $r = 0.3\lambda$ $r = 2.0$

ϕ	U_d from Eq. (5) Magnitude % Error	U_d from Eq. (5) Phase Diff. in Degrees	U_d from Eq. (7) Magnitude % Error	U_d from Eq. (7) Phase Diff. in Degrees
0	-0.479	-0.285	0.273	-1.070
20	-0.479	-0.293	0.275	-1.062
40	-0.480	-0.317	0.279	-1.038
60	-0.474	-0.360	0.287	-1.001
80	-0.449	-0.421	0.298	-0.952
100	-0.386	-0.497	0.307	-0.891
120	-0.266	-0.574	0.306	-0.816
140	-0.091	-0.620	0.278	-0.723
160	0.068	-0.596	0.198	-0.614
180	0.030	-0.494	0.030	-0.494
200	0.068	-0.596	0.198	-0.614
220	-0.091	-0.620	0.278	-0.723
240	-0.266	-0.574	0.306	-0.815
260	-0.386	-0.497	0.307	-0.891
280	-0.449	-0.421	0.297	-0.952
300	-0.474	-0.360	0.287	-1.001
320	-0.480	-0.317	0.279	-1.038
340	-0.479	-0.293	0.275	-1.062

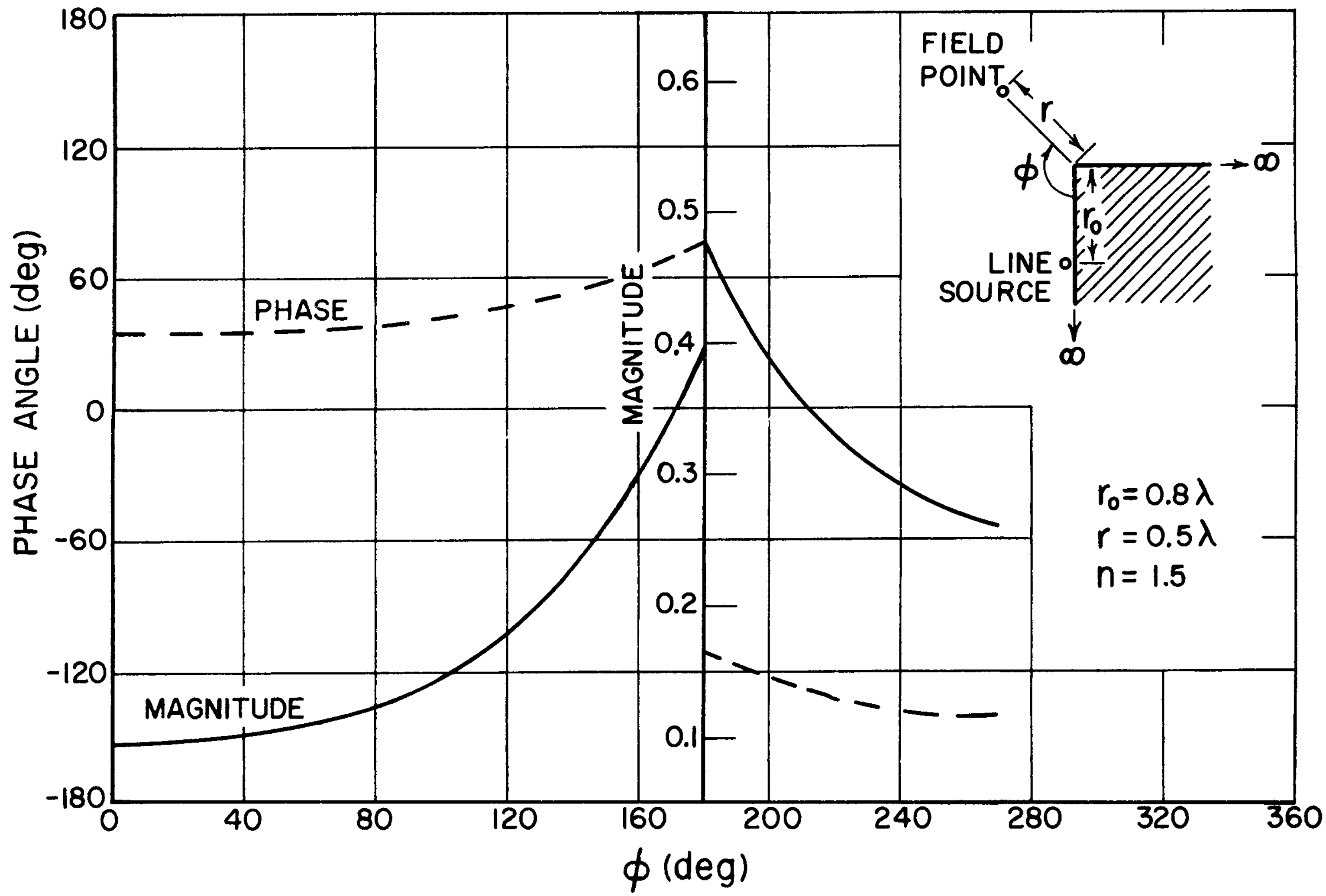

Fig. 7. Diffracted field U_d (exact formulation).

TABLE 5

$r_o = 0.8\lambda$ $\quad r = 0.5\lambda$ $\quad n = 1.5$

ϕ	U_d from Eq. (5) Magnitude % Error	U_d from Eq. (5) Phase Diff. in Degrees	U_d from Eq. (7) Magnitude % Error	U_d from Eq. (7) Phase Diff. in Degrees
0	-1.213	-0.528	0.641	-2.941
20	-0.836	-0.505	0.990	-2.781
40	-0.671	-0.540	1.108	-2.534
60	-0.991	-0.660	0.745	-2.336
80	-0.870	-0.783	0.814	-2.120
100	-0.635	-0.942	0.923	-1.914
120	-0.537	-1.112	0.746	-1.708
140	-0.146	-1.218	0.686	-1.478
160	0.230	-1.178	0.525	-1.228
180	0.068	-0.813	0.068	-0.813
200	0.051	-0.962	0.288	-0.997
220	-0.308	-0.989	0.333	-1.201
240	-0.572	-0.927	0.417	-1.445
260	-0.787	-0.869	0.566	-1.789

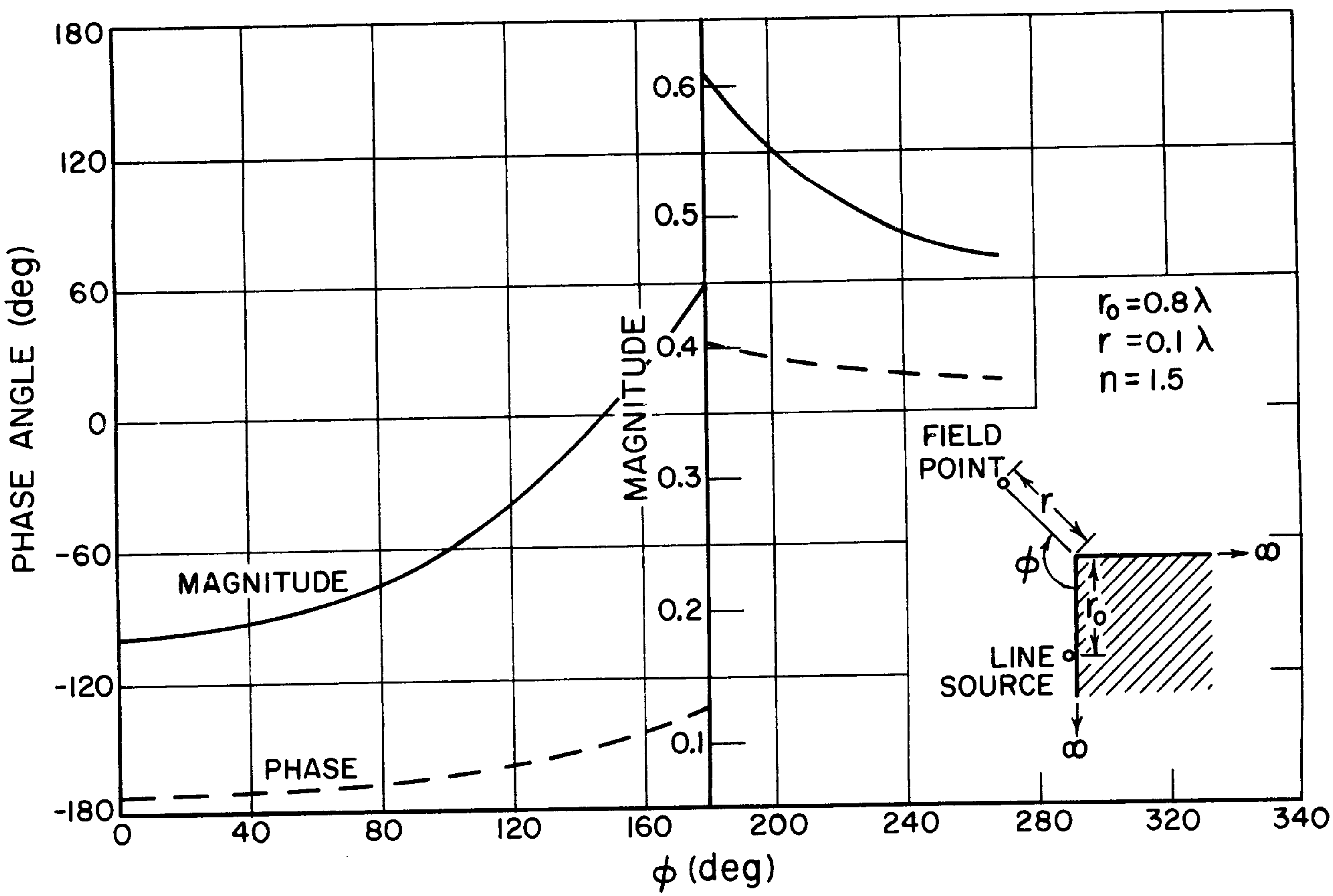

Fig. 8. Diffracted field U_d (exact formulation).

TABLE 6

$r_o = 0.8\lambda$ $r = 0.1\lambda$ $n = 1.5$

ϕ	U_d from Eq. (5) Magnitude % Error	U_d from Eq. (5) Phase Diff. in Degrees	U_d from Eq. (7) Magnitude % Error	U_d from Eq. (7) Phase Diff. in Degrees
0	-0.705	-1.471	1.275	-2.468
20	-0.680	-1.485	1.277	-2.449
40	-0.603	-1.522	1.282	-2.394
60	-0.471	-1.577	1.281	-2.307
80	-0.280	-1.637	1.264	-2.191
100	-0.035	-1.680	1.215	-2.049
120	0.235	-1.683	1.118	-1.883
140	0.472	-1.621	0.957	-1.696
160	0.570	-1.478	0.717	-1.490
180	0.040	-1.246	0.040	-1.246
200	0.165	-1.409	0.280	-1.420
220	0.085	-1.540	0.469	-1.615
240	-0.055	-1.621	0.682	-1.822
260	-0.150	-1.658	1.008	-2.029

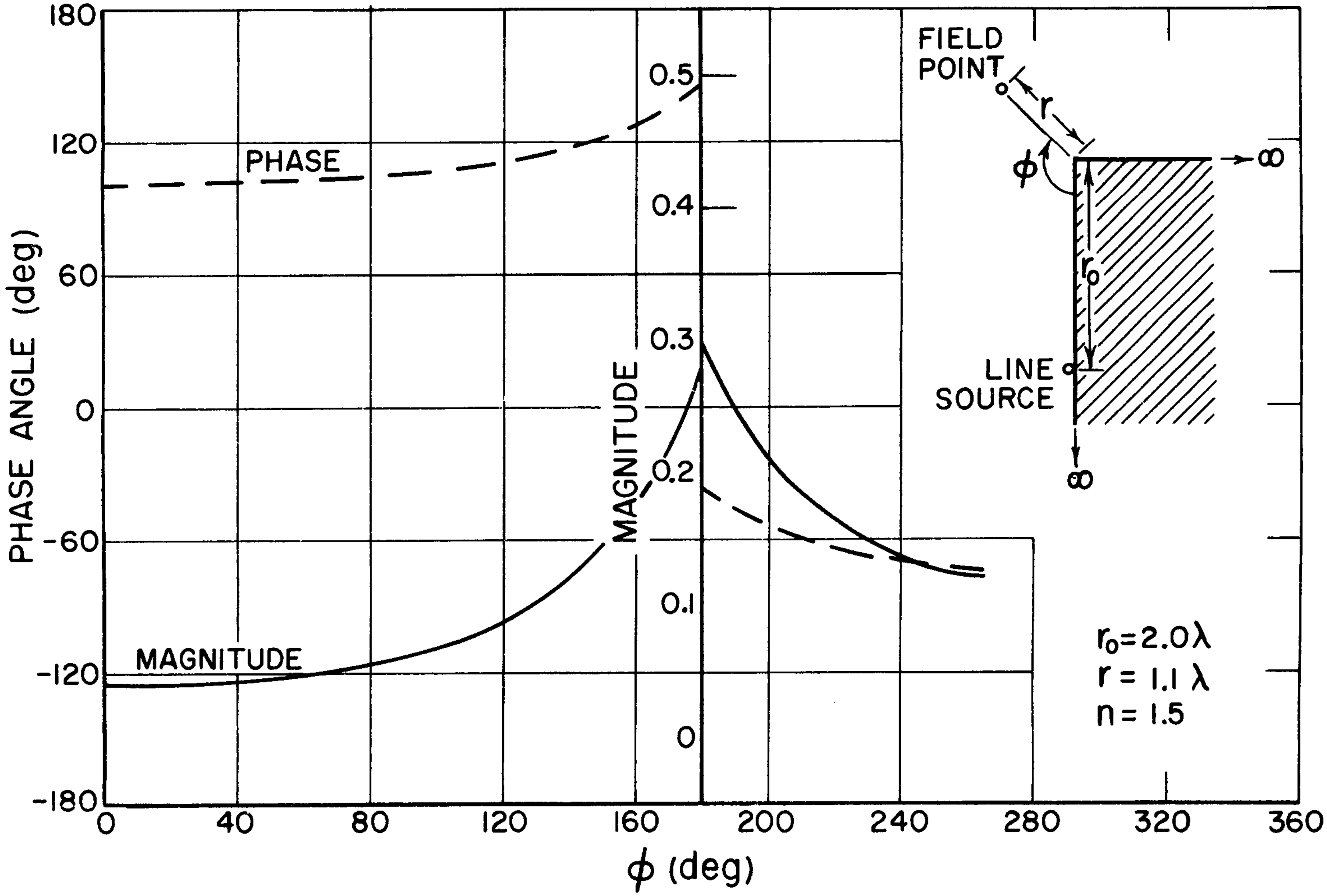

Fig. 9. Diffracted field U_d (exact formulation).

TABLE 7

$r_o = 2.0\lambda$ $\quad r = 1.1\lambda$ $\quad n = 1.5$

	U_d from Eq. (5)		U_d from Eq. (7)	
ϕ	Magnitude % Error	Phase Error in Degrees	Magnitude % Error	Phase Error in Degrees
0	-0.288	-0.040	0.199	-1.388
20	-0.291	-0.047	0.207	-1.375
40	-0.303	-0.064	0.162	-1.093
60	-0.328	-0.091	0.172	-1.018
80	-0.364	-0.137	0.189	-0.949
100	-0.398	-0.209	0.209	-0.880
120	-0.391	-0.317	0.231	-0.801
140	-0.277	-0.444	0.237	-0.699
160	-0.043	-0.503	0.187	-0.561
180	0.024	-0.345	0.024	-0.345
200	-0.030	-0.417	0.163	-0.460
220	-0.246	-0.350	0.156	-0.640
240	-0.349	-0.247	0.112	-0.640
260	-0.371	-0.187	0.131	-0.794

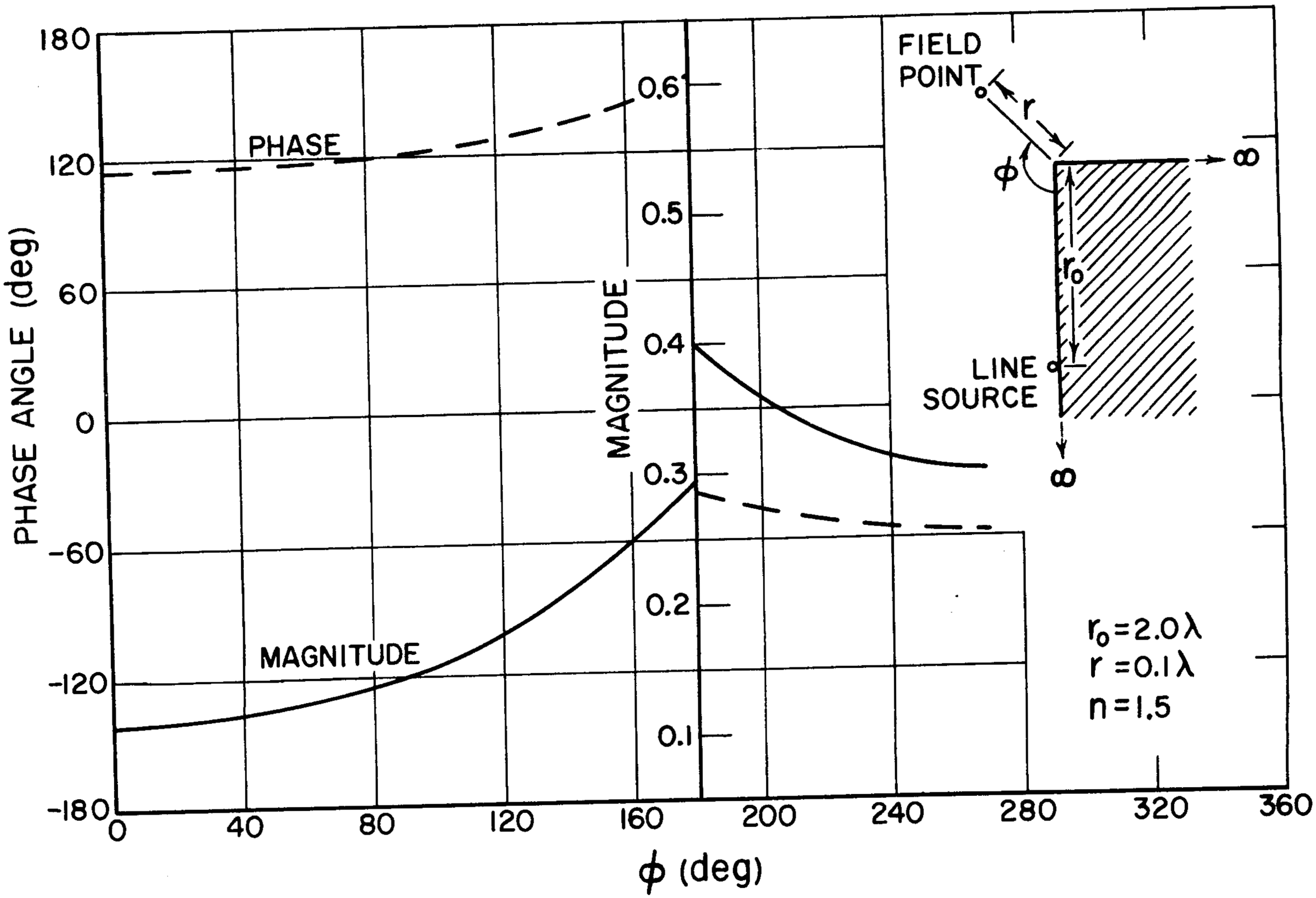

Fig. 10. Diffracted field U_d (exact formulation).

TABLE 8

$r_o = 2.0\lambda$ $r = 0.1\lambda$ $n = 1.5$

ϕ	U_d from Eq. (5) Magnitude % Error	U_d from Eq. (5) Phase Diff. in Degrees	U_d from Eq. (7) Magnitude % Error	U_d from Eq. (7) Phase Diff. in Degrees
0	-0.386	-0.611	0.412	-1.022
20	-0.376	-0.618	0.416	-1.017
40	-0.346	-0.637	0.425	-1.003
60	-0.291	-0.666	0.436	-0.978
80	-0.210	-0.699	0.443	-0.941
100	-0.102	-0.727	0.437	-0.891
120	0.022	-0.736	0.410	-0.827
140	0.136	-0.714	0.353	-0.749
160	0.191	-0.653	0.258	-0.659
180	-0.029	-0.535	-0.029	-0.535
200	0.018	-0.609	0.070	-0.614
220	-0.031	-0.666	0.141	-0.700
240	-0.104	-0.699	0.219	-0.790
260	-0.152	-0.713	0.347	-0.878

IV. CONCLUSIONS

Two approximate formulations have been used for the diffraction of a cylindrical wave by a wedge. In this report the accuracy of each formulation is checked by numerical comparison with the exact solution. The comparison can be made because the exact solution is practical to evaluate in the region where the approximate formulations have their largest errors. The accuracy of each of the approximate formulations is found to be quite good; the error is generally less than a few per cent.

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APPENDIX

The Scatran computer subprogram for the calculation of Bessel functions of any order is presented in this Appendix. Hankel functions of the second kind may be obtained by use of the following identity:

$$(9) \qquad H_{\nu}^{(2)}(x) = \frac{-j}{\sin \nu\pi} \left[e^{j\nu\pi} J_{\nu}(x) - J_{-\nu}(x) \right].$$

```
          SUBROUTINE (BESL)=BESSEL.(P,X)-
C         -
C         BESL IS THE BESSEL FUNCTION OF THE FIRST KIND, OF ORDER P,
                                     EVALUATED AT AN ARGUMENT OF X. -
C         -
          PRECISION (2,DPP,DPX,DPK,DPTEMP,DPBESL,DPCONS)-
          BESLX=X-
          BESLP=P-
          PROVIDED (X.GE.0.), TRANSFER TO (LEN)-
          BESL=0.-
          NORMAL EXIT -
C         GAMMA(Z)=(GAMMA (Z+1))/Z-
C         BESL(-N,X)=((-1).P.N)*BESL(N,X)-
  LEN     QIN=1.-
          PP=P+1.-
          PROVIDED (PP.G.0.), TRANSFER TO (BEGIN)-
          DEN=1.-
          PQ=PP-
  LENC    DEN=DEN*PQ-
          PQ=PQ+1.-
          PROVIDED (PQ.L.0.), TRANSFER TO (LENO)-
          PROVIDED (PQ.G.0.), TRANSFER TO (SX)-
          QIN=(-1.).P.(-P)-
          P=-P-
          TRANSFER TO (BEGIN)-
  SX      TEMP=(GAMMA.(PQ))/DEN-
          TRANSFER TO (NEGAM)-
```

```
BEGIN        TEMP=GAMMA.(P+1.)-
NEGAM        DPTEMP=TEMP-
             DPTEMP=1./DPTEMP-
             DPBESL=DPTEMP-
             DPP=P-
             DPX=X-
             DPCONS=(DPX/2.).P.DPP-
              DO THROUGH (LOOP),K=1,1, PROVIDED
                       (.ABS.(DPTEMP*DPCONS).G.1..X.-6)-
             DPK=K-
             DPTEMP=-DPTEMP*DPX*(1./DPK)*(1./(4.*(DPP+DPK)))*DPX-
              UNIVERSAL (CPT,BESLX,TEMP,BESLP,BESLL,QIN)-
              PROVIDED (K.LE.10),DPT(K)=DPTEMP-
             DPBESL=DPBESL+DPTEMP-
LOOP          CONTINUE -
             DPBESL=DPBESL*DPCONS-
             BESLL=DPBESL-
             BESL=BESLL*QIN-
              NORMAL EXIT -
              END SUBPROGRAM -
C            -
```