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W. M. WONHAM

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CENTER FOR DYNAMICAL SYSTEMS

DIVISION OF APPLIED MATHEMATICS

BROWN UNIVERSITY

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W. M. WONHAM†

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W. M. Wonham

1. Introduction.

Consider the linear control system described by the formal stochastic differential equation

$$\dot{x} = Ax - Bu + C\dot{w}_1 + G(x)\dot{w}_2 \tag{1.1}$$

In (1.1), u is the control and $\dot{w}_1$, $\dot{w}_2$ are independent Gaussian white noise disturbances.* The elements of the matrix G are assumed to be linear in x ; and so the term $G(x)\dot{w}_2$ represents a disturbance of which the intensity is roughly proportional to the deviation of x from the origin $x = 0$. Equivalently, the disturbance can be regarded as a wideband random perturbation of the system matrix A .

Now consider the problem of choosing a feedback control $u = \phi(x)$ such that, in the steady state, the expected quadratic cost

$$\mathcal{E}\{x'Mx + u'Nu\} \tag{1.2}$$

*A precise interpretation of (1.1) is given in §2.

is a minimum. If $G(x) \equiv 0$, the solution of this problem is well known [1], [2]. The optimal control always exists, and is a linear function of x which is independent of the intensity of the additive disturbance $C\dot{w}_1$. In the present note it is shown that an optimal control exists for the more general system (1.1), provided the state-dependent noise $G(x)\dot{w}_2$ is sufficiently small. The optimal control is again linear, but is now rather critically dependent on the coefficients of G. Examples are provided to show that instability may result if this dependence is ignored.

The problem is stated precisely in §2; the proof of existence is given in §3 and §4; and some examples studied in §§5,6. We conclude with some remarks on the interpretation of (1.1) and discuss alternative optimization problems which are closely related.

2. Statement of the problem.

To make (1.1) precise we assume that x is an n-vector with stochastic differential

$$dx = Axdt - Budt + Cdw_1 + G(x)dw_2 \quad . \tag{2.1}$$

In (2.1), A, B and C are real constant matrices of dimension $n \times n$, $n \times m$ and $n \times d_1$ respectively; $G(x)$ is an $n \times d_2$ matrix with (i, j)th element

$$g_{ij}(x) = \sum_{k=1}^{n} g_{ijk} x_k \tag{2.2}$$

where the coefficients g_{ijk} are constants. It is assumed that (A, B) is controllable, and that CC' is positive definite: that is, $d_1 \geqq n$ and C is of rank n. The latter assumption obviates fussy discussion about possible degeneracy of the ergodic measure (see below); it would actually be enough to assume that (A, C) is controllable. Finally, w_1 and w_2 are independent Wiener processes of dimension d_1, d_2 respectively.

In the following, E denotes Euclidean n-space; a prime $(')$ the transpose of a vector or matrix; and $|\cdot|$ the Euclidean norm.

In (2.1) let $u = \phi(x)$, where ϕ is defined on E and satisfies a uniform Lipschitz condition

$$|\phi(x) - \phi(y)| \leqq k|x - y| \qquad (x, y \in E) \tag{2.3}$$

With this choice of u, (2.1) becomes a stochastic differential equation of Itô's type [3]:

$$dx(t) = Ax(t)dt - B\phi[x(t)]dt + Cdw_1(t) + G[x(t)]dw_2(t) \tag{2.4}$$

If $x(0)$ is a random variable independent of the w_1, w_2 increments then (2.4), defined for $t \geqq 0$, determines a diffusion process

$$X_\phi = \{x_\phi(t) : t \geqq 0\} .$$

Diffusion processes are discussed extensively in [4]; a brief summary can be found in [5].

Of interest here is the case when X_ϕ is positive recurrent (for the definition of this term see [5]). Under this condition it is known that there exists a unique ergodic probability measure μ_ϕ defined on the Borel sets of E : that is, if the distribution of $x(0)$ is μ_ϕ then so is that of $x(t)$ for all $t > 0$. Let Φ be the class of <u>admissible</u> control functions ϕ , with the properties

(i) ϕ satisfies (2.3) for some constant k

(ii) X_ϕ is ergodic

(iii) The corresponding ergodic measure μ_ϕ is such that

$$\mathcal{E}_\phi\{|x|^2\} \equiv \int_E |x|^2 \mu_\phi(dx) < \infty \tag{2.5}$$

Now define

$$L(x, u) = x'Mx + u'Nu \tag{2.6}$$

where M, N are constant symmetric positive definite matrices of dimension $n \times n$, $m \times m$ respectively.

Our problem is the following: <u>find a control</u> $\overset{\circ}{\phi} \in \Phi$ <u>which is optimal in the sense that</u>

$$\mathcal{E}_{\overset{\circ}{\phi}}\{L(x, \overset{\circ}{\phi})\} = \min\, [\mathcal{E}_\phi\{L(x, \phi)\} \;:\; \phi \in \Phi]$$

3. Existence of an admissible control.

In this section it will be shown that Φ is nonempty provided the coefficients g_{ijk} of (2.2) are sufficiently small. This result will follow from the stability theorem stated below.

Let $V = V(x)$ be of class $C^{(2)}$ on E and let $\mathcal{L}_u$ denote the elliptic operator

$$\mathcal{L}_u V(x) \equiv \frac{1}{2} \operatorname{tr}\{[C + G(x)]' V_{xx}(x)[C + G(x)]\} + (Ax - Bu)' V_x(x) \qquad (3.1)$$

In (3.1), tr denotes trace, V_x the vector $[\partial V/\partial x_i]$ and V_{xx} the matrix $[\partial^2 V/\partial x_i \partial x_j]$. The operator $\mathcal{L}_\phi$, obtained by setting $u = \phi(x)$ in (3.1), is the differential generator of X_ϕ [4].

The following theorem is an immediate consequence of (2.6) and the results of [6].

Theorem 3.1

If there exist a function $V(x)$ of class $C^{(2)}$ on E, and a positive number λ, such that

$$V(x) \to \infty \qquad (|x| \to \infty) \qquad (3.2)$$

and

$$\mathcal{L}_\phi V(x) \leqq \lambda - L[x, \phi(x)] \qquad (x \in E) \qquad (3.3)$$

then $\phi \in \Phi$.

To apply the theorem set

$$\phi(x) = Kx \tag{3.4a}$$

$$V(x) = x'Px \tag{3.4b}$$

where K, P are constant $m \times n$ (resp. $n \times n$) matrices, to be determined so that

$$\mathcal{L}_\phi V(x) = \lambda - L[x, \phi(x)] \qquad (x \in E) \quad . \tag{3.5}$$

Let G_k denote the $n \times d_2$ matrix with (i, j)th element g_{ijk}; and let $\Pi(P)$ be the symmetric $n \times n$ matrix with elements

$$[\Pi(P)]_{k\ell} = \mathrm{tr}(G_k' P G_\ell) \tag{3.6}$$

Then a brief calculation shows that (3.4) determines a solution of (3.5) if and only if

$$\lambda = \mathrm{tr}(C'PC) \tag{3.7}$$

and

$$\Pi(P) + (A - BK)'P + P(A - BK) + M + K'NK = 0 \tag{3.8}$$

By our assumption of controllability, K can be chosen so that all eigenvalues of the matrix A-BK have negative real parts. With K so chosen, the following lemma shows that (3.8) has a unique positive definite solution P provided $\sum_k |G_k|^2$ is sufficiently small. This together with Theorem 3.1 implies that $\phi \in \Phi$.

Lemma 3.1

If $Q > 0$ and A is stable, the equation

$$\Pi(P) + A'P + PA + Q = 0 \tag{3.9}$$

has a unique solution $P > 0$ provided

$$d_2\left(\sum_{k=1}^{n} |G_k|^2\right) \left| \int_0^\infty e^{tA'} e^{tA} dt \right| < 1 \tag{3.10}$$

Here and below $P > 0$ (≥ 0) means P is positive (semi) definite; $P_1 > P_2$ means $P_1 - P_2 > 0$, etc.

Proof.

Eq. (3.9) is equivalent to the equation

$$P = R + T(P) \tag{3.11}$$

where

$$R = \int_0^\infty e^{tA'} Q e^{tA} dt$$

and

$$T(P) = \int_0^\infty e^{tA'}\Pi(P)e^{tA}dt \tag{3.12}$$

We observe that $\Pi(P)$ is a linear function of P and $\Pi(P) \geqq 0$ if $P \geqq 0$; it follows that $T(P)$ has the same properties. Define

$$P_1 = R \;;\quad P_{\nu+1} = R + T(P_\nu) \qquad \nu = 1, 2, \ldots \,.$$

The sequence P_ν is monotone nondecreasing ; it is bounded if, for some $\theta \in (0, 1)$,

$$|T(P)| \leq \theta|P| \tag{3.13}$$

If (3.13) holds, it follows by a result on positive operators (e.g. [7], p. 189, Theorem 1) that

$$P = \lim P_\nu \qquad (\nu \to \infty)$$

exists; and $P \geqq R > 0$. Since T is a contraction, P is unique. It is easily checked that (3.13) is a consequence of (3.10).

4. Existence of an optimal control.

It will be shown that an optimal control Φ^o exists whenever (3.10) holds, and that Φ^o is linear. We use dynamic programming and the well known method of approximation in policy space [8]. This approach was suggested by the work of Howard, who studied a similar problem for Markov chains [9]. The result depends on the following optimality theorem.

Theorem 4.1

Suppose there exists $\phi^\circ \in \Phi$, *a function* $v(x)$ *of class* $C^{(2)}$ *on* E , *and a positive number* λ , *with the properties:*

(i) *For every* $\phi \in \Phi$

$$\mathcal{E}_\phi\{|v(x)| + |x||v_x(x)| + |x|^2|v_{xx}(x)|\} < \infty \tag{4.1}$$

(ii)
$$\mathcal{L}_{\phi^\circ}v(x) + L[x, \phi^\circ(x)] = \lambda \quad (x \in E) \tag{4.2}$$

(iii) *For every* m-*vector* u ,

$$\mathcal{L}_u v(x) + L(x, u) \geq \lambda \quad (x \in E) \tag{4.3}$$

Then ϕ° *is optimal. Futhermore*

$$\lambda = \mathcal{E}_{\phi^\circ}\{L(x, \phi^\circ)\} \tag{4.4}$$

Combining (4.2) and (4.3) we obtain the appropriate version of *Bellman's equation*

$$\min_u \{\mathcal{L}_u v(x) + L(x, u)\} = \lambda \tag{4.5}$$

To prove Theorem 4.1 we need

Lemma 4.1

Let X be a diffusion process determined by (2.4), with differential generator $\mathcal{L}$ *and ergodic measure* μ . *If* $v(x)$ *is a function of class* $C^{(2)}$ *such that*

$$\mathcal{E}_{\mu}\{|v(x)| + |x||v_x(x)| + |x|^2|v_{xx}(x)|\} < \infty$$

then

$$\mathcal{E}_{\mu}\{\mathcal{L}v(x)\} = 0 \quad .$$

A proof is given in the Appendix.

To prove Theorem 4.1 observe that if $\phi \in \Phi$ then, by (4.2) and (4.3),

$$\lambda \leq \mathcal{L}_{\phi}v(x) + L[x, \phi(x)] \qquad (x \in E)$$

Taking expectations with respect to μ_{ϕ} on both sides, and applying Lemma 4.1, we obtain

$$\lambda \leq \mathcal{E}_{\phi}\{L(x, \phi)\}$$

Again by Lemma 4.1, (4.2) implies

$$\lambda = \mathcal{E}_{\phi^{\circ}}\{L(x, \phi^{\circ})\}$$

and the result follows.

To compute an optimal control we seek a solution of Bellman's equation, in the form

$$v(x) = x'Px \tag{4.6}$$

Substitution shows that (4.5) holds if and only if P satisfies (3.8), with

$$K = N^{-1}B'P \tag{4.7}$$

The control determined by (4.5) is

$$\phi^{\circ}(x) = Kx \tag{4.8}$$

We show next that (3.8) and (4.7) can be solved for a unique positive definite matrix P. For $\nu = 1, 2, \ldots$ let P_ν be a solution of (3.8) with $K = K_\nu$ and define

$$K_{\nu+1} = N^{-1}B'P_\nu \tag{4.9}$$

By Lemma 3.1, we can choose K_1 so that P_1 exists. It will be shown that if K_2 is defined by (4.9) then P_2 exists and $0 < P_2 \leq P_1$. Write $v_\nu(x) = x'P_\nu x$, $\phi_\nu = K_\nu x$ and $\mathcal{L}_\nu = \mathcal{L}_{\phi(\nu)}$. It can be verified directly that (4.9) is equivalent to the condition

$$\mathcal{L}_{\nu+1}v_\nu(x) + L[x, \phi_{\nu+1}(x)] \leq \mathcal{L}_u v_\nu(x) + L(x, u) \quad (x \in E) \tag{4.10}$$

for all m-vectors u. That is, $\phi_{\nu+1}$ is determined by the minimizing operation (4.5) applied to v_ν . Setting $\nu = 1$ and $u = \phi_1(x)$ in (4.10), and using (3.8), we see that

$$- Q \equiv \Pi(P_1) + (A - BK_2)'P_1 + P_1(A - BK_2) + M + K_2'NK_2$$
$$\leqq 0 . \tag{4.11}$$

Write $A_2 = A - BK_2$. Since $P_1 > 0$ satisfies (4.11) it follows (by a standard Liapunov theorem) that A_2 is stable. Hence

$$P_1 = \int_0^\infty e^{tA_2'}[M + K_2'NK_2 + \Pi(P_1) + Q]e^{tA_2}dt \tag{4.12}$$

Now P_2 is to be determined by (3.8) with $K = K_2$, or

$$P_2 = \int_0^\infty e^{tA_2'}[M + K_2'NK_2 + \Pi(P_2)]e^{tA_2}dt \quad . \tag{4.13}$$

As in the proof of Lemma 3.1, we solve (4.13) by successive approximations. Setting $P_2^{(1)} = 0$ we have

$$P_2^{(2)} = \int_0^\infty e^{tA_2'}(M + K_2'NK_2)e^{tA_2}dt$$
$$\leqq P_1 ;$$

and similarly $P_2^{(\kappa)} \leqq P_1$ $(\kappa = 2, 3,\ldots)$. Since the $P_2^{(\kappa)}$ are nondecreasing and bounded,

$$P_2 \equiv \lim P_2^{(\kappa)} \quad (\kappa \to \infty) \tag{4.14}$$

exists and satisfies (4.13). Thus $P_2 \leqq P_1$, and $M > 0$ implies $P_2 > 0$.

It is not asserted that the solution of (4.13) is unique; however, we may now proceed by induction and define

$$P_\nu = \lim P_\nu^{(\kappa)} \quad (\kappa \to \infty) \qquad \nu = 1, 2, \ldots \; .$$

In this way we obtain a sequence $\{P_\nu\}$ with $0 < P_{\nu+1} \leqq P_\nu$. Then

$$\begin{aligned} P &= \lim P_\nu \qquad (\nu \to \infty) \\ K &= N^{-1}B'P \end{aligned} \tag{4.15}$$

exist and satisfy (3.8) and (4.7).

Define

$$\begin{aligned} \phi^\circ(x) &= Kx \; . \\ v(x) &= x'Px \\ \lambda &= \mathrm{tr}(C'PC) \end{aligned} \tag{4.16}$$

Theorem 4.1 will be applied to show that ϕ° is optimal. By construction, ϕ° satisfies (4.2) and (4.3). Furthermore, if $\phi \in \Phi$ then (2.5) and (4.16) imply the truth of (4.1). The existence of ϕ° is now established.

We observe that Φ° is unique in the class of linear controls; for if Φ is another optimal linear control and $\hat{\lambda}$, $\hat{P}$ are the corresponding quantities dete·mined as before, then by (4.4) $\hat{\lambda} = \lambda$, and by (4.16)

$$\mathrm{tr}(C'\hat{P}C) = \mathrm{tr}(C'PC) \quad . \tag{4.17}$$

Since $\hat{P}$, P are independent of C, (4.17) holds for all C, and from this it easily follows that $\hat{P} = P$. Uniqueness of Φ is a consequence of (4.7).

5. Example 1.

The following artificial example is of interest because it illustrates the qualitative dependence of the control law on the intensity of the state-dependent noise. Let

$$dx_i = ax_i dt - bu_i dt + c dw_{1i} + g|x| dw_{2i} \ , \ i = 1, \ldots, n \ . \tag{5.1}$$

and

$$L(x, u) = |x|^2 + |u|^2 \quad .$$

In (5.1) the matrix $G(x) = g|x|I$ is not linear in x (cf.(2.2)); nevertheless the methods used above apply equally well here, and (3.8), (4.7) become

$$g^2(\mathrm{tr}\, P)I + (aI-bK)'P + P(aI-bK) + I + K'K = 0$$

$$K = bP \quad .$$

This gives $P = pI$, $K = bpI$, and $\lambda = nc^2p$, where

$$\begin{aligned} p &= (2b^2)^{-1}\{2a + ng^2 + [(2a + ng^2)^2 + 4b^2]^{\frac{1}{2}}\} \\ &\sim nb^{-2}g^2 \ , \quad g \to \infty \ . \end{aligned} \tag{5.2}$$

For large g, $\phi^\circ(x) \sim nb^{-1}g^2x$, and the optimal control depends rather critically on noise intensity.

Now suppose that for some k , $u = \phi(x) = kx$ in (5.1). Solution of (3.5) and application of (4.7) yield

$$\begin{aligned} \lambda_\phi &= \mathcal{E}_\phi\{|x|^2 + |u|^2\} \\ &= nc^2(1 + k^2)[2(bk - a) - ng^2]^{-1} \end{aligned}$$

provided

$$bk - a > ng^2/2 \quad . \tag{5.3}$$

If this inequality fails (i.e. control is not sufficiently vigorous) then instability results, in the sense that either $\lambda_\phi = +\infty$ or λ_ϕ is not defined: that is, X_ϕ is no longer ergodic. Using the methods of [5] one can show that X_ϕ is ergodic (i.e. μ_ϕ exists) if and only if $bk - a > (n-2)g^2/2$.

6. Example 2.

Our next example illustrates the fact that an admissible linear control can fail to exist if the intensity of state-dependent noise is large. Let

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} , \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix} , \quad G(x) = \sqrt{\gamma} \begin{pmatrix} x_1 & x_2 \\ -x_2 & x_1 \end{pmatrix} \tag{6.1}$$

where $\gamma > 0$ is a constant. Then $\Pi(P) = \gamma \operatorname{tr}(P) I$ and (cf.(3.12))

$$T(P) = \gamma \operatorname{tr}(P) \int_0^\infty e^{t(A-BK)'} e^{t(A-BK)} dt \tag{6.2}$$

Let $K = (k_1, k_2)$ and denote the integral in (6.2) by S . S exists (i.e. $A - BK$ is stable) if and only if $k_1 > 0$, $k_2 > 0$. A brief calculation shows that $\inf\{\operatorname{tr} S : k_1, k_2 > 0\} = 1$. Iterating the the operator T we then find

$$\begin{aligned} T^{(\nu)}(P) &= \gamma^\nu (\operatorname{tr} P)(\operatorname{tr} S)^{\nu-1} S \\ &> \gamma^\nu (\operatorname{tr} P) S , \end{aligned}$$

and so $\Sigma T^{(\nu)}$ converges only if $\gamma < 1$. This shows that the construction used in the proof of existence fails if $\gamma \geqq 1$. For the present example it is easy to verify directly that (3.8) has for some K a solution $P > 0$, if and only if $\gamma < 1$.

7. An alternative interpretation of (1.1).

It is worth emphasizing that the choice of Itô's equation (2.1) as a precise version of (1.1) is somewhat arbitrary. We shall discuss briefly an alternative version of (1.1) which may be more appropriate in engineering applications. Eq. (1.1) is a purely formal equation since the "derivatives" $\dot{w}_1$, $\dot{w}_2$ do not exist. In writing (1.1), we usually have in mind a physical system perturbed by noise with a power spectral density which is essentially constant within the frequency passband of the system. However, total noise power is presumably finite, and this fact is overlooked in adopting the precise model (2.1). Thus the question arises whether the diffusion process determined by (2.1) adequately reflects the properties of the physical random process of which (1.1) is a rough description. This question has been discussed in a precise fashion by Stratonovich [10], [11] and by Wong and Zakai [12]. It turns out that the proper Itô equation to associate with (1.1) will depend on what definition is adopted of the formal stochastic integral

$$J = \int_a^b G[x(t)]\dot{w}(t) \quad . \tag{6.1}$$

Let $\{t_\nu\}$ be a partition of the interval $[a, b]$. On the basis of results of [10]-[12] it is natural to adopt for (6.1) the definition

$$J = \text{l.i.m.} \sum_\nu G\left[\frac{x(t_{\nu+1}) + x(t_\nu)}{2}\right] [w(t_{\nu+1})-w(t_\nu)] \tag{6.2}$$

as $\max_{\nu}(t_{\nu+1} - t_{\nu}) \to 0$. Let us now suppose that $x(t)$ has the Itô stochastic differential

$$dx(t) = f(x)dt + G(x)dw \tag{6.3}$$

where $G(x) = [g_{ij}(x)]$. Then it can be shown [10] that

$$J = \frac{1}{2} \int_a^b G_x[x(t)] \cdot G[x(t)]dt + \int_a^b G[x(t)]dw(t) \tag{6.4}$$

where the second integral in (6.4) is an Itô stochastic integral, and $G_x \cdot G$ is the vector with i^{th} component

$$\sum_{jk} (\partial g_{ij}/\partial x_k) g_{kj} \tag{6.5}$$

This result means that an alternative natural interpretation of (1.1) is that the process $x(t)$ has the Itô stochastic differential

$$dx = [Ax - Bu + \frac{1}{2} G_x(x) \cdot G(x)]dt + Cdw_1 + G(x)dw_2 \tag{6.6}$$

Eq. (6.6) differs from (2.1) by the presence of an additional drift term contributed by the coefficient of the state-dependent noise.

Suppose that $G(x)$ has the form (2.2). Then (6.6) can be written

$$dx = \hat{A}xdt - Budt + Cdw_1 + G(x)dw_2$$

where $\hat{A}$ is a modified system matrix with elements

$$\hat{a}_{ij} = a_{ij} + \frac{1}{2} \sum_{k\ell} g_{ik\ell} g_{\ell kj} \quad .$$

After this modification the discussion of §§2-6 remains unchanged.

In light of this discussion consider again Example 1 . Here $G(x) = g|x|I$, and

$$G_x(x) \cdot G(x) = g^2 x$$

Thus $\hat{A} = aI + (g^2/2)I$ and the previous results hold with this replacement. With the new model,

$$\phi^\circ(x) \sim (n + 1)b^{-1}g^2 x \quad (g \to \infty) \;;$$

that is, the optimal control gain is somewhat higher than previously. Suppose next that $u = \phi(x) = kx$. Then (cf.§5) $\lambda_\phi < \infty$ if and only if

$$bk - a > (n + 1)g^2/2$$

Comparing this result with (5.3) we see that the choice of mathematical model may be critical in an assessment of the stability properties of the physical system of interest.

8. Alternative problems.

A variety of linear regulator problems with linearly state-dependent noise can be discussed by methods similar to the foregoing. If the index of performance is expectation of a quadratic functional, and if no a priori bound is placed on magnitude of the control vector, then in general the optimal control (when it exists) is linear in x and depends on noise intensity.

To mention one interesting variant, let

$$dx = Axdt - Budt + G(x)dw \tag{8.1}$$

and consider the problem of minimizing

$$\mathcal{E}_x\{\int_0^\infty [x(t)'Mx(t) + u(t)'Nu(t)]dt\} \tag{8.2}$$

If $u = \Phi(x)$ and $\Phi(0) = 0$ then (8.1) admits the null solution $x(t) \equiv 0$ (see e.g. [13]). The functional (8.2) is finite provided $x = 0$ is globally asymptotically stable in an appropriate sense. By a slight extension of the methods of [13] one can show that X_Φ is stable if and only if a continuous function $V(x)$ exists such that

(i) $V(x) > 0 \quad (x \neq 0)\ ; \quad V(0) = 0$

(ii) $V(x) \to +\infty$ as $|x| \to \infty$

(iii) $\mathcal{L}_\Phi V(x) \leq -|x|^2 \ , \quad x \neq 0$.

Call Φ admissible if X_Φ is stable. Just as in §3 we find that $\Phi(x) = Kx$ is admissible if (3.8) has a positive solution P, and this is so whenever $G(x)$ is restricted by the inequality (3.10). Under these conditions the optimal linear control is determined exactly as before.

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Appendix: Proof of Lemma 4.1

Let $\mathcal{E}_x$ denote expectation on the paths of X when $x(0) = x \in E$. Let $t > 0$ be fixed and write

$$w(x) = \mathcal{E}_x\{v[x(t)]\} \quad .$$

We show first that w exists a.e.$[\mu]$ and

$$\mathcal{E}_\mu\{w\} = \mathcal{E}_\mu\{v\} \quad . \tag{1}$$

If v is a simple function, (1) is obvious. If $v \geqq 0$ and v_n are simple functions with $v_n \uparrow v$, then

$$w_n(x) = \mathcal{E}_x\{v_n[x(t)]\}$$

is measurable and $w_n \uparrow w$. By monotone convergence

$$\begin{aligned} \mathcal{E}_\mu\{w\} &= \mathcal{E}_\mu\{\lim \ w_n\} = \lim \ \mathcal{E}_\mu\{w_n\} \\ &= \lim \ \mathcal{E}_\mu\{v_n\} = \mathcal{E}_\mu\{v\} \quad . \end{aligned}$$

The general result follows by applying the argument to the positive and negative parts of v.

Now let v be of class $C^{(2)}$ and of compact support. By the Itô-Dynkin formula [4]

$$\mathcal{E}_\mu\{\mathcal{E}_x\{\int_o^t \mathcal{L}v[x(s)]ds\}\}$$

$$= \mathcal{E}_\mu\{\mathcal{E}_x\{v[x(t)] - v(x)\}$$

$$= 0 \quad .$$

Since $\mathcal{L}v[x(s)]$ is bounded and almost surely continuous (in s) there follows, by dominated convergence,

$$\mathcal{E}_\mu\{\mathcal{L}v(x)\} = \mathcal{E}_\mu\{\mathcal{E}_x\{\lim_{t\downarrow 0} t^{-1}\int_o^t \mathcal{L}v[x(s)]ds\}$$

$$= \lim_{t\downarrow 0} \mathcal{E}_\mu\{\mathcal{E}_x\{t^{-1}\int_o^t \mathcal{L}v[x(s)]ds\}\}$$

$$= 0 \quad .$$

In general, suppose $v(x)$ satisfies the integrability condition of the hypothesis. Then for any $\epsilon > 0$ there exists a smooth function $\tilde{v}(x)$ of compact support such that

$$|\mathcal{E}_\mu\{\mathcal{L}v(x)\} - \mathcal{E}_\mu\{\mathcal{L}\tilde{v}(x)\}| < \epsilon \ ;$$

that is, $|\mathcal{E}_\mu\{\mathcal{L}v(x)\}| < \epsilon$.

References

[1] Florentin, J. J., Optimal control of continuous-time, Markov, stochastic systems, J. Electronics and Control, 10, 1961, pp. 473-488.

[2] Wonham, W. M., Stochastic Problems in Optimal Control, RIAS TR 63-14, May, 1963.

[3] Itô, K., On Stochastic Differential Equations, Mem. Amer. Math. Soc. 4, 1951.

[4] Dynkin, E. B., Markov Processes, Academic Press, New York, 1965.

[5] Wonham, W. M., Liapunov criteria for weak stochastic stability, J. Diff. Eqns., 2(2), 1966; pp. 195-207.

[6] Wonham, W. M., A Liapunov method for the estimation of statistical averages, J. Diff. Eqns., 1966, to appear.

[7] Kantorovich, L. V., and Akilov, G. P., Functional Analysis in Normed Spaces, MacMillan, New York, 1964.

[8] Bellman, R., Dynamic Programming, Princeton U. P., 1957.

[9] Howard, R. A., Dynamic Programming and Markov Processes, Wiley, New York, 1960.

[10] Stratonovich, R. L., A new form of representation of stochastic integrals and equations (in Russ.), Vestnik Mosk. Univ.(1) 1964, pp. 3-12.

[11] Stratonovich, R. L., Conditional Markov Processes and Their Applications in the Theory of Optimal Control (in Russ.), Izd. Mosk. Univ., 1966.

[12] Wong, E. and Zakai, M., On the relation between ordinary and stochastic differential equations, Int. J. Engrg. Sci. 3, 1965, pp. 213-229.

[13] Khasminskii, R. Z., On the stability of the trajectory of Markov processes, Appl. Math. Mech. 26(6), 1962, pp. 1554-1565.