ON MASS TRANSFER BETWEEN TWO PARALLEL STREAMS*

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Abstract

The convective and diffusive mass transfer between two parallel streams of gases is studied in this paper. When the two gases in the freestreams are of different densities, the flow field is disturbed due to mass diffusion. Solutions for mass concentration, flow velocity, and density are obtained in closed forms. Mass transfer between two parallel streams in transition regime is also studied, slip boundary condition is used, its solution shows a smooth transition between the free-molecule solution and the continuum solution.

1. Introduction

We consider two parallel streams of gases moving horizontally in the same direction. The x-axis is drawing horizontally in the direction of motion of the freestreams and y-axis vertically upwards, the origin is taken as the point at which the two flows are supposed first to come into contact. When the two flows are moving at different velocities, the laminar boundary layer between parallel streams has been studied by Lock[1]. In this paper we will study the simultaneous convective and diffusive mass transfer between the two parallel streams of gases. In order to investigate the main features of mass transfer between two flows, we consider that the freestreams of the two flows are moving at same speed and have same pressure and temperature.

Let the suffix A refer to flow conditions in the upper freestream and B to those in the lower, then $u_A = u_B = U$, $p_A = p_B$ and $T_A = T_B$. The boundary layer approximation is used in the analysis, since the concentration profile changes very rapidly inside a layer of small thickness, we will neglect the term $\partial^2 c / \partial x^2$ in comparison with $\partial^2 c / \partial y^2$. We also assume that the flows are subsonic in the two freestreams.

The effect of mass transfer on the flow field will be studied in this paper. When the gases in the two freestreams are of different densities, $\rho_A \neq \rho_B$, diffusion between two gases causes the mass of the gas mixture to move towards the side of lighter gas; this vertical motion causes the streamlines of the mean mass flow to deflect. Solutions of the mass diffusion equation, the continuity equation and the equation of motion are obtained to describe the change of concentration, density, and velocity of the flow field; we also obtain solution to describe streamlines as function of space variables.

We will also study mass transfer between two parallel streams in transition regime. A slip boundary condition is used to obtain the analytical solution which possesses a smooth transition between the free-molecule solution and the continuum solution. The thickness of the diffusion layers is calculated. All solutions obtained in the paper are in closed form. Analogous to mass diffusion in transition regime, a similarity solution is obtained for heat transfer in transition regime between two parallel streams of different temperature.

2. Mass Transfer between Two Streams of Gases with Equal Densities

If the densities in the freestreams of the two parallel flows are equal $\rho_A = \rho_B$, then the density of gas mixture would remain constant throughout the flow field. We use the non-dimensional variables

$$\xi = xU/D \qquad \text{and} \qquad \eta = yU/D$$

where D is the mass diffusion coefficient. The equation of mass diffusion can be written as

$$\frac{\partial c}{\partial \xi} = \frac{\partial^2 c}{\partial \eta^2} \tag{1}$$

where c is the concentration of gas A. The boundary conditions are:

$$\begin{aligned} c(\xi = 0, \eta > 0) &= c(\xi > 0, \eta \to \infty) = 1 \\ c(\xi = 0, \eta < 0) &= c(\xi > 0, \eta \to -\infty) = 0 \end{aligned} \tag{2}$$

Consider that the concentration and the diffusion mass flux be continuous across the streamline passing through the origin, the exact solution of the problem can be obtained by means of Laplace transformation method

$$\begin{aligned} c &= 1/2 + (1/2)\mathrm{erf}\left[\eta/(2\xi^{1/2})\right] \qquad \text{for} \quad \eta > 0 \\ c &= 1/2 - (1/2)\mathrm{erf}\left[-\eta/(2\xi^{1/2})\right] \qquad \text{for} \quad \eta < 0 \end{aligned} \tag{3}$$

The mass diffusion flux of gas A across the streamline passing through the origin is

$$\dot{m} = -(1/2)\, \rho_A U (\pi \xi)^{-1/2} \tag{4}$$

3. Mass Transfer between Two Streams of Gases with Different Densities

If the gases in the freestreams of the two parallel flows are of different densities $\rho_A \neq \rho_B$, then when the two flows come into contact, the density of gas mixture will change from ρ_A to ρ_B through a mass-diffusion layer. In addition to the diffusion of mass from one stream to another, the variation of mass density will disturb the flow field. As we have assumed that the two gases in the freestreams of the two flows are at the same temperature $T_A = T_B$, and the same pressure, $p_A = p_B$, from the equation of state

$$p = n k T \tag{5}$$

we obtain $n_A = n_B = N$ and $\rho_A / \rho_B = m_A / m_B$ where m denotes the particles mass. The x-component of the flow velocity, U, is virtually uneffected due to mass diffusion. As the freestream Mach numbers are less than one, we can consider that the temperature remains constant throughout the flow field. The governing equations are

$$\begin{aligned} \rho U \frac{\partial c}{\partial x} + \rho v \frac{\partial c}{\partial y} &= \frac{\partial}{\partial y}\left(\rho D \frac{\partial c}{\partial y}\right) \\ U \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial y} + \rho \frac{\partial v}{\partial y} &= 0 \\ \rho U \frac{\partial v}{\partial x} + \rho v \frac{\partial v}{\partial y} &= -\frac{\partial p}{\partial y} \end{aligned} \tag{6}$$

We shall define the stream function, which satisfies the equation of continuity in (6), in non-dimensional form as

$$\partial\psi/\partial\eta = \rho/\rho_A \qquad \text{and} \qquad \partial\psi/\partial\xi = -\rho v/(\rho_A U)$$

where $\xi = x D_A / U$ and $\eta = y D_A / U$. We may, in von Mises's fashion [(2)], take ξ and ψ as independent variables instead of x and y. Assume that D varies inversely as a second power of the density ρ , then we can write (6) as

$$\frac{\partial c}{\partial \xi} = \frac{\partial^2 c}{\partial \psi^2} \tag{7}$$

$$\frac{\partial}{\partial \xi}(\rho_A/\rho) = \frac{\partial}{\partial \psi}(v/U) \tag{8}$$

$$\frac{\partial}{\partial \xi}(v/U) = -\frac{1}{\rho_A U^2}\frac{\partial p}{\partial \psi} \tag{9}$$

The boundary conditions for c are

$$c(\xi = 0,\ \psi > 0) = c(\xi > 0,\ \psi \to \infty) = 1$$

and

$$c(\xi = 0,\ \psi < 0) = c(\xi > 0,\ \psi \to -\infty) = 0$$

The concentration c is described here by the same differential equation and boundary conditions as in the simple case descussed in section 2. Thus we can write the solution for c as

$$c = 1/2 + (1/2)\,\mathrm{erf}\,[\psi/(2\xi^{1/2})] \quad \text{for } \psi > 0$$

$$(10)$$

$$c = 1/2 - (1/2)\,\mathrm{erf}\,[-\psi/(2\xi^{1/2})] \quad \text{for } \psi < 0$$

As we have obtained the solution for c as function of ξ and ψ, we shall now study solutions for other dependent variables as functions of ξ and ψ, then we will find the expression for ψ as function of ξ and η. The density of the gas mixture ρ related to n and c by

$$n/N = (\rho/\rho_A)\left[c + (m_A/m_B)(1-c)\right]$$

Making use of this relation and equation of state (5) we can write the two differential equations, (8) and (9), as

$$\frac{\partial}{\partial \xi}(v/U) = -\frac{kT}{m_A U^2}\frac{\partial}{\partial \psi}(n/N)$$

$$\frac{\partial}{\partial \psi}(v/U) = \left(\frac{m_A}{m_B} - \frac{m_A - m_B}{m_B}c\right)\frac{\partial}{\partial \xi}(N/n) + \frac{m_B - m_A}{m_A}\frac{N}{n}\frac{\partial c}{\partial \xi} \tag{11}$$

Now we write $v = v_0 + v'$ and $n = N + n'$ where v_0 is defined as

$$v_0 = \frac{m_B - m_A}{m_B} U \frac{\partial c}{\partial \psi}$$

$$= \frac{m_B - m_A}{m_B}\frac{U}{2(\pi\xi)^{1/2}}\exp(-\psi^2/4\xi) \tag{12}$$

If $|v'/v_0| << 1$ and $|n'/N| << 1$ then equation (11) can be linearized. Later we will show that indeed $|v'/v_0|$ and $|n'/N|$ are very small compared with unity in the continuum regime [equation (15), (17) and (18)]. The linearized equations are

$$\frac{\partial}{\partial \xi}\left(v_0/U\right) = -\frac{kT}{m_A U^2}\frac{\partial}{\partial \psi}\left(n'/N\right)$$

$$\frac{\partial}{\partial \psi}\left(v'/U\right) = \frac{m_A - m_B}{m_B}\,\frac{n'}{N}\,\frac{\partial c}{\partial \xi} + \left(\frac{m_A - m_B}{m_B}\,c - \frac{m_A}{m_B}\right)\frac{\partial}{\partial \xi}\left(n'/N\right) \tag{13}$$

The exact solutions of (13) are

$$\frac{n'}{N} = \frac{m_B - m_A}{m_B}\,\frac{m_A U^2}{kT}\,\frac{\psi}{4\pi^{1/2}\xi^{3/2}}\exp\left(-\psi^2/4\xi\right)$$

$$\frac{v'}{U} = \left(\frac{m_A - m_B}{m_B}\right)^2 \frac{m_A U^2}{kT}\,\frac{(-1)}{8\pi^{1/2}\xi^{3/2}}\left[1 - \operatorname{erf}\frac{\psi}{(2\xi)^{1/2}} + \left(\frac{\psi^2}{2\xi} - 1\right)\exp\left(\frac{-\psi^2}{4\xi}\right)\left(\frac{m_A + m_B}{m_A - m_B} - \operatorname{erf}\frac{\psi}{2\xi^{1/2}}\right)\right]$$

$$\text{for} \quad \psi > 0 \tag{14}$$

$$\frac{v'}{U} = \left(\frac{m_A - m_B}{m_B}\right)^2 \frac{m_A U^2}{kT}\,\frac{(-1)}{8\pi^{1/2}\xi^{3/2}}\left[1 + \operatorname{erf}\frac{-\psi}{(2\xi)^{1/2}} + \left(\frac{\psi^2}{2\xi} - 1\right)\exp\left(\frac{-\psi^2}{4\xi}\right)\left(\frac{m_A + m_B}{m_A - m_B} + \operatorname{erf}\frac{-\psi}{2\xi^{1/2}}\right)\right]$$

$$\text{for} \quad \psi < 0$$

Using the solutions for c and n'/N we can calculate the density variation

$$\frac{\rho}{m_A N} = \frac{2 + 2(1 - m_A/m_B)\,m_A U^2 \psi\,(4kT)^{-1}\pi^{-1/2}\xi^{-3/2}\exp\left[-\psi^2/(4\xi)\right]}{1 + m_A/m_B + (1 - m_A/m_B)\operatorname{erf}\left[\psi/(2\xi^{1/2})\right]} \quad \text{for } \psi > 0$$

$$\frac{\rho}{m_A N} = \frac{2 + 2(1 - m_A/m_B)\,m_A U^2 \psi\,(4kT)^{-1}\pi^{-1/2}\xi^{-3/2}\exp\left[-\psi^2/(4\xi)\right]}{1 + m_A/m_B - (1 - m_A/m_B)\operatorname{erf}\left[-\psi/(2\xi^{1/2})\right]} \quad \text{for } \psi < 0 \tag{15}$$

Note that solution (14) (15) are valid only when the linearization assumptions are fulfilled. The upper bounds for $|n'/N|$ and $|v'/U|$ are

$$|n'/N| \le |m_A - m_B|\,(m_A/m_B)\,U^2 (2kT)^{-1}(2\pi e)^{-1/2}\,\xi^{-1}$$

$$|v'/U| < (m_A - m_B)^2 (m_A/m_B^2)\,U^2 (4kT)^{-1}\,\pi^{-1/2}\xi^{-3/2}\left(1 + 2e^{-3/2}\right) \tag{16}$$

where e is the base of natural logarithms. From (12) and (16), we obtain that the ratio $|v'/v_0|$ can be estimated by

$$|v'/v_0| \sim |m_A - m_B| (m_A/m_B)\, U^2 (2kT\xi)^{-1} \left(1 + 2e^{-3/2}\right) \tag{17}$$

From kinetic theory, the diffusion coefficient[3] is related to the mean free path ℓ and the speed of sound a by

$$D \cong 0.75\, \gamma^{-1/2}\, \ell\, a$$

where γ is the ratio of specific heats. Using this relation we can write

$$m_A U^2 (2kT\xi)^{-1} \cong 0.4\, \gamma^{1/2}\, M \ell / x \tag{18}$$

where M is the flow Mach number. In the region where $x \gg M\ell$, equations (15), (17) and (18) show that the linearization assumptions $|v'/v_0| \ll 1$ and $|n'/N| \ll 1$ are justified. Therefore equation (12) can be used to represent the vertical component of the flow velocity; and integration of (12) will give solution for ψ as function of ξ and η. In the freestreams $\eta = \eta_0 = \psi$ for the upper flow and $\eta = \eta_0 = (m_B/m_A)\psi$ for the lower flow. For $\xi \geq 0$ and $\eta_0 > 0$

$$\eta - \eta_0 = \frac{m_B - m_A}{m_B} \left(\frac{\xi}{\pi}\right)^{1/2} \exp\left(\frac{-\eta_0^2}{4\xi}\right) + \frac{m_A - m_B}{2 m_B}\, \eta_0\, \mathrm{erfc}\left(\frac{\eta_0}{2\xi^{1/2}}\right) \tag{19}$$

and for $\xi \geq 0$ and $\eta_0 < 0$

$$\eta - \eta_0 = \frac{m_B - m_A}{m_B}\left(\frac{\xi}{\pi}\right)^{1/2}\exp\left(-\frac{m_A^2\,\eta^2}{4\,m_B^2\,\xi}\right) + \frac{m_A - m_B}{2\,m_B}\,\frac{m_A}{m_B}\,\eta_0\,\mathrm{erfc}\left(\frac{-m_A\,\eta_0}{2 m_B\,\xi^{1/2}}\right) \tag{20}$$

The streamline passing through the origin is

$$\eta = \left(1 - m_A/m_B\right)\left(\xi/\pi\right)^{1/2} \tag{21}$$

The mass of the flow moves vertically towards the side of lighter gas. The speed of this vertical mass movement is directly proportional to the ratio $\left(1 - m_A/m_B\right)$ which equals to $\left(1 - \rho_A/\rho_B\right)$. When the two streams are at the same densities, v_0, n' and v' are all identically zero; the problem reduces to the simple case treated in section 2.

4. Mass Diffusion in Transition Regime

In sections 2 and 3, the problems are described by the boundary condition that the concentration and the mass diffusion flux be continuous across the streamline passing through the origin. The solutions for the flow field [equations (12), (14)] obtained in section 3 are valid in the continuum regime where the distance x is considerably larger than the mean free path. The diffusion mass flux across the streamline passing through the origin [equation (4)] also tend to infinity near the origin. In the regime where x is very much smaller than the mean free path, intermolecular collisions can be neglected, the flow is called a free-molecule flow[4,5]. In the free-molecule regime, if the particle distribution functions are Maxwellian in the freestreams, then the mass flux of gas A across the interface between the two parallel streams is

$$\dot{m}_{\text{free-molecule regime}} = -\,N m_A\,(kT)^{1/2}(2\pi m_A)^{-1/2}$$

If we introduce a non-dimensional mass flux

$$Q = -\dot{m}/N \ (2\pi)^{1/2} (m_A k T)^{-1/2} \tag{22}$$

then in the two limiting regimes we have

$$Q_{\text{free-molecule regime}} = 1 \tag{23}$$

$$Q_{\text{continuum regime}} = U m_A^{1/2} (2kT\xi)^{-1/2} \tag{24}$$

Between the free molecule and the continuum regime, the flow may be called transition flow. Here we will study the mass transfer between two parallel streams in transition regime by mean of a "slip" boundary condition. This slip boundary condition may be stated as that the concentration has a jump Δc across the streamline passing through the origin, but the mass flux across that streamline is continuous and its magnitude is proportional to the concentration jump Δc.

To study the mass transfer in transition regime we will consider a simple case in which the gases in the two streams are of equal densities.

In addition to equation (1) and the boundary condition (2) we may write the slip condition in non-dimensional form as

$$\left(\frac{\partial c}{\partial \eta}\right)_{\eta = 0_\pm} = \frac{A}{2} \Delta c = \frac{A}{2} \left[(c)_{\eta = 0+} - (c)_{\eta = 0-} \right] \tag{25}$$

where A is a constant to be determined later. The exact solution of equation (1) subject to boundary conditions (2) and (25) can be obtained by the method of Laplace transformation

$$
\begin{aligned}
c &= 1-(1/2)\exp(-\eta^2/4\xi)\left\{F[\eta/(2\xi^{1/2})] - F[A\xi^{1/2}+\eta/(2\xi^{1/2})]\right\} \quad \text{for } \eta>0 \\
c &= (1/2)\exp(-\eta^2/4\xi)\left\{F[-\eta/(2\xi^{1/2})] - F[A\xi^{1/2}-\eta/(2\xi^{1/2})]\right\} \quad \text{for } \eta<0
\end{aligned}
\tag{26}
$$

where $$F(Z) = \exp(Z^2)\,\mathrm{erfc}\,Z \tag{27}$$

From (26) we can calculate the concentration jump

$$\Delta c = F(A\xi^{1/2}) \tag{28}$$

and the non-dimensional mass flux (22)

$$Q = UA(\pi m_A)^{1/2}(2kT)^{-1/2}F(A\xi^{1/2})$$

By requiring that this solution reduce to equation (23) in the limit of free-molecule regime, we can determine the constant A

$$A = (2kT)^{1/2}(\pi m_A)^{-1/2}U^{-1}$$

Therefore

$$Q = F(A\xi^{1/2}) \tag{29}$$

In the limit of continuum regime (29) reduces to (24). This means making use of the slip boundary condition we can obtain a solution (29) for mass flux across the streamline passing throught the origin, this solution possesses a smooth transition between the free-molecule solution and the continuum solution [Fig. 1]. From (29) we can calculate that in the region $x > 100\ M\ell$, the no-slip solution (24) is accurate to within 0.8

percent; the influence of concentration slip can be neglected in that region.

The concentration profile changes smoothly from $c = 1$ in the upper main stream to $c = (1/2)(1 + \Delta c)$ at $y = 0_+$, it jumps to $c = (1/2)(1 - \Delta c)$ at $y = 0_-$, then it decreases smoothly to $c = 0$ in the lower main stream. The concentration profile actually changes very rapidly from $c = 1$ to $c = (1/2)(1 + \Delta c)$ and from $c = (1/2)(1 - \Delta c)$ to $c = 0$ in a narrow layer outside of which the concentration changes asymptotically and the influence of mass diffusion is inperceptible. If we call the region in which 99 percent of concentration change takes place the mass diffusion layer, then the thickness of mass diffusion layer, δ, may be defined as the distance from the streamline $y = 0$ for which

$$0.99 = \frac{c - (1/2)(1+\Delta c)}{1 - (1/2)(1+\Delta c)}$$

then from (26) we obtain that when $y = \delta$

$$0.01 = \exp(-\eta^2/4\xi)\,\frac{F[\eta/(2\xi^{1/2})] - F[A\xi^{1/2} + \eta/(2\xi^{1/2})]}{F(0) - F(A\xi^{1/2})} \tag{30}$$

where F is defined by equation (27). When ξ tends to infinity, equation (30) reduces to

$$0.01 = \operatorname{erfc}\left[\eta/(2\xi^{1/2})\right]$$

or

$$\eta/\xi^{1/2} = 3.643 \tag{31}$$

When ξ tends to zero, equation (30) reduces to

$$0.01 = \mathrm{erfc}\left[\eta/(2\xi^{1/2})\right]$$

or $$\eta / \xi^{1/2} = 3.213 \tag{32}$$

From (31) and (32), we can express the thickness of mass diffusion layer as

$$\delta / x = \epsilon \left[D/(Ux)\right]^{1/2} \tag{33}$$

where the value of ϵ is between 3.21 and 3.64, ϵ equals to 3.21 at the free molecule regime and 3.64 the continuum regime.

5. Heat Transfer between Two Streams

The solution obtained in section 4 can be used to study the heat transfer between two parallel streams in transition regime. If we consider that the two main streams have the same velocities and the same densities but they have different temperatures T_A and T_B. In the transition regime we may use the slip condition[6] that the temperature has a jump across the interface between the two streams, but the heat flux across the interface is continuous and its magnitude is proportional to the temperature jump. Write θ for $(T - T_B)/T_A - T_B)$, ξ' and η' for xU/α and yU/α where α is the thermal diffusivity, the problem can be described by the following differential equation and boundary conditions

$$\frac{\partial \theta}{\partial \xi'} = \frac{\partial^2 \theta}{\partial \eta'^2} \tag{34}$$

$$\theta(\xi'=0,\ \eta'>0)=\theta(\xi'>0,\ \eta'\to\infty)=1$$

$$\theta(\xi'=0,\ \eta'<0)=\theta(\xi'>0,\ \eta'\to-\infty)=0 \qquad (35)$$

and

$$\left(\frac{\partial\theta}{\partial\eta'}\right)_{\eta'=0_\pm}=\frac{A'}{2}\left[(\theta)_{\eta'=0_+}-(\theta)_{\eta'=0_-}\right] \qquad (36)$$

Here θ must satisfy the same differential equation and the same boundary conditions as c for mass transfer in transition regime. Dropping the dashes, we can write the similar solution for θ as

$$\theta=1-(1/2)\exp(-\eta^2/4\xi)\left\{F[\eta/(2\xi^{1/2})]-F[A\xi^{1/2}+\eta/(2\xi^{1/2})]\right\} \quad \text{for } \eta>0$$

$$\theta=(1/2)\exp(-\eta^2/4\xi)\left\{F[-\eta/(2\xi^{1/2})]-F[A\xi^{1/2}-\eta/(2\xi^{1/2})]\right. \quad \text{for } \eta<0 \qquad (37)$$

and

$$\frac{2}{A}\left(\frac{\partial\theta}{\partial\eta}\right)_{\eta=0}=F(A\xi^{1/2}) \qquad (38)$$

By requiring that the heat flux across the interface calculated by this solution must reduce to that calculated in the free-molecule regime, we can determine the constant A

$$A=4(3\gamma U)^{-1}(2kT_A)^{1/2}(\pi m_A)^{-1/2}\left[1+(T_B/T_A)^{1/2}\right]^{-1} \qquad (39)$$

The non-dimensional heat flux is

$$Q'=-q\left[\rho c_p U(T_A-T_B)A/2\right]^{-1}=F(A\xi^{1/2})$$

which is represented by the same curve in Figure 1 as the non-dimensional mass flux plotted versus $A^2\xi$.

This method can also be applied to study radiative transfer for parallel streams of radiating gases[7]. Solutions for radiative transfer in transition regime are quite similar to those obtained in this paper.

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Figure Captions

Figure 1 Mass diffusion flux between two parallel streams in transition regime.

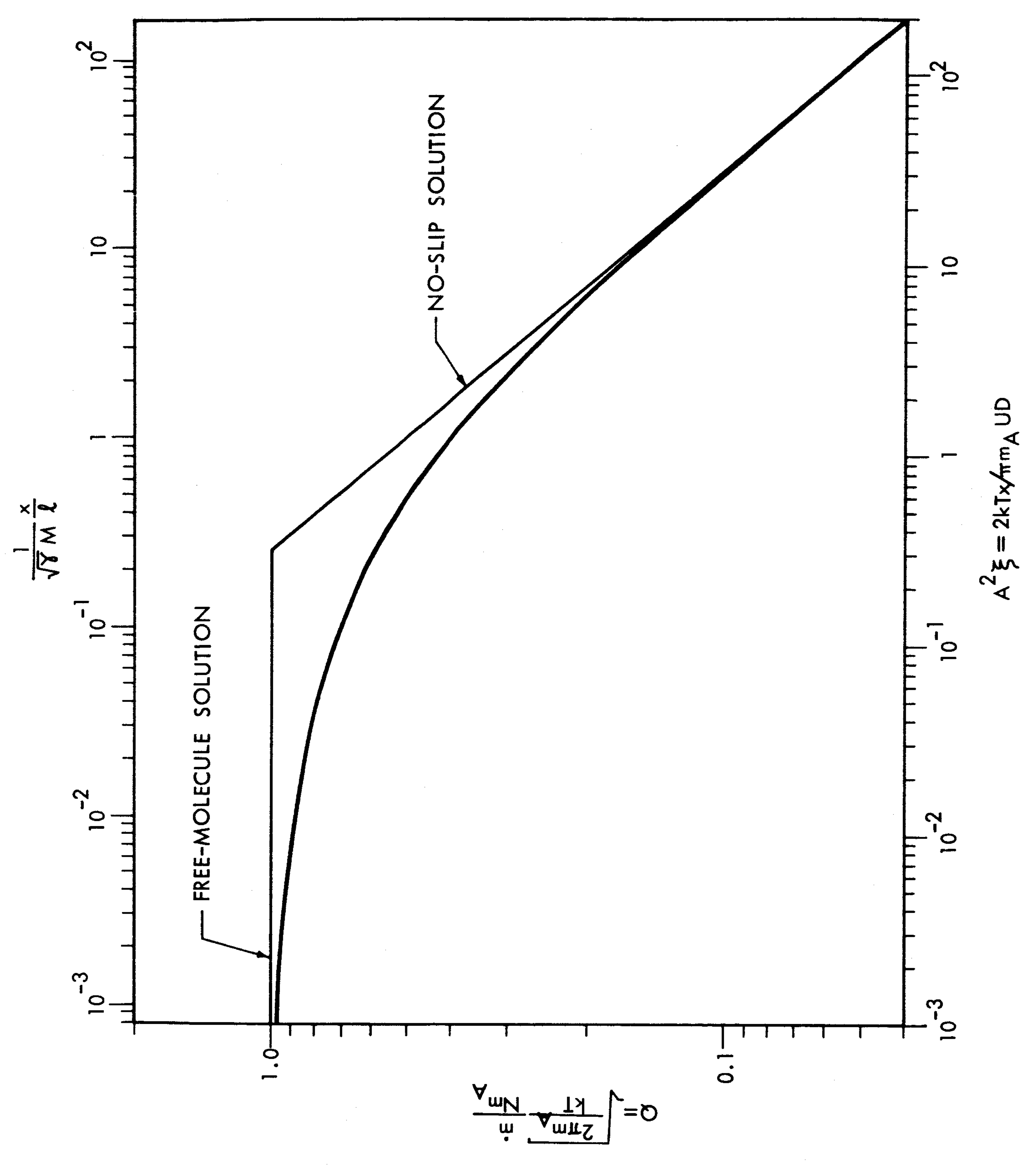

Figure 1 (Y. C. Whang)