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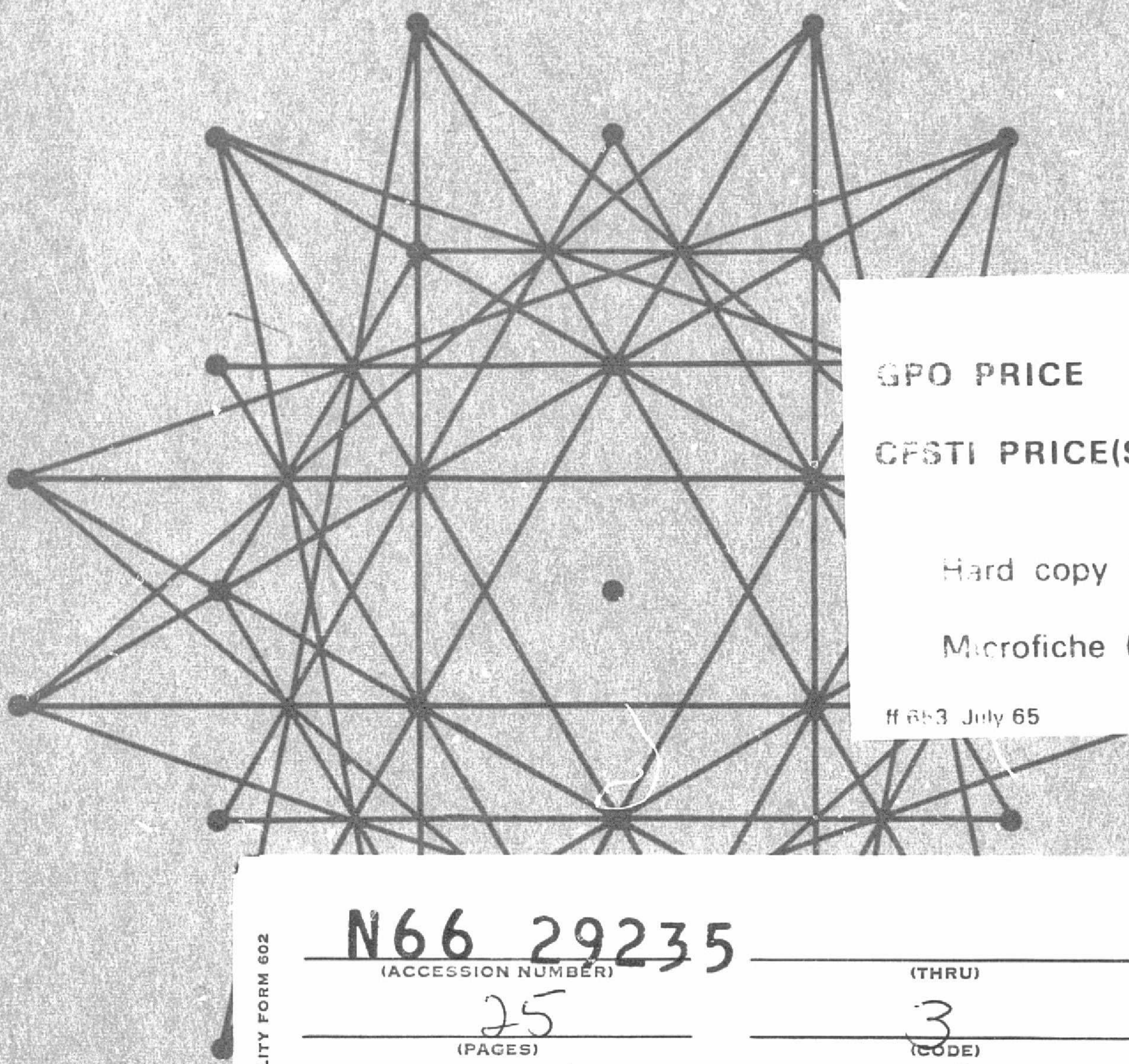
MEASURES OF DISTANCE FROM A RANDOMLY LOCATED POINT TO NEIGHBORING LATTICE POINTS FOR RECTANGULAR AND HEXAGONAL POINT LATTICES

Michael F. Dacey



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TECHNICAL REPORT NO. 3

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OFFICE OF NAVAL RESEARCH

GEOGRAPHY BRANCH

Department of Geography
Northwestern University
Evanston, Illinois

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ABSTRACT

This report lists the probability density function and distribution function of distance from a point randomly located in a polygon to the *i*th nearest corner for the following polygons: square, diamond, and rectangle. Also, low order moments and moment constants are tabulated for these order distances. The three lowest orders of distance for the square and the diamond pertain identically to the three lowest orders of distance for the square and hexagonal lattices, and the two lowest orders of distance for the rectangle apply identically to the two lowest orders of distances for the primitive rectangular lattice.

MEASURES OF DISTANCE FROM A RANDOMLY LOCATED POINT TO NEIGHBORING LATTICE POINTS FOR RECTANGULAR AND HEXAGONAL POINT LATTICES

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Some geographic location problems may be formulated, for a first crude approximation, in the context of (i) distance from a point randomly located in a polygon to the ith nearest corner of that polygon or (ii) distance from a point randomly located in a two-dimensional point lattice to the ith nearest lattice point. The formulations (i) for the square and isosceles triangle are identical to the formulations (ii) for, respectively, the square and hexagonal point lattices where $i = 1, 2, 3$, and the formulation (i) for the rectangle is identical to the formulation (ii) for the primitive rectangular point lattice where $i = 1, 2$. There has been need for properties of several of these distances. Because of certain relations obtaining between mathematical properties of these problems, a systematic approach readily yielded numerical values for the collection of distances.

This report lists the probability density function and distribution function of distance from a point randomly located in a polygon to the ith nearest corner for the following polygons: square, diamond (which includes the isosceles triangle as a special case), and rectangle. Also, low order moments and moment constants are tabulated for these order distances. The three lowest orders of distance for the square and the diamond pertain identically to the three lowest orders of distance for the square and hexagonal lattices, and the two lowest orders of distance for the rectangle apply identically to the two lowest orders of distances for the primitive rectangular lattice.

Since this work was initiated some results have been located. Persson [1964] reported the density function, distribution function, mean and variance for the square lattice. Essed [1957] used numerical methods to obtain means and variances. Holgate [1965] gave properties of first order distance for the hexagonal lattice. For sake of completeness, all results are repeated here.

While the derivation of properties is algebraically tedious, only elementary principles of geometric probabilities are required to obtain the density functions of order distance. Accordingly, derivations are omitted and only results are reported.

REFERENCES

Essed, F. E. 1957. Estimation of standing timber, Medelingen van de Landbouwhogeschool te Wageningen, 57, 1-60.

Holgate, P. 1965. The distance from a random point to the nearest point of a closely packed lattice, Biometriks, 52, 261-263.

Persson, O. 1964. Distance Methods, Studia Forestalia Suecica, No. 15.

APPENDIX

The computation of the moments for order distances and mean order distance requires evaluation of integrals of the form

$$\int r^n \cos^{-1}(m/r)\,dr, \quad \int r^n \sin^{-1}(m/r)\,dr \tag{1}$$

These integrals occur frequently in two-dimensional geometric probabilities. Although they are not difficult to evaluate, the computations are somewhat tedious. A listing for small integer values of n may facilitate reporting by geographers the numerical results of problems involving the same integrals.

Integrating (1) by parts gives

$$\int r^n \begin{Bmatrix} \sin^{-1} \\ \cos^{-1} \end{Bmatrix} \left(\frac{m}{r}\right) dr = \frac{r^{n+1}}{n+1} \begin{Bmatrix} \sin^{-1} \\ \cos^{-1} \end{Bmatrix} \left(\frac{m}{r}\right) \begin{Bmatrix} + \\ - \end{Bmatrix} \frac{m}{n+1} \int \frac{r^n\,dr}{(r^2 - m^2)^{1/2}}$$

Also

$$\int \frac{r^n\,dr}{(r^2 - m^2)^{1/2}} = \frac{r^{n-1}}{n} (r^2 - m^2)^{1/2} + \frac{m^2(n-1)}{n} \int \frac{r^{n-2}\,dr}{(r^2 - m^2)^{1/2}}$$

Repeated integration by parts of (1) gives the following results for n = 1(1)5

$$\begin{aligned} \int r^n \cos^{-1}(m/r) &= (r^{n+1}/n+1)\cos^{-1}(m/r) - R(n, m) \\ \int r^n \sin^{-1}(m/r) &= (r^{n+1}/n+1)\sin^{-1}(m/r) + R(n, m) \end{aligned} \tag{2}$$

where

$R(1, m) = m(r^2 - m^2)^{1/2}/2$

$R(2, m) = mr(r^2 - m^2)^{1/2}/6 + m^3 \log_e(r + (r^2 - m^2)^{1/2})/6$

$$R(3, m) = m(r^2 + 2m^2)(r^2 - m^2)^{1/2}/12$$

$$R(4, m) = m\left[2r^3(r^2 - m^2)^{1/2} + 3rm^2(r^2 - m^2)^{1/2} + 3m^4\log_e(r + (r^2 - m^2)^{1/2})\right]/40$$

$$R(5, m) = \left[m(r^2 - m^2)^{1/2}/6\right]\left[(r^2 - m^2)^2/5 + 2m^2(r^2 - m^2)/3 + m^4\right]$$

These results can also be used for integrals involving arc sec and arc csc. Also, the following identity may be useful

$$\log_e\left[r + (r^2 - m)^{1/2}\right] = \left[\cosh^{-1}(r/m)\right]/m.$$

The numerical evaluation of moments of order distance is simplified because, over the required intervals, the sum of terms not containing R(n, m) is zero. Accordingly, computational fomulas for moments may be obtained directly from the density functions. The results (Table 7) were used for computation of moments (Tables 8 and 9).

TABLE 1. FREQUENCY FUNCTIONS $f_i(r)$ FOR DISTANCE r FROM A POINT RANDOMLY PLACED IN SQUARE OF UNIT AREA TO THE ith NEAREST CORNER

First Corner	
$f_1(r) = 2\pi r$	$0 < r < 1/2$
$= 4r\left(\frac{\pi}{2} - 2 \arccos \frac{1}{2r}\right)$	$1/2 < r < 1/\sqrt{2}$
Second Corner	
$f_2(r) = 8r \arccos \frac{1}{2r}$	$1/2 < r < 1/\sqrt{2}$
$= 8r\left(\frac{\pi}{4} - \arccos \frac{1}{\sqrt{2}r}\right)$	$1/\sqrt{2} < r < 1$
Third Corner	
$f_3(r) = 8r\left(\frac{\pi}{4} + \arccos \frac{1}{\sqrt{2}r} - \arccos \frac{1}{2r}\right)$	$1/\sqrt{2} < r < 1$
$= 8r\left(\frac{\pi}{2} - \arccos \frac{1}{2r} - \arccos \frac{1}{r}\right)$	$1 < r < \sqrt{5}/2$
Fourth Corner	
$f_4(r) = 4r\left(2 \arccos \frac{1}{2r} - \frac{\pi}{2}\right)$	$1/\sqrt{2} < r < \sqrt{5}/2$
$= 4r\left(\frac{\pi}{2} - 2 \arccos \frac{1}{r}\right)$	$\sqrt{5}/2 < r < \sqrt{2}$

The $f_i(r)$ are also the frequency functions for distance r from a point randomly located in the plane containing a hexagonal lattice defined by primitive vectors of unit length to the ith nearest lattice point for $i = 1, 2, 3$.

TABLE 2. DISTRIBUTION FUNCTIONS $F_i(r)$ FOR DISTANCE r FROM A POINT RANDOMLY PLACED IN SQUARE OF UNIT AREA TO THE ith NEAREST CORNER

First Corner

$$F_1(r) = \pi r^2 \qquad 0 < r < 1/2$$

$$= \pi r^2 - 4r^2 \arccos \frac{1}{2r} + (4r^2 - 1)^{1/2} \qquad 1/2 < r < 1/\sqrt{2}$$

Second Corner

$$F_2(r) = 4r^2 \arccos \frac{1}{2r} - (4r^2 - 1)^{1/2} \qquad 1/2 < r < 1/\sqrt{2}$$

$$= \pi r^2 - 4r^2 \arccos \frac{1}{\sqrt{2}r} + (8r^2 - 4)^{1/2} - 1 \qquad 1/\sqrt{2} < r < 1$$

Third Corner

$$F_3(r) = \pi r^2 + 4r^2 \left[\arccos \frac{1}{\sqrt{2}r} - \arccos \frac{1}{2r}\right] \qquad 1/\sqrt{2} < r < 1$$

$$+ (4r^2 - 1)^{1/2} - (8r^2 - 4)^{1/2} - 1$$

$$= \pi r^2 - 4r^2 \left[\arccos \frac{1}{2r} + \arccos \frac{1}{r}\right] \qquad 1 < r < \sqrt{5}/2$$

$$+ (4r^2 - 1)^{1/2} + 4(r^2 - 1)^{1/2} - 3$$

Fourth Corner

$$F_4(r) = 4r^2 \arccos \frac{1}{2r} - \pi r^2 - (4r^2 - 1)^{1/2} + 1 \qquad 1/\sqrt{2} < r < \sqrt{5}/2$$

$$= \pi r^2 - 4r^2 \arccos \frac{1}{r} + 4(r^2 - 1)^{1/2} - 3 \qquad \sqrt{5}/2 < r < \sqrt{2}$$

The $F_i(r)$ are also the frequency functions for distance r from a point randomly located in the plane containing a hexagonal lattice defined by primitive vectors of unit length to the ith nearest lattice point for i = 1, 2, 3.

TABLE 3. FREQUENCY FUNCTIONS $f_i(r)$ FOR DISTANCE r FROM A POINT RANDOMLY PLACED IN A DIAMOND WITH SIDES OF UNIT LENGTH TO THE ith NEAREST CORNER

First Corner

$f_1(r) = 4\pi r/\sqrt{3}$	$0 < r < 1/2$
$= 4\sqrt{3}r\left(\frac{\pi}{3} - 2 \arccos \frac{1}{2r}\right)$	$1/2 < r < 1/\sqrt{3}$

Second Corner

$f_2(r) = 8\sqrt{3}r \arccos \frac{1}{2r}$	$1/2 < r < 1/\sqrt{3}$
$= 8\sqrt{3}r\left(\frac{\pi}{3} - \arccos \frac{1}{2r}\right)$	$1/\sqrt{3} < r < 1$

Third Corner

$f_3(r) = 8\sqrt{3}r\left(\arccos \frac{1}{2r} - \frac{\pi}{6}\right)$	$1/\sqrt{3} < r < \sqrt{3}/2$
$= 8\sqrt{3}r\left(\arccos \frac{1}{2r} - \arccos \frac{\sqrt{3}}{2r} - \frac{\pi}{6}\right)$	$\sqrt{3}/2 < r < 1$

Fourth Corner

$f_4(r) = \frac{8r}{\sqrt{3}} \arccos \frac{\sqrt{3}}{2r}$	$\sqrt{3}/2 < r < 1$
$= \frac{8r}{\sqrt{3}}\left(\frac{\pi}{3} - \arccos \frac{\sqrt{3}}{2r}\right)$	$1 < r < \sqrt{3}$

The $f_i(r)$ are also the frequency functions for distance r from a point randomly located in the plane containing a square lattice defined by primitive vectors of unit length to the ith nearest lattice point for $i = 1, 2, 3$.

TABLE 4. DISTRIBUTION FUNCTIONS $F_i(r)$ FOR DISTANCE r FROM A POINT RANDOMLY PLACED IN A DIAMOND WITH SIDES OF UNIT LENGTH TO THE ith NEAREST CORNER

First Corner

$$F_1(r) = 2\pi r^2/\sqrt{3} \qquad 0 < r < 1/2$$

$$= 2\left(\frac{\pi r^2}{\sqrt{3}} - 2\sqrt{3}\, r^2 \arccos \frac{1}{2r} + \sqrt{3}\, (r^2 - 1/4)^{1/2}\right) \qquad 1/2 < r < 1/\sqrt{3}$$

Second Corner

$$F_2(r) = 2\sqrt{3}\left(2r^2 \arccos \frac{1}{2r} - (r^2 - 1/4)^{1/2}\right) \qquad 1/2 < r < 1/\sqrt{3}$$

$$= 2\sqrt{3}\left(\frac{2\pi r^2}{3} - 2r^2 \arccos^{-1} \frac{1}{2r} + (r^2 - 1/4)^{1/2} - \frac{1}{\sqrt{3}}\right) \qquad 1/\sqrt{3} < r < 1$$

Third Corner

$$F_3(r) = 2\sqrt{3}\left(2r^2 \arccos \frac{1}{2r} - (r^2 - 1/4)^{1/2} - \frac{\pi r^2}{3} + \frac{1}{2\sqrt{3}}\right) \qquad 1/\sqrt{3} < r < \sqrt{3}/2$$

$$= 2\sqrt{3}\left(2r^2 \left[\arccos \frac{1}{2r} - \arccos \frac{\sqrt{3}}{2r}\right] - (r^2 - 1/4)^{1/2} + \sqrt{3}\, (r^2 - 3/4)^{1/2} - \frac{\pi r^2}{3} + \frac{1}{2\sqrt{3}}\right) \qquad \sqrt{3}/2 < r < 1$$

Fourth Corner

$$F_4(r) = \frac{2}{\sqrt{3}}\left(2r^2 \arccos \frac{\sqrt{3}}{2r} - \sqrt{3}\, (r^2 - 3/4)^{1/2}\right) \qquad \sqrt{3}/2 < r < 1$$

$$= \frac{2}{\sqrt{3}}\left(\frac{2\pi r^2}{3} - 2r^2 \arccos \frac{\sqrt{3}}{2r} + \sqrt{3}\, (r^2 - 3/4)^{1/2} - \sqrt{3}\right) \qquad 1 < r < \sqrt{3}$$

The $F_i(r)$ are also the distribution functions for distance r from a point randomly located in the plane containing a square lattice defined by primitive vectors of unit length to the ith nearest lattice point for i = 1, 2, 3.

TABLE 5. FREQUENCY FUNCTIONS $f_i(r)$ FOR DISTANCE r FROM A POINT RANDOMLY LOCATED IN A RECTANGLE WITH SIDES OF LENGTHS b and 1 TO THE ith NEAREST CORNER

Definitions: $0 < b \leq 1$, $B = \sqrt{b^2 + 1}$

First Corner for $0 < b \leq 1$

$$f_1(r) = 2\pi r/b \qquad 0 < r < b/2$$

$$= \frac{4r}{b}\left(\frac{\pi}{2} - \arccos\frac{b}{2r}\right) \qquad b/2 < r < 1/2$$

$$= \frac{4r}{b}\left(\frac{\pi}{2} - \arccos\frac{b}{2r} - \arccos\frac{1}{2r}\right) \qquad 1/2 < r < B/2$$

Second Corner for $0 < b \leq \frac{1}{2}$

$$f_2(r) = \frac{4r}{b}\arccos\frac{b}{2r} \qquad b/2 < r < b$$

$$= \frac{4r}{b}\left(\arccos\frac{b}{2r} - \arccos\frac{b}{r}\right) \qquad b < r < 1/2$$

$$= \frac{4r}{b}\left(\arccos\frac{b}{2r} - \arccos\frac{b}{r} + \arccos\frac{1}{2r}\right) \qquad 1/2 < r < B/2$$

$$= \frac{4r}{b}\left(\frac{\pi}{2} - 2\arccos\frac{B}{2r} - \arccos\frac{b}{r}\right) \qquad B/2 < r < B^2/2$$

Second Corner for $1/2 \leq b \leq 1/\sqrt{3}$

$$f_2(r) = \frac{4r}{b}\arccos\frac{b}{2r} \qquad b/2 < r < 1/2$$

$$= \frac{4r}{b}\left(\arccos\frac{b}{2r} + \arccos\frac{1}{2r}\right) \qquad 1/2 < r < b$$

$$= \frac{4r}{b}\left(\arccos\frac{b}{2r} - \arccos\frac{b}{r} + \arccos\frac{1}{2r}\right) \qquad b < r < B/2$$

$$= \frac{4r}{b}\left(\frac{\pi}{2} - 2\arccos\frac{B}{2r} - \arccos\frac{b}{r}\right) \qquad B/2 < r < B^2/2$$

Second Corner for $1/\sqrt{3} \leq b \leq 1$

$$f_2(r) = \frac{4r}{b} \arccos \frac{b}{2r} \qquad b/2 < r < 1/2$$

$$= \frac{4r}{b}\left(\arccos \frac{b}{2r} + \arccos \frac{1}{2r}\right) \qquad 1/2 < r < B/2$$

$$= \frac{4r}{b}\left(\frac{\pi}{2} - 2 \arccos \frac{B}{2r}\right) \qquad B/2 < r < b$$

$$= \frac{4r}{b}\left(\frac{\pi}{2} - 2 \arccos \frac{B}{2r} - \arccos \frac{b}{r}\right) \qquad b < r < B^2/2$$

Third Corner for $0 < b \leq \sqrt{3}/2$

$$f_3(r) = \frac{4r}{b}\left(\frac{\pi}{2} + 2 \arccos \frac{B}{2r} - \arccos \frac{b}{2r} - \arccos \frac{1}{2r}\right) \qquad B/2 < r < B^2/2$$

$$= \frac{4r}{b}\left(\pi - \arccos \frac{b}{2r} - \arccos \frac{b}{r} - \arccos \frac{1}{2r}\right) \qquad B^2/2 < r < \frac{1}{2}\sqrt{4b^2 + 1}$$

$$= \frac{4r}{b}\left(\frac{\pi}{2} - \arccos \frac{b}{2r}\right) \qquad \frac{1}{2}\sqrt{4b^2 + 1} < r < 1$$

$$= \frac{4r}{b}\left(\frac{\pi}{2} - \arccos \frac{b}{2r} - \arccos \frac{1}{r}\right) \qquad 1 < r < \frac{1}{2}\sqrt{b^2 + 4}$$

Third Corner for $\sqrt{3}/2 \leq b \leq 1$

$$f_e(r) = \frac{4r}{b}\left(\frac{\pi}{2} + 2 \arccos \frac{B}{2r} - \arccos \frac{b}{2r} - \arccos \frac{1}{2r}\right) \qquad B/2 < r < B^2/2$$

$$= \frac{4r}{b}\left(\pi - \arccos \frac{b}{2r} - \arccos \frac{b}{r} - \arccos \frac{1}{2r}\right) \qquad B^2/2 < r < 1$$

$$= \frac{4r}{b}\left(\pi - \arccos \frac{b}{2r} - \arccos \frac{1}{r} - \arccos \frac{b}{r} - \arccos \frac{1}{2r}\right) \qquad 1 < r < \frac{1}{2}\sqrt{4b^2 + 1}$$

$$= \frac{4r}{b}\left(\frac{\pi}{2} - \arccos \frac{b}{2r} - \arccos \frac{1}{r}\right) \qquad \frac{1}{2}\sqrt{4b^2 + 1} < r < \frac{1}{2}\sqrt{b^2 + 4}$$

Fourth Corner for $0 < b \leq 1$

$$f_4(r) = \frac{4r}{b}\left(\arccos\frac{b}{2r} + \arccos\frac{1}{2r} - \frac{\pi}{2}\right) \qquad B/2 < r < \frac{1}{2}\sqrt{4b^2+1}$$

$$= \frac{4r}{b}\left(\arccos\frac{b}{2r} - \arccos\frac{b}{r}\right) \qquad \frac{1}{2}\sqrt{4b^2+1} < r < \frac{1}{2}\sqrt{b^2+4}$$

$$= \frac{4r}{b}\left(\frac{\pi}{2} - \arccos\frac{b}{r} - \arccos\frac{1}{r}\right) \qquad \frac{1}{2}\sqrt{b^2+4} < r < B$$

The $f_i(r)$ are also the frequency functions for distance r from a point randomly located in the plane containing a primitive rectangular lattice defined by primitive vectors of lengths b and 1, $b < 1$, to the ith nearest lattice point for $i = 1, 2$.

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TABLE 6. DISTRIBUTION FUNCTIONS $F_i(r)$ FOR DISTANCE r FROM A POINT RANDOMLY PLACED IN A RECTANGLE WITH SIDES OF LENGTHS b AND 1 TO THE ith NEAREST CORNER

Definitions: $0 < b \le 1$, $A = \sqrt{4b^2 + 1}$, $B = \sqrt{b^2 + 1}$, $C = \sqrt{b^2 + 4}$

First Corner for $0 < b \le 1$

$$F_1(r) = \pi r^2/b \qquad 0 < r < b/2$$

$$= \frac{4}{b}\left[\frac{\pi r^2}{4} - \frac{r^2}{2}\arccos\frac{b}{2r} + \frac{b}{8}(4r^2 - b^2)^{1/2}\right] \qquad b/2 < r < 1/2$$

$$= \frac{4}{b}\left[\frac{\pi r^2}{4} - \frac{r^2}{2}\arccos\frac{b}{2r} - \frac{r^2}{2}\arccos\frac{1}{2r} + \frac{b}{8}(4r^2 - b^2)^{1/2} + \frac{1}{8}(4r^2 - 1)^{1/2}\right] \qquad 1/2 < r < B/2$$

Second Corner for $0 < b \le \frac{1}{2}$

$$F_2(r) = \frac{4}{b}\left[\frac{r^2}{2}\cos^{-1}\frac{b}{2r} - \frac{b}{8}(4r^2 - b^2)^{1/2}\right] \qquad b/2 < r < b$$

$$= \frac{4}{b}\left[\frac{r^2}{2}\cos^{-1}\frac{b}{2r} - \frac{r^2}{2}\cos^{-1}\frac{b}{r} - \frac{b}{8}(4r^2 - b^2)^{1/2} + \frac{b}{2}(r^2 - b^2)^{1/2}\right] \qquad b < r < 1/2$$

$$= \frac{4}{b}\left[\frac{r^2}{2}\cos^{-1}\frac{b}{2r} - \frac{r^2}{2}\cos^{-1}\frac{b}{r} + \frac{r^2}{2}\cos^{-1}\frac{1}{2r} - \frac{b}{8}(4r^2 - b^2)^{1/2} + \frac{b}{2}(r^2 - b^2)^{1/2} - \frac{1}{8}(4r^2 - 1)^{1/2}\right] \qquad 1/2 < r < B/2$$

$$= \frac{4}{b}\left[\frac{\pi r^2}{4} - r^2\cos^{-1}\frac{B}{2r} - \frac{r^2}{2}\cos^{-1}\frac{b}{r} + \frac{B}{4}(4r^2 - B^2)^{1/2} + \frac{b}{2}(r^2 - b^2)^{1/2} - \frac{b}{4}\right] \qquad B/2 < r < B^2/2$$

<u>Second Corner</u> for $1/2 \leq b \leq 1/\sqrt{3}$

$$F_2(r) = \frac{4}{b}\left[\frac{r^2}{2}\cos^{-1}\frac{b}{2r} + \frac{b}{8}(4r^2 - b^2)^{1/2}\right] \qquad b/2 < r < 1/2$$

$$= \frac{4}{b}\left[\frac{r^2}{2}\cos^{-1}\frac{b}{2r} + \frac{r^2}{2}\cos^{-1}\frac{1}{2r} + \frac{b}{8}(4r^2 - b^2)^{1/2} - \frac{1}{8}(4r^2 - 1)^{1/2}\right] \qquad 1/2 < r < b$$

$$= \frac{4}{b}\left[\frac{r^2}{2}\cos^{-1}\frac{b}{2r} - \frac{r^2}{2}\cos^{-1}\frac{b}{r} + \frac{r^2}{2}\cos^{-1}\frac{1}{2r} - \frac{b}{8}(4r^2 - b^2)^{1/2} + \frac{b}{2}(r^2 - b^2)^{1/2} - \frac{1}{8}(4r^2 - 1)^{1/2}\right] \qquad b < r < B/2$$

$$= \frac{4}{b}\left[\frac{\pi r^2}{4} - r^2\cos^{-1}\frac{B}{2r} - \frac{r^2}{2}\cos^{-1}\frac{b}{r} + \frac{B}{4}(4r^2 - B^2)^{1/2} + \frac{b}{2}(r^2 - b^2)^{1/2} - \frac{b}{4}\right] \qquad B/2 < r < B^2/2$$

<u>Second Corner</u> for $1/\sqrt{3} \leq b \leq 1$

$$F_2(r) = \frac{4}{b}\left[\frac{r^2}{2}\cos^{-1}\frac{b}{2r} - \frac{b}{8}(4r^2 - b^2)^{1/2}\right] \qquad b/2 < r < 1/2$$

$$= \frac{4}{b}\left[\frac{r^2}{2}\cos^{-1}\frac{b}{2r} + \frac{r^2}{2}\cos^{-1}\frac{1}{2r} - \frac{b}{8}(4r^2 - b^2)^{1/2} - \frac{1}{8}(4r^2 - 1)^{1/2}\right] \qquad 1/2 < r < B/2$$

$$= \frac{4}{b}\left[\frac{\pi r^2}{4} - r^2\cos^{-1}\frac{B}{2r} + \frac{B}{4}(4r^2 - B^2)^{1/2} - \frac{b}{4}\right] \qquad B/2 < r < b$$

$$= \frac{4}{b}\left[\frac{\pi r^2}{4} - r^2\cos^{-1}\frac{B}{2r} - \frac{r^2}{2}\cos^{-1}\frac{b}{r} + \frac{B}{4}(4r^2 - B^2)^{1/2} + \frac{b}{2}(r^2 - b^2)^{1/2} - \frac{b}{4}\right] \qquad b < r < B^2/2$$

Third Corner for $0 < b \leq \sqrt{3}/2$

$$F_3(r) = \frac{4}{b}\left[\frac{\pi r^2}{4} + r^2 \cos^{-1}\frac{B}{2r} - \frac{r^2}{2}\cos^{-1}\frac{b}{2r} - \frac{r^2}{2}\cos^{-1}\frac{1}{2r} - \frac{B}{4}(4r^2 - B^2)^{1/2} + \frac{b}{8}(4r^2 - b^2)^{1/2} + \frac{1}{8}(4r^2 - 1)^{1/2} - \frac{b}{4}\right] \qquad B/2 < r < B^2/2$$

$$= \frac{4}{b}\left[\frac{\pi r^2}{2} - \frac{r^2}{2}\cos^{-1}\frac{b}{2r} - \frac{r^2}{2}\cos^{-1}\frac{b}{r} - \frac{r^2}{2}\cos^{-1}\frac{1}{2r} + \frac{b}{8}(4r^2 - b^2)^{1/2} + \frac{b}{2}(r^2 - b^2)^{1/2} + \frac{1}{8}(4r^2 - 1)^{1/2} - \frac{3b}{4}\right] \qquad B^2/2 < r < A/2$$

$$= \frac{4}{b}\left[\frac{\pi r^2}{4} - \frac{r^2}{2}\cos^{-1}\frac{b}{2r} + \frac{b}{8}(4r^2 - b^2)^{1/2} - \frac{b}{4}\right] \qquad A/2 < r < 1$$

$$= \frac{4}{b}\left[\frac{\pi r^2}{4} - \frac{r^2}{2}\cos^{-1}\frac{b}{2r} - \frac{r^2}{2}\cos^{-1}\frac{1}{r} + \frac{b}{8}(4r^2 - b^2)^{1/2} + \frac{1}{2}(r^2 - 1)^{1/2} - \frac{b}{4}\right] \qquad 1 < r < C/2$$

Third Corner for $\sqrt{3}/2 \leq b \leq 1$

$$F_3(r) = \frac{4}{b}\left[\frac{\pi r^2}{4} + r^2 \cos^{-1}\frac{B}{2r} - \frac{r^2}{2}\cos^{-1}\frac{b}{2r} - \frac{r^2}{2}\cos^{-1}\frac{1}{2r} - \frac{B}{4}(4r^2 - B^2)^{1/2} + \frac{b}{8}(4r^2 - b^2)^{1/2} + \frac{1}{8}(4r^2 - 1)^{1/2} - \frac{b}{4}\right] \qquad B/2 < r < B^2/2$$

$$= \frac{4}{b}\left[\frac{\pi r^2}{2} - \frac{r^2}{2}\cos^{-1}\frac{b}{2r} - \frac{r^2}{2}\cos^{-1}\frac{b}{r} - \frac{r^2}{2}\cos^{-1}\frac{1}{2r} + \frac{b}{8}(4r^2 - b^2)^{1/2} + \frac{b}{2}(r^2 - b^2)^{1/2} + \frac{1}{8}(4r^2 - 1)^{1/2} - \frac{3b}{4}\right] \qquad B^2/2 < r < 1$$

$$= \frac{4}{b}\left[\frac{\pi r^2}{4} - \frac{r^2}{2}\cos^{-1}\frac{b}{2r} - \frac{r^2}{2}\cos^{-1}\frac{1}{r} - \frac{r^2}{2}\cos^{-1}\frac{b}{r}\right. \qquad 1 < r < A/2$$

$$- \frac{r^2}{2}\cos^{-1}\frac{1}{2r} + \frac{b}{8}(4r^2 - b^2)^{1/2} + \frac{1}{2}(r^2 - 1)^{1/2}$$

$$\left. + \frac{b}{2}(r^2 - b^2)^{1/2} + \frac{1}{8}(4r^2 - 1)^{1/2} - \frac{3b}{4}\right]$$

$$= \frac{4}{b}\left[\frac{\pi r^2}{4} - \frac{r^2}{2}\cos^{-1}\frac{b}{2r} - \frac{r^2}{2}\cos^{-1}\frac{1}{r}\right. \qquad A/2 < r < C/2$$

$$\left. + \frac{b}{8}(4r^2 - b^2)^{1/2} + \frac{1}{2}(r^2 - 1)^{1/2} - \frac{b}{4}\right]$$

<u>Fourth Corner for $0 < b \leq 1$</u>

$$F_4(r) = \frac{4}{b}\left[\frac{r^2}{2}\cos^{-1}\frac{b}{2r} + \frac{r^2}{2}\cos^{-1}\frac{1}{2r} - \frac{\pi r^2}{4}\right. \qquad B/2 < r < A/2$$

$$\left. - \frac{b}{8}(4r^2 - b^2)^{1/2} - \frac{1}{8}(4r^2 - 1)^{1/2} + \frac{b}{4}\right]$$

$$= \frac{4}{b}\left[\frac{r^2}{2}\cos^{-1}\frac{b}{2r} - \frac{r^2}{2}\cos^{-1}\frac{b}{r} - \frac{b}{8}(4r^2 - b^2)^{1/2}\right. \qquad A/2 < r < C/2$$

$$\left. + \frac{b}{2}(r^2 - b^2)^{1/2} - \frac{b}{4}\right]$$

$$= \frac{4}{b}\left[\frac{\pi r^2}{4} - \frac{r^2}{2}\cos^{-1}\frac{b}{r} - \frac{r^2}{2}\cos^{-1}\frac{1}{r} + \frac{b}{2}(r^2 - b^2)^{1/2}\right. \qquad C/2 < r < B$$

$$\left. + \frac{1}{2}(r^2 - 1)^{1/2} - \frac{3b}{4}\right]$$

TABLE 7. MOMENTS $\mu_z'(r)$ FOR DISTANCE r FROM A POINT RANDOMLY LOCATED IN A DIAMOND OR RECTANGLE TO THE ith NEAREST CORNER

Definitions: $0 < b \leq 1,\ A^2 = 4b^2 + 1,\ B^2 = b^2 + 1,\ C^2 = b^2 + 4$

Diamond

First Corner

$$\mu_z'(r) = 8\sqrt{3}R(z+1, 1/2)\Big|_{1/2}^{1/\sqrt{3}}$$

Second Corner

$$\mu_z'(r) = 8\sqrt{3}\left[R(z+1, 1/2)\Big|_{1/\sqrt{3}}^{1} - R(z+1, 1/2)\Big|_{1/2}^{1/\sqrt{3}}\right]$$

Third Corner

$$\mu_z'(r) = 8\sqrt{3}\left[R(z+1, \sqrt{3}/2)\Big|_{\sqrt{3}/2}^{1} - R(z+1, 1/2)\Big|_{1/\sqrt{3}}^{1}\right]$$

Fourth Corner

$$\mu_z'(r) = \frac{8}{\sqrt{3}}\left[R(z+1, \sqrt{3}/2)\Big|_{1}^{\sqrt{3}} - R(z+1, \sqrt{3}/2)\Big|_{\sqrt{3}/2}^{1}\right]$$

Rectangle

First Corner

$$\mu_z'(r) = \frac{4}{b}\left[R(z+1, b/2)\Big|_{b/2}^{B/2} + R(z+1, 1/2)\Big|_{1/2}^{B/2}\right]$$

<u>Second Corner</u>

$$\mu_z'(r) = \frac{4}{b}\left[R(z + 1, b) \Bigg|_{b}^{B^2/2} + 2R(z + 1, B/2) \Bigg|_{B/2}^{B^2/2} - R(z + 1, b/2) \Bigg|_{b/2}^{B/2} - R(z + 1, 1/2) \Bigg|_{1/2}^{B/2} \right]$$

<u>Third Corner</u>

$$\mu_z'(r) = \frac{4}{b}\left[R(z + 1, b/2) \Bigg|_{B/2}^{C/2} + R(z + 1, 1/2) \Bigg|_{B/2}^{A/2} + R(z + 1, b) \Bigg|_{B^2/2}^{A/2} + R(z + 1, 1) \Bigg|_{1}^{C/2} - 2R(z + 1, B/2 \Bigg|_{B/2}^{B^2/2} \right]$$

<u>Fourth Corner</u>

$$\mu_z'(r) = \frac{4}{b}\left[R(z + 1, b) \Bigg|_{A/2}^{B} + R(z + 1, 1) \Bigg|_{C/2}^{B} - R(z + 1, b/2) \Bigg|_{B/2}^{C/2} - R(z + 1, 1/2) \Bigg|_{B/2}^{A/2} \right]$$

The symbol $R(z + 1, b) \Big|_{r_\ell}^{r_u}$ means $R(z + 1, b)$ evaluated at r_u minus $R(z + 1, b)$ evaluated at r_ℓ where r_u and r_ℓ replace r in $R(z + 1, b)$ as listed in the Appendix.

TABLE 8. MOMENTS AND MOMENT CONSTANTS OF DISTANCE FROM A POINT RANDOMLY PLACED IN A RECTANGLE WITH SIDES OF LENGTHS b AND 1 TO THE *ith* NEAREST CORNER

i	1	2	3	4	1	2	3	4
		b = .1				b = .2		
μ'_1	.25319	.26772	.75059	.07540	.26046	.30473	.75256	.15319
μ'_2	.08417	.08915	.58418	.05892	.08667	.10640	.58693	.12133
μ'_3	.03157	.03352	.46970	.04753	.03255	.04098	.47272	.09906
μ'_4	.01264	.01350	.38848	.03943	.01308	.01687	.39155	.08307
μ_2	.02006	.01748	.02079	.05323	.01883	.01354	.02058	.09787
μ_3	.00010	.00029	.00000	.03506	.00017	.00030	.00003	.05049
μ_4	.00072	.00053	.00078	.02701	.00063	.00033	.00076	.03780
β_1	.001	.016	.000	8.151	.004	.037	.000	2.720
β_2	1.779	1.751	1.799	9.532	1.791	1.806	1.789	3.946
		b = .3				b = .4		
μ'_1	.27053	.35197	.75642	.23559	.28273	.40472	.76294	.32457
μ'_2	.09083	.13448	.59218	.19075	.09667	.17240	.60093	.27067
μ'_3	.03428	.05492	.47832	.15882	.03687	.07665	.48750	.23148
μ'_4	.01385	.02362	.39712	.13557	.01504	.03529	.40606	.20252
μ_2	.01765	.01060	.02000	.13525	.01673	.00860	.01886	.16532
μ_3	.00016	.00013	.00012	.05016	.00007	-.00009	.00025	.03632
μ_4	.00058	.00021	.00071	.04018	.00054	.00015	.00062	.03978
β_1	.005	.013	.002	1.017	.001	.012	.010	.029
β_2	1.847	1.891	1.765	2.197	1.940	2.009	1.743	1.455
		b = .5				b = .6		
μ'_1	.29662	.45975	.77300	.42179	.31188	.51465	.78752	.52860
μ'_2	.10417	.21875	.61458	.36458	.11333	.27173	.63493	.47600
μ'_3	.04044	.10711	.50183	.32235	.04514	.14661	.52368	.43746
μ'_4	.01675	.05370	.41991	.29093	.01912	.08056	.44128	.40963
μ_2	.01619	.00738	.01706	.18668	.01606	.00687	.01474	.19658
μ_3	-.00006	-.00025	.00039	.01109	-.00022	-.00030	.00044	-.02198
μ_4	.00054	.00012	.00051	.04130	.00056	.00012	.00040	.04846
β_1	.001	.156	.030	.019	.012	.285	.061	.064
β_2	2.058	2.223	1.757	1.185	2.184	2.468	1.840	1.254

TABLE 8 CONTINUED

i	1	2	3	4	1	2	3	4
		b = .7				b = .8		
μ_1'	.32828	.56753	.80745	.64610	.34563	.61682	.83365	.77514
μ_2'	.12417	.32915	.66418	.60842	.13667	.38840	.70493	.76533
μ_3'	.05113	.19459	.55638	.58374	.05855	.24933	.60445	.76904
μ_4'	.02231	.11700	.47432	.56981	.02651	.16296	.52533	.78542
μ_2	.01640	.00706	.01221	.19098	.01721	.00793	.00996	.16449
μ_3	-.00040	-.00022	.00037	-.05613	-.00058	-.00002	.00018	-.07921
μ_4	.00062	.00013	.00030	.06228	.00071	.00016	.00021	.07699
β_1	.037	.143	.074	.452	.066	.001	.033	1.410
β_2	2.298	2.541	1.999	1.708	2.385	2.485	2.159	2.846
		b = .9				b = 1.0		
μ_1'	.36377	.66121	.86697	.91641	.38260	.69956	.90818	1.07045
μ_2'	.15083	.44648	.76018	.95025	.16667	.50001	.83333	1.16667
μ_3'	.06759	.30774	.67385	1.00219	.07840	.36509	.77219	1.29305
μ_4'	.03195	.21637	.60357	1.07369	.03889	.27223	.72222	1.45556
μ_2	.01850	.00929	.00854	.11044	.02029	.01062	.00854	.02081
μ_3	-.00075	.00025	-.00002	-.07105	-.00089	.00045	-.00014	-.00036
μ_4	.00083	.00022	.00016	.07236	.00101	.00029	.00017	.00104
β_1	.088	.076	.001	3.747	.095	.167	.029	.015
β_2	2.436	2.519	2.190	5.932	2.451	2.591	2.278	2.410

Values for the first two (three) corners also describe distance from a point randomly located in a primitive rectangular lattice defined by primitive vectors of lengths b and 1 to the ith nearest lattice point for i = 1, 2 (1, 2, 3) where b < 1 (b = 1).

TABLE 9. MOMENTS AND MOMENT CONSTANTS OF DISTANCE FROM A POINT RANDOMLY PLACED IN A DIAMOND WITH SIDES OF UNIT LENGTH TO THE ith NEAREST CORNER

i	1	2	3	4
Crude Moments				
μ_1'	.35102	.67813	.79481	1.17456
μ_2'	.13889	.47222	.63889	1.41667
μ_3'	.05873	.33780	.51905	1.75526
μ_4'	.02593	.24815	.42593	2.23333
Central Moments				
μ_2	.01567	.01236	.00717	.03708
μ_3	-.00102	.00081	.00014	.00419
μ_4	.00059	.00038	.00012	.00344
Skewness				
β_1	.027	.344	.053	.345
Kurtosis				
β_2	2.405	2.499	2.424	2.499

Values for the first three corners also describe distance from a point randomly located in a hexagonal lattice defined by primitive vectors of unit length to the ith nearest lattice point for i = 1, 2, 3.