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ON NONCONSERVATIVE STABILITY PROBLEMS OF ELASTIC SYSTEMS WITH SLIGHT DAMPING[1]

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ABSTRACT

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A linear two-degree-of-freedom system with slight viscous damping and subjected to [redacted] nonconservative loading is analyzed with the aim of studying the effects of damping on stability of equilibrium. It is found that in such systems multiple ranges of stability and instability may exist in a richer variety than in corresponding systems without damping. Further, for certain systems, instability either by divergence (static buckling) or by flutter may occur first as the compressive load increases, depending upon the ratio of the damping coefficients in the two degrees of freedom. It is shown finally that systems exist for which the destabilizing effect of slight viscous damping cannot be completely removed whatever the ratio of the (positive) damping coefficients.

[redacted]

Introduction

Ziegler's [1] discovery of the destabilizing effect of linear viscous damping in a nonconservative elastic system provided an impetus for further studies of this remarkable phenomenon. In particular, Bolotin [2] found that the destabilizing effect in an elastic system with two degrees of freedom is highly dependent on the ratio of the damping coefficients and that it could be eliminated for a certain particular value of this ratio.

More recently, the influence of damping in nonconservative systems was discussed by Leipholz [3] and also by Herrmann and Jong [4]. References to further work on this subject are given in [2] and [3]. In [4] attention was focussed on establishing a generic relationship between critical loadings for no damping, for slight damping, as well as for vanishing damping. It was found that while the presence of small linear viscous damping may have a destabilizing effect, proper interpretation of the limiting process of vanishing damping leads to the same critical load as for no damping.

The conclusions arrived at in [4] were based on the analysis of a system in which the (nonconservative) loading was completely specified and no neighboring equilibrium position existed, i. e. stability was lost by flutter. It was shown by Herrmann and Bungay [5] for a system without damping, however, that by varying a loading parameter, the otherwise identical system could lose stability either by flutter or by divergence (neighboring equilibrium position exists) or by both (at different loads) depending upon the value of this parameter.

In view of the results obtained in [1], [2] and [4], it appears to be desirable to extend the analysis of [5] by including linear viscous damping considered in a particular system in [1] and [2]. To this end the same model of a system with two degrees of freedom as in [5] is investigated, but damping is also included.

The linearized equations of motion permit a detailed study of the nature of the roots of the characteristic equation associated with small motions in the vicinity of the static equilibrium position, using the theory of equations [6] in addition to the Routh-Hurwitz criteria [7].

The results of this investigation indicate that multiple ranges of stability and instability may occur also in the presence of slight damping and that the variety of possibilities is even richer than in the absence of damping. The critical loads, as already shown in [2], are highly dependent on the ratio of the damping coefficients but, in addition, two features, not known heretofore, are shown to exist: Firstly, for otherwise identical systems the existence of neighboring equilibrium may depend on the ratio of the damping coefficients. This means that by changing this ratio two otherwise identical systems may lose stability by either divergence (static instability) or by flutter (dynamic instability). The critical loads are different in the two cases. Secondly, for a class of elastic systems the elimination of the destabilizing effect of damping by an appropriate choice of the ratio of the damping coefficients is not possible. It was found that for certain systems negative damping would be required in order to make the critical load in the presence of slight damping to be identical to that in the absence of ~~xxxxx~~ damping.

In conclusion it should be emphasized that the investigation presented here had solely the purpose of indicating the existence of certain types of behavior of a simple model of a nonconservative system. Whether or not linear viscous damping is realistic for actual systems and whether the loadings considered are realizable are questions deferred to subsequent studies.

The Model

We consider a double pendulum, Fig. 1, composed of two rigid weightless bars of equal length ℓ, which carry concentrated masses $m_1 = 2m$, $m_2 = m$. The generalized coordinates φ_1 and φ_2 are taken to be small in the usual sense. A load P is applied at the free end at an angle $\alpha\varphi_2$, as shown in Fig. 1. At the hinges, the restoring moments $c\varphi_1 + b_1\dot{\varphi}_1$ and $c(\varphi_2 - \varphi_1) + b_2(\dot{\varphi}_2 - \dot{\varphi}_1)$ are induced. The damping coefficients b_1 and b_2 are taken as positive and no gravitational effects are included.

The kinetic energy T, the dissipation function D, the potential energy V, and the generalized forces Q_1 and Q_2 are:

$$T = \frac{1}{2} m\ell^2(3\dot{\varphi}_1^{\,2} + 2\dot{\varphi}_1\dot{\varphi}_2 + \dot{\varphi}_2^{\,2})$$

$$D = \frac{1}{2} b_1\dot{\varphi}_1^{\,2} + \frac{1}{2} b_2(\dot{\varphi}_1^{\,2} - 2\dot{\varphi}_1\dot{\varphi}_2 + \dot{\varphi}_2^{\,2})$$

$$V = \frac{1}{2} c(2\varphi_1^{\,2} - 2\varphi_1\varphi_2 + \varphi_2^{\,2})$$

$$Q_1 = P\ell(\varphi_1 - \alpha\varphi_2)$$

$$Q_2 = P\ell(1 - \alpha)\varphi_2$$

Lagrange's equations in the form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\phi}_i}\right) + \frac{\partial D}{\partial \dot{\phi}_i} - \frac{\partial T}{\partial \varphi_i} + \frac{\partial V}{\partial \varphi_i} = Q_i \qquad (i = 1,2)$$

are employed to establish the linear equations of motion

$$3m\ell^2\ddot{\varphi}_1 + (b_1 + b_2)\dot{\phi}_1 - (P\ell - 2c)\varphi_1 + m\ell^2\ddot{\varphi}_2 - b_2\dot{\phi}_2 + (\alpha P\ell - c)\varphi_2 = 0$$

$$m\ell^2\ddot{\varphi}_1 - b_2\dot{\phi}_1 - c\varphi_1 + m\ell^2\ddot{\varphi}_2 + b_2\dot{\phi}_2 - [(1-\alpha)P\ell - c]\varphi_2 = 0$$

which, upon stipulating solutions of the form

$$\varphi_i = A_i e^{\omega t} \qquad (i = 1,2)$$

yield the characteristic equation

$$p_o\Omega^4 + p_1\Omega^3 + p_2\Omega^2 + p_3\Omega + p_4 = 0$$

with the coefficients

$$p_o = 2$$

$$p_1 = B_1 + 6B_2$$

$$p_2 = 2(2-\alpha)\left[-F + \frac{7+B_1B_2}{2(2-\alpha)}\right]$$

$$p_3 = (1-\alpha)(B_1 + 2B_2)\left[-F + \frac{\beta+1}{(1-\alpha)(\beta+2)}\right]$$

$$p_4 = (1-\alpha)\left[F - \frac{3}{2}\left(1 + \sqrt{\frac{5/9 - \alpha}{1-\alpha}}\right)\right]\left[F - \frac{3}{2}\left(1 - \sqrt{\frac{5/9 - \alpha}{1-\alpha}}\right)\right]$$

and the dimensionless quantities

$$\Omega = \ell\left(\frac{m}{c}\right)^{1/2}\omega$$

$$B_i = \frac{b_i}{\ell(cm)^{1/2}} \qquad (i = 1,2)$$

$$F = \frac{P\ell}{c}$$

$$\beta = \frac{B_1}{B_2}$$

Root Domains of the Characteristic Equation

The type of motion of the system and therefore the problem concerning stability is closely related to the nature of the roots of the characteristic equation. It was found in [4] that small damping rather than vanishing or large damping is the cause of the destabilizing effect, and thus only small damping ($B_1 \ll 1$) will be considered in the sequel.

Let us introduce first the following quantities:

$$H = \frac{1}{6} p_o p_2 - \frac{1}{16} p_1^2$$

$$\cong \frac{2}{3} (\alpha-2)\left[F - \frac{7}{2(2-\alpha)}\right]$$

$$I = p_o p_4 - \frac{1}{4} p_1 p_3 + \frac{1}{12} p_2^2$$

$$\cong \frac{1}{12} [4(\alpha^2 - 10\alpha + 10)F^2 + 4(25\alpha - 32)F + 73]$$

$$J = \frac{1}{6} p_o p_2 p_4 + \frac{1}{48} p_1 p_2 p_3 - \frac{1}{16} p_o p_3^2 - \frac{1}{16} p_1^2 p_4 - \frac{1}{216} p_2^3$$

$$\cong - \frac{1}{216} [(8\alpha^3 + 96\alpha^2 - 336\alpha + 224)F^3 -$$

$$- (348\alpha^2 - 1{,}464\alpha + 1{,}032)F^2 -$$

$$- (1{,}362\alpha - 1{,}212)F - 161]$$

$$K = p_o^2 I - 12H^2$$

$$\cong - 4[(\alpha-1)^2+1]\left\{F - \frac{1}{2[(\alpha-1)^2+1]}\left[(8-\alpha)+6.325\sqrt{-(\alpha-0.345)(\alpha-1.305)}\right]\right\} \times$$

$$\times \left\{F - \frac{1}{2[(\alpha-1)^2+1]}\left[(8-\alpha)-6.325\sqrt{-(\alpha-0.345)(\alpha-1.305)}\right]\right\}$$

$$\Delta = I^3 - 27J^2 \cong \frac{1}{8} p_4 K^2$$

$$X = p_1p_2p_3 - p_op_3^2 - p_1^2p_4$$

$$= B_2^2 \Big\{(1-\alpha)[\beta^2+12\beta+4-8\alpha(\beta+2)]F^2 - [2(\beta^2+7\beta+6) +$$

$$+ 2(1-\alpha)(\beta^2+11\beta-10) + (1-\alpha)(\beta^2+8\beta+12)B_1B_2]\ F +$$

$$+ [4\beta^2+33\beta+4+(\beta^2+7\beta+6)B_1B_2]\Big\}$$

$$\cong B_2^2 \Big\{(1-\alpha)[\beta^2+12\beta+4-8\alpha(\beta+2)]\ F^2 -$$

$$- 2[\beta^2+7\beta+6+(1-\alpha)(\beta^2+11\beta-10)]\ F +$$

$$+ (4\beta^2+33\beta+4)\Big\}$$

where $p_o, \ldots, p_4$ and other symbols have been previously defined.

It is known from the theory of equations [6] that:

(a) when $\Delta < 0$, the characteristic equation has two real and two complex roots;

(b) when $\Delta > 0$ and both H and K are negative, the four roots are all real;

(c) when $\Delta > 0$ and at least one of H and K is positive or zero, the four roots are all complex.

These criteria lead to the different root domains shown in Fig. 2. The domain marked by crosses xxxxxxxxxxx indicates the existence of four real roots; that marked by dots xxxxxxxxxxxxxx corresponds to two real and two complex roots; and that marked by horizontal dashes or by diagonal lines indicates the existence of four complex roots. The more detailed nature of the roots and the related stable and unstable behavior of the system may be deduced from the following.

(A) <u>Domain $\Delta > 0$, $H < 0$, $K < 0$</u>:

This domain is marked by crosses xxxxxxxxxxx in Fig. 2. In it p_o, p_1, and p_4 are always positive; p_2 is always negative. Applying the well-known Descartes' rule of signs, regardless of the sign of p_3, it is seen that in this domain the four real roots of the characteristic equation are always pairs of two positive and two negative ones. Consequently, this is throughout a region of instability by divergent motion.

(B) Domains $\Delta < 0$:

These domains are marked by dots in Fig. 2. Let the two real and two complex roots in these domains be represented by

$$\Omega = \begin{cases} \rho_1 \pm i\rho_2 \\ r_1 , r_2 \end{cases}$$

From the relations between roots and coefficients in the theory of equations [6] and the definition of the expression X in the Routh-Hurwitz criterion [7], the following relationships hold

$$2(\rho_1 + r_1) = -\frac{p_1}{p_o} = -\frac{1}{2}(B_1 + 6B_2) < 0$$

$$2[\rho_1(r_1^2 - r_2^2) + r_1(\rho_1^2 + \rho_2^2)] = -\frac{p_3}{p_o} = -\frac{1}{2}p_3$$

$$(\rho_1^2 + \rho_2^2)(r_1^2 - r_2^2) = \frac{p_4}{p_o} = \frac{1}{2}p_4$$

$$4\rho_1 r_1 \left\{[(\rho_1 + r_1)^2 + \rho_2^2 - r_2^2]^2 + 4\rho_2^2 r_2^2\right\} = \frac{X}{p_o^3} = \frac{1}{8}X$$

As p_4 is always negative in these three domains, the third equation above indicates that

$$r_2^2 > r_1^2$$

which, in turn, shows that the two real roots are of opposite sign. Hence, these three domains are also regions of instability. Again, recalling that $p_4 < 0$, it is seen from the above four equations that the real part of the conjugate complex roots will be negative if $X > 0$ or if $X < 0$ and $p_3 < 0$, but will be positive if $X < 0$ and $p_3 > 0$. Whence it follows that divergent motion will prevail in this region, of the type as sketched in Fig. 3(a) if $X > 0$ or if $X < 0$ and $p_3 < 0$, or as in Fig. 3(b) if $X < 0$ and $p_3 > 0$. It is noted that if the system is undamped ($B_i = 0$), ρ_1 and r_1 will vanish identically. The undamped system will therefore undergo divergent motion of the type as sketched in Fig. 3(c). By definition, in all above cases the system is unstable.

(C) Domain $K > 0$:

This domain is marked by horizontal dashes in Fig. 2. Let us denote the four complex roots in this domain by

$$\Omega = \begin{cases} \gamma_1 \pm i\gamma_2 \\ \delta_1 \pm i\delta_2 \end{cases}$$

Then, as before, the following relationships are obtained:

$$2(\gamma_1 + \delta_1) = -\frac{p_1}{p_o} = -\frac{1}{2}(B_1 + 6B_2) < 0$$

$$4\gamma_1\delta_1[(\gamma_1 + \delta_1)^2 + (\gamma_2 + \delta_2)^2][(\gamma_1 + \delta_1)^2 + (\gamma_2 - \delta_2)^2] = \frac{X}{p_o^3} = \frac{1}{8}X$$

which indicate that γ_1 and δ_1 (the real parts of the two pairs of conjugate complex roots) will be both negative if $X > 0$, but of opposite sign if $X < 0$.

Now, within this domain, we have

$$K \cong 8p_4 - p_2^2 > 0$$

or

$$p_4 > \frac{1}{8}p_2^2$$

which, in turn, leads to

$$X < -\frac{1}{8}(4p_3 - p_1p_2)^2 \leq 0$$

or

$$X < 0$$

Therefore, the real parts of the two pairs of conjugate complex roots are of opposite sign. The nature of these four roots indicates that in this domain the system will flutter.

(D) Domain $\Delta > 0$, $H > 0$, $K < 0$:

This domain is marked by diagonal lines in Fig. 2. As the four roots are all complex, the signs of the real parts of the roots will also be governed by the signs of X as asserted above. Thus, the system will vibrate with decreasing amplitude (asymptotic stability) if the values of α and F are in those parts of this domain where $X > 0$. However, the system will flutter if the values of α and F are in those parts where $X < 0$.

Further separation of stability from instability in the present domain is governed solely by the sign of X. This is illustrated for the four cases of $\beta = 0$, 1, 11.071, and ∞ as shown in Figs. 4, 5, 6, and 7 where the regions shaded by diagonal lines are regions of stability; those shaded by horizontal dashes are regions of flutter; those shaded by small triangles are regions of divergent motion, of the type shown in Fig. 3(a); those shaded by dots are regions of divergent motion of the type shown in Fig. 3(b); and those shaded by crosses are regions of divergent motion, in which the time increase of the generalized coordinates is of the exponential type.

It is to be noted that in the present domain ($\Delta > 0$, $H > 0$, and $K < 0$), if the damping effects vanish, the four complex roots of the characteristic equation will all be pure imaginary and distinct. Thus the undamped system executes steady state vibrations and is stable throughout the domain, as found in [5].

Nature of Boundaries Separating Different Root Domains

In this section the boundaries given by $X = 0$, $p_4 = 0$, and $K = 0$ will be examined. For the sake of convenience, the term "boundaries given by $X = 0$" will be restricted to mean only those parts of the curves given by $X = 0$ which lie in the domain $\Delta > 0$, $H > 0$, and $K < 0$.

(A) Boundaries $X(\alpha, F, \beta) = 0$:

On these boundaries the characteristic equation has, by definition of X [7], two roots equal in magnitude but opposite in sign. These two roots are

$$\Omega_{1,2} = \pm \left(-\frac{p_3}{p_1}\right)^{1/2}$$

where p_1 is positive for positive damping. It is found that the curves $p_3 = 0$, $p_4 = 0$, and $X = 0$ have a common point of intersection which is given by

$$\alpha = \alpha' = \frac{\beta^2 + 3\beta + 1}{2\beta^2 + 5\beta + 2}$$

$$F = F' = \frac{2\beta + 1}{\beta + 1}$$

Further, as $p_3 = 0$ and $X = 0$ have only one point of intersection at (α', F') on $p_4 = 0$, it is evident that along the boundaries given by $X = 0$ p_3 is always positive. This can be seen from Figs. 4, 5, 6, and 7. Consequently, $\Omega_{1,2}$ are two distinct pure imaginary roots. The sum of the other two conjugate complex roots is $-\frac{p_1}{p_o} = -\frac{1}{2} p_1$ which is negative (for positive damping). Hence, along the boundaries given by $X = 0$ the characteristic equation has two pure imaginary roots equal in magnitude but opposite in sign and two conjugate complex roots with negative real part. Thus, the system will execute steady state vibrations as a result of some initial disturbance. It is only in this case that the damped, nonconservative system can undergo such motions.

(B) Point of Intersection of $X = 0$, $p_3 = 0$, $p_4 = 0$:

At this common intersection point denoted by (α', F'), the characteristic equation has two **zero roots.** The other two roots, being given by

$$p_o\Omega^2 + p_1\Omega + p_2 = 0,$$

are two conjugate complex roots with negative real part. The two zero roots will induce two terms of the form $c_1 + c_2t$ in the general solution of φ_i. Thus, the system will execute divergent motion in which the increase of φ_i is linear with respect to the time. This point (α', F') is the only one at which the stability region for the damped, nonconservative system is open.

(C) Points of Intersection of $p_4 = 0$, $X = 0$, $S = 0$:

Let us introduce the quantity

$$S = p_1p_2 - p_op_3$$

which is one of the expressions entering the Routh-Hurwitz criterion. Then,

$$X = p_3S - p_1^2p_4$$

It can be shown that the curves $p_4 = 0$, $S = 0$, and $X = 0$ have two points of common intersection, denoted by (α'', F'') and (α''', F'''), where

$$\left.\begin{matrix}\alpha' \\ \alpha''\end{matrix}\right\} = \frac{(5\beta^2+228\beta+1,440) \mp [(5\beta^2+180\beta+800)^2 - 6,400\beta(\beta+6)]^{1/2}}{16(15\beta+112)}$$

$$F' = \frac{5(\beta+8)}{2(\beta+10-4\alpha')}$$

$$F'' = \frac{5(\beta+8)}{2(\beta+10-4\alpha'')}$$

These two points usually exist when β is finite, but the point (α'', F'') approaches infinity as $\beta \to \infty$. At the point (α', F'), the characteristic equation has: one zero root, one positive real root equal to $(-\frac{p_3}{p_o})^{1/2}$, two negative real roots equal to $-(-\frac{p_3}{p_o})^{1/2}$ and $-\frac{p_1}{p_o}$; therefore, the system will execute divergent motions. At the point (α'', F''), the four roots are: one zero root, two pure imaginary roots equal to $\pm(-\frac{p_3}{p_1})^{1/2}$, and one negative real root equal to $-\frac{p_1}{p_o}$; hence, after the initial disturbance, the system will execute steady state vibrations about a certain position which in general is not the position whose stability is being studied.

(D) Boundaries $p_4 = 0$, Excluding the Points (α', F'), (α', F''), and (α'', F''):

Along these boundaries the characteristic equation has one zero root and three other roots given by

$$p_o\Omega^3 + p_1\Omega^2 + p_2\Omega + p_3 = 0$$

where, by the theory of equations and for small damping $(B_1 \ll 1)$, the three roots will be all real if $p_2 < 0$, but one real and two complex if $p_2 \geq 0$. In the range of either $F < F'$ or $\alpha > \alpha''$ along $p_4 = 0$, the four roots are found to be: one zero root, one negative real root, and two conjugate complex roots with negative real part. The nature of these four roots indicates that after the initial disturbance the system may execute transient vibrations and then come to rest at a position which in general is not the position whose stability is being studied. This phenomenon can be interpreted as a stabilizing effect of viscous damping because the same system with no damping would execute divergent motion.

The curves $p_2 = 0$ (i.e. $H = 0$), $p_4 = 0$, and $K = 0$ have two common intersection points at (0.423, 2.219) and (1.182, 4.281). In the range of $F' < F \leq 2.219$ along $p_4 = 0$ the four roots are: one zero root, one positive real root, and two conjugate complex roots with negative real part. In the range $2.219 < F < 3$ along $p_4 = 0$, the four roots are: one zero root, one positive real root, and two negative real roots. Thus, in the range of $F' \leq F < 3$ along $p_4 = 0$, the system will execute divergent motions.

In the range $F'' < F \leq 4.281$ along $p_4 = 0$, the four roots are: one zero root, one negative real root, and two conjugate complex roots with positive real part and thus flutter will occur. In the range $F > 4.281$ along $p_4 = 0$, the four roots are: one zero root, one negative real root and two positive real roots; hence, the system will undergo divergent motions.

(E) Boundary $K \cong 8p_4 - p_2^2 = 0$:

The exact curve of $K = 0$ is

$$K = 8p_4 - p_2^2 - (p_1p_3 - \frac{1}{2} p_1^2p_2 + \frac{3}{64} p_1^4) = 0$$

As B_1 and hence p_1 and p_3 are assumed small, of the order of 10^{-3} [4], the last three terms in parentheses are higher order terms and may be neglected. Thus

$$K \cong 8p_4 - p_2^{\,2} = 0$$

is a boundary curve which is very close to the exact curve $K = 0$. Substituting $\frac{1}{8}\, p_2^{\,2}$ for p_4 in X, we have

$$X = -\frac{1}{8}\,(p_1 p_2 - 4p_3)^2 \leq 0$$

which indicates that the system will be unstable when α and F are on the boundary curve given by $K = 8p_4 - p_2^{\,2} = 0$ except at the point where X vanishes and p_3 is positive (steady state vibrations). The instability mechanism will, on the whole, be of the flutter type, except at the points where the exact expressions of K and H are all negative (divergence).

Influence of Damping Ratio on Instability Mechanisms

In the preceding sections it was established that stability is possible only in the region ($\Delta > 0$, $H > 0$, and $K < 0$) which is marked by diagonal lines in Fig. 2. In this region the sign of X governs the type of motion; i.e., the system is stable if $X \geq 0$ and unstable if $X < 0$.

Critical loads for divergence, if any, are given by $p_4 = 0$; i.e., they are

$$F_{div} = \frac{3}{2}\left(1 \pm \sqrt{\frac{5/9 - \alpha}{1 - \alpha}}\right)$$

On the other hand, critical loads for flutter, if any, are always given by $X = 0$; i. e., they are

$$F_{flu} = \frac{2(\beta^2+9\beta-2) - \alpha(\beta^2+11\beta-10) \pm (\beta+6)\sqrt{(\beta^2-22\beta+1)\alpha^2 + 33\beta\alpha - 9\beta}}{8(\beta+2)(\alpha-1)(\alpha-\alpha_0)}$$

where $1 \neq \alpha \neq \alpha_0$, and

$$\alpha_o = \frac{\beta^2+12\beta+4}{8(\beta+2)}$$

The two vertical lines $\alpha = 1$ and $\alpha = \alpha_o$ (Figs. 4 to 7) are asymptotes of $X = 0$. For $\alpha = 1$ the critical load is given by

$$F_{flu}\Big|_{\alpha=1} = \frac{4\beta^2+33\beta+4}{2(\beta^2+7\beta+6)}$$

which was studied in [4]. For $\alpha = \alpha_o$ the critical load for flutter, if any, becomes

$$F_{flu}\Big|_{\alpha=\alpha_o} = \frac{4(\beta+2)(4\beta^2+33\beta+4)}{\beta^4+7\beta^3-50\beta^2-332\beta+24}$$

The curves of critical loads for $\beta = 0$, 1, 11.071 and ∞ are illustrated in Figs. 4, 5, 6, and 7.

For $\alpha = 0$ (conservative case) in Fig. 4 the point (0,-1), which is an intersection point of two branches of the curves given by $X = 0$, is itself on the boundary given by $X = 0$; therefore, this point corresponds to steady state vibrations of the system. The point (0,-1) is thus also a point representing stability rather than a point which indicates an isolated critical load for the conservative system ($\alpha = 0$) with damping. However, depending on the ratio of damping coefficients, a nonconservative system ($\alpha \neq 0$) may have multiple critical loads for flutter, in addition to those for divergence, at the same value of α anywhere in the range $\alpha \lessgtr 0$, except for $\frac{5}{9} < \alpha \leq 1$ where critical loads for flutter only will occur. Fig. 4 illustrates that for $\beta = 0$ flutter will occur for any α, except $\alpha = 0$, while Fig. 5 shows that the smallest range of α, in which flutter is possible, becomes minimum ($\frac{5}{9} < \alpha < 1.305$) when the damping coefficients are identical (i.e. $\beta = 1$).

It was found in [5] that the presence or absence of neighboring equilibrium positions was strongly influenced by the behavior of the nonconservative loading and also by the constraints of the system. A further result of the present study is that the ratio of the damping coefficients may exert an analogous influence, and may thus render the static criterion inapplicable for systems in which without damping the critical load could be determined statically. For instance, it is seen that in the range $\frac{1}{2} < \alpha < \frac{5}{9}$ the static stability criterion is applicable if $\beta = \infty$ (see Fig. 7), but breaks down if $\beta = 0$ (see Fig. 4).

Similarly to applicability, the sufficiency of the static stability criterion (in the sense of supplying all critical loads) also depends on the ratio of damping coefficients. To exemplify this feature, let us examine again Figs. 4 and 7. In the range $\alpha < \frac{1}{2}$ we note that the static stability criterion is sufficient if $\beta = \infty$, but proves to be insufficient if $\beta = 0$. The equation $p_4 = 0$ expresses, in fact, the static stability criterion; i.e., the condition of the static equilibrium of the system in the vicinity of its neutral configuration. Thus the static stability criterion is implied in the kinetic stability criterion which is usually sufficient in determining all critical loads for the nonconservative system.

It is possible to identify the range of α in which flutter cannot occur and thus the application of the kinetic criterion is not required. This range will, however, depend on the ratio of the damping coefficients. To determine this range, we consider the expression F_{flu} derived in this section. Flutter cannot occur if the quantity $(\beta^2 - 22\beta + 1)\alpha^2 + 33\beta\alpha - 9\beta$ appearing under the square root in that expression is negative. Thus flutter may occur in the following ranges

$\alpha \geq \alpha_1$ and $\alpha \leq \alpha_2$ if $\beta > a_1$ or $\beta < a_2$

or

$\alpha_1 > \alpha > \alpha_2$ if $a_1 > \beta > a_2$

where

$$a_{1,2} = 11 \pm \sqrt{120} = \begin{cases} 21.954 \\ 0.046 \end{cases}$$

and

$$\alpha_{1,2} = \frac{-33\beta \pm \sqrt{\beta(36\beta^2+297\beta+36)}}{2(\beta-a_1)(\beta-a_2)}, \qquad (a_1 \neq \beta \neq a_2)$$

If $\beta = a_1$ or $\beta = a_2$, the range in which the kinetic stability criterion must be considered will be only $\alpha \geq \frac{3}{11}$. Consequently, if there exist any ranges of α which are outside the above specified ranges, the static stability criterion alone will be sufficient to determine all the critical loads, despite the nonconservativeness of the loading. However, according to the preceding section, if $\alpha < \alpha'$ or $\alpha > \alpha''$ the static stability criterion will definitely be applicable but not necessarily sufficient in determining all critical loads.

Possibility of Elimination of Destabilizing Effects

Critical loads for flutter in the undamped system analyzed in [5] are given by the equation $K(\alpha,F,B_1) = 0$ with the terms due to small damping neglected, i.e., by the equation

$$K(\alpha,F) = -[4(\alpha^2-2\alpha+2)F^2 + 4(\alpha-8)F + 41] = 0$$

Critical loads for flutter in the damped system analyzed here are given by

$$X(\alpha,F,\beta) = 0$$

whose loci constitute, in fact, a family of curves in the α - F plane with β as the parametric constant. Different curves of the critical load for flutter will be obtained if different values are assigned to β in $X(\alpha,F,\beta) = 0$.

To study the interrelation between the curves of critical loads given by $K(\alpha,F) = 0$ and $X(\alpha,F,\beta) = 0$, let us examine the envelope of the family of curves defined by $X(\alpha,F,\beta) = 0$. It is known [8] that, if an envelope exists, it must satisfy

$$X(\alpha,F,\beta) = 0$$

and

$$\frac{\partial}{\partial\beta} X(\alpha,F,\beta) = 0$$

Elimination of β in these two equations yields

$$(F-2)[(1-\alpha)F-2][4(1-\alpha)F-5]^2 \cdot K(\alpha,F) = 0$$

where $K(\alpha,F)$ is as defined before. However, this equation may contain some curves which are other than the envelope [8]. Deleting these, the true envelope is found as given by

$$[(1-\alpha)F-2] \cdot K(\alpha,F) = 0$$

Thus, the curve for critical flutter loads of the system with no damping is a branch of the envelope of the family of curves of the critical flutter loads of the same system with damping. This remarkable relation shows a significant connection between the two governing equations of the critical loads for flutter of the undamped and the damped systems.

In consequence of the above relation, it appears possible to eliminate the destabilizing effect of damping on the critcal loads for flutter in the damped system if we choose the value of β which defines a curve of the family $X(\alpha,F,\beta) = 0$ tangent to $K(\alpha,F) = 0$ (the envelope) at the given value of α. Eliminating F in $X(\alpha,F,\beta) = 0$ and $\frac{\partial}{\partial\beta} X(\alpha,F,\beta) = 0$, we find that this value of β is given by the positive, real root of the quintic

$$\begin{aligned}&(8\alpha-3)(7\alpha-3)(4\alpha-3)\beta^5 - (896\alpha^4 - 5{,}936\alpha^3 + 8{,}196\alpha^2 - 3{,}870\alpha + \\ &+594)\beta^4 - (12{,}800\alpha^4 - 60{,}928\alpha^3 + 82{,}680\alpha^2 - 38{,}664\alpha + 5{,}832)\beta^3 - \\ &- (80{,}128\alpha^4 - 365{,}280\alpha^3 + 502{,}416\alpha^2 - 234{,}576\alpha + 34{,}992)\beta^2 - \\ &- (353{,}280\alpha^4 - 1{,}480{,}320\alpha^3 + 1{,}925{,}856\alpha^2 - 874{,}800\alpha + 128{,}304)\beta - \\ &- (838{,}656\alpha^4 - 2{,}941{,}056\alpha^3 + 3{,}411{,}072\alpha^2 - 1{,}469{,}664\alpha + 209{,}952) \\ &= 0\end{aligned}$$

and the critical load for flutter in this case is given by

$$F = \frac{(15-32\alpha)\beta^2 + (24-128\alpha)\beta + (84-496\alpha)}{2[(6-17\alpha+8\alpha^2)\beta^2 + (24-92\alpha+32\alpha^2)\beta + (120-484\alpha+256\alpha^2)]}$$

which will be identical to the critical loads for flutter of the same system with no damping.

For example, if the elimination of the destabilizing effect of damping for the case $\alpha = 1$ is desired, β must be equal to the positive, real root of the quintic

$$\beta^5 + 6\beta^4 - 86\beta^3 - 884\beta^2 - 2{,}616\beta - 2{,}448 = 0$$

i.e.,

$$\beta = 4 + 5\sqrt{2} = 11.071$$

which, together with $\alpha = 1$, yields

$$F = \frac{7}{2} - \sqrt{2} = 2.086$$

The critical load for $\alpha = 1$ in the undamped system determined in [1,4,5] is identical to the value we obtained above. The complete elimination of the destabilizing effect for this case is thus attained as is illustrated in Fig. 6. For $\alpha = \frac{3}{4}$, a similar procedure will show that the destabilizing effect is completely removed when $\beta = \infty$. This is illustrated in Fig. 7.

The possibility of complete elimination of the destabilizing effect depends on the existence of a positive, real root in the foregoing quintic. The range of α where the elimination of the destabilizing effect is of interest to us is, of course, $0.423 \leq \alpha \leq 1.305$. However, it is found that in the range

$$\frac{3}{7} < \alpha < \frac{3}{4}$$

the quintic has no positive, real root. Thus in this range the system will always experience some destabilizing for whatever value of β in its range $0 \leq \beta \leq \infty$.

For instance, let us consider the case $\alpha = 0.6$, where the critical load for the system with no damping is

$$F_e = \frac{5}{58}(37 - 6\sqrt{5}) = 2.033$$

While the critical load for the system with damping is given by

$$F_d = \frac{1.4\beta^2 + 11.4\beta + 2 - (\beta+6)\sqrt{0.36\beta^2 + 2.88\beta + 0.36}}{(3.2\beta+6.4)(\alpha_o - 0.6)}$$

where

$$\alpha_o = \frac{\beta^2 + 12\beta + 4}{8(\beta+2)}$$

The ratio of F_d to F_e versus β is plotted in Fig. 8. It is noted that the value of F_d/F_e increases as β increases and approaches $29/5(37-6\sqrt{5}) = 0.984$, instead of 1, as the upper limit when β approaches infinity; i.e., the destabilizing effect of damping is at least 1.6% if the value of α is kept at 0.6. The discovery of this novel phenomenon is a further indication of the rather peculiar effects associated with damping in nonconservative systems.

In the range $1.182 < \alpha < 1.305$ the undamped system has multiple critical loads for flutter given by $K(\alpha,F) = 0$. However, an investigation of the roots of the quintic shows that for any α in the range $1.182 \leq \alpha \leq 1.285$ there is only one positive, real root which defines a curve of the family $X(\alpha,F,\beta) = 0$ tangent to the lower part of $K(\alpha,F) = 0$. Thus, in the range $1.182 \leq \alpha \leq 1.285$ the damped system has no critical load which is given by the upper part of $K(\alpha,F) = 0$.

As an alternative, the possibility of eliminating the effects of damping could also be studied by equating the frequencies first and then the critical forces, obtained with and without damping. The frequency of the undamped system is given by

$$\mathrm{Im}\Omega = \frac{1}{2}\sqrt{7-2(2-\alpha)F}$$

while the frequency of the system with damping is given by [7]

$$\mathrm{Im}\Omega = \sqrt{\frac{p_3}{p_1}} = \sqrt{\frac{(B_1+B_2)-(1-\alpha)(B_1+2B_2)F}{B_1+6B_2}}$$

Equating the two expressions and eliminating F in $K(\alpha,F) = 0$, leads to

$$28(\alpha-\tfrac{3}{7})(\alpha-\tfrac{3}{4})\beta^2+4(16\alpha^2-33\alpha+9)\beta+4(182\alpha^2-297\alpha+81) = 0$$

which in turn gives the range

$$\frac{3}{7} < \alpha < \frac{3}{4}$$

in which elimination of the damping effect is not possible for positive damping.

Fig. 9 illustrates the function $\beta(\alpha)$ which insures elimination of damping effects. For completeness the required values of negative β in the range $\frac{3}{7} < \alpha < \frac{3}{4}$ have also been indicated.

Concluding Remarks

The foregoing stability analysis of a simple, linear system with two degrees of freedom with slight viscous damping and subjected to nonconservative forces leads to several conclusions concerning the existence of certain features. These included multiple ranges of stability and instability, nonremovable destabilizing effects due to damping and the influence of damping on instability mechanisms.

It should be emphasized that these conclusions are consequences of the model selected for study. They are believed to be of considerable interest in themselves but no question is raised here as to the possibility of realization of systems which would be representable by the model studied. This question will be treated in separate studies, together with the associated problem concerning the validity of initial assumptions, which can be resolved only by systematic experiments.

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CAPTIONS OF FIGURES

1. Two-degree-of freedom model
2. General nature of roots of the characteristic equation
3. Types of divergent motion
4. Critical loads and instability mechanisms for $\beta = 0$
5. Critical loads and instability mechanisms for $\beta = 1$
6. Critical loads and instability mechanisms for $\beta = 11.071$
7. Critical loads and instability mechanisms for $\beta = \infty$
8. Critical load versus ratio of damping coefficients for $\alpha = 0.6$
9. Appropriate values of β versus values of α for complete elimination of destabilizing effect

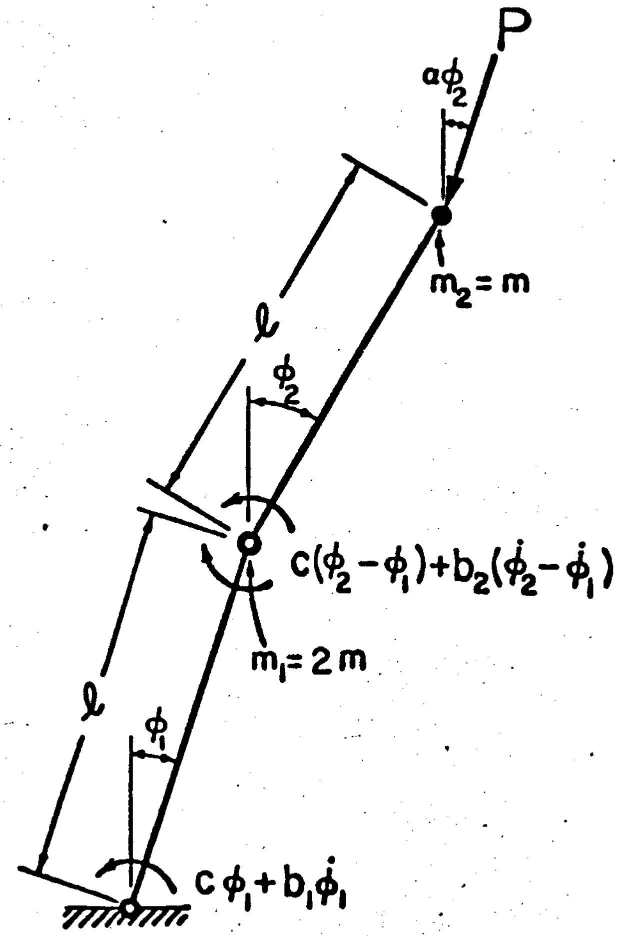
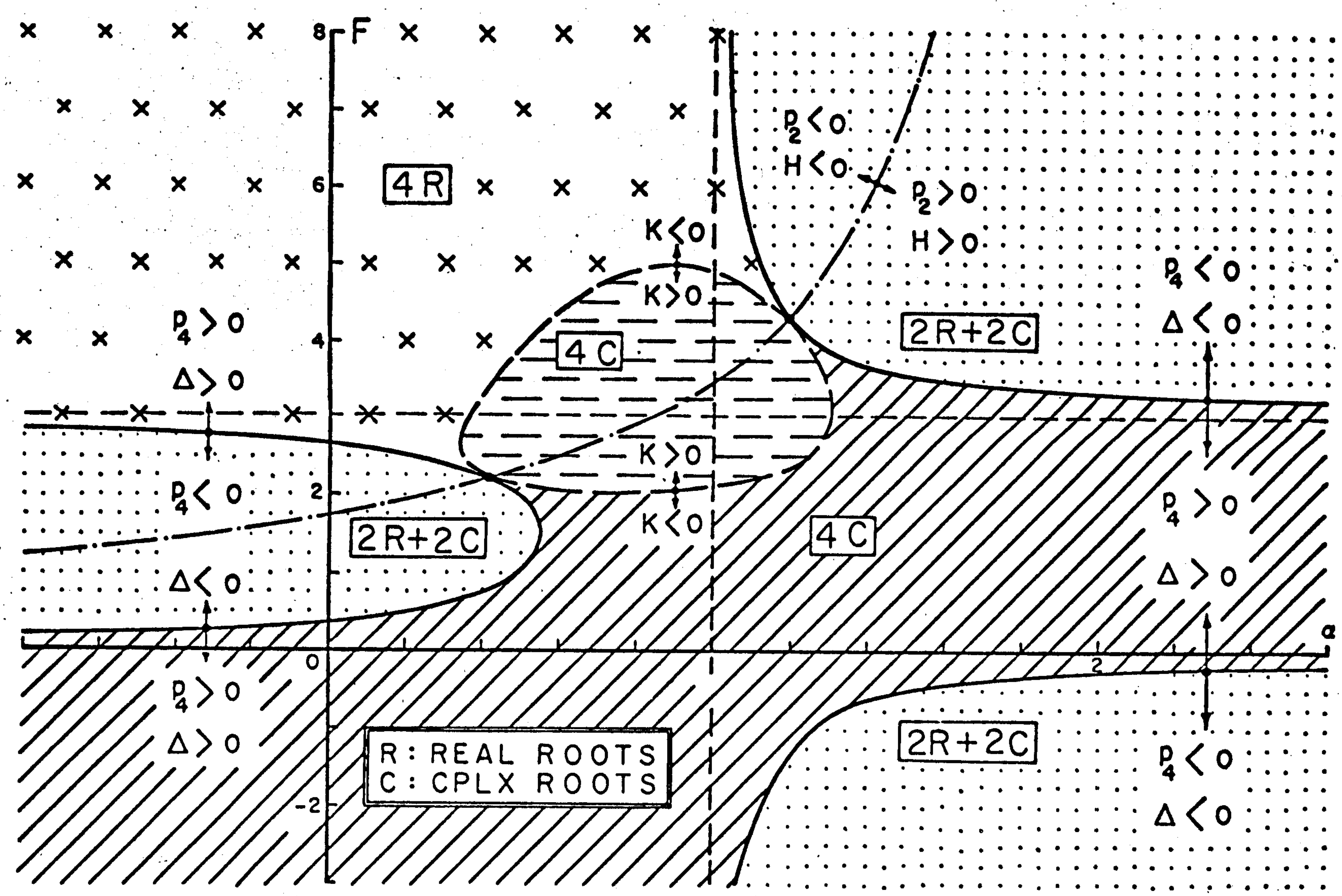

P
$a\phi_2$
$m_2 = m$
ℓ
ϕ_2
$c(\phi_2 - \phi_1) + b_2(\dot{\phi}_2 - \dot{\phi}_1)$
$m_1 = 2m$
ℓ
ϕ_1
$c\phi_1 + b_1\dot{\phi}_1$

Fig. 1


F
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6
4
2
0
-2
2
α
4R
4C
4C
2R+2C
2R+2C
2R+2C
K<0
K>0
K>0
K<0
p2<0
H<0
p2>0
H>0
p4>0
Δ>0
p4<0
Δ<0
p4>0
Δ>0
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Δ<0
p4>0
Δ>0
p4<0
Δ<0
R: REAL ROOTS
C: CPLX ROOTS

Fig. 2

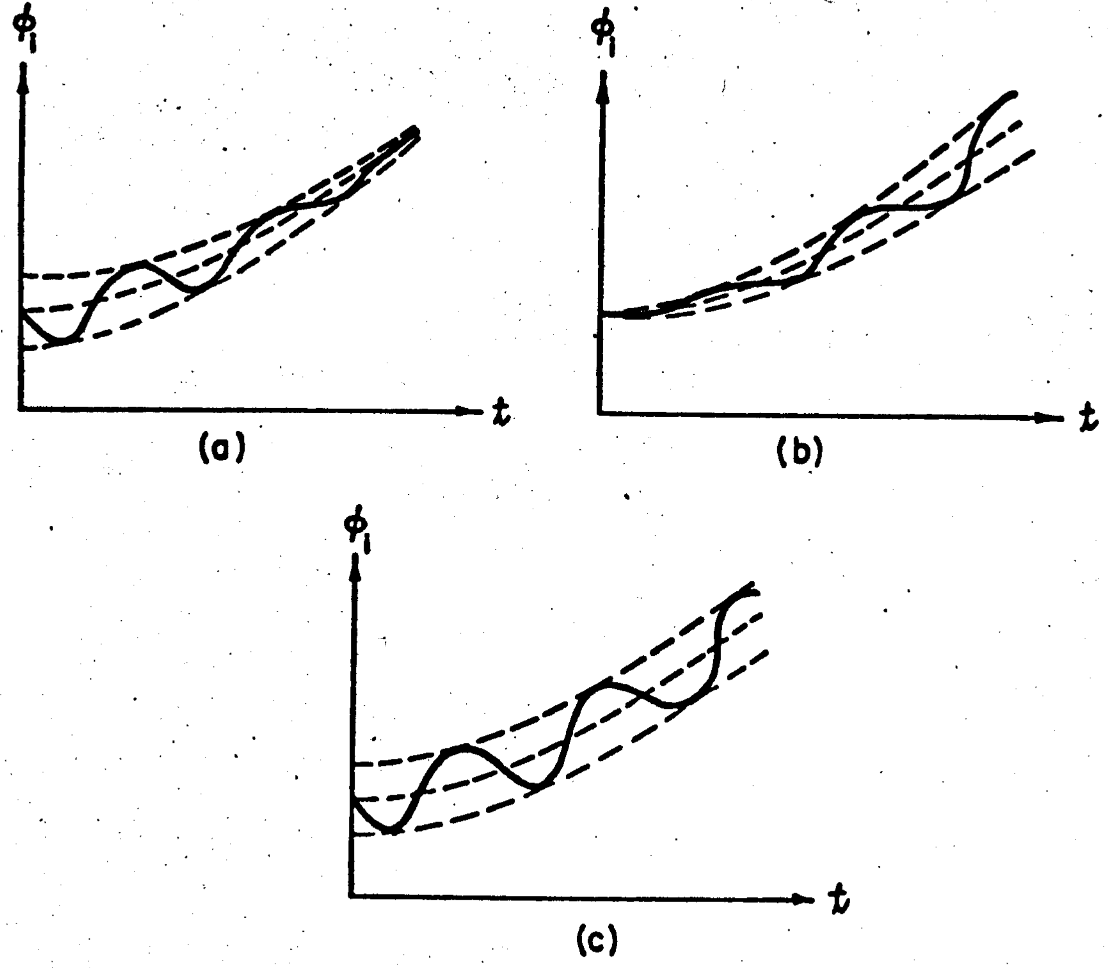

Fig. 3

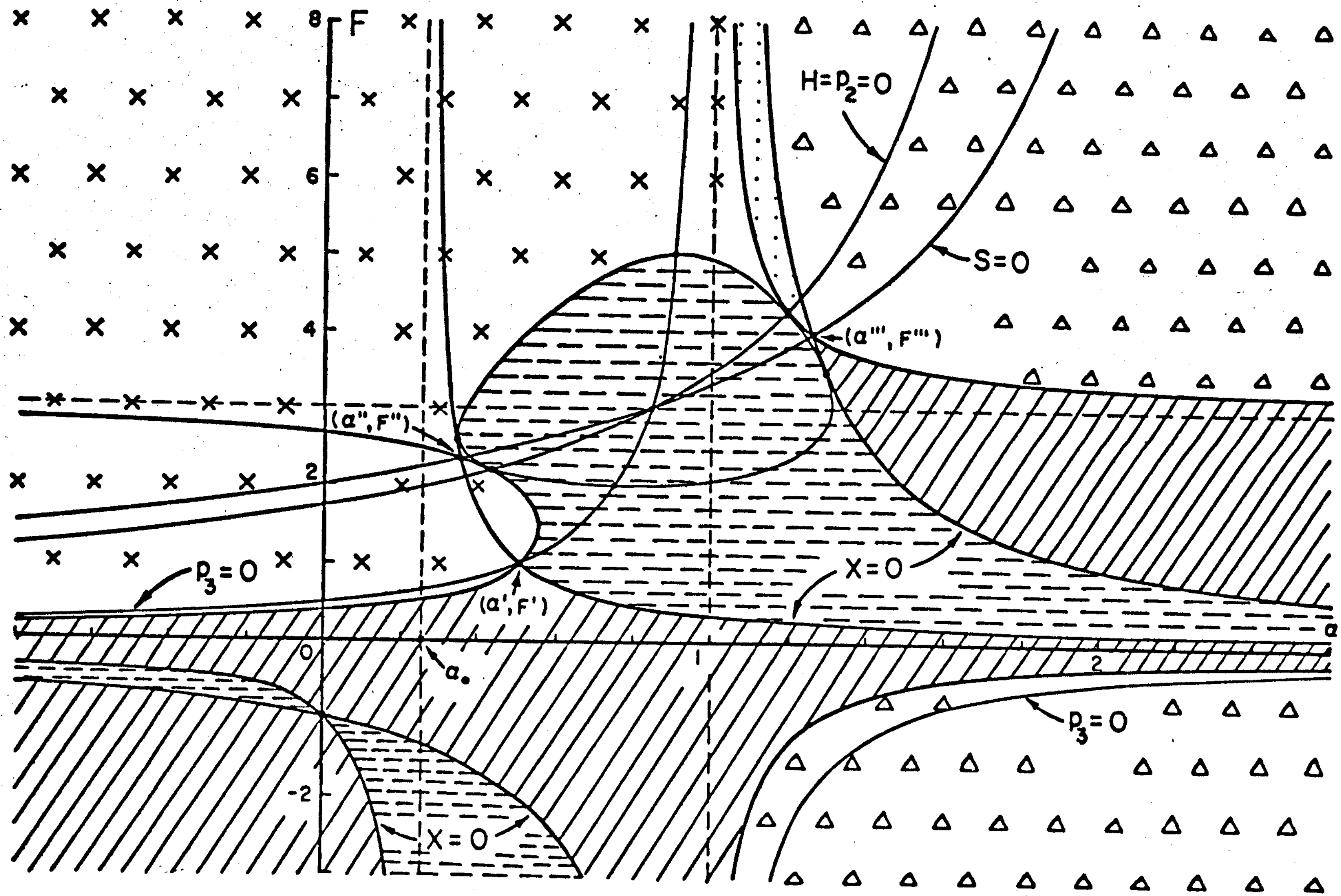

Fig. 4

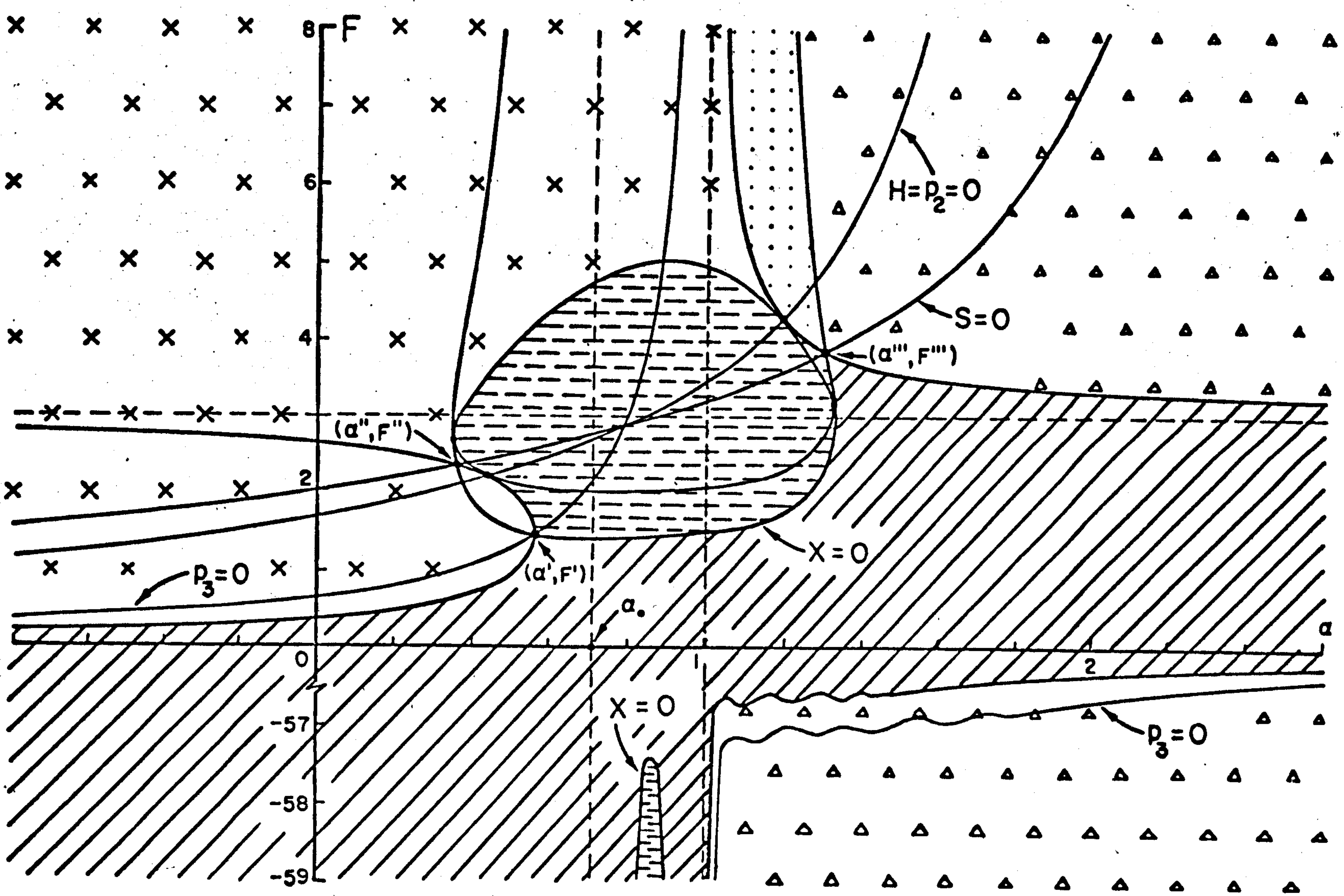

Fig. 5

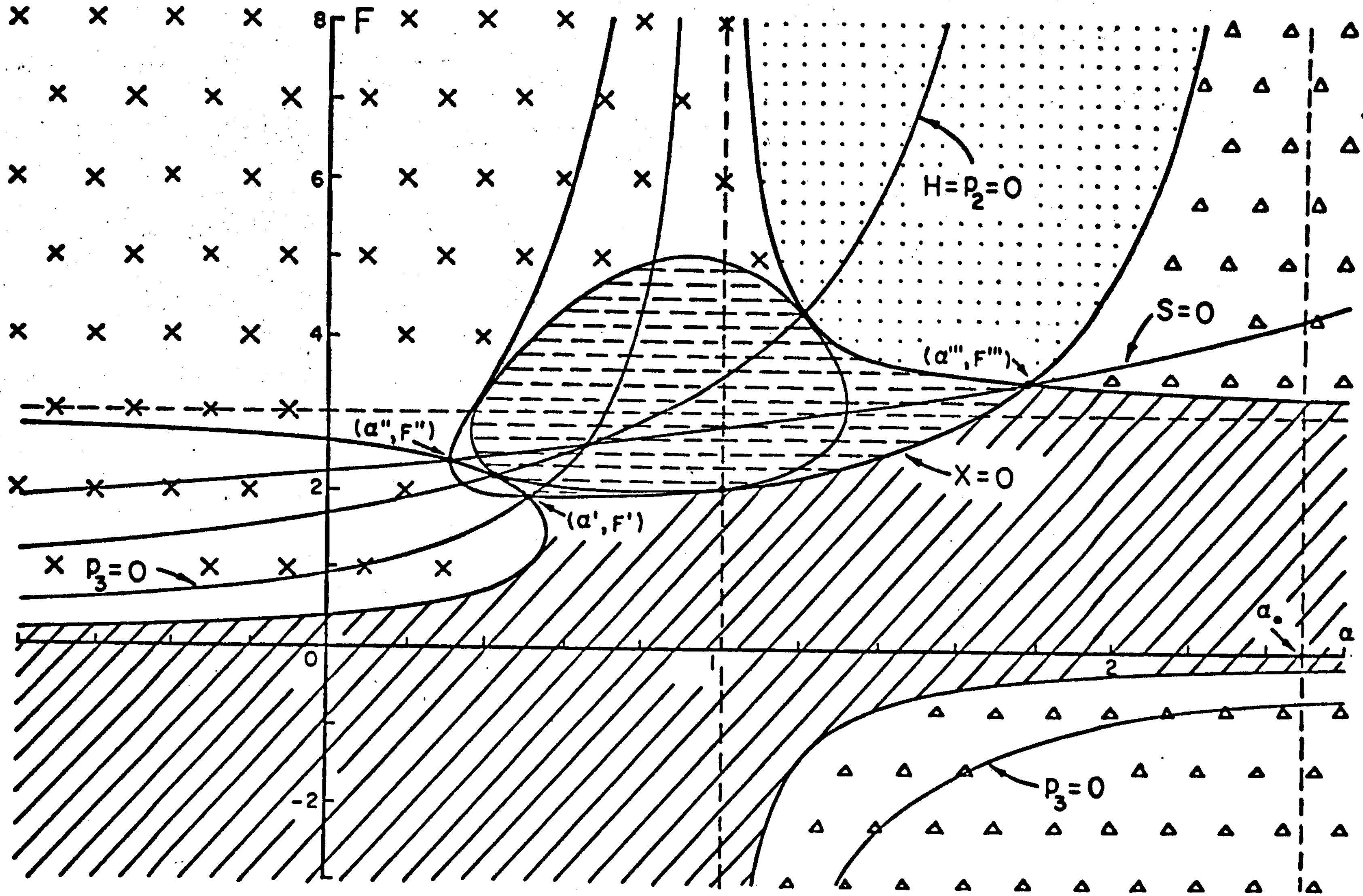

Fig. 6

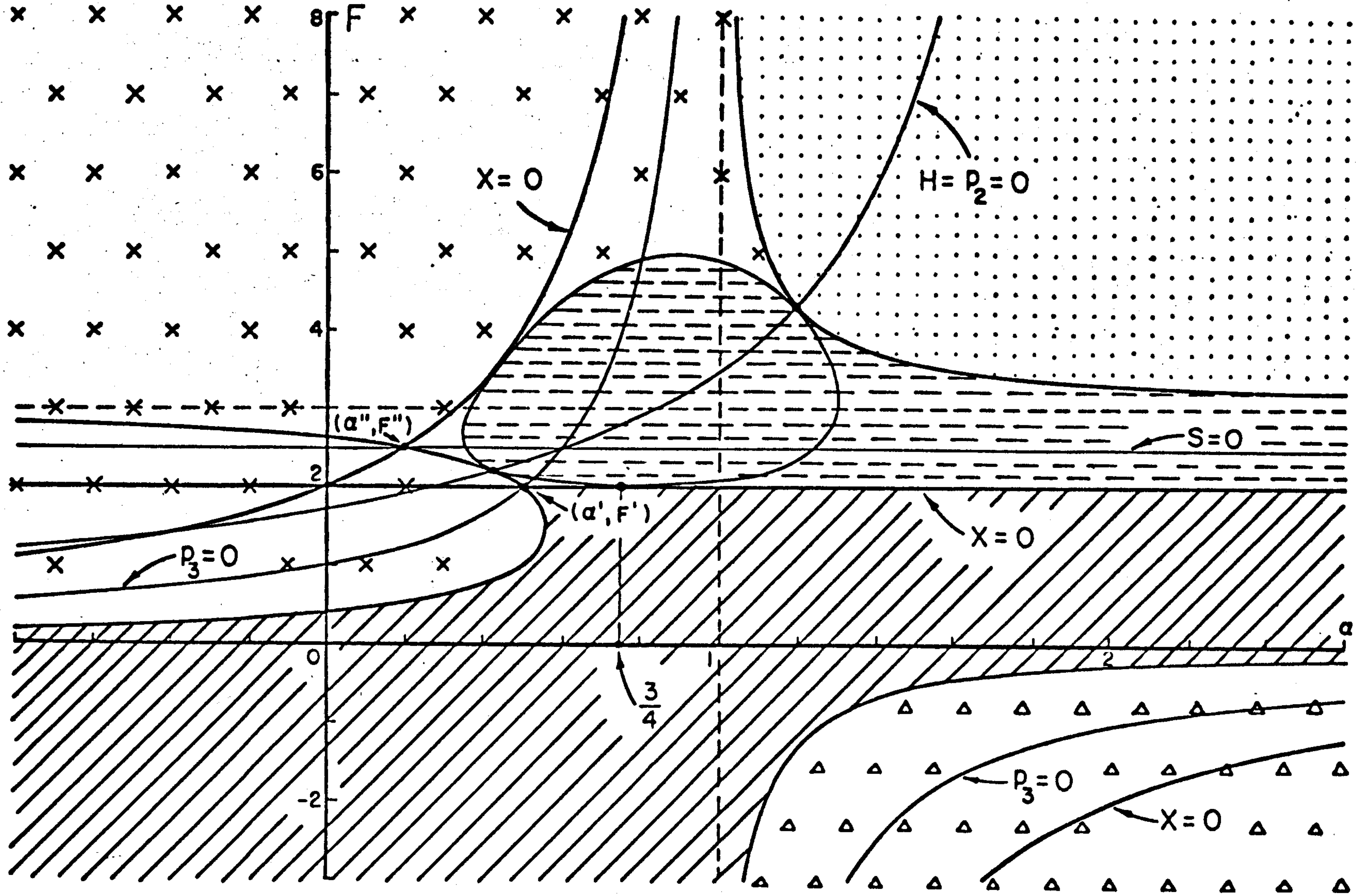

Fig. 7

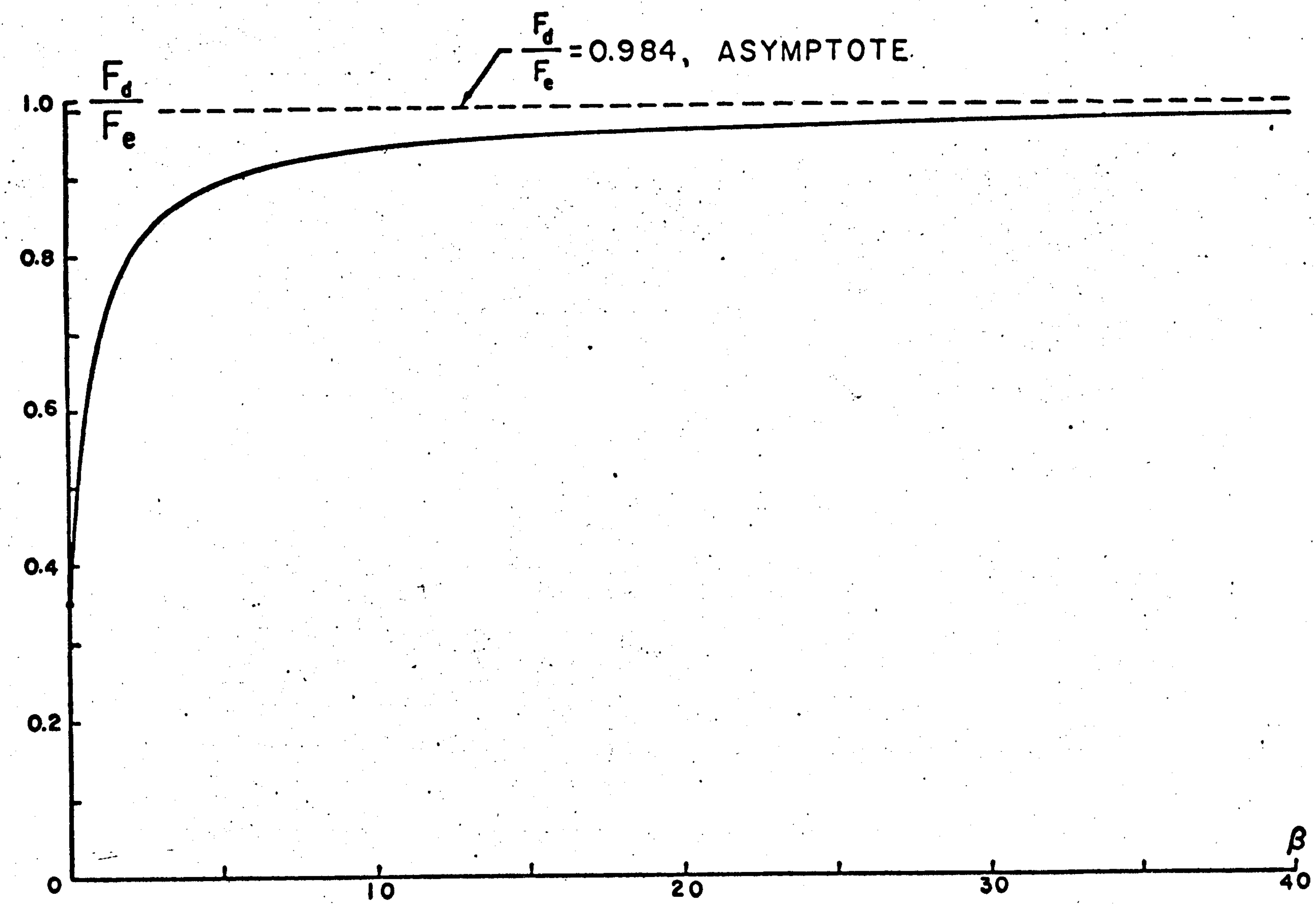

Fig. 8

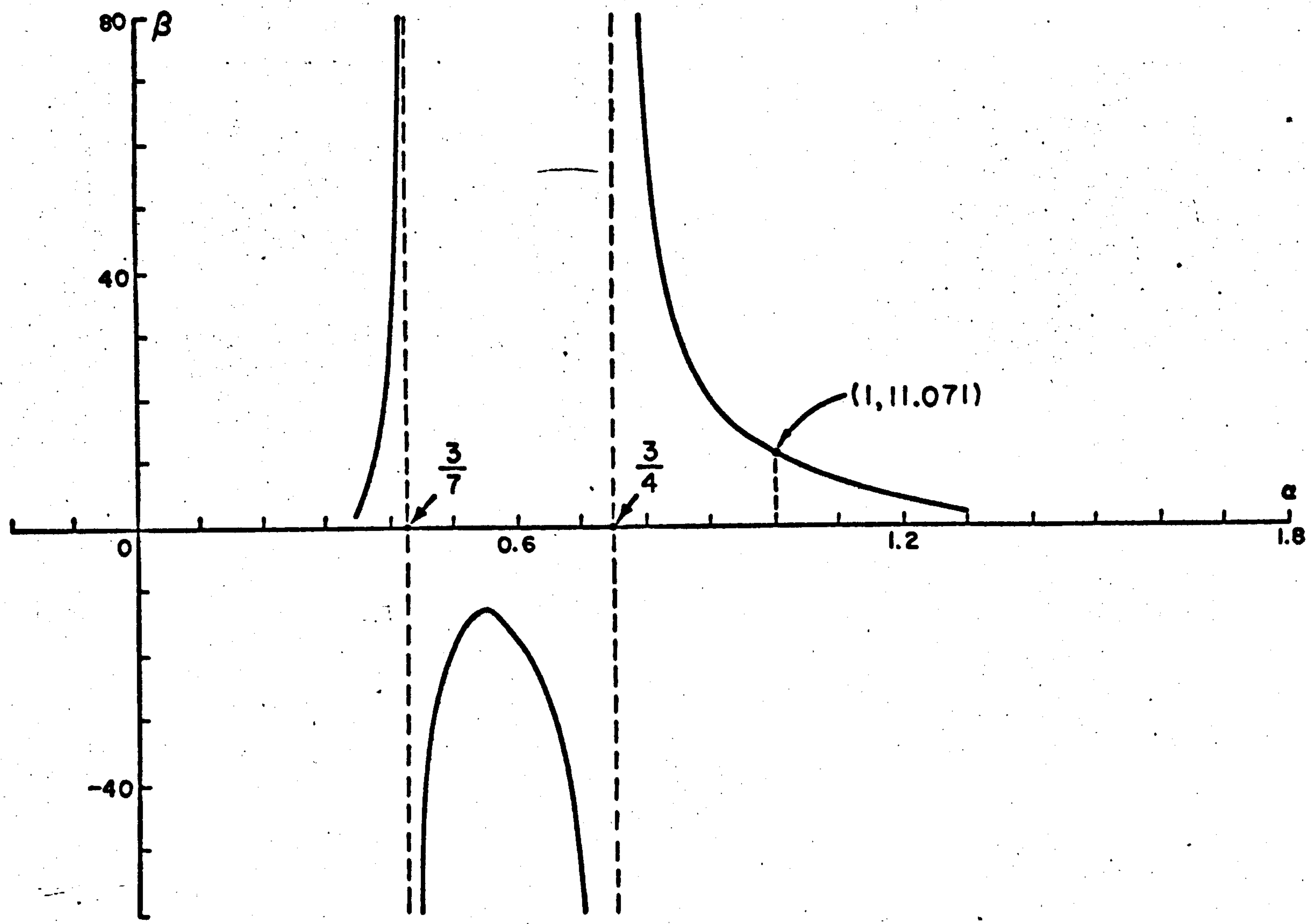

Fig. 9