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EFFECTS OF SURFACE RANDOMNESS OF RADAR BACKSCATTER FROM A SPHERICAL SURFACE

Technical Report EE-132

by

Donald H. Lenhert

Ahmed Erteza

This work performed for
National Aeronautics and
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ABSTRACT

With the recently available antenna arrays and receivers that are capable of receiving both components of power reflected from distant bodies, the possibility of using radar to determine the surface characteristics of inaccessible bodies becomes a matter for consideration. However, to accomplish this, it is necessary to have available a vector solution for the scattering of electromagnetic waves from distant rough surfaces. This paper is concerned with the obtaining of such a solution.

This analysis is based on the concept of differential reflectivity as published by Erteza, Doran, and Lenhert [1965]. The specific problem considered in this paper is, first, the integral formulation of the direct- and cross-polarized instantaneous back-scattered power from an arbitrary homogeneous rough sphere. The second part of the problem is the approximate solution of these integrals for the case of a statistically rough sphere with a gaussian roughness and with an exponential covariance function. The source is considered to be an ideal conical source which behaves as a short dipole at all frequencies. The transmitted waveform is a pulse modulated sinusoid of frequency ω_o and pulse width T. The effects of multiple scattering and shadowing are assumed negligible.

The integral expressions for the direct- and cross-polarized power obtained for the case of a pulsed source shows that, if the surface roughness is not a function of φ, the cross-polarized power is identically zero. An approximate solution is obtained for the time-averaged expected values of direct- and cross-polarized received power from a normally distributed surface (in height from an average sphere) for the condition that the ratio of standard deviation to correlation distance is much less than one over the square root of the radius of the sphere in wavelengths. Comparison of this solution with experimental data indicates that the moon must have a roughness characterized by a much larger value of this ratio. The minimum value of dielectric constant obtained from this analysis equals $1.82\epsilon_o$.

This analysis shows that the $Re[\vec{E} \times \vec{H}^*]$ can be used for the time averaged power from a pulsed source only in a portion of the return pulse and then only if the standard deviation of heights is greater than one-tenth of a wavelength. The analysis also shows that any statistical matching with experimental data must take into consideration the angular variation of the reflection coefficients. This paper is concluded with recommendations for future research.

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CHAPTER 1

INTRODUCTION

1.1 Motivation and General Aspects of the Scattering of Electromagnetic Radiation From a Rough Surface

The problems of scattering of waves from rough surfaces have been studied continuously since the late 1890's. An excellent reference and introduction to the scattering of electromagnetic waves from rough surfaces is presented by Beckmann and Spizzichino [1963] which covers both theoretical work and applications. As indicated in this book, a general and exact vector solution to the problem is as yet unavailable due to the difficulties in satisfying Maxwell's equations and the boundary conditions across an interface. In the case of smooth bodies of revolution, exact vector solutions for the reflection of an electromagnetic plane wave from the surface are available in only a few cases. In these cases, the incident plane wave is expanded into an infinite set of waves whose character is dependent upon the shape of the body and which are centered on the body. Then the boundary conditions are satisfied for each wave of the infinite set in one step for the entire surface, and the reflected waves are summed up to obtain the reflected fields. This method requires the separability of the wave equation, which is possible in only eleven coordinate systems. Also, excluding the plane surface, the infinite series obtained are difficult to handle and, except in the simplest cases, require the use of high speed digital computers to obtain numerical values.

This approach does not appear feasible for the obtaining of a general solution to the problem of scattering from rough surfaces. An additional complication in the solution of scattering from rough surfaces by use of the boundary conditions is the necessity to take into consideration not only the incident field in the absence of the surface but also the effects due to shadowing and diffraction, as well as multiple scattering.

With the advent of radar, the problem of scattering of electromagnetic waves from rough surfaces, particularly from terrains, became of special interest. As more high powered and sensitive radars became available, the scattered field from the moon could be detected. With this development, the possibility of using radar to determine the surface characteristics of inaccessible bodies became a matter for consideration. However, to accomplish this it is necessary to have available a more general solution of scattering of electromagnetic waves from distant rough surfaces. Also, with recently available antenna arrays and receivers that receive both the transmitted polarization (called the direct-polarized component) and the polarization orthogonal to the transmitted polarization (called the cross-polarized component), it is necessary to obtain a vector solution to the scattering problem. With such a solution, it is of interest to determine the effects of surface characteristics (both roughness and electromagnetic) upon the ratios of these two components.

With the possibility of making a manned landing on the lunar surface within the next few years, some information as to the surface characteristics is needed to design such landing vehicles. One source of such information is the data taken from earth-based radars. The majority of this data has been taken using highly directive pulsed radars receiving both polarizations; therefore, to determine the possible surface characteristics, it is desirable to have a vector solution to the problem of scattering of narrow beam pulsed electromagnetic radiation from a rough sphere. The remainder of this paper is concerned with the obtaining of such a solution.

1.2 Specific Problem Considered and Purpose of Investigation

The specific problem considered in this paper is, first, the integral formulation of the direct- and cross-polarized instantaneous back-scattered power from an arbitrary homogeneous rough sphere. The second part of the problem is the approximate solution of these integrals for the case of a statistically rough sphere with a gaussian roughness and with an exponential correlation function. The source is considered to be an ideal conical source whose radiated field within the cone is constant and equal to the maximum of a short dipole, and zero outside the cone. The transmitted waveform is a sinusoid of frequency ω_o and amplitude modulated with an ideal pulse containing an integral number of cycles of the transmitted frequency. The phasing is such that the transmitted waveform is a continuous function of time. The reflecting sphere is assumed to be at a long distance from the source, and the radius of curvature at any point is large compared to the wavelength. Also, the effects of multiple scattering and shadowing are assumed negligible.

1.3 Discussion of the Idealized Problem

There are several gross simplifications and strong assumptions implicit in the idealized problem described in Section 1.2 which merit discussion. First, the transmitted waveform is amplitude modulated with an ideal pulse such that the waveform is a continuous function of time, but contains a discontinuity in slope which is not physically realizable. For analysis of problems where the propagation or reflection characteristics are of primary importance, it is common practice to use physically unrealizable sources. Another physically unrealizable assumption is that the source is an ideal conical one. This assumption is made to simplify the analysis and yet show the effect of beam limiting. If a specified antenna pattern were given, it could easily be included in the description of the incident field.

The reflecting rough sphere is assumed to be homogeneous and isotropic which, for the case of the moon, is not the actual case. The more realistic case of an inhomogeneous body would considerably complicate the reflection coefficients of the surface; these are, even in the case of the Fresnel reflection, very difficult to integrate. Another assumption is that the sphere is nonconductive. The effect of conductivities of naturally occurring terrains on the reflection coefficients at radar frequencies is negligible. Consequently, it is felt that this assumption has a reasonable justification.

In the analysis presented in this paper, multiple scattering and shadowing are neglected. The effect of multiple scattering could be included by properly modifying the incident field; however, this would require the evaluation of several additional integrals, and was considered too complicated for a first analysis. The effect of shadowing was neglected to avoid the introduction of additional complication to an already complicated problem. This effect may be taken into account in the case of a statistically rough surface by statistically modifying the amplitude of the received power [Beckmann, 1965].

In considering a statistically rough surface and averaging over an ensemble of such surfaces, the random process is assumed stationary, thus implying that an ensemble average is equivalent to a time average. However, when attempting to relate the results of this analysis to experimentally obtained data from the moon, it should be realized that, while time-averaged data are available, they are not equivalent to the ensemble average obtained in this analysis. The assumption of equivalence is generally made in the literature.

The effect of the earth's atmosphere and ionosphere is neglected in the formulation of this problem. The Faraday rotation of the transmitted and reflected signals which occurs in the ionosphere causes an interchange of power between the observed direct- and cross-polarized components.

Experimentally obtained data must be analyzed carefully to eliminate the effect of the Faraday rotation before attempting to obtain the surface characteristics.

1.4 Previous Investigations

A large number of papers has been published on the subject of the scattering of electromagnetic waves from rough surfaces, especially in the last ten years. Many experimental data have been accumulated and many theories have been developed to explain and predict measured data. However, none of the theories is general and rigorous at the same time. Most of the methods make one or more of the following assumptions [Beckmann and Spizzichino, 1963].

1) The dimensions of scattering elements are much larger or much smaller than the wavelength of the incident radiation.
2) The radius of curvature is much larger than the wavelength of the incident radiation.
3) Shadowing effects are neglected.
4) Only the far field is calculated.
5) Multiple scattering is neglected.
6) The density of the scatterers is not considered.
7) The treatment is restricted to a particular model of surface roughness.
8) The surface is perfectly conducting.

The results of the various treatments are limited to the conditions under which their initial assumptions are considered valid. The reader is referred to Beckmann and Spizzichino [1963], Evans [1961], Janza [1963] or Fung [1965], each of which gives excellent summaries of the previous work and extensive bibliographies.

By far the largest number of rough surface scattering theories is based on the Kirchhoff approximation of the boundary conditions required for the evaluation of the Helmholtz integral in the scalar case, or the Stratton-Chu integral in the vector case. The most recent of such theories, using a statistical description of the surface, was made by Fung [1965]. The basic assumptions of Fung are discussed so that the differences between his analysis and the one presented in this work may be observed.

Fung made the basic assumption of the Kirchhoff approximation; namely, the radius of curvature is much greater than the wavelength of the incident radiation. In addition the following assumptions were made:

1) The surface is perfectly conducting, i.e., the reflection coefficient is independent of angle of incidence.

2) There is no shadowing of one part of the surface by another.

3) There is no multiple reflection.

4) The random surface is continuous in the mean and differentiable over a finite region.

5) The variation of the angle of incidence over the domain of integration in the case of pulse radar is negligible.

6) The radius of correlation is much smaller than the dimensions of the illuminated area.

7) The illuminated area is pulse limited.

8) The average surface is that of a plane.

9) The time-averaged expected power, in the pulsed radar case, can be obtained by using $1/2\ \mathrm{Re}\ [\vec{E} \times \vec{H}^*]$.

10) Only the direct-polarized component of power is of interest.

The results of this analysis were applied to experimentally obtained pulse radar return from the moon.

In the analysis of the present paper, assumptions 1, 5, 7, 8, 9, and 10 are removed or modified. The method used here does not use the Kirchhoff approximation, but rather the concept of differential reflectivity to obtain the integral equations. Both methods make the assumption of the general form of the tangent plane approximation.

1.5 Summary of Chapter Development

This introductory chapter contains statement of the problem and basic assumptions of the model used. Chapter 2 contains a development and delineation of restrictions of the concept of differential reflectivity. This concept is the basis of the analysis made here.

In Chapter 3 the integral equations are developed for the direct- and cross-polarized instantaneous powers reflected from an arbitrary rough sphere. This development requires the inverse Fourier transform to obtain the instantaneous powers. In conjunction with this chapter, Appendix A presents the coordinate system transforms necessary to convert the power reflected from an incremental area of the surface into a receiver-based coordinate system, allowing the separation of the two components of power. Also, Appendix B discusses the false poles of the reflection coefficients and the development of a power series expansion valid for small slopes.

In Chapter 4, the sphere is taken to be statistically rough with a gaussian distribution of heights and with an exponential correlation function. Ensemble averages of the integrals developed in Chapter 3 are taken and the integrals are solved approximately and time averaged. In conjunction with this chapter, Appendix C presents the derivation of the expected values of the various statistical terms encountered. Appendix D presents the generalized integration by parts necessary for the q - q' integrations. In Appendix E, the generalized ψ integration is accomplished by an infinite series expansion and then by exact integration. Also several special ψ integrals are considered in this Appendix. Appendix F presents the partial derivatives of the correlation function needed in the final result.

In Chapter 5, the results are discussed with a view toward a physical interpretation of the mathematical analysis. Particular emphasis is placed on the possible separation of the electromagnetic and statistical properties of the surface with a view to explain the radar backscatter obtained from the lunar surface. A discussion of the more important results of this analysis, and some suggestions for future research, conclude Chapter 5.

CHAPTER 2

CONCEPT OF DIFFERENTIAL REFLECTIVITY

2.1 Development of the Concept of Differential Reflectivity

The concept of differential reflectivity as published by Erteza, Doran, and Lenhert [1965][1] will be reviewed and some of the restrictions imposed in that paper will be removed. Differential reflectivity, $\hat{\sigma}(\vec{r}_1, \vec{r}_o, \omega)$, is a dyadic quantity which, when multiplied by a differential surface area and the steady-state vector field incident on that area, yields an expression for the contribution of that surface element to the scattered field at an arbitrary observation point. Consequently the differential reflectivity is a function of the following variables:

1) Location and orientation of the surface with respect to the observation point.

2) Properties of the two media separated by the surface.

3) Frequency of the incident radiation, ω.

For the case of steady-state incident radiation, the reflected Hertzian potential field, $\vec{\Pi}_r$, may be described by

$$\vec{\Pi}_r(\vec{r}_1, \omega, t) = \iint_{S_o} \hat{\sigma}(\vec{r}_1, \vec{r}_o, \omega) \cdot \vec{\Pi}_i(\vec{r}_o, \omega, t)dS_o \qquad (2\text{-}1)$$

[1] The Concept of Differential Reflectivity was conceived and formulated by Dr. A. Erteza.

where

$\vec{r}_1$ = radius vector from the origin to the observation point

$\vec{r}_o$ = radius vector from the origin to the surface point

$\vec{\Pi}_i(\vec{r}_o, \omega, t)$ = incident vector field at surface element ds_o

$\hat{\sigma}(\vec{r}_1, \vec{r}_o, \omega)$ = dyadic differential reflectivity

S_o = illuminated surface

The case of a pulsed sinusoidal source will be considered in Section 2.3.

It should be noted that Weyl's method of expansion of the spherical waves into plane waves [Stratton, 1941, pp. 577-582] cannot be used unless the source is spherically symmetric; however, in this paper an ideal conical source is assumed. An ideal conical source is defined as one whose electric and magnetic fields are uniform over any spherical surface centered on the source within the cone and identically zero outside of the cone. This method can also be used for any arbitrary source or antenna pattern by including an antenna factor in the integration of (2-1).

Considering only one member of the family of steady-state waves, a derivation of the theory involving the concept of differential reflectivity will now be shown. Let the Hertz vector due to the component steady-state incident waves be described by

$$\vec{\Pi}_i(\vec{r}_o, \omega, t) = \vec{a}_\pi \, C_o \frac{e^{ikR_o}}{R_o} e^{-i\omega t} \qquad (2\text{-}2)$$

for all points $\vec{r}_o$ on the surface S_o, where

$\vec{a}_\pi$ = unit vector in the $\vec{\Pi}_i$ direction (determined by polarization)

$C_o = C_o(\omega)$ and relates to the amplitude of the source

$k = \omega/c$ = propagation constant in the incident medium

$R_o = |\vec{r}_o - \vec{r}_s|$

$\vec{r}_s$ = radius vector from the origin to the source point

It should be noted that by the complex notation of (2-2) we imply that the real time function $\vec{\Pi}_i$ is the real part of the right-hand side of the equation; however, for the sake of brevity this notation will be omitted except when it is necessary for clarity.

For the case of reflection from a plane, if the origin is taken in the infinite plane surface (x-y plane) of which S_o is a region, and the source has rectangular coordinates $(0, 0, z_s)$ as shown in figure 2-1, so that

$$R_o = |\vec{r}_o - \vec{r}_s| = \sqrt{x_o^2 + y_o^2 + z_s^2}$$

then (2-2) can be written as

$$\vec{\Pi}_i(\vec{r}_o, \omega, t) = \vec{a}_\pi C_o e^{-i\omega t} \iint_{S_o} \frac{e^{ikR'_o}}{R'_o} \delta(x_o - x'_o) \cdot$$

$$\cdot \delta(y_o - y'_o) dx'_o dy'_o \qquad (2\text{-}3)$$

where

$$R'_o = \sqrt{x_o'^2 + y_o'^2 + z_s^2}$$

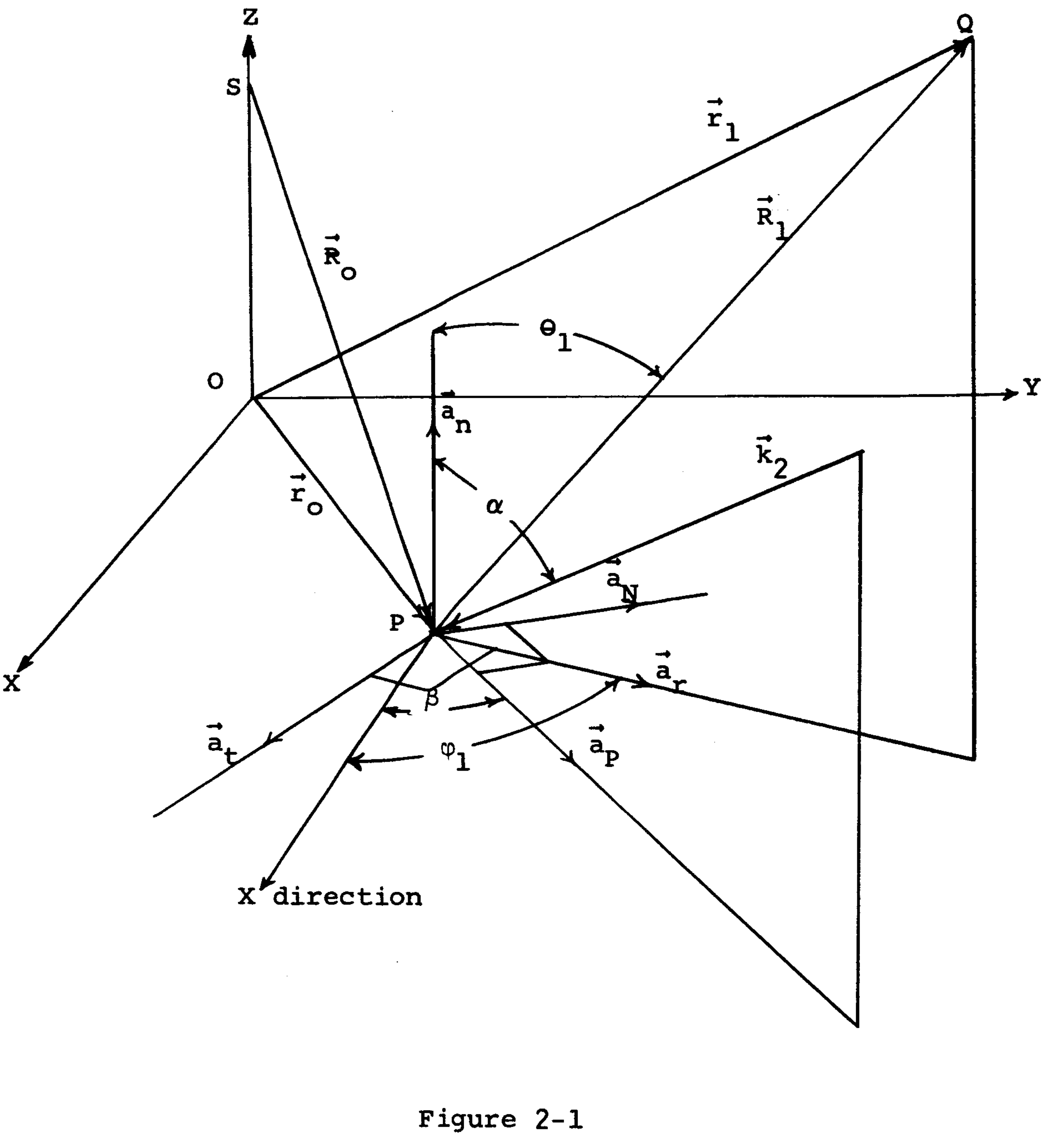

Figure 2-1
Reflection Geometry

Using the Fourier integral expansion one may write

$$\delta(x_o - x_o')\delta(y_o - y_o') = \frac{1}{4\pi^2}\iint_{-\infty}^{\infty} e^{iu(x_o-x_o')+iv(y_o-y_o')}\,dudv \qquad (2\text{-}4)$$

so that the incident field at each point, $\vec{r}_o$, of the surface is given by

$$\vec{\Pi}_i(\vec{r}_o, \omega, t) = \frac{e^{-i\omega t}}{4\pi^2}\iint_{S_o}\frac{e^{ikR_o'}}{R_o'}\left[\iint_{-\infty}^{\infty}\vec{a}_\pi C_o e^{iu(x_o-x_o')+iv(y_o-y_o')}\cdot\right.$$

$$\left.\cdot\, dudv\right]dx_o'dy_o' \qquad (2\text{-}5)$$

If now

$$r = \sqrt{x_o^2 + y_o^2 + z_o^2}\ ,$$

the following integral is obtained by analytic continuation.

$$\vec{\Pi}_i(\vec{r}, \omega, t) = \frac{e^{-i\omega t}}{4\pi^2}\iint_{S_o}\frac{e^{ikR_o'}}{R_o'}\left[\iint_{-\infty}^{\infty}\vec{a}_\pi C_o e^{iW_i}dudv\right]dx_o'dy_o' \qquad (2\text{-}6)$$

where

$$W_i = u(x_o - x_o') + v(y_o - y_o') - z_o\sqrt{K^2 - (u^2 + v^2)}$$

Expression (2-6) can be interpreted as the field due to an infinite collection of plane waves, symmetrically distributed about the local normal to the incident wave front, which combine at a point on the reflecting surface to yield the net incident field due to the original source. The propagation constant associated with each of the plane waves is determined so that the entire collection adds to a two-dimensional delta function at the point in question.

Consider now a Hertzian plane wave having a propagation vector with components $\left[u, v, \sqrt{k^2 - (u^2 + v^2)}\right]$ and a polarization in the direction $\vec{a}_\pi$ to be reflected from the surface. The reflected plane wave will be described through the use of a dyadic reflection coefficient $\hat{V}(u, v)$. At an observation point $Q(\vec{r}_1)$, the total reflected field due to an illumination of the surface S_o by the infinite set of plane waves is given by

$$\vec{\Pi}_r(\vec{r}_1, \omega, t) = e^{-i\omega t} \iint_{S_o} \left[\frac{1}{4\pi^2} \iint_{-\infty}^{\infty} \hat{V}(u, v) e^{iW_r} du dv\right] \bullet \left[\vec{a}_\pi C_o \frac{e^{ikR_o'}}{R_o'}\right] dx_o' dy_o' \tag{2-7}$$

where

$$W_r = u(x_1 - x_o') + v(y_1 - y_o') + z_1\sqrt{k^2 - (u^2 + v^2)}$$

which, when compared with (2-1), yields the differential reflectivity as

$$\hat{\sigma}(\vec{r}_1, \vec{r}_o, \omega) = \frac{1}{4\pi^2} \iint_{-\infty}^{\infty} \hat{V}(u, v) e^{iW_r} du dv \tag{2-8}$$

The form of the components of the reflection coefficient $\hat{V}$ will depend on the nature of the surface S_o in the general case.

If the surface is spherical these components may be derived from Mie's solution for a plane wave incident on a sphere [Stratton, 1941, pp. 563-567]. For an infinite plane surface such as is being considered here, they reduce to the ordinary Fresnel reflection coefficients.

2.2 Evaluation of $\vec{\Sigma}$

For the purposes of computation it is useful to evaluate the vector quantity

$$\vec{\Sigma} = \frac{1}{4\pi^2} \iint_{-\infty}^{\infty} \hat{V}(u, v) \cdot \vec{a}_\pi C_o e^{iW_r} dudv = \hat{\sigma} \cdot \vec{a}_\pi C_o \qquad (2\text{-}9)$$

Referring to figure 2-1, the two coordinate systems having their origin at the point P on the reflecting surface are defined as follows: The Q system is defined by the surface normal and the direction to the receiver at point Q with the orthogonal set of unit vectors $\vec{a}_n$, $\vec{a}_r$ and $\vec{a}_t$; and the K system defined by the surface normal and the propagation vector $\vec{k}$ with the orthogonal set of unit vectors $\vec{a}_P$, $\vec{a}_N$, $\vec{a}_n$. Here $\vec{a}_n$ is the positive unit normal to the reflecting surface, $\vec{a}_r$ is the unit vector in the direction of the projection of $\vec{R}_1$ on the tangent plane through P, and $\vec{a}_t = \vec{a}_r \times \vec{a}_n$ (also in the tangent plane). The unit vector $\vec{a}_P$ is in the direction opposite to that of the projection of $\vec{k}$ on the tangent plane and $\vec{a}_N = \vec{a}_n \times \vec{a}_P$. In the general problem the surface normal described in the primary reference system (x, y, z) changes direction as one traverses the surface S_o under consideration; therefore these two additional coordinate systems are needed.

For the case of Π-plane waves reflected from an infinite plane, the off-diagonal terms of $\hat{V}$ (i.e., V_{ij} where i, j = P, N, or n and $i \neq j$) can be shown to be zero by converting $\vec{\Pi}$ to $\vec{E}$ and $\vec{H}$ and the diagonal terms are

$$V_{PP}(\alpha) = -V_{nn}(\alpha) = -\frac{(\mu_2/\mu_1)n^2 \cos\alpha - \sqrt{n^2 - \sin^2\alpha}}{(\mu_2/\mu_1)n^2 \cos\alpha + \sqrt{n^2 - \sin^2\alpha}}$$

$$V_{NN}(\alpha) = \frac{(\mu_1/\mu_2) \cos\alpha - \sqrt{n^2 - \sin^2\alpha}}{(\mu_1/\mu_2) \cos\alpha + \sqrt{n^2 - \sin^2\alpha}} \tag{2-10}$$

where

α = angle of incidence = $\sin^{-1} \frac{\sqrt{u^2 + v^2}}{k}$

n = index of refraction = k_1/k

μ_1 = permeability of the reflecting medium

μ_2 = permeability of the incident medium (free space)

Thus V_{PP} and V_{NN} are identical with the Fresnel reflection coefficients in this case. The vector (2-9) may be resolved as

$$\vec{\Sigma} = \frac{1}{4\pi^2} \iint_{-\infty}^{\infty} \vec{a}_P V_{PP}(\alpha) c_P e^{iW_r} du dv + \frac{1}{4\pi^2} \iint_{-\infty}^{\infty} \vec{a}_N V_{NN}(\alpha) c_N e^{iW_r} du dv$$

$$+ \frac{1}{4\pi^2} \iint_{-\infty}^{\infty} \vec{a}_n V_{nn}(\alpha) c_n e^{iW_r} du dv = \vec{\Sigma}_P + \vec{\Sigma}_N + \vec{\Sigma}_n \tag{2-11}$$

where

$$\vec{a}_\pi C_o = \vec{a}_P C_P + \vec{a}_N C_N + \vec{a}_n C_n$$

In order to perform the integration of (2-11), u and v are converted into cylindrical coordinates. Let

$$u = \lambda \cos \beta = k \sin \alpha \cos \beta$$

$$v = \lambda \sin \beta = k \sin \alpha \sin \beta$$

$$x_1 - x_o' = \rho_1 \cos \varphi_1$$

$$y_1 - y_o' = \rho_1 \sin \varphi_1$$

$$\rho_1 = R_1 \sin \Theta_1$$

$$z_1 = R_1 \cos \Theta_1$$

$$\Theta_1 = \text{angle between } \vec{a}_R \text{ and } \vec{a}_n$$

$$\vec{a}_R = \vec{R}_1/|\vec{R}_1| \qquad (2\text{-}12)$$

Substitution (2-12) into (2-11) yields

$$\vec{\Sigma}_n = \vec{a}_n \frac{C_n}{4\pi^2} \int_0^\infty V_{nn}[\alpha(\lambda)] \cdot \left[\int_0^{2\pi} e^{[i\lambda\rho_1\cos(\varphi_1-\beta)+iz_1k\cos\alpha]} d\beta \right] \lambda d\lambda \qquad (2\text{-}13)$$

which by Stratton [1941, p. 412] becomes

$$\vec{\Sigma}_n = \vec{a}_n \frac{C_n}{2\pi} \int_0^\infty V_{nn}[\alpha(\lambda)] J_o(\lambda\rho_1)\, e^{[iz_1k\cos\alpha]} \lambda d\lambda \qquad (2\text{-}14)$$

Also by the same method one can obtain

$$\vec{\Sigma}_T = \vec{\Sigma}_P + \vec{\Sigma}_N = \frac{\vec{a}_t c_t}{4\pi} \int_0^\infty [f_1(\lambda) J_o(\lambda\rho_1) + f_2(\lambda) J_2(\lambda\rho_1)] \cdot$$

$$\cdot\ e^{iz_1 k\cos\alpha} \lambda d\lambda + \frac{\vec{a}_r c_r}{4\pi} \int_0^\infty [f_1(\lambda) J_o(\lambda\rho_1)$$

$$- f_2(\lambda) J_2(\lambda\rho_1)]\ e^{iz_1 k\cos\alpha} \lambda d\lambda \qquad (2\text{-}15)$$

where

$$f_1(\lambda) = V_{PP}[\alpha(\lambda)] + V_{NN}[\alpha(\lambda)]$$

$$f_2(\lambda) = V_{PP}[\alpha(\lambda)] - V_{NN}[\alpha(\lambda)]$$

$$c_t = (\vec{a}_t \cdot \vec{a}_\pi) c_o = (\vec{a}_t \cdot \vec{a}_P) c_P + (\vec{a}_t \cdot \vec{a}_N) c_N$$

$$c_r = (\vec{a}_r \cdot \vec{a}_\pi) c_o = (\vec{a}_r \cdot \vec{a}_P) c_P + (\vec{a}_r \cdot \vec{a}_N) c_N$$

Approximate evaluation of $\vec{\Sigma}_n$ and $\vec{\Sigma}_T$, as given in (2-14) and (2-15), may be made by use of the saddle-point method [Brekhovskikh, 1960, pp. 245-255]. This method consists of first converting the λ integration to α integration over a contour Γ_o, converting the Bessel functions to Hankel functions, and then expanding the Hankel functions in their asymptotic representation over the appropriate contour Γ_1. The location of the saddle point, α_o, is found and the contour Γ_1 is then deformed to the contour Γ which passes through the saddle point along the path of steepest descent (i.e., a path of constant phase). This new contour Γ is represented by new variables defined by

$$f(\alpha) = f(\alpha_o) - s^2 \tag{2-16}$$

and s varies over all real values from $-\infty$ to $+\infty$.

In deforming of the contour Γ_1 to Γ, the following items must be taken into account:

1) Restriction of region of possible saddle-point locations so that the asymptotic representation of the Hankel function may be used.

2) The residues of any poles of V_{PP} and V_{NN} crossed in the deformation of the contour.

3) The effect of crossing a branch point of V_{PP} and V_{NN} in the deforming of the contour.

These effects are assumed to be negligible in this section and the exact limitations are determined in Section 2.3. Now the integrand, excluding the exponential, is rewritten in terms of the new variable s, expanded in a power series in s, and integrated. The only difficulty arising in these manipulations is that a pole close to the saddle point will limit the radius of convergence of the power series. This will again be neglected in this section and the restriction delineated in Section 2.3. If ρ is assumed to be sufficiently large, only the first term of the integral of the power series need be considered.

The saddle-point method will be used to evaluate $\vec{\Sigma}_n$ and $\vec{\Sigma}_T$. First consider $\vec{\Sigma}_n$ as given in (2-14): the conversion to α integration requires a contour Γ_o as shown in figure 2-2

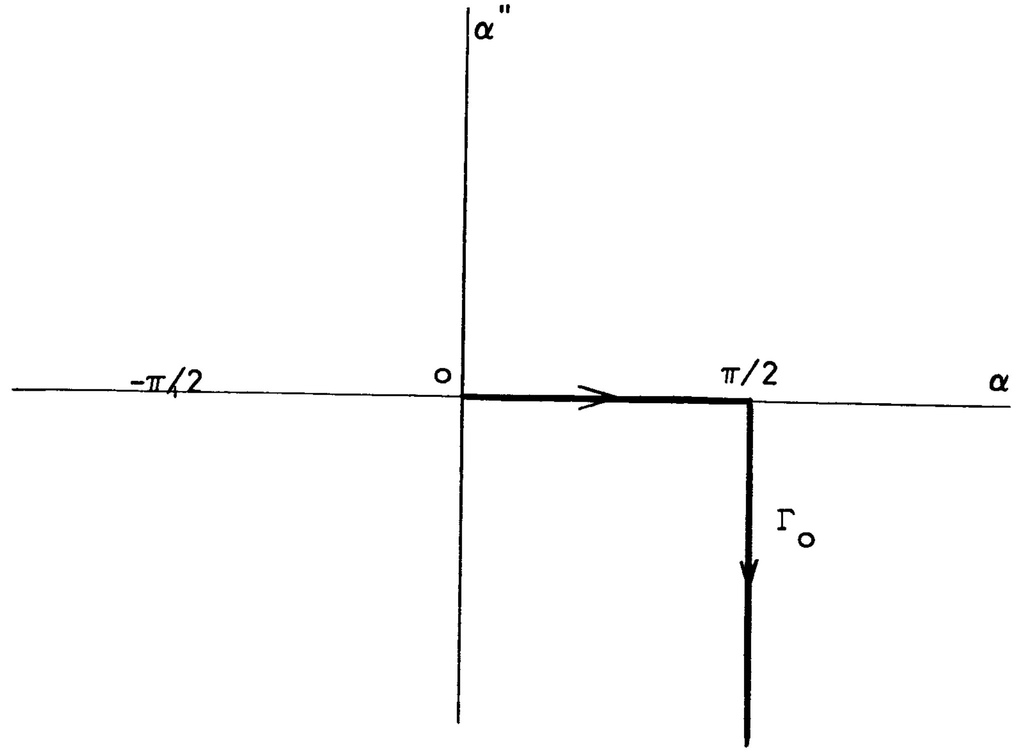

Figure 2-2

Integration Contour Γ_o

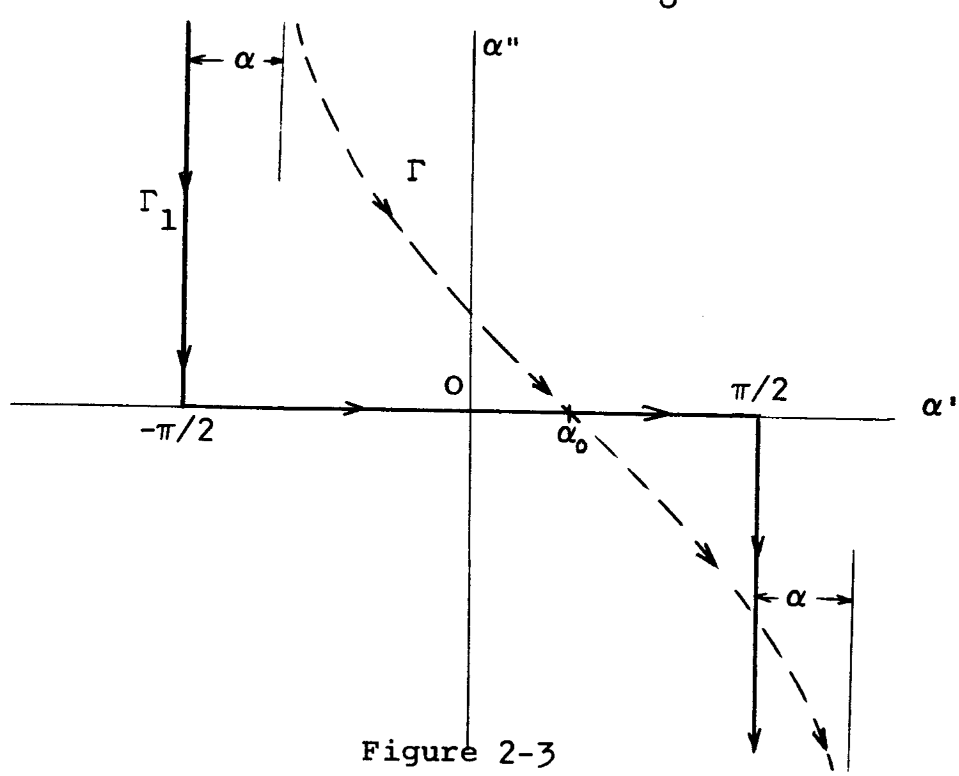

Figure 2-3

Integration Contours Γ_1 and Γ

to be used since k may be complex. Using (2-12), then (2-14) becomes

$$\vec{\Sigma}_n = \frac{\vec{a}_n C_n k^2}{2\pi} \int_{\Gamma_o} V_{nn}(\alpha) J_o(kR_1 \sin \theta_o \sin \alpha) \cdot$$

$$e^{ikR_1 \cos\theta_1 \cos\alpha} \sin \alpha \cos \alpha d\alpha \qquad (2\text{-}17)$$

The expansion of the Bessel function into Hankel functions of the same kind will be made avoiding the apparent pole of the Hankel functions at the origin.

$$J_o(z) = \frac{1}{2} [H_o^{(1)}(z) - H_o^{(1)}(-z)]$$

$$J_2(z) = \frac{1}{2} [H_2^{(1)}(z) - H_2^{(1)}(-z)] \qquad (2\text{-}18)$$

Since the integrand of (2-17) is an odd function of α, substituting (2-18) into (2-17) and changing the contour to Γ_1 as shown by the solid line in figure 2-3 yields

$$\vec{\Sigma}_n = \frac{\vec{a}_n C_n k^2}{4\pi} \int_{\Gamma_1} V_{nn}(\alpha) H_o^{(1)}(kR_1 \sin \theta_1 \sin \alpha) \cdot$$

$$\cdot \, e^{ikR_1 \cos\theta_1 \cos\alpha} \sin \alpha \cos \alpha d\alpha \qquad (2\text{-}19)$$

Under the assumption that $kR_1 \sin^2 \theta >> 1$, the large argument expansion of the Hankel function can be used, keeping only the first term.

$$\vec{\Sigma}_n = \frac{\vec{a}_n C_n k^2}{2\pi} \sqrt{\frac{2}{\pi k R_1 \sin \theta_1}} \int_{\Gamma_1} V_{nn}(\alpha) \cdot$$

$$e^{[ikR_1 \cos(\theta_1 - \alpha) - i\pi/4]} \cos \alpha \sqrt{\sin \alpha}\, d\alpha \qquad (2\text{-}20)$$

The saddle point of (2-20) is $\alpha_o = \theta_1$; then, by the saddle-point method of integration, the contour is changed to Γ as shown by the dotted line in figure 2.3. The effects of deforming the contour through any poles and remaining on the same sheet are assumed negligible (see Section 2.3). Under the assumption that $kR_1 >> 1$, saddle-point integration of (2-20) yields

$$\vec{\Sigma}_n = \frac{-\vec{a}_n C_n k}{2\pi i} V_{PP}(\theta_1) \frac{e^{ikR_1}}{R_1} \cos \theta_1 \qquad (2\text{-}21)$$

By the same method (2-15) becomes

$$\vec{\Sigma} = \frac{\vec{a}_t k C_t}{2\pi i} V_{NN}(\theta_1) \frac{e^{ikR_1}}{R_1} \cos \theta_1 + \frac{\vec{a}_r k C_r}{2\pi i} V_{PP}(\theta_1) \cdot$$

$$\cdot \frac{e^{ikR_1}}{R_1} \cos \theta_1 \qquad (2\text{-}22)$$

Thus by (2-21) and (2-22), (2-9) becomes

$$\vec{\Sigma} = \hat{\sigma} \cdot \vec{a}_\pi C_o = \frac{\omega \cos \theta_1}{2\pi i c} \frac{e^{ikR_1}}{R_1} [-\vec{a}_n C_n V_{PP}(\theta_1)$$

$$+ \vec{a}_r C_r V_{PP}(\theta_1) + \vec{a}_t C_t V_{NN}(\theta_1)] \qquad (2\text{-}23)$$

and substituting (2-23) into (2-7) yields

$$\vec{\Pi}_r(\vec{r}_1, \omega, t) = \frac{\omega e^{-i\omega t}}{2\pi c i} \iint_{S_o} \frac{e^{ikR_1} e^{ikR_o}}{R_1 R_o} [-\vec{a}_n c_n V_{PP}(\theta_1)$$

$$+ \vec{a}_r c_r V_{PP}(\theta_1) + \vec{a}_t c_t V_{NN}(\theta_1)] \cdot$$

$$\cdot \cos\theta_1 \, dx_o' dy_o' \qquad (2\text{-}24)$$

This is the reflected steady-state $\vec{\Pi}$ field from a plane surface provided $kR_1 \sin^2\theta_1 >> 1$, no poles or branch points are crossed in the deformation of contour, and no pole is close to the saddle point. These limitations will be removed in the next section.

2.3 Removal and Delineation of Restrictions

This section will discuss each of the restrictions imposed in the previous sections to evaluate $\vec{\Sigma}_n$ and $\vec{\Sigma}_T$ and to obtain $\vec{\Pi}_r$. These restrictions were:

1) Saddle point is not close to origin
2) No poles are crossed in deforming the contour
3) No pole is sufficiently close to saddle point
4) No branch points are crossed
5) Reflecting surface is a plane
6) Steady-state reflection

Each of these restrictions will be considered individually and the restrictions on the index of refraction n determined such that (2-24) is a valid approximation for the general case.

2.3.1 Saddle Point Near Origin

In Section 2.2 it was seen that the expansion of the Bessel functions in terms of Hankel functions in the evaluation of the components of $\vec{\Sigma}$ was not valid unless $kR_1 \sin^2 \theta_1 >> 1$. In this section this restriction is examined more closely and the conditions for its relaxation will be determined. First consider (2-17) for the n component of $\vec{\Sigma}$.

$$\vec{\Sigma}_n = \frac{\vec{a}_n C_n k^2}{2\pi} \int_{\Gamma_o} V_{nn}(\alpha) J_o(kR_1 \sin \theta_1 \sin \alpha) \cdot$$

$$\cdot\, e^{[ikR_1 \cos\theta_1 \cos\alpha]} \cos \alpha \sin \alpha d\alpha \qquad (2\text{-}17)$$

If $kR_1 \sin \theta_1 << 1$, then by using the first term in the expansion of the Bessel function, (2-17) becomes

$$\vec{\Sigma}_n \cong \frac{\vec{a}_n C_n k^2}{2\pi} \int_{\Gamma_o} V_{nn}(\alpha)\, e^{[ikR_1 \cos\theta_1 \cos\alpha]} \cos \alpha \sin \alpha d\alpha \qquad (2\text{-}25)$$

Letting $\alpha = \alpha' - i\alpha''$ and $F(\alpha) = F(\alpha', \alpha'')$, the integral of $F(\alpha)$ over contour Γ_o can be represented as

$$\int_{\Gamma_o} F(\alpha) d\alpha = \int_o^{\pi/2} F(\alpha', o) d\alpha' - i \int_o^{\infty} F(\pi/2, \alpha'') d\alpha'' \qquad (2\text{-}26)$$

Conversion of (2-25) to the form of (2-26) and using the trigonometric identities

$$\sin \alpha = \sin (\alpha' - i\alpha'') = \sin \alpha' \cosh \alpha'' - i \cos \alpha' \sinh \alpha''$$

$$\cos \alpha = \cos (\alpha' - i\alpha'') = \cos \alpha' \cosh \alpha'' + i \sin \alpha' \sinh \alpha'' \tag{2-27}$$

yields

$$\sum_n \vec{\Sigma} = \frac{\vec{a}_n C_n k^2}{2\pi} \left[\int_0^{\pi/2} V_{nn}(\alpha', o)\, e^{[ikR_1 \cos\theta_1 \cos\alpha']} \cdot \cos \alpha' \sin \alpha' d\alpha' + \int_0^{\infty} V_{nn}(\pi/2, \alpha'') \cdot e^{[-kR_1 \cos\theta_1 \sinh\alpha'']} \sinh \alpha'' \cosh \alpha'' d\alpha'' \right] \tag{2-28}$$

Let the first integral in (2-28) be called I_1, the second I_2 and $w = \cos \alpha'$; then

$$I_1 = \int_0^1 V_{nn}(\cos^{-1} w, o)\, e^{[ikR_1 \cos\theta_1 w]} \, w dw \tag{2-29}$$

Integrating (2-29) by parts and neglecting all but the first term (assuming $kR_1 \cos \theta_1 >> 1$ and $\cos \theta_1 \cong 1$), yields

$$I_1 = \frac{V_{nn}(\cos^{-1} 1, o)}{ikR_1} e^{ikR_1} \tag{2-30}$$

Since in I_2, $V_{nn}(\pi/2, \alpha'')$ is approximately constant for very small α'', then setting the derivative with respect to α'' of the remaining portion of the integrand of I_2 equal to zero, the maximum is found to occur at $\alpha'' = 1/kR_1$. The maximum value of the integrand is $V_{nn}(\pi/2)e^{-1}/kR_1$. Since the integrand is approximately zero at $10/kR_1$, then

$$I_2 < \int_0^{10/kR_1} \frac{V_{nn}(\pi/2)e^{-1}}{kR_1} d\alpha'' = \frac{3.7}{(kR_1)^2} \tag{2-31}$$

which is negligible with respect to I_1. Thus, for θ_1 very close to zero

$$\vec{\Sigma}_n = \frac{\vec{a}_n C_n k}{2\pi i} V_{nn}(0^\circ) \frac{e^{ikR_1}}{R_1} \tag{2-32}$$

which is the same as the saddle-point solution for $\theta_1 \rightarrow 0$ as given in (2-21).

By the same method, $\vec{\Sigma}_T$ is

$$\vec{\Sigma}_T = \frac{\vec{a}_t C_t k}{4\pi i} [V_{PP}(0^\circ) + V_{NN}(0^\circ)] \frac{e^{ikR_1}}{R_1}$$

$$+ \frac{\vec{a}_r C_r k}{4\pi i} [V_{PP}(0^\circ) + V_{NN}(0^\circ)] \frac{e^{ikR_1}}{R_1} \tag{2-33}$$

which, since $V_{PP}(0^\circ) = V_{NN}(0^\circ)$ is the same as (2-22) for $\theta_1 \rightarrow 0$. Thus, since $\vec{\Sigma}$ is an analytic function, the saddle-point solution as given in (2-23) is valid for all θ_1 if $kR_1 >> 1$.

2.3.2 Effect of Branch Point and Branch Cut

There exists a branch point in both V_{PP} and V_{NN} at

$$\sqrt{n^2 - \sin^2 \alpha_b} = 0 \qquad (2\text{-}34)$$

where α_b is the location of the branch point. The cut from this branch point will be taken as $\operatorname{Im}\sqrt{n^2 - \sin^2 \alpha} = 0$. The sheet $\operatorname{Im}\sqrt{n^2 - \sin^2 \alpha} > 0$ will be called the upper sheet and $\operatorname{Im}\sqrt{n^2 - \sin^2 \alpha} < 0$, the lower sheet. This branch cut is chosen since the contour Γ_1 must pass over the upper sheet so that the retransmitted wave in the reflecting medium does not have an infinite amplitude as $z \to -\infty$.

Now the location of the branch point with respect to the contour Γ will be determined. The contour Γ is defined by

$$\cos(\theta_1 - \alpha_c) = 1 + is^2 \qquad (2\text{-}35)$$

Since the contour is deformed the most for $\theta_1 = \pi/2$ (its maximum value), (2-35) becomes

$$\sin \alpha_c = 1 + is^2 \qquad \text{for } \theta_1 = \pi/2 \qquad (2\text{-}36)$$

If it is assumed that an n exists where the branch point is on the contour, then $\alpha_c = \alpha_b$. For (2-34) to be true then the radicand must be zero. Squaring (2-36) and substituting into (2-34) yields

$$n^2 - 1 + s^4 - 2is^2 = 0$$

$$\mathrm{Re}(n^2) - 1 + s^4 = 0 \quad \text{and} \quad \mathrm{Im}(n^2) - 2s^2 = 0 \tag{2-37}$$

since for free space with a conductivity $\sigma_2 = 0$

$$n^2 = \mu_r \epsilon_r (1 + i\gamma) \tag{2-38}$$

where

$\mu_r = \mu_1/\mu_2$

$\epsilon_r = \epsilon_1/\epsilon_2$

ϵ_1 = dielectric constant of reflectivity medium

ϵ_2 = dielectric constant of free space

$\gamma = \dfrac{\sigma_1}{\omega\epsilon_1}$

σ_1 = conductivity of reflecting medium

Then no solution can exist for $\mu_r\epsilon_r > 1$. This condition will be assumed for the solution obtained in this paper. Therefore the branch point is always above the contour Γ. In view of the fact that V_{PP} and V_{NN} are analytic functions of α in the region bounded by Γ and Γ_1 and the branch cut if crossed will be crossed twice, no correction need be made to the saddle-point integration.

2.3.3 Location of Poles of Reflection Coefficients

The location of the poles of V_{PP} and V_{NN} will be found to determine if the contour has been deformed across a pole and consequently the value of the residue of any such pole will be added to (2-23). Consider first V_{NN} whose poles α_N are determined by the solution of the equation

$$\mu_r \cos \alpha_N = -\sqrt{n^2 - \sin^2 \alpha_N} \tag{2-39}$$

If it is assumed that $|n^2| > 1$, no poles exist for $\mu_r = 1$. However, for $\mu_r > 1$, the poles are determined by squaring (2-39) and obtaining

$$\cos \alpha_N = \pm \sqrt{\frac{n^2 - 1}{\mu_r^2 - 1}} \quad \text{and} \quad \sin \alpha_N = \pm \sqrt{\frac{n^2 - \mu_r^2}{\mu_r^2 - 1}} \tag{2-40}$$

Hence assuming $\epsilon_r > \mu_r$, there are four values for $\alpha_N = \alpha_N' + i\alpha_N''$ where α_N' is restricted to the area of interest, namely from $-\pi$ to π.

P_1: $0 < \alpha_N' < \pi/2$, $\alpha_N'' < 0$

P_2: Symmetrical to P_1 (with respect to the origin)

P_3: $\pi/2 < \alpha_N' < \pi$, $\alpha_N'' > 0$

P_4: Symmetrical to P_3 (with respect to the origin) (2-41)

It should be noted that of these four poles, two are not true poles but were obtained in the squaring process. Squaring $\sin \alpha_q$ from (2-40) and substituting into (2-39), it is seen that $\cos \alpha_q$ in (2-40) must have the negative sign. Thus, only P_3 and P_4 are true poles. Again, using (2-39), it may be seen that these poles lie on the lower sheet; therefore they will not modify (2-23) due to residues or affect its validity due to being too close to the saddle point, so long as $|n^2| > 1$ and $\epsilon_r > \mu_r$.

Now consider V_{PP} whose poles α_P are determined by the solution to the equation

$$\frac{n^2}{\mu_r} \cos \alpha_p = -\sqrt{n^2 - \sin^2 \alpha_p} \tag{2-42}$$

The squaring of (2-42) yields

$$\cos \alpha_p = \frac{-\mu_r}{\sqrt{n^2 + \mu_r}} \sqrt{\frac{n^2 - 1}{n^2 - \mu_r}}$$

$$\text{and } \sin \alpha_p = \pm\sqrt{\frac{n^2}{n^2 + \mu_r}} \sqrt{\frac{n^2 - \mu_r^2}{n^2 - \mu_r}} \tag{2-43}$$

where the minus sign on the cosine equation was determined by using the sine term squared in (2-42) in order that only the two correct solutions to (2-42) be obtained. Again assuming $\epsilon_r > \mu_r$, the two true poles are

P_1: $\pi/2 < \alpha_p' < \pi$, $\alpha_p'' < 0$

P_2: Symmetrical to P_1 (with respect to the origin)

Using (2-42) it can be determined that these poles lie on the upper sheet. Therefore it must be determined if these poles lie between Γ and Γ_1. By referring to figure 2-3, it is determined that pole P_2 will not lie between Γ and Γ_1 due to its location; however, pole P_1 might be crossed. Pole P_1 corresponds to the positive sign on the sine term in (2-43). Since the contour Γ has had its maximum deformation when the saddle point $\Theta_1 = \pi/2$, this case will be considered to determine if some values of μ_r, ϵ_r, and γ will cause the pole to lie on the contour. Setting α_c in (2-36) equal to α_p, it is seen that

$$\text{Re} \sin \alpha_p = 1 \tag{2-45}$$

Converting (2-43) into polar form

$$\sin \alpha_p = \left| \sqrt{\frac{n^2}{n^2 + \mu_r}} \sqrt{\frac{n^2 - \mu_r^2}{n^2 - \mu_r}} \right| e^{i\varphi} \tag{2-46}$$

where

$$0 < \varphi < \pi/2$$

However, when the magnitude of $\sin \alpha_p$ was determined, it was found to be less than one so long as $|n^2| > 1$ and $\epsilon_r > \mu_r$. Consequently pole P_1 does not lie between Γ and Γ_1, but it may be arbitrarily close to the saddle point. This effect will be examined in the next section.

2.3.4 Pole Near the Saddle Point

In the saddle point integration, the variable α was replaced by s and the entire integrand, excluding the exponential, was expanded in a power series in s. It is necessary that this power series be convergent. The power series will converge inside a circle of radius s_o, on whose boundary the pole is located. For the saddle-point method, it is necessary that the entire range of significant values of s, for which the integrand is not yet very small, occupy a small portion of the circle of convergence near its center. Using the upper limit s_1 of significant values of s as the value for which the exponential decreases to a value of e^{-1}, thus

$$s_1 \cong 1/\sqrt{kR_1} \tag{2-47}$$

Then the condition that $|s_1/s_o|^2 << 1$ becomes

$$kR_1|s_o|^2 >> 1 \tag{2-48}$$

From the previous section it was seen that the only pole that could approach the saddle point was pole P_1 of V_{PP} which approaches the saddle point only when $\theta_1 \to \pi/2$. The location of the pole in the s plane is determined by

$$\cos(\theta_1 - \alpha_P) = 1 + is_o^2 \tag{2-49}$$

Letting $w = \sqrt{kR_1}\, s_o$, $\theta_1 = \pi/2$, and using (2-49) and (2-43) it is found that [Brekhovskikh, 1960]

$$|w|^2 \cong \frac{kR_1}{2|n^2|} \tag{2-50}$$

Thus for (2-48) to be satisfied and consequently (2-24) be correct it is necessary that $kR_1 >> 2|n^2|$.

2.3.5 Effect of Nonplanar Reflecting Surface

In the derivation of the differential reflectivity it was assumed that the reflecting surface was a plane. If the body is not a plane, two effects must be either neglected or taken into account. These are: (1) the modification of the incident field due to multiple reflections in the incident medium, and (2) modification of the reflected field due to the retransmitted wave in the reflecting medium intersecting the surface at another point. These two effects are illustrated in figure 2-4 at points B and C, respectively. The effect at B can be negated, in the case of a smooth body, by requiring convexity [Erteza, Doran, and Lenhert, 1965] or, in the case of a rough surface, by assuming that multiple reflections are negligible, as will be done in this work.

The case at C is more difficult. However, if the radius of curvature is very large or the reflecting body is lossy, then the contribution at C due to illumination at A can be made negligibly small. For the case of a smooth spherical surface the results obtained by Erteza, Doran, and Lenhert [1965] for steady-state full illumination matched with those obtained using Mie's reflection coefficients for the radius of the spherical surface greater than 100 wavelengths. At this point the approximations made in Section 2 can give errors of the order of one percent; thus it cannot be stated that the concept of differential reflectivity gives an incorrect answer for a smaller radius of curvature. Consequently it will be assumed that the differential reflectivity is a good approximation for the rough surface case so long as the roughness is not extreme.

2.3.6 Pulsed Source

If the source is considered to be an elementary dipole whose current is a pulsed sinusoid of angular frequency ω_o and pulse width T, it is necessary to replace the incident time varying radiation with an infinite set of steady-state waves obtained by means of the Fourier transform of the resulting $\vec{\Pi}$ field. The resulting $\vec{\Pi}$ field at the reflecting surface is [Van Bladel, 1963, p. 194]

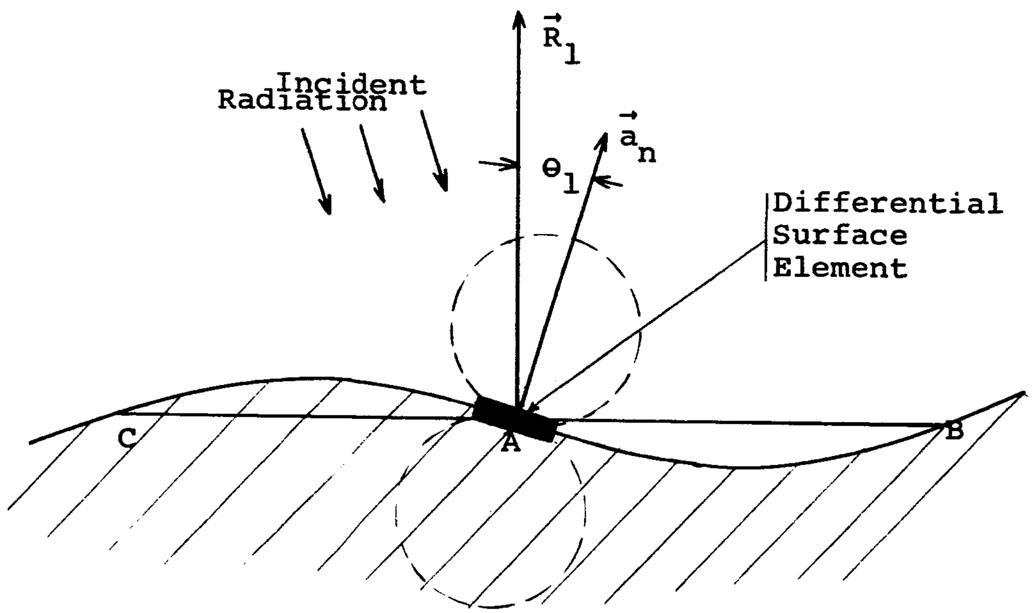

Figure 2-4
Reflection From Non-Planar Surface

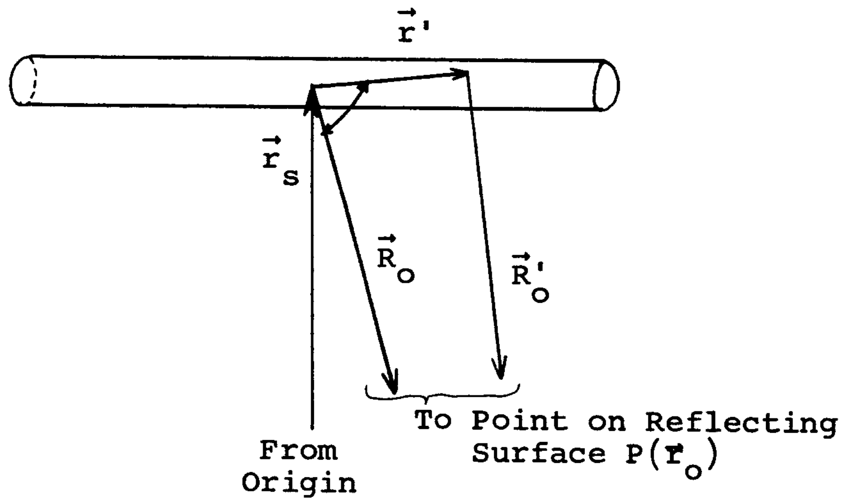

Figure 2-5
Configuration of Source

$$\vec{\Pi}_i(\vec{r}_o, \omega_o, t) = \frac{1}{4\pi\epsilon_o} \int_0^t dt' \int_V \frac{\vec{j}(\vec{r}', t' - R_o'/c)}{R_o'} dV' \qquad (2\text{-}51)$$

where $\vec{j}(\vec{r}', t' - R_o'/c)$ is the current density in the antenna

$\vec{R}_o' = \vec{R}_o - \vec{r}'$

$\vec{r}'$ = vector from the center of the antenna (defined by $\vec{r}_s$) to a point in the antenna

These vectors are shown in figure 2-5. Let the current density be defined by

$$\vec{j}(\vec{r}', t' - R_o'/c) = \vec{a}_\pi j(\vec{r}') \sin \omega_o(t' - R_o'/c) \cdot$$
$$\cdot [u(t' - R_o'/c) - u(t' - R_o'/c - T)] \qquad (2\text{-}52)$$

which can be expressed in complex form as

$$\vec{j}(\vec{r}', t' - R_o'/c) = \text{Im}\Big\{-\vec{a}_\pi j(\vec{r}')\, e^{-i\omega_o(t' - R_o'/c)} \cdot$$
$$[u(t' - R_o'/c) - u(t' - R_o'/c - T)]\Big\} \qquad (2\text{-}53)$$

The series expansion of $e^{ik_2R_o'}/R_o'$ is [Stratton, 1941, p. 431]

$$\frac{e^{ik_2R_o'}}{R_o'} = ik_2 \sum_{n=0}^{\infty} (2n + 1)P_n(\cos \gamma)j_n(k_2r')h_n^{(1)}(k_2R_o) \qquad (2\text{-}54)$$

where γ is as shown in figure 2-5

$P_n(\cos \gamma)$ are the Legendre polynomials

$j_n(k_2 r')$ are the spherical Bessel functions

$h_n^{(1)}(k_2 R_o)$ are the spherical Hankel functions of the first kind

Under the conditions that $k_2 r' \ll 1$ and $k_2 R_o \gg 1$, (2-54) can be approximated by the first term ($n = 0$) and the R_o' in the argument of the unit steps may be replaced by R_o. Substitution of (2-53) and the first term of (2-54) into (2-51) gives

$$\vec{\Pi}_i(\vec{r}_o, \omega_o, t) = \text{Im}\left\{\frac{-\vec{a}_\pi}{4\pi\epsilon_o} \frac{e^{ik_2 R_o}}{R_o} \int_0^t e^{-i\omega_o t'} [u(t' - R_o/c) - u(t' - R_o/c - T)] \left[\int_V j(r')dV'\right] dt'\right\} \quad (2\text{-}55)$$

The assumptions of a thin, straight-wire antenna of length ℓ with uniform current density on any transverse cross section and with current distribution $I(\xi)$ over its length yield

$$\int_V j(\vec{r}')dV' = \int_{-\ell/2}^{\ell/2} I(\xi)d\xi = 4\pi\epsilon_o C_1 \quad (2\text{-}56)$$

The substitution of (2-56) into (2-55) gives

$$\vec{\Pi}_i(\vec{r}_o, \omega_o, t) = \text{Im}\left\{\vec{a}_\pi C_1 \frac{e^{ik_2 R_o}}{R_o} \left[-\int_0^t e^{-i\omega_o t'} u(t' - R_o/c) - u(t' - R_o/c - T)dt'\right]\right\} \quad (2\text{-}57)$$

Taking the Fourier and inverse Fourier transforms of (2-57) and rearranging yields

$$\vec{\Pi}_i(\vec{r}_o, \omega_o, t) = \text{Im}\, \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\vec{a}_\pi e^{ik_2 R_o} e^{-i\omega t}}{R_o} \left\{ C_1 \int_{-\infty}^{\infty} e^{i\omega t_1} \cdot \right.$$

$$\cdot \left[-\int_o^{t_1} e^{-i\omega_o t'} [u(t' - R_o/c) \right.$$

$$\left. \left. - u(t' - R_o/c - T)] dt' \right] dt_1 \right\} d\omega \qquad (2\text{-}58)$$

Converting (2-58) to the real part and taking the Fourier transform of the t'-integral yields

$$\vec{\Pi}_i(\vec{r}_o, \omega_o, t) = \text{Re}\left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\vec{a}_\pi e^{ik_2 R_o} e^{-i\omega t}}{R_o} \left[\frac{C_1}{-\omega} \int_{-\infty}^{\infty} e^{i\omega t_1} \cdot \right. \right.$$

$$\left. \left. [u(t_1 - R_o/c) - u(t_1 - R_o/c - T)] dt_1 \right] d\omega \right\}$$

(2-59)

Since the development of the reflected fields of Section 2.1 used an amplitude of $C_o(\omega)$ for the steady-state wave, it is necessary to obtain $C_o(\omega)$ for each of the steady-state waves from (2-59) and then sum these waves to obtain the time varying incident or reflected fields. Then

$$\vec{\Pi}_i(\vec{r}_o, \omega_o, t) = \text{Re}\left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{C_o(\omega)\, e^{ik_2 R_o} e^{-i\omega t}}{R_o} d\omega \right\} \qquad (2\text{-}60)$$

where

$$c_o(\omega) = \frac{c_1 f(\omega, \omega_o, T)}{-\omega}$$

$$f(\omega, \omega_o, T) = \int_{-\infty}^{\infty} [u(t_1 - R_o/c) - u(t_1 - R_o/c - T)] \cdot$$

$$\cdot\ e^{i\omega t_1}\, dt_1$$

$$= \frac{e^{-i(\omega - \omega_o)T} - 1}{i(\omega - \omega_o)}$$

Thus the pulsed case shown in (2-60) is of the same form as the steady-state case of (2-2) with only $c_o(\omega)$ redefined. Consequently,

$$\vec{\Pi}_r(\vec{r}_1, \omega_o, t) = \frac{1}{2\pi ci} \int_{S_o} \left\{\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\omega e^{ik(R_o+R_1)}}{R_o R_1} e^{-i\omega t} \cdot \right.$$

$$[-\vec{a}_n c_n V_{PP}(\theta_1) + \vec{a}_r c_r V_{PP}(\theta_1)$$

$$\left. + \vec{a}_t c_t V_{NN}(\theta_1)] \cos \theta_1 d\omega \right\} dS \qquad (2\text{-}61)$$

where c_n, c_r, and c_t are related to $c_o(\omega)$ as previously defined but $c_o(\omega)$ is now defined as shown in (2-60).

2.4 Conclusion

In summary, the concept of differential reflectivity developed in this chapter was found to be valid for reflection from a smooth convex surface of either large radius of curvature or composed of a lossy material so long as $kR_o >> |n^2|$ and $\mu_r < \epsilon_r$. In the case of a rough surface where $\vec{a}_n$, $\vec{a}_r$,

$\vec{a}_t$ vary with position, it is necessary to assume that multiple reflections both in the incident and reflecting medium are negligible. This can be forced by requiring a large radius of curvature of the surface and possibly lossy material in the reflecting surface. Again, in the rough surface case, it is necessary to assume $kR_o >> |n^2|$ and $\mu_r < \epsilon_r$. With these assumptions the concept yields the equation for the reflected Hertzian potential as

$$\vec{\Pi}_r(\vec{r}_1, \omega_o, t) = \frac{1}{4\pi^2 ci} \int_{S_o} \int_{-\infty}^{\infty} \frac{\omega e^{ik(R_o+R_1)}}{R_oR_1} e^{-i\omega t}[-\vec{a}_n c_n V_{PP}(\theta_1) + \vec{a}_r c_r V_{PP}(\theta_1) + \vec{a}_t c_t V_{NN}(\theta_1)] \cos \theta_1 d\omega dS \quad (2\text{-}62)$$

where S_o is the illuminated surface.

CHAPTER 3

APPLICATION OF THE CONCEPT OF DIFFERENTIAL REFLECTIVITY TO A ROUGH SPHERE

3.1 Definition of Coordinate Systems

In this section the coordinate systems necessary for the application of the concept of differential reflectivity are developed for a rough sphere whose average radius is a. The variation in the radius of the rough sphere at a point θ, φ is denoted by $H(\theta, \varphi)$. From Chapter 2, using $\vec{r}_o = r_o \vec{a}_{ro}$ as the vector from the origin to a point on the reflecting surface, the equation of the surface is given by

$$\Psi = r_o - \left[a + H(\theta, \varphi)\right] \equiv 0 \tag{3-1}$$

The unit outward surface normal vector $\vec{a}_n$ is found by normalizing the gradient of (3-1) in normal spherical coordinates $(\vec{a}_{ro}, \vec{a}_\theta, \vec{a}_\varphi)$.

$$\vec{a}_n = \frac{\nabla\Psi}{|\nabla\Psi|} = \left[\frac{\partial\Psi}{\partial r_o} \vec{a}_{ro} + \frac{1}{r_o} \frac{\partial\Psi}{\partial\theta} \vec{a}_\theta + \frac{1}{r_o \sin\theta} \frac{\partial\Psi}{\partial\varphi} \vec{a}_\varphi\right] \frac{1}{|\nabla\Psi|} \tag{3-2}$$

It should be noted at this point that if the sphere is perfectly smooth, $\vec{a}_n$ is identical with $\vec{a}_{ro}$. Performing the indicated operations of (3-2) on Ψ as defined in (3-1) yields

$$\vec{a}_n = \left[\vec{a}_{ro} - \frac{1}{r_o} \frac{\partial H(\theta, \varphi)}{\partial\theta} \vec{a}_\theta - \frac{1}{r_o \sin\theta} \frac{\partial H(\theta, \varphi)}{\partial\varphi} \vec{a}_\varphi\right] J^{-1} \tag{3-3}$$

where

$$J = \sqrt{1 + \left[\frac{1}{r_o} \frac{\partial H(\theta, \varphi)}{\partial \theta}\right]^2 + \left[\frac{1}{r_o \sin \theta} \frac{\partial H(\theta, \varphi)}{\partial \varphi}\right]^2}$$

It is now necessary to define several orthogonal coordinate systems so that $\vec{\Pi}_r$, and consequently the direct- and cross-polarized power, may be calculated for the case where the transmitter and receiver are coincident. These coordinate systems are shown in figure 3-1. The coordinate system used and defined in Chapter 2 with unit vectors $(\vec{a}_n, \vec{a}_r, \vec{a}_t)$ has its origin at the reflecting point P. The reference, or inertial coordinate system $(\vec{a}_x, \vec{a}_y, \vec{a}_z)$, has its origin at the center of the sphere with $\vec{a}_z$ in the direction of the transmitter and $\vec{a}_x$ in the direction of a linearly polarized $\vec{\Pi}_i$ (i.e., $\vec{a}_x = \vec{a}_\pi$). Another coordinate system $(\vec{a}_R, \vec{a}_q, \vec{a}_\varphi)$ has its origin at the reflection point P with $\vec{a}_R$ in the direction of the receiver (in the case of the sphere with receiver also on z axis it lies in the meridian plane), $\vec{a}_\varphi$ the standard spherical unit vector as used previously in this section, and $\vec{a}_q = \vec{a}_\varphi \times \vec{a}_R$ (also in meridian plane defined by $\vec{a}_{ro}$ and $\vec{a}_\theta$). Several additional quantities shown in figure 3-1 need to be defined. These are: $\vec{a}_\ell$ is a unit vector in the direction of the projection of $\vec{a}_n$ on the meridian plane, θ_n is the angle from $\vec{a}_{ro}$ to $\vec{a}_\ell$ in the meridian plane, and φ_n is the angle from $\vec{a}_n$ to $\vec{a}_\ell$ in the plane perpendicular to the meridian plane containing $\vec{a}_n$ and $\vec{a}_\varphi$.

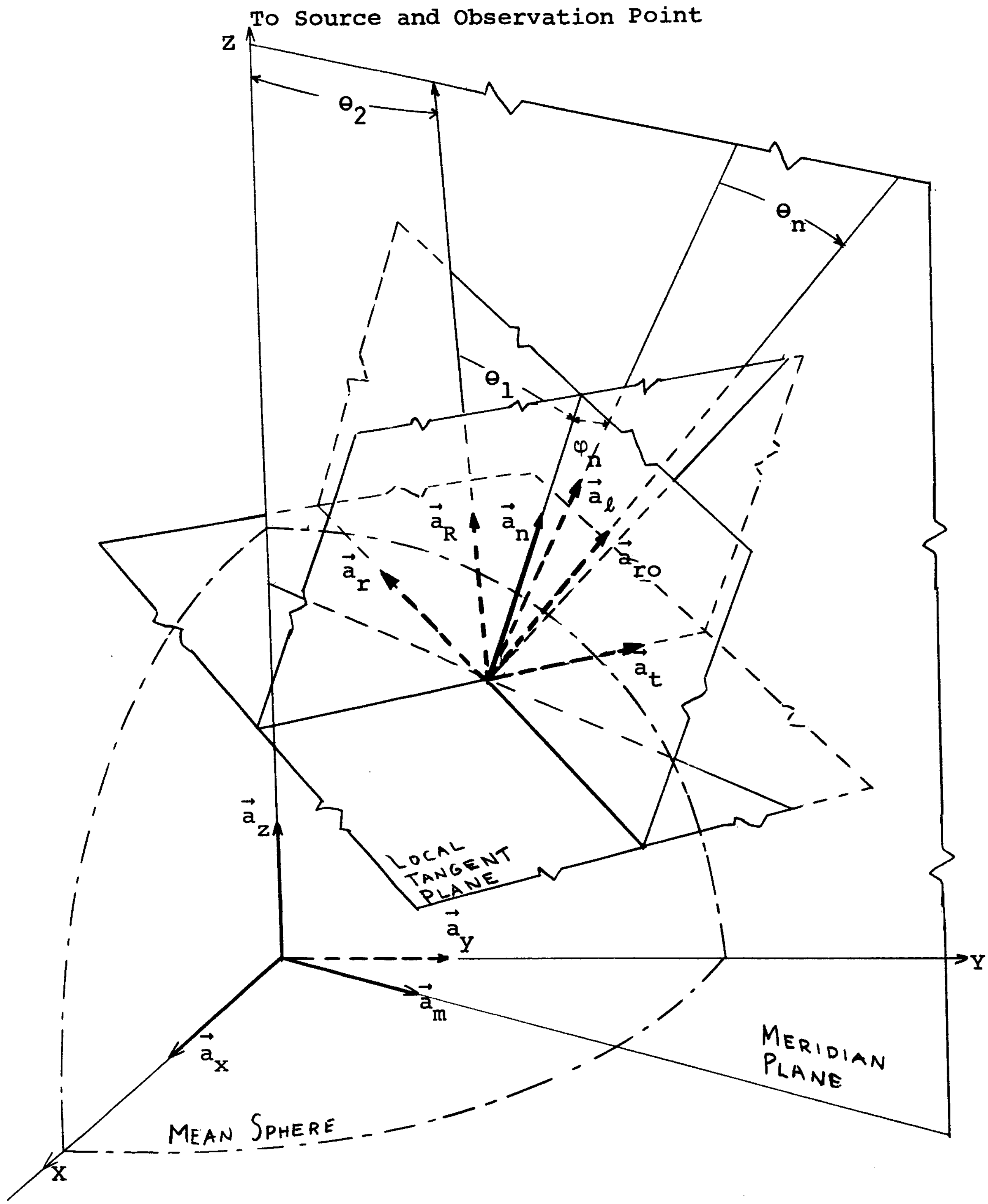

Figure 3-1

Reflection Geometry for a Sphere

Matrix notation for the coordinate unit vectors is used so that transformations between the various coordinate systems may be more easily made. The coordinate column vectors are defined as

$$\vec{A}_1 = \begin{bmatrix} \vec{a}_n \\ \vec{a}_r \\ \vec{a}_t \end{bmatrix}; \quad \vec{A}_2 = \begin{bmatrix} \vec{a}_R \\ \vec{a}_q \\ \vec{a}_\varphi \end{bmatrix}; \quad \vec{A}_3 = \begin{bmatrix} \vec{a}_x \\ \vec{a}_y \\ \vec{a}_z \end{bmatrix}; \quad \vec{A}_4 = \begin{bmatrix} \vec{a}_{ro} \\ \vec{a}_\theta \\ \vec{a}_\varphi \end{bmatrix} \qquad (3\text{-}4)$$

The (3 x 3) coordinate transformation matrices A_{ij} are defined as

$$\vec{A}_i = A_{ij}\vec{A}_j \qquad (3\text{-}5)$$

and are calculated in Appendix A.

3.2 Determination of $\vec{E}_r$ and $\vec{H}_r$

The reflected power will be determined in Section 3.3 from the reflected electric and magnetic field intensities ($\vec{E}_r$ and $\vec{H}_r$) by application of the Poynting theorem. Therefore it is necessary to now determine $\vec{E}_r$ and $\vec{H}_r$ from $\vec{\Pi}_r$ as defined in (2-62). Assuming that $(k_2R_1)^{-2}$ is negligible with respect to $(k_2R_1)^{-1}$, consider only the integrand [Stratton, 1941, p. 435]

$$\delta\vec{E}_r(\vec{r}_1, \omega_o, \omega) = -k^2\vec{a}_R \times [\vec{a}_R \times \delta\vec{\Pi}_r(\vec{r}_1, \omega_o, \omega)]$$

$$\delta\vec{H}_r(\vec{r}_1, \omega_o, \omega) = \omega k\epsilon_2[\vec{a}_R \times \delta\vec{\Pi}_r(\vec{r}_1, \omega_o, \omega)] \qquad (3\text{-}6)$$

where $\delta\vec{\Pi}_r$ is the integrand of (2-62). Let

$$\vec{V} = -(\vec{a}_n \cdot \vec{a}_x)V_{PP}(\theta_1)\vec{a}_n + (\vec{a}_r \cdot \vec{a}_x)V_{PP}(\theta_1)\vec{a}_r + (\vec{a}_t \cdot \vec{a}_x)V_{NN}(\theta_1)\vec{a}_t \tag{3-7}$$

then (2-62) can be rewritten for the monostatic case (i.e., $R_o = R_1 = R$)

$$\vec{\Pi}_r(\vec{r}_1, \omega_o, t) = \frac{1}{4\pi^2 ci} \int_{S_o} \int_{-\infty}^{\infty} \frac{C_o \omega e^{i2kr} e^{-i\omega t}}{R^2} \vec{V} \cos \theta_1 d\omega dS \tag{3-8}$$

The vector $\vec{V}$ can be written in matrix form as

$$\vec{V} = V_1\vec{A}_1 \tag{3-9}$$

where V_1 is the row matrix

$$V_1 = [-(\vec{a}_n \cdot \vec{a}_x)V_{PP}(\theta_1), (\vec{a}_r \cdot \vec{a}_x)V_{PP}(\theta_1), (\vec{a}_t \cdot \vec{a}_x)V_{NN}(\theta_1)] \tag{3-10}$$

Since the vector operations in (3-7) operate only on $\vec{V}$, transformation to (x, y, z) coordinates by use of (3-4) and (3-5) yields the following terms:

$$\vec{E}_1 = \vec{a}_R \times (\vec{a}_R \times \vec{V}) = V_1A_{12}e_1A_{23}\vec{A}_3$$

$$\vec{H}_1 = \vec{a}_R \times \vec{V} = V_1A_{12}h_1A_{23}\vec{A}_3 \tag{3-11}$$

where e_1 and h_1 are (3 x 3) matrices determined by $\vec{a}_R \times (\vec{a}_R \times \vec{A}_2)$ and $\vec{a}_R \times \vec{A}_2$, respectively, and are

$$e_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad h_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \tag{3-12}$$

Then (3-6) and (3-9) are combined, using the notation of (3-11); the results are

$$\vec{E}_r(\vec{r}_1,\ \omega_o,\ t) = \frac{-1}{4\pi^2 c^3 i} \int_{S_o} \int_{-\infty}^{\infty} \frac{C_o \omega^3 e^{i2kR}\ e^{-i\omega t}}{R^2}\ \vec{E}_1 \cos\ \theta_1 d\omega dS$$

$$\vec{H}_r(\vec{r}_1,\ \omega_o,\ t) = \frac{\epsilon_2}{4\pi^2 c^2 i} \int_{S_o} \int_{-\infty}^{\infty} \frac{C_o \omega^3 e^{i2kR}\ e^{-i\omega t}}{R^2}\ \vec{H}_1 \cos\ \theta_1 d\omega dS \tag{3-13}$$

With the transmitter and receiver located at (0, 0, D) in the (x, y, z) coordinate system and assuming D >> a, then $\theta_2 \cong 0$. This assumption greatly simplifies the calculation of $\vec{E}_1$ and $\vec{H}_1$.

The substitution of (A-13) and (A-14) into (3-11) yields

$$\vec{E}_1 = \frac{V_1}{\sin\theta_1} \left[\begin{array}{c|c|c} \cos\theta_1 \sin\theta_1 & \cos\varphi_n \sin(\theta-\theta_n)\sin\theta_1 & \sin\varphi_n \sin\theta_1 \\ \sin^2\theta_1 & -\cos\varphi_n \sin(\theta-\theta_n)\cos\theta_1 & -\sin\varphi_n \cos\theta_1 \\ 0 & -\sin\varphi_n & \cos\varphi_n \sin(\theta-\theta_n) \end{array}\right] \cdot$$

$$\cdot \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \end{bmatrix} \vec{A}_3 \tag{3-14}$$

with V_1 being

$$V_1 = \frac{1}{\sin\theta_1}[(\sin\varphi\sin\varphi_n\sin\theta_1-\cos\varphi\cos\varphi_n\sin(\theta-\theta_n)\sin\theta_1)V_{PP}(\theta_1),$$
$$(\sin\varphi\sin\varphi_n\cos\theta_1-\cos\varphi\cos\varphi_n\sin(\theta-\theta_n)\cos\theta_1)V_{PP}(\theta_1),$$
$$(-\cos\varphi\sin\varphi_n-\sin\varphi\cos\varphi_n\sin(\theta-\theta_n))V_{NN}(\theta_1)] \tag{3-15}$$

The matrix multiplication of (3-14) using (3-15) gives

$$\vec{E}_1 = \frac{1}{\sin^2\theta_1}\{\vec{a}_x[-V_{PP}(\theta_1)\cos 2\theta_1(\cos\varphi_n\sin(\theta-\theta_n)\cos\varphi-\sin\varphi_n\sin\varphi)^2$$
$$-V_{NN}(\theta_1)(\sin\varphi\cos\varphi_n\sin(\theta-\theta_n)+\sin\varphi_n\cos\varphi)^2]$$
$$+\vec{a}_y[-(V_{PP}(\theta_1)\cos 2\theta_1-V_{NN}(\theta_1))(\sin\varphi\cos\varphi\cos^2\varphi_n\sin^2(\theta-\theta_n)$$
$$+\cos^2\varphi\sin\varphi_n\cos\varphi_n\sin(\theta-\theta_n)-\sin^2\varphi\sin\varphi_n\cos\varphi_n\sin(\theta-\theta_n)$$
$$-\sin\varphi\cos\varphi\sin^2\varphi_n)]\} \tag{3-16}$$

Also

$$\vec{H}_1 = \frac{\gamma_1}{\sin\theta_1}\begin{bmatrix} \cos\theta_1\sin\theta_1 & \cos\varphi_n\sin(\theta-\theta_n)\sin\theta_1 & \sin\varphi_n\sin\theta_1 \\ \sin^2\theta_1 & -\cos\varphi_n\sin(\theta-\theta_n)\cos\theta_1 & -\sin\varphi_n\cos\theta_1 \\ 0 & -\sin\varphi_n & \cos\varphi_n\sin(\theta-\theta_n) \end{bmatrix} \cdot$$
$$\cdot \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \end{bmatrix} \vec{A}_3 \tag{3-17}$$

which gives upon multiplication

$$\vec{H}_1 = \frac{1}{\sin^2\theta_1}\{\vec{a}_x[-(V_{PP}(\theta_1)\cos 2\theta_1 - V_{NN}(\theta_1))(\sin\varphi\cos\varphi\cos^2\varphi_n \cdot$$

$$\cdot \sin^2(\theta-\theta_n)+\cos^2\varphi\sin\varphi_n\cos\varphi_n\sin(\theta-\theta_n)-\sin^2\varphi\sin\varphi_n\cos\varphi_n \cdot$$

$$\cdot \sin(\theta-\theta_n)-\sin\varphi\cos\varphi\sin^2\varphi_n)]+\vec{a}_y[V_{PP}(\theta_1)\cos 2\theta_1(\cos\varphi_n \cdot$$

$$\cdot \sin(\theta-\theta_n)\cos\varphi-\sin\varphi_n\sin\varphi)^2+V_{NN}(\theta_1)(\sin\varphi\cos\varphi_n\sin(\theta-\theta_n)$$

$$+\sin\varphi_n\cos\varphi)^2]\} \qquad (3\text{-}18)$$

Inspection of (3-16) and (3-18) shows that

$$E_{1x} = -H_{1y} \qquad \text{and} \qquad E_{1y} = H_{1x} \qquad (3\text{-}19)$$

Inspecting $\vec{E}_1$, both the x and y components appear to have poles at $\theta_1 = 0$. These poles are shown in Appendix B to be apparent poles. From (B-3) and (B-8), (3-16) becomes

$$\vec{E}_1 = \vec{a}_x\{-V_{PP}(\theta_1)\cos 2\theta_1+V_2(\theta_1)[\sin\varphi\cos\varphi_n\sin(\theta-\theta_n)$$

$$+\sin\varphi_n\cos\varphi]^2\}-\vec{a}_y V_2(\theta_1)[\sin\varphi\cos\varphi\cos^2\varphi_n\sin^2(\theta-\theta_n)$$

$$+\cos^2\varphi\sin\varphi_n\cos\varphi_n\sin(\theta-\theta_n)-\sin^2\varphi\sin\varphi_n\cos\varphi_n\sin(\theta-\theta_n)$$

$$-\sin\varphi\cos\varphi\sin^2\varphi_n] \qquad (3\text{-}20)$$

where

$$V_2(\theta_1) = \frac{2\cos\theta_1\left[\mu_r(n^2-1)\cos\theta_1+(n^2-\mu_r^2)\sqrt{n^2-\sin^2\theta_1}\right]}{\left[n^2\cos\theta_1+\mu_r\sqrt{n^2-\sin^2\theta_1}\right]\left[\mu_r\cos\theta_1+\sqrt{n^2-\sin^2\theta_1}\right]}$$

Since the surface variation is given in terms of $H(\theta,\varphi)$ and θ_n, φ_n are related to the partial derivatives of $H(\theta, \varphi)$, (3-20) will be converted to $H(\theta, \varphi)$ and its partial derivatives by means of the following definitions:

$$X_1 = \frac{H(\theta,\varphi)}{a}\ ; \qquad X_3 = \frac{1}{a}\frac{\partial H(\theta,\varphi)}{\partial\theta}\ ; \qquad X_5 = \frac{1}{a\sin\theta}\frac{\partial H(\theta,\varphi)}{\partial\varphi}$$

$$X_2 = \frac{H(\theta',\varphi')}{a}\ ; \qquad X_4 = \frac{1}{a}\frac{\partial H(\theta',\varphi')}{\partial\theta'}\ ; \qquad X_6 = \frac{1}{a\sin\theta'}\frac{\partial H(\theta',\varphi')}{\partial\varphi'}$$

$$Y_1 = X_1\ ; \qquad Y_3 = \frac{a}{r_o}X_3\ ; \qquad Y_5 = \frac{a}{r_o}X_5$$

$$Y_2 = X_2\ ; \qquad Y_4 = \frac{a}{r_o}X_4\ ; \qquad Y_6 = \frac{a}{r_o}X_6 \qquad (3\text{-}21)$$

Also define

$$V_{rc}(\theta_1) = \cos\theta_1 V_2(\theta_1)$$

$$V_{rd}(\theta_1) = \cos\theta_1\cos2\theta_1 V_{PP}(\theta_1)$$

$$\cos\theta_1 E_{1x} = -\cos\theta_1 H_{1y} = -V_{rd}(\theta_1)+V_{rc}(\theta_1)E_a$$

$$\cos\theta_1 E_{1y} = \cos\theta_1 H_{1x} = V_{rc}(\theta_1)E_b \qquad (3\text{-}22)$$

Now convert E_a and E_b to Y's by use of (A-18)

$$E_a = \frac{1}{J^2}[\sin^2\theta\sin^2\varphi - 2Y_3\sin\theta\cos\theta\sin^2\varphi - 2Y_5\sin\theta\sin\varphi\cos\varphi$$

$$+Y_3^2\cos^2\theta\sin^2\varphi + 2Y_3Y_5\cos\theta\sin\varphi\cos\varphi + Y_5^2\cos^2\varphi] \quad (3\text{-}23)$$

$$E_b = \frac{1}{2J^2}[\sin^2\theta\sin 2\varphi - 2Y_3\sin\theta\cos\theta\sin 2\varphi - 2Y_5\sin\theta\cos 2\varphi$$

$$+Y_3^2\cos^2\theta\sin 2\varphi + 2Y_3Y_5\cos\theta\cos 2\varphi - Y_5^2\sin 2\varphi] \quad (3\text{-}24)$$

3.3 Determination of Direct- and Cross-Polarized Power

In this section, the instantaneous power is calculated in integral form and separated into the direct- and cross-polarized components. From Poynting's Theorem the instantaneous power $\vec{S}(t)$ across an interface per unit area is given as

$$\vec{S}(t) = \text{Re}[\vec{E}_r(t)] \times \text{Re}[\vec{H}_r(t)] \quad (3\text{-}25)$$

With the receiving antenna located on the z axis, the only component of power that will be seen by the receiver is the z component. Expanding (3-25) and considering only the z component gives

$$S_z(t) = \text{Re}[E_{rx}(t)]\text{Re}[H_{ry}(t)] - \text{Re}[E_{ry}(t)]\text{Re}[H_{rx}(t)] \quad (3\text{-}26)$$

The direct-polarized component of power (S_d) is that component whose $\vec{E}_r$ field has the same polarization as the transmitted $\vec{E}$ field (i.e., the x component of $\vec{E}_r$). The cross-polarized component of power (S_c) is that component having an $\vec{E}_r$ field orthogonal to the transmitted polarization (i.e., the y component of $\vec{E}_r$). The identification of these components in (3-26) yields

$$s_d(t) = \mathrm{Re}[E_{rx}(t)]\mathrm{Re}[H_{ry}(t)]$$

$$s_c(t) = -\mathrm{Re}[E_{ry}(t)]\mathrm{Re}[H_{rx}(t)] \tag{3-27}$$

Using

$$\mathrm{Re}[E_{rx}(t)] = \frac{1}{2}[E_{rx}(t) + E^*_{rx}(t)] \tag{3-28}$$

and the Fourier transforms

$$\mathcal{F}[E_{rx}(t)] = E_{rx}(\omega)$$

$$\mathcal{F}[E^*_{rx}(t)] = E^*_{rx}(-\omega)$$

$$\mathcal{F}[E_{rx}(t)H_{ry}(t)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{rx}(\omega_1)H_{ry}(\omega - \omega_1)d\omega_1 \tag{3-29}$$

then (3-27) becomes

$$s_d(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega t}\left\{\frac{1}{8\pi}\int_{-\infty}^{\infty}[E_{rx}(\omega_1)H_{ry}(\omega - \omega_1) + E_{rx}(\omega_1)\cdot\right.$$

$$\cdot H^*_{ry}(\omega_1 - \omega) + E^*_{rx}(-\omega_1)H_{ry}(\omega - \omega_1) + E^*_{rx}(-\omega_1)\cdot$$

$$\left. H^*_{ry}(\omega_1 - \omega)]d\omega_1\right\}d\omega$$

$$s_c(t) = \frac{-1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega t}\left\{\frac{1}{8\pi} \int_{-\infty}^{\infty} [E_{ry}(\omega_1)H_{rx}(\omega - \omega_1) + E_{ry}(\omega_1) \cdot \right.$$

$$\cdot H^*_{rx}(\omega_1 - \omega) + E^*_{ry}(-\omega_1)H_{rx}(\omega - \omega_1) + E^*_{ry}(-\omega_1) \cdot$$

$$\left. H^*_{rx}(\omega_1 - \omega)]d\omega_1\right\}d\omega \tag{3-30}$$

Now consider the pulsed case where $c_o(\omega)$ is defined in (2-60) and substitute into (3-13) after taking the Fourier transform. The results are

$$\vec{E}_r(\vec{r}_1, \omega_o, \omega) = \frac{C_1}{2\pi c^3 i} \int_{S_o} \frac{\omega^2 e^{i2kR}}{R^2} \vec{E}_1 \cos\theta_1 f(\omega, \omega_o, T)dS$$

$$\vec{H}_r(\vec{r}_1, \omega_o, \omega) = \frac{-C_1\epsilon_2}{2\pi c^2 i} \int_{S_o} \frac{\omega^2 e^{i2kR}}{R^2} \vec{H}_1 \cos\theta_1 f(\omega, \omega_o, T)dS \tag{3-31}$$

The substitution of (3-22) into (3-31) yields for the x and y components

$$E_{rx}(\vec{r}_1, \omega_o, \omega_1) = \frac{C_1}{2\pi c^3} \int_{S_o} \frac{\omega_1^2\, e^{i2\omega_1 R/c}}{iR^2}[-V_{rd}(\theta_1) + V_{rc}(\theta_1)E_a] \cdot$$

$$\cdot\, f(\omega_1, \omega_o, T)dS$$

$$E_{ry}(\vec{r}_1, \omega_o, \omega_1) = \frac{C_1}{2\pi c^3} \int_{S_o} \frac{\omega_1^2\, e^{i2\omega_1 R/c}}{iR^2} f(\omega_1, \omega_o, T)V_{rc}(\theta_1)E_b dS$$

$$H_{rx}(\vec{r}_1, \omega_o, \omega - \omega_1) = \frac{-c_1\epsilon_2}{2\pi c^2} \int_{S_o'} \frac{(\omega - \omega_1)^2 e^{i2(\omega-\omega_1)R'/c}}{iR'^2} \cdot$$

$$\cdot f(\omega - \omega_1, \omega_o, T) V_{rc}(\theta_1') E_b' dS'$$

$$H_{ry}(\vec{r}_1, \omega_o, \omega - \omega_1) = \frac{c_1\epsilon_2}{2\pi c^2} \int_{S_o'} \frac{(\omega - \omega_1)^2 e^{i2(\omega-\omega_1)R'/c}}{iR'^2} \cdot$$

$$\cdot f(\omega - \omega_1, \omega_o, T)[-V_{rd}(\theta_1')$$

$$+ V_{rc}(\theta_1') E_a'] dS' \qquad (3\text{-}32)$$

Substituting (3-32) into (3-30) and changing the order of integration gives, under the assumption that the Fresnel reflection coefficients are not functions of frequency (i.e., zero conductivity),

$$s_d(t) = \frac{c_1^2\epsilon_2}{4\pi^2 c^5} \int_{S_o} \int_{S_o'} \frac{[-V_{rd}(\theta_1) + V_{rc}(\theta_1)E_a]}{R^2} \cdot$$

$$\cdot \frac{[-V_{rd}(\theta_1') + V_{rc}(\theta_1')E_a']}{R'^2} \left\{\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega t} \cdot\right.$$

$$\cdot \left[\frac{1}{8\pi} \int_{-\infty}^{\infty} -\omega_1^2(\omega-\omega_1)^2 e^{i2\omega_1 R/c} f(\omega_1, \omega_o, T) e^{i2(\omega-\omega_1)R'/c} \cdot\right.$$

$$\cdot f(\omega-\omega_1, \omega_o, T) + \omega_1^2(\omega-\omega_1)^2 e^{i2\omega_1 R/c} f(\omega_1, \omega_o, T) \cdot$$

$$\cdot\ e^{i2(\omega-\omega_1)R'/c} \quad f^*(\omega_1-\omega,\omega_o,T)+\omega_1^2(\omega-\omega_1)^2\ \cdot$$

$$\cdot\ e^{i2\omega_1 R/c} \quad f^*(-\omega_1,\omega_o,T)\ e^{i2(\omega-\omega_1)R'/c} \quad f(\omega-\omega_1,\omega_o,T)$$

$$-\omega_1^2(\omega-\omega_1)^2\ e^{i2\omega_1 R/c} \quad f^*(-\omega_1,\omega_o,T)\ e^{i2(\omega-\omega_1)R'/c}\ \cdot$$

$$\cdot\ f^*(\omega_1-\omega,\omega_o,T) \quad d\omega_1\Big]d\omega\Big\}dSdS'$$

$$S_c(t) = \frac{c_1^2\epsilon_2}{4\pi^2c^5}\int_{S_o}\int_{S_o'}\frac{V_{rc}(\Theta_1)V_{rc}(\Theta_1')E_bE_b'}{R^2R'^2}\Big\{\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega t}\ \cdot$$

$$\cdot\ \Big[\frac{1}{8\pi}\int_{-\infty}^{\infty} -\omega_1^2(\omega-\omega_1)^2\ e^{i2\omega_1 R/c} \quad f(\omega_1,\omega_o,T)\ \cdot$$

$$\cdot\ e^{i2(\omega-\omega_1)R'/c} \quad f(\omega-\omega_1,\omega_o,T)+\omega_1^2(\omega-\omega_1)^2\ e^{i2\omega_1 R/c}\ \cdot$$

$$\cdot\ f(\omega_1,\omega_o,T)\ e^{i2(\omega-\omega_1)R'/c} \quad f^*(\omega_1-\omega,\omega_o,T)+\omega_1^2(\omega-\omega_1)^2\ \cdot$$

$$\cdot\ e^{i2\omega_1 R/c} \quad f^*(-\omega_1,\omega_o,T)\ e^{i2(\omega-\omega_1)R'/c} \quad f(\omega-\omega_1,\omega_o,T)$$

$$-\omega_1^2(\omega-\omega_1)^2\ e^{i2\omega_1 R/c} \quad f^*(-\omega_1,\omega_o,T)\ e^{i2(\omega-\omega_1)R'/c}\ \cdot$$

$$\cdot\ f^*(\omega_1-\omega,\omega_o,T) \quad d\omega_1\Big]d\omega\Big\}dSdS' \qquad (3\text{-}33)$$

Let

$$f_1(t) = \mathcal{F}^{-1}\left\{e^{i2\omega R/c}\, f(\omega, \omega_o, T)\right\}$$

$$f_2(t) = \mathcal{F}^{-1}\left\{e^{i2\omega R'/c}\, f(\omega, \omega_o, T)\right\} \qquad (3\text{-}34)$$

Using (3-29) and

$$\mathcal{F}\left\{\frac{d^2 g(t)}{dt^2}\right\} = -\omega^2 \mathcal{F}\left\{g(t)\right\}$$

$$\mathrm{Im}[g(t)] = \frac{1}{2i}[g(t) - g^*(t)] \qquad (3\text{-}35)$$

gives

$$\mathrm{Im}\left[\frac{d^2 f_1(t)}{dt^2}\right]\mathrm{Im}\left[\frac{d^2 f_2(t)}{dt^2}\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega t}\left\{\frac{1}{8\pi}\int_{-\infty}^{\infty}\left[-\omega_1^2(\omega-\omega_1)^2 f_1(\omega_1)\cdot\right.\right.$$

$$\cdot\, f_2(\omega-\omega_1)+\omega_1^2(\omega-\omega_1)^2 f_1^*(-\omega_1)f_2(\omega-\omega_1)$$

$$+\omega_1^2(\omega-\omega_1)^2 f_1(\omega_1)f_2^*(\omega_1-\omega)-\omega_1^2(\omega-\omega_1)^2\cdot$$

$$\left.\left. f_1^*(-\omega_1)f_2^*(\omega_1-\omega)\right]d\omega_1\right\}d\omega \qquad (3\text{-}36)$$

Realizing that (3-34) substituted into (3-36) is identical to the terms inside the brace of both equations of (3-33), it is necessary to evaluate

$$\mathrm{Im}\left[\frac{d^2 f_1(t)}{dt^2}\right]\mathrm{Im}\left[\frac{d^2 f_2(t)}{dt^2}\right] = -\frac{1}{4}\left\{\frac{d^2 f_1}{dt^2}\,\frac{d^2 f_2}{dt^2} - \left[\frac{d^2 f_1}{dt^2}\right]\left[\frac{d^2 f_2}{dt^2}\right]^*\right.$$

$$- \left[\frac{d^2 f_1}{dt^2}\right]^* \left[\frac{d^2 f_2}{dt^2}\right] + \left[\frac{d^2 f_1}{dt^2}\right]^* \left[\frac{d^2 f_2}{dt^2}\right]^* \Big\} \qquad (3\text{-}37)$$

Taking the inverse Fourier transform of (3-34) yields

$$f_1(t) = e^{-i\omega_o \eta_1} \, U(\eta_1)$$

$$f_2(t) = e^{-i\omega_o \eta_2} \, U(\eta_2) \qquad (3\text{-}38)$$

where

$$\eta_1 = t - 2R/c$$

$$\eta_2 = t - 2R'/c$$

$$U\left[\frac{w_1}{w_2}\right] = u\left[\frac{w_1}{w_2}\right] - u\left[\frac{w_1 - T}{w_2}\right]$$

then

$$\frac{d^2 f_1}{dt^2} = -\omega_o^2 \, e^{-i\omega_o \eta_1} \, L(\eta_1) \qquad (3\text{-}39)$$

where

$$L(\eta_1) = U(\eta_1) + \frac{i2}{\omega_o} \Delta(\eta_1) - \frac{1}{\omega_o^2} \Delta'(\eta_1)$$

$$\Delta\left[\frac{w_1}{w_2}\right] = \delta\left[\frac{w_1}{w_2}\right] - \delta\left[\frac{w_1 - T}{w_2}\right]$$

$$\Delta'\left[\frac{w_1}{w_2}\right] = \delta'\left[\frac{w_1}{w_2}\right] - \delta'\left[\frac{w_1 - T}{w_2}\right]$$

$\delta(x)$ is the Dirac delta function

$$\delta'(x) = \frac{d\delta(x)}{dx}$$

$$\text{Im}\left[\frac{d^2 f_1}{dt^2}\right]\text{Im}\left[\frac{d^2 f_2}{dt^2}\right] = -\frac{\omega_o^4}{4}\Big[e^{-i\omega_o(\eta_1+\eta_2)} L(\eta_1)\, L(\eta_2)$$

$$- e^{-i\omega_o(\eta_1-\eta_2)} L(\eta_1)L^*(\eta_2) - e^{-i\omega_o(\eta_2-\eta_1)} \cdot$$

$$\cdot L^*(\eta_1)L(\eta_2) + e^{i\omega_o(\eta_1+\eta_2)} \cdot$$

$$\cdot L^*(\eta_1)L^*(\eta_2)\Big] \qquad (3\text{-}40)$$

which can be expressed as

$$\text{Im}\left[\frac{d^2 f_1}{dt^2}\right]\text{Im}\left[\frac{d^2 f_2}{dt^2}\right] = \frac{-\omega_o^4}{4}\sum_{j=1}^{4} e^{-ia_j\omega_o\eta_1} \cdot$$

$$\cdot e^{-ib_j\omega_o\eta_2} L_j(\eta_1)L_j(\eta_2) \qquad (3\text{-}41)$$

where

$a_1 = +1 \quad a_2 = +1 \quad a_3 = -1 \quad a_4 = -1$

$b_1 = +1 \quad b_2 = -1 \quad b_3 = +1 \quad b_4 = -1$

$$L_j(\eta_1) = U(\eta_1) + \frac{2\Delta(\eta_1)}{(-ia_j\omega_o)} + \frac{\Delta'(\eta_1)}{(-ia_j\omega_o)^2}$$

$$L_j(\eta_2) = U(\eta_2) + \frac{2\Delta(\eta_2)}{(-ib_j\omega_o)} + \frac{\Delta'(\eta_2)}{(-ib_j\omega_o)^2}$$

In order to integrate (3-33), it is necessary to express dS and dS'. These are

$$dS = Jr_o^2 \sin\theta d\theta d\varphi$$

$$dS' = J'(r_o')^2 \sin\theta' d\theta' d\varphi' \qquad (3\text{-}42)$$

where

$$r_o = a(1 + X_1) \quad \text{and} \quad r_o' = a(1 + X_2)$$

then E_a and E_b can be expressed as

$$E_a dS = \frac{a^2}{J} E_c \sin\theta d\theta d\varphi$$

$$E_b dS = \frac{a^2}{2J} E_d \sin\theta d\theta d\varphi \qquad (3\text{-}43)$$

Using (3-21) gives

$$E_c = (1+X_1)^2\sin^2\theta\sin^2\varphi - 2(1+X_1)X_3\sin\theta\cos\theta\sin^2\varphi$$

$$-2(1+X_1)X_5\sin\theta\sin\varphi\cos\varphi + X_3^2\cos^2\theta\sin^2\varphi + 2X_3X_5\cos\theta\sin\varphi\cos\varphi$$

$$+X_5^2\cos^2\varphi$$

$$E_d = \Big[(1+X_1)^2\sin^2\theta\sin2\varphi - 2(1+X_1)X_3\sin\theta\cos\theta\sin2\varphi - 2(1+X_1)X_5 \cdot$$

$$\sin\theta\cos2\varphi + X_3^2\cos^2\theta\sin2\varphi + 2X_3X_5\cos\theta\cos2\varphi - X_5^2\sin2\varphi\Big] \qquad (3\text{-}44)$$

Considering an ideal conical source of vertex angle $2\theta_a$ pointed at the center of the moon, the illuminated surface S_o is defined by

$$0 \le \theta \le \theta_a \qquad 0 \le \theta' \le \theta_a$$

$$-\pi \le \varphi \le \pi \qquad -\pi \le \varphi' \le \pi \tag{3-45}$$

Then the substitution of (3-42) through (3-45) into (3-33) yields

$$S_d(t) = -4B_1 \sum_{j=1}^{4} a_j b_j \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{0}^{\theta_a}\int_{0}^{\theta_a} \left[-J(1+X_1)^2 V_{rd}(\theta_1) + \frac{V_{rc}(\theta_1)}{J} E_c\right]\left[-J'(1+X_2)^2 V_{rd}(\theta_1') + \frac{V_{rc}(\theta_1')}{J} E_c'\right] \cdot \frac{e^{-i\omega_o(a_j\eta_1+b_j\eta_2)}}{R^2R'^2} L_j(\eta_1)L_j(\eta_2)\sin\theta d\theta \sin\theta' d\theta' d\varphi d\varphi'$$

$$S_c(t) = -B_1 \sum_{j=1}^{4} a_j b_j \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{0}^{\theta_a}\int_{0}^{\theta_a} \frac{V_{rc}(\theta_1)}{J}\frac{V_{rc}(\theta_1')}{J'} E_d E_d' \cdot \frac{e^{-i\omega_o(a_j\eta_1+b_j\eta_2)}}{R^2R'^2} L_j(\eta_1)L_j(\eta_2)\sin\theta d\theta \sin\theta' d\theta' d\varphi d\varphi' \tag{3-46}$$

where

$$B_1 = \frac{C_1^2 \epsilon_2 a^4 \omega_o^4}{64\pi^2 c^5}$$

3.4 Discussion of the Direct- and Cross-Polarized Power Integrals

The integral equations (3-46) for the direct- and cross-polarized power are valid under the following assumptions:

1) The radius of curvature is much larger than the wavelength of incident radiation.
2) Shadowing effects are neglected.
3) Multiple scattering is neglected.
4) Only the far field is calculated ($k_2R_o >> 2|n^2|$)
5) The transmitted pulse contains an integral number of cycles.
6) The transmitting antenna behaves as a short dipole at all frequencies.
7) $\mu_r < \epsilon_r$

These assumptions are just those needed for the validity of the concept of differential reflectivity; however assumption 6 needs some further discussion. Since both the direct- and cross-polarized components are very dependent upon the frequency description of the pulsed source, it is necessary to have a complete frequency description of the radiated signal before attempting to match experimentally obtained data with the solution of these integrals. This frequency description is lacking in most experimental situations.

If $H(\theta, \varphi)$ is known explicitly the two components of power could be obtained by solution of the integrals in (3-46). Also, the effect of shadowing could be taken into account by suitable modification of the limits of the integrals such that only the illuminated regions were integrated.

An examination of E_c and E_d given in (3-44) indicates that, if $H(\theta, \varphi)$ does not vary with φ, the cross-polarized power vanishes due to φ and φ' integrations. In the event that $H(\theta, \varphi)$ is identically zero, the φ and φ' integrations become very simple and the θ and θ' integrations can be separated into identical integrals. Thus these integrals contain also the results for pulsed source reflections from a smooth sphere. However, as with all electromagnetic reflection problems, the integrations can be accomplished only by infinite series or by some approximate method.

Further, these integrals contain the solution for all time with the unit steps and delta functions in the integrand limiting the integration to the proper range. If a transmitted pulse had been assumed which contained continuous derivatives, then the amplitude of the terms multiplying the delta functions would be zero.

CHAPTER 4

EVALUATION OF EXPECTED POWER FOR A GAUSSIAN SURFACE

4.1 Determination of Expected Values

In this chapter, the deviation of surface heights, $H(\theta,\varphi)$, from the average sphere of radius a is considered to be a random variable. It is assumed that $H(\theta,\varphi)$ is a gaussian separable random process, real and continuous in the mean over the surface of the sphere, with zero mean, variance σ^2 and normalized covariance function $\rho(\theta',\theta,\varphi',\varphi)$. Since it is desired that $H(\theta,\varphi)$ represent a continuous surface, it is required that $H(\theta,\varphi)$ be three times mean square differentiable. This is assured if ρ has continous partial derivatives up to and including the fourth order. [Hoffman, 1955; Moyal, 1949, p. 167]. Also, since it is desired that $H(\theta,\varphi)$ be a stationary random process, the covariance function ρ will depend only on the distance d between the two points (θ,φ) and (θ',φ'). An exponential covariance function will be assumed, but $\exp(-\alpha|d|)$ is not satisfactory since it does not possess the required continuous derivatives. Consequently rather than modify $\exp(-\alpha|d|)$ at $d=0$, the form $\exp(-\alpha d^2)$ will be assumed for ρ. On the surface of the sphere, the distance d is given by $a\gamma$, where γ is the smaller spherical angle between the two points. Therefore

$$\rho(\theta,\theta',\varphi,\varphi') = \exp\left(\frac{-a^2\gamma^2}{\delta^2}\right) \tag{4-1}$$

where δ is the correlation distance.

If $\delta \ll a$, and approximating γ^2 by

$$\gamma^2 \cong 2(1 - \cos \gamma) \tag{4-2}$$

where $\cos \gamma = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos \psi$

$$\psi = \varphi' - \varphi$$

then

$$\rho = \exp\left[-\beta f_3(\theta,\theta',\psi)\right] \tag{4-3}$$

where $\beta = 2a^2/\delta^2$

$$f_3(\theta,\theta',\psi) = 1 - \cos \theta \cos \theta' - \sin \theta \sin \theta' \cos \psi.$$

The ρ defined in (4-3) differs from the original ρ of (4-1) only for very small values of ρ.

Under the above conditions, the joint probability density of $H(\theta,\varphi)$ and $H(\theta',\varphi')$ is

$$p\left[H(\theta\varphi),\ H(\theta'\varphi')\right] = \frac{1}{2\pi\sigma^2\sqrt{1-\rho^2}} \cdot$$

$$\cdot\left\{-\left[\left[H(\theta,\varphi)\right]^2 + \left[H(\theta',\varphi')\right]^2 - 2\rho H(\theta,\varphi)H(\theta',\varphi')\right]/2\sigma^2(1-\rho^2)\right\} \tag{4-4}$$

Letting

$$\xi_1 = H(\theta,\varphi)/\sigma$$
$$\xi_2 = H(\theta',\varphi')/\sigma$$

then

$$p(\xi_1,\xi_2) = \frac{\exp\left\{-\left[\xi_1^2 + \xi_1^2 - 2\rho\xi_1\xi_2\right]/2(1-\rho^2)\right\}}{2\pi\sqrt{1-\rho^2}} \tag{4-5}$$

Under the assumption that the distance to the sphere is much greater than the radius of the sphere (i.e., $D \gg a$), then, as far as the phase term goes,

$$R \cong D - \left[a + H(\theta,\varphi)\right] . q \tag{4-6}$$

Let

$$\zeta_1 = t - \frac{2(D - aq)}{c} \qquad \zeta_2 = t - \frac{2(D - aq')}{c}$$

$$\sigma_1 = \frac{2q\sigma}{c} \qquad \sigma_2 = \frac{2q'\sigma}{c}$$

then

$$\eta_1 = \zeta_1 + \sigma_1\xi_1 \qquad \eta_2 = \zeta_2 + \sigma_2\xi_2 \tag{4-7}$$

Now consider the following expected values

$$\langle X_k X_\ell \, e^{-i\omega_o(a_j\eta_1 + b_j\eta_2)} L_j(\eta_1)L_j(\eta_2)\rangle = e^{-i\omega_o(a_j\zeta_1 + b_j\zeta_2)}\, {}_N(k,\ell) \tag{4-8}$$

where

$${}_N(k,\ell) = \langle X_k X_\ell \, e^{-i\omega_o(a_j\sigma_1\xi_1 + b_j\sigma_2\xi_2)} L_j(\sigma_1\xi_1 + \zeta_1)L_j(\sigma_2\xi_2 + \zeta_2)\rangle$$

where the notation $\langle\ \rangle$ indicates the ensemble averages.

Realizing that

$$\frac{\partial U(\sigma_1\xi_1 + \zeta_1)}{\partial\zeta_1} = \Delta(\sigma_1\xi_1 + \zeta_1)$$

$$\frac{\partial\Delta(\sigma_1\xi_1 + \zeta_1)}{\partial\zeta_1} = \Delta'(\sigma_1\xi_1 + \zeta_1) \tag{4-9}$$

then

$$L_j(\sigma_1\xi_1 + \zeta_1)L_j(\sigma_2\xi_2 + \zeta_2) =$$

$$\sum_{n=1}^{3} e_n \frac{\partial^{n-1}}{\partial\zeta_2^{n-1}} \sum_{m=1}^{3} g_m \frac{\partial^{m-1}}{\partial\zeta_1^{m-1}} \left[U(\sigma_1\xi_1+\zeta_1)U(\sigma_2\xi_2+\zeta_2)\right] \tag{4-10}$$

where $e_1 = 1 \quad e_2 = \dfrac{2}{(-i\omega_o b_j)}$; $g_1 = 1$; $g_2 = \dfrac{2}{(-i\omega_o a_j)}$

$$e_3 = \frac{1}{(-i\omega_o b_j)^2} \qquad g_3 = \frac{1}{(-i\omega_o a_j)^2}$$

which upon interchanging expectation and summations yields

$$N^{(k,\ )} = \sum_{n=1}^{3} e_n \frac{\partial^{n-1}}{\partial \zeta_2^{n-1}} \sum_{m=1}^{3} g_m \frac{\partial^{m-1}}{\partial \zeta_1^{m-1}} \langle X_i\ e^{-i\omega_o(a_j\sigma_1\xi_1+b_j\sigma_2\xi_2)} \cdot$$

$$\cdot\ U(\sigma_1\xi_1 + \zeta_1)U(\sigma_2\xi_2 + \zeta_2)\rangle \tag{4-11}$$

For the $X_o = 1$ case, consider n=1, m=1

$$N_{11}^{(o,o)} = \langle e^{-i\omega_o(a_j\sigma_1\xi_1+b_j\sigma_2\xi_2)} U(\sigma_1\xi_1+\zeta_1)U(\sigma_2\xi_2+\zeta_2)\rangle$$

$$= \int_{-\zeta_1/\sigma_1}^{(T-\zeta_1)/\sigma_1} e^{-i\omega_o a_j\sigma_1\xi_1} \int_{-\zeta_2/\sigma_2}^{(T-\zeta_2)/\sigma_2} e^{-i\omega_o b_j\sigma_2\xi_2} \cdot$$

$$\cdot\ p(\xi_1,\xi_2)\ d\xi_2\ d\xi_1 \tag{4-12}$$

If $\sigma < 100$ meters, the error functions which appear in each integration can be reasonably approximated as

$$\frac{1}{2}\left[\Phi(x_1 + iy) - \Phi(x_2 + iy)\right] \cong u(x_1) - u(x_2) \tag{4-13}$$

Then (4-12) becomes

$$N_{11}^{(o,o)} = e^{-Q_j}\ U\left(\frac{\zeta_1}{\sigma_1\sqrt{2}}\right)\ U\left(\frac{\zeta_2}{\sigma_2\sqrt{2(1-\rho^2)}}\right)$$

$$= \langle e^{-i\omega_o(a_j\sigma_1\xi_1 + b_j\sigma_2\xi_2)}\rangle\ \langle U(\sigma_1\xi_1 + \zeta_1)\cdot$$

$$\cdot U(\sigma_2\xi_2 + \zeta_2)\rangle \tag{4-14}$$

where

$$Q_j = \frac{pg}{2}\left[q^2 + q'^2 + 2a_jb_jqq'\rho\right]$$

$$p = 2k_2a$$

$$g = \frac{2k_2\sigma^2}{a}$$

With this justification, $N^{(k,\ell)}$ will be approximated by

$$N^{(k,\ell)} = \langle X_k X_\ell \, e^{-i\omega_o(a_j\sigma_1\xi_1 + b_j\sigma_2\xi_2)} \rangle \langle L_j(\sigma_1\xi_1 + \zeta_1) \cdot$$

$$\cdot \, L_j(\sigma_2\xi_2 + \zeta_2) \rangle$$

$$= M_{k,\ell} \langle L_j(\sigma_1\xi_1 + \zeta_1) L_j(\sigma_2\xi_2 + \zeta_2) \rangle \qquad (4\text{-}15)$$

The $M_{k,\ell}$ terms are calculated in Appendix C. It should be noted that the coefficient of the $M_{k,o}$ terms for k = 1, 2 is a factor g larger than that of the k = 0 term and for k = 3, 4, 5, and 6 is a factor βg larger than that of the k = 0 term. Therefore, for an infinite series expansion of the reflection coefficients in the integrand of the power integrals of Chapter 3 to be a convergent, a sufficient condition would be $g\beta < 1$. This assumption will be made for the remainder of this paper. Now the ensemble average of the direct- and cross-polarized powers from (3-46) can be written as

$$\langle S_d(t) \rangle = -4B_1 \sum_{j=1}^{4} a_j b_j \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \int_{o}^{\Theta_a} \int_{o}^{\Theta_a} \langle \left[-J(1 + X_1)^2 \cdot \right.$$

$$\left. \cdot \, V_{rd}(\Theta_1) + \frac{V_{rc}(\Theta_1)}{J} E_c \right] \left[-J'(1+X_2)^2 V_{rd}(\Theta_1') + \frac{V_{rc}(\Theta_1')}{J'} E_c' \right] \cdot$$

$$\cdot \, \frac{e^{-i\omega_o(a_j\sigma_1\xi_1 + b_j\sigma_2\xi_2)}}{R^2 R'^2} \rangle \langle L_j(\eta_1) L_j(\eta_2) \rangle \, e^{-i\omega_o(a_j\zeta_1 + b_j\zeta_2)} \cdot$$

$$\cdot \, \sin\Theta d\Theta \sin\Theta' d\Theta' \, d\varphi d\varphi'$$

$$\langle S_c(t) \rangle = -B_1 \sum_{j=1}^{4} a_j b_j \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \int_{o}^{\Theta_a} \int_{o}^{\Theta_a} \langle \left[\frac{V_{rc}(\Theta_1)}{J} \frac{V_{rc}(\Theta_1')}{J'} \right]$$

$$\cdot \left[E_d E_d' \frac{e^{-i\omega_o(a_j\sigma_1\xi_1+b_j\sigma_2\xi_2)}}{R^2 R'^2} \right] \rangle \langle L_j(\eta_1) L_j(\eta_2) \rangle \cdot$$

$$\cdot\ e^{-i\omega_o(a_j\zeta_1+b_j\zeta_2)} \sin\Theta d\Theta \sin\Theta' d\Theta'\ d\varphi d\varphi' \qquad (4\text{-}16)$$

Since D >> a, then, as far as the amplitude is concerned, $R \cong D$ for all Θ. Neglect terms containing more than two X coefficients and use the relations

$$q = \cos\Theta \qquad \sqrt{1-q^2} = \sin\Theta$$

$$q = \cos\Theta' \qquad \sqrt{1-q'^2} = \sin\Theta'$$

$$\alpha = \cos\Theta_a$$

$$\langle \left[-J(1+X_1)^2 V_{rd}(\Theta_1) + \frac{V_{rc}(\Theta_1)}{J} E_c \right] \left[-J'(1+X_2)^2 V_{rd}(\Theta_1') + \frac{V_{rc}(\Theta_1')}{J'} E_c' \right] e^{-i\omega_o(a_j\sigma_1\xi_1+b_j\sigma_2\xi_2)} \rangle = \sum_{k=o}^{6} \sum_{\ell=o}^{k} F_{1d}^{(k,\ell)}(q) \cdot$$

$$\cdot\ F_{2d}^{(k,\ell)}(q')\ g_{1d}^{(k,\ell)}(\varphi,\varphi')\ M_{k,\ell}$$

$$\langle \frac{V_{rc}(\Theta_1)}{J} \frac{V_{rc}(\Theta_1')}{J'} E_d E_d'\ e^{-i\omega_o(a_j\sigma_1\xi_1+b_j\sigma_2\xi_2)} \rangle$$

$$= \sum_{k=o}^{6} \sum_{\ell=o}^{k} F_{1c}^{(k,\ell)}(q)\ F_{2c}^{(k,\ell)}(q')\ g_{1c}^{(k,\ell)}(\varphi,\varphi')\ M_{k,\ell} \qquad (4\text{-}17)$$

then (4-16) becomes

$$\langle S_d(t) \rangle = -\frac{4B_1}{D^4} \sum_{j=1}^{4} \sum_{k=o}^{6} \sum_{\ell=o}^{k} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \int_{\alpha}^{1} \int_{\alpha}^{1} F_a^{(k,\ell)}(q,\varphi) \cdot$$

$$\cdot\ F_b^{(k,\ell)}(q',\varphi')\ M_{k,\ell}\ \langle L_j(\eta_1) L_j(\eta_2) \rangle\ e^{-i\omega_o(a_j\zeta_1+b_j\zeta_2)} dq dq' d\varphi d\varphi'$$

$$\langle s_c(t)\rangle = -\frac{B_1}{D^4}\sum_{j=1}^{4}\sum_{k=o}^{6}\sum_{\ell=o}^{k} a_j b_j \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{\alpha}^{1}\int_{\alpha}^{1} F_{1c}^{(k,\ell)}(q)\ F_{2c}^{(k,\ell)}(q')\ \cdot$$

$$\cdot\ g_{1c}^{(k,\ell)}\ (\varphi,\varphi')\ M_{k,\ell}\ \langle L_j(\eta_1)L_j(\eta_2)\rangle\ e^{-i\omega_o(a_j\zeta_1+b_j\zeta_2)}\, dq dq'\ d\varphi d\varphi' \tag{4-18}$$

Further, realizing that the φ and φ' integrations are around a circle, the limits could be $(-\pi,\pi)$, $(0,2\pi)$ or $(k-\pi, k+\pi)$ for any k. Therefore, let the limits on the φ' integration be $(\varphi-\pi,\ \varphi+\pi)$ and define $\psi = \varphi' - \varphi$. This substitution is very helpful in the integrations since $M_{k,\ell}$ is only a function of ψ. Then (4-18) becomes

$$\langle s_d(t)\rangle = -\frac{4B_1}{D^4}\sum_{j=1}^{4}\sum_{k=o}^{6}\sum_{\ell=o}^{k} a_j b_j \int_{-\pi}^{\pi}\int_{\alpha}^{1}\int_{\alpha}^{1}\int_{-\pi}^{\pi} F_a^{(k,\ell)}(q,\varphi)\ \cdot$$

$$\cdot\ F_b^{(k,\ell)}(q',\varphi,\psi)\ M_{k,\ell}\ \langle L_j(\eta_1)L_j(\eta_2)\rangle\ e^{-i\omega_o(a_j\zeta_1+b_j\zeta_2)}\, d\varphi dq\ dq' d\psi$$

$$\langle s_c(t)\rangle = -\frac{B_1}{D^4}\sum_{j=1}^{4}\sum_{k=o}^{6}\sum_{\ell=o}^{k} a_j b_j \int_{-\pi}^{\pi}\int_{\alpha}^{1}\int_{\alpha}^{1}\int_{-\pi}^{\pi} F_{1c}^{(k,\ell)}(q) F_{2c}^{(k,\ell)}(q')\ \cdot$$

$$\cdot\ g_{2c}^{(k,\ell)}(\varphi,\psi) M_{k,\ell}\langle L_j(\eta_1)L_j(\eta_2)\rangle\ e^{-i\omega_o(a_j\zeta_1+b_j\zeta_2)}\, d\varphi dq\ dq' d\psi \tag{4-19}$$

where the functional relations for $F_a^{(k,\ell)}\ (q,\varphi)$ $F_b^{(k,\ell)}\ (q,\varphi,\psi)$ $F_{1c}^{(k,\ell)}\ (q)$, $F_{2c}^{(k,\ell)}\ (q')$ and $g_{2c}^{(k,\ell)}\ (\varphi,\psi)$ are given in Appendix D. These terms were calculated using the following identities

$$Y_3(1+X_1) = X_3 \qquad Y_5(1+X_1) = X_5$$

$$Y_4(1+X_2) = X_4 \qquad Y_6(1+X_2) = X_6$$

$$X_o = 1 \qquad V_{rm}(q) = (1-q^2)V_{rc}(q) \qquad (4\text{-}20)$$

and the substitution of (3-44), (B-16) and (B-18) into the first expectation term of the integrands of (4-16).

4.2 Integration of Expected Powers

In order to use the expected values defined in (4-19) by the integrals, it is necessary to integrate them by some method. These equations are not very amenable to computer solution in their present form due to the highly oscillatory terms in the integrand, however, a reasonable approximate solution for a specific range can be obtained by the following methods. For the φ integration, in the $\langle s_d(t)\rangle$ integral only $F_a^{(k,\ell)}$ and $F_b^{(k,\ell)}$ and in $\langle s_c(t)\rangle$ only $g_{2c}^{(k,\ell)}$ are functions of φ and, in all cases, the φ variation is in simple trigonometric form. The φ integration of (4-19) yields the following results.

$$\langle s_d(t)\rangle = \frac{-2B_1\pi}{D^4} \sum_{j=1}^{4} \sum_{k=o}^{6} \sum_{\ell=o}^{k} \int_{-\pi}^{\pi} \int_{\alpha}^{1} \int_{\alpha}^{1} F_{1d}^{(k,\ell)}(q)\, F_{2d}^{(k,\ell)}(q') \cdot$$

$$\cdot\, M_{k,\ell} \langle L_j(\eta_1) L_j(\eta_2)\rangle\, e^{-i\omega_o(a_j\zeta_1 + b_j\zeta_2)}\, dq\, dq'\, d\psi - \frac{B_1\pi}{D^4} \sum_{j=1}^{4} \sum_{k=o}^{6} \cdot$$

$$\cdot \sum_{\ell=o}^{k} a_j b_j \int_{-\pi}^{\pi} \int_{\alpha}^{1} \int_{\alpha}^{1} F_{1r}^{(k,\ell)}(q)\, F_{2r}^{(k,\ell)}(q')\, g_{3r}^{(k,\ell)}(\psi)\, M_{k,\ell} \cdot$$

$$\cdot\, \langle L_j(\eta_1) L_j(\eta_2)\rangle\, e^{-i\omega_o(a_j\zeta_1 + b_j\zeta_2)}\, dq\, dq'\, d\psi$$

$$\langle s_c(t)\rangle = \frac{-B_1\pi}{D^4} \sum_{j=1}^{4} \sum_{k=o}^{6} \sum_{\ell=o}^{k} a_j b_j \int_{-\pi}^{\pi} \int_{\alpha}^{1} \int_{\alpha}^{1} F_{1c}^{(k,\ell)}(q) F_{2c}^{(k,\ell)}(q') \cdot$$

$$\cdot\ g_{3c}^{(k,\ell)}(\psi)\ M_{k,\ell}\langle L_j(\eta_1)L_j(\eta_2)\rangle\ e^{-i\omega_o(a_j\zeta_1+b_j\zeta_2)}dq\ dq'\ d\psi \tag{4-21}$$

The values of $F_{1d}^{(k,\ell)}(q)$, $F_{2d}^{(k,\ell)}(q')$ $F_{1r}^{(k,\ell)}(q)$, $F_{2r}^{(k,\ell)}(q')$, $F_{1c}^{(k,\ell)}(q)$, $F_{2c}^{(k,\ell)}(q')$, $g_{3c}^{(k,\ell)}(\psi)$, and $g_{3r}^{(k,\ell)}(\psi)$ are listed in Appendix D, where

$$V_{rs}(q) = -\ 2V_{rd}(q) + V_{rm}(q) \tag{4-22}$$

It is necessary to solve an integral of the form

$$I^{(k,\ell)} = \int_\alpha^1 \int_\alpha^1 F_1^{(k,\ell)}(q)\ F_2^{(k,\ell)}(q')\ M_{k,\ell}\ \langle L_j(\eta_1)L_j(\eta_2)\rangle\ \cdot$$

$$\cdot\ e^{-i\omega_o(a_j\zeta_1\ +\ b_j\zeta_2)}\ dq\ dq' \tag{4-23}$$

This integral is solved in Appendix D. From (D-16)

$$I^{(k,\ell)} \cong \Big[\ G_1^{(k,\ell)}(1,1,\psi)\ e^{-i\omega_o t_o(a_j+b_j)} - G_3^{(k,\ell)}(q_1,1,\psi)\ \cdot$$

$$\cdot\ e^{-i\omega_o b_j t_o} - G_4^{(k,\ell)}(1,q_1,\psi)\ e^{-i\omega_o a_j t_o} + G_2^{(k,\ell)}(q_2,q_2,\psi)\Big]U_a\ \cdot$$

$$\cdot\ + \Big[G_2^{(k,\ell)}(q_1,q_1,\psi) - G_2^{(k,\ell)}(q_1,q_2,\psi) - G_2^{(k,\ell)}(q_2,q_1,\psi)\ \cdot$$

$$\cdot\ + G_2^{(k,\ell)}(q_2,q_2,\psi)\Big]U_b + \Big[G_1^{(k,\ell)}(\alpha,\alpha,\psi)\ e^{-i\omega_o(t_o-t_1)(a_j+b_j)}\ \cdot$$

$$\cdot\ - G_3^{(k,\ell)}(q_2,\alpha,\psi)\ e^{-i\omega_o b_j(t_o-t_1)} - G_4^{(k,\ell)}(\alpha,q_2,\psi)\ \cdot$$

$$\cdot\ e^{-i\omega_o a_j(t_o-t_1)} + G_2^{(k,\ell)}(q_2,q_2,\psi)\ \Big]\ U_c \tag{4-24}$$

Then (4-21) becomes

$$\overline{\langle S_d(t)\rangle} = \frac{-2B_1\pi}{D^4}\sum_{j=1}^{4}\sum_{k=o}^{6}\sum_{\ell=o}^{k} a_j b_j \int_{-\pi}^{\pi} I_d^{(k,\ell)}\,d\psi - \frac{B_1\pi}{D^4}\cdot$$

$$\cdot\sum_{j=1}^{4}\sum_{k=o}^{6}\sum_{\ell=o}^{k}\int_{-\pi}^{\pi} g_{3r}^{(k,\ell)}(\psi)\, I_r^{(k,\ell)}\,d\psi$$

$$\overline{\langle S_c(t)\rangle} = \frac{-B_1\pi}{D^4}\sum_{j=1}^{4}\sum_{k=o}^{6}\sum_{\ell=o}^{k} a_j b_j \int_{-\pi}^{\pi} g_{3c}^{(k,\ell)}(\psi)\, I_c^{(k,\ell)}\,d\psi$$

(4-25)

where the subscript on $I^{(k,\ell)}$ identifies the $F_1^{(k,\ell)}(q)$ and $F_2^{(k,\ell)}(q')$ functions from Appendix D.

Since the time averaged (over one cycle) power is desired and $G^{(k,\ell)}$ terms are slowly varying compared to a cycle of the transmitted power, the time averaging may be done by just eliminating those terms of $I^{(k,\ell)}$ which have a high frequency term. Thus, for $I^{(k,\ell)}$ considering the terms separately. Terms containing $\exp[-i\omega_o t(a_j+b_j]$ will vanish for j = 1 and 4; and terms containing $\exp[-i\omega_o b_j t]$ will vanish for all j. Then

$$\overline{\langle S_d(t)\rangle} = \frac{-2B_1\pi}{D^4}\sum_{k=o}^{6}\sum_{\ell=o}^{k}\int_{-\pi}^{\pi}\Bigg\{\Bigg[\sum_{j=2}^{3} a_j b_j\, G_{1d}^{(k,\ell)}(1,1,\psi) + \cdot$$

$$\cdot\sum_{j=1}^{4} a_j b_j\, G_{2d}^{(k,\ell)}(q_1,q_1,\psi)\Bigg]U_a + \sum_{j=1}^{4} a_j b_j\Big[G_{2d}^{(k,\ell)}(q_1,q_1,\psi)\,\cdot$$

$$\cdot - G_{2d}^{(k,\ell)}(q_1,q_2,\psi) - G_{2d}^{(k,\ell)}(q_2,q_1,\psi) + G_{2d}^{(k,\ell)}(q_2,q_2,\psi)\Big]U_b\,\cdot$$

$$\cdot + \Bigg[\sum_{j=2}^{3} a_j b_j\, G_{1d}^{(k,\ell)}(\alpha,\alpha,\psi) + \sum_{j=1}^{4} a_j b_j\, G_{2d}^{(k,\ell)}(q_2,q_2,\psi)\Bigg]U_c\Bigg\}\,d\psi\ .$$

$$\cdot \frac{-B_1 \pi}{D^4} \sum_{k=o}^{6} \sum_{\ell=o}^{k} \int_{-\pi}^{\pi} \left\{ \left[\sum_{j=2}^{3} a_j b_j \, G_{1r}^{(k,\ell)}(1,1,\psi) + \sum_{j=1}^{4} a_j b_j \cdot \right.\right.$$

$$\left. \cdot \, G_{2d}^{(k,\ell)}(q_1,q_1,\psi) \right] U_a + \sum_{j=1}^{4} a_j b_j \left[G_{2r}^{(k,\ell)}(q_1,q_1,\psi) - G_{2r}^{(k,\ell)} \cdot \right.$$

$$\left. \cdot \, (q_1,q_2,\psi) - G_{2r}^{(k,\ell)}(q_1,q_2,\psi) + G_{2r}^{(k,\ell)}(q_2,q_2,\psi) \right] U_b + \cdot$$

$$\left. \cdot \left[\sum_{j=2}^{3} a_j b_j \quad G_{1r}^{(k,\ell)} \ (\alpha,\alpha,\psi) + \sum_{j=1}^{4} a_j b_j \, G_{2r}^{(k,\ell)}(q_2,q_2,\psi) \right] U_c \right\} \cdot$$

$$\cdot \quad g_{3r}^{(k,\ell)}(\psi) \; d\psi$$

$$\overline{\langle s_c(t) \rangle} = \frac{-B_1 \pi}{D^4} \sum_{k=o}^{6} \sum_{\ell=o}^{k} \int_{-\pi}^{\pi} g_{3c}^{(k,\ell)}(\psi) \left\{ \left[\sum_{j=2}^{3} a_j b_j \, G_{1c}^{(k,\ell)}(1,1,\psi) \cdot \right.\right.$$

$$\left. \cdot + \sum_{j-1}^{4} a_j b_j \, G_{2c}^{(k,\ell)}(q_1,q_1,\psi) \right] U_a + \sum_{j=1}^{4} a_j b_j \left[G_{2c}^{(k,\ell)}(q_1,q_1,\psi) \cdot \right.$$

$$\left. \cdot - G_{2c}^{(k,\ell)}(q_1,q_2,\psi) - G_{2c}^{(k,\ell)}(q_2,q_1,\psi) + G_{2c}^{(k,\ell)}(q_2,q_2,\psi) \right] U_b \cdot$$

$$\left. + \left[\sum_{j=2}^{3} a_j b_j \, G_{1c}^{(k,\ell)}(\alpha,\alpha,\psi) + \sum_{j=1}^{4} a_j b_j \, G_{2c}^{(k,\ell)}(q_2 q_2,\psi) \right] U_c \right\} d\psi \tag{4-26}$$

If $D_2 >> 1$ and the correlation distance is much smaller than the dimensions of the illuminated area defined by the unit steps then

$$e^{-Q_2} = e^{-Q_3} >> e^{-Q_1} = e^{-Q_4} \tag{4-27}$$

and

$$\left| G_2^{(k,\ell)}(q_2,q_1,\psi) + G_2^{(k,\ell)}(q_1,q_2,\psi) \right| << \left| G_2^{(k,\ell)}(q_1,q_1,\psi) \cdot \right.$$

$$\left. \cdot \quad G_2^{(k,\ell)}(q_2,q_2,\psi) \right| \tag{4-28}$$

The use of (4-27) and (4-28) further reduces (4-26) to

$$\overline{\langle S_d(t)\rangle} = \frac{-2B_1\pi}{D^4} \sum_{j=2}^{3} a_j b_j \sum_{k=o}^{6} \sum_{\ell=o}^{k} \int_{-\pi}^{\pi} \Big\{ \Big[G_{1d}^{(k,\ell)}(1,1,\psi) + \cdot$$

$$\cdot\ G_{2d}^{(k,\ell)}(q_1,q_1,\psi) \Big]\ U_a + \Big[G_{2d}^{(k,\ell)}(q_1,q_1,\psi)\ \cdot$$

$$\cdot + G_{2d}^{(k,\ell)}(q_2,q_2,\psi) \Big]\ U_b + \Big[G_{1d}^{(k,\ell)}(\alpha,\alpha,\psi) + G_{2d}^{(k,\ell)}\ \cdot$$

$$\cdot\ (q_2,q_2,\psi) \Big]\ U_c \Big\} d\psi - \frac{B_1\pi}{D^4} \sum_{j=2}^{3} a_j b_j \sum_{k=o}^{6} \sum_{\ell=o}^{k} \int_{-\pi}^{\pi} \Big\{ \Big[G_{1r}^{(k,\ell)}\ \cdot$$

$$\cdot\ (1,1,\psi) + G_{2r}^{(k,\ell)}(q_1,q_1,\psi) \Big]\ U_a + \Big[G_{2r}^{(k,\ell)}(q_1,q_1,\psi) + G_{2r}^{(k,\ell)}\ \cdot$$

$$\cdot\ (q_2,q_2,\psi) \Big]\ U_b + \Big[G_{1r}^{(k,\ell)}(\alpha,\alpha,\psi) + G_{2r}^{(k,\ell)}(q_2,q_2,\psi) \Big]\ U_c \Big\}\ \cdot$$

$$\cdot\ g_{3r}^{(k,\ell)}(\psi)\ d\psi$$

$$\overline{\langle S_c(t)\rangle} = \frac{-B_1\pi}{D^4} \sum_{j=2}^{3} a_j b_j \int_{-\pi}^{\pi} g_{3c}^{(k,\ell)}(\psi) \Big\{ \Big[G_{1c}^{(k,\ell)}(1,1,\psi) + G_{2c}^{(k,\ell)}\ \cdot$$

$$\cdot\ (q_1,q_1,\psi) \Big]\ U_a + \Big[G_{2c}^{(k,\ell)}(q_1,q_1,\psi) + G_{2c}^{(k,\ell)}(q_2,q_2,\psi) \Big]\ U_b\ \cdot$$

$$+ \Big[G_{1c}^{(k,\ell)}(\alpha,\alpha,\psi) + G_{2c}^{(k,\ell)}(q_1,q_1,\psi) \Big]\ U_c\ d\psi \qquad (4\text{-}29)$$

Since the $G^{(k,\ell)}$ terms in (4-29) contain large numbers of terms containing derivatives of products of several terms, some method of ordering the terms is necessary. Since $\overline{\langle S_d(t)\rangle}$ and $\overline{\langle S_c(t)\rangle}$ are real quantities and the $M_{k,\ell}$ terms are real for k=0, ℓ=0 and k> ℓ, $\ell \neq 0$, and imaginary for ℓ=0 [See (D-16)] then it would be expected that the imaginary terms would cancel out for k=0, ℓ=0, and $k \geq \ell$, $\ell \neq 0$ and the real terms would cancel out for ℓ=0. Inspection of (D-16) and noting the summation over j, this conjecture proves correct. The $G^{(k,\ell)}$ terms will be ordered by investigating the coefficients of each in terms of the factors $\beta \gg 1$ and $pg \gg 1$. The coefficient of the $M_{o,o}$ term is 1. The maximum coefficient of the $M_{k,o}$ terms is βg and of the $M_{k,\ell}$ terms ($k \geq \ell$, $\ell \neq 0$) is $\beta^2 g^2$. The second derivative of $M_{o,o}$ has a maximum coefficient of $(\beta pg)^2$ due to the derivative of the correlation function. Therefore in G_1 the second derivative of $M_{o,o}$, the first derivative of $M_{k,o}$ and the $M_{k,\ell}$ ($k \geq \ell$, $\ell \neq 0$) terms are of equivalent order in terms of the coefficient $g\beta/p$. If a matrix were constructed with the j^{th} column determined by the j^{th} partial terms of G_1, and the rows determined by the number of X's in $M_{k,\ell}$ then one would find that the diagonals of the matrix proceeding upward from the left are of the same order with the largest term being in the upper left hand corner and these diagonals are alternately zero and non-zero.

The use of this ordering technique, (E-20), and evaluation the first two non-zero terms yields

$$\sum_{k=o}^{6} \sum_{\ell=o}^{k} \int_{-\pi}^{\pi} G_{1d}^{(k,\ell)} (1,1,\psi)\, d\psi = \frac{2\pi V_{rs}^{2}(1)}{p^2} \left[1 - \frac{g\beta}{p}\right] \cdot$$

$$+ 4\pi \frac{g\beta}{p} V'_{rs}(1)\, V_{rs}(1) + O(g^2\beta^2/p^4) \qquad (4\text{-}30)$$

where $O(g^2\beta^2/p^4)$ means of the order of $g^2\beta^2/p^4$

This evaluation requires the use of tables of Appendix D, Appendix C, and differentiation of the product of the terms contained in these places. The terms containing $1/(1-q_1^2)$ cancel out and a limiting process must be done as indicated in Appendix E.

The integrations of $G_{1r}^{(k,\ell)}$ and $G_{2r}^{(k,\ell)}$ are much more complicated due to there being $(1-q_1^2)$ and cos $n\psi$ terms. Considering the $M_{o,o}$ term, an indication of the problem can be seen. These terms contain a $(1-q_1^2)^2$ cos 2ψ coefficient. The derivatives of the correlation function are the only way in which terms which will cancel this coefficient may be obtained. Reference to Appendix F, which contains the derivatives of ρ, shows that to obtain a term of the form $\cos 2\psi/(1-q_1^2)^2$ requires the product of two or more partial derivatives of ρ and the sum of the orders of the partial must be at least 4. Therefore, at least the fourth partial of $M_{o,o}$ must be taken. By the same reasoning at least the 3rd partial of $M_{k,o}$ and 2nd partial of $M_{k,\ell}$ $(\ell \neq 0)$ must be taken. Consequently, the determination of the nonvanishing terms of the integrand requires long tedious, but elementary, differentiation and algebraic manipulation. The integration of these manipulations yields, keeping only the largest term,

$$\sum_{k=o}^{6} \sum_{\ell=o}^{k} \int_{-\pi}^{\pi} G_{1r}^{(k,\ell)}(1,1,\psi)\, g_{3r}^{(k,\ell)}(\psi)\, d\psi = o\left(\frac{g^2\beta^2}{p^4}\right)$$

$$\sum_{k=o}^{6} \sum_{\ell=c}^{k} \int_{-\pi}^{\pi} G_{1c}^{(k,\ell)}(1,1,\psi)\, g_{3c}^{(k,\ell)}(\psi)\, d\psi \cong 124\pi \frac{g^2\beta^2}{p^2} V_{rc}^2(1) \qquad (4\text{-}31)$$

For the G_2 terms another ordering criteria can be added. From (E-27) it can be seen that (1-cos ψ) terms reduces the coefficient by at least $1/\beta D_2$ and $(1-\rho)$ terms reduces the coefficient by at least $1/D_2$. Thus the maximum coefficient that each derivative will have after integration can be obtained and is shown in Table 4-1. Then retaining only the first term

$$\sum_{k=o}^{6} \sum_{\ell=o}^{k} \int_{-\pi}^{\pi} G_{2d}^{(k,\ell)}(q_1,q_1,\psi) \cong \frac{g^2\beta^2 v_{rs}^2(q_1)}{p^4} \sqrt{\frac{2\pi}{\beta(pgq_1^2+2)}} \cdot \left[\frac{3q_1^{\,4}}{(1-q_1^2)^{5/2}}\right]$$

$$\sum_{k=o}^{6} \sum_{\ell=o}^{k} \int_{-\pi}^{\pi} G_{2r}^{(k,\ell)}(q_1,q_1,\psi)\; g_{3r}^{(k,\ell)}(\psi)\; d\psi \cong \frac{g^2\beta^2 v_{rc}^2(q_1)}{p^4} \cdot \sqrt{\frac{2\pi}{\beta(pgq_1^2+2)}} \left[\frac{3q_1^{\,4}}{(1-q_1^{\,2})^{1/2}}\right]$$

$$\sum_{k=o}^{6} \sum_{\ell=o}^{k} \int_{-\pi}^{\pi} G_{2c}^{(k,\ell)}(q_1,q_1,\psi)\; g_{3c}^{(k,\ell)}(\psi)\; d\psi \cong \frac{g^2\beta^2 v_{rc}^{\,2}(q_1)}{p^4} \cdot \sqrt{\frac{2\pi}{\beta(pgq_1^2+2)}} \left[\frac{3q_1^{\,4}}{(1=q_1^{\,2})^{1/2}}\right] \qquad (4\text{-}32)$$

The substitution of (4-30), (4-31), and (4-32) into (4-29) and the summation over j gives

Table 4-1

Maximum Coefficient of $\frac{1}{p^n} \frac{\partial^n M_{k,\ell}}{\partial q^i \partial q'^{(n-i)}}$

(k,ℓ) \ n	4	5	6	7	8
(0,0)	$g^2\beta^2/p^2$		$g^3\beta^3/p^3$		$g^4\beta^4/p^4$
(1,0) (2,0)		$g^3\beta^2/p^3$		$g^4\beta^3/p^4$	
(3,0) (4,0)		$g^3\beta^3/p^3$		$g^4\beta^4/p^4$	
(5,0) (6,0)		$g^3\beta^3/p^3$		$g^3\beta^3/p^4$	
(1,1) (2,1) (2,2)	$g^3\beta^2/p^3$		$g^4\beta^3/p^4$		$g^5\beta^4/p^5$
(3,1) (4,1) (3,2) (4,2)	$g^3\beta^2/p^3$		$g^4\beta^3/p^4$		$g^5\beta^4/p^5$
(5,1) (6,1) (5,2) (6,2)	$g^3\beta^3/p^3$		$g^4\beta^4/p^4$		$g^5\beta^5/p^5$
(3,3) (3,4) (4,4)	$g^3\beta^3/p^3$		$g^4\beta^4/p^4$		$g^5\beta^5/p^5$
(5,3) (5,4) (6,3) (6,4)	$g^4\beta^3/p^2$		$g^5\beta^4/p^3$		$g^6\beta^5/p^4$
(5,5) (6,5) (5,5)	$g^4\beta^4/p^2$		$g^5\beta^5/p^3$		$g^6\beta^6/p^4$

Note: $pg > 1$ $\beta < p$

$$\langle \overline{S_d(t)} \rangle \cong \frac{4B_1\pi}{p^2D^4}\left\{\left[2\pi V_{rs}^2(1)\ (1-\frac{g\beta}{p}) + \frac{4\pi g\beta}{p} V'_{rs}(1)V_{rs}(1) + \frac{g^2\beta^2}{p^2}\cdot\right.\right.$$

$$\cdot\sqrt{\frac{2\pi}{\beta}}\left(\frac{3q_1^4}{\sqrt{pgq_1^2+2}}\right)\left(\frac{V_{rs}^2(q_1)}{(1-q_1^2)^{5/2}} + \frac{V_{rc}^2(q_1)}{2(1-q_1^2)^{1/2}}\right)\Bigg]\cdot$$

$$\cdot U_a + \frac{g^2\beta^2}{p^2}\sqrt{\frac{2\pi}{\beta}}\left[\frac{3q_1^4}{\sqrt{pgq_1^2+2}}\left(\frac{V_{rc}^2(q_1)}{(1-q_1^2)^{5/2}} + \frac{V_{rc}^2\ (q_1)}{2(1-q_1^2)^{1/2}}\right) + \right.$$

$$\left. + \frac{3q_2^4}{\sqrt{pgq_2^2+2}}\left(\frac{V_{rs}^2(q_2)}{(1-q_2^2)^{5/2}} + \frac{V_{rc}^2(q_2)}{2(1-q_2^2)^{1/2}}\right)\right] U_b$$

$$+ \left[2\pi\ V_{rs}^2(\alpha)\ (1-\frac{g\beta}{p}) + \frac{4\pi g\beta}{p} V'_{rs}(\alpha)\ V_{rs}(\alpha)\right.$$

$$\left. + \frac{g^2\beta^2}{p^2}\sqrt{\frac{2\pi}{\beta}}\left(\frac{3q_2^4}{\sqrt{pgq_2^2+2}}\right)\left(\frac{V_{rs}^2(q_2)}{(1-q_2^2)^{5/2}} + \frac{V_{rc}^2(q_2)}{2(1-q_2^2)^{1/2}}\right)\right]U_c$$

$$\langle \overline{S_c(t)} \rangle \cong \frac{2B_1\pi g^2\beta^2}{D^4p^4}\left[124\pi\ V_{rc}^2(1) + V_{rc}^2(q_1)\right.$$

$$\left.\cdot\sqrt{\frac{2\pi}{\beta(1-q_1^2)}}\left(\frac{3q_1^4}{\sqrt{pgq_1^2+2}}\right)\right]U_a + \sqrt{\frac{2\pi}{\beta}}\left[\frac{3q_1^4\ V_{rc}^2\ (q_1)}{\sqrt{(1-q_1^2)(pgq_1^2+2)}} + \right.$$

$$+ \frac{3q_2^4 v_{rc}^2(q_2)}{\sqrt{(1-q_2^2)(pgq_2^2+2)}} \Bigg] U_b + \Bigg[124\pi \, v_{rc}^2(\alpha) + \sqrt{\frac{2\pi}{\beta(1-q_2^2)}} \; \cdot$$

$$\cdot \left(\frac{3q_2^4 \, v_{rc}^2 \, (q_2)}{\sqrt{pgq_2^2 + 2}} \right) \Bigg] U_c \tag{4-33}$$

It is necessary to evaluate the C_1^2 term included in B_1 in terms of normal radar parameters in order to obtain the power received by the radar. The matching of the power in the main lobe with that from a short dipole yields [Erteza, Doran, and Lenhert, 1965]

$$C_1^2 = \frac{P_T \, G_T \, c^3}{2\pi \, \epsilon_2 \omega_o^2} \tag{4-34}$$

where P_T = peak power radiated by the antenna

G_T is the gain of the transmitting antenna over an isotropic antenna

Then

$$B_1 = \frac{P_T \, G_T \, a^4 k_2^2}{128 \, \pi^3} \tag{4-35}$$

The received power, P_r, is given by

$$P_r = \frac{\langle \overline{S(t)} \rangle \, G_R \, \lambda^2}{4\pi} \tag{4-36}$$

where G_R = gain of receiving antenna over that of an isotropic antenna

λ = wavelength at frequency ω_o

Then, the substitution of (4-35) and (4-33) into (4-36) yields in terms of σ, δ, and a

$$P_{rd} = \frac{P_T G_T G_R \lambda^2 (\pi a^2)}{(4\pi)^3 D^4} \left\{ \left[\frac{V_{rs}^2(1)}{4} \left(1 - \frac{2\sigma^2}{\delta^2}\right) + \left(\frac{\sigma}{\delta}\right)^2 V'_{rs}(1) V_{rs}(1) \right.\right.$$

$$\left. + \left(\frac{\sigma}{\delta}\right)^4 \frac{3q_1^{\ 4}}{\sqrt{2\pi\beta(4k_2^{\ 2}\sigma^2 q_1^{\ 2} + 2)}} \left(\frac{V_{rs}^{\ 2}(q_1)}{(1-q_1^{\ 2})^{5/2}} + \frac{V_{rc}^{\ 2}(q_1)}{2(1-q_1^{\ 2})^{1/2}} \right) \right] U_a$$

$$+ \left(\frac{\sigma}{\delta}\right)^4 \frac{1}{\sqrt{2\pi\beta}} \left[\frac{3q_1^{\ 4}}{\sqrt{4k_2^{\ 2}\sigma^2 q_1^{\ 2} + 2}} \left(\frac{V_{rs}^{\ 2}(q_1)}{(1-q_1^{\ 2})^{5/2}} + \frac{V_{rc}^{\ 2}(q_1)}{2(1-q_1^{\ 2})^{1/2}} \right) \right.$$

$$\left. + \frac{3q_2^{\ 4}}{\sqrt{4k_2^{\ 2}\sigma^2 q_2^{2} + 2}} \left(\frac{V_{rs}^{\ 2}(q_2)}{(1-q_2^{\ 2})^{5/2}} + \frac{V_{rc}^{\ 2}(q_2)}{2(1-q_2^{\ 2})^{1/2}} \right) \right] .$$

$$\cdot U_b + \left[\frac{V_{rs}^{\ 2}(\alpha)}{4} \left(1 - \frac{2\sigma^2}{\delta^2}\right) + \left(\frac{\sigma}{\delta}\right)^2 V'_{rs}(\alpha)\, V_{rs}(\alpha) \right.$$

$$\left. \left. + \left(\frac{\sigma}{\delta}\right)^4 \frac{3q_2^{\ 4}}{\sqrt{4k_2^{\ 2}\sigma^2 q_2^{\ 2} + 2}} \left(\frac{V_{rs}^{\ 2}(q_2)}{(1-q_2^{\ 2})^{5/2}} + \frac{V_{rc}^{\ 2}(q_2)}{2(1-q_2^{\ 2})^{1/2}} \right) \right] U_c \right\}$$

$$P_{rc} = \frac{P_T G_T G_R \lambda^2 (\pi_a^{\ 2})}{(4\pi)^3 D^4} \left(\frac{\sigma}{\delta}\right)^4 \left\{ \left[32 V_{rc}^{\ 2} + \frac{3q_1^{\ 2}\, V_{rc}^{\ 2}(q_1)}{2\sqrt{2\pi\beta(1-q_1^{\ 2})(4k_2^{\ 2}\sigma^2 q_1^{\ 2} + 2)}} \right] U_a \right.$$

$$+ \frac{1}{2\sqrt{2\pi\beta}} \left[\frac{3q_1^4 \, v_{rc}^2(q_1)}{\sqrt{(1-q_1^2)(4k_2^2\sigma^2 q_1^2 + 2)}} + \frac{3q_2^4 v_{rc}^2(q_2)}{\sqrt{(1-q_2^2)(4k_2^2\sigma^2 q_2^2 + 2)}} \right] U_b$$

$$+ \left[32 \, v_{rc}^2(\alpha) + \frac{3q_2^2 \, v_{rc}^2(q_2)}{2\sqrt{2\pi\beta(1-q_2^2)(4k_2^2\sigma^2 q_1^2 + 2)}} \right] U_c \Bigg\} \qquad (4\text{-}37)$$

where P_{rd} = direct-polarized received power

P_{rc} = cross-polarized received power

4.3 Discussion of Integral Solution

The approximate solution obtained to the integral equations (3-46) for the direct-and cross-polarized received power is given in (4-37) and requires several additional assumptions over those given in Section 3.4. These are:

1. The deviation in surface heights can be represented by a separable qaussian random process, real and continuous in the mean over the surface of the sphere with zero mean, variance σ^2 and exponential normalized covariance function ρ.

2. The standard deviation of $H(\Theta,\varphi)$ is less than 100 meters and greater than a wavelength.

3. $g\beta < 1$; [i.e., $(\sigma/\delta)^2 < 1/4k_2 a$]

4. The correlation distance δ is much smaller than the dimensions of the illuminated area.

5. The conductivity has negligible effect on the reflection coefficients.

None of these additional assumptions may be relaxed without some increase in complexity of the resulting solution , which is already complicated. However, some of them may be relaxed without exceeding complication. The first assumption was necessary to be able to define the expected values of the random variables and consequently could only be relaxed if another suitable random description of the surface could be found. The stationarity of the surface (σ^2 not a function of Θ or φ) appears most realistic, but could be modified. The form of the correlation function could also be modified, but would require a function which possessed finite derivatives. Also, the requirement that $\sigma < 100$ meters could be relaxed, but only with a great deal of complication. This would require the integration of error functions of complex arguments which are available only in the form of infinite series or tabulated values. All of these changes would only modify the expected value functions.

The assumption that σ was greater than a wavelength was made to reduce the infinite series $S(s,v)$ to a single term and this single term can be numerically shown to be less than 5% in error for any value of s or v. This infinite series could be tabulated and its value used from a table without a major increase in complication.

The fourth assumption was again made to reduce the complexity of the resulting equations and could be relaxed by adding only the two G terms in the region U_b. This would not modify the results greatly, adding only hyperbolic sines and cosines, but would increase the already tedious calculations somewhat.

Assumption 3 is the most critical and is a sufficient condition for the convergence of the two series obtained. The first being the infinite series expansion of the reflection coefficients and the second being the infinite series obtained by the repeated integration by parts of the q-q' integrals. These series were investigated to determine if a more relaxed condition for convergence could be obtained, but none could be justified due to the extremely long and tedious calculations. When the new computer language FORMAC (Formula Manipulation Compiler) becomes generally available on accessible machines, it is recommended that the G functions be calculated by this means to determine if a more realistic convergence criterion can be obtained. FORMAC will differentiate and manipulate series of algebraic quantities without resorting to the requirement of using numerical values.

A possible method for reducing the complexity of the determination of the convergence criteria lies in an infinite series expansion of the reflection coefficients. This expansion appears to place the most serious limitation on the convergence of the series of integrals. Since the Fresnel reflection coefficients are slowly varying functions of angle, some transformation or expansion may exist which allows the determination of a more rapidly convergencing series for larger values of σ/δ. An investigation of the first few terms indicates the convergence criterion of the present series may be $k_2 \sigma^2/\delta < 1$.

CHAPTER 5

RESULTS, INTERPRETATIONS, AND CONCLUSIONS

5.1 Results and Interpretations

The integral expressions for the direct-and cross-polarized power are derived (3-46), for the case of a pulsed source. If the steady state values were desired the L_j terms would be replaced by unity. These integral expressions are valid under the assumptions discussed in Section 3.4 for an arbitrary rough sphere. These expressions show that if $H(\theta,\varphi)$ is not a function of φ, then the cross-polarized power is identically zero.

The approximate solutions are obtained for the expected values of the direct-and cross-polarized received powers from a normally distributed surface (in height from a mean sphere), (4-37), under the condition that the square of the ratio of the standard deviation of height to the correlation distance is less than one over the 8π times the radius of the sphere in wavelengths (i.e., $\sigma^2/\delta^2 < 1/4k_2a$). This solution shows different characteristics in the three regions defined by the step functions U_a, U_b, and U_c. These regions are illustrated in figure 5-1. The first or the nose region ($U_a \neq 0$) is defined by the conditions that the leading edge of the pulse in space has intercepted the sphere and the trailing edge has not yet reached the sphere. The second region or mid-region ($U_b \neq 0$) is defined by the conditions that the leading edge of the pulse in space has not yet reached the intersection of the sphere and beam edge and the trailing edge has intercepted the sphere. This region may be non-existent for a sufficiently long pulse or a sufficiently narrow beam angle. The third region or tail region ($U_c \neq 0$) is defined by the conditions that the leading edge has passed the intersection of the beam edge and the sphere and the trailing edge has not yet reached it.

The characteristics of the return in each of the above regions are discussed separately. The effect of these results on the estimation of the statistical and electromagnetic properties of the surface are discussed. The results are then compared to experimentally obtained data from the lunar surface.

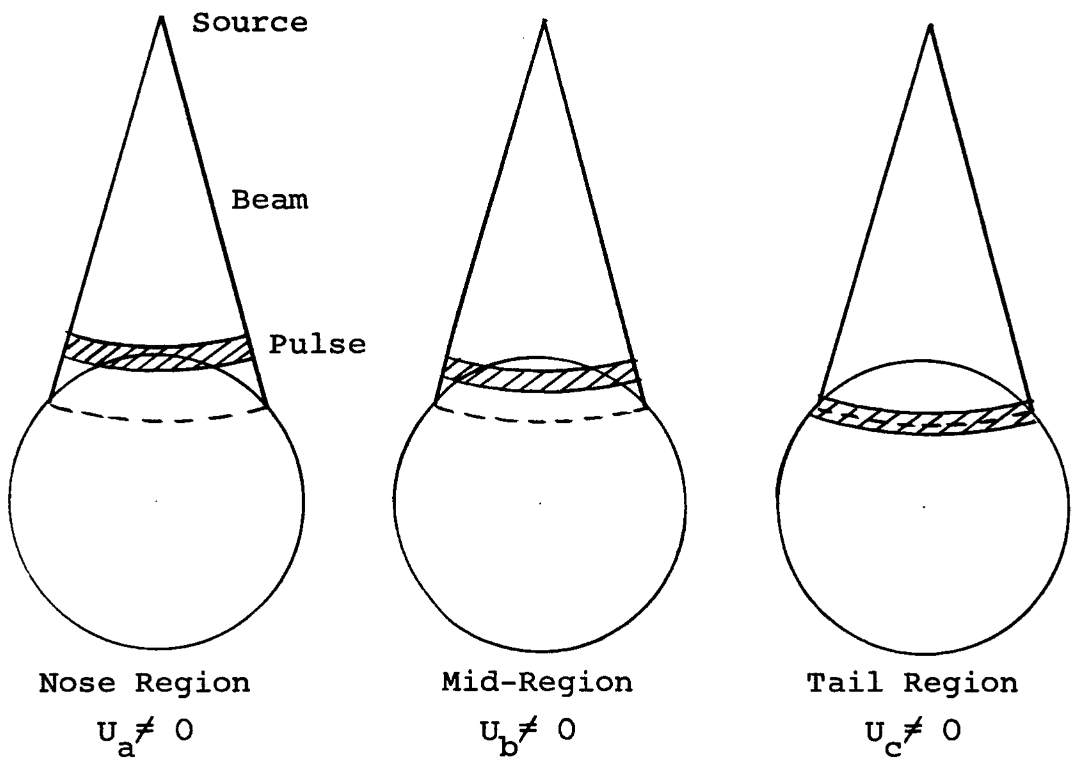

Figure 5-1

Diagram Showing Three Regions of Return

Figure 5-2 shows the theoretically derived expected powers for an index of refraction of n=1.5, a relative permeability μ_r=1.0, a=1.7 x 10^6m, ω_o=425 Mcs and T=2ms. Also shown in figure 5-2 is the experimentally obtained data from the moon [Mathis, 1963] for the same a, ω_o, and T.

In the nose region, the amplitude of the cross-polarized received power is at least 150 db below the amplitude of the direct-polarized power for the parameter values of figure 5-2. The amplitude of the direct-polarized return is determined by the reflection coefficient $V_{rs}^2(1)$ of the slowly varying (with location) surfaces for which the approximate solution is valid. Thus, the index of refraction may be determined by using only this value. The rapid rise of the pulse will not be observed in the receiver output. Consequently, the pulse width must be considerably longer than the rise time of the receiver in order for the pulse to reach its maximum value. The amplitude of the cross-polarized return is determined by both the reflection coefficient $V_{rc}^2(1)$ and the surface roughness properties (i.e., σ^4/δ^4). With present day radars this signal could not be received for the lunar situation. However, referring to figure 5-2, the cross-polarized return of the experimental data is only 12 db below the direct-polarized return. Evans [1961] states that in most experimental lunar data the cross-polarized return is from 12 to 15 db below the direct-polarized return. Since the amplitude of both the direct- and cross-polarized

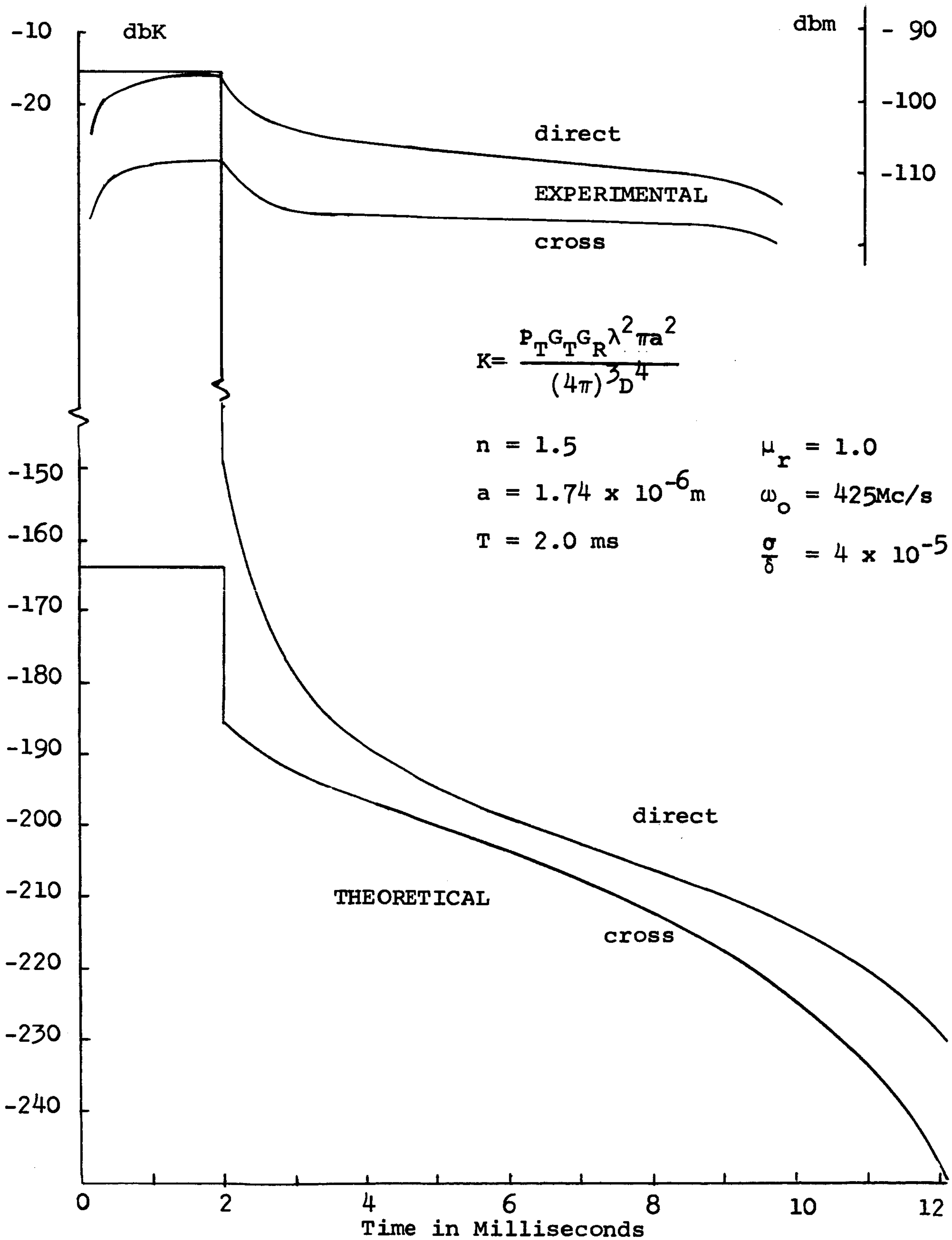

Figure 5-2

Comparison of Theoretical and Experimental Data returns are essentially the steady-state values for full illumination of the moon, a difference in the frequency description between the theoretical and experimental antennas cannot account for the discrepencies in the cross-polarized return. Therefore, the surface of the moon must have a standard deviation of heights to correlation distance (i.e., σ/δ) much greater than 10^{-4}.

In the mid-region, the amplitudes of the direct- and cross-polarized differ by only 2 to 7 db except in the early portion of the mid-region. The cross-polarized amplitude in this range is down by over 30 db from the nose region cross-polarized power. Thus this amplitude is even harder to detect. The amplitude of the signals in this range, particularly the direct-polarized power, are very dependent upon the frequency description of the source. Because of this dependence, it is extremely important that the frequency description of the source used in an experiment be known before attempting to explain the experimental data using the results of this work.

Also in this region, the description of the covariance function will affect the amplitude variation of the signals. Figure 5-2 shows a 30 to 40 db variation within this region.

When the index of refraction was varied from 1.5 to 6, the shape of the curve for either polarization in the mid-region did not change appreciably from that shown in figure 5-2, but the amplitude did. The shape of the curve is determined by the change in the reflection coefficient across the region and the assumed covariance function. The variation of the reflection coefficients across the region was about 15 db. If a solution had been obtained for a perfectly conducting surface, this variation would have been non-existant and the change in shape would have been due entirely to the covariance function. Fung [1965] made this assumption and matched statistics to the lunar return. However, from the above discussion, it is evident that this procedure will give erroneous statistics due to neglecting of the reflection coefficient variation.

In the tail region, both the direct- and cross-polarized powers show a large step, except in the case of full illumination of the surface. This step is due to the use of the ideal conical beam. If a more realistic source description were used the lower limits of integration would be 0 and the reflection coefficients would vanish.

Considering the approximate solution in all ranges, some of the terms seem to converge even for larger ratios of σ/δ (i.e., $1/4\ k_2 a < (\sigma/\delta)^2 < 1$). If this condition is assumed valid for the problem, then additional terms must be taken into account in the integration of the G_2 terms as indicated in Table 4-1. These additional terms would change the shape of the curve in the mid-region and possibly the separation of the direct- and cross-polarized powers in this region. With such a solution, just comparing the amplitudes of the steady-state solutions (i.e., the G_1 terms since the G_2 terms will not appear), it would be necessary to have a σ/δ of the order of 1/10 to be able to match the experimental lunar returns in the nose region.

Using just the first term in the nose region of the direct polarized return, a minimum value for the dielectric constant of the moon using Mathis' data is $\epsilon_{ave} = 1.82\ \epsilon_0$.

5.2 Conclusions and Suggestions for Future Investigation

The most important results of this investigation are as follows:

1. The integral expressions are obtained for the direct- and cross-polarized reflected power from a rough sphere.

2. If the deviation of surface heights from the average sphere is not a function of φ then the cross-polarized power is zero.

3. A rigorous field theory formulation of the pulsed return from a slightly irregular sphere. This result clearly shows the characteristic shape of the lunar backscatter.

4. If $(\sigma/\delta)^2 < 1/4\ k_2 a$, the value of the index of refraction can be obtained from using the direct-polarized received power in the nose region. An estimate of the statistical properties may be obtained by using the amplitude

of the cross-polarized received power in this region in conjunction with the amplitude of the direct polarized received power. The shape of the received power curves in the mid-regions indicates the validity of the assumed covariance function.

5. The $Re[\vec{E} \times \vec{H}^*]$ can be used for the time averaged power from a pulsed source in the mid-region only if $4k_2^2\sigma^2 > 3$. The $Re[\vec{E} \times \vec{H}^*]$ is not valid for the time averaged power in the nose region.

6. The amplitude of the cross-polarized return is a function of both electromagnetic and surface roughness properties. If the statistical model used in this investigation is realistic for explaining lunar return data, then $(\sigma/\delta)^2$ must be much greater than $1/4k_2a$ for the lunar surface, very likely of the order of 1/10.

The analysis of this problem, while interesting in itself, does not give sufficient information for estimating the electromagnetic and surface roughness properties of the lunar surface. Therefore, this analysis may be more useful as a first step toward solving other cases which could give a better estimate of the lunar surface. For example, the convergence criterion of the approximate solution to the integral expressions of received power is possibly more strict than necessary. This criterion could be investigated to determine if it could be relaxed by carrying more terms and using a FORMAC computer routine to do the tedious algebra as discussed in Section 4.3. Relaxation of the convergence criterion might be obtained through an investigation of other methods of obtaining the expected values of the reflection coefficients than that of the series expansion used in Appendix B.

Another area of recommended study is another approximate evaluation of the integral expressions for received power for larger values of σ/δ than those discussed in this work.

The direct- and cross-polarized received power in the mid-region is critically dependent upon the frequency discriptions of the antennas used to obtain experimental data be investigated before attempting to draw definitive conclusions based on this work.

It is recommended that future work be applied to the extension of this formulation to the solution of problems involving bistatic configuration of the source and receiver.

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APPENDIX A

COORDINATE SYSTEM TRANSFORMATIONS

The derivation of the rough sphere coordinate system ($\vec{a}_n$, $\vec{a}_r$, $\vec{a}_t$) will be made in this appendix. Also the transformations between the several appropriate coordinate systems will be derived (figure A-1). The first of these coordinate systems is the inertial coordinate system ($\vec{a}_x$, $\vec{a}_y$, $\vec{a}_z$). This coordinate system has its origin at the center of the sphere. The receiver is on the z-axis and the x-axis is in the direction of the linearly polarized $\vec{\Pi}$ (i.e., $\vec{a}_x = \vec{a}_\pi$). The standard spherical coordinate system ($\vec{a}_{ro}$, $\vec{a}_\theta$, $\vec{a}_\varphi$) will also be used. The last coordinate system ($\vec{a}_R$, $\vec{a}_q$, $\vec{a}_\varphi$) considered has its origin at the point of reflection as does the $\vec{a}_n$, $\vec{a}_r$, $\vec{a}_t$ coordinate system. The $\vec{a}_R$ unit vector is in the direction of the receiver and lies in the meridian plane, $\vec{a}_\varphi$ is the same as the spherical unit vector, and $\vec{a}_q = \vec{a}_\varphi \times \vec{a}_R$. These vectors and the associated angles are shown in figures A-2 through A-5. Using matrix notation, define the coordinate column vectors as follows

$$\vec{A}_1 = \begin{bmatrix} \vec{a}_n \\ \vec{a}_r \\ \vec{a}_t \end{bmatrix}; \quad \vec{A}_2 = \begin{bmatrix} \vec{a}_R \\ \vec{a}_q \\ \vec{a}_\varphi \end{bmatrix}; \quad \vec{A}_3 = \begin{bmatrix} \vec{a}_x \\ \vec{a}_y \\ \vec{a}_z \end{bmatrix}; \quad \vec{A}_4 = \begin{bmatrix} \vec{a}_{ro} \\ \vec{a}_\theta \\ \vec{a}_\varphi \end{bmatrix} \qquad \text{(A-1)}$$

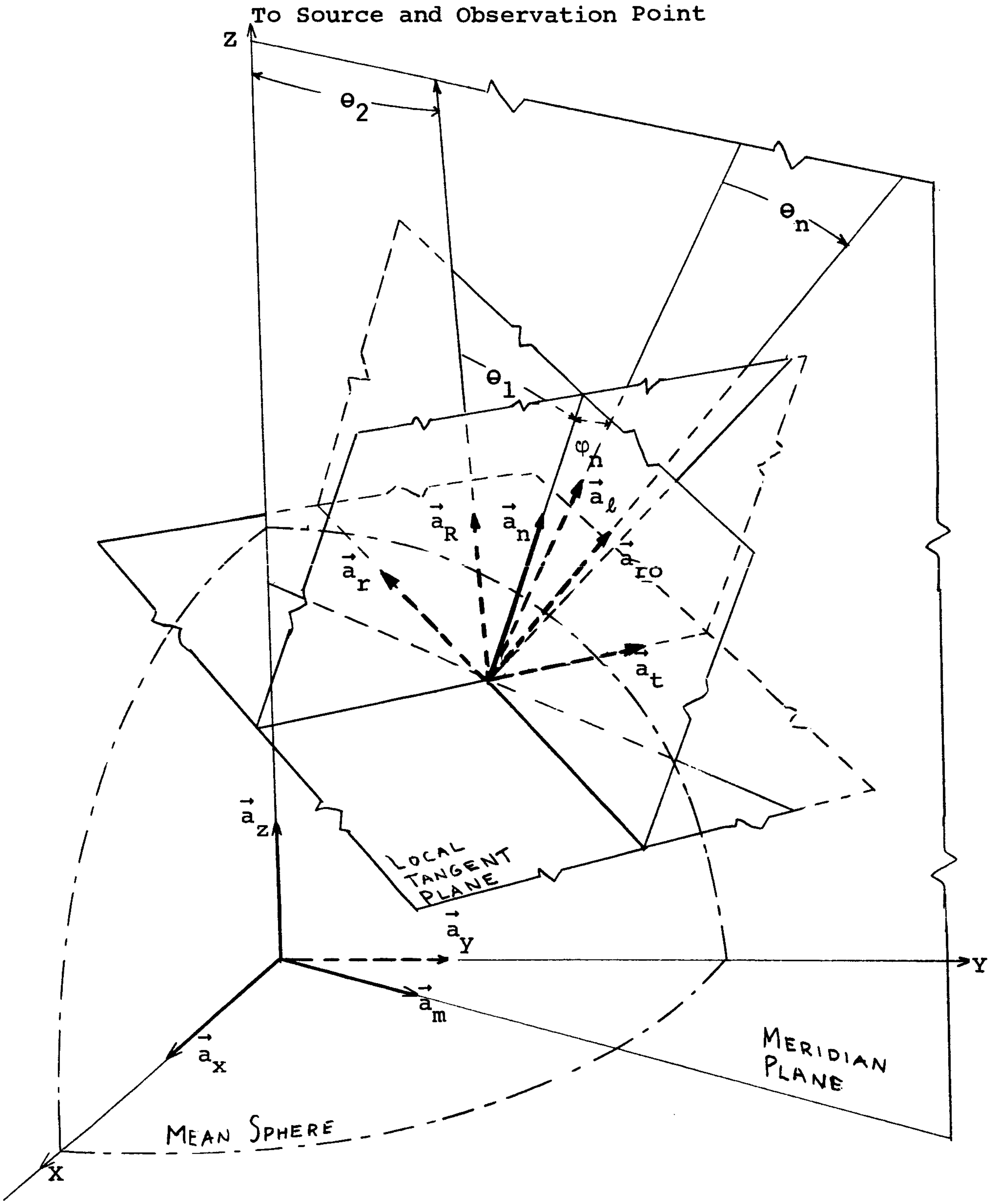

Figure A-1

Reflection Geometry for a Sphere

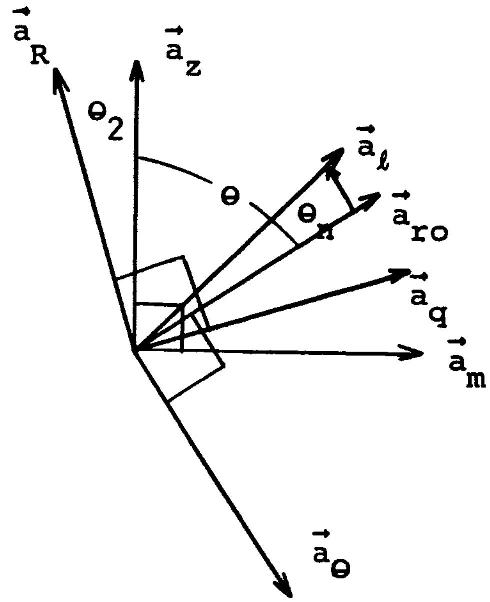

Figure A-2
Vectors in Meridian Plane

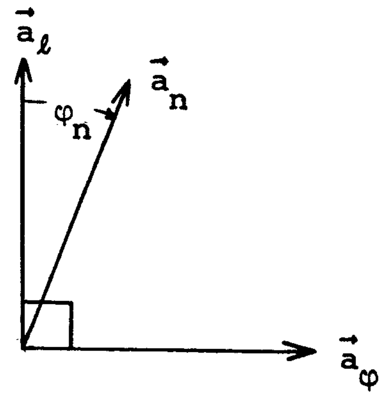

Figure A-3
Vectors in Plane Normal to Meridian Plane

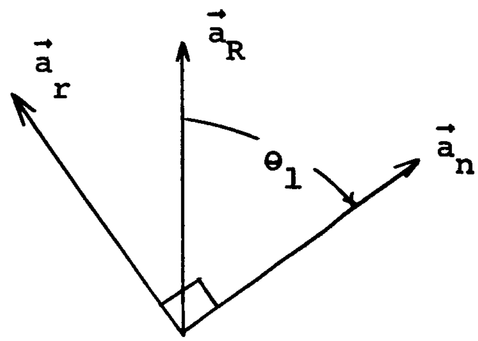

Figure A-4
Vectors in $\vec{a}_n$ - $\vec{a}_R$ Plane

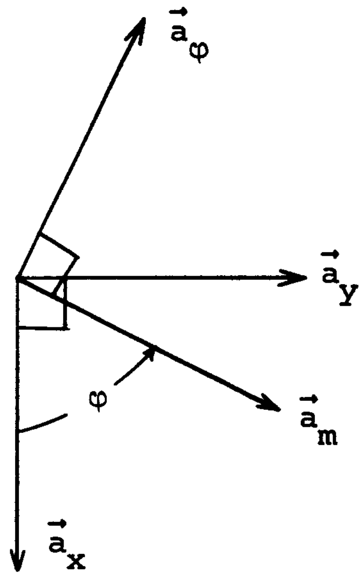

Figure A-5
Vectors in x - y Plane

The 3 x 3 coordinate transformation matrices A_{ij} are defined as

$$\vec{A}_i = A_{ij}\vec{A}_j \qquad (A-2)$$

with the elements being the dot product of the two appropriate vectors and the property

$$A_{ij} = A_{ji}^T \qquad (A-3)$$

where the T indicates the transpose.

From figure A-2, the following relationships can be obtained

$$\vec{a}_z = \cos\theta_2\vec{a}_R+\sin\theta_2\vec{a}_q = \cos\theta\vec{a}_{ro}-\sin\theta\vec{a}_\theta$$

$$\vec{a}_m = -\sin\theta_2\vec{a}_R+\cos\theta_2\vec{a}_q = \sin\theta\vec{a}_{ro}+\cos\theta\vec{a}_\theta$$

$$\begin{aligned}\vec{a}_\ell &= \cos\theta_n\vec{a}_{ro}-\sin\theta_n\vec{a}_\theta \\ &= \cos(\theta+\theta_2-\theta_n)\vec{a}_R+\sin(\theta+\theta_2-\theta_n)\vec{a}_q \\ &= \cos(\theta-\theta_n)\vec{a}_z+\sin(\theta-\theta_n)\vec{a}_m \qquad (A-4)\end{aligned}$$

which yields the dot products

$$(\vec{a}_R\cdot\vec{a}_z) = \cos\theta_2 \qquad (\vec{a}_\ell\cdot\vec{a}_{ro}) = \cos\theta_n$$

$$(\vec{a}_R\cdot\vec{a}_\ell) = \cos(\theta+\theta_2-\theta_n) \qquad (\vec{a}_\ell\cdot\vec{a}_q) = \sin(\theta+\theta_2-\theta_n)$$

$$(\vec{a}_R\cdot\vec{a}_{ro}) = \cos(\theta+\theta_2) \qquad (\vec{a}_\ell\cdot\vec{a}_m) = \sin(\theta-\theta_n)$$

$$(\vec{a}_R\cdot\vec{a}_m) = -\sin\theta_2 \qquad (\vec{a}_\ell\cdot\vec{a}_\theta) = -\sin\theta_n$$

$$(\vec{a}_R \cdot \vec{a}_\theta) = -\sin(\theta+\theta_2) \qquad (\vec{a}_{ro} \cdot \vec{a}_q) = \sin(\theta+\theta_2)$$

$$(\vec{a}_z \cdot \vec{a}_\ell) = \cos(\theta-\theta_n) \qquad (\vec{a}_{ro} \cdot \vec{a}_m) = \sin\theta$$

$$(\vec{a}_z \cdot \vec{a}_{ro}) = \cos\theta \qquad (\vec{a}_q \cdot \vec{a}_m) = \cos\theta_2$$

$$(\vec{a}_z \cdot \vec{a}_q) = \sin\theta_2 \qquad (\vec{a}_q \cdot \vec{a}_\theta) = \cos(\theta+\theta_2)$$

$$(\vec{a}_z \cdot \vec{a}_\theta) = -\sin\theta \qquad (\vec{a}_m \cdot \vec{a}_\theta) = \cos\theta \tag{A-5}$$

Figures A-3 through A-5 yield the following dot products

$$(\vec{a}_\ell \cdot \vec{a}_n) = \cos\varphi_n \qquad (\vec{a}_x \cdot \vec{a}_m) = \cos\varphi$$

$$(\vec{a}_\varphi \cdot \vec{a}_n) = \sin\varphi_n \qquad (\vec{a}_y \cdot \vec{a}_m) = \sin\varphi$$

$$(\vec{a}_R \cdot \vec{a}_n) = \cos\theta_1 \qquad (\vec{a}_x \cdot \vec{a}_\varphi) = -\sin\varphi$$

$$(\vec{a}_R \cdot \vec{a}_r) = \sin\theta_1 \qquad (\vec{a}_y \cdot \vec{a}_\varphi) = \cos\varphi \tag{A-6}$$

By using

$$\vec{a}_\varphi = -\sin\varphi\vec{a}_x+\cos\varphi\vec{a}_y$$

$$\vec{a}_m = \cos\varphi\vec{a}_x+\sin\varphi\vec{a}_y$$

$$\vec{a}_n = \cos\varphi_n\vec{a}_\ell+\sin\varphi_n\vec{a}_\varphi \tag{A-7}$$

the following dot products can be obtained

$$(\vec{a}_R \cdot \vec{a}_x) = -\sin\theta_2\cos\varphi \qquad (\vec{a}_\theta \cdot \vec{a}_x) = \cos\theta\cos\varphi$$

$$(\vec{a}_R \cdot \vec{a}_y) = -\sin\theta_2\sin\varphi \qquad (\vec{a}_\theta \cdot \vec{a}_y) = \cos\theta\sin\varphi$$

$$(\vec{a}_q \cdot \vec{a}_x) = \cos\theta_2\cos\varphi \qquad (\vec{a}_{ro} \cdot \vec{a}_x) = \sin\theta\cos\varphi$$

$$(\vec{a}_q \cdot \vec{a}_y) = \cos\theta_2 \sin\varphi \qquad (\vec{a}_{ro} \cdot \vec{a}_y) = \sin\theta\sin\varphi$$

$$(\vec{a}_n \cdot \vec{a}_z) = \cos(\theta-\theta_n)\cos\varphi_n$$

$$(\vec{a}_n \cdot \vec{a}_x) = \cos\varphi\cos\varphi_n\sin(\theta-\theta_n)-\sin\varphi\sin\varphi_n$$

$$(\vec{a}_n \cdot \vec{a}_y) = \sin\varphi\cos\varphi_n\sin(\theta-\theta_n)+\cos\varphi\sin\varphi_n$$

$$(\vec{a}_n \cdot \vec{a}_{ro}) = \cos\varphi_n\cos\theta_n$$

$$(\vec{a}_n \cdot \vec{a}_\theta) = -\cos\varphi_n\sin\theta_n$$

$$(\vec{a}_n \cdot \vec{a}_q) = \cos\varphi_n\sin(\theta+\theta_2-\theta_n) \qquad \text{(A-8)}$$

The relationship

$$\vec{a}_r = \frac{\vec{a}_R}{\sin\theta_1} - \frac{\cos\theta_1\vec{a}_n}{\sin\theta_1} \qquad \text{(A-9)}$$

yields

$$(\vec{a}_r \cdot \vec{a}_x) = [-\sin\theta_2\cos\varphi-\cos\varphi\cos\theta_1\cos\varphi_n\sin(\theta-\theta_n)$$
$$+\sin\varphi\cos\theta_1\sin\varphi_n]/\sin\theta_1$$

$$(\vec{a}_r \cdot \vec{a}_y) = [-\sin\theta_2\sin\varphi-\sin\varphi\cos\theta_1\cos\varphi_n\sin(\theta-\theta_n)$$
$$-\cos\varphi\cos\theta_1\sin\varphi_n]/\sin\theta_1$$

$$(\vec{a}_r \cdot \vec{a}_z) = [\cos\theta_2-\cos\theta_1\cos\varphi_n\cos(\theta-\theta_n)]/\sin\theta_1$$

$$(\vec{a}_r \cdot \vec{a}_{ro}) = [\cos(\theta+\theta_2)-\cos\theta_n\cos\varphi_n\cos\theta_1]/\sin\theta_1$$

$$(\vec{a}_r \cdot \vec{a}_\theta) = [-\sin(\theta+\theta_2)+\sin\theta_n\cos\varphi_n\cos\theta_1]/\sin\theta_1$$

$$(\vec{a}_r \cdot \vec{a}_\varphi) = [-\cos\theta_1 \sin\varphi_n]/\sin\theta_1$$

$$(\vec{a}_r \cdot \vec{a}_q) = [-\cos\theta_1 \cos\varphi_n \sin(\theta+\theta_2-\theta_n)]/\sin\theta_1 \qquad \text{(A-10)}$$

Finally using $\vec{a}_t = \vec{a}_r \times \vec{a}_n$ gives

$$(\vec{a}_t \cdot \vec{a}_x) = [-\cos\theta_2 \cos\varphi \sin\varphi_n$$

$$-\sin\varphi \cos\varphi_n \sin(\theta+\theta_2-\theta_n)]/\sin\theta_1$$

$$(\vec{a}_t \cdot \vec{a}_y) = [-\sin\varphi \cos\theta_2 \sin\varphi_n$$

$$+\cos\varphi \cos\varphi_n \sin(\theta+\theta_2-\theta_n)]/\sin\theta_1$$

$$(\vec{a}_t \cdot \vec{a}_z) = [-\sin\varphi_n \sin\theta_2]/\sin\theta_1$$

$$(\vec{a}_t \cdot \vec{a}_{ro}) = [-\sin(\theta+\theta_2)\sin\varphi_n]/\sin\theta_1$$

$$(\vec{a}_t \cdot \vec{a}_\theta) = [-\cos(\theta+\theta_2)\sin\varphi_n]/\sin\theta_1$$

$$(\vec{a}_t \cdot \vec{a}_\varphi) = [\cos\varphi_n \sin(\theta+\theta_2-\theta_n)]/\sin\theta_1$$

$$(\vec{a}_t \cdot \vec{a}_q) = -\sin\varphi_n/\sin\theta_1 \qquad \text{(A-11)}$$

One additional useful relationship is obtained.

$$\cos\theta_1 = \cos\varphi_n \cos(\theta+\theta_2-\theta_n) \qquad \text{(A-12)}$$

The dot products obtained above allow the determination of all of the transformation matrices A_{ij}. The pertinent ones are given below.

$$A_{12} = \frac{1}{\sin\theta_1}\begin{bmatrix} \cos\theta_1\sin\theta_1 & \cos\varphi_n\sin(\theta+\theta_2-\theta_n)\cdot\sin\theta_1 & \sin\varphi_n\sin\theta_1 \\ \sin^2\theta_1 & -\cos\theta_1\cos\varphi_n\cdot\sin(\theta+\theta_2-\theta_n) & -\cos\theta_1\sin\varphi_n \\ 0 & -\sin\varphi_n & \sin(\theta+\theta_2-\theta_n)\cdot\cos\varphi_n \end{bmatrix} \quad \text{(A-13)}$$

$$A_{23} = \begin{bmatrix} -\sin\theta_2\cos\varphi & -\sin\theta_2\sin\varphi & \cos\theta_2 \\ \cos\theta_2\cos\varphi & \cos\theta_2\sin\varphi & \sin\theta_2 \\ -\sin\varphi & \cos\varphi & 0 \end{bmatrix} \quad \text{(A-14)}$$

$$A_{34} = \begin{bmatrix} \sin\theta\cos\varphi & \cos\theta\cos\varphi & -\sin\varphi \\ \sin\theta\sin\varphi & \cos\theta\sin\varphi & \cos\varphi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \quad \text{(A-15)}$$

From these three transformation matrices any other transformation matrix can be obtained.

Now to relate θ_n and φ_n to the Y's defined in (3-21), the dot products defined above are used. Then

$$\cos\varphi_n\cos\theta_n = \frac{1}{J}$$

$$\cos\varphi_n\sin\theta_n = \frac{Y_3}{J}$$

$$\sin\varphi_n = \frac{-Y_5}{J} \quad \text{(A-16)}$$

From (A-16) it is found that

$$\cos\varphi_n = \frac{\pm\sqrt{1+Y_3^2}}{J} \qquad \sin\theta_n = \frac{Y_3}{\pm\sqrt{1+Y_3^2}} \qquad \cos\theta_n = \frac{1}{\pm\sqrt{1+Y_3^2}} \tag{A-17}$$

where all three must have the same sign. To determine the sign, consider the $\cos\varphi_n$ term, with φ_n restricted to the interval $[-\pi/2, \pi/2]$. Then $\cos\varphi_n$ must be positive, as must all other terms. With the use of the trigonometric identities for sum and difference angles, the following summary is obtained

$$\cos\theta_n = \frac{1}{\sqrt{1+Y_3^2}} \qquad \cos\theta_n' = \frac{1}{\sqrt{1+Y_4^2}}$$

$$\sin\theta_n = \frac{Y_3}{\sqrt{1+Y_3^2}} \qquad \sin\theta_n' = \frac{Y_4}{\sqrt{1+Y_4^2}}$$

$$\cos\varphi_n = \frac{\sqrt{1+Y_3^2}}{J} \qquad \cos\varphi_n' = \frac{\sqrt{1+Y_4^2}}{J'}$$

$$\sin\varphi_n = \frac{-Y_5}{J} \qquad \sin\varphi_n' = \frac{-Y_6}{J'}$$

$$\sin(\theta-\theta_n) = \frac{\sin\theta - Y_3\cos\theta}{\sqrt{1+Y_3^2}} \qquad \sin(\theta'-\theta_n') = \frac{\sin\theta' - Y_4\cos\theta'}{\sqrt{1+Y_4^2}}$$

$$\cos\theta_1 = \cos\varphi_n \cos(\theta-\theta_n) \qquad \cos\theta_1' = \cos\varphi_n' \cos(\theta'-\theta_n')$$

$$= \frac{\cos\theta + Y_3 \sin\theta}{J} \qquad = \frac{\cos\theta' + Y_4 \sin\theta'}{J'}$$

$$J = \sqrt{1+Y_3^2+Y_5^2} \qquad J' = \sqrt{1+Y_4^2+Y_6^2} \tag{A-18}$$

APPENDIX B

REFLECTION COEFFICIENTS

The equations for $\vec{E}_1$ and $\vec{H}_1$ as given in (3-16) and (3-18) show apparent poles at $\theta_1 = 0$. This appendix will investigate these poles, show that they are false poles and develop new expressions for $\vec{E}_1$ and $\vec{H}_1$. Also included in this appendix will be an expansion of newly derived reflection coefficients in a power series of the height variation and its derivatives. Consider first the reflection coefficient of E_{1y} (denoted by $V_2(\theta_1)$).

$$V_2(\theta_1) = \frac{V_{PP}(\theta_1)\cos 2\theta_1 - V_{NN}(\theta_1)}{\sin^2\theta_1} \tag{B-1}$$

Substituting (2-11) for $V_{PP}(\theta_1)$ and $V_{NN}(\theta_1)$ yields

$$V_2(\theta_1) = \frac{-1}{\sin^2\theta_1}\left[\frac{n^2\cos\theta_1 - \mu_r\sqrt{n^2-\sin^2\theta_1}}{n^2\cos\theta_1 + \mu_r\sqrt{n^2-\sin^2\theta_1}}(2\cos^2\theta_1 - 1) + \frac{\mu_r\cos\theta_1 - \sqrt{n^2-\sin^2\theta_1}}{\mu_r\cos\theta_1 + \sqrt{n^2-\sin^2\theta_1}}\right] \tag{B-2}$$

which on putting over a common denominator and simplifying gives

$$V_2(\theta_1) = \frac{2\cos\theta_1\left[\mu_r(n^2-1)\cos\theta_1+(n^2-\mu_r^2)\sqrt{n^2-\sin^2\theta_1}\right]}{\left[n^2\cos\theta_1+\mu_r\sqrt{n^2-\sin^2\theta_1}\right]\left[\mu_r\cos\theta_1+\sqrt{n^2-\sin^2\theta_1}\right]} \tag{B-3}$$

Examination of (B-3) shows that no poles exist for θ_1 real and $|n^2| > 1$, therefore E_{1y} and H_{1x} are regular at $\theta_1 = 0$.

Next consider E_{1x} which from (3-16) is

$$E_{1x} = \frac{-1}{\sin^2\theta_1}[V_{PP}(\theta_1)\cos 2\theta_1(\cos\varphi_n\sin(\theta-\theta_n)\cos\varphi-\sin\varphi_n\sin\varphi)^2$$

$$+V_{NN}(\sin\varphi\cos\varphi_n\sin(\theta-\theta_n)+\sin\varphi_n\cos\varphi)^2] \tag{B-4}$$

Upon squaring, converting $\cos\varphi$ to $\sin\varphi$ and $\sin\varphi$ to $\cos\varphi$ in the V_{PP} coefficient, (B-4) becomes

$$E_{1x} = \frac{-1}{\sin^2\theta_1}\{V_{PP}(\theta_1)\cos 2\theta_1[\cos^2\varphi_n\sin^2(\theta-\theta_n)+\sin^2\varphi_n]$$

$$-[V_{PP}(\theta_1)\cos 2\theta_1-V_{NN}(\theta_1)][\sin\varphi\cos\varphi_n\sin(\theta-\theta_n)$$

$$+\sin\varphi_n\cos\varphi]^2\} \tag{B-5}$$

Since from (A-12)

$$\cos\theta_1 = \cos\varphi_n\cos(\theta-\theta_n) \tag{B-6}$$

then

$$\sin^2\theta_1 = \sin^2\varphi_n+\cos^2\varphi_n\sin^2(\theta-\theta_n) \tag{B-7}$$

Substituting (B-7) and (B-1) into (B-5) yields

$$E_{1x} = -V_{PP}(\theta_1)\cos 2\theta_1$$

$$+V_2(\theta_1)[\sin\varphi\cos\varphi_n\sin(\theta-\theta_n)+\sin\varphi_n\cos\varphi]^2 \qquad \text{(B-8)}$$

In order to express the terms containing θ_1 variation in terms of $H(\theta, \varphi)$ and its various partial derivatives several new quantities are defined.

$$V_{rc}(\theta_1) = V_2(\theta_1)\cos\theta_1$$

$$V_{rd}(\theta_1) = V_{PP}(\theta_1)\cos 2\theta_1\cos\theta_1 \qquad \text{(B-9)}$$

using the definition of (3-1) and

$$\frac{a}{r_o} = \frac{1}{1+\frac{H(\theta,\varphi)}{a}} \cong 1- \frac{H(\theta,\varphi)}{a} \qquad \text{(B-10)}$$

then

$$Y_1 = X_1 \quad Y_3 = \frac{a}{r_o}X_3 \cong X_3-X_1X_3 \quad Y_5 = \frac{a}{r_o}X_5 \cong X_5-X_1X_5$$

$$Y_2 = X_2 \quad Y_4 = \frac{a}{r_o}X_4 \cong X_4-X_2X_4 \quad Y_6 = \frac{a}{r_o}X_6 \cong X_6-X_2X_6 \qquad \text{(B-11)}$$

Assuming that the reflection coefficients can be expressed as a power series in $\cos\theta_1$ then

$$V_{rc}(\theta_1) = \sum_{f=0}^{\infty} b_f\cos^f\theta_1 \qquad \text{(B-12)}$$

where

$$\cos\theta_1 = \frac{q+Y_3\sqrt{1-q^2}}{J}$$

$$q = \cos\theta$$

Using the binomial theorem on (B-12) yields

$$\frac{V_{rc}(\theta_1)}{J} = \sum_{f=0}^{\infty} \sum_{g=0}^{f} \frac{b_f\, C_g^f\, Y_3^g\, q^{(f-g)}(1-q^2)^{g/2}}{J^{f+1}} \qquad \text{(B-13)}$$

where C_g^f is the binomial coefficient.

Under the assumption that $Y_3^2 + Y_5^2 < 1$, then the denominator of (B-13) can be expanded by use of the binomial theorem as

$$\frac{1}{J^{f+1}} = \sum_{h=0}^{\infty} \sum_{i=0}^{h} (-1)^h\, C_h^{(f+1)/2}\, C_i^h\, Y_3^{2(h-i)}\, Y_5^{2i} \qquad \text{(B-14)}$$

where $C_h^{(f+1)/2}$ is expressed in terms of the gamma function.
Then (B-13) becomes upon the substitution of (B-14)

$$\frac{V_{rc}(\theta_1)}{J} = \sum_{f=0}^{\infty} \sum_{g=0}^{f} \sum_{h=0}^{\infty} \sum_{i=0}^{h} (-1)^h\, b_f\, C_g^f\, C_h^{(f+1)/2} \cdot$$

$$\cdot\, C_i^h\, Y_3^{g+2h-2i}\, Y_5^{2i}\, q^{(f-q)}(1-q^2)^{g/2} \qquad \text{(B-15)}$$

From the series (B-12)

$$V_{rc}(q) = \sum_{f=0}^{\infty} b_f q^f \qquad V'_{rc}(q) = \sum_{f=1}^{\infty} f b_f q^{f-1}$$

$$V''_{rc}(q) = \sum_{f=2}^{\infty} f(f-1) b_f q^{f-2} \qquad V_{rc}^{(n)}\, q = \sum_{f=n} \frac{f!}{(f-n)!} b_f q^{f-n} \qquad \text{(B-16)}$$

Expressing (B-15) in terms of increasing powers of Y gives

$$\frac{V_{rc}(\theta_1)}{J} = V_{rc}(q) + Y_3(1-q^2)^{1/2} V'_{rc}(q) + \frac{Y_3^2}{2}\Big[(1-q^2)V''_{rc}(q)$$

$$-qV'_{rc}(q) - V_{rc}(q)\Big] - \frac{Y_5^2}{2}\Big[qV'_{rc}(q) + V_{rc}(q)\Big] + \ldots \qquad \text{(B-17)}$$

By the same method as (B-15) was obtained

$$J\, V_{rd}(\theta_1) = \sum_{f=0}^{\infty} \sum_{g=0}^{f} \sum_{h=0}^{\infty} \sum_{i=0}^{h} (-1)^h\, d_f\, C_g^f\, C_h^{(f-1)/2}\, C_i^h \,\cdot$$

$$\cdot\; Y_3^{g+2h-2i}\, Y_5^{2i}\, q^{(f-g)}(1-q^2)^{g/2} \qquad \text{(B-18)}$$

Expressing (B-18) in terms of increasing powers of Y gives

$$J\, V_{rd}(\theta_1) = V_{rd}(q) + Y_3(1-q^2)^{1/2} V'_{rd}(q) + \frac{Y_3^2}{2}\Big[(1-q^2)V''_{rd}(q)$$

$$-qV'_{rd}(q) + V_{rd}(q)\Big] - \frac{Y_5^2}{2}\Big[qV'_{rd}(q) - V_{rd}(q)\Big] + \ldots \qquad \text{(B-19)}$$

APPENDIX C

CALCULATION OF ENSEMBLE AVERAGES

The ensemble averages are calculated by an orthogonal expansion of the original random variable using the Karhunen-Loève representation theorem [Hoffman, 1955]. Under the assumptions of Chapter 4 on $H(\theta,\varphi)$, there exists the bilinear representation

$$\sigma^2\rho(\theta',\theta,\varphi',\varphi) = \sum_{m,n=1}^{\infty} \lambda_{mn}^{-1} \Phi_{mn} (\theta',\varphi') \qquad \text{(C-1)}$$

where the $\Phi_{mn}(\theta,\varphi)$ and λ_{mn} are the eigenfunctions and eigenvalues of the integral equation

$$\Phi(\theta,\varphi) = \lambda \iint_S \sigma^2\rho(\theta',\theta,\varphi',\varphi)\ \Phi(\theta',\varphi')d\theta'\ d\varphi' \qquad \text{(C-2)}$$

According to the Karhunen-Loève representation theorem, there exists for every (θ,φ) element of the surface, the expansion

$$H(\theta,\varphi) = \text{l.i.m.} \sum_{m,n} \lambda_{mn}^{-1/2} \Phi_{mn}(\theta,\varphi)h_{mn} \qquad \text{(C-3)}$$

in terms of the orthogonal process $\{h_{mn}\}$ with

$$\langle h_{mn}\rangle = 0 \quad \langle h_{mn}h_{pg}\rangle = \delta_{mp}\ \delta_{pg} \qquad \text{(C-4)}$$

where the notation $\langle\ \rangle$ means ensemble average and δ_{mp} is the standard Dirac delta function. The use of (C-4) and (C-3) gives the variance as

$$\sigma^2 = \langle H^2(\theta,\varphi)\rangle = \sum_{m,n} \lambda^{-1}_{m,n} \Phi_{mn}^{\ 2}(\theta,\varphi) \tag{C-5}$$

The ensemble averages needed in Chapter 4 will now be calculated. First let the characteristic function of $f_{mn}(\theta,\varphi)$ with respect to the random variable h_{mn} be denoted by

$$\chi\left[f_{mn}(\theta,\varphi)\right] = \langle \exp\left[i\, f_{mn}(\theta,\varphi)h_{mn}\right]\rangle \tag{C-6}$$

Let

$$c^{(j)} = -a_j 2k_2 qH(\theta,\varphi) - b_j 2k_2 q'\, H'(\theta',\varphi') \tag{C-7}$$

where $q = \cos\theta$, $q' = \cos\theta'$, $a_j = \pm 1$ and $b_j = \pm 1$

Using (C-3), (C-7) becomes

$$c^{(j)} = \sum_{m,n} c^{(j)}_{mn}\, h_{mn} \tag{C-8}$$

where

$$c^{(j)}_{mn} = -2k_2\lambda_{mn}^{-1/2}\left[b_j q'\Phi_{mn}(\theta',\varphi') + a_j q\, \Phi_{mn}(\theta,\varphi)\right]$$

Since the h_{mn} are orthogonal

$$\langle e^{ic^{(j)}}\rangle = \prod_{m,n} \langle e^{ic^{(j)}_{mn}h_{mn}}\rangle = \prod_{m,n} \chi(c^{(j)}_{mn}) \tag{C-9}$$

and

$$\langle h_{pq}e^{ic^{(j)}}\rangle = -i\frac{d}{dc^{(j)}_{pq}}\left\{\prod_{mn} \chi(c^{(j)}_{mn})\right\} \tag{C-10}$$

If $H(\theta,\varphi)$ is a gaussian process with zero mean and variance σ^2 (not a function of θ or φ), then (C-9) becomes

$$\langle e^{iC^{(j)}}\rangle = e^{-\sum \frac{\left[C_{mn}^{(j)}\right]^2}{2}} = e^{-Q_j} = M_{0,0} \qquad \text{(C-11)}$$

where

$$Q_j = \frac{pg}{2}\left[q^2 + q'^2 + 2a_j b_j q q' \rho(q,q',\psi)\right]$$

$$p = 2k_2 a$$

$$g = \frac{2k_2\sigma^2}{a}$$

$$\psi = \varphi' - \varphi$$

and

$$\langle h_{pq} e^{iC^{(j)}}\rangle = i\, C_{pq}^{(j)}\, e^{-Q_j} \qquad \text{(C-12)}$$

Using the definitions of the X's (3-21),(C-8), (4-15)

$$\frac{\partial H(\theta,\varphi)}{\partial\theta} = \sum_{m,n} \lambda_{mn}^{-1/2} \frac{\partial}{\partial\theta}\left[\Phi_{mn}(\theta,\varphi)\, h_{mn}\right], \qquad \text{(C-13)}$$

and

$$\frac{\partial \Phi_{mn}(\theta,\varphi)}{\partial\theta}\, \Phi_{mn}(\theta,\varphi) = \frac{1}{2}\frac{\partial}{\partial\theta}\left[\Phi_{mn}^2(\theta,\varphi)\right] \qquad \text{(C-14)}$$

$$M_{1,0} = -\, ig\, a_j \left[q + a_j b_j\, q' \rho \right] e^{-Q_j}$$

$$M_{2,0} = -\, ig\, b_j \left[b_j a_j q \rho + q' \right] e^{-Q_j}$$

$$M_{3,0} = -igb_j q' \frac{\partial \rho}{\partial \theta} e^{-Q_j}$$

$$M_{4,0} = -iga_j q \frac{\partial \rho}{\partial \theta'} e^{-Q_j}$$

$$M_{5,0} = \frac{-igb_j q'}{\sqrt{1-q^2}} \frac{\partial \rho}{\partial \varphi} e^{-Q_j}$$

$$M_{6,0} = \frac{-iga_j q}{\sqrt{1-q'^2}} \frac{\partial \rho}{\partial \varphi'} e^{-Q_j} \quad \text{(C-15)}$$

Considering the next series of terms

$$\langle h_{st} h_{pq} e^{ic^{(j)}} \rangle = - \frac{d}{dc_{st}^{(j)}} \frac{d}{dc_{pq}^{(j)}} \langle e^{ic^{(j)}} \rangle \quad \text{(C-16)}$$

which for the gaussian case becomes

$$\langle h_{st} h_{pq}\, e^{ic^{(j)}} \rangle = (\delta_{ps}\, \delta_{qt} - c_{st}^{(j)}\, c_{pq}^{(j)}) e^{-Q_j} \quad \text{(C-17)}$$

Inspection of (C-17) using (C-4), (C-11) and (C-12) yields an alternate form

$$\langle x_a x_b e^{ic^{(j)}} \rangle = \langle x_a x_b \rangle \langle e^{ic^{(j)}} \rangle + \frac{\langle x_a e^{ic^{(j)}} \rangle \langle x_b e^{ic^{(j)}} \rangle}{\langle e^{ic^{(j)}} \rangle} \quad \text{(C-18)}$$

Bartlett [1956, p. 140] shows that

$$\langle \frac{\partial X(t)}{\partial t} \rangle = \frac{\partial}{\partial t} \langle X(t) \rangle , \qquad \langle \frac{\partial X(t)}{\partial t} X(s) \rangle = \frac{\partial}{\partial t}(\sigma^2 \rho),$$

and

$$\langle \frac{\partial X(t)}{\partial t} \frac{\partial X(s)}{\partial s} \rangle = \frac{\partial^2(\sigma^2\rho)}{\partial t \partial s} \qquad \text{(C-19)}$$

Then $\langle X_a X_b \rangle$ is

$$\langle x_1^2 \rangle = \sigma^2/a^2 \qquad \langle x_3^2 \rangle = \frac{\sigma^2}{a^2} \frac{\partial^2 \rho}{\partial\theta\partial\theta'} \Big|_{\theta'=\theta}$$

$$\langle x_1 x_2 \rangle = \frac{\sigma^2 \rho}{a^2} \qquad \langle x_3 x_4 \rangle = \frac{\sigma^2}{a^2} \frac{\partial^2 \rho}{\partial\theta\partial\theta'}$$

$$\langle x_1 x_4 \rangle = \frac{\sigma^2}{a^2} \frac{\partial \rho}{\partial \theta'} \qquad \langle x_3 x_6 \rangle = \frac{\sigma^2}{a^2\sqrt{1-q'^2}} \frac{\partial^2 \rho}{\partial\theta\partial\varphi'}$$

$$\langle x_1 x_6 \rangle = \frac{\sigma^2}{a^2\sqrt{1-q'^2}} \frac{\partial \rho}{\partial \varphi'} \qquad \langle x_4^2 \rangle = \frac{\sigma^2}{a^2} \frac{\partial^2 \rho}{\partial\theta\partial\theta'} \Big|_{\theta=\theta'}$$

$$\langle x_2^2 \rangle = \frac{\sigma^2}{a^2} \qquad \langle x_4 x_5 \rangle = \frac{\sigma^2}{a^2\sqrt{1-q^2}} \frac{\partial^2 \rho}{\partial\theta'\partial\varphi}$$

$$\langle x_2 x_3 \rangle = \frac{\sigma^2}{a^2} \frac{\partial \rho}{\partial \theta} \qquad \langle x_5^2 \rangle = \frac{\sigma^2}{a^2(1-q^2)} \frac{\partial^2 \rho}{\partial\varphi\partial\varphi'} \Big|_{\varphi'=\varphi}$$

$$\langle x_2 x_5 \rangle = \frac{\sigma^2}{a^2\sqrt{1-q^2}} \frac{\partial \rho}{\partial \varphi} \qquad \langle x_5 x_6 \rangle = \frac{\sigma^2}{a^2\sqrt{1-q^2}\sqrt{1-q'^2}} \frac{\partial^2 \rho}{\partial \varphi \partial \varphi'}$$

$$\langle x_6{}^2 \rangle = \frac{\sigma^2}{a^2(1-q'^2)} \frac{\partial^2 \rho}{\partial \varphi \partial \varphi'} \Big|_{\varphi=\varphi'} \tag{C-20}$$

All other terms are zero due to the assumption that σ^2 is not a function of Θ or φ. The substitution of (C-15) and (C-20) into (C-18) gives

$$M_{1,1} = \langle x_1{}^2 \, e^{ic^{(j)}} \rangle = \left[\frac{\sigma^2}{a^2} - g^2(q^2 + q'^2\rho^2 + 2a_j b_j qq'\rho) \right] e^{-Q_j}$$

$$M_{2,1} = \langle x_1 x_2 e^{ic^{(j)}} \rangle = \left[\frac{\sigma^2}{a^2} \rho - g^2(a_j b_j qq' + \rho(q^2+q'^2) + a_j b_j qq'\rho^2) \right] e^{-Q_j}$$

$$M_{3,1} = \langle x_1 x_3 e^{ic^{(j)}} \rangle = -g^2 \frac{\partial \rho}{\partial \Theta} \left[a_j b_j qq' + q'^2 \rho \right] e^{-Q_j}$$

$$M_{4,1} = \frac{\partial \rho}{\partial \Theta'} \left[\frac{\sigma^2}{a^2} - g^2(q^2 + a_j b_j qq' \rho) \right] e^{-Q_j}$$

$$M_{5,1} = \frac{-g^2}{\sqrt{1-q^2}} \frac{\partial \rho}{\partial \varphi} \left[a_j b_j q q' + q'^2 \rho \right] e^{-Q_j}$$

$$M_{6,1} = \frac{1}{\sqrt{1-q'^2}} \frac{\partial \rho}{\partial \varphi'} \left[\frac{\sigma^2}{a^2} - g^2(q^2 + a_j b_j qq'\rho) \right] e^{-Q_j}$$

$$M_{2,2} = \left[\frac{\sigma^2}{a^2} - g^2(q^2\rho^2 + q'^2 + 2a_j b_j q q'\rho) \right] e^{-Q_j}$$

$$M_{3,2} = \frac{\partial\rho}{\partial\theta}\left[\frac{\sigma^2}{a^2} - g^2(q'^2 + a_j b_j q\, q'\rho)\right]e^{-Q_j}$$

$$M_{4,2} = -g^2 \frac{\partial\rho}{\partial\theta'}\left[a_j b_j q\, q' + q^2\rho\right]e^{-Q_j}$$

$$M_{5,2} = \frac{1}{\sqrt{1-q^2}} \frac{\partial\rho}{\partial\varphi}\left[\frac{\sigma^2}{a^2} - g^2(q'^2 + a_j b_j q\, q'\rho)\right]e^{-Q_j}$$

$$M_{6,2} = \frac{-g^2}{\sqrt{1-q'^2}} \frac{\partial\rho}{\partial\varphi'}\left[a_j b_j q\, q' + q^2\rho\right]e^{-Q_j}$$

$$M_{3,3} = \left[\frac{\sigma^2}{a^2} \frac{\partial^2\rho}{\partial\theta\partial\theta'}\bigg|_{\theta'=\theta} -g^2 q'^2 \left(\frac{\partial\rho}{\partial\theta}\right)^2\right]e^{-Q_j}$$

$$M_{4,3} = \left[\frac{\sigma^2}{a^2} \frac{\partial^2\rho}{\partial\theta\partial\theta'} - g^2 a_j b_j q\, q' \frac{\partial\rho}{\partial\theta} \frac{\partial\rho}{\partial\theta'}\right] e^{-Q_j}$$

$$M_{5,3} = \frac{-g^2 q'^2}{\sqrt{1-q^2}} \frac{\partial\rho}{\partial\theta} \frac{\partial\rho}{\partial\varphi} e^{-Q_j}$$

$$M_{6,3} = \frac{1}{\sqrt{1-q'^2}}\left[\frac{\sigma^2}{a^2} \frac{\partial^2\rho}{\partial\theta\partial\varphi'} - g^2 a_j b_j q\, q' \frac{\partial\rho}{\partial\theta} \frac{\partial\rho}{\partial\varphi'}\right]e^{-Q_j}$$

$$M_{4,4} = \left[\frac{\sigma^2}{a^2} \frac{\partial^2\rho}{\partial\theta\partial\theta'}\bigg|_{\theta=\theta'} - g^2 q^2 \left(\frac{\partial\rho}{\partial\theta'}\right)^2\right]e^{-Q_j}$$

$$M_{5,4} = \frac{1}{\sqrt{1-q^2}} \left[\frac{\sigma^2}{a^2} \frac{\partial^2\rho}{\partial\theta'\partial\varphi} = g^2 a_j b_j q\, q' \frac{\partial\rho}{\partial\theta'} \frac{\partial\rho}{\partial\varphi}\right] e^{-Q_j}$$

$$M_{6,4} = \frac{-g^2\, q^2}{\sqrt{1-q'^2}} \frac{\partial\rho}{\partial\theta'} \frac{\partial\rho}{\partial\varphi'} e^{-Q_j}$$

$$M_{5,5} = \frac{1}{(1-q^2)} \left[\frac{\sigma^2}{a^2} \frac{\partial^2\rho}{\partial\varphi\partial\varphi'}\bigg|_{\varphi'=\varphi} - g^2 q'^2 \left(\frac{\partial\rho}{\partial\varphi}\right)^2\right] e^{-Q_j}$$

$$M_{6,5} = \frac{1}{\sqrt{1-q^2}\sqrt{1-q'^2}} \left[\frac{\sigma^2}{a^2} \frac{\partial^2\rho}{\partial\varphi\partial\varphi'} - g^2 a_j b_j q\, q' \frac{\partial\rho}{\partial\varphi} \frac{\partial\rho}{\partial\varphi'}\right] e^{-Q_j}$$

$$M_{6,6} = \frac{1}{(1-q'^2)} \left[\frac{\sigma^2}{a^2} \frac{\partial^2\rho}{\partial\varphi\partial\varphi'}\bigg|_{\varphi=\varphi'} - g^2 q^2 \left(\frac{\partial\rho}{\partial\varphi'}\right)^2\right] e^{-Q_j} \qquad \text{(C-21)}$$

APPENDIX D

q - q' INTEGRATION

Consider an integral of the form

$$I^{(k,\ell)} = \int_{\alpha}^{1}\int_{\alpha}^{1} F_1^{(k,\ell)}(q)\, F_2^{(k,\ell)}(q')\, M_{k,\ell}(q,q',\psi)\, \cdot$$

$$\cdot\ \langle L_j(\eta_1)L_j(\eta_2)\rangle\, e^{-i\omega_o(a_j\zeta_1+b_j\zeta_2)}\, dq\, dq' \qquad \text{(D-1)}$$

which on using (4-10) and (4-14) becomes

$$I^{(k,\ell)} = \int_{\alpha}^{1} F_2^{(k,\ell)}(q')\, e^{-i\omega_o b_j\zeta_2} \sum_{m=1}^{3} e_m \frac{\partial^{m-1}}{\partial\zeta_2^{m-1}}\, U\left(\frac{\zeta_2}{\sigma_2\sqrt{2(1-\rho^2)}}\right) \cdot$$

$$\cdot \left\{\int_{\alpha}^{1} F_1^{(k,\ell)}(q)\, e^{-i\omega_o a_j\zeta_1}\, M_{k,\ell}\left[U\left(\frac{\zeta_1}{\sigma_1\sqrt{2}}\right)\right.\right.$$

$$+ \frac{2}{(-ia_j\omega_o)}\frac{\partial}{\partial\zeta_1}\, U\left(\frac{\zeta_1}{\sigma_1\sqrt{2}}\right) +$$

$$\left.\left. + \frac{1}{(-ia_j\omega_o)^2}\frac{\partial^2}{\partial\zeta_1^2}\, U\left(\frac{\zeta_1}{\sigma_1\sqrt{2}}\right)\right] dq\right\} dq' \qquad \text{(D-2)}$$

Letting

$$F_3^{(k,\ell)} = F_1^{(k,\ell)}(q)\, M_{k,\ell}(q,q',\psi) \qquad \text{(D-3)}$$

and considering only the q integral inside the braces of (D-2) which can be written as

$$I_o^{(k,\ell)} = I_1^{(k,\ell)} + 2I_2^{(k,\ell)} + I_3^{(k,\ell)}$$

$$= \int_\alpha^1 F_3^{(k,\ell)} \, e^{-i\omega_o a_j \zeta_1} \, U\left(\frac{\zeta_1}{\sigma_1\sqrt{2}}\right) dq$$

$$+ 2\int_\alpha^1 \frac{F_3^{(k,\ell)} e^{-i\omega_o a_j \zeta_1}}{(-ia_j\omega_o)} \frac{\partial}{\partial\zeta_1} U\left(\frac{\zeta_1}{\sigma_1\sqrt{2}}\right) dq$$

$$+ \int_\alpha^1 \frac{F_3^{(k,\ell)} \, e^{-i\omega_o a_j \zeta_1}}{(-ia_j\omega_o)^2} \frac{\partial^2}{\partial\zeta_1^2} U\left(\frac{\zeta_1}{\sigma_1\sqrt{2}}\right) dq \qquad \text{(D-4)}$$

The integrations will be carried out by repeated integration by parts using the same dv.

$$dv = e^{-i\omega_o a_j \zeta_1} \, dq \qquad v = \frac{e^{-i\omega_o a_j \zeta_1}}{(-ia_j p)}$$

$$\int_\alpha^1 u\,dv = uv\Big|_\alpha^1 - \int_\alpha^1 v\,du \qquad \text{(D-5)}$$

where

$$\frac{\partial\zeta_1}{\partial q} = \frac{2a}{c} \qquad \text{and} \qquad p = 2k_2 a$$

Realizing that terms of the form $x\,\delta(x)$ are identically zero then

$$\frac{\partial}{\partial q} U\left(\frac{\zeta_1}{\sigma_1\sqrt{2}}\right) = \frac{\partial}{\partial \zeta_1} U\left(\frac{\zeta_1}{\sigma_1\sqrt{2}}\right) \frac{\partial \zeta_1}{\partial q} \tag{D-6}$$

The intervals of (D-4) become

$$I_1^{(k,\ell)} = \left. \frac{F_3^{(k,\ell)} U\left(\frac{\zeta_1}{\sigma_1\sqrt{2}}\right) e^{-i\omega_o a_j \zeta_1}}{(-ia_j p)} \right|_{\alpha}^{1}$$

$$\underbrace{- \frac{1}{(-ia_j p)} \int_{\alpha}^{1} \frac{\partial F_3^{(k,\ell)}}{\partial q} U\left(\frac{\zeta_1}{\sigma_1\sqrt{2}}\right) e^{-i\omega_o a_j \zeta_1} dq}_{I_4^{(k,\ell)}}$$

$$\underbrace{- \frac{1}{(-ia_j \omega_o)} \int_{\alpha}^{1} F_3^{(k,\ell)} \frac{\partial}{\partial \zeta_1} U\left(\frac{\zeta_1}{\sigma_1\sqrt{2}}\right) e^{-i\omega_o a_j \zeta_1} dq}_{-I_2^{(k,\ell)}}$$

$$I_2^{(k,\ell)} = \left. \frac{F_3^{(k,\ell)} \Delta\left(\frac{\zeta_1}{\sigma_1\sqrt{2}}\right) e^{-i\omega_o a_j \zeta_1}}{(-i\omega_o a_j)(-ia_j p)(\sqrt{2}\sigma_1)} \right|_{\alpha}^{1}$$

$$\underbrace{- \frac{1}{(-i\omega_o a_j)(-ia_j p)} \int_{\alpha}^{1} \frac{\partial F_3^{(k,\ell)}}{\partial q} \frac{\partial}{\partial \zeta_1} U\left(\frac{\zeta_1}{\sigma_1\sqrt{2}}\right) e^{-i\omega_o a_j \zeta_1} dq}_{I_5^{(k,\ell)}}$$

$$- \underbrace{\frac{1}{(-ia_j\omega_o)^2} \int_\alpha^1 F_3^{(k,\ell)} \frac{\partial^2}{\partial \zeta_1^2} U\left(\frac{\zeta_1}{\sigma_1\sqrt{2}}\right) e^{-i\omega_o a_j \zeta_1} dq}_{-I_3^{(k,\ell)}} \qquad \text{(D-7)}$$

Integrating $I_4^{(k,\ell)}$ four times more by parts

$$I_4^{(k,\ell)} = \left[- \frac{\frac{\partial F_3^{(k,\ell)}}{\partial q}}{(-ia_jp)^2} + \frac{\frac{\partial^2 F_3^{(k,\ell)}}{\partial q^2}}{(-ia_jp)^3} - \frac{\frac{\partial^3 F_3^{(k,\ell)}}{\partial q^3}}{(-ia_jp)^4} + \frac{\frac{\partial^4 F_3^{(k,\ell)}}{\partial q^4}}{(-ia_jp)^5} \right] \cdot$$

$$\cdot \; U\left(\frac{\zeta_1}{\sigma_1\sqrt{2}}\right) e^{-i\omega_o a_j \zeta_1} \Bigg|_\alpha^1$$

$$+ \underbrace{\frac{1}{(-ia_j\omega_o)(-ia_jp)} \int_\alpha^1 \frac{\partial F_3^{(k,\ell)}}{\partial q} \frac{\partial}{\partial \zeta_1} U\left(\frac{\zeta_1}{\sigma_1\sqrt{2}}\right) e^{-i\omega_o a_j \zeta_1} dq}_{-I_5^{(k,\ell)}}$$

$$- \underbrace{\frac{1}{(-ia_jp)^3} \int_\alpha^1 e^{-i\omega_o a_j \zeta_1} \left[\frac{\frac{\partial^2 F_3^{(k,\ell)}}{\partial q^2}}{(-ia_jp)^3} - \frac{\frac{\partial^3 F_3^{(k,\ell)}}{\partial q^3}}{(-ia_jp)^4} + \frac{\frac{\partial^4 F_3^{(k,\ell)}}{\partial q^4}}{(-ia_jp)^5} \right] \cdot \frac{\partial}{\partial q} U\left(\frac{\zeta_1}{\sigma_1\sqrt{2}}\right) dq}_{I_6^{(k,\ell)}}$$

$$- \underbrace{\frac{1}{(-ia_jp)^5} \int_\alpha^1 e^{-i\omega_o a_j \zeta_1} \frac{\partial^5 F_3^{(k,\ell)}}{\partial q^5} U\left(\frac{\zeta_1}{\sigma_1\sqrt{2}}\right) dq}_{I_7^{(k,\ell)}} \qquad \text{(D-8)}$$

The evaluation of $I_6^{(k,\ell)}$ requires the following results [Friedman, 1956]

$$\frac{\partial}{\partial q} U\left(\frac{\zeta_1}{\sigma_1\sqrt{2}}\right) = [\delta(q-q_1) - \delta'(q-q_2)] = \frac{2a}{c}\left[\frac{\Delta\left(\frac{\zeta_1}{\sigma_1\sqrt{2}}\right)}{\sigma_1\sqrt{2}}\right]$$

$$\int_\alpha^1 f(q)[\delta(q'q_1) - \delta(q-q_2)]dq = f(q)[u(1-q) - u(\alpha-q)]\Big|_{q_1}^{q_2} \quad \text{(D-9)}$$

where

$$q_1 = \frac{D}{a} - \frac{ct}{2a} \qquad q_2 = \frac{D}{a} - \frac{c(t-T)}{2a}$$

Since the derivatives of the reflection coefficients are bounded, increasing only slightly with each differentiation, and the derivatives of $M_{k,\ell}$ have a maximum coefficient of $pg\beta$, then

$$\left|\frac{1}{(-ia_jp)^5}\frac{\partial^5 F_3^{(k,\ell)}}{\partial q^5}\right| << \left|\frac{1}{(-ia_jp)^2}\frac{\partial^2 F_3^{(k,\ell)}}{\partial q^2}\right|$$

$$\text{for } g\beta < 1 \qquad p >> 1 \qquad \text{(D-10)}$$

which allows us to neglect $I_7^{(k,\ell)}$ compared to $I_6^{(k,\ell)}$. Then using

$$I_0^{(k,\ell)} \cong F_4^{(k,\ell)} \, U\left(\frac{\zeta_1}{\sigma_1\sqrt{2}}\right) e^{-i\omega_0 a_j \zeta_1}\Big|_\alpha^1$$

$$- F_5^{(k,\ell)}[u(1-q) - u(\alpha-q)]\, e^{-i\omega_o a_j \zeta_2}\Big|_{q_2}^{q_1}$$

$$+ \frac{F_3^{(k,\ell)}\, e^{-i\omega_o a_j \zeta_1}[\delta(q-q_1) - \delta(q-q_2)]}{(-i\omega_o a_j)^2}\Bigg|_{\alpha}^{1} \qquad \text{(D-11)}$$

where

$$F_4^{(k,\ell)} = \left[\frac{F_1^{(k,\ell)} M_{k,\ell}}{(-ia_j p)} - \frac{F_1^{(k,\ell)} \frac{\partial M_{k,\ell}}{\partial q} + \frac{\partial F_1^{(k,\ell)}}{\partial q} M_{k,\ell}}{(-ia_j p)^2} + F_5^{(k,\ell)}\right]$$

$$F_5^{(k,\ell)} = \left[\frac{F_1^{(k,\ell)} \frac{\partial^2 M_{k,\ell}}{\partial q^2} + \frac{2\partial F_1^{(k,\ell)}}{\partial q} \frac{\partial M_{k,\ell}}{\partial q} + \frac{\partial^2 F_1^{(k,\ell)}}{\partial q^2} M_{k,\ell}}{(-ia_j p)^3}\right.$$

$$- \frac{F_1^{(k,\ell)} \frac{\partial^3 M_{k,\ell}}{\partial q^3} + \frac{3\partial F_1^{(k,\ell)}}{\partial q} \frac{\partial^2 M_{k,\ell}}{\partial q^2}}{(-ia_j p)^4}$$

$$+ \frac{\frac{3\partial^2 F_1^{(k,\ell)}}{\partial q^2} \frac{\partial M_{k,\ell}}{\partial q} + \frac{\partial^3 F_1^{(k,\ell)}}{\partial q^3} M_{k,\ell}}{(-ia_j p)^4}$$

$$+ \frac{F_1^{(k,\ell)} \frac{\partial^4 M_{k,\ell}}{\partial q^3} + \frac{4\partial F_1^{(k,\ell)}}{\partial q} \frac{\partial^3 M_{k,\ell}}{\partial q^3} + \frac{6\partial^2 F_1^{(k,\ell)}}{\partial q^2} \frac{\partial^2 M_{k,\ell}}{\partial q^2}}{(-ia_j p)^5}$$

$$+ \frac{\frac{4\partial^3 F_1^{(k,\ell)}}{\partial q^3} \frac{\partial M_{k,\ell}}{\partial q} + \frac{\partial^4 F_1^{(k,\ell)}}{\partial q^4} M_{k,\ell}}{(-ia_j p)^5} \Bigg] .$$

Letting

$$F_6^{(k,\ell)} = F_2^{(k,\ell)} F_4^{(k,\ell)} U\left(\frac{\zeta_1}{\sigma_1\sqrt{2}}\right)$$

$$F_7^{(k,\ell)} = F_2^{(k,\ell)} F_5^{(k,\ell)}[u(1-q) - u(\alpha-q)]$$

$$F_8^{(k,\ell)} = \frac{F_2^{(k,\ell)} F_3^{(k,\ell)}[\delta(q-q_1) - \delta(q-q_2)]}{(-ia_j\omega_o)^2} \qquad \text{(D-12)}$$

then the substitution of (D-11), (D-12) into (D-2) gives

$$I^{(k,\ell)} = \int_\alpha^1 F_6^{(k,\ell)} e^{-i\omega_o b_j \zeta_2}\left[U\left(\frac{\zeta_2}{\sigma_2\sqrt{2(1-\rho^2)}}\right) + \frac{2}{(-ib_j\omega_o)} \frac{\partial}{\partial \zeta_2} \cdot\right.$$

$$\left. \cdot\, U\left(\frac{\zeta_2}{\sigma_2\sqrt{2(1-\rho^2)}}\right) + \frac{1}{(-ib_j\omega_o)^2} \frac{\partial^2}{\partial \zeta_2^2} U\left(\frac{\zeta_2}{\sigma_2\sqrt{2(1-\rho^2)}}\right)\right] dq' \, \cdot$$

$$\cdot\, e^{-i\omega_o a_j \zeta_1}\Big|_\alpha^1 - \int_\alpha^1 F_7^{(k,\ell)} e^{-i\omega_o b_j \zeta_2} \, \cdot$$

$$\cdot \left[U\left(\frac{\zeta_2}{\sigma_2\sqrt{2(1-\rho^2)}}\right) + \frac{2}{(-ib_j\omega_o)} \frac{\partial}{\partial \zeta_2} U\left(\frac{\zeta_2}{\sigma_2\sqrt{2(1-\rho^2)}}\right)\right.$$

$$+ \frac{1}{(-ib_j\omega_o)^2} \frac{\partial^2}{\partial \zeta_2^2} U\left(\frac{\zeta_2}{\sigma_2\sqrt{2(1-\rho^2)}}\right) \Bigg] dq' \; e^{-i\omega_o a_j \zeta_1} \Bigg|_{q_2}^{q_1}$$

$$+ \int_\alpha^1 F_8^{(k,\ell)} \; e^{-i\omega_o b_j \zeta_2} \left[U\left(\frac{\zeta_2}{\sigma_2\sqrt{2(1-\rho^2)}}\right) + \frac{2}{(-ib_j\omega_o)} \frac{\partial}{\partial \zeta_2} \cdot \right.$$

$$\left. \cdot \; U\left(\frac{\zeta_2}{\sigma_2\sqrt{2(1-\rho^2)}}\right) \right] dq' \; e^{-i\omega_o a_j \zeta_1} \Bigg|_\alpha^1 \qquad \text{(D-13)}$$

The integrals in (D-13) are of the same form as that of (D-4), so using (D-7) and (D-8) on (D-13) yields upon neglecting the integral similar to $I_7^{(k,\ell)}$

$$I^{(k,\ell)} = \left[\frac{F_6^{(k,\ell)}}{(-ib_j p)} - \frac{\frac{\partial F_6^{(k,\ell)}}{\partial q'}}{(-ib_j p)^2} + \frac{\frac{\partial^2 F_6^{(k,\ell)}}{\partial q'^2}}{(-ib_j p)^3} - \frac{\frac{\partial^3 F_6^{(k,\ell)}}{\partial q'^3}}{(-ib_j p)^4} + \frac{\frac{\partial^4 F_6^{(k,\ell)}}{\partial q'^4}}{(-ib_j p)^5} \right] \cdot$$

$$\cdot \; U\left(\frac{\zeta_2}{\sigma_2\sqrt{2(1-\rho^2)}}\right) e^{-i\omega_o b_j \zeta_2} \Bigg|_\alpha^1 e^{-i\omega_o a_j \zeta_1} \Bigg|_\alpha^1 - \left[\frac{\frac{\partial^2 F_6^{(k,\ell)}}{\partial q'^2}}{(-ib_j p)^3} \right.$$

$$\left. - \frac{\frac{\partial^3 F_6^{(k,\ell)}}{\partial q'^3}}{(-ib_j p)^4} + \frac{\frac{\partial^4 F_6^{(k,\ell)}}{\partial q'^4}}{(-ib_j p)^5} \right] [u(1-q') - u(\alpha-q')] \; \cdot$$

$$e^{-i\omega_o b_j \zeta_2} \Bigg|_{q_2}^{q_1} e^{-i\omega_o a_j \zeta_1} \Bigg|_\alpha^1 - \left[\frac{F_7^{(k,\ell)}}{(-ib_j p)} - \frac{\frac{\partial F_7^{(k,\ell)}}{\partial q'}}{(-ib_j p)^2} \right.$$

$$+ \frac{\frac{\partial^2 F_7^{(k,\ell)}}{\partial q'^2}}{(-ib_j p)^3} - \frac{\frac{\partial^3 F_7^{(k,\ell)}}{\partial q'^3}}{(-ib_j p)^4} + \frac{\frac{\partial^4 F_7^{(k,\ell)}}{\partial q'^4}}{(-ib_j p)^5}\Bigg] U\left(\frac{\zeta_2}{\sigma_2\sqrt{2(1-\rho^2)}}\right) e^{-i\omega_o b_j \zeta_2}\Big|_{\alpha}^{1} .$$

$$\cdot\, e^{-i\omega_o a_j \zeta_1}\Big|_{q_2}^{q_1} + \left[\frac{\frac{\partial^2 F_7^{(k,\ell)}}{\partial q'^2}}{(-ib_j p)^3} - \frac{\frac{\partial^3 F_7^{(k,\ell)}}{\partial q'^3}}{(-ib_j p)^4} + \frac{\frac{\partial^4 F_7^{(k,\ell)}}{\partial q'^4}}{(-ib_j p)^5}\right] [u(1-q')$$

$$- u(\alpha-q')] e^{-i\omega_o b_j \zeta_2}\Big|_{q_2}^{q_1} e^{-i\omega_o a_j \zeta_1}\Big|_{q_2}^{q_1} + \frac{F_6^{(k,\ell)}}{(-i\omega_o b_j)^2} [\delta(q'-q_1)$$

$$- \delta(q'-q_2)] e^{-i\omega_o b_j \zeta_2}\Big|_{\alpha}^{1} e^{-i\omega_o a_j \zeta_1}\Big|_{\alpha}^{1} - \frac{F_7^{(k,\ell)}}{(-i\omega_o b_j)^2} [\delta(q'-q_1)$$

$$- \delta(q'-q_2)] e^{-i\omega_o b_j \zeta_2}\Big|_{q_2}^{q_1} e^{-i\omega_o a_j \zeta_1}\Big|_{\alpha}^{1} + \Bigg[\frac{F_8^{(k,\ell)}}{(-ib_j p)} - \frac{\frac{\partial F_8^{(k,\ell)}}{\partial q'}}{(-ib_j p)^2}$$

$$+ \frac{\frac{\partial^2 F_8^{(k,\ell)}}{\partial q'^2}}{(-ib_j p)^3} - \frac{\frac{\partial^3 F_8^{(k,\ell)}}{\partial q'^3}}{(-ib_j p)^4} + \frac{\frac{\partial^4 F_8^{(k,\ell)}}{\partial q'^4}}{(-ib_j p)^5}\Bigg] U\left(\frac{\zeta_2}{\sigma_2\sqrt{2(1-\rho^2)}}\right) e^{-i\omega_o b_j \zeta_2}\Big|_{\alpha}^{1} .$$

$$\cdot\, e^{-i\omega_o a_j \zeta_1}\Big|_{\alpha}^{1} - \left[\frac{\frac{\partial^2 F_8^{(k,\ell)}}{\partial q'^2}}{(-ib_j p)^3} - \frac{\frac{\partial^3 F_8^{(k,\ell)}}{\partial q'^3}}{(-ib_j p)^4} + \frac{\frac{\partial^4 F_8^{(k,\ell)}}{\partial q'^4}}{(-ib_j p)^5}\right] [u(1-q')$$

$$- u(\alpha-q')] \; e^{-i\omega_o b_j \zeta_2}\Big|_{q_2}^{q_1} \; e^{-i\omega_o a_j \zeta_1}\Big|_{\alpha}^{1} + \frac{F_8^{(k,\ell)}[\delta(q'-q_1) - \delta(q'-q_2)]}{(-ib_j\omega_o)^2} \cdot$$

$$\cdot \; e^{-i\omega_o b_j \zeta_2}\Big|_{\alpha}^{1} \; e^{-i\omega_o a_j \zeta_1}\Big|_{\alpha}^{1} \qquad \text{(D-14)}$$

Define

$$U_a = [u(1-q_1) - u(1-q_2)] = [u(t_o) - u(t_o - T)]$$

$$U_b = [u(1-q_2) - u(\alpha-q_1)] = [u(t_o - T) - u(t_o - t_1)]$$

$$U_c = [u(\alpha-q_1) - u(\alpha-q_2)] = [u(t_o - t_1) - u(t_o - t_1 - T)]$$

$$t_o = t - \frac{2(D-a)}{c} \qquad t_1 = \frac{2a}{c}(1-\alpha)$$

$$\Delta_a = [\delta(1-q_1) - \delta(1-q_2)]$$

$$\Delta_c = [\delta(\alpha-q_1) - \delta(\alpha-q_2)]$$

$$a_j^2 = b_j^2 = 1$$

$$U_i U_j = 0 \text{ unless } i = j, \text{ where } i,j = a, b, c$$

$$e^{-i\omega_o T} \equiv 1 \qquad \text{(D-15)}$$

The Δ_a and Δ_c which appear in (D-15) are necessary for the continuity of the equation and to make the values of the integral vanish at the proper places. In the present work these terms will be neglected by the simple expediency of not evaluating the integral at the four points of apparent trouble. Therefore (D-14) becomes upon substitution of $F_6^{(k,\ell)}$ and $F_7^{(k,\ell)}$

$$I^{(k,\ell)} = \Big[G_1^{(k,\ell)}(1,1,\psi)e^{-i\omega_o t_o(a_j+b_j)} - G_3^{(k,\ell)}(q_1,1,\psi) \cdot$$

$$\cdot e^{-i\omega_o b_j t_o} - G_4^{(k,\ell)}(1,q_1,\psi)e^{-i\omega_o a_j t_o}$$

$$+ G_2^{(k,\ell)}(q_1,q_1,\psi\Big]U_a + \Big[G_2^{(k,\ell)}(q_1,q_1,\psi)$$

$$- G_2^{(k,\ell)}(q_1,q_2,\psi) - G_2^{(k,\ell)}(q_2,q_1,\psi) + G_2^{(k,\ell)} \cdot$$

$$\cdot (q_2,q_2,\psi)\Big]U_b + \Big[G_1^{(k,\ell)}(\alpha,\alpha,\psi)e^{-i\omega_o(t_o-t_1)(a_j+b_j)}$$

$$- G_3^{(k,\ell)}(q_2,\alpha,\psi)e^{-i\omega_o b_j(t_o-t_1)} - G_4^{(k,\ell)}(\alpha,q_2,\psi) \cdot$$

$$\cdot e^{-i\omega_o a_j(t_o-t_1)} + G_2^{(k,\ell)}(q_2,q_2,\psi)\Big]U_c \qquad \text{(D-16)}$$

where

$$F^{(k,\ell)} = F_1^{(k,\ell)}(q)F_2^{(k,\ell)}(q')M_{k,\ell}(q,q',\psi) ,$$

$$G_1^{(k,\ell)}(q,q',\psi) = -\frac{a_j b_j F^{(k,\ell)}}{p^2} + \frac{i}{p^3}\left[b_j \frac{\partial F^{(k,\ell)}}{\partial q} + a_j \frac{\partial F^{(k,\ell)}}{\partial q'}\right]$$

$$+ \frac{1}{p^4}\left[\frac{\partial^2 F^{(k,\ell)}}{\partial q'\partial q} + a_j b_j \left(\frac{\partial^2 F^{(k,\ell)}}{\partial q'^2} + \frac{\partial^2 F^{(k,\ell)}}{\partial q^2}\right)\right]$$

$$- \frac{i}{p^5} \left[b_j \frac{\partial^3 F^{(k,\ell)}}{\partial q^3} + a_j \frac{\partial^3 F^{(k,\ell)}}{\partial q' \partial q^2} + b_j \frac{\partial^3 F^{(k,\ell)}}{\partial q'^2 \partial q} \right.$$

$$\left. + a_j \frac{\partial^3 F^{(k,\ell)}}{\partial q'^3} \right] - \frac{1}{p^6} \left[a_j b_j \frac{\partial^4 F^{(k,\ell)}}{\partial q^4} + \frac{\partial^4 F^{(k,\ell)}}{\partial q' \partial q^3} \right.$$

$$\left. + \frac{\partial^4 F^{(k,\ell)}}{\partial q'^3 \partial q} + a_j b_j \frac{\partial^4 F^{(k,\ell)}}{\partial q'^4} \right] + \frac{i}{p^7} \left[a_j \frac{\partial^5 F^{(k,\ell)}}{\partial q'^4 \partial q} \right.$$

$$\left. + b_j \frac{\partial^5 F^{(k,\ell)}}{\partial q'^4 \partial q} \right] + G_2^{(k,\ell)}(q,q',\psi)$$

$$G_2^{(k,\ell)}(q,q',\psi) = - \frac{a_j b_j}{p^6} \frac{\partial^4 F^{(k,\ell)}}{\partial q^2 \partial q'^2} + \frac{i}{p^7} \left[b_j \frac{\partial^5 F^{(k,\ell)}}{\partial q'^2 \partial q^3} \right.$$

$$\left. + a_j \frac{\partial^5 F^{(k,\ell)}}{\partial q'^3 \partial q^2} \right] + \frac{1}{p^8} \left[a_j b_j \frac{\partial^6 F^{(k,\ell)}}{\partial q'^4 \partial q^2} + \frac{\partial^6 F^{(k,\ell)}}{\partial q'^3 \partial q^3} \right.$$

$$\left. + a_j b_j \frac{\partial^6 F^{(k,\ell)}}{\partial q'^2 \partial q^4} \right] - \frac{i}{p^9} \left[a_j \frac{\partial^7 F^{(k,\ell)}}{\partial q'^3 \partial q^4} \right.$$

$$\left. + b_j \frac{\partial^7 F^{(k,\ell)}}{\partial q'^4 \partial q^3} \right] + \frac{a_j b_j}{p^{10}} \frac{\partial^8 F^{(k,\ell)}}{\partial q'^4 \partial q^4}$$

$$G_3^{(k,\ell)}(q,q',\psi) = \frac{a_j b_j}{p^4} \frac{\partial^2 F^{(k,\ell)}}{\partial q^2} - \frac{i}{p^5} \left[b_j \frac{\partial^3 F^{(k,\ell)}}{\partial q^3} + a_j \frac{\partial^3 F^{(k,\ell)}}{\partial q' \partial q^2} \right]$$

$$- \frac{1}{p^6} \left[a_j b_j \frac{\partial^4 F^{(k,\ell)}}{\partial q^4} + \frac{\partial^4 F^{(k,\ell)}}{\partial q' \partial q^3} \right]$$

$$+ \frac{i}{p^7} a_j \frac{\partial^5 F^{(k,\ell)}}{\partial q' \partial q^4} + G_2(q,q',\psi)$$

$$G_4^{(k,\ell)}(q,q',\psi) = \frac{a_j b_j}{p^4} \frac{\partial^2 F^{(k,\ell)}}{\partial q^2} - \frac{i}{p^5}\left[a_j \frac{\partial^3 F^{(k,\ell)}}{\partial q'^3} + b_j \frac{\partial^3 F^{(k,\ell)}}{\partial q'^2 \partial q}\right]$$

$$- \frac{1}{p^6}\left[a_j b_j \frac{\partial^4 F^{(k,\ell)}}{\partial q'^4} + \frac{\partial^4 F^{(k,\ell)}}{\partial q'^3 \partial q}\right]$$

$$+ \frac{i b_j}{p^7} \frac{\partial^5 F^{(k,\ell)}}{\partial q'^4 \partial q} + G_2(q,q',\psi)$$

Table D-1 lists the functions $F_a^{(k,\ell)}(q,\varphi)$ and $F_b^{(k,\ell)}(q',\varphi,\psi)$ needed for the integrand of (4-19). Table D-2 lists the functions $F_{1c}^{(k,\ell)}(q)$, $F_{2c}^{(k,\ell)}(q')$, and $g_{2c}^{(k,\ell)}(\varphi,\psi)$ needed for the integrand of (4-19) and the function $g_{3c}^{(k,\ell)}(\psi)$ needed in (4-21). Table D-3 lists the functions $F_{1d}^{(k,\ell)}(q)$, $F_{2d}^{(k,\ell)}(q')$, $F_{1r}^{(k,\ell)}(q)$, $F_{2r}^{(k,\ell)}(q')$ and $g_{3r}^{(k,\ell)}(\psi)$ needed for the integrand of (4-21).

TABLE D-1

INTEGRAND FUNCTIONS FOR (4-19)

k, ℓ	$F_a^{(k,\ell)}(q,\varphi)$	$F_b^{(k,\ell)}(q',\varphi,\psi)$
0,0	$-V_{rd}(q) + V_{rm}(q)\sin^2\varphi$	$-V_{rd}(q') + V_{rm}(q')\sin^2(\varphi+\psi)$
1,0	$2\left[-V_{rd}(q) + V_{rm}(q)\sin^2\varphi\right]$	$-V_{rd}(q') + V_{rm}(q')\sin^2(\varphi+\psi)$
2,0	$-V_{rd}(q) + V_{rm}(q)\sin^2\varphi$	$2\left[-V_{rd}(q') + V_{rm}(q')\sin^2(\varphi+\psi)\right]$
3,0	$-(1-q^2)^{1/2}V'_{rd}(q)$ $+(1-q^2)^{1/2}V'_{rm}(q)\sin^2\varphi]$	$-V_{rd}(q') + V_{rm}(q')\sin^2(\varphi+\psi)$
4,0	$-V_{rd}(q) + V_{rm}(q)\sin^2\varphi$	$\left[-(1-q'^2)^{1/2}V_{rd}(q') + (1-q'^2)^{1/2}V'_{rm}(q')\sin^2(\varphi+\psi)\right]$

TABLE D-1 (Continued)

INTEGRAND FUNCTIONS FOR (4-19)

k,ℓ	$F_a^{(k,\ell)}(q,\varphi)$	$F_b^{(k,\ell)}(q',\varphi,\psi)$
5,0	$-2(1-q^2)^{1/2} V_{rc}(q)\sin\varphi \cos\varphi$	$-V_{rd}(q') + V_{rm}(q')\sin^2(\varphi+\psi)$
6,0	$-V_{rd}(q) + V_{rm}(q)\sin^2\varphi$	$-2(1-q')^{1/2} V_{rc}(q')\sin(\phi+\psi)\cos(\varphi+\psi)$
1,1	$-V_{rd}(q) + V_{rm}(q)\sin^2\varphi$	$-V_{rd}(q') + V_{rm}(q')\sin^2(\varphi+\psi)$
2,1	$2\left[-V_{rd}(q) + V_{rm}(q)\sin^2\varphi\right]$	$2\left[-V_{rd}(q') + V_{rm}(q')\sin^2(\varphi+\psi)\right]$
3,1	$-1(1-q^2)^{1/2} V'_{rd}(q)$ $+ (1-q^2)^{1/2} V'_{rm}(q)\sin^2\varphi$	$-V_{rd}(q') + V_{rm}(q')\sin^2(\varphi+\psi)$
4,1	$2\left[-V_{rd}(q) + V_{rm}(q)\sin^2\varphi\right]$	$-(1-q'^2)^{1/2} V'_{rd}(q')$ $+(1-q'^2)^{1/2} V'_{rm}(q')\sin^2(\varphi+\psi)$

TABLE D-1 (Continued)

INTEGRAND FUNCTIONS FOR (4-19)

k,ℓ	$F_a^{(k,\ell)}(q,\varphi)$	$F_b^{(k,\ell)}(q',\varphi,\psi)$
5,1	$-2(1-q^2)^{1/2} V_{rc}(q)\sin\varphi\cos\varphi$	$2\left[-V_{rd}(q') + V_{rm}(q')\sin^2(\varphi+\psi)\right]$
6,1	$2\left[-V_{rd}(q) + V_{rm}(q)\sin^2\varphi\right]$	$-2(1-q'^2)^{1/2} V_{rc}(q')\sin(\varphi+\psi)\cos(\varphi+\psi)$
2,2	$-V_{rd}(q) + V_{rm}(q)\sin^2\varphi$	$-V_{rd}(q') + V_{rm}(q')\sin^2(\varphi+\psi)$
3,2	$\left[-(1-q^2)^{1/2} V'_{rd}(q) + (1-q^2)^{1/2} V'_{rm}(q)\sin^2\varphi\right]$	$2\left[-V_{rd}(q') + V_{rm}(q')\sin^2(\varphi+\psi)\right]$
4,2	$-V_{rd}(q) + V_{rm}(q)\sin^2\varphi$	$-(1-q'^2)^{1/2} V'_{rd}(q') + (1-q'^2)^{1/2} V'_{rm}(q')\sin^2(\varphi+\psi)$
5,2	$-2(1-q^2)^{1/2} V_{rc}(q)\sin\varphi\cos\varphi$	$2\left[-V_{rd}(q') + V_{rm}(q')\sin^2(\varphi+\psi)\right]$

TABLE D-1 (Continued)

INTEGRAND FUNCTIONS FOR (4-19)

k,ℓ	$F_a^{(k,\ell)}(q,\varphi)$	$F_b^{(k,\ell)}(q',\varphi,\psi)$
6,2	$2\left[-V_{rd}(q) + V_{rm}(q)\sin^2\varphi\right]$	$-2(1-q'^2)^{1/2}\ V_{rc}(q')\sin(\varphi+\psi)\cos(\varphi+\psi)$
3,3	$\frac{1}{2}\left[-(1-q^2)V''_{rd}(q) + qV'_{rd}(q)\right.$ $-V_{rd}(q) + (1-q^2)^2\ V''_{rm}(q)\sin^2\varphi$ $\left.-qV'_{rm}\sin^2\varphi + V_{rm}(q)\sin^2\varphi\right]$	$-V_{rd}(q') + V_{rm}(q')\sin^2(\varphi+\psi)$
4,3	$-(1-q^2)^{1/2}\ V'_{rd}(q)$ $+(1-q^2)^{1/2}\ V'_{rm}(q)\sin^2\varphi$	$-(1-q'^2)^{1/2}\ V'_{rd}(q')$ $+(1-q'^2)^{1/2}\ V'_{rm}(q')\sin^2(\varphi+\psi)$
5,3	$2\left[qV_{rc}(q) - (1-q^2)V'_{rc}(q)\right]\cdot$ $\cdot\ \sin\varphi\ \cos\varphi$	$-V_{rd}(q') + V_{rm}(q')\sin^2(\varphi+\psi)$

TABLE D-1 (Continued)

INTEGRAND FUNCTIONS FOR (4-19)

k,ℓ	$F_a^{(k,\ell)}(q,\varphi)$	$F_b^{(k,\ell)}(q',\varphi,\psi)$
6,3	$-(1-q^2)^{1/2} V'_{rd}(q)$ $+(1-q^2)^{1/2} V'_{rm}(q)\sin^2\varphi$	$-2(1-q'^2)^{1/2} V_{rc}(q')\sin(\varphi+\psi) \cdot$ $\cdot \cos(\varphi+\psi)$
4,4	$-V_{rd}(q) + V_{rm}(q)\sin^2\varphi$	$\frac{1}{2}\Big[-(1-q'^2)V''_{rd}(q') + q'V'_{rd}(q')$ $-V_{rd}(q') + (1-q'^2)^2 V''_{rm}(q')\sin^2(\varphi+\psi)$ $-q'V_{rm}(q')\sin^2(\varphi+\psi)+V_{rm}(q')\sin^2(\varphi+\psi)\Big]$
4,5	$-2(1-q^2)^{1/2} V_{rc}(q)\sin\varphi\cos\varphi$	$-(1-q'^2)^{1/2} V'_{rd}(q') + (1-q'^2)^{1/2} \cdot$ $\cdot V'_{rm}(q')\sin^2(\varphi+\psi)$
4,6	$-V_{rd}(q) + V_{rm}(q)\sin^2\varphi$	$2\Big[q'V_{rc}(q') - (1-q'^2)V'_{rc}(q')\Big] \cdot$ $\sin(\varphi+\psi)\cos(\varphi+\psi)$

TABLE D-1 (Continued)

INTEGRAND FUNCTIONS FOR (4-19)

k,ℓ	$F_a^{(k,\ell)}(q,\varphi)$	$F_b^{(k,\ell)}(q',\varphi,\psi)$
5,5	$-V_{rd}(q) + qV'_{rd}(q) + V_{rc}(q) \cdot \cos^2\varphi$	$-V_{rd}(q') + V_{rm}(q')\sin^2(\varphi+\psi)$
6,5	$-2(1-q^2)^{1/2}\ V_{rc}(q)\sin\varphi\ \cos\varphi$	$-2(1-q'^2)^{1/2}\ V_{rc}(q')\sin(\varphi+\psi) \cdot \cos(\varphi+\psi)$
6,6	$-V_{rd}(q) + V_{rm}(q)\sin^2\varphi$	$-V_{rd}(q') + q'V'_{rd}(q') + V_{rc}(q')\cos^2(\varphi+\psi)$

TABLE D-2

INTEGRAND FUNCTIONS FOR (4-19) and (4-21)

k,ℓ	$F_{1c}^{(k,\ell)}(q)$	$F_{2c}^{(k,\ell)}(q')$	$g_{2c}^{(k,\ell)}(\varphi,\psi)$	$g_{3c}^{(k,\ell)}(\psi)$
0,0	$V_{rm}(q)$	$V_{rm}(q')$	$\sin 2\varphi \sin 2(\varphi+\psi)$	$\cos 2\psi$
1,0	$2V_{rm}(q)$	$V_{rm}(q')$	$\sin 2\varphi \sin 2(\varphi+\psi)$	$\cos 2\psi$
2,0	$V_{rm}(q)$	$2V_{rm}(q')$	$\sin 2\varphi \sin 2(\varphi+\psi)$	$\cos 2\psi$
3,0	$(1-q^2)^{1/2} V'_{rm}(q)$	$V_{rm}(q')$	$\sin 2\varphi \sin 2(\varphi+\psi)$	$\cos 2\psi$
4,0	$V_{rm}(q)$	$(1-q'^2)^{1/2} V'_{rm}(q')$	$\sin 2\varphi \sin 2(\varphi+\psi)$	$\cos 2\psi$
5,0	$-2V_{rc}(q)(1-q^2)^{1/2}$	$V_{rm}(q')$	$\cos 2\varphi \sin 2(\varphi+\psi)$	$\sin 2\psi$
6,0	$V_{rm}(q)$	$-2V_{rc}(q')(1-q'^2)^{1/2}$	$\sin 2\varphi \cos 2(\varphi+\psi)$	$-\sin 2\psi$
1,1	$V_{rm}(q)$	$V_{rm}(q')$	$\sin 2\varphi \sin 2(\varphi+\psi)$	$\cos 2\psi$
2,1	$2V_{rm}(q)$	$2V_{rm}(q')$	$\sin 2\varphi \sin 2(\varphi+\psi)$	$\cos 2\psi$
3,1	$(1-q^2)^{1/2} V'_{rm}(q)$	$V_{rm}(q')$	$\sin 2\varphi \sin 2(\varphi+\psi)$	$\cos 2\psi$

TABLE D-2 (Continued)

k,ℓ	$F_{1c}^{(k,\ell)}(q)$	$F_{2c}^{(k,\ell)}(q')$	$g_{2c}^{(k,\ell)}(\varphi,\psi)$	$g_{3c}^{(k,\ell)}(\psi)$
4,1	$2V_{rm}(q)$	$(1-q'^2)^{1/2} V'_{rm}(q')$	$\sin2\varphi \sin2(\varphi+\psi)$	$\cos2\psi$
5,1	$-2V_{rc}(q)(1-q^2)^{1/2}$	$V_{rm}(q')$	$\cos2\varphi \sin2(\varphi+\psi)$	$\sin2\psi$
6,1	$2V_{rm}(q)$	$-2V_{rc}(q')(1-q'^2)^{1/2}$	$\sin2\varphi \cos2(\varphi+\psi)$	$-\sin2\psi$
2,2	$V_{rm}(q)$	$V_{rm}(q')$	$\sin2\varphi \sin2(\varphi+\psi)$	$\cos2\psi$
3,2	$(1-q^2)^{1/2} V'_{rm}(q)$	$2V_{rm}(q')$	$\sin2\varphi \sin2(\varphi+\psi)$	$\cos2\psi$
4,2	$V_{rm}(q)$	$(1-q'^2)^{1/2} V'_{rm}(q')$	$\sin2\varphi \sin2(\varphi+\psi)$	$\cos2\psi$
5,2	$-2V_{rc}(q)(1-q^2)^{1/2}$	$2V_{rm}(q')$	$\cos2\varphi \sin2(\varphi+\psi)$	$\sin2\psi$
6,2	$V_{rm}(q)$	$-2V_{rc}(q')(1-q')^2$	$\sin2\varphi \cos2(\varphi+\psi)$	$-\sin2\psi$
3,3	$\frac{1}{2}\Big[(1-q^2)^2 V''_{rc}(q)$ $- 5q(1-q^2)V'_{rc}(q)$ $- V_{rc}(q)(1-3q^2)\Big]$	$V_{rm}(q')$	$\sin2\varphi \sin2(\varphi+\psi)$	$\cos2\psi$

TABLE D-2 (Continued)

k, ℓ	$F_{1c}^{(k,\ell)}(q)$	$F_{2c}^{(k,\ell)}(q')$	$g_{2c}^{(k,\ell)}(\varphi,\psi)$	$g_{3c}^{(k,\ell)}(\psi)$
4,3	$(1-q^2)^{1/2} V'_{rm}(q)$	$(1-q'^2)^{1/2} V'_{rm}(q')$	$\sin 2\varphi \sin 2(\varphi+\psi)$	$\cos 2\psi$
5,3	$2\left[qV_{rc}(q) - (1-q^2)V'_{rc}(q)\right]$	$V_{rm}(q')$	$\cos 2\varphi \sin 2(\varphi+\psi)$	$\sin 2\psi$
6,3	$(1-q^2)^{1/2} V'_{rm}(q)$	$-2V_{rc}(q')(1-q'^2)^{1/2}$	$\sin 2\varphi \cos 2(\varphi+\psi)$	$-\sin 2\psi$
4,4	$V_{rm}(q)$	$\frac{1}{2}\left[(1-q'^2)^2 V''_{rc}(q') - 5q'(1-q'^2)V'_{rc}(q') - V_{rc}(q')(1-3q'^2)\right]$	$\sin 2\varphi \sin 2(\varphi+\psi)$	$\cos 2\psi$
5,4	$-2V_{rc}(q)(1-q^2)^{1/2}$	$(1-q'^2)^{1/2} V'_{rm}(q')$	$\cos 2\varphi \sin 2(\varphi+\psi)$	$\sin 2\psi$
6,4	$V_{rm}(q)$	$2\left[q'V_{rc}(q') - (1-q'^2)V'_{rc}(q')\right]$	$\sin 2\varphi \cos 2(\varphi+\psi)$	$-\sin 2\psi$
5,5	$-\frac{1}{2}\left[q(1-q^2)V'_{rc}(q) + (3-q^2)V_{rc}(q)\right]$	$V_{rm}(q')$	$\sin 2\varphi \sin 2(\varphi+\psi)$	$\cos 2\psi$

TABLE D-2 (Continued)

k,ℓ	$F_{1c}^{(k,\ell)}(q)$	$F_{2c}^{(k,\ell)}(q')$	$g_{2c}^{(k,\ell)}(\varphi,\psi)$	$g_{3c}^{(k,\ell)}(\psi)$
6,5	$-2V_{rc}(q)(1-q^2)^{1/2}$	$-2V_{rc}(q')(1-q'^2)^{1/2}$	$\cos 2\varphi \cos 2(\varphi+\psi)$	$\cos 2\psi$
6,6	$V_{rm}(q)$	$-\frac{1}{2}\left[q'(1-q'^2)V'_{rc}(q') + (3-q'^2)V_{rc}(q')\right]$	$\sin 2\varphi \sin 2(\varphi+\psi)$	$\cos 2\psi$

TABLE D-3

INTEGRAND FUNCTIONS FOR (4-21)

k, ℓ	$F_{1d}^{(k,\ell)}(q)$	$F_{2d}^{(k,\ell)}(q')$	$F_{1r}^{(k,\ell)}(q)$	$F_{2r}^{(k,\ell)}(q')$	$g_{3r}^{(k,\ell)}(\psi)$
0,0	$V_{rs}(q)$	$V_{rs}(q')$	$V_{rm}(q)$	$V_{rm}(q')$	$\cos 2\psi$
1,0	$2V_{rs}(q)$	$V_{rs}(q')$	$2V_{rm}(q)$	$V_{rm}(q')$	$\cos 2\psi$
2,0	$V_{rs}(q)$	$2V_{rs}(q')$	$V_{rm}(q)$	$2V_{rm}(q')$	$\cos 2\psi$
3,0	$(1-q^2)^{1/2} V'_{rs}(q)$	$V_{rs}(q')$	$(1-q^2)^{1/2} V'_{rm}(q)$	$V_{rm}(q')$	$\cos 2\psi$
4,0	$V_{rs}(q)$	$(1-q'^2)^{1/2} V'_{rs}(q')$	$V_{rm}(q)$	$(1-q'^2)^{1/2} \cdot$ $\cdot V'_{rm}(q')$	$\cos 2\psi$
5,0	0	0	$-2(1-q^2)^{1/2} \cdot$ $\cdot V_{rc}(q)$	$V_{rm}(q')$	$\sin 2\psi$
6,0	0	0	$V_{rm}(q)$	$-2(1-q'^2)^{1/2} \cdot$ $\cdot V_{rc}(q')$	$-\sin 2\psi$

TABLE D-3 (Continued)

k,ℓ	$F_{1d}^{(k,\ell)}(q)$	$F_{2d}^{(k,\ell)}(q')$	$F_{1r}^{(k,\ell)}(q)$	$F_{2r}^{(k,\ell)}(q')$	$g_3^{(k,\ell)}(\psi)$
1,1	$V_{rs}(q)$	$V_{rs}(q')$	$V_{rm}(q)$	$V_{rm}(q')$	$\cos 2\psi$
2,1	$2V_{rs}(q)$	$2V_{rs}(q')$	$2V_{rm}(q)$	$2V_{rm}(q')$	$\cos 2\psi$
3,1	$(1-q^2)^{1/2} V'_{rs}(q)$	$V_{rs}(q')$	$2(1-q^2)^{1/2} V'_{rm}(q)$	$V_{rm}(q)$	$\cos 2\psi$
4,1	$2V_{rs}(q)$	$(1-q'^2)^{1/2} \cdot V'_{rs}(q')$	$2V_{rm}(q)$	$(1-q'^2)^{1/2} \cdot V'_{rm}(q')$	$\cos 2\psi$
5,1	0	0	$-2(1-q^2)^{1/2} \cdot V_{rc}(q)$	$V_{rm}(q')$	$\sin 2\psi$
6,1	0	0	$2V_{rm}(q)$	$-2(1-q'^2)^{1/2} \cdot V_{rc}(q')$	$-\sin 2\psi$
2,2	$V_{rs}(q)$	$V_{rs}(q')$	$V_{rm}(q)$	$V_{rm}(q')$	$\cos 2\psi$

TABLE D-3 (Continued)

k, ℓ	$F_{1d}^{(k,\ell)}(q)$	$F_{2d}^{(k,\ell)}(q')$	$F_{1r}^{(k,\ell)}(q)$	$F_{2r}^{(k,\ell)}(q')$	$g_{3r}^{(k,\ell)}(\psi)$
3,2	$(1-q^2)^{1/2} V'_{rs}(q)$	$2V_{rs}(q')$	$(1-q^2)^{1/2} V'_{rm}(q)$	$2V_{rm}(q')$	$\cos 2\psi$
4,2	$V_{rs}(q)$	$(1-q'^2)^{1/2} \cdot V'_{rs}(q')$	$V_{rm}(q)$	$(1-q'^2)^{1/2} \cdot V'_{rm}(q')$	$\cos 2\psi$
5,2	0	0	$-2(1-q^2)^{1/2} \cdot V_{rc}(q)$	$2V_{rm}(q')$	$\sin 2\psi$
6,2	0	0	$2V_{rm}(q)$	$-2(1-q'^2) \cdot V_{rc}(q')$	$-\sin 2\psi$
3,3	$\frac{1}{2}\left[(1-q^2)V''_{rs}(q) - qV'_{rs}(q) + V_{rs}(q)\right]$	$V_{rs}(q')$	$\frac{1}{2}\left[(1-q^2)V''_{rm}(q) - qV'_{rm}(q) + V_{rm}(q)\right]$	$V_{rm}(q')$	$\cos 2\psi$
4,3	$(1-q^2)^{1/2} V'_{rs}(q)$	$(1-q'^2)^{1/2} \cdot V'_{rs}(q')$	$(1-q^2)^{1/2} \cdot V'_{rm}(q)$	$(1-q'^2)^{1/2} \cdot V'_{rm}(q')$	$\cos 2\psi$

TABLE D-3 (Continued)

k, ℓ	$F_{1d}^{(k,\ell)}(q)$	$F_{2d}^{(k,\ell)}(q')$	$F_{1r}^{(k,\ell)}(q)$	$F_{2r}^{(k,\ell)}(q')$	$g_{3r}^{(k,\ell)}(\psi)$
5,3	0	0	$2\left[qV_{rc}(q) - (1-q^2)V'_{rc}(q)\right]$	$V_{rm}(q')$	$\sin 2\psi$
6,3	0	0	$(1-q^2)^{1/2} V'_{rm}(q)$	$-2(1-q'^2)^{1/2} \cdot V_{rc}(q')$	$-\sin 2\psi$
4,4	$V_{rs}(q)$	$\frac{1}{2}\left[(1-q'^2)V''_{rs}(q') - q'V'_{rs}(q') + V_{rs}(q')\right]$	$V_{rm}(q)$	$\frac{1}{2}\left[(1-q'^2)V''_{rm}(q') - q'V'_{rm}(q') + V_{rm}(q')\right]$	$\cos 2\psi$
5,4	0	0	$-2(1-q^2)^{1/2} \cdot V_{rc}(q)$	$(1-q'^2)^{1/2} V'_{rm}(q')$	$\sin 2\psi$
6,4	0	0	$V_{rm}(q)$	$2\left[q'V_{rc}(q') -(1-q'^2)V'_{rc}(q')\right]$	$-\sin 2\psi$

TABLE D-3 (Continued)

k, ℓ	$F_{1d}^{(k,\ell)}(q)$	$F_{2d}^{(k,\ell)}(q')$	$F_{1r}^{(k,\ell)}(q)$	$F_{2r}^{(k,\ell)}(q')$	$g_{3r}^{(k,\ell)}(\psi)$
5,5	$V_{rs}(q) + 2qV_{rd}(q)$	$V_{rs}(q')$	$V_{rc}(q)$	$V_{rm}(q')$	$\cos 2\psi$
6,5	0	0	$-2(1-q^2)^{1/2} \cdot V_{rc}(q)$	$-2(1-q'^2)^{1/2} \cdot V_{rc}(q')$	$\cos 2\psi$
6,6	$V_{rs}(q)$	$V_{rs}(q') + 2q'V_{rd}(q')$	$V_{rm}(q)$	$V_{rc}(q')$	$\cos 2\psi$

APPENDIX E

ψ INTEGRATION

Consider an integral of the following form

$$I(q,q',m,s,a_jb_j) = \int_{-\pi}^{\pi} \cos m\psi \, \rho^s \, e^{-Q_j} \, d\psi \tag{E-1}$$

and let

$$D_1 = \beta \sqrt{1-q^2}\sqrt{1-q'^2} \qquad D_2 = pgqq'$$

Using the definitions of ρ from (4-3) and Q_j from (4-14), then

$$I(q,q',m,s,a_jb_j) = \rho^s(\psi=\pi/2)e^{-Q_j(\rho=0)} \cdot \int_{-\pi}^{\pi} \cos m\psi \, e^{sD_1\cos\psi} e^{-a_jb_jD_2\rho} \, d\psi \tag{E-2}$$

The expansion of the second exponential term in the integrand of (E-2) in a power series yields

$$I(q,q',m,s,a_jb_j) = \rho^s(\psi=\pi/2)e^{-Q_j(\rho=0)} \sum_{n=0}^{\infty} \frac{\left[-a_jb_jD_2\rho(\psi=\pi/2)\right]^n}{n!} \cdot \int_{-\pi}^{\pi} \cos m\psi \, e^{(s+n)D_1\cos\psi} \, d\psi \tag{E-3}$$

Recognizing the symmetry of the integrand of (E-3) and using Abramowitz [1965]

$$\int_0^{\pi} \cos m\psi \; e^{\alpha \cos \psi} d\psi = \pi I_m(\alpha) \qquad \text{(E-4)}$$

where $I_m(\alpha)$ is the modified Bessel function of order m, then

$$I(q,q',m,s,a_j b_j) = 2\pi\rho^s(\psi=\pi/2)e^{-Q_j(\rho=0)} \sum_{n=0}^{\infty} \frac{\left[-a_j b_j D_2 \rho(\psi=\pi/2)\right]^n}{n!}$$

$$\cdot I_m\left[(s+n)D_1\right] \qquad \text{(E-5)}$$

Since $I_m(0) = 0$ for $m \neq 0$, and $D_1 = 0$ for q or q' equal 1

$$I(1,1,m,s,a_j b_j) = \begin{cases} 0 & \text{for } m \neq 0 \\ 2\pi & \text{for } m = 0 \end{cases} \qquad \text{(E-6)}$$

If $D_1 >> m$ and the first term of the large argument expansion of the modified Bessel function is used, then (E-5) becomes

$$I(q,q',m,s,a_j b_j) = B_2(q,q')S(s,\nu=1/2)+M(s,m) \qquad \text{(E-7)}$$

where

$$B_2(q,q',s) = \sqrt{\frac{2\pi}{D_1}} \; \rho^s(\psi=0)e^{-Q_j(\psi=0)}$$

$$S(s,\nu) = e^{+a_j b_j D_2 \rho(\psi=0)} \sum_{n=n_1}^{\infty} \frac{\left[-a_j b_j D_2 \rho(\psi=0)\right]^n}{n! \; (n+s)^{\nu}}$$

$$n_1 = \begin{cases} 0 & \text{if } s \neq 0 \\ 1 & \text{if } s = 0 \end{cases}$$

$$M(s,m) = \begin{cases} 2\pi e^{-Q_j(\rho=0)} & \text{if } s = 0 \text{ and } m = 0 \\ 0 & \text{otherwise} \end{cases}$$

If $D_2\rho(\psi=0) >> 1$, (E-7) further reduces to

$$I(q,q',m,s,a_jb_j) \cong \frac{B_2(q,q')}{\sqrt{D_2\rho(\psi=0) + s}} \tag{E-8}$$

which is the first term of the saddle point solution of (E-1).

Now consider another somewhat more general form of a ψ integral for the case of $q = q' = q_1$.

$$I_a(q_1,k,\ell,r,s,t,a_jb_j) = \int_{-\pi}^{\pi} \frac{(\cos k\,\psi)(1-\cos\psi)^{\ell}(1-\rho)^r\rho^s e^{-Q_j}d\psi}{(1-q_1^2)^t} \tag{E-9}$$

For this case

$$D_1 = \beta(1-q_1^2) \qquad D_2 = pgq_1^2$$

$$\rho(\psi = 0) = 1 \qquad e^{-Q_j(\psi=0)} = e^{-D_2(1+a_jb_j)}$$

$$\rho(\psi = \tfrac{\pi}{2}) = e^{-D_1} \qquad e^{-Q_j(\psi=\frac{\pi}{2})} = e^{-D_2(1+a_jb_je^{-D_1})} \tag{E-10}$$

Two separate cases will be considered. First for q_1 approaching one so that D_1 is small and secondly for q_1 such that D_1 is very large.

For D_1 small, using the power series expansion of ρ then

$$(1-\rho) = -\sum_{u=1}^{\infty} \frac{(-\beta)^u(1-q_1^2)^u(1-\cos\psi)^u}{u!} \tag{E-11}$$

The r^{th} power of $(1-\rho)$ can be expressed in the following form

$$(1-\rho)^r = \sum_{u=0}^{\infty} d_u^{\ r} [\beta (1-q_1^{\ 2}) \ (1-\cos \psi)]^{r+u} \qquad (E-12)$$

where $d_o^r - 1,\ d_1^1 = -\frac{1}{2}$

Then

$$I_a(q_1 \to 1, k, \ell, r, s, t, a_j b_j) = \sum_{u=0}^{\infty} d_u^r \frac{\beta^{r+u}}{(1-q_1^2)^{t-r-u}} \int_{-\pi}^{\pi} \cos k\psi (1-\cos\psi)^{\ell+r+u} \rho^s e^{-Q_j} d\psi \qquad (E-13)$$

Using the binomial expansion on the $(1-\cos \psi)$ term and the Fourier expansion on the powers of $\cos \psi$, one obtains

$$(1-\cos \psi)^{\ell+r+u} = \sum_{m=0}^{\ell+r+u} a_m^{(\ell+r+u)} \cos m\psi \qquad (E-14)$$

and

$$\cos k\psi \ (1-\cos \psi)^{\ell+r+u} = \sum_{m=0}^{\ell+r+u} \frac{a_m^{(\ell+r+u)}}{2} [\cos (k+m)\psi + \cos (k-m)\psi] \qquad (E-15)$$

The substitution of (E-15) into (E-13) gives

$$I_a(q_1 \to 1, k, \ell, r, s, t, a_j b_j) = \sum_{u=0}^{\infty} \sum_{m=o}^{\ell+r+u} \frac{d_u^r \ \beta^{r+u} a_m^{(\ell+r+u)}}{(1-q^2)^{t-r-u} \ 2} \cdot$$

$$\cdot \left\{ \int_{-\pi}^{\pi} \cos(k+m)\psi \ \rho^s e^{-Q_j} d\psi + \int_{-\pi}^{\pi} \cos(k-m) \ \rho^s e^{-Q_j} d\psi \right\} \qquad (E-16)$$

From the results of (E-5) realizing that as $q_1 \to 1$, $\rho(\psi/2) \to 1$ and using the small argument expansion of the modified Bessel function, (E-16) becomes

$$I_a(q_1 \to, k, \ell, r, s, t, a_j b_j) = 2\pi \sum_{u=0}^{\infty} \sum_{m=0}^{\ell+r+u} \frac{d_u^r \, \beta^{r+u} a_m^{(\ell+r+u)}}{(1-q_1^2)^{\frac{t-r-u}{2}}} \cdot$$

$$\cdot \left\{ \frac{\beta^{k+m}(1-q_1^2)^{k+m}}{2^{k+m}} \sum_{v=0}^{\infty} \frac{\beta^{2v}(1-q_1^2)^{2v}}{2^{2v} v!(v+k+m)!} \cdot \right.$$

$$\cdot \left[e^{a_j b_j D_2} \sum_{n=0}^{\infty} \frac{[-a_j b_j D_2]^n}{n!} (n+s)^{k+m+2v} \right]$$

$$+ \frac{\beta^{|k-m|}(1-q_1^2)^{|k-m|}}{2^{|k-m|}} \cdot \sum_{v=0}^{\infty} \frac{\beta^{2v}(1-q_1^2)^{2v}}{2^{2v} v!(v+|k-m|)!} \cdot$$

$$\left. \left[e^{a_j b_j D_2} \sum_{n=0}^{\infty} \frac{[-a_j b_j D_2]^n}{n!} (n+s)^{|k-m|+2v} \right] \right\} \qquad \text{(E-17)}$$

Using the expansion

$$(n+s)^m = \sum_{i=0}^{m} b_i^{(m,s)} \frac{n!}{(n-i)!}$$

where $b_o^{(m,s)} = s^m$ (E-18)

The substitution of (E-18) into (E-17) yields

$$I_a(q_1 \to 1, k, \ell, r, s, t, a_j b_j) = 2\pi \sum_{u=o}^{\infty} \sum_{m=0}^{\ell+r+u} \frac{d_u^r \beta^{r+u} a_m^{(\ell+r+u)}}{2(1-q_1^2)^{t-r-u}} \cdot$$

$$\cdot \left\{ \frac{\beta^{k+m}(1-q_1^2)^{k+m}}{2^{k+m}} \sum_{v=0}^{\infty} \frac{\beta^{2v}(1-q^2)^{2v}}{2^{2v}\, v!\,(v+k+m)!} \cdot \right.$$

$$\cdot \left[\sum_{i=0}^{k+m+2v} b_i^{(k+m+2v,s)} (-a_j b_j D_2)^i \right]$$

$$+ \frac{\beta^{|k-m|}(1-q_1^2)^{|k-m|}}{2^{|k-m|}} \sum_{v=0}^{\infty} \frac{\beta^{2v}}{2^{2v}} \frac{(1-q_1^2)^{2v}}{v!(v+|-m|)!} \cdot$$

$$\left. \cdot \left[\sum_{i=0}^{|k-m|+2v} b_i^{(|k-m|+2v,s)} (-a_j b_j D_2)^i \right] \right\} \qquad \text{(E-19)}$$

Realizing that as $q_1 \to 1$ any terms containing $(1-q_1^2)$ in the numerator will vanish then

$$I_a(1,k,\ell,r,s,t,a_j b_j) = \begin{cases} 0 & \text{for } r > t \text{ or } k-\ell > t \\ \pi\, a_k^{(\ell+r)} \beta^r & r=t \quad k > 0 \\ 2\pi\, a_o^{(\ell+r)} \beta^r & r=t \quad k \doteq 0 \end{cases} \qquad \text{(E-20)}$$

For the remaining cases all terms containing $(1-q_1^2)$ terms in the denominator are expected to cancel out in the integral of Chapter 4. For D_1 large, the binomial expansion will be used in the $(1-\rho)$ term and the large argument expansion will be used for the modified Bessel function. It can be numerically shown that the $a_m^{(\ell)}$ terms have the property

$$\sum_{m=0}^{\ell} m^{2w} a_m \equiv 0 \text{ for } 0 < w < \ell \tag{E-21}$$

This property is probably due to a property of the binomial coefficients. i.e.,

$$\sum_{j=0}^{n} (-1)^j C_j^n j^v = \begin{cases} 0 & \text{for } v < n-1 \\ (-1)^r r! & \text{for } v=n \end{cases} \tag{E-22}$$

Then using (E-21)

$$\sum_{m=0}^{\ell} \frac{a_m}{2} [(m+\ell)^{2w} + (m-\ell)^{2w}] = \sum_{m=0}^{\ell} m^{2\ell} a_m \quad \text{for } w \le \ell \tag{E-23}$$

Then I_a becomes

$$I_a(q_1,k,\ell,r,s,t,a_j b_j) = \sum_{u=0}^{r} (-1)^u \frac{C_u^r B_2(q_1,q_1,s+u) \sum_{m=0}^{\ell} m^{2\ell} a_m}{\ell!\, 2^\ell \beta^\ell (1-q_1^2)^\ell}$$

$$\cdot S[(s+u), (\ell + \tfrac{1}{2})] \tag{E-24}$$

If $D_2 >> 1$

$$S[(s+u),\ (\ell+\tfrac{1}{2})] \cong \frac{1}{(D_2+s+u)^{\ell+\frac{1}{2}}} = \sum_{j=0}^{\infty} \frac{(-1)^j\ \Gamma(\ell+\frac{1}{2}+j)}{j!\ \Gamma(\ell+\frac{1}{2})}\ .$$

$$\cdot \left(\frac{s+u}{D_2}\right)^j \quad \frac{1}{D_2^{\ell+1/2}} \tag{E-25}$$

Substituting (E-25) into (E-24) and expanding (s+u) in a binomial expansion

$$I_a(q_1,k,\ell,r,s,t,a_jb_j) = \sum_{j=0}^{\infty} \sum_{i=0}^{j} \frac{B_2(q_1,q_1,s+u) \sum_{m=0}^{\ell} m^{2\ell} a_m}{\ell!\ 2^\ell \beta^\ell (1-q_1{}^2)^\ell\ D_2^{\ell+\frac{1}{2}}} \cdot$$

$$\cdot \frac{(-1)^j\ \Gamma(\ell+\frac{1}{2}+j)\ C_i^j\ s^{j-i}}{j!\ \Gamma(\ell+\frac{1}{2})\ D_2^j} \sum_{u=0}^{r} (-1)^u\ C_u^r u^i \tag{E-26}$$

Since the last summation is 0 for $j < r$ by (E-22) and if $D_2 >> 1$ using only the first term of the j series

$$I_a(q_1,k,\ell,r,s,t,a_jb_j) \cong \frac{B_2(q_1,q_1,s) \sum_{m=0}^{\ell} m^{2\ell} a_m\ \Gamma(\ell+\frac{1}{2}+r)}{\ell!\ 2^\ell \beta^\ell (1-q_1{}^2)^\ell\ D_2^{\ell+r+\frac{1}{2}}\ \Gamma(\ell+\frac{1}{2})} \tag{E-27}$$

If r=0, then

$$I_a(q_1,k,\ell,o,s,t,a_jb_j) \cong \frac{B_2(q_1,q_1,s) \sum_{m=0}^{\ell} m^{2\ell}\ a_m}{\ell!\ 2^\ell\ \beta^\ell (1-q_1{}^2)^\ell\ (D_2+s)^{\ell+\frac{1}{2}}} \tag{E-28}$$

APPENDIX F

PARTIAL DERIVATIVES OF COVARIANCE FUNCTION

This appendix contains the partial derivatives needed to evaluate the G functions of Appendix D. From (4-3)

$$\rho = e^{-\beta[1-qq' - \sqrt{1-q^2}\ \sqrt{1-q'^2}\ \cos\psi]} \qquad \text{(F-1)}$$

The following list contains the partials of ρ after setting $q = q' = q_1$.

$$\left.\frac{\partial\rho}{\partial q}\right|_{q=q'=q_1} = \left.\frac{\partial\rho}{\partial q'}\right|_{q=q'=q_1} = \beta q_1(1-\cos\psi)\rho$$

$$\left.\frac{\partial^2\rho}{\partial q^2}\right|_{q=q'=q_1} = \left.\frac{\partial^2\rho}{\partial q'^2}\right|_{q=q'=q_1} = \rho\left[\beta^2 q_1^{\ 2}(1-\cos\psi)^2 - \frac{\beta\cos\psi}{(1-q_1^{\ 2})}\right]$$

$$\left.\frac{\partial^2\rho}{\partial q\partial q'}\right|_{q=q'=q_1} = \rho\left\{\beta^2 q_1^{\ 2}(1-\cos\psi)^2 + \beta\left[1 + \frac{q_1^{\ 2}\cos\psi}{(1-q_1^{\ 2})}\right]\right\}$$

$$\left.\frac{\partial^3\rho}{\partial q^3}\right|_{q=q'=q_1} = \left.\frac{\partial^3\rho}{\partial q'^3}\right|_{q=q'=q_1} = \rho\left[\beta^3 q_1^{\ 3}(1-\cos\psi)^3 - \frac{3\beta^2 q_1\cos\psi(1-\cos\psi)}{(1-q_1^{\ 2})} - \frac{3\beta q_1\cos\psi}{(1-q_1^{\ 2})^2}\right]$$

$$\left.\frac{\partial^3\rho}{\partial q^2\partial q'}\right|_{q=q'=q_1} = \left.\frac{\partial^3\rho}{\partial q\partial q'^2}\right|_{q=q'=q_1} = \rho\Big\{\beta^3 q_1^{\,3}(1-\cos\psi)^3 - \beta^2 q_1(1-\cos\psi)\cdot$$

$$\cdot\left[2 - \cos\psi + \frac{q_1^{\,2}\cos\psi}{(1-q_1^{\,2})}\right] + \frac{\beta\, q_1\cos\psi}{(1-q_1^{\,2})^2}\Big\}$$

$$\left.\frac{\partial^4\rho}{\partial q^4}\right|_{q=q'=q_1} = \left.\frac{\partial^4\rho}{\partial q'^4}\right|_{q=q'=q_1} = \rho\Big\{\beta^4 q_1^{\,4}(1-\cos\psi)^4 - \frac{6\beta^3 q_1^{\,2}(1-\cos\psi)^2\cos\psi}{(1-q_1^{\,2})^3}$$

$$+ 3\beta^2\left[-4\;\frac{q_1^{\,2}\cos\psi(1-\cos\psi) + 3\cos^2\psi}{(1-q_1^{\,2})^2}\right] - \frac{3\beta(1+4q_1^{\,2})\cos\psi}{(1-q_1^{\,2})^3}\Big\}$$

$$\left.\frac{\partial^4\rho}{\partial q^3\partial q'}\right|_{q=q'=q_1} = \left.\frac{\partial^4\rho}{\partial q\partial q'^3}\right|_{q=q'=q_1} = \rho\Big\{\beta^4 q_1^{\,4}(1-\cos\psi)^4$$

$$+ 3\beta^3 q_1^{\,2}(1-\cos\psi)^3 - \frac{3\beta^2\cos\psi}{(1-q_1^{\,2})}\left[1 + \frac{q_1^{\,2}\cos\psi}{1-q_1^{\,2}}\right]$$

$$+ \frac{3\beta q_1^{\,2}\cos\psi}{(1-q_1^{\,2})^3}\Big\}$$

$$\left.\frac{\partial^4 \rho}{\partial q^2 \partial q'^2}\right|_{q=q'=q_1} = \rho\Big\{\beta^4 q_1^{\,4}(1-\cos\psi)^4 + 2\beta^3 q_1^{\,2}(1-\cos\psi)^2 \cdot$$

$$\cdot\left[2 - \cos\psi + \frac{q_1 \cos\psi}{1-q_1^{\,2}}\right]$$

$$+\beta^2\left[+\frac{4q_1^{\,2}(1-\cos\psi)\cos\psi + \cos^2\psi}{(1-q_1^{\,2})^2} + \left(1 + \frac{q_1^{\,2}\cos\psi}{1-q_1^{\,2}}\right)^2\right]$$

$$+ \frac{\beta\cos\psi}{(1-q_1^{\,2})^3}\Big\} \qquad \text{(F-2)}$$