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A Class of Linear Functional Equations

by

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*This research was supported in part by National Aeronautics and Space Administration under Contract No. NGR 40-002-015, in part by the United States Army Research Office, Durham, under Contract No. DA-31-124-ARO-D-270 and in part by the Office of Aerospace Research, United States Air Force, under AFOSR Grant No. 693-66.

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This report is a summary of some unpublished results of K. Meyer, C. Perelló and the author concerning a class of autonomous linear functional equations which includes as special cases autonomous linear functional differential equations of retarded and neutral type as well as functional difference equations.

Let R^n be a real or complex n-dimensional linear vector space of column vectors with norm $|\cdot|$ and let $C_r([-r,0],R^n)$ be the Banach space of continuous functions mapping $[-r,0]$ into R^n with the norm $\|\varphi\|_r$ for φ in C_r defined by $\|\varphi\|_r = \max\{|\varphi(\theta)|,\ \theta \text{ in } [-r,0]\}$. If g, f are continuous linear mappings of C_r into R^n, then there exist $n \times n$ matrices μ, η whose elements are of bounded variation on $[-r,0]$ such that

$$
(1)\qquad
\begin{aligned}
g(\varphi) &= \int_{-r}^{0} [d\mu(\theta)]\varphi(\theta) \\
f(\varphi) &= \int_{-r}^{0} [d\eta(\theta)]\varphi(\theta)
\end{aligned}
$$

for all φ in C_r. We shall suppose that the measure μ is nonatomic at 0 and more specifically that there is a continuous nondecreasing function $\delta(s)$, $0 \leqq s \leqq r$, such that $\delta(0) = 0$ and

$$(2) \qquad \left| \int_{-s}^{0} [d\mu(\theta)]\varphi(\theta) \right| \leqq \delta(s)\|\varphi\|_s$$

for all φ in C_r.

For any φ in C_r, define $\gamma(\varphi) = \varphi(0)-g(\varphi)$. For any continuous function h mapping $[0,\infty)$ into R^n and any fixed element φ in C_r, consider the functional integral equation

$$(3) \qquad \begin{aligned} x(t) &= \varphi(t) \quad , \quad -r \leqq t \leqq 0 \, , \\ x(t) &= \gamma(\varphi)+g(x_t) + \int_0^t f(x_s)ds + \int_0^t h(s)ds \quad , \quad t \geqq 0, \end{aligned}$$

where, for each fixed $t \geqq 0$, x_t is in C_r and is defined by $x_t(\theta) = x(t+\theta)$, $-r \leqq \theta \leqq 0$. By a solution of (3), we will always mean a continuous function satisfying the above relation.

For $g \equiv 0$, equation (3) is equivalent to the functional differential equation of retarded type

$$(4) \qquad \dot{x}(t) = f(x_t) + h(t)$$

with the initial condition at $t = 0$ given by φ. If $f \equiv 0$ and $h \equiv 0$, equation (3) is a functional difference equation of retarded type and, in particular, includes difference equations. For both f and g not identically zero, equation (3) corresponds to a retarded equation of neutral type. In fact, formal differentiation of the equation yields

$$\dot{x}(t) = g(\dot{x}_t) + f(x_t) + \dot{h}(t) \tag{5}$$

where $\dot{x}_t$ is defined as $\dot{x}_t(\theta) = \dot{x}(t+\theta)$, $-r \leqq \theta \leqq 0$. Also, if one begins with (5) and defines a solution with initial function φ at 0 to be a continuous function satisfying (5) almost everywhere, then an integration yields (3) with $\gamma(\varphi) = \varphi(0)-g(\varphi)$.

This latter remark is precisely the reason for considering the equation (3) rather than (5). If one attempts to discuss equation (5) directly, then the first problem that is encountered is a precise definition of a solution and a precise definition of the topology to be induced on the space in which the solution will lie. Such a topology will necessarily include the first derivative of x in some way; whereas, if we consider equation (3), the simpler space C_r can be employed.

If h in (3) is identically zero, we will say equation (3) is homogeneous and, otherwise, it is nonhomogeneous.

<u>THEOREM 1</u>. For any given φ in C_r, there is a unique function $x(\varphi)$ defined and continuous on $[-r,\infty)$ such that $x(\varphi)$ satisfies (3) on

$[0,\infty)$. Furthermore, there is a constant $\beta > 0$ such that

(6) $$\|x_t(\varphi)\| \leq e^{\beta t}\,[\|\varphi\| + \int_0^t |h(s)|\,ds]\ , \quad t \geq 0.$$

This theorem is proved by using the nonatomic property of μ at 0 together with the contraction mapping principle to first show that (3) has a solution on a small interval to the right of $t = 0$. An application of a result on the continuation of the solution then allows one to obtain the estimate (6) for $t \geq 0$.

If h is identically zero and $x = x(\varphi)$ is a solution of the homogeneous equation

(7) $$\begin{aligned} x_0 &= \varphi \\ x(t) &= \gamma(\varphi) + g(x_t) + \int_0^t f(x_s)\,ds\ , \quad t \geq 0, \end{aligned}$$

then it follows from the uniqueness of the solution that $x_t(\varphi)$ is a continuous linear mapping of C_r into C_r for each fixed $t \geq 0$, and $x_t(\varphi)$ satisfies the semigroup property. If we define the linear operator $T(t)$ by

(8) $$x_t(\varphi) \overset{\text{def}}{=} T(t)\varphi\ , \quad t \geq 0\ ,$$

then we can prove

THEOREM 2. The family of linear operators $\{T(t),\ t \geqq 0\}$ mapping C_r into C_r is a strongly continuous semigroup on $[0,\infty)$ with $T(0) = I$. If, in addition the function $\delta(s)$ in (2) satisfies $\lim_{s\to 0}\delta(s)/s = 0$, then the infinitesimal generator A of $T(t)$ is given by

$$(A\varphi)(\theta) = \begin{cases} \dot\varphi(\theta) , & -r \leqq \theta < 0 , \\ g(\dot\varphi) + f(\varphi) , & \theta = 0 \end{cases}$$

and the domain of A, $\mathscr{D}(A)$ consists of all functions φ in C_r with a continuous first derivative and $\dot\varphi(0) = g(\dot\varphi) + f(\varphi)$.

It is interesting to note that if φ is in $\mathscr{D}(A)$, then $T(t)\varphi$ is actually a continuously differentiable solution of the functional differential equation (5) with $h \equiv 0$.

It is easy to show that the spectrum of A, $\sigma(A)$, consists of only point spectrum and that λ is in $\sigma(A)$ if and only if λ satisfies the characteristic equation

$$\det \Delta(\lambda) = 0 \quad , \quad \Delta(\lambda) = \lambda I - \int_{-r}^{0} \lambda e^{\lambda\theta} d\mu(\theta) - \int_{-r}^{0} e^{\lambda\theta} d\eta(\theta) . \tag{9}$$

Also, because A is a closed operator and a root λ_0 of (8) has finite multiplicity, one can show that the resolvent operator $(A-\lambda I)^{-1}$ has a pole of finite order at λ_o and, thus, the generalized eigenspace of λ_o has finite dimension. If $\mathfrak{N}(A)$ and $\mathscr{R}(A)$ denote, respectively, the null space and range of an operator A and the generalized eigenspace of λ_o is given by $\mathfrak{N}(A-\lambda_o I)^k$, then it can be shown that the space C_r is decomposed as a direct sum of the subspaces $P \overset{\text{def}}{=} \mathfrak{N}(A-\lambda_o I)^k$, $Q \overset{\text{def}}{=} \mathscr{R}(A-\lambda_o I)^k$ each of which is invariant under both A and $T(t)$, $t \geqq 0$. When C_r is decomposed in this way, we shall say C_r is decomposed by λ_o as $C_r = P \oplus Q$ and write any element φ in C_r as $\varphi = \varphi^P + \varphi^Q$, φ^P in P, φ^Q in Q. If Φ is a basis for P, then there is a matric B such that $A\Phi = \Phi B$ and, thus, $\Phi(\theta) = \Phi(0)e^{B\theta}$, $-r \leqq \theta \leqq 0$. Also, one easily shows that $T(t)\Phi(\theta) = \Phi(0)e^{B(t+\theta)}$, $-r \leqq \theta \leqq 0$, which implies that the solutions of (3) on the generalized eigenspace of a solution of (9) can be defined on $(-\infty,\infty)$ and that the action of the semigroup $T(t)$ on this subspace is essentially the same as an ordinary differential equation. The decomposition outlined here plays the same role as the Jordan canonical form in ordinary differential equations.

In the applications, it is necessary to have an explicit representation for the projection operator E_{λ_o} associated with the above decomposition. This can be obtained from the formula

$$E_{\lambda_o}\varphi = \frac{1}{2\pi i}\int_c (A-\lambda I)^{-1}\varphi \, d\lambda$$

where c is a circle in the complex plane which contains no point in

$\sigma(A)$ except λ_o. As in the case of retarded functional differential equations (see [1] or [2]) the projection operator E_{λ_o} can also be obtained in the following way. Let $R^{n'}$ be the n-dimensional real or complex space of row vectors and define the operator A^* with $\mathscr{D}(A^*) \subset C([0,r],R^n)$ given by all ψ in $C([0,r],R^n)$ which are continuously differentiable with $\dot{\psi}(0) = \int_{-r}^{0} \dot{\psi}(-\theta)d\mu(\theta) - \int_{-r}^{0} \psi(-\theta)d\eta(\theta)$ and for ψ in $\mathscr{D}(A^*)$,

$$(A^*\psi)(s) = \begin{cases} -\dot{\psi}(s) & \text{for } 0 < s \leqq r\,, \\ -\int_{r}^{0} \dot{\psi}(-\theta)d\mu(\theta) + \int_{-r}^{0} \psi(-\theta)d\eta(\theta) - & \text{for } s = 0. \end{cases}$$

For any ψ in $C([0,r],R^{n'})$, $\dot{\psi}$ continuous, and any φ in $C([-r,0],R^n)$, define

$$(\psi,\varphi) = \psi(0)\varphi(0) - \int_{-r}^{0} [\, \frac{d}{d\zeta} \int_{0}^{\zeta} \psi(s-\zeta)d\mu(\theta)\varphi(s)ds]_{\zeta=\theta} - \int_{-r}^{0}\int_{0}^{\theta} \psi(s-\theta)d\eta(\theta)\varphi(s)ds.$$

For ψ in $\mathscr{D}(A^*)$, φ in $\mathscr{D}(A)$, if follows that $(\psi,A\varphi) = (A^*\psi,\varphi)$. To obtain the projection operator E_{λ_o}, one proceeds as follows: if we let $\Phi = (\varphi_1,\ldots\varphi_p)$ be a basis for the generalized eigenspace $P = \mathfrak{N}(A-\lambda_o I)^k$ of λ_o and let $\Psi = \mathrm{col}(\psi_1,\ldots,\psi_p)$ be a basis for $\mathfrak{N}(A^*-\lambda_o I)^k$, then $E_{\lambda_o}\varphi = \Phi(\Psi,\varphi)$ for all φ in $C([-r,0],R^n)$.

Another important relation in ordinary differential equations is the variation of constants formula. By using the fact the solution

$x^*(t,h)$ of (3) with $\varphi = 0$ is a continuous linear mapping of $C([0,t],R^n)$ into R^n and the Reisz representation theorem, it follows that

$$x^*(t,f) = \int_0^t [d_s W(t,s)]h(s)$$

where $W(t,t) = 0$, $W(t,s)$ is of bounded variation in s for s in $[0,t]$ and $W(t,s)$ is continuous from the right in s for s in $(0,t)$. One can also show that $W(t,s)$ is continuous from the right at $s = 0$, $W(t,s) = W(t-s,0)$ and $-W(t,0)$ is the matrix solution of (3) with $\varphi = 0$ and f equal to the identity matrix. Because equation (3) is linear, it follows that the solution $x = x(\varphi)$ satisfies

$$x(t) = [T(t)\varphi](0) + \int_0^t [d_s V(t-s)]h(s)$$

where we have defined $V(t) = W(t,0)$. Using the fact that $V(t) = 0$ for $-r \leqq t \leqq 0$, we also obtain

$$x_t(\theta) = [T(t)\varphi](\theta) + \int_0^t d_s V_{t-s}(\theta)]f(s) \ , \ -r \leqq \theta \leqq 0 \ ,$$

which can be written more compactly as

$$x_t = T(t)\varphi + \int_0^t [d_s V_{t-s}]h(s) \ . \tag{10}$$

Equation (10) is called the <u>variation of constants formula</u> for (3).

If λ_o is a solution of (8) and C is decomposed by λ_o as $P \oplus Q$, then it can also be shown that

$$x_t^P = T(t)\varphi^P + \int_o^t [d_s V_{t-s}^P]h(s) ,$$

$$x_t^Q = T(t)\varphi^Q + \int_o^t [d_s V_{t-s}^Q]h(s) .$$

If g is identically zero in (3), then we have seen that equation (3) is equivalent to (4) and the variation of constants formula (10) can be written as

$$x_t = T(t)\varphi + \int_o^t T(t-s)K_o f(s)ds$$

where $T(t)K_o$ is the solution of (4) with initial value at 0 given by $K_o(\theta) = 0$ for $-r \leq \theta < 0$, $K_o(0) = I$, the identity matrix. This is the standard manner of writing the variation of constants formula for (4) as given in [3] and [4]. For the equation of neutral type; i.e., f,g not identically zero, this formula also coincides with the one given for some special cases in [3].

To apply these results in special applications, it is necessary to obtain precise estimates of $T(t)\varphi^Q$ and V_t^Q. We are in the process of obtaining these estimates by using the general representation theorems for a semigroup in terms of the inverse Laplace transform of the resolvent of the infinitesimal generator. An easier way would be to prove that

$$\sigma(T(t)) = e^{\overline{t\sigma(A)}} + \{0\} ,$$

a fact which seems to be true for the particular A given above.

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