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TITLE – Relationships Between Geographic and Inertial Coordinates of Position

TM – 66-2012-1

FILING CASE NO(S) – 310

DATE – March 17, 1966

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FILING SUBJECT(S) – Navigation
(ASSIGNED BY AUTHOR(S) –

ABSTRACT

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Relationships between geographic (longitude, geodetic latitude, altitude) and inertial (x, y, z) coordinates of position are examined and computational techniques given. The distinction between geodetic latitude, geocentric latitude and geocentric declination are shown: errors resulting from misuse of these angles are given.

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BELLCOMM, INC.

SUBJECT: Relationships Between Geographic and Inertial Coordinates of Position - Case 310

DATE: March 15, 1966

FROM: W. G. Heffron
S. B. Watson

TM-66-2012-1

TECHNICAL MEMORANDUM

This memorandum examines the relationships between geographic coordinates (altitude, geodetic (or geocentric) latitude of the sub-vehicle point, longitude) and (x, y, z) components of inertial position. Formulas are developed for conversion from one system to the other. Special emphasis is placed on the distinction between geodetic latitude (which applies only for the sub-vehicle point), the geocentric latitudes of the sub-vehicle point and the geocentric declination (latitude) of the vehicle itself. It is shown that the numerical differences between these three latitudes are small. Despite the small differences, conversion from geographic coordinates to x, y, z coordinates can lead to serious errors if the distinctions between the three types of latitude are not preserved. This is shown by a numerical example.

The quantities of interest are:

(x,y,z) coordinates[1,2] of the vehicle. The z axis is the Earth's North Pole axis of the present instant, the x axis is towards the first point of ♈ and the y axis completes a right-handed set. See Figure 1.

(w,z) coordinates of the vehicle in the meridian plane of the vehicle. $w = \sqrt{x^2 + y^2}$, z is as above.

(u,v) coordinates of the sub-vehicle point in the vehicle meridian plane. The line between (u,v) and (w,z) is perpendicular to the reference ellipsiod.[3] The vehicle is at the zenith relative to the (u,v) point.

R radius to the vehicle $R = (x^2 + y^2 + z^2)^{1/2}$.

ϕ geodetic latitude, which is the angle between the equatorial plane and a line passing through (u,v) perpendicular to the ellipsoid (the zenith ray).

ψ geocentric latitude of the sub-vehicle point, which is the angle between the equatorial plane and the radius vector to the sub-vehicle point (u,v).

δ geocentric declination of the vehicle, which is the angle between the equatorial plane and the radius vector to the vehicle point (x,y,z) or (w,z).

λ longitude (earth-fixed) of the vehicle,which is the angle between the Greenwich meridian and the vehicle meridian, positive eastward from Greenwich.

GHA♈ Greenwich Hour Angle, the angle between the x-z inertial plane and the meridian of Greenwich.

H altitude, which is the distance along the zenith ray from the sub-vehicle point to the vehicle.

a,b major and minor axes of the reference ellipsoid.

α the angle $\phi-\delta$

D a distance which closely approximates the altitude.

The relationships are developed from a computational viewpoint. First the computation of (x,y,z) given H, ϕ and λ (or ψ or δ in place of the standard coordinate ϕ) is considered, and then the inverse problem is treated. The latter is shown to involve transcendental functions in computing ϕ and ψ, so approximate methods are developed and numerical comparisons are given.

<u>Computation of x,y,z</u>

The equation of the reference ellipse (see Figure 2) is

$$\left(\frac{u}{a}\right)^2 + \left(\frac{v}{b}\right)^2 = 1 \qquad 1.$$

From this it can be deduced that

$$\tan \phi = \left(\frac{v}{u}\right) \left(\frac{a}{b}\right)^2 \qquad 2.$$

And so, by definition of the angle ψ, the relationship between ψ and ϕ is found to be

$$\tan \phi = \left(\frac{a}{b}\right)^2 \tan \psi \qquad 3.a.$$

or

$$\tan \psi = \left(\frac{b}{a}\right)^2 \tan \phi \qquad 3.b.$$

These formulas are independent of the altitude of the vehicle, since only the sub-vehicle point is involved.

If ϕ or ψ is given, the other can be easily found, and the ratio $\frac{v}{u}$ formed. Substitution of this in equation 1, and solving for u, gives

$$u = \frac{a}{\left[1 + \left(\frac{a}{b}\tan\psi\right)^2\right]^{1/2}} = \frac{a}{\left[1 + \left(\frac{b}{a}\tan\phi\right)^2\right]^{1/2}} \qquad 4.$$

and then

$$v = u \tan \psi = u \; \frac{b}{a}^2 \tan \phi \qquad 5.$$

It can also be deduced that the intercept of the zenith ray with the equatorial plane occurs at $u(1 - \left(\frac{b}{a}\right)^2)$ since this ray has slope $\tan \phi$ and passes through (u,v).

Using these relationships, then

$$w = u + H \cos \phi \qquad 6.a.$$

$$z = v + H \sin \phi \qquad 6.b.$$

and

$$\tan \delta = \frac{z}{w} \qquad 7.$$

The equatorial plane component w is resolved into x and y components using the longitude λ and the Greenwich Hour Angle GHA. The standard formula for GHA is (where the xyz system is defined for January 0.5, 1950)

$$GHA♈° = 100.075543 + 0.9856473460D + 2.9015x10^{-13}D^2 + \omega t \quad 8.$$

where D = number of integral days past the given epoch, t is the time in seconds past the 0 hr at Greenwich, and ω is given by

$$\omega = \frac{360}{86164.09972 + 4.49075 \times 10^{-8}D} \text{ deg/sec} \quad 9.$$

and so

$$x = w \cos (\lambda° + GHA♈°) \quad 10.$$

$$y = w \sin (\lambda° + GHA♈°) \quad 11.$$

and finally,

$$R = (x^2 + y^2 + z^2)^{1/2} = (w^2 + z^2)^{1/2} \quad 12.$$

Computation of λ, ϕ, H

Given x,y,z, the longitude is first found using the GHA formula and the equation

$$\lambda° = \arctan \left(\frac{y}{x}\right)^{\circ} - GHA♈° \quad 13.$$

H and ϕ are more difficult to find. First w is computed and then equations 6a and 6b are put in the form

$$u = w - H \cos \phi \quad 14.$$

$$v = z - H \sin \phi \quad 15.$$

If H and ϕ are correct, the equations for the ellipse and for tan ϕ (equation 1 and 2) will be satisfied, i.e.

$$\left(\frac{w-H\cos\phi}{a}\right)^2 + \left(\frac{z-H\sin\phi}{b}\right)^2 = 1 \quad 16.$$

and

$$\tan \phi = \left(\frac{a}{b}\right)^2 \frac{z-H\sin\phi}{w-H\cos\phi} \quad 17.$$

These are put in the following form

$$f(\phi,H) = 0 = \frac{w^2}{a^2} + \frac{z^2}{b^2} - 2H\left(\frac{w\cos\phi}{a^2} + \frac{z\sin\phi}{b^2}\right) + H^2\left(\frac{\cos^2\phi}{a^2} + \frac{\sin^2\phi}{b^2}\right) - 1 \qquad 18.$$

$$g(\phi,H) = 0 = \frac{z\cos\phi}{b^2} - \frac{w\sin\phi}{a^2} - H\sin\phi\cos\phi\left(\frac{1}{b^2} - \frac{1}{a^2}\right) \qquad 19.$$

Obviously these equations are transcendental and require either use of approximations or an iterative procedure. The approximations are discussed below and then an iterative technique (for which one iteration appears sufficient) is given. The approximations are discussed first because they are good starting points for any iterative procedure.

Approximate H and ϕ

Numerical computations as well as analysis show that the distance D (see Figure 2) is an excellent approximation to H. Figure 3 bears this out, showing the difference always to be less than 36 meters. Using the Fischer reference ellipsoid[3] for which a = 6 378 166 m., b = 6 356 784 m. and f = 1/298.3, D is given by

$$\left(\frac{w}{a}\right)^2 + \left(\frac{z}{b}\right)^2 = \left(\frac{R}{R-D}\right)^2 \qquad 20.$$

or

$$D = R\left(1 - \left[\left(\frac{w}{a}\right)^2 + \left(\frac{z}{b}\right)^2\right]^{-1/2}\right) \qquad 21.a.$$

or

$$D = R - a\left[1 - \left[\left(\frac{a}{b}\right)^2 - 1\right]\sin^2\theta\right]^{-1/2} \qquad 21.b.$$

and D is thus a first approximation to H.

To develop an approximation to ϕ, a starting point is to rewrite $g(\phi,H)$ in terms of the small angle $\alpha = \phi - \delta$, and D. The angle δ is known from equation 7, and D is as just given. That α is small is shown in Figure 4, which gives the true α for several latitudes and altitudes. (The difference between ϕ and ψ is equal to α for H = 0.) The function $g(\phi,H)$ is used because it is more sensitive to α than the $f(\phi,H)$ function. Using small angle assumptions and making several simplifications, the approximation to α, called α_*, is given by

$$\alpha_* = \frac{\left(\frac{a}{b}\right)^2 - 1}{\left(\frac{a}{b}\right)^2 + 1} \; \frac{\left(1 - \frac{D}{R}\right)\sin 2\delta}{1 - \frac{\left(\frac{a}{b}\right)^2 - 1}{\left(\frac{a}{b}\right)^2 + 1}\left(1 - 2\frac{D}{R}\right)\cos 2\delta} \quad \text{(radians)} \qquad 22.$$

or

$$\alpha_* = (0.19239808^\circ) \; \frac{\left(1 - \frac{D}{R}\right)\sin 2\delta}{1 - (.0033579801)\left(1 - 2\frac{D}{R}\right)\cos 2\delta} \text{(degrees)} \qquad 23.$$

and the approximation to ϕ is

$$\phi = \delta + \alpha_* \qquad 24.$$

Using a 27 significant bit digital computer for calculations, the error in ϕ from this formula occurs in the 26th and 27th bits, i.e., is less than $2\text{x}10^{-6}$ degrees. Thus, this is for most purposes a sufficiently accurate formula for ϕ.

<u>An Iterative Technique</u>

If greater precision is required a Newton-Raphson technique may be used. One form is the matrix equation

$$\begin{bmatrix} \phi \\ H \end{bmatrix} = \begin{bmatrix} \delta+\alpha_* \\ D \end{bmatrix} - [A] \begin{bmatrix} g(\delta+\alpha_*,D) \\ f(\delta+\alpha_*,D) \end{bmatrix} \qquad 25.$$

where the matrix A is the inverse of the sensitivity matrix, i.e.

$$A^1 = \begin{bmatrix} \frac{\partial g}{\partial \phi} & \frac{\partial g}{\partial H} \\ \frac{\partial f}{\partial \phi} & \frac{\partial f}{\partial H} \end{bmatrix}_{(\delta+\alpha_*D)} \qquad 26.$$

Analysis shows that the diagonal terms greatly dominate the A matrix, and that a sufficient set of equations is

$$\phi = \delta + \alpha_* - \frac{g(\delta+\alpha_*,D)}{\frac{\partial g}{\partial \phi}\Big|_{(\delta+\alpha_*,D)}} \qquad 27.$$

$$H = D - \frac{f(\ + \ _*,D)}{\frac{\delta f}{\delta \phi}\Big|_{(\delta+\alpha_*,D)}} \qquad 28.$$

For a 27 significant bit digital computer the error in H lies in the 24th through 27th bits (less than 0.25 meters) and the error in ϕ now lies in the 26th and 27th bits and is often zero.

Misapplication Errors

Unfortunately the fact that the difference between ϕ, ψ and δ is always less than 0.2° leads to a tendancy to ignore the distinction between them. This can lead to errors in (w,z) and (x,y,z). It is important that the altitude be projected along the zenith ray for correct computation of (x,y,z). Figure 5 shows the correct application and two possible misapplications. At point P, the correct geocentric latitude of the sub-vehicle point has been computed but the altitude has been projected along the extension of the radius to the sub-vehicle point. For an altitude of 100 NM, the error is 619 meters, and for an altitude of 200,000 NM the error is 1,238,848 meters (about 2/3 the radius of the moon). At point Q the given "latitude = 45°" has been falsely interpreted as geocentric declination of the vehicle rather than the geodetic latitude of the sub-vehicle point (this latter being the standard meaning of latitude). Point Q is in error by 21,448 meters, and this error is independent of altitude. None of these errors should be tolerated in most applications.

Conclusions

Relatively straightforward conversion formulas between geographic and inertial components of position have been developed. The conversion from inertial to geographic coordinates involves transcendental equations, but the accuracy of the formulas given is demonstrated. The need for distinction between the geodetic latitude, the geocentric latitude of the sub-vehicle point and the geocentric declination of the vehicle is demonstrated, even though the angles differ by less than 0.2°.

W. G. Heffron

2012-WGH-mrr
SBW-jvd

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Attachment
References

REFERENCES

1. "The American Practical Navigator", N. Bowditch et al, U. S. Navy Hydrographic Office, U.S. Government Printing Office.

2. "Project Apollo Coordinate System Standards", NASA Standard SE 008-001-1, OMSF, NASA, June 1965

3. "An Astrogeodetic World Datum from Geoidal Heights Based on the Flattening f-1/298.3", Jane Fischer, Journal of Geophysical Research, Vol. 65 No. 7, page 2067.

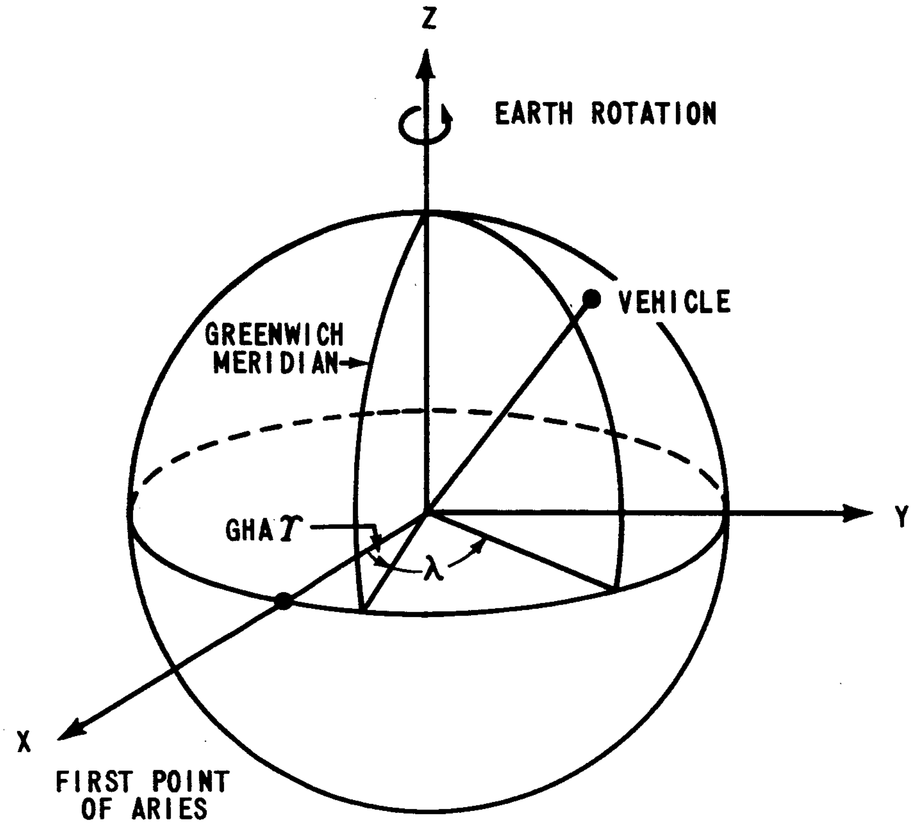

FIGURE I BASIC COORDINATE AXES

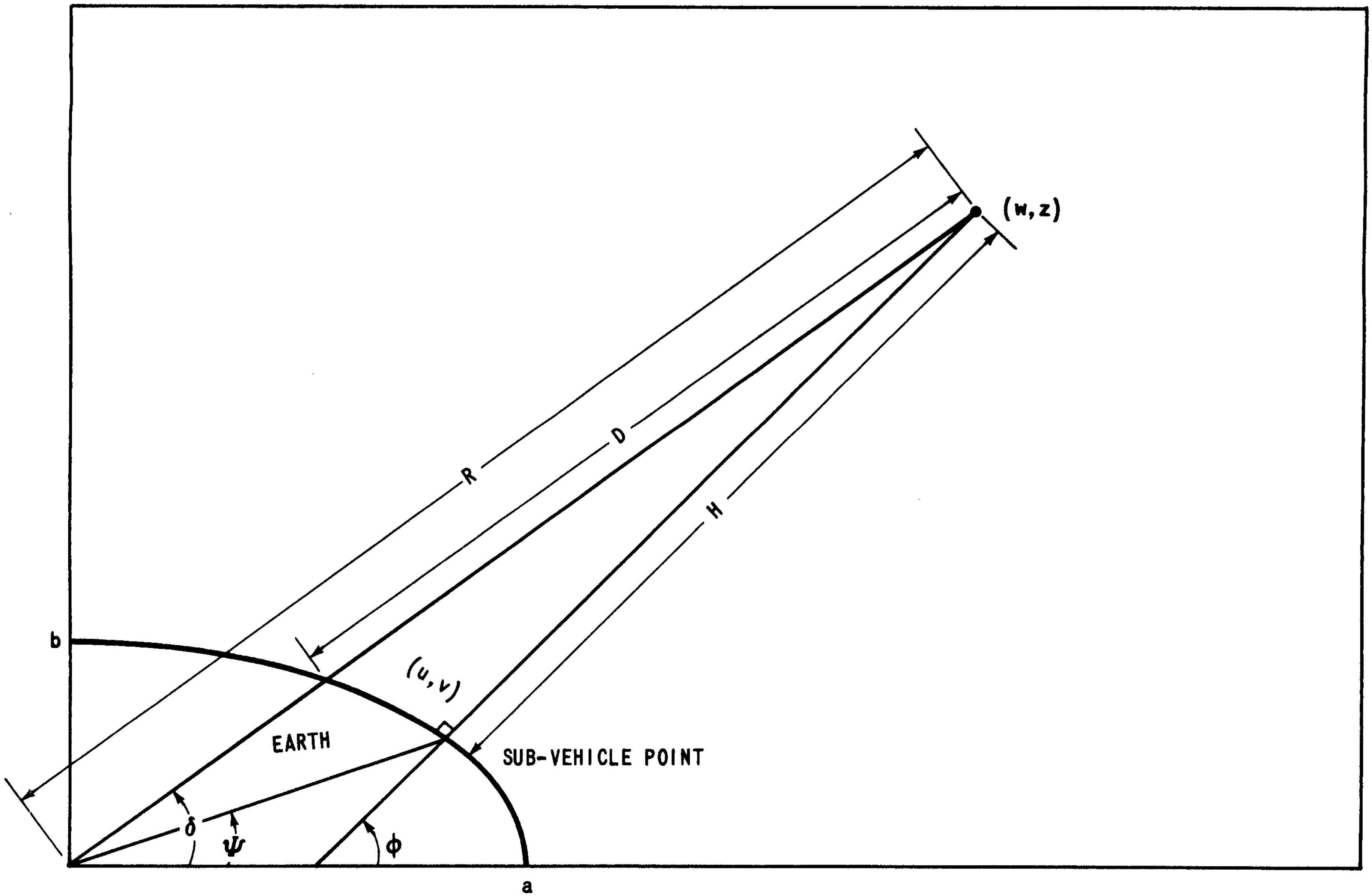

FIGURE 2 GEODETIC AND GEOCENTRIC LATITUDES ASSOCIATED WITH POINT (w,z)

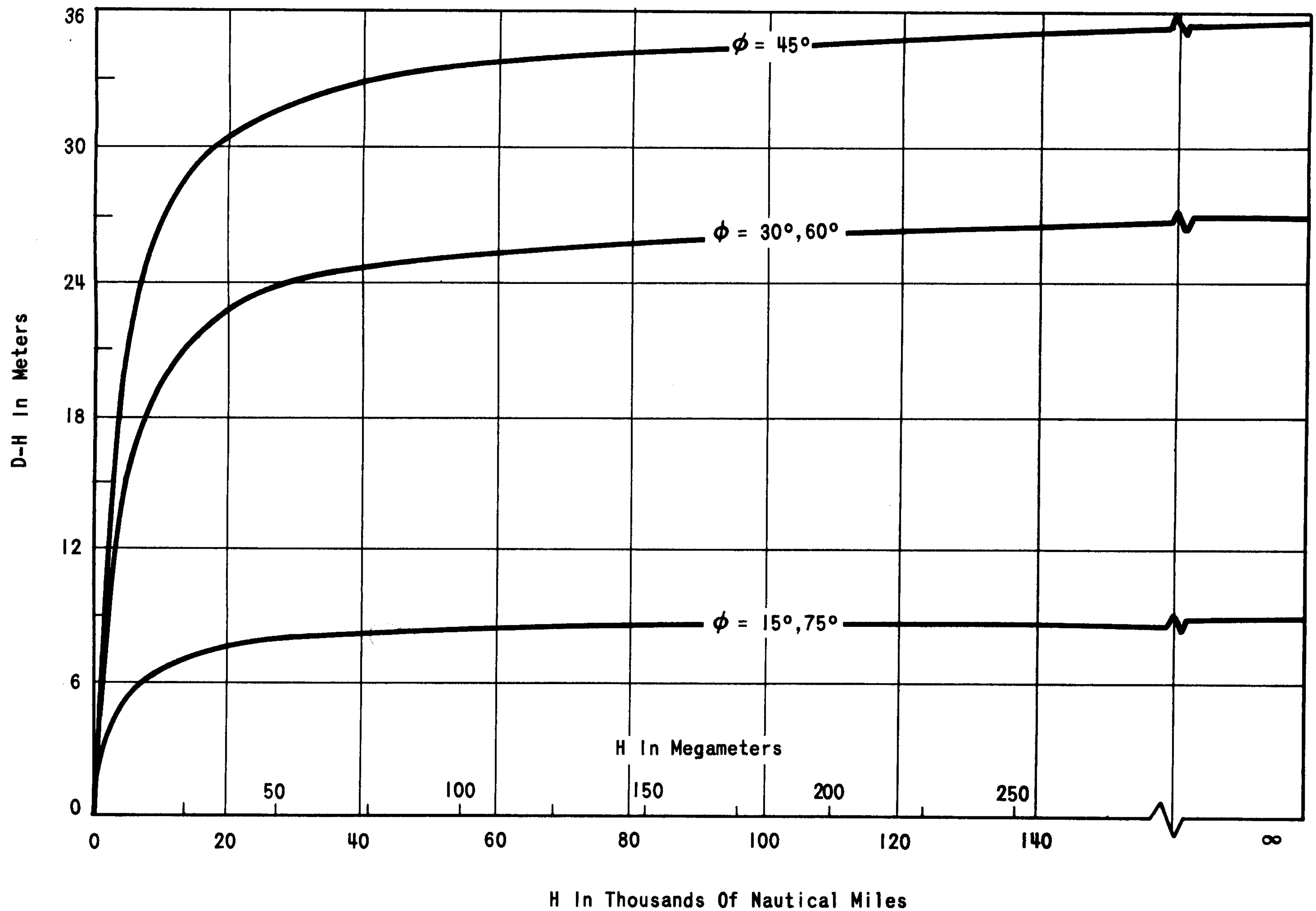

FIGURE 3 DIFFERENCE BETWEEN D AND H

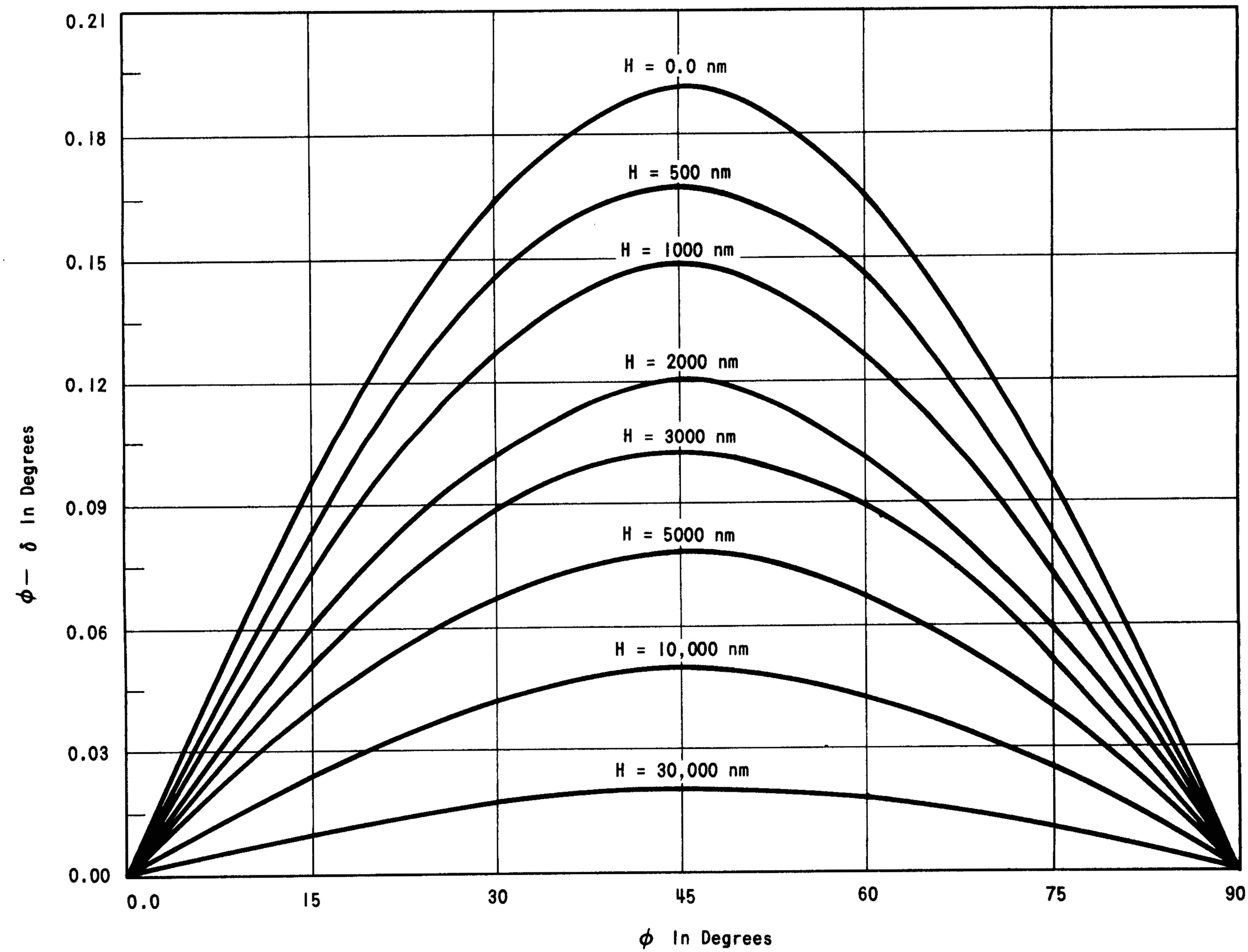

FIGURE 4 DIFFERENCES BETWEEN TRUE ϕ AND TRUE δ

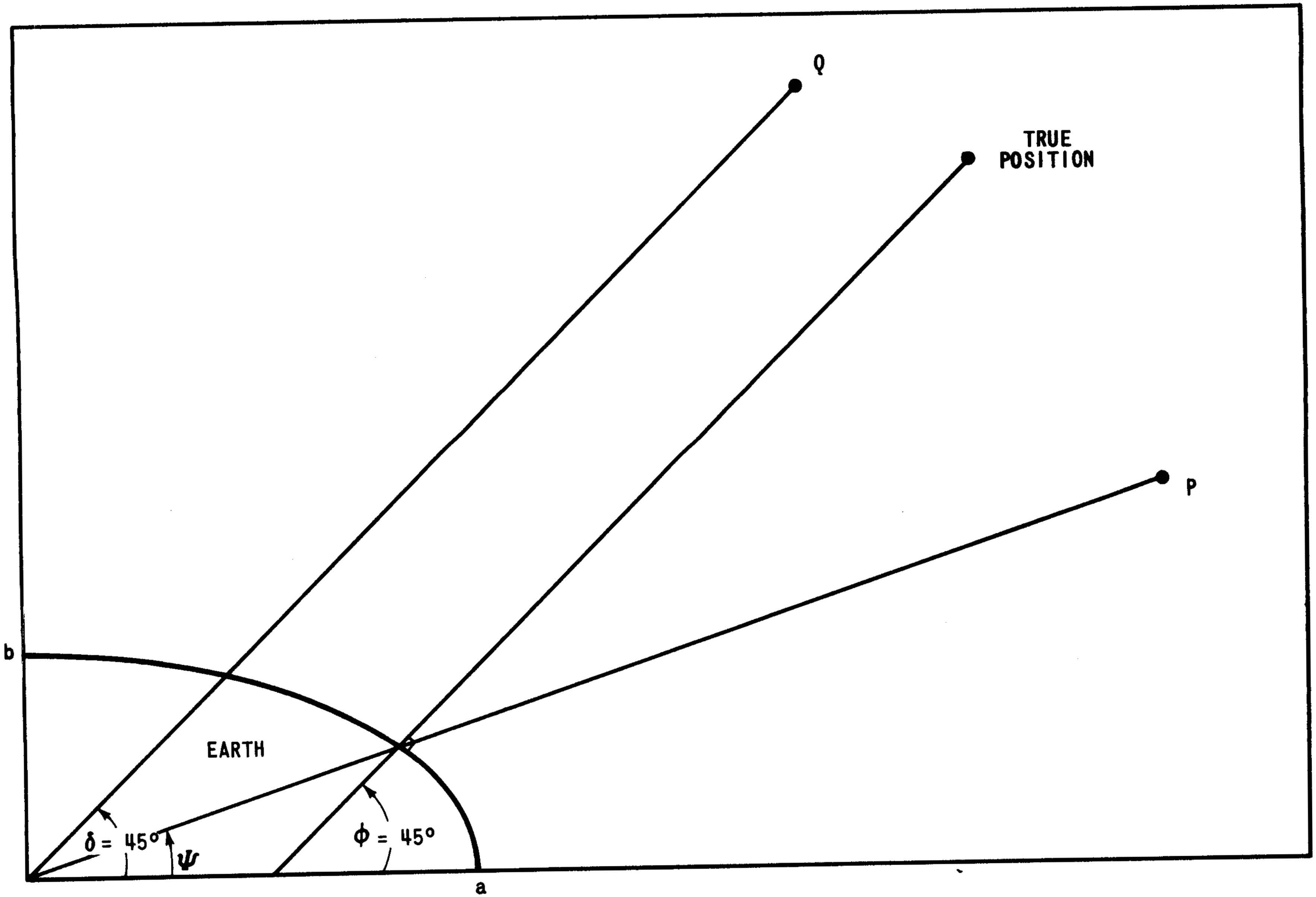

FIGURE 5 POSSIBLE MISAPPLICATIONS