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AROD REAL TIME DATA EXTRAPOLATION

PREPARED BY

RADIO FREQUENCY TRACKING LABORATORY

H. M. SUMMER, TECHNICAL DIRECTOR

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GEORGE C. MARSHALL SPACE FLIGHT CENTER

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

HUNTSVILLE, ALABAMA

APPROVED BY

C. H. Weaver
Head Professor
Electrical Engineering

SUBMITTED BY

H. M. Summer
Project Leader

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AROD REAL TIME DATA EXTRAPOLATION

by J. C. Sanderlin & J. T. Hannon

I. INTRODUCTION

AROD is a space vehicle tracking system in which tracking data are extracted from c-w, radio transmissions between the vehicle and sets of four ground-based transponders. Ideally, continuous communications are required between the vehicle and each of at least three transponders. In addition to improving tracking accuracy, the fourth transponder permits sequential switching of individual transponders between sets while maintaining communications with three transponders at all times.

In practice, radio transmissions are performed in the presence of many sources of interference which may, and often do, result in an interruption of communications for a finite period of time.

The receivers in both the vehicle and the transponders are phase-locked. For the purposes of this report, phase-locked receivers may be characterized by two features, namely, acquisition time and response time.

The acquisition time, T_{ac}, of a phase-locked receiver is dependent upon many factors. The most important of these factors are contained in the proportionality

$$T_{ac} = \frac{(\text{initial error})\ (\text{error rate})}{(\text{loop bandwidth})\ (\text{signal strength})} \tag{1}$$

A phase-locked receiver exhibits limited response under two conditions.

Suppose a step change is made in the phase of the input signal. The automatic phase control (APC) loop integrates the error step so that after some time T_d the VCO phase is approximately equal to that of the incoming signal. Suppose the receiver is phase-locked to an incoming signal with zero error rate, and the signal is suddenly removed. This is also a step change in the error, which is integrated as in the previous case, so that a change in the VCO phase will be observed only after a finite delay. If the signal is reapplied immediately after it is removed, the reacquisition time should be short since the numerator of (1) approaches zero as the period of signal interruption approaches zero.

For a receiver with given loop bandwidth the reacquisition time, T_{rac}, after a signal interruption of duration T_i is determined by

$$T_{rac} \propto \frac{T_i \, \partial \phi_s / \partial t}{|P_s|} \qquad (2)$$

where ϕ_s = signal phase,

$\partial \phi_s / \partial t$ = error rate,

$|P_s|$ = signal strength.

To obtain adequate sensitivity the AROD receivers use narrow bandwidths so that one term in the denominator of (1) is always small.

Consider the relation of the variables in (2) to the distance, s, between the vehicle and a transponder. Small signal strengths correspond to large distances, as do low error rates. These two variables thus tend to compensate, for large s. A high probability of signal interruption

corresponds to small signal strengths, as do longer average durations of interruption. Hence, for large s, the probability of signal interruption is great, and signal interruptions will probably be accompanied by relatively large reacquisition times.

Large signal strengths and large error rates correspond to small distances. Hence, these two terms compensate for small, as well as large, distances. The probability of signal interruption is minimum for minimum distance, and the average duration of interruptions should also be minimum.

We may conclude that there exists a possibility that a signal interruption may occur during any part of the trajectory in which the vehicle is "visible" to the transponder. Further, the probability of interruption is greatest when the vehicle is near the horizon, and these interruptions will be accompanied by the maximum reacquisition times.

Since continuous data are required, a technique for reducing the data lost due to signal interruptions is desirable. This report presents the preliminary investigation of a real time data extrapolation technique designed to minimize reacquisition time following a signal interruption, as well as to "replace" the data lost during the interruption.

II. FINITE DIFFERENCE OPERATORS, INTERPOLATION, AND EXTRAPOLATION

A brief introduction to finite differences, and finite difference operators is followed by a short development of the Gregory-Newton interpolation formulas, of which one is suitable for use in extrapolation.

A. **Finite Difference Operators**: A function f of a variable x is defined to be an ordered collection of number pairs x, $f(x)$, of which no two have the same first term.

A collection (of points, functions, etc.) is referred to as a set (of points, functions, etc.). An ordered set is a collection in which the elements are arranged according to some rule, hence, to order a set, is to arrange its elements according to some rule. Let M be a set of points, and let N be a set of points such that every point of N is also a point of M. If M contains one or more points not contained in N, then N is said to be a subset of M.

Let M be a set of points, and let f be a function. If for every x in the set M, there exists a real number y such that $y = f(x)$, then the function f is said to be defined on the set M.

The interval $[a,b]$ is the set of all points x such that

$$a \leq x \leq b. \tag{1}$$

From the interval $[a,b]$ select an ordered subset of equally spaced points $\{x_i\}_{i=0}^{n}$ such that

$$a = x_0 < x_1 < \cdots < x_{r-1} < x_r < x_{r+1} < \cdots < x_n = b \qquad (2)$$

where for every non-negative integer i, such that $0 \leq i < n$,

$$x_{i+1} - x_i = h. \qquad (3)$$

Equation (3) may also be written $x_{i+1} = x_i + h$. Repeated application of this relation and/or induction yields

$$
\begin{aligned}
x_{i+2} &= x_{i+1} + h = x_{i+h} + h = x_{i+2h} , \\
&\;\;\vdots \qquad\;\;\; \vdots \qquad\;\;\; \vdots \qquad\;\;\; \vdots \\
x_{i+k} &= x_{i+k-1} + h = x_{i+h-2} + 2h = \cdots = x_{i+kh} ,
\end{aligned}
\qquad (4)
$$

which is valid for all non-negative integers i such that $0 \leq i < n$. In the last equation of (4), let $i = 0$ and $k = n$ to obtain an expression for the greatest member of the set x_n in terms of the smallest member and the constant difference h. Hence,

$$x_n = x_0 + nh. \qquad (5)$$

Only bounded sets of the form (2) will be considered, for if the set is unbounded, there exists at least point-wise knowledge of the function for all x; and there is no need for extrapolation.

Let f be a function defined on the set $\{x_i\}_{i=0}^{n}$ (see equation 2), then for every x_i there exists a real number y_i such that $y_i = f(x_i)$. Further, there exists the set $\{y_i\}_{i=0}^{n}$,

$$y_0 = f(a) = f(x) \; ,$$

$$y_1 = f(x_1) \; ,$$

$$\vdots \qquad \vdots$$

$$y_{k-1} = f(x_{k-1}) \; ,$$

$$y_k = f(x_k) \; ,$$

$$y_{k+1} = f(x_{k+1}) \; ,$$

$$\vdots \qquad \vdots$$

$$y_n = f(x_n) \; .$$

$$(6)$$

Note that there exists a one-to-one correspondence between the terms of $\{x_i\}_{i=0}^{n}$ and the terms of $\{y_i\}_{i=0}^{n}$. For each member x_j of the set $\{x_i\}_{i=0}^{n}$ there exists a member of the set $\{y_i\}_{i=0}^{n}$ such that $y_i = f(x_i)$. Hence, there exists a one-to-one correspondence between the members of the sets; however, observe that the set $\{y_i\}_{i=0}^{n}$ is not ordered.

The forward difference operator Δ is defined by the relation

$$\Delta f(x_i) = f(x_{i+1}) - f(x_i) \tag{9}$$

for all x_i such that $f(x)$ is defined on some set containing x_i. The relation, from (4) ,

$$\Delta f(x_i) = f(x_i + h) - f(x_i) \; ; \tag{8}$$

and the relation, from (6) ,

$$\Delta y_i = y_{i+1} - y_i \tag{9}$$

are both equivalent to (7). These three notations may be used interchangeably; however, the notation of (9) will be used unless there is a possibility of ambiguity. For any two functions $y_i = f(x_i)$, and $u = g(x_i)$ defined on the set $\left\{ x_i \right\}_{i=0}^{n}$, obtain from (9)

$$\Delta (y_i + u_i) = (y_i + 1 + u_{i+1}) - (y_i + u_i) ,$$

$$= (y_{i+1} - y_i) + (u_{i+1} - u_i) , \qquad (10)$$

$$= \Delta y_i + \Delta u_i ,$$

and for c a constant

$$\Delta (cy_i) = cy_{i+1} - cy_i ,$$

$$= c (y_{i+1} - y_i) , \qquad (11)$$

$$= c \Delta y_i .$$

Repeated application of (9) and/or induction yields

$$\Delta y_i = y_{i+1} - y_i ,$$

$$\Delta^2 y_i = \Delta (\Delta y_i) = \Delta y_{i+1} - \Delta y_i ,$$

$$\Delta^3 y_i = \Delta^2 (\Delta y_i) = \Delta^2 y_{i+1} - \Delta^2 y_i , \qquad (12)$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$\Delta^k y_i = \Delta^{k-1} (\Delta y_i) = \Delta^{k-1} y_{i+1} - \Delta^{k-1} y_i .$$

Hence, for any two non-negative integers n and k

$$\Delta^k(\Delta^n y_i) = \Delta^{n+k} y_i .$$ (13)

Equations (10) and (11) are the defining relations for a linear operator. Hence the forward difference operator Δ is a linear operator, further, equations (11) and (12) assign an additive law of exponents to the operator Δ.

The effect of the operator Δ upon a set of ordered number pairs $\left\{x_i, y_i\right\}_{i=1}^{11}$ is shown in Table 1. Note that successive differences may be computed from any starting point. The use of the word "forward" in the descriptive title of the operator Δ is a result of the fact that only successive differences may be computed with Δ.

Let f be a function defined on the set $\left\{x_i\right\}_{i=0}^{n}$. Define the shift operator E to be that operator which advances $y_i = f(x_i)$ by one increment, hence

$$E y_i = y_{i+1} .$$ (14)

Observe, for any two functions $y_i = f(x_i)$ and $u_i = g(x_i)$ defined on the set $\left\{x_i\right\}_{i=0}^{n}$, that

$$E(y_i + u_i) = y_{i+1} + u_{i+1} = E y_i + E u_i$$ (15)

for c a constant

$$E(c y_i) = c y_{i+1} = c E y_i ,$$ (16)

and

$$Ey_i = y_{i+1}$$

$$E^2 y_i = Ey_{i+1} = y_{i+2}$$

$$\begin{matrix} \bullet & \bullet \\ \bullet & \bullet \\ \bullet & \bullet \end{matrix} \qquad (17)$$

$$E^n y_i = y_{i+n} \bullet$$

From (15), (16) and (17) conclude that the shift operator E is a linear operator with an additive law of exponents.

Observe that there exists an inverse shift operator E^{-1} such that

$$E^{-1} (Ey_i) = y_i. \qquad (18)$$

Substitution of (14) into (18) yields

$$E^{-1} y_{i+1} = y_i \qquad (19)$$

Make the change of notation $j = i+1$, hence, from (19) write

$$E^{-1} y_j = y_{j-1} \bullet \qquad (20)$$

Equations (14) and (20) are the defining relations for the forward and reverse shift operators respectively.

Consider again the ordered set of number pairs $\left\{ x_j, y_j \right\}_{j=0}^{n}$ defined in equations (2), (3) and (6). The ordering of the set is completely arbitrary, hence let the set be reordered such that

$$a = x_o < \cdots < x_{n-1-i} < x_{n-1+1} < \cdots < x_{n-1} < x_n = b \qquad (21)$$

where for every j such that $o < j \le n$

$$x_{n-j} - x_{n-j-1} = h , \qquad (22)$$

and for a function $y = f(x)$ defined on $\left\{ x_i \right\}_{i=o}^{n}$ there exists the corresponding set $\left\{ y_j' \right\}_{j=o}^{n}$

$$y_o' = f(x_n) = f(b) ,$$
$$y_o' = f(x_{n-1}) ,$$
$$\vdots \qquad \vdots$$
$$y_{1-1}' = f(x_{n-k+1}) ,$$
$$y_k' = f(x_{n-k}) , \qquad (23)$$
$$y_{k+1}' = f(x_{n-k-1}) ,$$
$$\vdots \qquad \vdots$$
$$y_n' = f(x_o) = f(a) .$$

The prime on y_j' does not denote differentiation with respect to a variable, but is used to discriminate between the members of the set (4) and the set (23). Equations (21), (22) and (23) are analagous to (2), (3) and (4) respectively. Equation (21) is identical in form to (2), except that the reference term is n-th term, instead of the initial term. Equation (22) may be obtained from (3) by letting

$$i+1 = n-j , \quad i = n-j-1. \qquad (24)$$

Equation (23) is a reordering of (4) where $y'_0 = f(x_n)$ is the reference instead of $y_0 = f(x_0)$.

Given the set of ordered number pairs $\left\{x_j, y'_j\right\}_{j=0}^{n}$, define the backward difference operator δ such that

$$\delta y_j = y_j - y_{j-1} \tag{25}$$

It is readily shown that the operator δ is a linear operator having an additive law of exponents. Observe that the operator δ forms only successive differences backward from $f(x_n)$. A table similar to Table 1 can be formed using the operator δ on a finite data set, where Δ would be replaced with δ and the remainder of the table being essentially unchanged. The important difference lies in the definition of the terms, since in the table formed by δ, the first differences δy_0 is actually the difference Δy_{11} of Table 1. The result is to place the final differences of the set $\left\{f(x_i)\right\}_{i=0}^{n}$ at the beginning of the set $\left\{f(x_i)\right\}_{i=0}^{n}$.

In comparing the operations of Δ and δ on the set $\left\{x_i\right\}_{i=0}^{n}$ the compact notation of (9), used above may be confusing, hence the notation (8) will be used. Hence from (8) and (23), equation (25) would be written in the form $\delta y_j = f(x_{n-j}) = f(x_{n-j+1}) - f(x_{n-j})$.

From (7) write $\Delta f(x_i) = f(x_{i+1}) - f(x_i)$, and solve for $f(x_{i+1})$. Hence

$$f(x_{i+1}) = f(x_i) + \Delta f(x_i) = (1 + \Delta) \, f(x_i) \tag{26}$$

where 1 is the identity operator. From (14) write $f(x_{i+1}) = Ef(x_i)$, substitute into (26) to obtain

$$Ef(x_i) = (1+\Delta) \, f(x_i) \; . \tag{27}$$

From (27) one may write

$$E = (1+\Delta) \tag{28}$$

where the operand is implicit, and it is understood that without an operand the operator is of no significance. From (25) write

$$f(x_{n-j}) = f(x_{n-j}) - f(x_{n-j+1}) \quad \text{and solve for} \quad f(x_{n-j+1}) \quad \text{to obtain}$$

$$f(x_{n-j+1}) = f(x_{n-j}) - f(x_{n-j}) = (1-\delta) \, f(x_{n-j}) \; . \tag{29}$$

Observe from (23) that $f(x_{n-j+1}) = y_{j-1}$, and from (20) $y_{j-1} = E^{-1} y_j$. Substitute into (29) to obtain

$$f(x_{n-j+1}) = E^{-1} \, f(x_{n-j}) = (1-\delta) \, f(x_{n-j}) \tag{30}$$

whence the symbolic equality

$$E^{-1} = (1-\delta). \tag{31}$$

Also from (27) and (30), upon solving for $\Delta f(x)$ and $\delta f(x_{n-j})$ respectively; one obtains

$$\Delta f(x_i) = (E+1)\, f(x_i), \tag{32}$$

and

$$\delta f(x_{n-j}) = (1-E^{-1})\, f(x_{n-j}). \tag{33}$$

B. Interpolation: The general requirements for interpolation are considered; however, only the Gregory-Newton interpolation formulas are derived.

Let f be a function defined on the interval $[a,b]$. Consider the ordered set of number pairs $\{x_i,\, y_i\}_{i=0}^{n}$ satisfying the following conditions.

1. For every positive integer i such that $o \le i < n$, $x_{i+1} - x_i = h$.

2. The set $\{x_i\}_{i=0}^{n}$ consists of all real numbers x_i satisfying (1) above, and such that $a = x_0 < x_1 < \cdots < x_n = b$.

3. The set $\{y_i\}_{i=0}^{n}$ consists of the real numbers y_i such that for each x_i in the set $\{x_i\}_{i=0}^{n}$, $y_i = f(x_i)$.

Suppose that given only the set $\{x_i,\, y_i\}_{i=0}^{n}$ it is desired to obtain the value $f(x_k + rh)$ where r is a real number, and $r < 1$. It should be noted that there exists a value $f(x_k + rh)$ for arbitrary k and r only if the function f is defined on the entire interval $[a,b]$.

From (17) write

$$f(x_k) = E^k f(x_o).$$

(34)

Recall from the defining relation (14) for the operator E, that $Ef(x_k) = f(x_k+h)$, then for any real number r one may write

$$E^r f(x_k) = f(x_k+rh)$$

(35)

for any function $f(x)$ defined at $x = x_k+rh$. From (34) and (35) obtain

$$f(x_k+rh) = E^r f(x_k) = E^{k+r} f(x_o).$$

(36)

From (27) recall that $(1+\Delta) f(x_1) = Ef(x_1)$, hence from (36) write

$$f(x_k+rh) = (1+\Delta)^{k+r}f(x_o).$$

(37)

Since k is an integer, and r is a real number, the sum k+r is a real number. Let

$$k+r = R,$$

(38)

and write (37) in the form

$$f(x_k + rh) = (1+\Delta)^R f(x_o).$$

(39)

Upon expanding (39) using the binomial theorem one obtains

$$f(x_k + rh) = f(x_o) + R\Delta f(x_o) + \frac{R(R-1)}{2!} \Delta^2 f(x_o) + \cdots \tag{40}$$

which may be written in the more compact form

$$f(x_k + rh) = \sum_{m=0}^{R} \frac{R!}{m!\,(R-m)!} \Delta^m f(x_o) \tag{41}$$

where $R = k+r$. Equation (41) is a form of Newton's forward interpolation equation (Reference 5).

It has been shown that given an ordered set of number pairs $\{x_i\ y_i\}_{i=0}^{n}$, satisfying the three conditions stated previously, the value of the function may be determined (interpolated) at any point in the interval $[a,b]$ provided the function is defined on the entire interval $[a,b]$.

In interpolation between points near the end of the data set it is advantageous to use the inverted ordering of $\{y_j^{'}\}_{j=0}^{n}$ instead of the ordering $\{y_i\}_{i=0}^{n}$. There are several reasons for this preference; the given data set may be incomplete, or one may desire to compute the fewest number of differences necessary; however, the effect of round-off error is probably the most important reason. In sequential computations, such as those required to form Table 1, the effect of round-off error is an accumulative uncertainty in the computed values.

Suppose x_k is near the end of the set $\left\{x_i\right\}_{i=0}^{n}$. Since h is defined by the relation

$$x_{k+1} - x_k = h , \tag{42}$$

write

$$x_k + rh = x_{k+1} - (1-r)h. \tag{43}$$

Since r is a real number, so also is 1-r, let

$$r' = 1-r. \tag{44}$$

From (43) and (44) write

$$f(x_k + rh) = f(x_{k+1} - r'h). \tag{45}$$

Using the change of subscript (4), write

$$k = n-j-1, \quad k+1 = n-j. \tag{46}$$

This, with (45), yields

$$f(x_k + rh) = f(x_{n-j} - r'h) . \tag{47}$$

From (23), $f(x_{n-y}) = y'_j$. So, analogous to (35), write

$$E^{-r'} y'_j = f(x_{n-j} - r'h) . \tag{48}$$

From (48) and the definition of E^{-1}, obtain

$$E^{-j} E^{-1} y'_n = f(x_{n-j} - r'h). \tag{49}$$

From (49) and (31) write

$$f(x_{n-j} - r'h) = E^{-(j+r')} y'_n = (1-\delta)^{-(j+r')} y'_n . \tag{50}$$

Upon expanding (50) by the binomial theorem, and letting

$$j + r' = R' , \tag{51}$$

one obtains

$$\tag{52}$$
$$f(x_{n-j} - r'h) = (1-\delta)^{-R'} y'_n = y'_n + R'y'n + \frac{R'(R'+1)}{2!} y'_n+\ldots$$

Equation (52) may be written in the more compact form

$$f(x_{n-j}-r'h) = \sum_{m=0}^{R'} \frac{(n+m-1)!}{m!} \delta^m y'_n . \tag{53}$$

Observe from (45) that the two series (41) and (53) yield identical results for the two orderings $\left\{x_i, y_i\right\}_{i=0}^{n}$ and $\left\{x_j, y_j'\right\}_{j=0}^{n}$, respectively.

Equation (53) is the second of the two interpolation formulas to have been derived, and is known as the Gregory-Newton backward interpolation formula. As indicated, this formula is used to interpolate between values near the end of a data set.

C. Extrapolation: Let f be a function defined on the interval $[a,c]$, and let $[a,b]$ be a subinterval of $[a,c]$ such that $a < b < c$. Given the ordered set of number pairs $\left\{x_{n-j}, y_j'\right\}_{j=0}^{n}$ satisfying the conditions:

1. $x_{n-j} - x_{n-j-1} = h$ for every positive integer j such that $o < j \leq n$,

2. $\left\{x_{n-j}\right\}_{j=0}^{n}$ consists of the real numbers x_{n-j} satisfying (1) above, and such that $a = x_o < \cdots < x_{n-2} \leq x_{n-1} < x_n = b$,

3. $\left\{y_j'\right\}_{j=0}^{n}$ consists of the real numbers y_j' such that for each x_{n-j} in the set $\left\{x_{n-j}\right\}_{j=0}^{n}$ $y_j' = f(x_{n-j})$,

it is desired to extend, analytically, the data set $\left\{x_{n-j}, y_j\right\}_{j=0}^{n}$, which is defined on the interval $[a,b]$, into the interval $[b,c]$. This process is termed extrapolation.

The first step in extending the data set is to find the term $f(x_n + h)$. Use the formula (53) since $f(x_n)$ is at the end of the data set. Write

$$f(x_n + h) = Ef(x_n) = E^{-1} y_0' . \tag{54}$$

But $y_0' = E^{-n} y_n$, hence from (54) obtain

$$f(x_n + n) = E^{-(n+1)} y_n' . \tag{55}$$

From (31) $E^{-1} = (1-\delta)$, so that (55) may be written

$$f(x_n + h) = (1-\delta)^{n+1} y_n' . \tag{56}$$

Expand (56) to obtain

$$f(x_n + h) = y_{-1}' = \sum_{m=0}^{n+1} \frac{(n+m-1)!}{m!} \delta^m y_n' . \tag{57}$$

In like manner

$$f(x_n + 2h) = y_{-2}' = \sum_{m=0}^{n+2} \frac{(n+m-1)!}{m!} \delta^m y_n' . \tag{58}$$

By successive applications of (57) , the data set may be expanded throughout the interval $[c,d]$, with an increasing uncertainty in each successive point so obtained, provided that f is defined on the interval $[c,d]$.

III. COMPUTER PROGRAM TO PERFORM EXTRAPOLATIONS

From Table 1 it is evident that many calculations are necessary for one extrapolation, and that many extrapolations are necessary to show the trend of the deviation. To reduce labor and errors a computer program was written to perform the mathematical operations. The program was written in Fortran II for use in the IBM 1620 computer. The program was designed to use eleven data points from any function, take up to ten differences, and make up to 200 extrapolations. This program is shown in Table 2. A second program, designed to use data from arc-tangent functions only, was written to perform extrapolations as well as to compute the percent error of each extrapolated point. Table 3 shows this program. A flow diagram (Figure 1) illustrates the sequences of the programs.

In Fortran language certain mathematical notations have meanings different from those they ordinarily have in mathematics. To make the variables and subscripts fit the Fortran notation, changes are often necessary. The equal sign in Fortran literally means "replace the number on the left hand side of the statement by the quantity on the right hand side". Subscripted variables are noted by a parenthesis behind the numbers such as X(11) for X_{11}. Note that zero can never be used as a subscript.

The only variables requiring changes for use in the computer program were the variables denoting the sets of differences. Thus $\Delta_1 Y_1$, $\Delta_2 Y_2$, etc., were changed to D1Y(1), D2Y(2), etc., since the symbol Δ is not

readily available on the computer. Notice that the difference sets were designed for easy adaptibility to the computer by omitting all zeroes in the subscript notation.

Input data to the program consists of the interval size between data points (referred to as H), the value of X at the point where the extrapolations begin, (denoted $Y(1)$ to $Y(11)$). The subscript variable is N, and is set equal to eleven for the first calculation. The number of the extrapolation is M, and is set equal to zero at the first of the program. Storage of up to 200 terms for each set of differences is made available by the DIMENSION statement.

The first part of the program (statements 1-64) consists of taking the ten sets of differences for the eleven data points. After each set of differences is made the result must be checked to determine if it is to be last difference set. If it is, then all of the following variables representing sets of differences are set equal to zero. The computer goes through this part of the program only once.

The second part of the program is the section that makes the extrapolations (statements 65-75). The values for the extrapolated point, Y, and the corresponding value of X are calculated. The results are punched on a card. The subscript, N, and the extrapolation number, M, are then advanced by one increment for the next extrapolation. The value of M is checked since the computer stops after 200 extrapolations.

The third section of the program (statements 76-94) takes the sets of differences using the extrapolated point as the last data point.

As in the first section a check must be made after each difference to determine the final difference. When the last difference is found the program returns to section two to calculate the next extrapolation. Thus, sections two and three form a loop to perform the 200 extrapolations.

The only difference between the first and second programs is the second section. The second program makes the extrapolation in the same manner as the first, but after each extrapolation, the value of the arctangent at that point is computed and compared with the extrapolated value. From this the error is computed. Table 4 shows the output results from this program using data in increments of 0.1 from 6.0 to 5.0.

These two programs have proved to be efficient and versatile in the calculation of extrapolations for the functions used.

IV. EXTRAPOLATIONS OF SEVERAL FUNCTIONS

To test the extrapolation equation as well as the computer program, it was necessary to make extrapolations for well behaved curves and for those that are not so well behaved. Extrapolations were made of parabolic functions , cubic functions, circular functions, and on tabulated data such as log tables and interest tables. For the parabolic and cubic functions, which are polynomials, the extrapolations were exact and the curve may be extrapolated indefinitely. This is illustrated in Figures 1 and 2.

The accuracy of the extrapolations of a circular function depends on the significance of the data points, which in turn depends on the increment size. Figures 3A, 4A, and 5A show the deviation of the extrapolated circular functions from the actual curves for the data taken in different steps. The percent error versus number of extrapolations are illustrated in Figures 3B, 4B, and 5B for each associated circular function. Figures 6 and 7 present the percent deviation for extrapolations of tabulated log and interest data. Since the data points from the interest table have more significant figures than those of the log table , the deviation for any given number of extrapolations was less.

V. EXTRAPOLATIONS OF THE ARC-TANGENT FUNCTION

To estimate the extrapolation error for tracking data, the arc-tangent function, which is similar in form, was used. Figure 8 shows a plot of the arc-tangent curve as used in this system. The arc-tangent varies slowly until X approaches the cross-over point at the origin. Near this point the change becomes very rapid. Extrapolations were made using data samples along several points on the curve.

Two sets of extrapolations were made, one with the data samples taken at intervals of 1.0 and the other with the data samples taken at intervals of 0.1. Figures 9A through 29A are plots of the extrapolated points compared with the actual arc-tangent curve for data samples in intervals of 1.0 while Figures 30A through 42A are the same type curves for the data samples taken in intervals of 0.1. Figures 9B through 42B are plots of the percent error versus the number of extrapolations for each of the associated curves. On each curve is a notation showing where the data was taken and where the extrapolations started. The extrapolated curve is shown for that portion of the graph where it does not lie on top of the actual curve. For the part of the curve where the arc-tangent curve and the extrapolated curve seem to be the same, the difference is so slight that it cannot be shown on this curve. By referring to the error curve, the deviation for this part of the curve can be obtained. Referring to these curves it is evident that for data samples taken at almost any point, the extrapolated curve follows the arc-tangent

very closely until it nears the origin. For data samples taken a large distance from the origin, the error of the extrapolations was less than 1% for more than 100 extrapolations; however, for data taken very near the origin, the error in the extrapolation increased very rapidly. By comparing Figures 17A and 31A, it can be shown that the accuracy is improved by taking the data samples in smaller intervals, and for the same number of extrapolations the accuracy is greater for the smaller data increments (compare 17B and 31B.). Comparison of Figures 22A and 42A also reveals that the accuracy is better for smaller increments.

With actual tracking data the input signals will contain random noise which will tend to increase errors in the extrapolated points. Since there was no actual tracking data available, no conclusions are drawn as to the exact results of noise; however, if enough data samples are used, the increase in the error of the extrapolations due to random noise should be tolerable.

VI.   BLOCK DIAGRAMS OF THE COMPUTER

There are 2 types of computing systems which may be used to perform the calculations of the extrapolation computer. These are the analog type and the digital type. Each of these systems has several relative advantages and disadvantages with respect to the other. Both systems use a control gate and a buffer (or gate) in the input. The output of the computer is fed back through the control gate to the buffer. The control gate, however, is blocked by the data signal input. If a loss of input occurs, then the control gate conducts and the extrapolations begin.

A.   Analog Systems

Figure 43 is a diagram of an analog system which uses a series of differentiators to perform the difference function. The output of each differentiator is applied to another differentiator and also into a summing amplifier. This system can incorporate as many differentiators as desired. The output of the summing amplifier should produce a very accurate value of extrapolation, since differentiators are used in the system rather than some method of subtracting.

There are several problems associated with this system, such as the problems of drift and the inherent delay in the operational amplifiers. The biggest problem is that of the differentiator. These problems are probably amenable to solution in a laboratory environment, but the prospects of solutions for a space vehicle environment are not promising.

Another disadvantage arises in case the signal contains noise. In this case differentiation tends to enhance the noise while diminishing the signal.

B. Digital Systems

There are many methods by which the extrapolations may be accomplished digitally. The system in Figure 44 is a simple method of implementing the Gregory-Newton equation. Basically, this system uses a control gate, an amplitude-analog-to-digital converter, a method of shifting data points, a bank of binary subtracters, and a digital to analog converter. In this system the control gate performs the same function as in the analog system of blocking the extrapolated signal feed-back in the presence of the data signal. The output of the buffer is fed through the timing gate to the analog converter. The timing gate is controlled by the signals from the timing pulse generator. The analog convertor is designed to yield a number of pulses proportional to the amplitude of the analog input. The output of this converter is applied to the binary counter. The counter must consist of enough binaries to count the highest possible number of pulses. The system illustrated uses five binaries for a total of 32 pulses; in practice at least ten would probably be needed. This number is determined by the settings in the analog to digital converter required to achieve good resolution, so that a slight change in amplitude will be detected. After each counting period the binaries are reset for the next cycle. This is achieved by the reset pulse from the timing pulse generators. This first part of

the system could probably be achieved with little difficulty.

The next part of the system consists of the shift-registers and binary subtracters. The output of the five bit binary counter is applied through and-gates to five, two stage, shift-registers. There is one shift-register needed for each bit in the system. The two outputs from each shift-register are fed to a five bit subtracter in such a manner that the earlier signal is algebraically subtracted from the later one. The shift pulse for the register is generated by the timing pulse generator. The output of the subtracter is applied to a second shift-register, the output of which is fed to another five bit subtracter in the same manner. This process of shifting and subtracting is the method of obtaining differences, so that this process must be repeated as many times as is necessary for accurate results. It must be noted that the subtracter is really an adder-subtracter, depending on the algebraic signs of the two inputs. An extra bit must be carried to denote the algebraic sign, since these must be carried all the way through the system. The outputs of each subtracter, together with their signs, are added in the summing amplifier. This output is then converted to an analog signal by the digital-to-analog converter and fed back through the control gate as another extrapolated point.

There are several disadvantages to this system. The system would be very complex and would require an exact sequential timing system.

Each of the control pulses, the timing pulse, shift pulse, reset pulse, etc., would have to be sequenced and synchronized. In order to have the desired resolution to detect a small change in amplitude, many digital bits would be necessary.

VII. CONCLUSIONS

From the results of the extrapolations of the arc-tangent data, it is evident that extrapolations can be made with a high degree of accuracy along the slowly changing portion of the curve. Along the rapidly changing portion near the origin, the extrapolations deviate from the true data rapidly so that the accuracy is poor after three or four extrapolations. The extrapolation should reduce the probability of loss of phase-lock during the period of signal loss, even if the signal loss occurs near the origin.

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X	Y	$\Delta_1 Y$	$\Delta_2 Y$	$\Delta_3 Y$	$\Delta_4 Y$	$\Delta_5 Y$	$\Delta_6 Y$	$\Delta_7 Y$	$\Delta_8 Y$	$\Delta_9 Y$	$\Delta_{10} Y$
X_1	Y_1										
		$\Delta_1 Y_2$									
X_2	Y_2		$\Delta_2 Y_3$								
		$\Delta_1 Y_3$		$\Delta_3 Y_4$							
X_3	Y_3		$\Delta_2 Y_4$		$\Delta_4 Y_5$						
		$\Delta_1 Y_4$		$\Delta_3 Y_5$		$\Delta_5 Y_6$					
X_4	Y_4		$\Delta_2 Y_5$		$\Delta_4 Y_6$		$\Delta_6 Y_7$				
		$\Delta_1 Y_5$		$\Delta_3 Y_6$		$\Delta_5 Y_7$		$\Delta_7 Y_8$			
X_5	Y_5		$\Delta_2 Y_6$		$\Delta_4 Y_7$		$\Delta_6 Y_8$		$\Delta_8 Y_9$		
		$\Delta_1 Y_6$		$\Delta_3 Y_7$		$\Delta_5 Y_8$		$\Delta_7 Y_9$		$\Delta_9 Y_{10}$	
X_6	Y_6		$\Delta_2 Y_7$		$\Delta_4 Y_8$		$\Delta_6 Y_9$		$\Delta_8 Y_{10}$		$\Delta_{10} Y_{11}$
		$\Delta_1 Y_7$		$\Delta_3 Y_8$		$\Delta_5 Y_9$		$\Delta_7 Y_{10}$		$\Delta_9 Y_{11}$	
X_7	Y_7		$\Delta_2 Y_8$		$\Delta_4 Y_9$		$\Delta_6 Y_{10}$		$\Delta_8 Y_{11}$		
		$\Delta_1 Y_8$		$\Delta_3 Y_9$		$\Delta_5 Y_{10}$		$\Delta_7 Y_{11}$			
X_8	Y_8		$\Delta_2 Y_9$		$\Delta_4 Y_{10}$		$\Delta_6 Y_{11}$				
		$\Delta_1 Y_9$		$\Delta_3 Y_{10}$		$\Delta_5 Y_{11}$					
X_9	Y_9		$\Delta_2 Y_{10}$		$\Delta_4 Y_{11}$						
		$\Delta_1 Y_{10}$		$\Delta_3 Y_{11}$							
X_{10}	Y_{10}		$\Delta_2 Y_{11}$								
		$\Delta_1 Y_{11}$									
X_{11}	Y_{11}										

TABLE ONE: DIAGONAL DIFFERENCE TABLE

TABLE TWO

FORTRAN II PROGRAM FOR NEWTONS EXTRAPOLATION EQUATION

USING TEN DIFFERENCES AND 200 EXTRAPOLATIONS

```
1 DIMENSION X(200), Y(200), D1Y(200), D2Y(200), D3Y(200), D4Y(200),
 1 D5Y(200), D6Y(200), D7Y(200), D8Y(200), D9Y(200), DOY(200)
2 READ 03,H,X(11)
3 FORMAT (2F10.0)
4 READ 05, (Y(J),J=1,11)
5 FORMAT(4E15.0/4E15.0/3E15.0)
6 N=11
7 M = 0
8 DO 09 J=2,11
9 D1Y(J)=Y(J)-Y(J-1)
10 FORMAT(E20.8)
11 DO 12 J=2,11
12 PUNCH 10, D1Y(J)
13 DO 14 J=3,11
14 D2Y(J)=D1Y(J)-D1Y(J-1)
15 IF(ABSF(D2Y(N))-ABSF(D1Y(N)))16,56,56
16 DO 17 J=3,11
17 PUNCH 10, D2Y(J)
18 DO 19 J=4,11
19 D3Y(J)=D2Y(J)-D2Y(J-1)
```

TABLE TWO (CONTINUED)

```
20 IF(ABSF(D3Y(N))-ABSF(D2Y(N)))21,57,57

21 DO 22 J=4,11

22 PUNCH 10, D3Y(J)

23 DO 24 J=5,11

24 D4Y(J)=D3Y(J)-D3Y(J-1)

25 IF(ABSF(D4Y(N))-ABSF(D3Y(N)))26,58,58

26 DO 27 J=5,11

27 PUNCH 10, D4Y(J)

28 DO 29 J=6,11

29 D5Y(J)=D4Y(J)-D4Y(J-1)

30 IF(ABSF(D5Y(N))-ABSF(D4Y(N)))31,59,59

31 DO 32 J=6,11

32 PUNCH 10, D5Y(J)

33 DO 34 J=7,11

34 D6Y(J)=D5Y(J)-D5Y(J-1)

35 IF(ABSF(D6Y(N))-ABSF(D5Y(N)))36,60,60

36 DO 37 J=7,11

37 PUNCH 10, D6Y(J)

38 DO 39 J=8,11

39 D7Y(J)=D6Y(J)-D6Y(J-1)

40 IF(ABSF(D7Y(N))-ABSF(D6Y(N)))41,61,61

41 DO 42 J=8,11

42 PUNCH 10, D7Y(J)
```

TABLE TWO (CONTINUED)

```
43 DO 44 J=9,11

44 D8Y(J)=D7Y(J)-D7Y(J-1)

45 IF(ABSF(D8Y(N))-ABSF(D7Y(N)))46,62,62

46 DO 47 J=9,11

47 PUNCH 10, D8Y(J)

48 DO 49 J=10,11

49 D9Y(J)=D8Y(J)-D8Y(J-1)

50 IF(ABSF(D9Y(N))-ABSF(D8Y(N)))51,63,63

51 DO 52 J=10,11

52 PUNCH 10, D9Y(J)

53 DOY(11)=D9Y(11)-D9Y(10)

54 PUNCH 10, DOY(11)

55 IF(ABSF(DOY(N))-ABSF(D9Y(N)))95,64,64

56 D2Y(N)=0.0

57 D3Y(N)=0.0

58 D4Y(N)=0.0

59 D5Y(N)=0.0

60 D6Y(N)=0.0

61 D7Y(N)=0.0

62 D8Y(N)=0.0

63 D9Y(N)=0.0

64 DOY(N)=0.0

65 PUNCH 66, M, N, X(N), Y(N)
```

TABLE TWO (CONTINUED)

```
66 FORMAT(2I5,2E15.8)

72 X(N+1) =X(N) + H

73 Y(N+1)= Y(N)+D1Y(N)+D2Y(N)+D3Y(N)+D4Y(N)+D5Y(N)

   1+D6Y(N)+D7Y(N)+D8Y(N)+D9Y(N)+D0Y(N)

     N = N+1

     M = M+1

74 IF(M-101)76, 101, 75

75 IF(M-201)76, 105, 76

76 D1Y(N)=Y(N)-Y(N-1)

77 D2Y(N)=D1Y(N)-D1Y(N-1)

78 IF(ABSF(D2Y(N))-ABSF(D1Y(N)))79,56,56

79 D3Y(N)=D2Y(N)-D2Y(N-1)

80 IF(ABSF(D3Y(N))-ABSF(D2Y(N)))81,57,57

81 D4Y(N)=D3Y(N)-D3Y(N-1)

82 IF(ABSF(D4Y(N))-ABSF(D3Y(N)))83,58,58

83 D5Y(N)=D4Y(N)-D4Y(N-1)

84 IF(ABSF(D5Y(N))-ABSF(D4Y(N)))85,59,59

85 D6Y(N)=D5Y(N)-D5Y(N-1)

86 IF(ABSF(D6Y(N))-ABSF(D5Y(N)))87,60,60

87 D7Y(N)=D6Y(N)-D6Y(N-1)

88 IF(ABSF(D7Y(N))-ABSF(D6Y(N)))89,61,61

89 D8Y(N)=D7Y(N)-D7Y(N-1)

90 IF(ABSF(D8Y(N))-ABSF(D7Y(N)))91,62,62
```

TABLE TWO (CONTINUED)

```
91 D9Y(N)=D8Y(N)-D8Y(N-1)

92 IF(ABSF(D9Y(N))-ABSF(D8Y(N)))93,63,63

93 D0Y(N) =D9Y(N)-D9Y(N-1)

94 IF(ABSF(D0Y(N))-ABSF(D9Y(N)))65,64,64

95 TYPE 96

96 FORMAT(22HNOT ENOUGH DIFFERENCES)

97 PAUSE

98 GO TO 65

101 TYPE 102

102 FORMAT(18H100 EXTRAPOLATIONS)

103 PAUSE

104 GO TO 76

105 TYPE 106

106 FORMAT(18H200 EXTRAPOLATIONS)

107 PAUSE

108 GO TO 76

99 STOP

   END
```

TABLE THREE

FORTRAN II PROGRAM FOR NEWTONS EXTRAPOLATION EQUATION

FOR ARCTANGENT CURVE WITH ERROR CALCULATION

```
1 DIMENSION X(200), Y(200), D1Y(200), D2Y(200), D3Y(200), D4Y(200),
 1 D5Y(200), D6Y(200), D7Y(200), D8Y(200), D9Y(200), D0Y(200)
2 READ 03,H,X(11)
3 FORMAT (2F10.0)
4 READ 05, (Y(J),J=1,11)
5 FORMAT(4E15.0/4E15.0/3E15.0)
6 N=11
7 M = 0
8 DO 09 J=2,11
9 D1Y(J)=Y(J)-Y(J-1)
10 FORMAT(E20.8)
11 DO 12 J=2,11
12 PUNCH 10, D1Y(J)
13 DO 14 J=3,11
14 D2Y(J)=D1Y(J)-D1Y(J-1)
15 IF(ABSF(D2Y(N))-ABSF(D1Y(N)))16,56,56
16 DO 17 J=3,11
17 PUNCH 10, D2Y(J)
18 DO 19 J=4,11
19 D3Y(J)=D2Y(J)-D2Y(J-1)
```

```
20 IF(ABSF(D3Y(N))-ABSF(D2Y(N)))21,57,57

21 DO 22 J=4,11

22 PUNCH 10, D3Y(J)

23 DO 24 J=5,11

24 D4Y(J)=D3Y(J)-D3Y(J-1)

25 IF(ABSF(D4Y(N))-ABSF(D3Y(N)))26,58,58

26 DO 27 J=5,11

27 PUNCH 10, D4Y(J)

28 DO 29 J=6,11

29 D5Y(J)=D4Y(J)-D4Y(J-1)

30 IF(ABSF(D5Y(N))-ABSF(D4Y(N)))31,59,59

31 DO 32 J=6,11

32 PUNCH 10, D5Y(J)

33 DO 34 J=7,11

34 D6Y(J)=D5Y(J)-D5Y(J-1)

35 IF(ABSF(D6Y(N))-ABSF(D5Y(N)))36,60,60

36 DO 37 J=7,11

37 PUNCH 10, D6Y(J)

38 DO 39 J=8,11

39 D7Y(J)=D6Y(J)-D6Y(J-1)

40 IF(ABSF(D7Y(N))-ABSF(D6Y(N)))41,61,61

41 DO 42 J=8,11

42 PUNCH 10, D7Y(J)
```

TABLE THREE (CONTINUED)

```
43  DO 44  J=9,11

44  D8Y(J)=D7Y(J)-D7Y(J-1)

45  IF(ABSF(D8Y(N))-ABSF(D7Y(N)))46,62,62

46  DO 47  J=9,11

47  PUNCH 10, D8Y(J)

48  DO 49  J=10,11

49  D9Y(J)=D8Y(J)-D8Y(J-1)

50  IF(ABSF(D9Y(N))-ABSF(D8Y(N)))51,63,63

51  DO 52  J=10,11

52  PUNCH 10, D9Y(J)

53  DOY(11)=D9Y(11)-D9Y(10)

54  PUNCH 10, DOY(11)

55  IF(ABSF(DOY(N))-ABSF(D9Y(N)))95,64,64

56  D2Y(N)=0.0

57  D3Y(N)=0.0

58  D4Y(N)=0.0

59  D5Y(N)=0.0

60  D6Y(N)=0.0

61  D7Y(N)=0.0

62  D8Y(N)=0.0

63  D9Y(N)=0.0

64  DOY(N)=0.0

65  B = ATANF(X(N))
```

```
66 IF(B)69,67,69

67 ERR = 100.0

68 GO TO 70

69 ERR = ABSF((Y(N)-B)/B*100.0)

70 PUNCH 71, M , N, X(N), Y(N), B, ERR

71 FORMAT(2I5,4F15.0)

72 X(N+1) =X(N) + H

73 Y(N+1)= Y(N)+D1Y(N)+D2Y(N)+D3Y(N)+D4Y(N)+D5Y(N)

  1+D6Y(N)+D7Y(N)+D8Y(N)+D9Y(N)+D0Y(N)

    N = N+1

    M = M+1

74 IF(M-101)76, 101, 75

75 IF(M-201)76, 105, 76

76 D1Y(N)=Y(N)-Y(N-1)

77 D2Y(N)=D1Y(N)-D1Y(N-1)

78 IF(ABSF(D2Y(N))-ABSF(D1Y(N)))79,56,56

79 D3Y(N)=D2Y(N)-D2Y(N-1)

80 IF(ABSF(D3Y(N))-ABSF(D2Y(N)))81,57,57

81 D4Y(N)=D3Y(N)-D3Y(N-1)

82 IF(ABSF(D4Y(N))-ABSF(D3Y(N)))83,58,58

83 D5Y(N)=D4Y(N)-D4Y(N-1)

84 IF(ABSF(D5Y(N))-ABSF(D4Y(N)))85,59,59

85 D6Y(N)=D5Y(N)-D5Y(N-1)
```

TABLE THREE (CONTINUED)

```
86  IF(ABSF(D6Y(N))-ABSF(D5Y(N)))87,60,60

87  D7Y(N)=D6Y(N)-D6Y(N-1)

88  IF(ABSF(D7Y(N))-ABSF(D6Y(N)))89,61,61

89  D8Y(N)=D7Y(N)-D7Y(N-1)

90  IF(ABSF(D8Y(N))-ABSF(D7Y(N)))191,62,62

91  D9Y(N)=D8Y(N)-D8Y(N-1)

92  IF(ABSF(D9Y(N))-ABSF(D8Y(N)))93,63,63

93  DOY(N) =D9Y(N)-D9Y(N-1)

94  IF(ABSF(DOY(N))-ABSF(D9Y(N)))65,64,64

95  TYPE 96

96  FORMAT(22HNOT ENOUGH DIFFERENCES)

97  PAUSE

98  GO TO 65

101 TYPE 102

102 FORMAT(18H100 EXTRAPOLATIONS)

103 PAUSE

104 GO TO 76

105 TYPE 106

106 FORMAT(18H200 EXTRAPOLATIONS)

107 PAUSE

108 GO TO 76

99  STOP

    END
```

TABLE FOUR

COMPUTER RESULTS FOR EXTRAPOLATIONS MADE FROM AN ARCTANGENT CURVF
WITH DATA TAKEN FROM 6.0 TO 5.0 IN INCREMENTS OF 0.1

N	M	X(N)	Y(N)	ARCTAN X	ERROR
	11	50000000.F-07	13734007.E-07	13734007.F-07	00000000.E-99
1	12	49900000.E-07	13730232.E-07	13730154.E-07	56809268.F-11
2	13	49800000.E-07	13726456.E-07	13726285.E-07	12457850.E-10
3	14	49700000.E-07	13722679.E-07	13722402.E-07	20185970.F-10
4	15	49600000.E-07	13718901.E-07	13718503.E-07	29011911.F-10
5	16	49500000.F-07	13715122.E-07	13714590.E-07	38790805.E-10
6	17	49400000.E-07	13711342.E-07	13710661.F-07	49669377.E-10
7	18	49300000.F-07	13707561.E-07	13706717.F-07	61575649.F-10
8	19	49200000.E-07	13703779.E-07	13702757.E-07	74583530.E-10
9	20	49100000.F-07	13699996.E-07	13698782.F-07	88621017.F-10
10	21	49000000.E-07	13696212.E-07	13694792.F-07	10368905.E-09
11	22	48900000.E-07	13692427.E-07	13690786.E-07	11986163.E-09
12	23	48800000.E-07	13688641.E-07	13686763.E-07	13721286.E-09
13	24	48700000.E-07	13684854.E-07	13682726.E-07	15552456.E-09
14	25	48600000.E-07	13681066.E-07	13678672.E-07	17501698.E-09
15	26	48500000.F-07	13677277.E-07	13674602.E-07	19561812.E-09
16	27	48400000.F-07	13673487.F-07	13670516.F-07	21732903.F-09
17	28	48300000.F-07	13669696.E-07	13666414.F-07	24015078.F-09

TABLE FOUR (CONTINUED)

18	29	48200000.E-07	13665904.E-07	13662295.E-07	26415766.E-09
19	30	48100000.E-07	13662111.E-07	13658160.E-07	28927761.E-09
20	31	48000000.E-07	13658317.E-07	13654009.E-07	31551172.E-09
21	32	47900000.E-07	13654522.E-07	13649841.E-07	34293439.E-09
22	33	47800000.E-07	13650726.E-07	13645656.E-07	37154681.E-09
23	34	47700000.E-07	13646929.E-07	13641455.E-07	40127684.E-09
24	35	47600000.E-07	13643131.E-07	13637236.E-07	43227234.E-09
25	36	47500000.E-07	13639332.E-07	13633001.E-07	46438784.E-09
26	37	47400000.E-07	13635532.E-07	13628748.E-07	49777132.E-09
27	38	47300000.E-07	13631731.E-07	13624478.E-07	53235067.E-09
28	39	47200000.E-07	13627929.E-07	13620191.E-07	56812712.E-09
29	40	47100000.E-07	13624126.E-07	13615886.E-07	60517545.E-09
30	41	47000000.E-07	13620322.E-07	13611564.E-07	64342348.E-09
31	42	46900000.E-07	13616517.E-07	13607225.E-07	68287251.E-09
32	43	46800000.E-07	13612711.E-07	13602867.E-07	72367097.E-09
33	44	46700000.E-07	13608904.E-07	13598492.E-07	76567313.E-09
34	45	46600000.E-07	13605096.E-07	13594099.E-07	80895394.E-09
35	46	46500000.E-07	13601287.E-07	13589687.E-07	85358846.E-09
36	47	46400000.E-07	13597477.E-07	13585258.E-07	89943083.E-09
37	48	46300000.E-07	13593666.E-07	13580810.E-07	94662984.E-09
38	49	46200000.E-07	13589854.E-07	13576344.E-07	99511326.E-09
39	50	46100000.E-07	13586041.E-07	13571859.E-07	10449563.E-08
40	51	46000000.E-07	13582227.E-07	13567356.E-07	10960868.E-08

TABLE FOUR (CONTINUED)

41	52	45900000.E-07	13578412.E-07	13562834.E-07	11485800.E-08
42	53	45800000.E-07	13574596.E-07	13558293.E-07	12024375.E-08
43	54	45700000.E-07	13570779.E-07	13553733.E-07	12576608.E-08
44	55	45600000.E-07	13566961.E-07	13549154.E-07	13142517.E-08
45	56	45500000.E-07	13563142.E-07	13544556.E-07	13722118.E-08
46	57	45400000.E-07	13559322.E-07	13539939.E-07	14315426.E-08
47	58	45300000.E-07	13555501.E-07	13535302.E-07	14923198.E-08
48	59	45200000.E-07	13551679.E-07	13530645.E-07	15545452.E-08
49	60	45100000.E-07	13547856.E-07	13525969.E-07	16181465.E-08
50	61	45000000.E-07	13544032.E-07	13521273.E-07	16831995.E-08
51	62	44900000.E-07	13540207.E-07	13516558.E-07	17496318.E-08
52	63	44800000.E-07	13536381.E-07	13511822.E-07	18175935.E-08
53	64	44700000.E-07	13532554.E-07	13507065.E-07	18870864.E-08
54	65	44600000.E-07	13528726.E-07	13502289.E-07	19579643.E-08
55	66	44500000.E-07	13524897.E-07	13497492.E-07	20303771.E-08
56	67	44400000.E-07	13521067.E-07	13492675.E-07	21042528.E-08
57	68	44300000.E-07	13517236.E-07	13487837.E-07	21796675.E-08
58	69	44200000.E-07	13513404.E-07	13482978.E-07	22566231.E-08
59	70	44100000.E-07	13509571.E-07	13478098.E-07	23351217.E-08
60	71	44000000.E-07	13505737.E-07	13473197.E-07	24151654.E-08
61	72	43900000.E-07	13501902.E-07	13468275.E-07	24967562.E-08
62	73	43800000.E-07	13498066.E-07	13463331.E-07	25799707.E-08
63	74	43700000.E-07	13494229.E-07	13458366.E-07	26647365.E-08

TABLE FOUR (CONTINUED)

64	75	43600000.E-07	13490391.E-07	13453379.E-07	27511304.E-08
65	76	43500000.E-07	13486552.E-07	13448371.E-07	28390799.E-08
66	77	43400000.E-07	13482712.E-07	13443340.E-07	29287364.E-08
67	78	43300000.E-07	13478871.E-07	13438288.E-07	30199531.E-08
68	79	43200000.E-07	13475029.E-07	13433213.E-07	31128814.E-08
69	80	43100000.E-07	13471186.E-07	13428116.E-07	32074492.E-08
70	81	43000000.E-07	13467342.E-07	13422996.E-07	33037333.E-08
71	82	42900000.E-07	13463497.E-07	13417854.E-07	34016616.E-08
72	83	42800000.E-07	13459651.E-07	13412689.E-07	35013113.E-08
73	84	42700000.E-07	13455804.E-07	13407501.E-07	36026847.E-08
74	85	42600000.E-07	13451956.E-07	13402290.E-07	37057846.E-08
75	86	42500000.E-07	13448107.E-07	13397056.E-07	38106133.E-08
76	87	42400000.E-07	13444257.E-07	13391799.E-07	39171734.E-08
77	88	42300000.E-07	13440406.E-07	13386517.E-07	40256177.E-08
78	89	42200000.E-07	13436554.E-07	13381212.E-07	41357987.E-08
79	90	42100000.E-07	13432701.E-07	13375884.E-07	42477192.E-08
80	91	42000000.E-07	13428847.E-07	13370531.E-07	43615320.E-08
81	92	41900000.E-07	13424992.E-07	13365154.E-07	44771650.E-08
82	93	41800000.E-07	13421136.E-07	13359753.E-07	45946208.E-08
83	94	41700000.E-07	13417279.E-07	13354327.E-07	47139777.E-08
84	95	41600000.E-07	13413421.E-07	13348877.E-07	48351632.E-08
85	96	41500000.E-07	13409562.E-07	13343401.E-07	49583310.E-08
86	97	41400000.E-07	13405702.E-07	13337901.E-07	50833335.E-08

```
┌─────────┐           ╭─────────────────────╮
│         │           │  READ H, K(11)      │
│  START  │──────────▶│                     │──────
│         │           │  READ J(1) TO J(11) │
└─────────┘           ╰─────────────────────╯


         ╱╲                    ┌──────────────────┐
        ╱  ╲                   │  D4Y(J) = D3Y(J) │
 <    ╱ ABS(D4Y):╲   ◀──(24)◀──│                  │◀────
    ╱  ABS(D3Y)  ╲             │    - D3Y(J-1)    │
      ╲         ╱              └──────────────────┘
       ╲      ╱
        ╲    ╱
    ≥    ╲  ╱


   ┌──────────┐              ┌──────────────────┐
   │  DO 29   │              │  D5Y(J) = D4Y(J) │
   │          │──────▶( )───▶│                  │◀────
   │ J = 6, 11│              │    - D4Y(J-1)    │
   └──────────┘              └──────────────────┘


                        ≥
   ┌──────────┐        ╱╲
   │          │       ╱  ╲              ┌────────┐
( )│   STOP   │  <   ╱ABS(D5Y):╲   ◀────│ D5Y(N) │
   │          │    ╱  ABS(D4Y) ╲        │  - D...│
   └──────────┘      ╲         ╱        └────────┘
                      ╲      ╱


        ╭───────────────╮        ┌──────────────────┐
        │  PUNCH M, N,  │        │  Y(N+1) = Y(N) + │
        │               │───────▶│  D1Y(N) + D2Y(N) │
        │  X(N), Y(N):  │        │    + D3Y(N) +    │
        ╰───────────────╯        │  D4Y(N) + D5Y(N) │
                                 └──────────────────┘
```

47-1
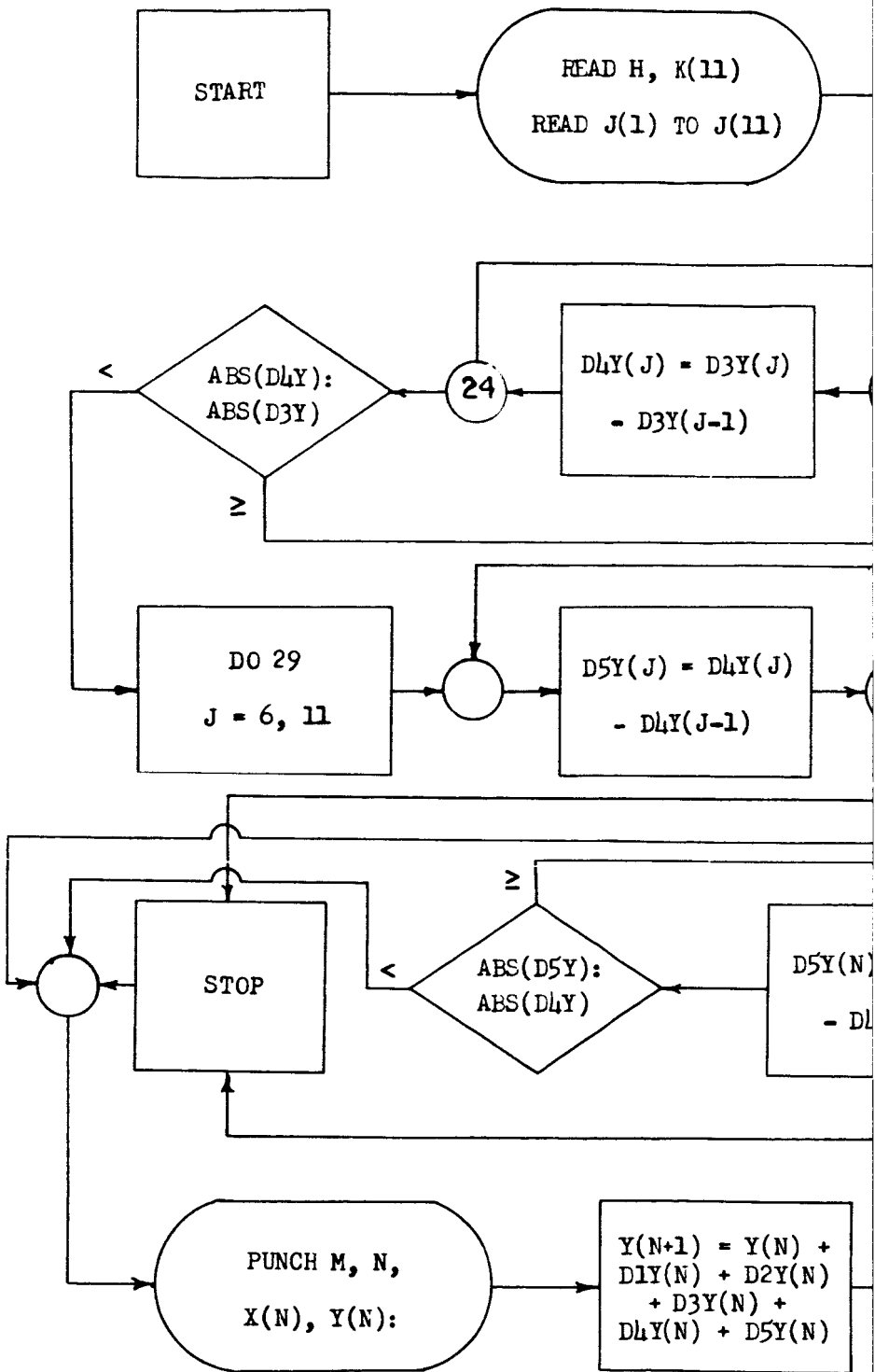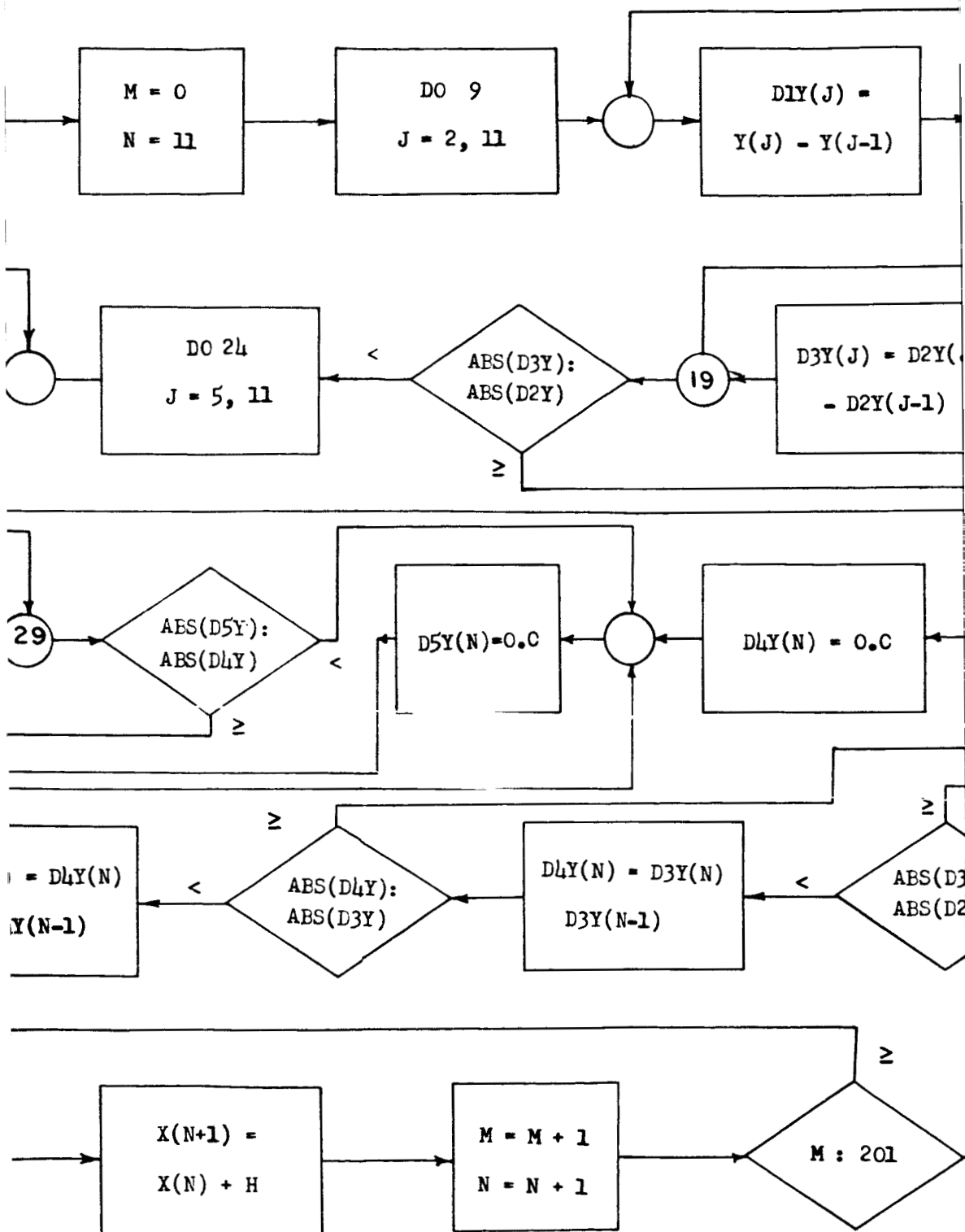
Figure 1.   Flow Diagram of Fortran II Program for Computing Extrapolati
Using Newton's Extrapolation Equation.

47-2

Figure 2-b. Cubic Curve from Equation
$$y = x^3 + 6x^2 + 3x + 10$$

X (Units)

Y(units)

Figure 2-a. Parabolic Curve from Equation
$$y = x^2/4$$

X (Units)
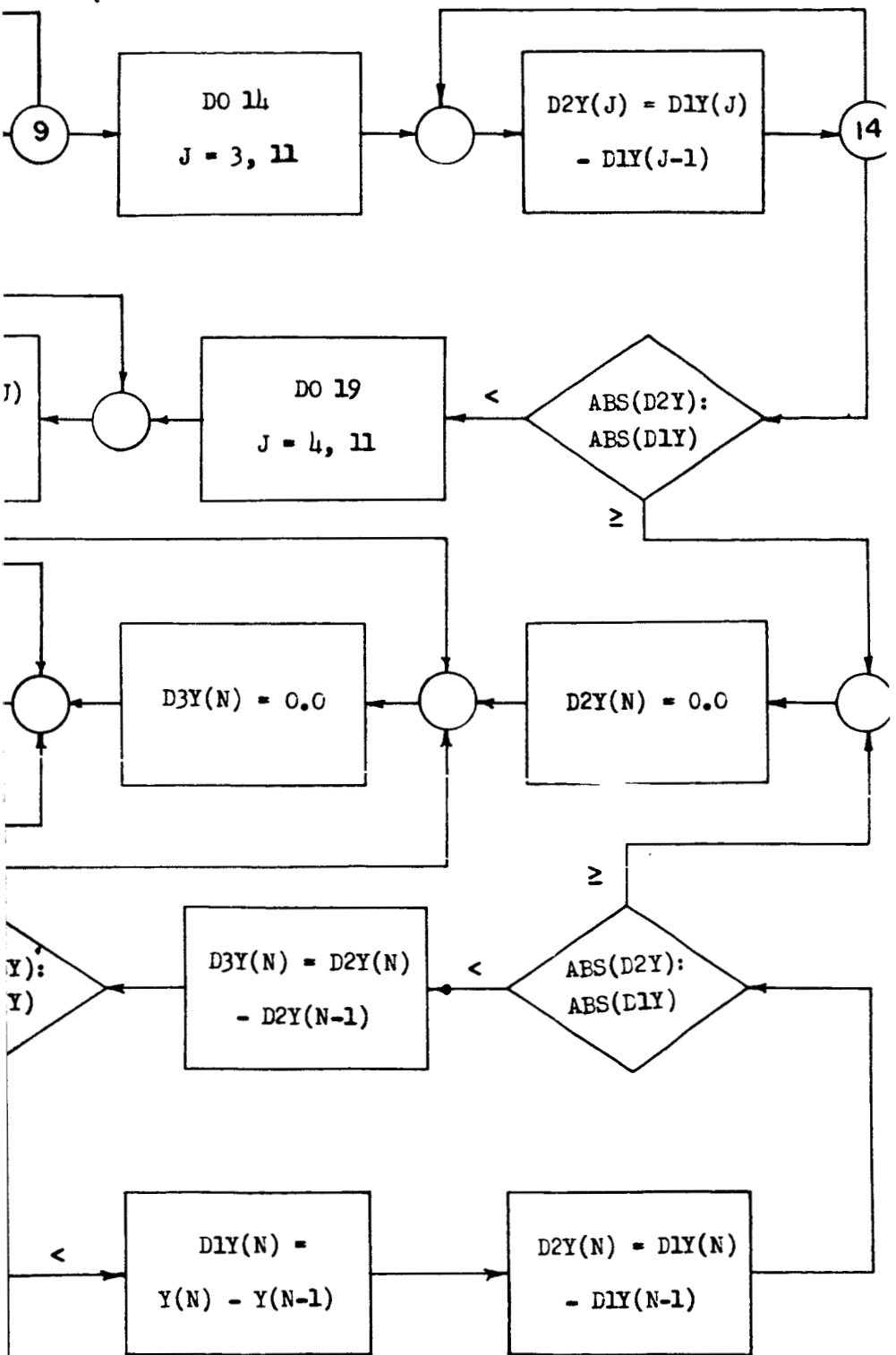
Y(units)
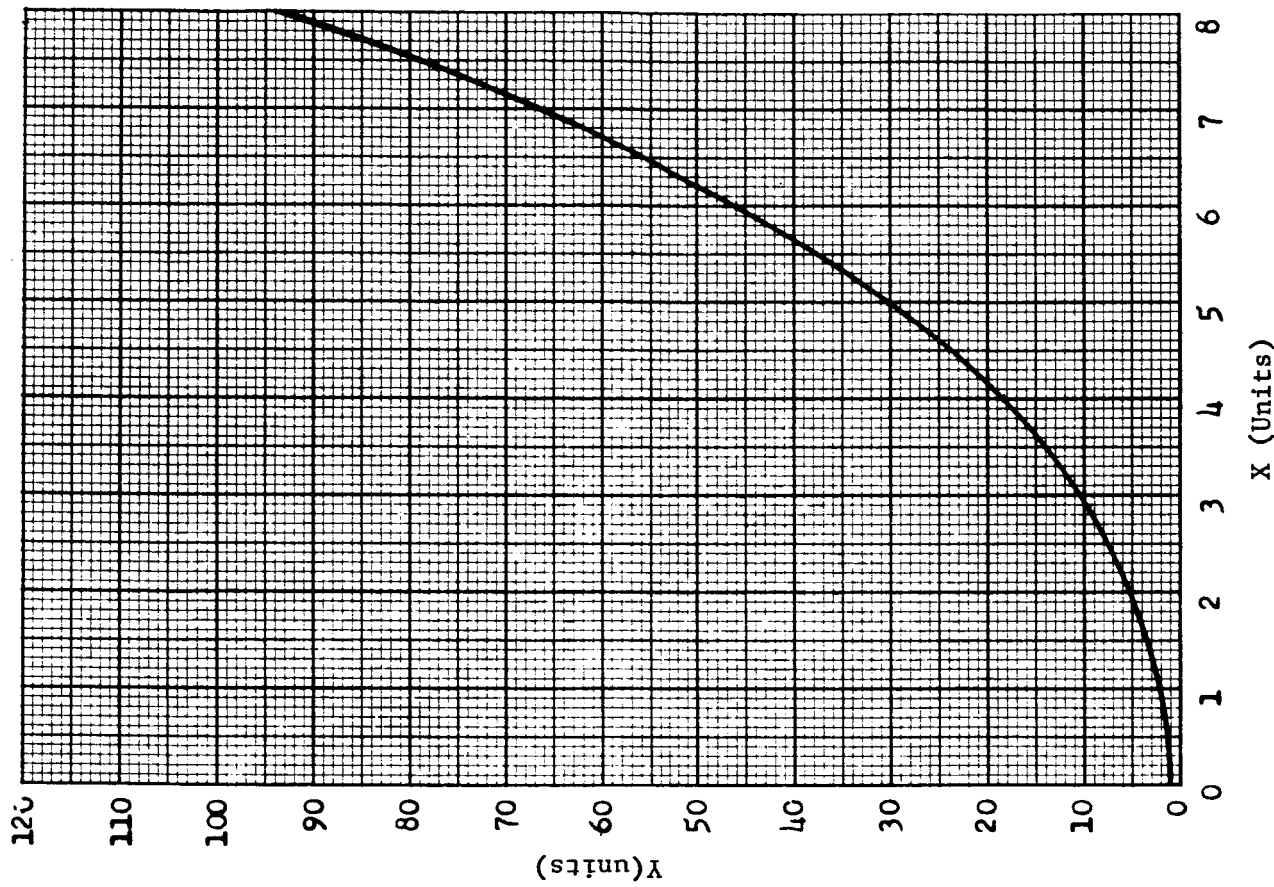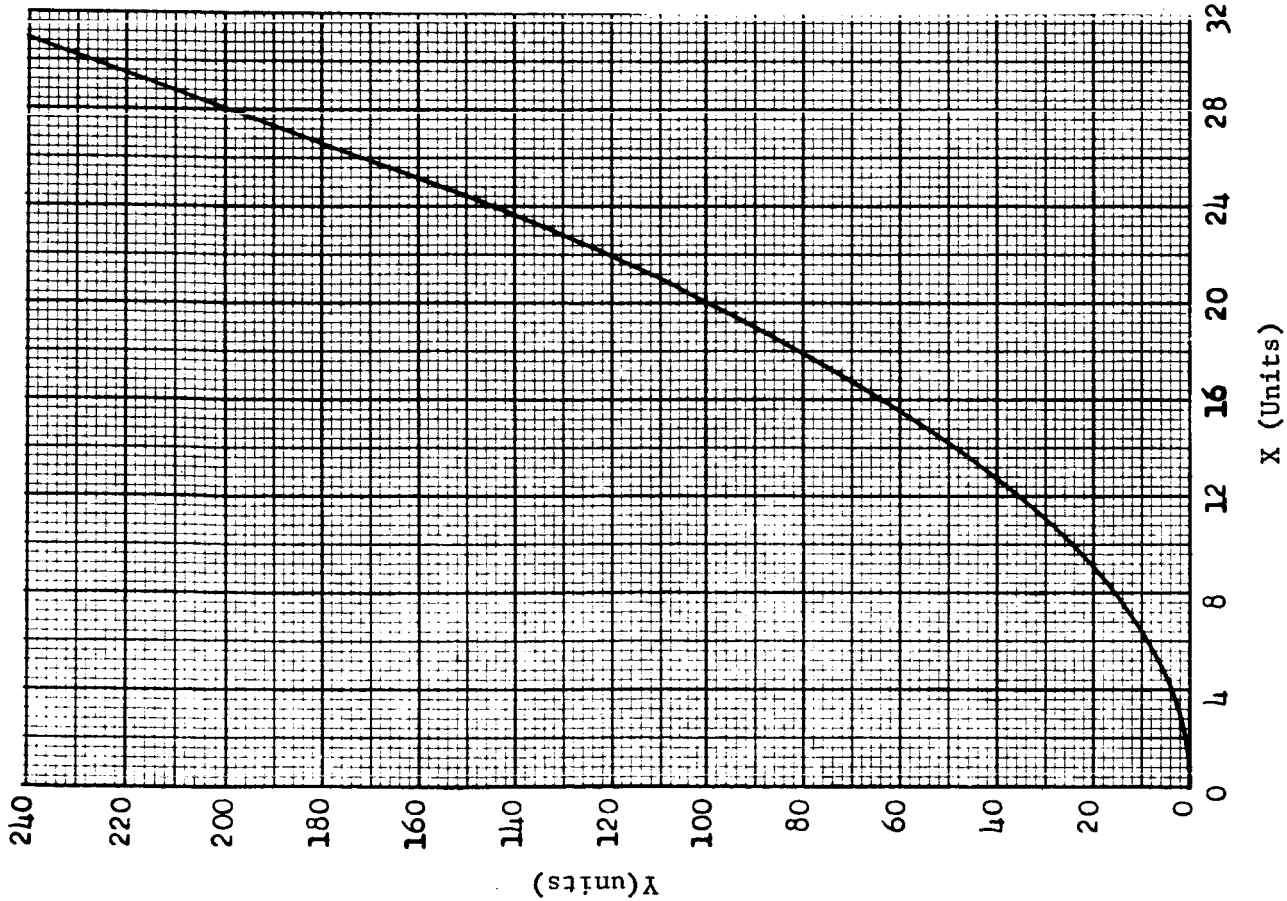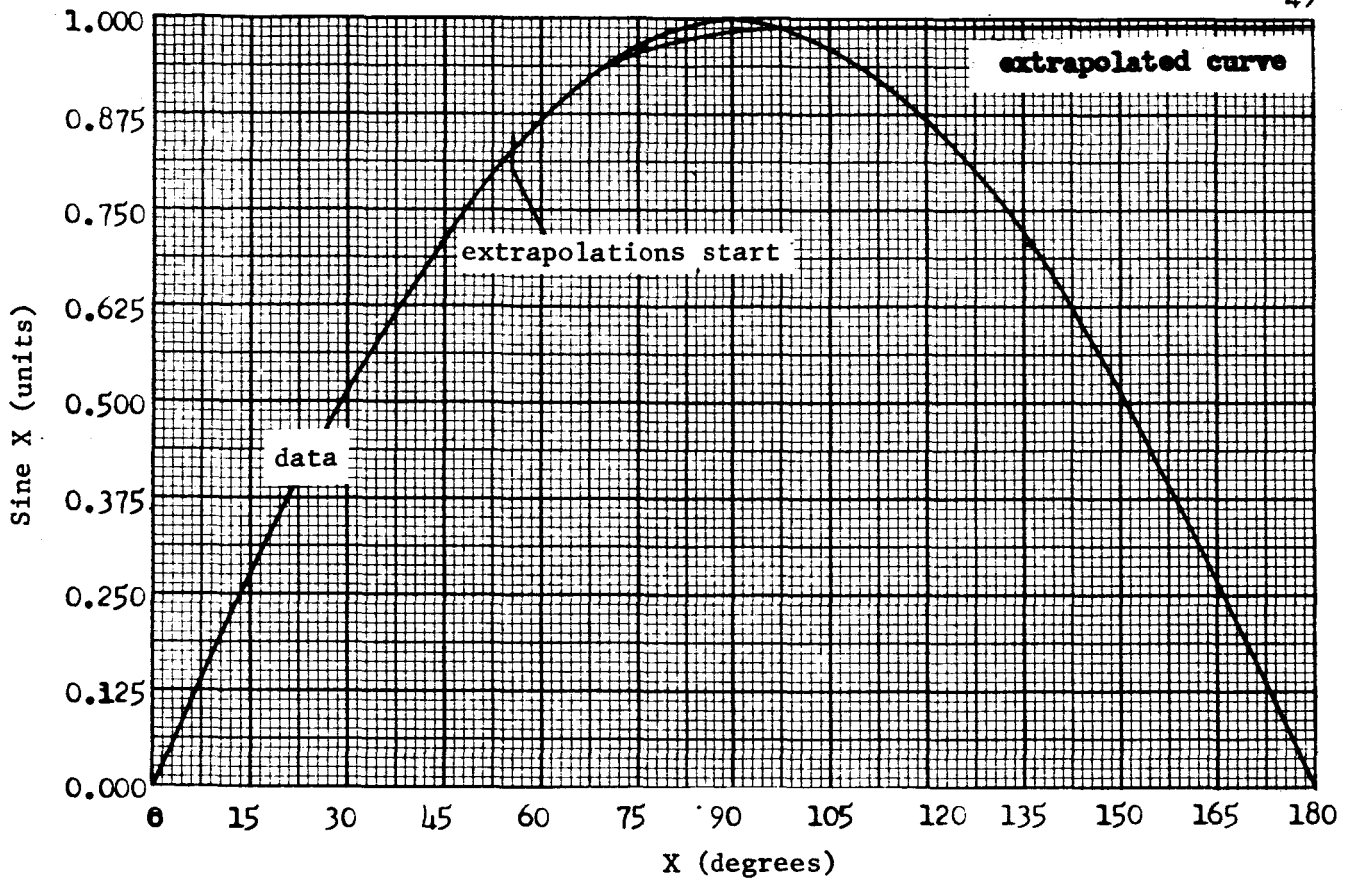
Figure 3-a. Sine x in Five Degree Increments.
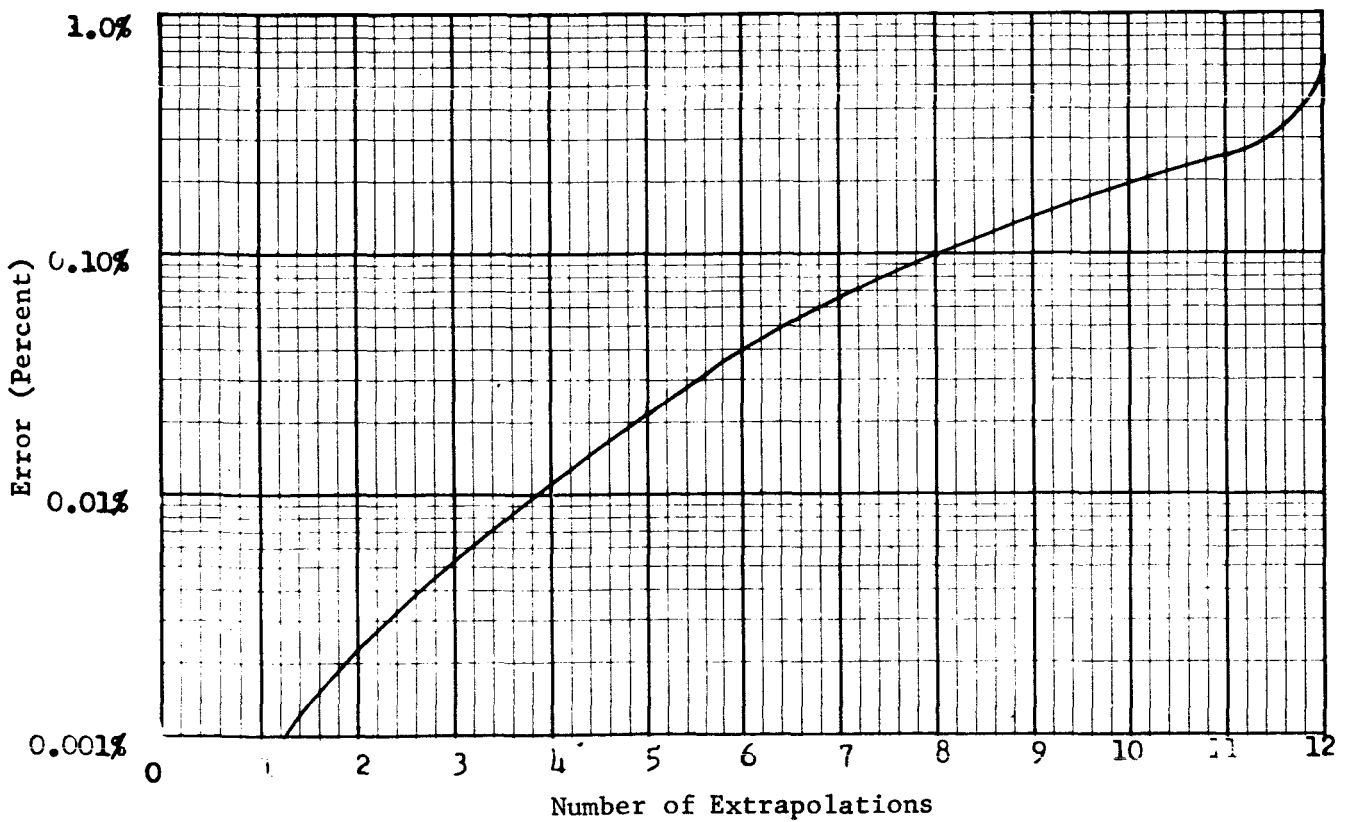
Figure 3-b. Percent Error Versus Number of Extrapolations.
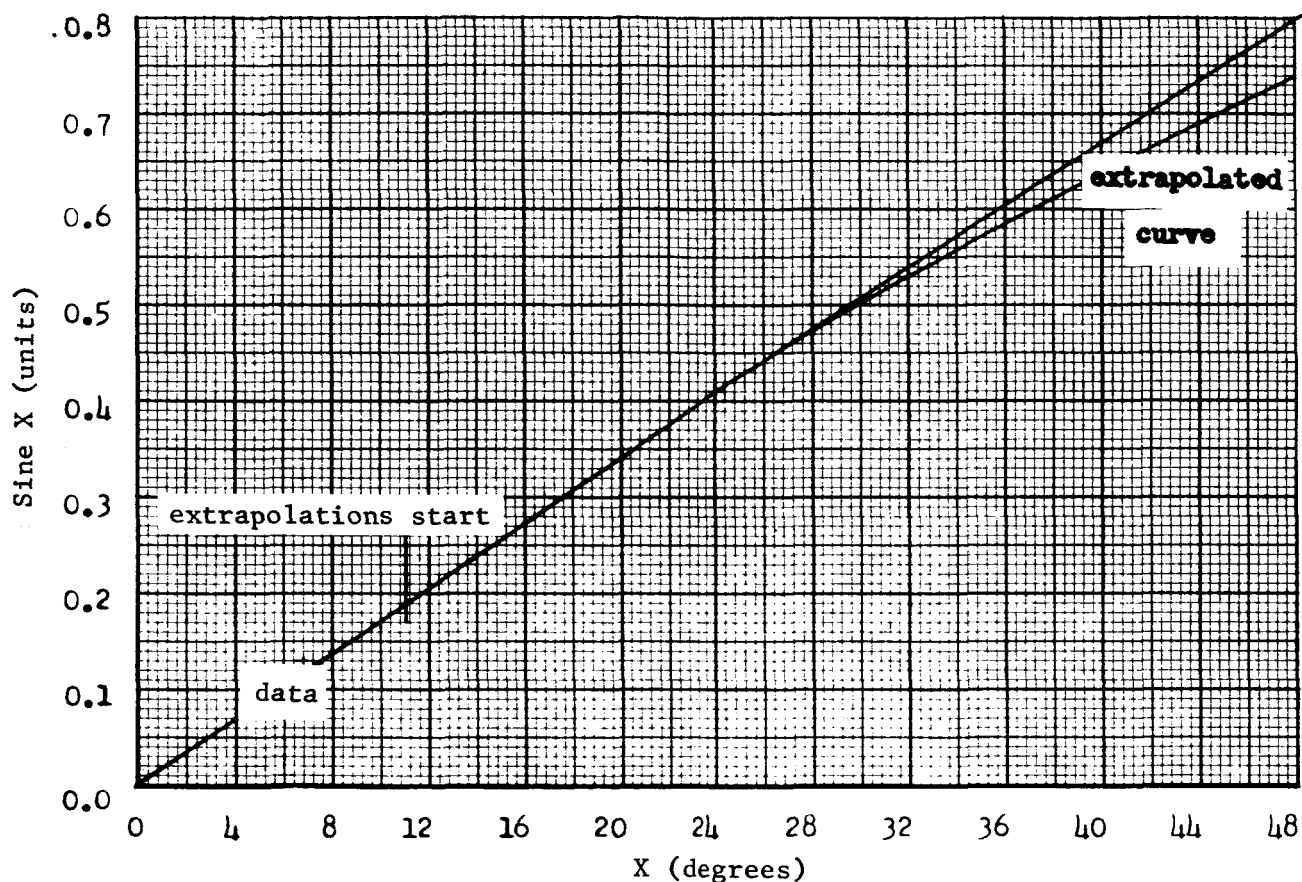
Figure 4-a.   Sine x in One Degree Increments.
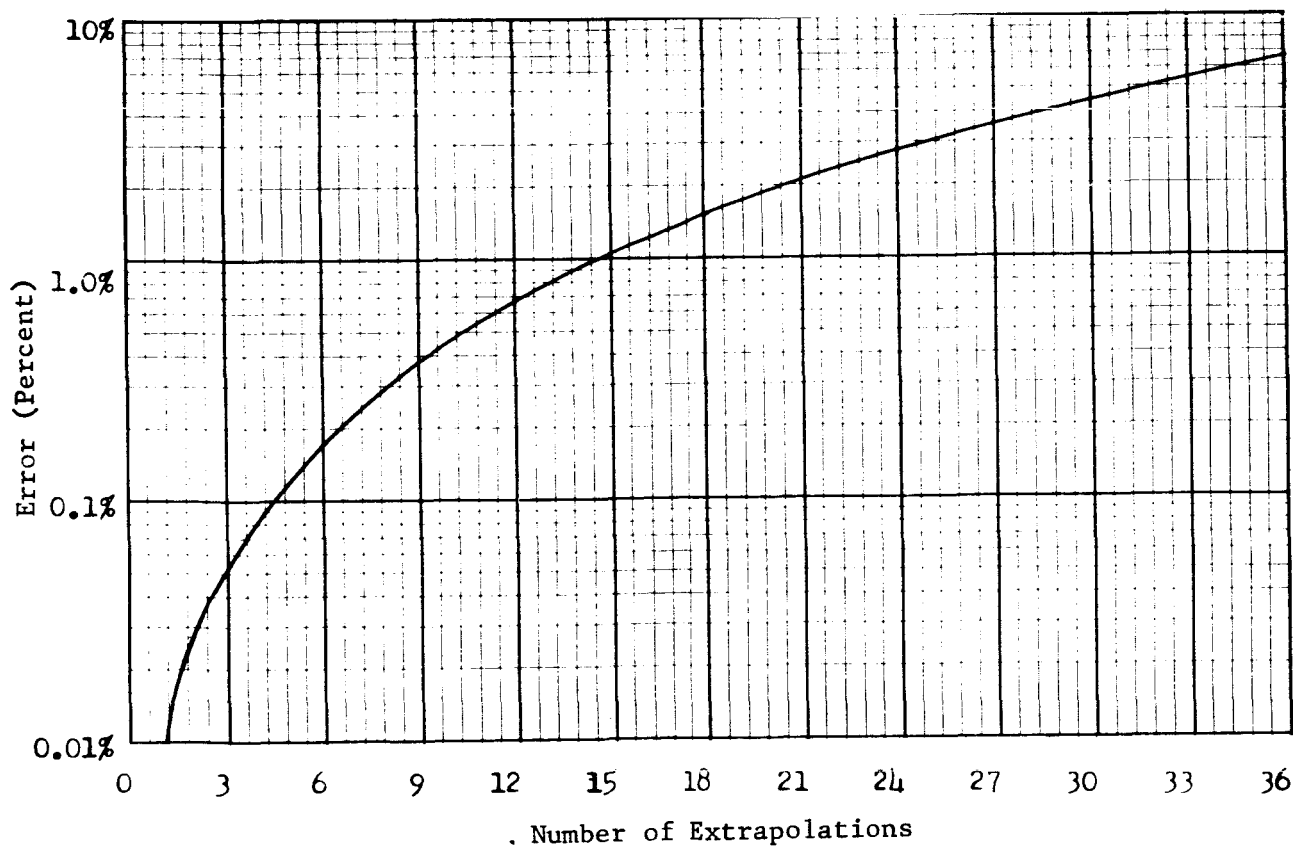
Figure 4-b.   Percent Error Versus Number of Extrapolations.
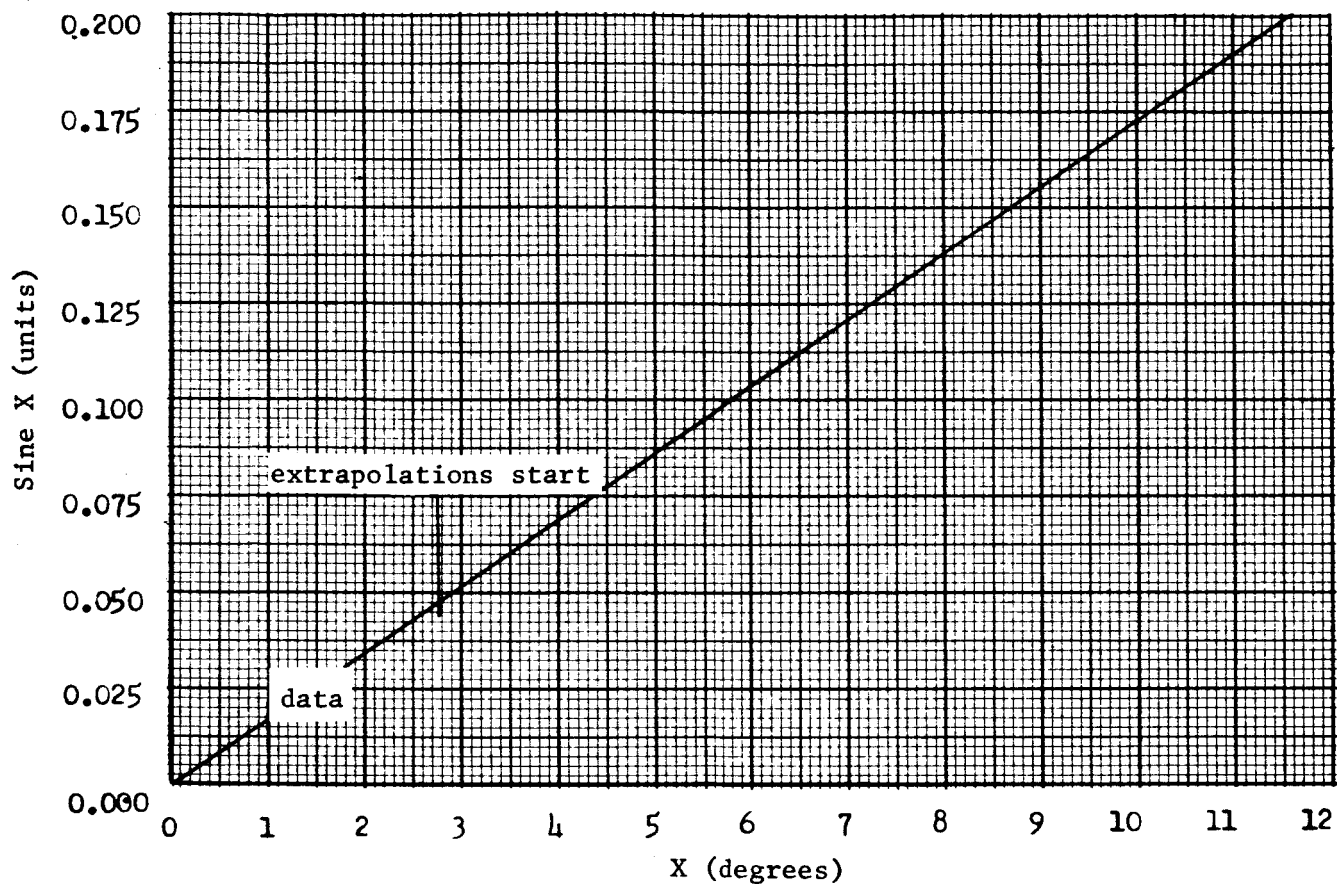
Figure 5-a. Sine X in 15 Minute Increments.
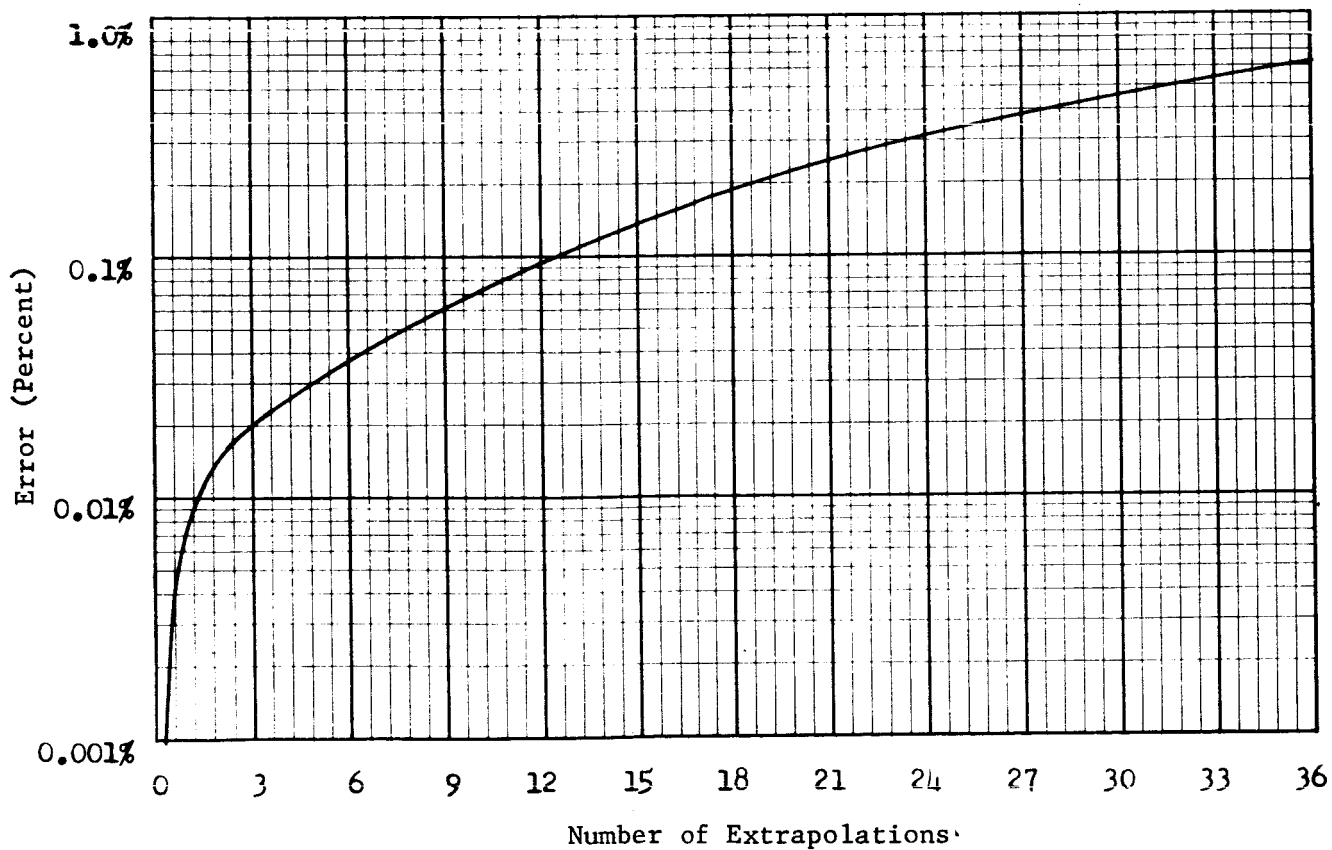
Figure 5-b. Percent Error Versus Number of Extrapolations.
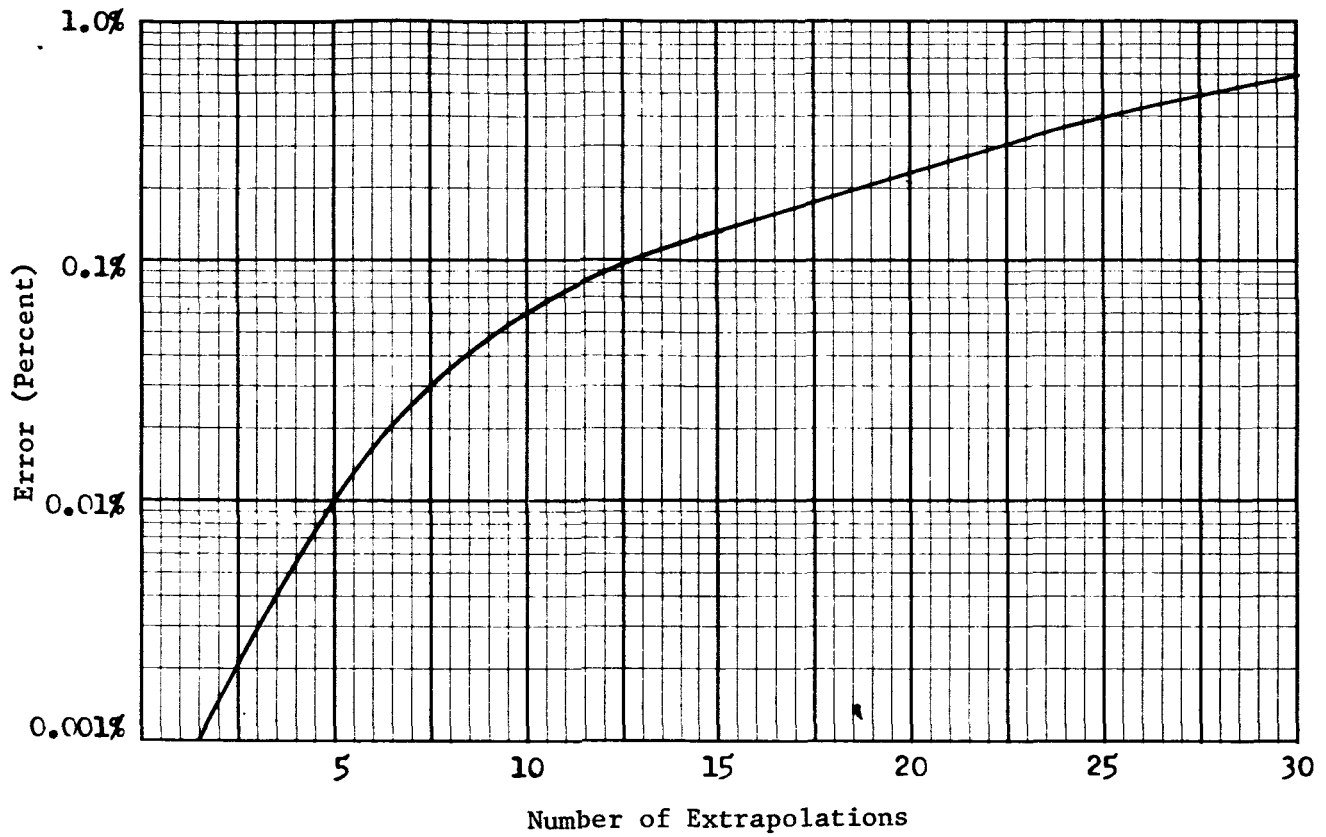
Figure 6.   Extrapolations Error for Log Table.
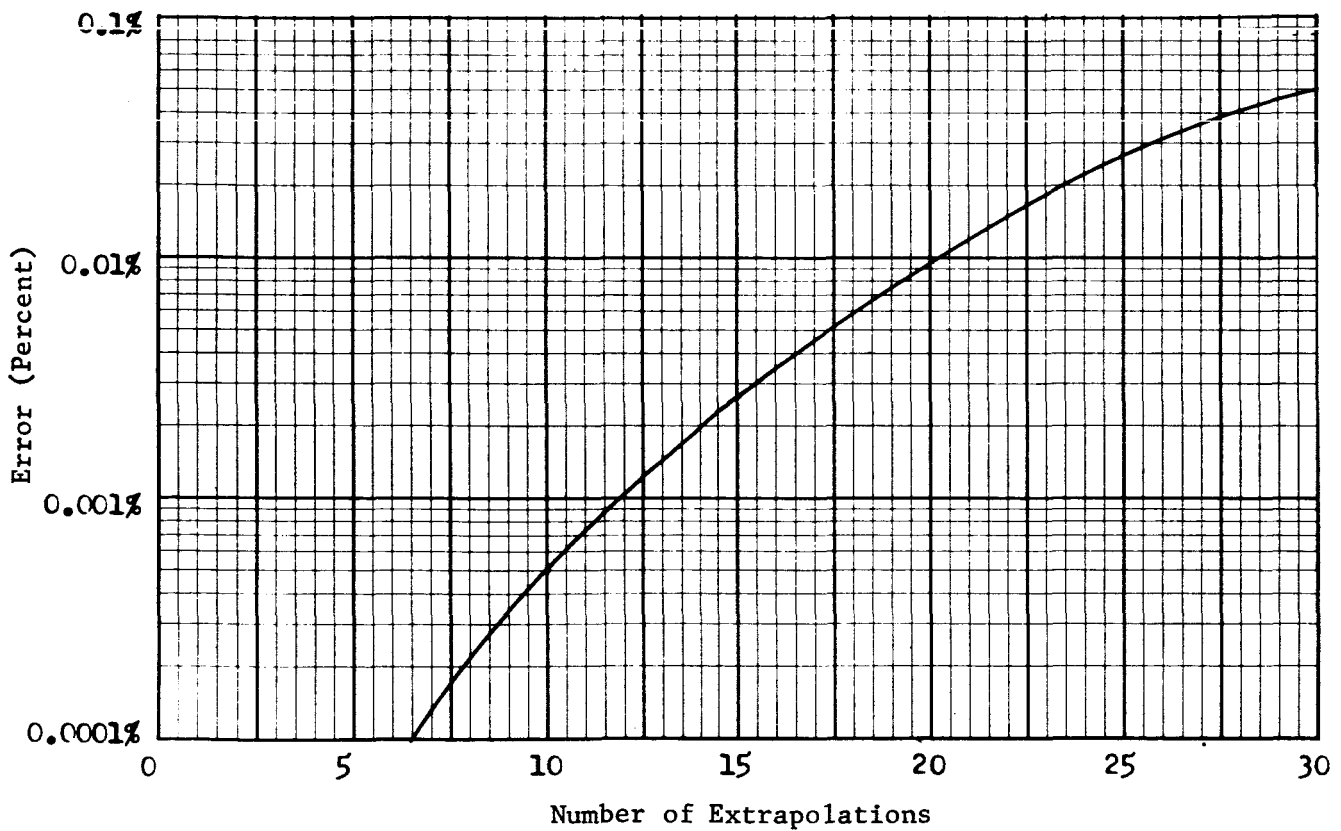
Figure 7.   Extrapolation Error for Interest Table.
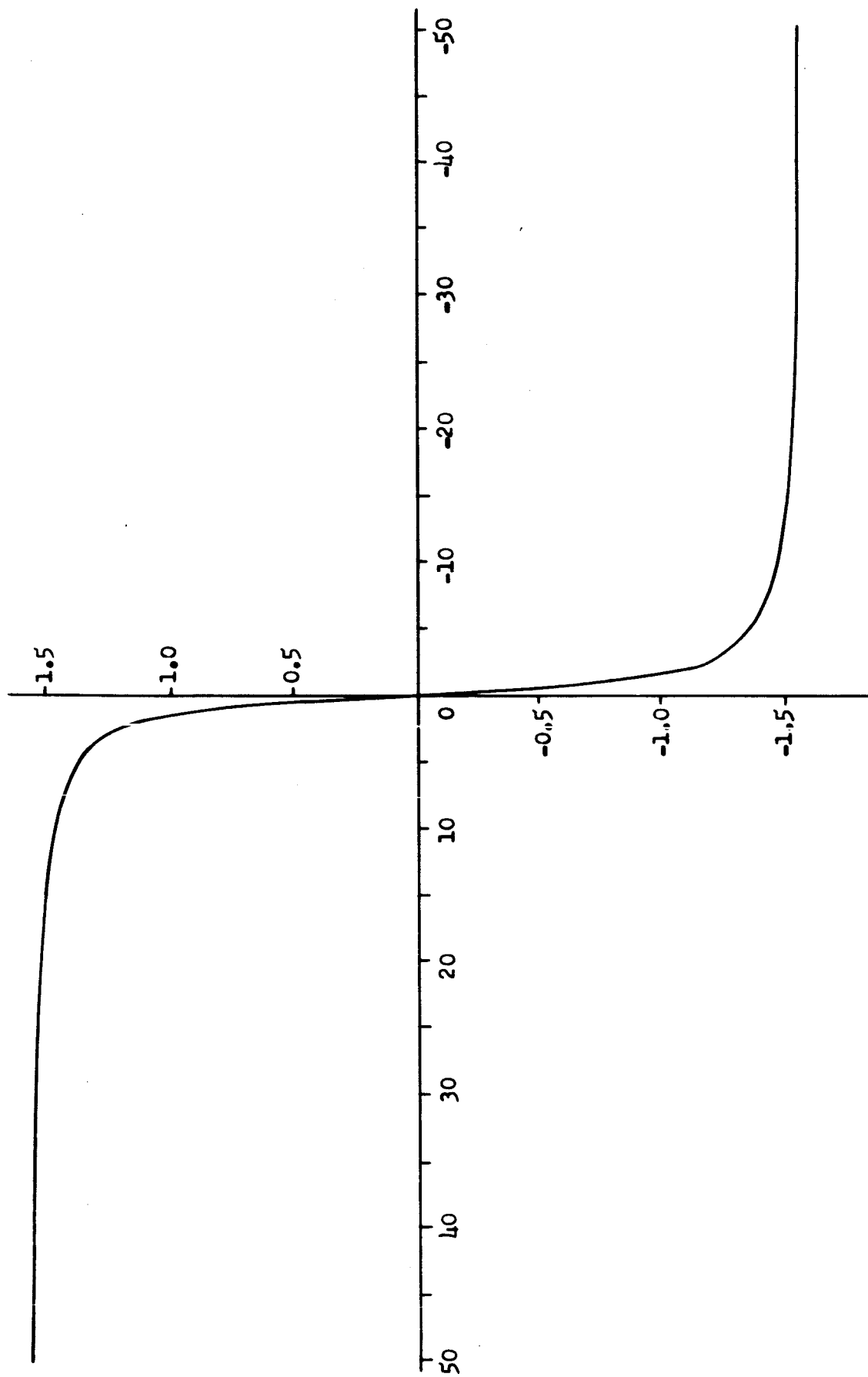
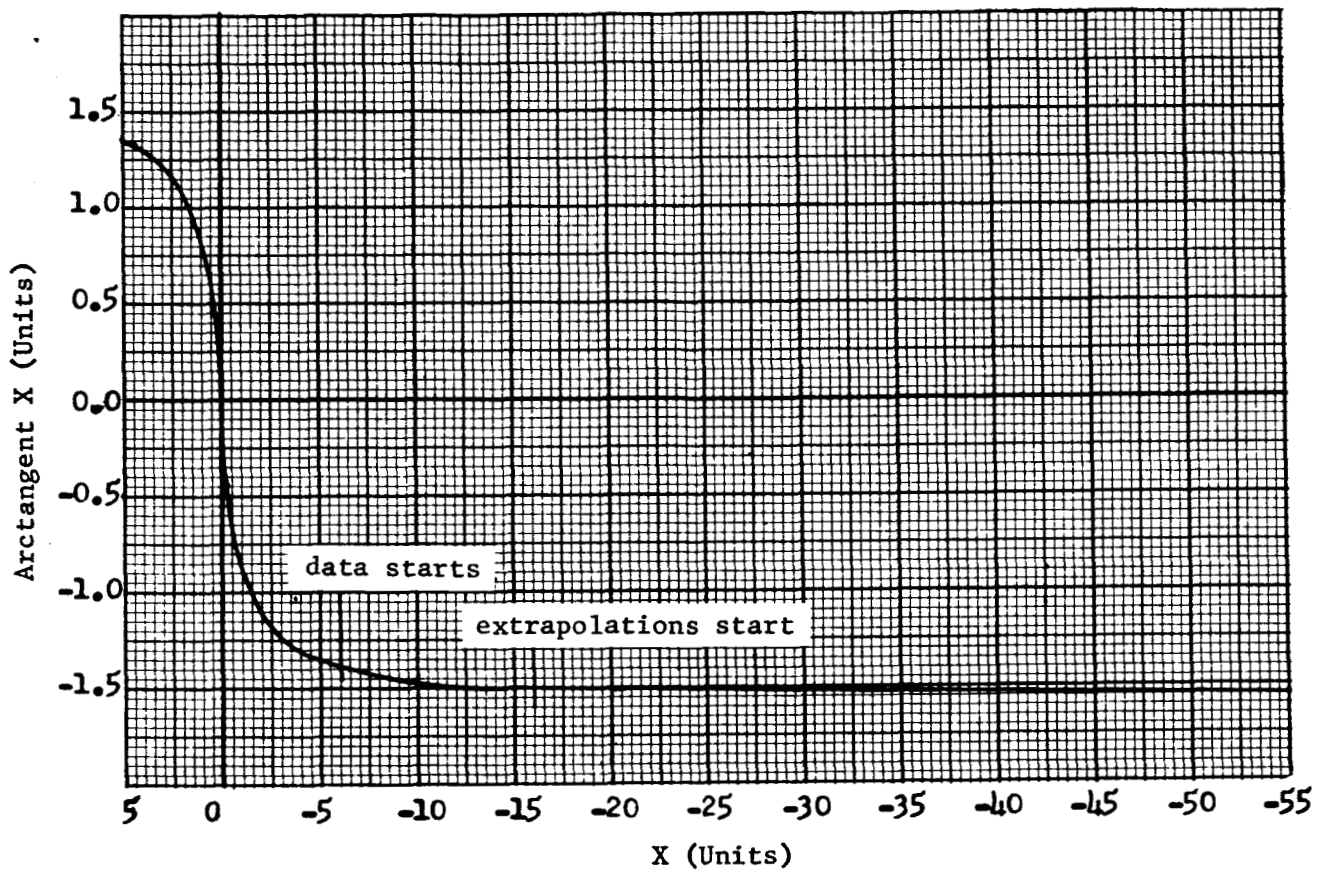Figure 8. Arctangent X Curve Used as Simulated Tracking Data for Computing Extrapolations.
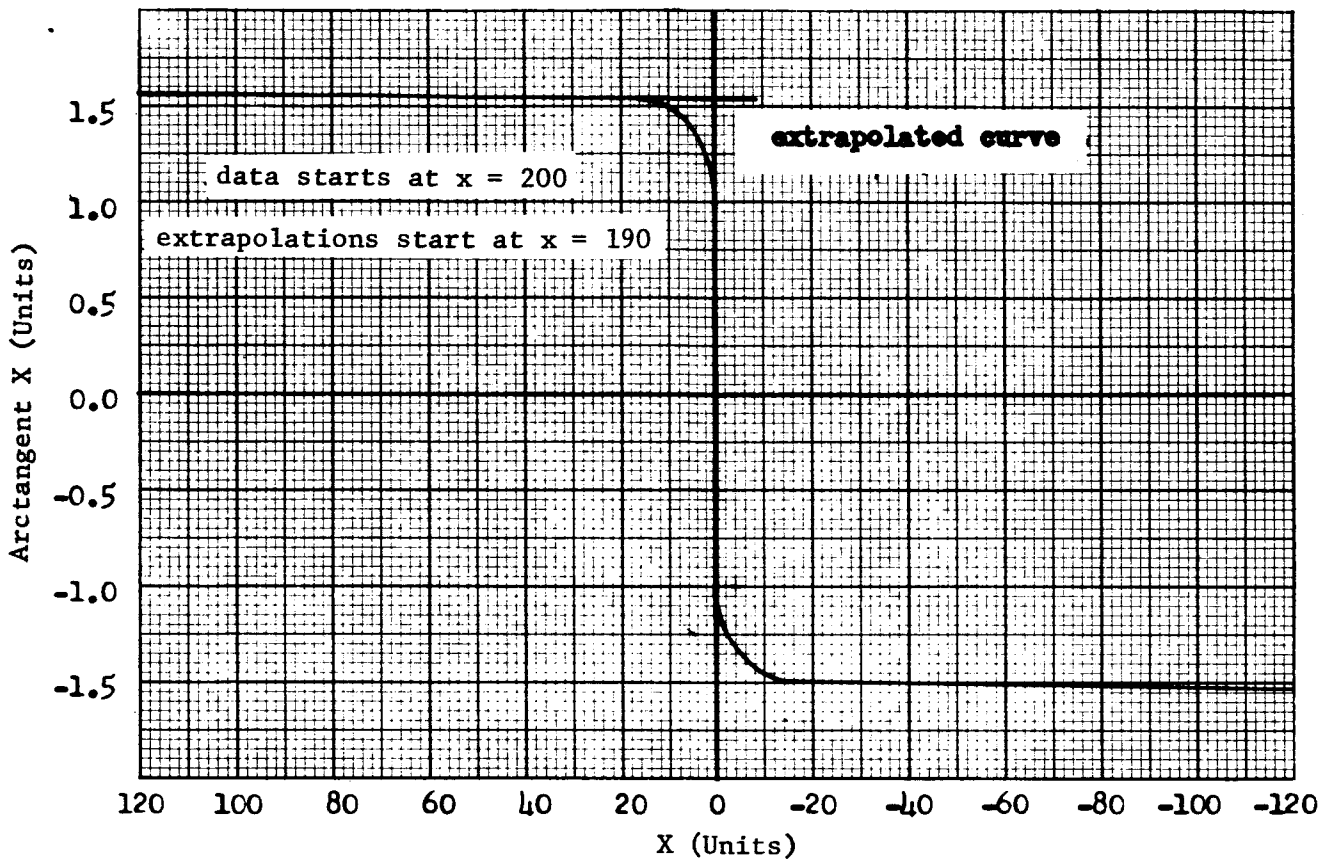
Figure 9-a. Arctangent X as a Function of X With Extrapolations from Data Increments of 1.0.
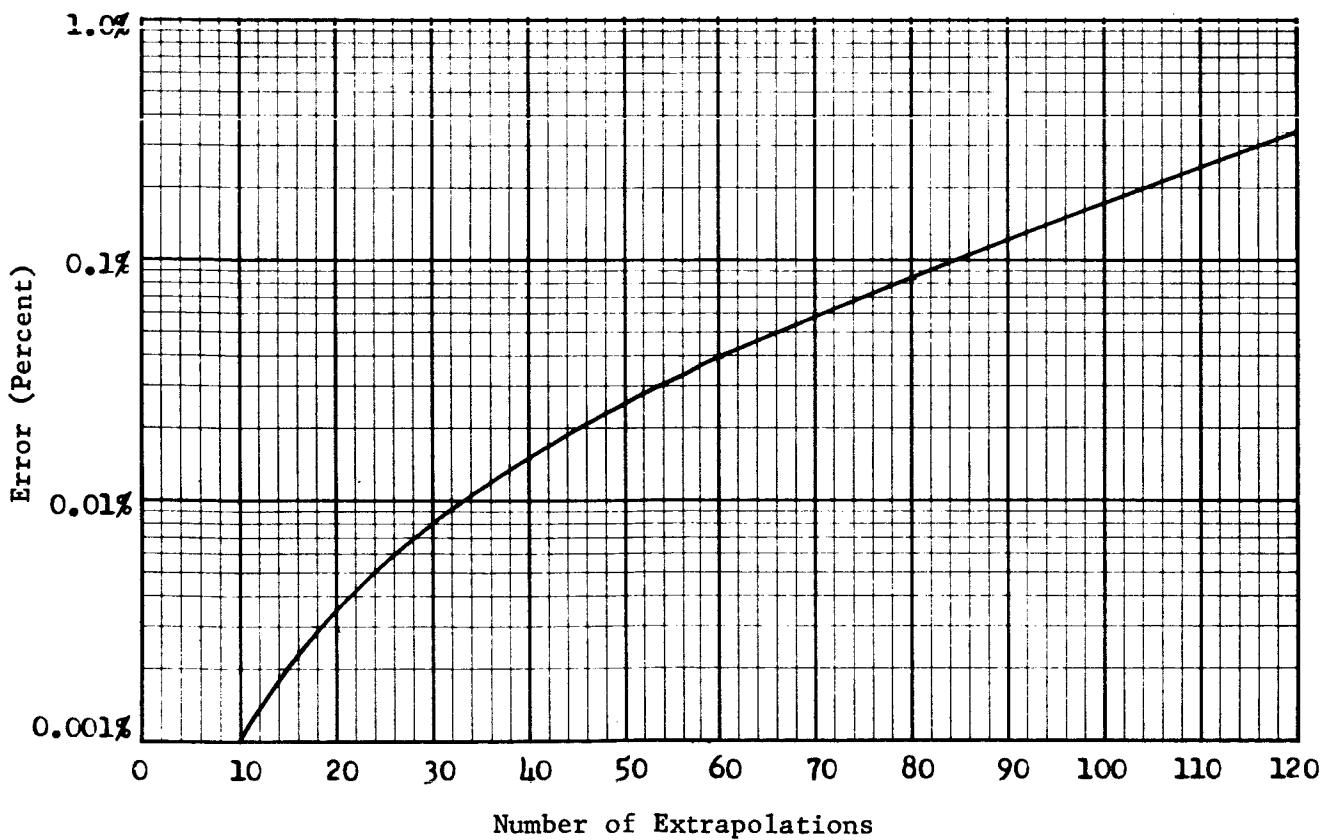
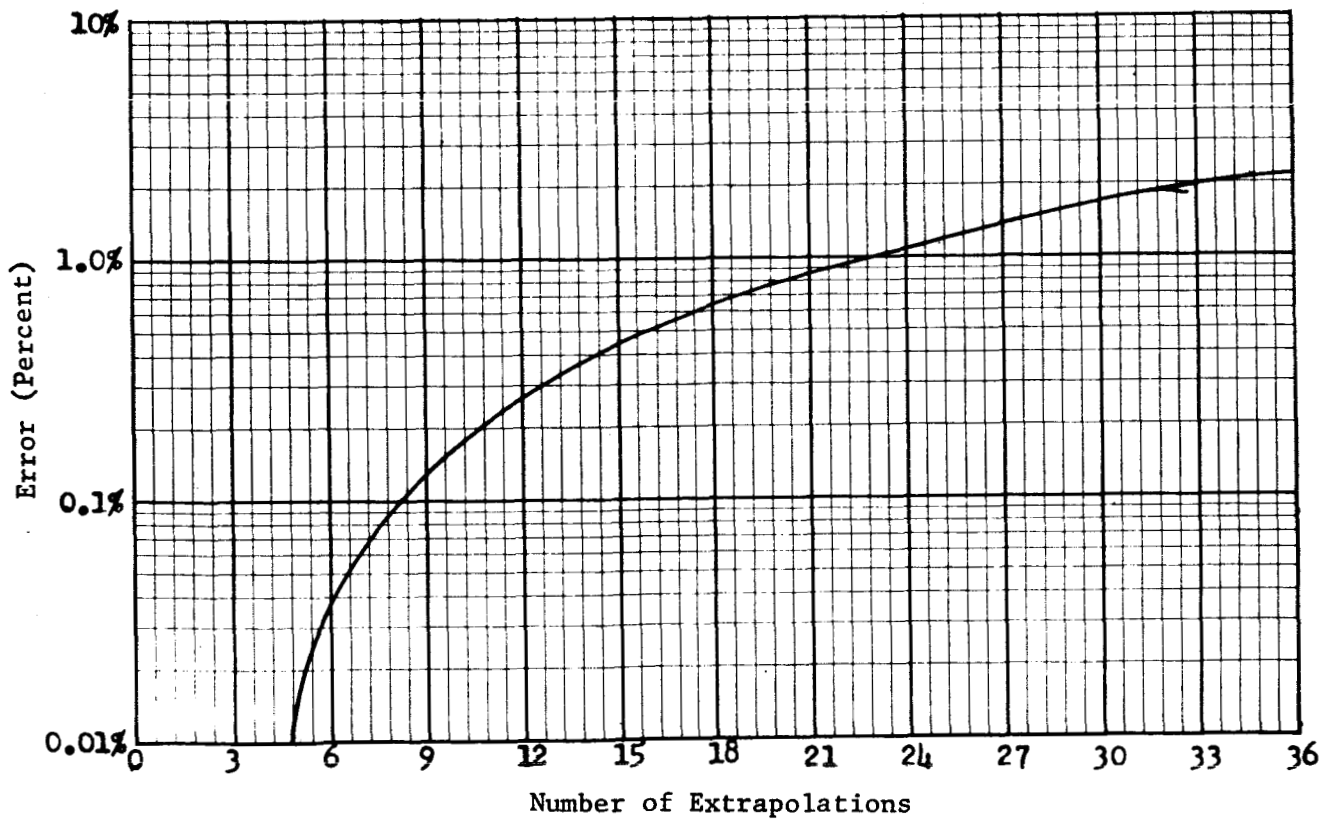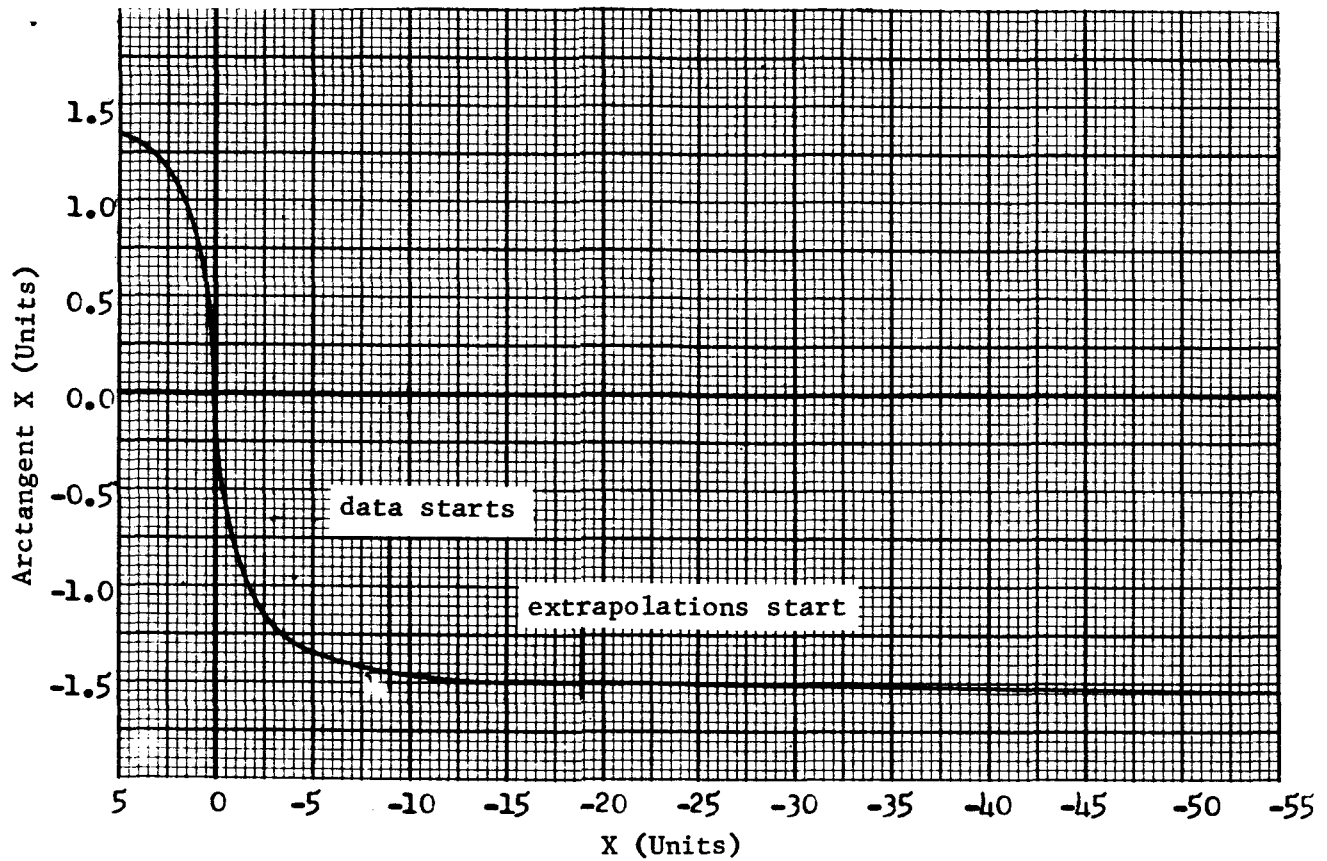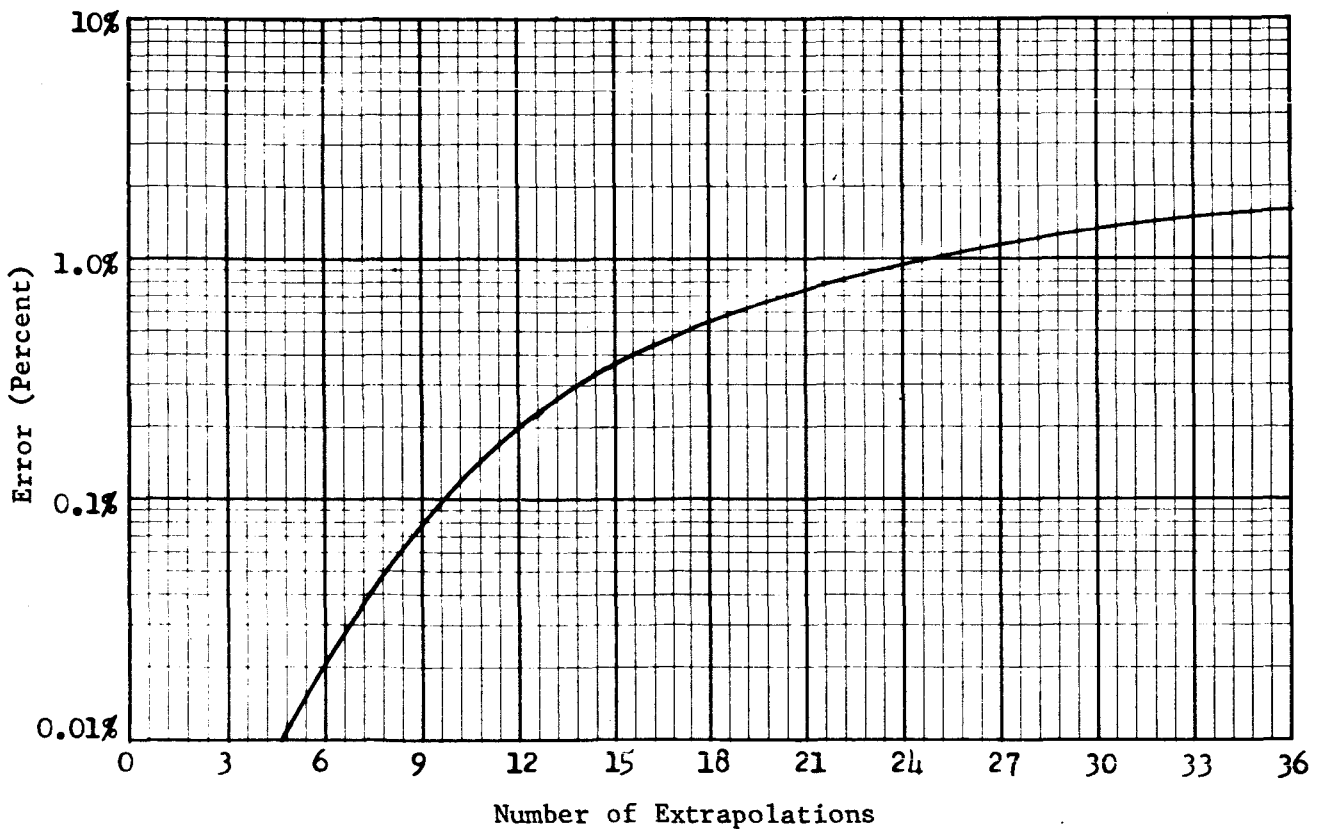
Figure 9-b. Percent Error Versus Number of Extrapolations.
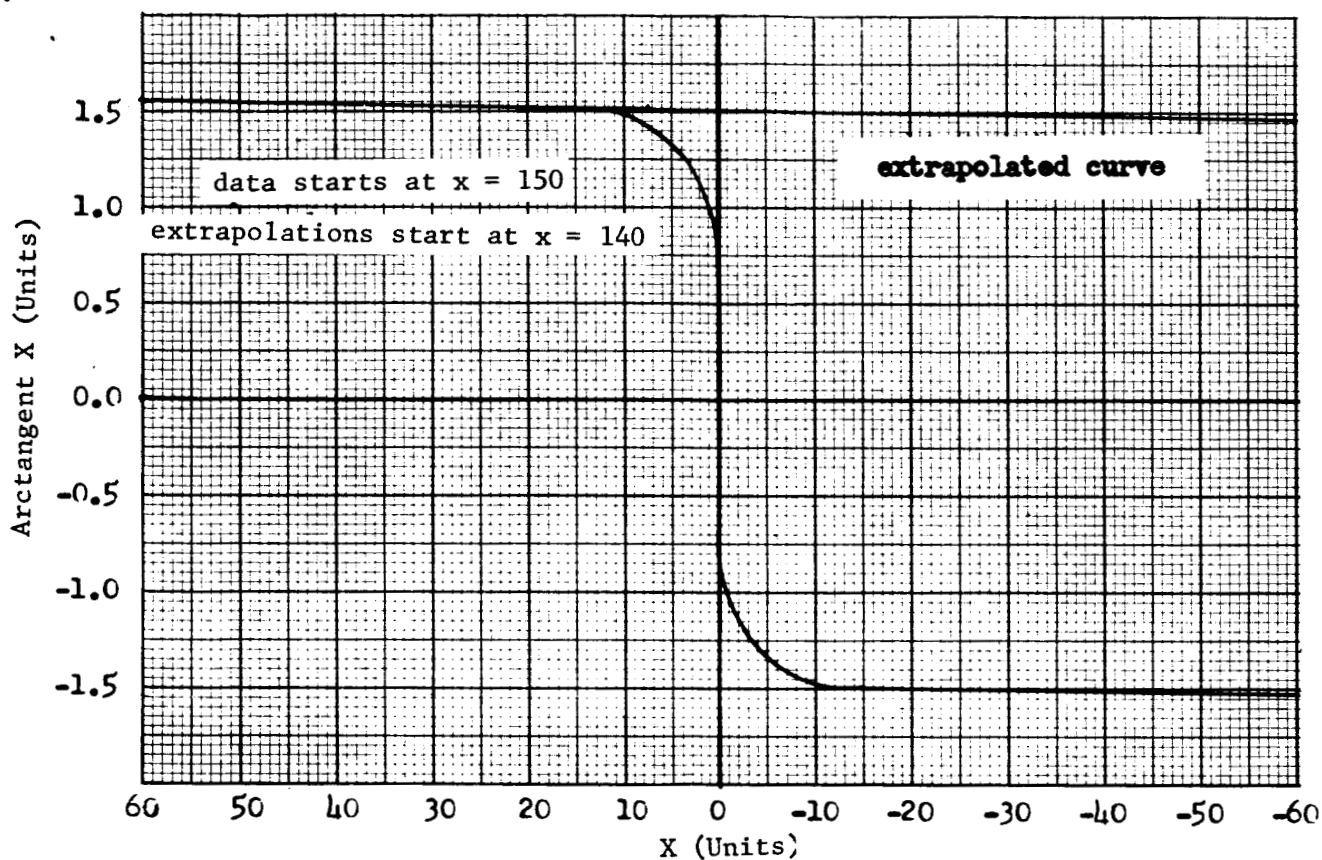
Figure 10-a. Arctangent X as a Function of X With Extrapolations from Data Increments.of 1.0.

In-figure labels:

data starts at x = 150

extrapolations start at x = 140

extrapolated curve

Y-axis: Arctangent X (Units): 1.5, 1.0, 0.5, 0.0, -0.5, -1.0, -1.5

X-axis: X (Units): 60, 50, 40, 30, 20, 10, 0, -10, -20, -30, -40, -50, -60
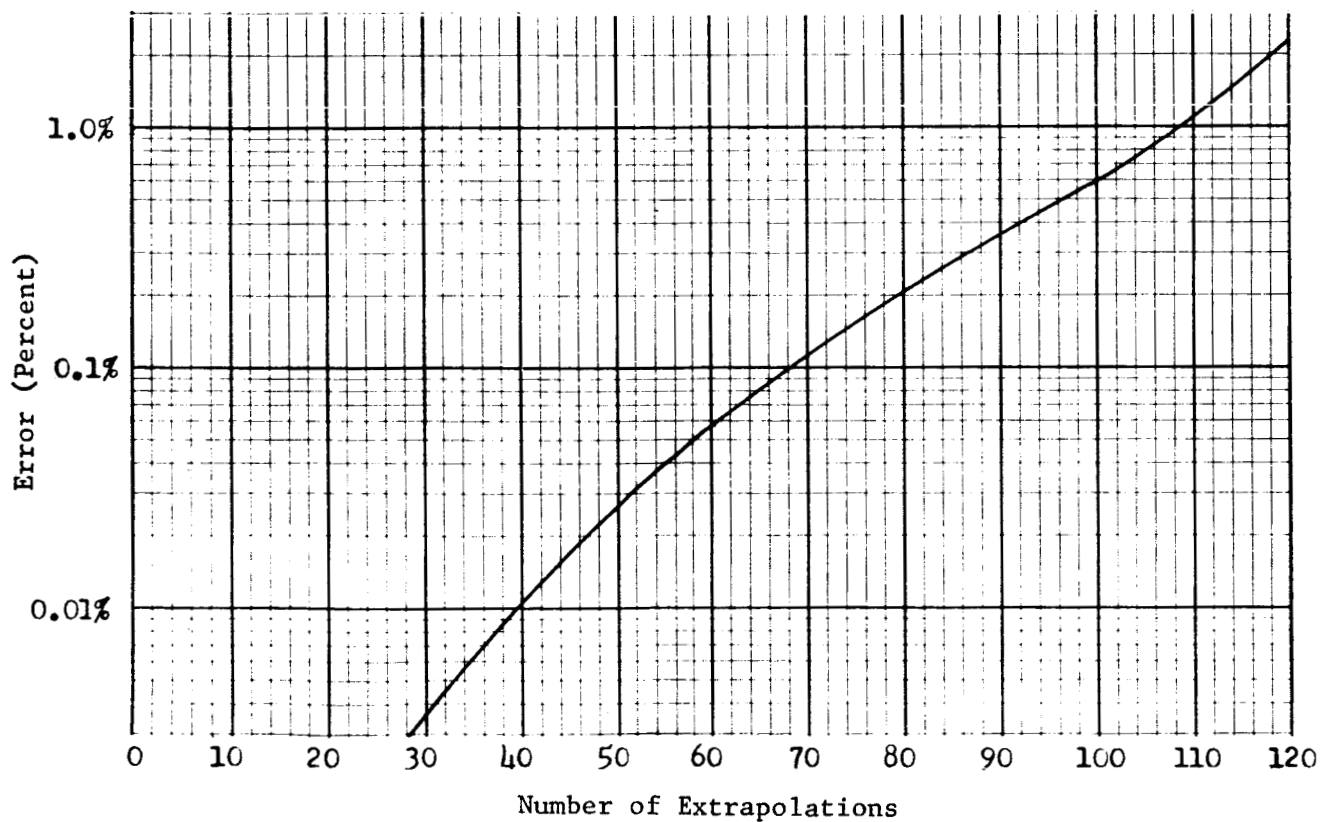
Figure 10-b. Percent Error Versus Number of Extrapolations.

Y-axis: Error (Percent): 1.0%, 0.1%, 0.01%

X-axis: Number of Extrapolations: 0, 10, 20, 30, 40, 50, 60, 70, 80, 90, 100, 110, 120
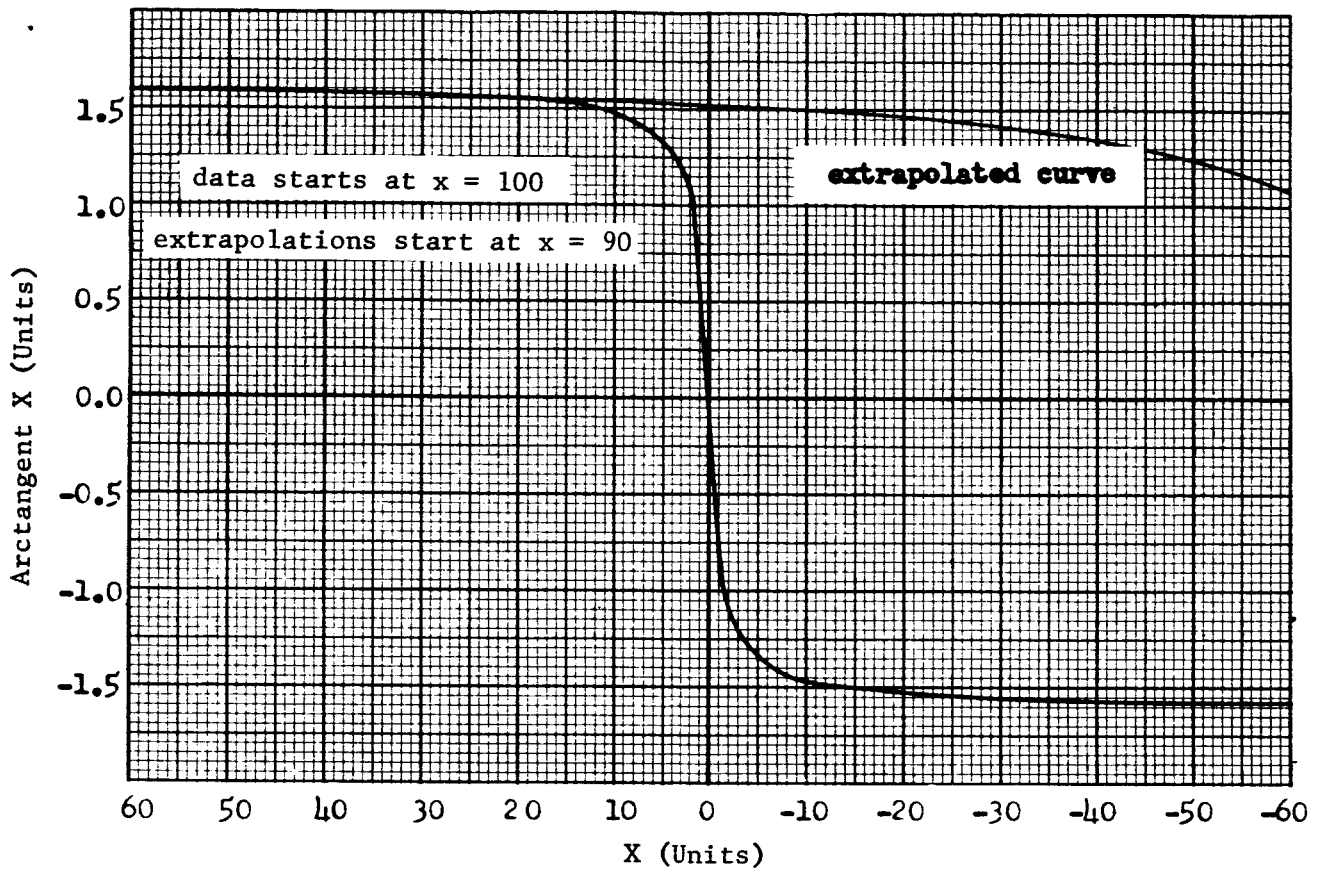
Figure 11-a. Arctangent X as a Function of X With Extrapolations from Data Increments of 1.0.
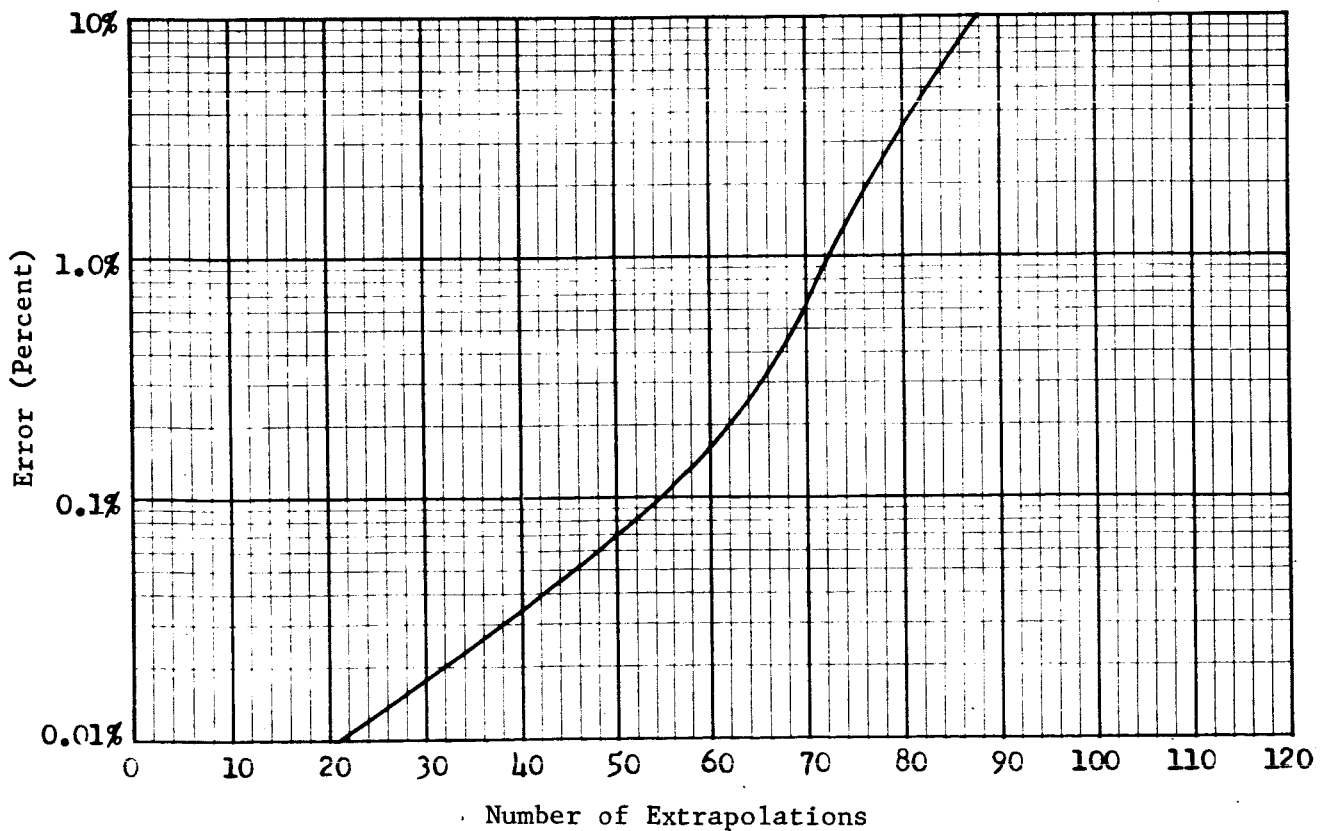
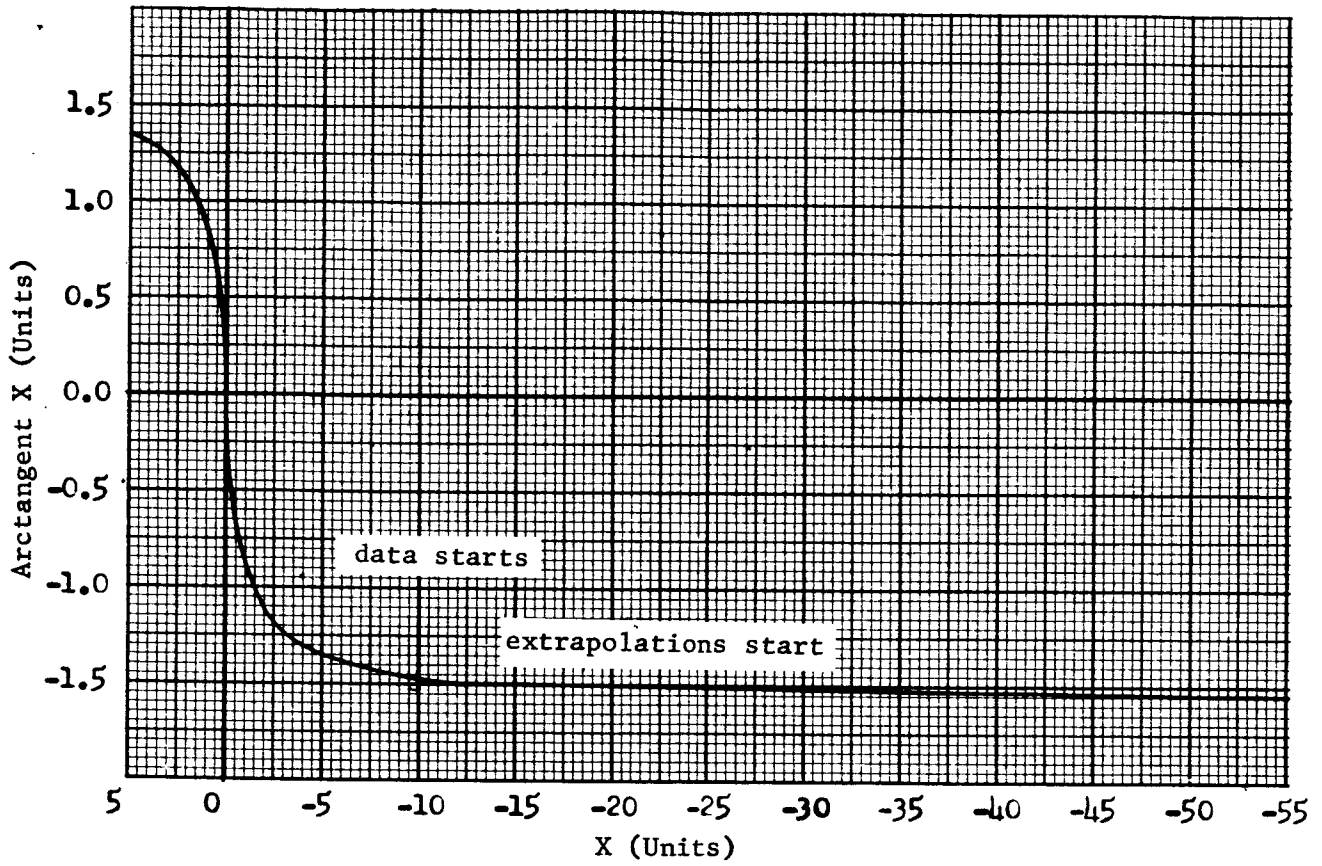Figure 11-b. Percent Error Versus Number of Extrapolations.
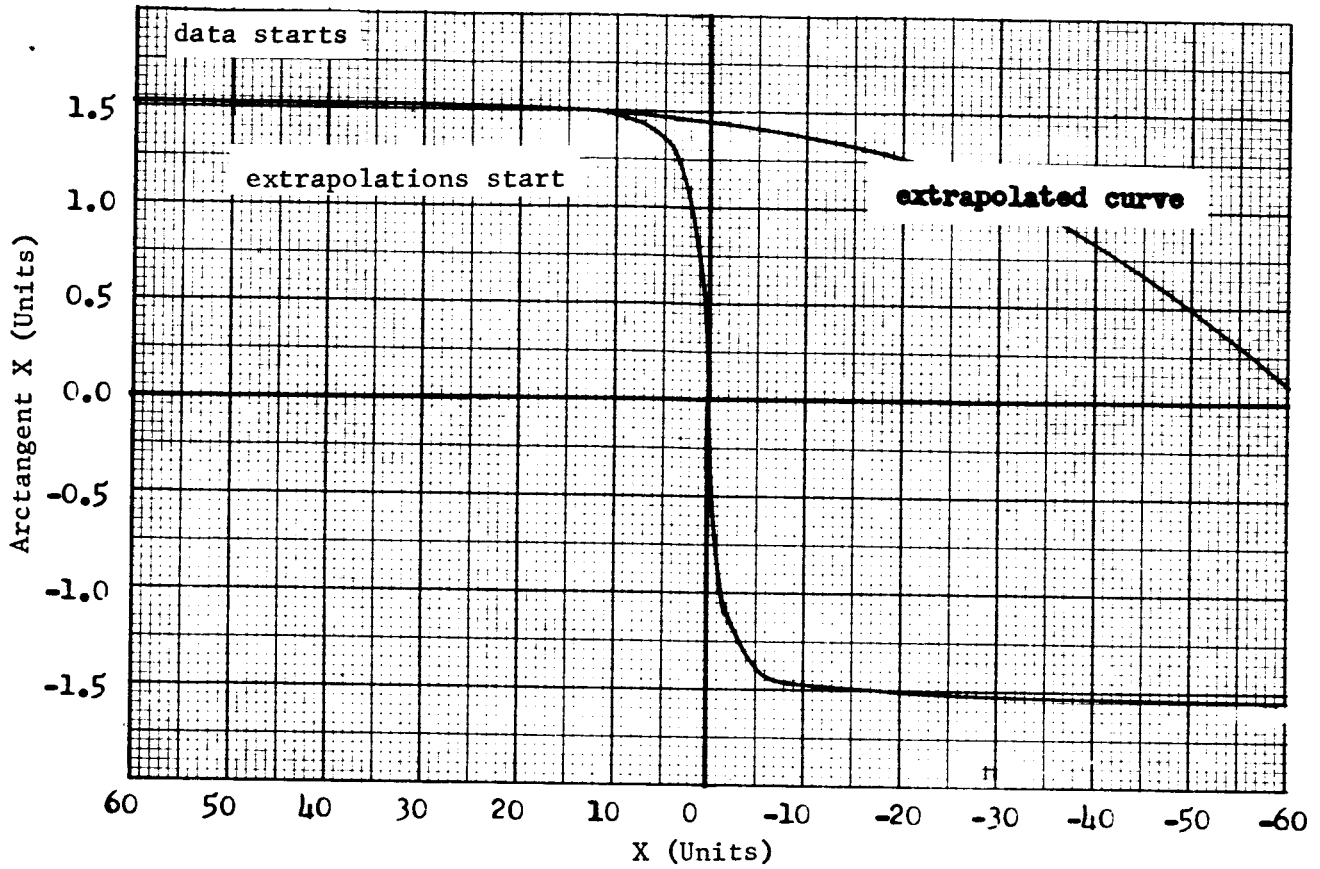
Figure 12-a. Arctangent X as a Function of X With Extrapolations from Data Increments of 1.0.
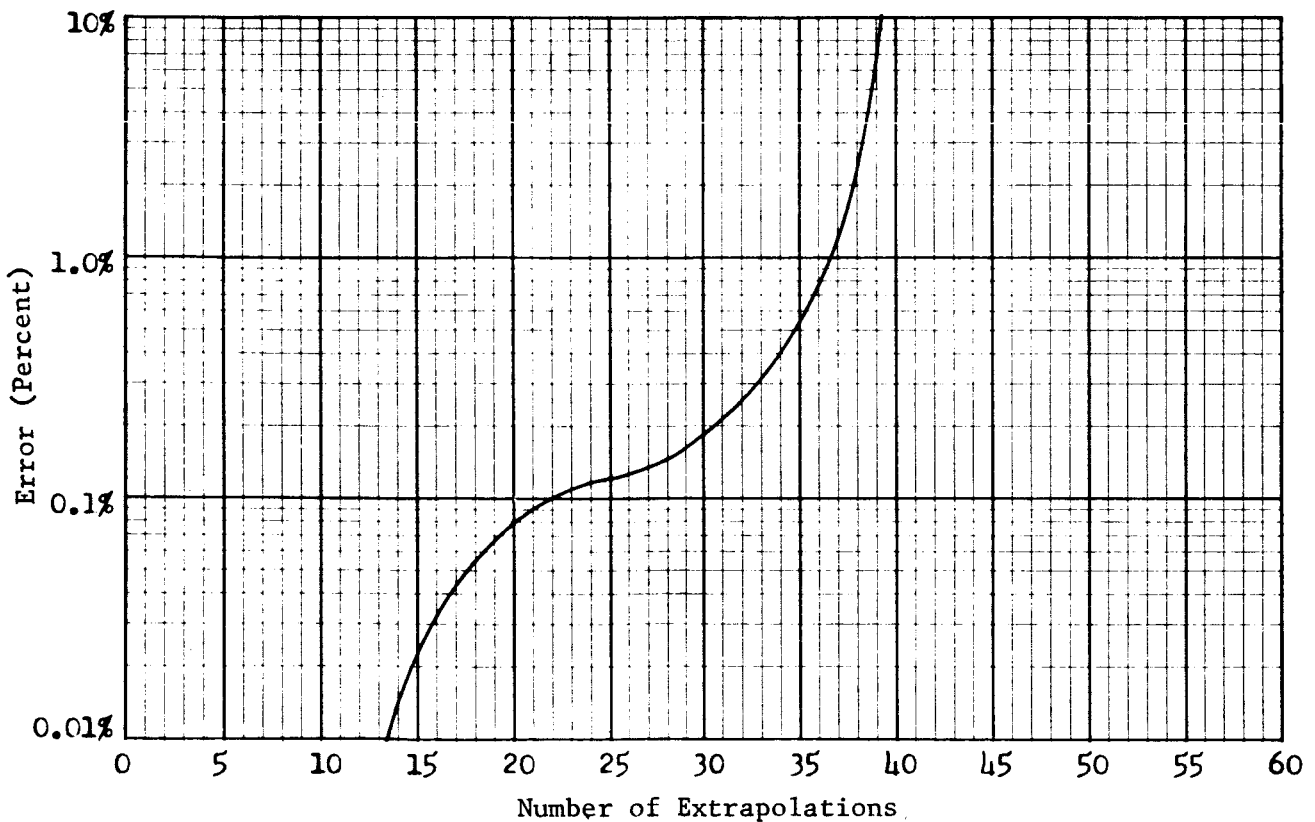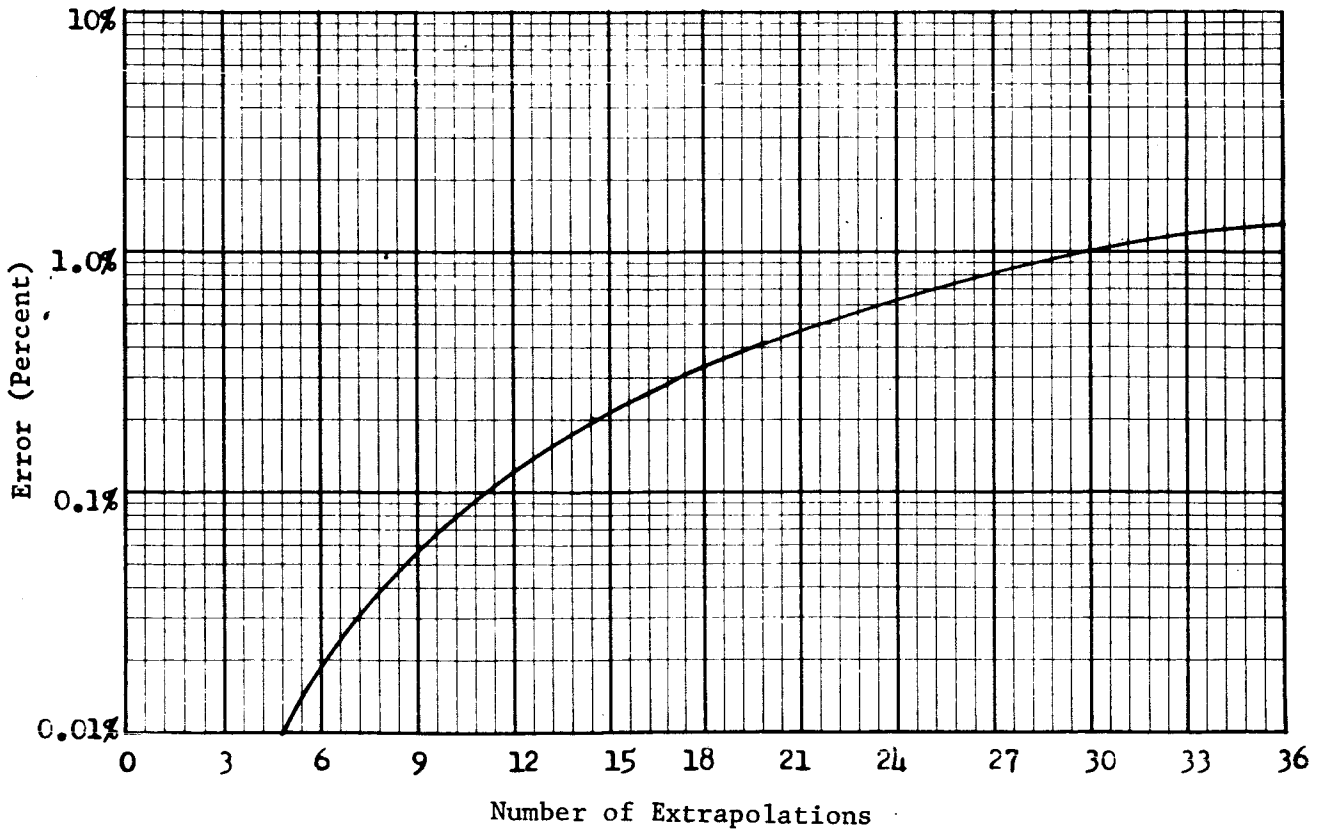

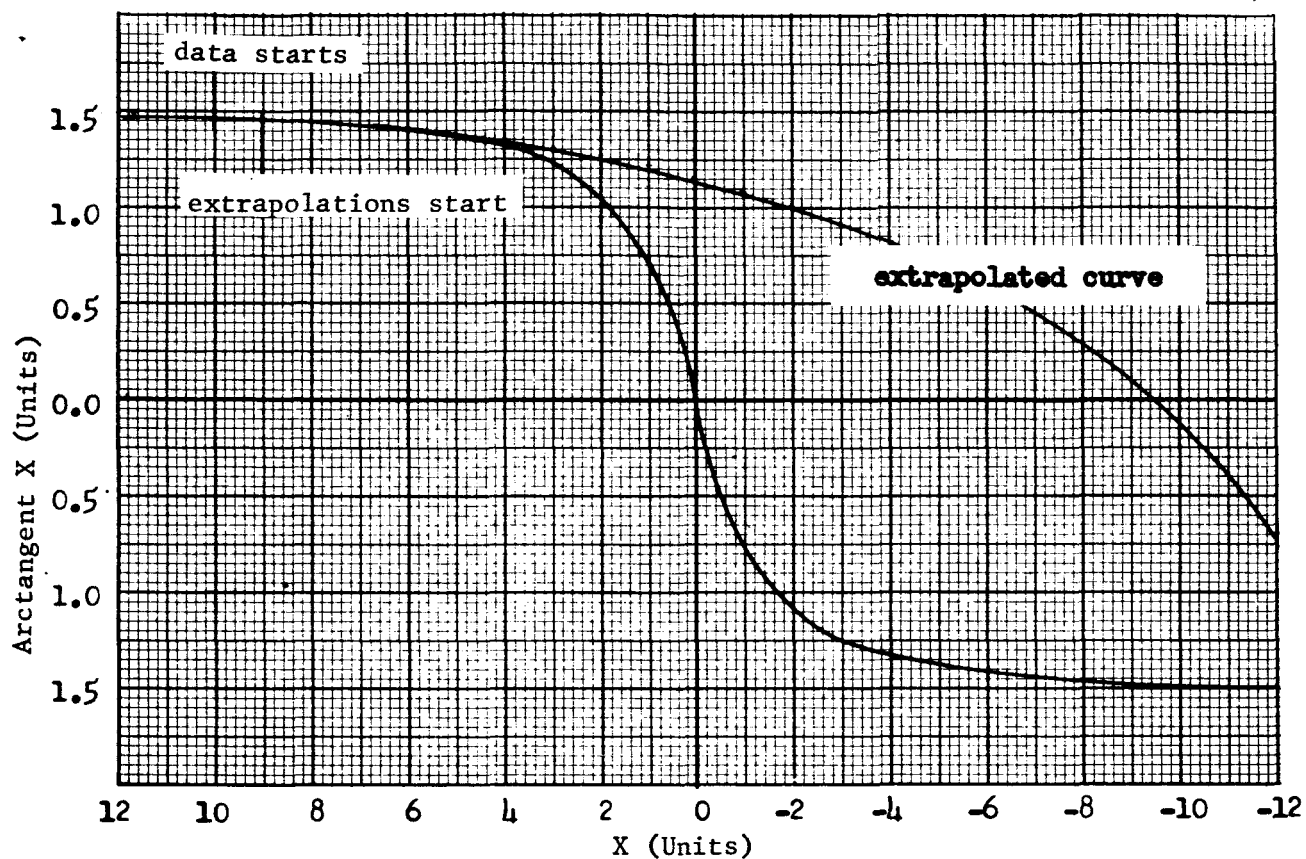Figure 12-b. Percent Error Versus Number of Extrapolations.
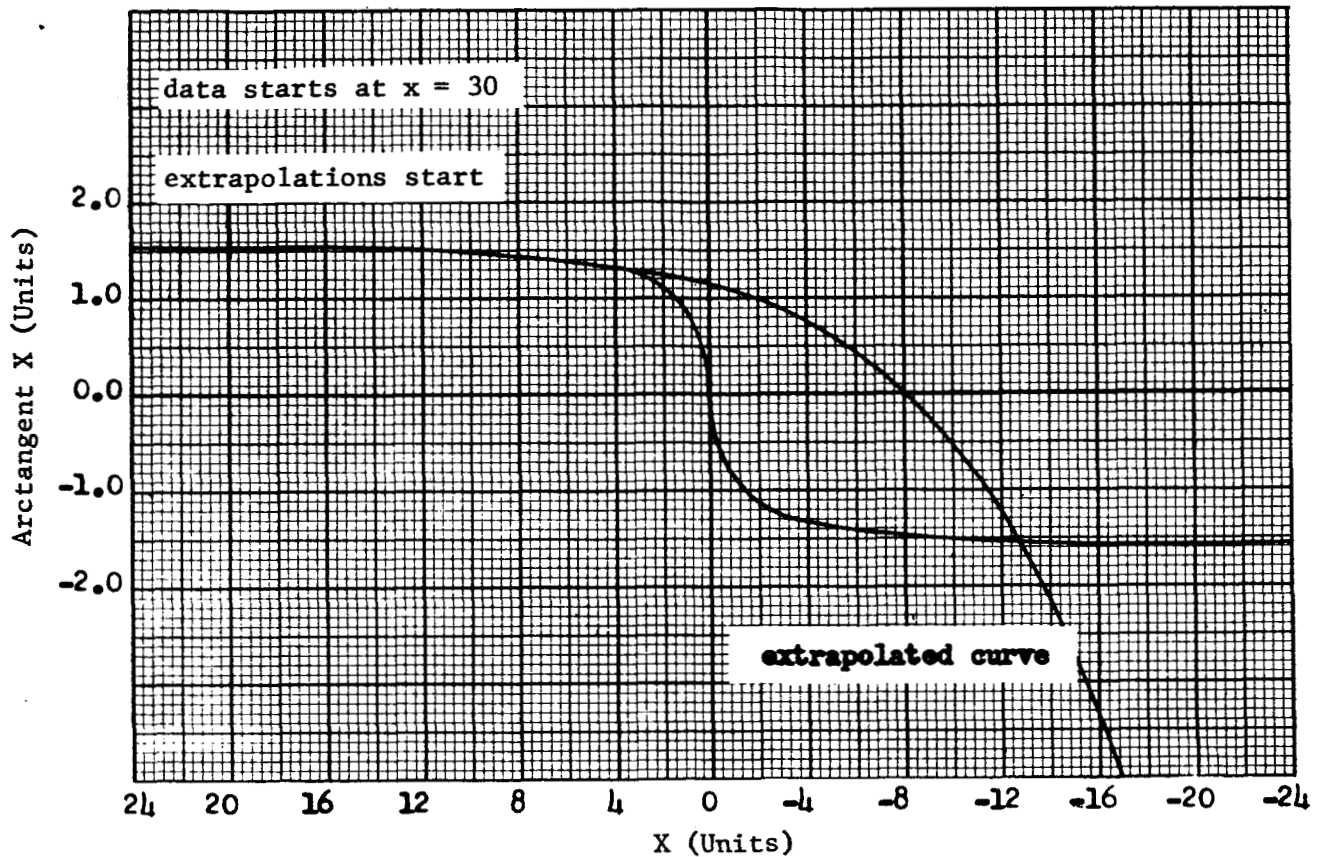
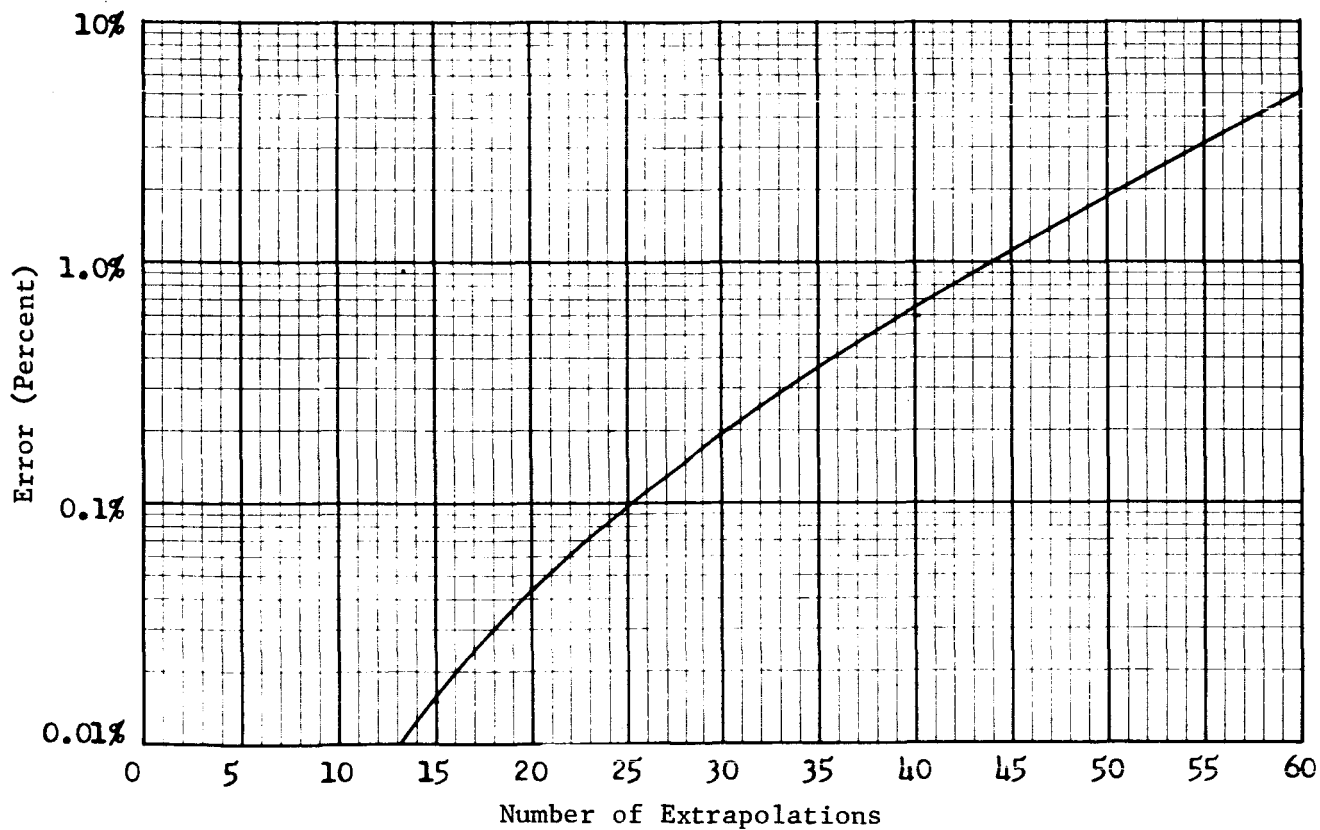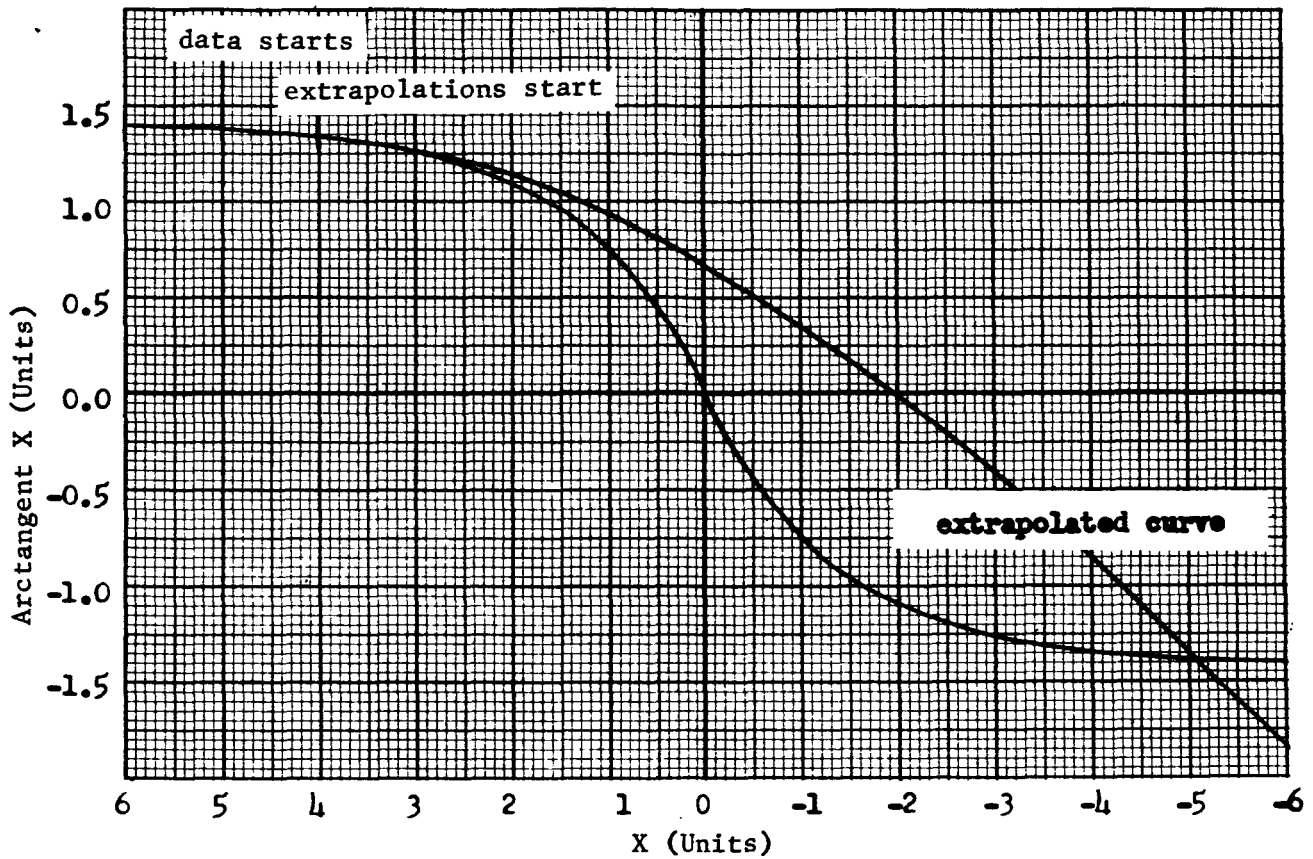Figure 13-a.   Arctangent X as a Function of X With Extrapolations from Data Increments of 1.0.
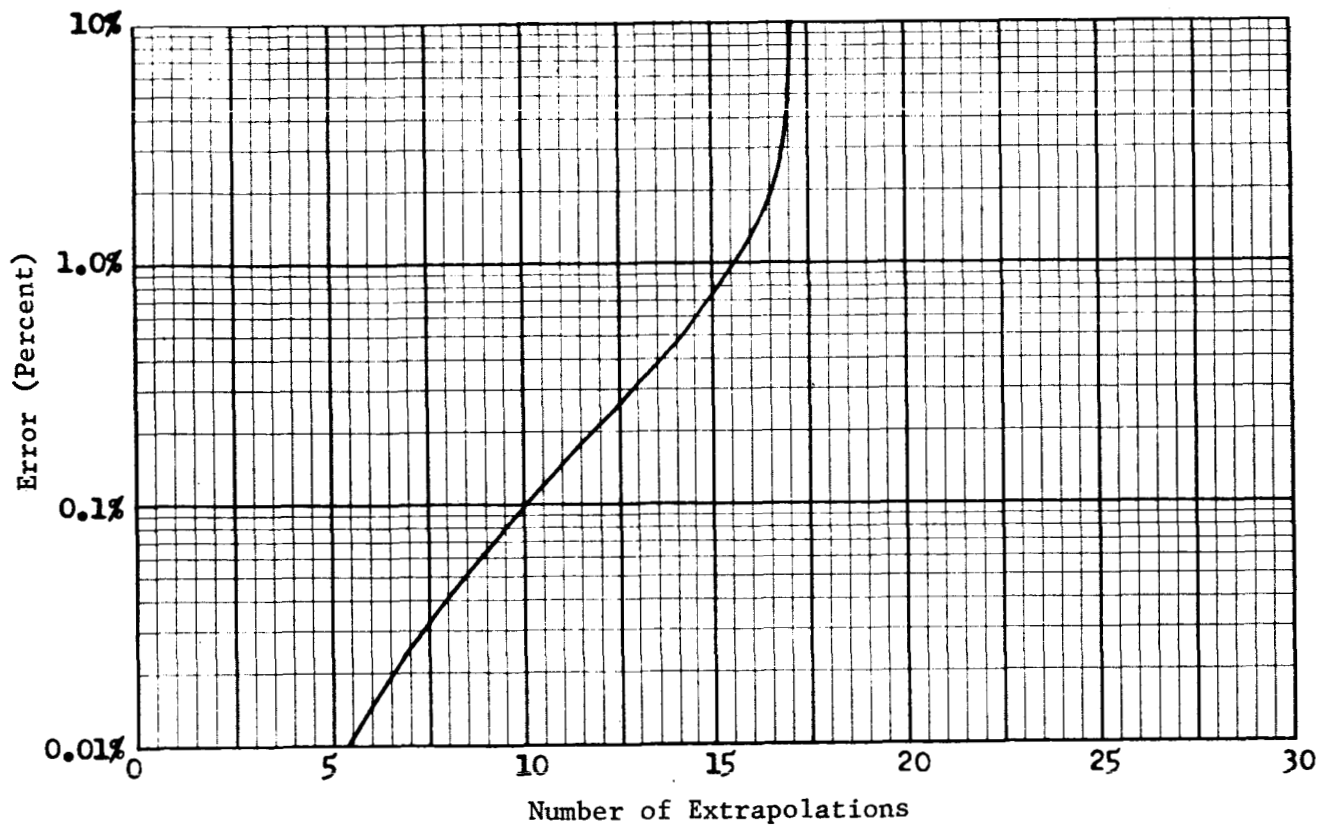
Figure 13-b.   Percent Error Versus Number of Extrapolations.

extrapolations start

1.5

1.0

data starts at x = 20

0.5

0.0

-0.5

extrapolated curve

-1.0

-1.5

Arctangent X (Units)

12   10   8   6   4   2   0   -2   -4   -6   -8   -1C   -12

X (Units)
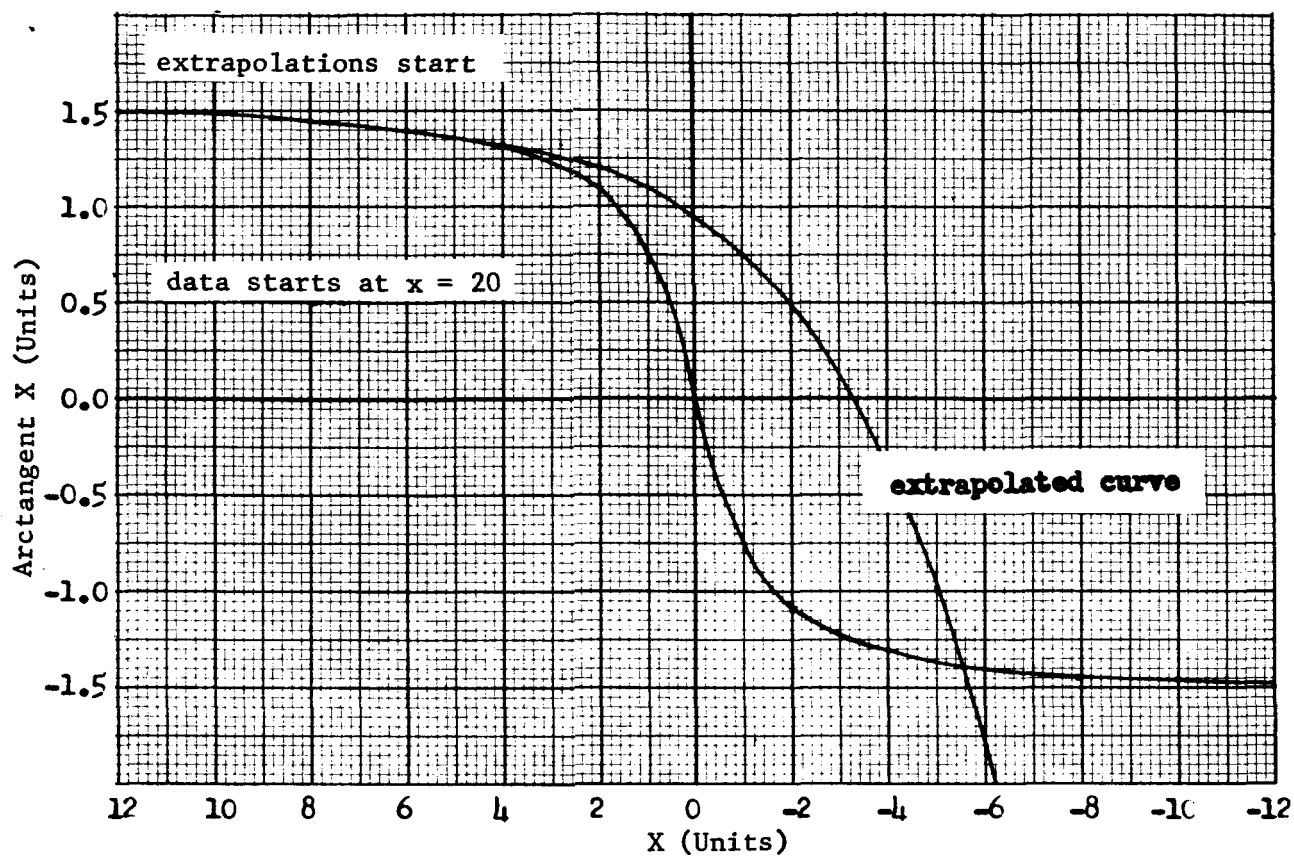
Figure 14-a.   Arctangent X as a Function of X With Extrapolations from
Data Increments of 1.0.

10%

1.0%

Error (Percent)

0.1%

0.01%

0   1   2   3   4   5   6   7   8   9   10   11   12
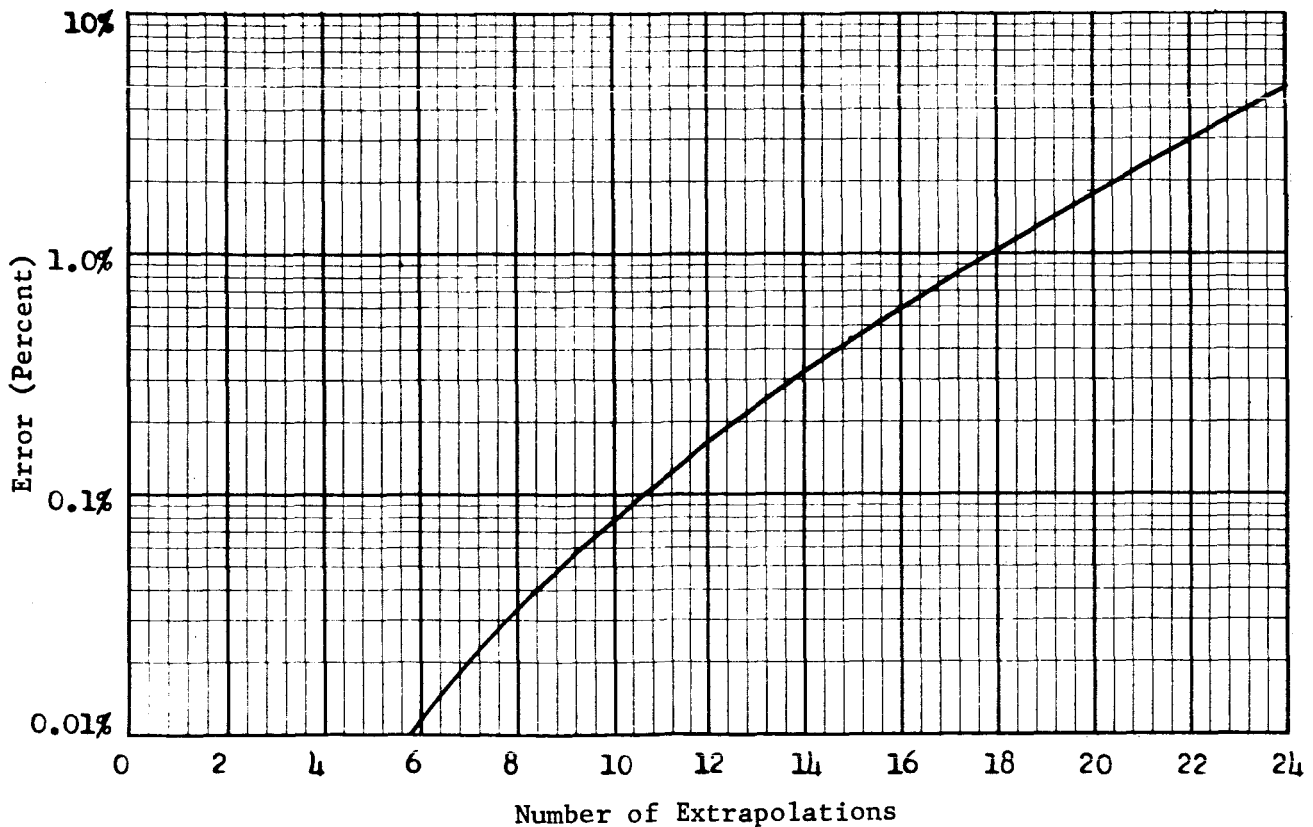
Number of Extrapolations
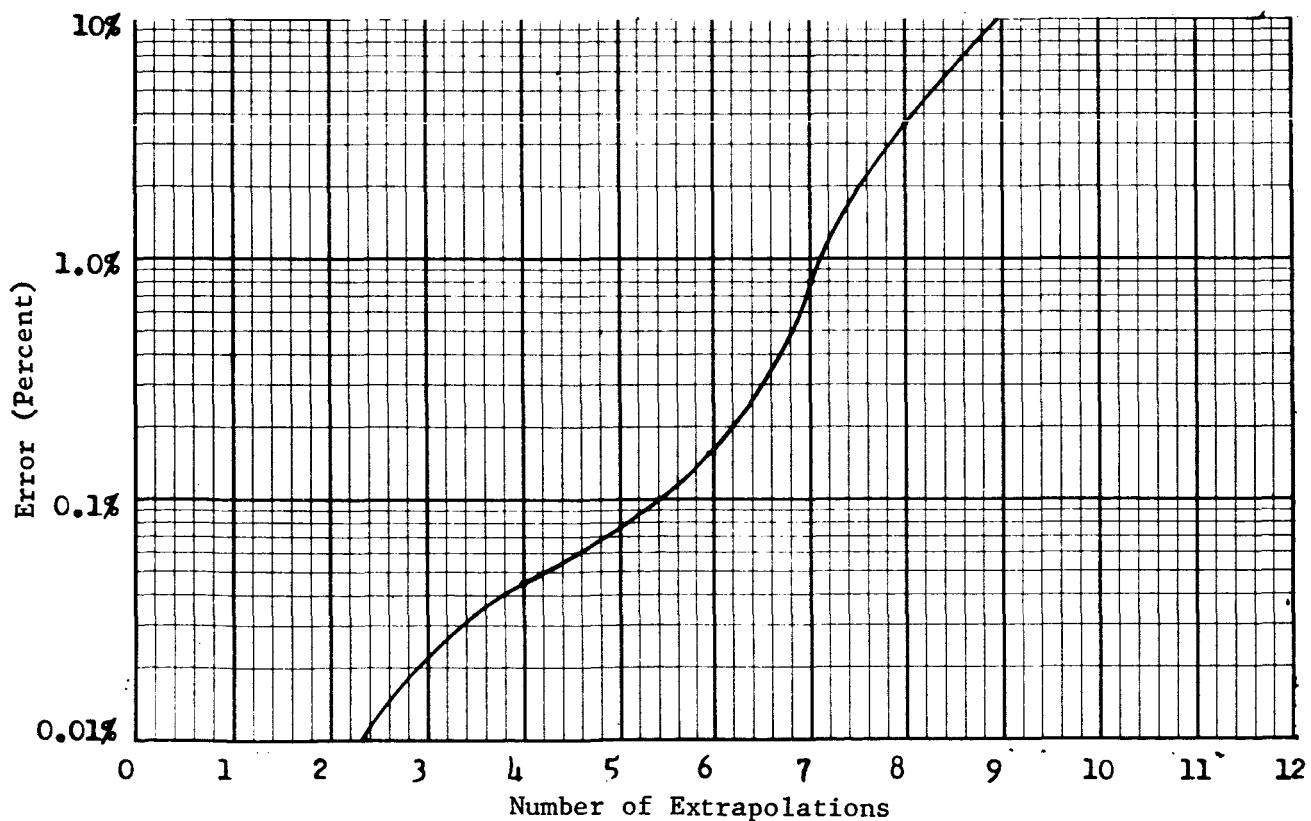
Figure 14-b.   Percent Error Versus Number of Extrapolations.
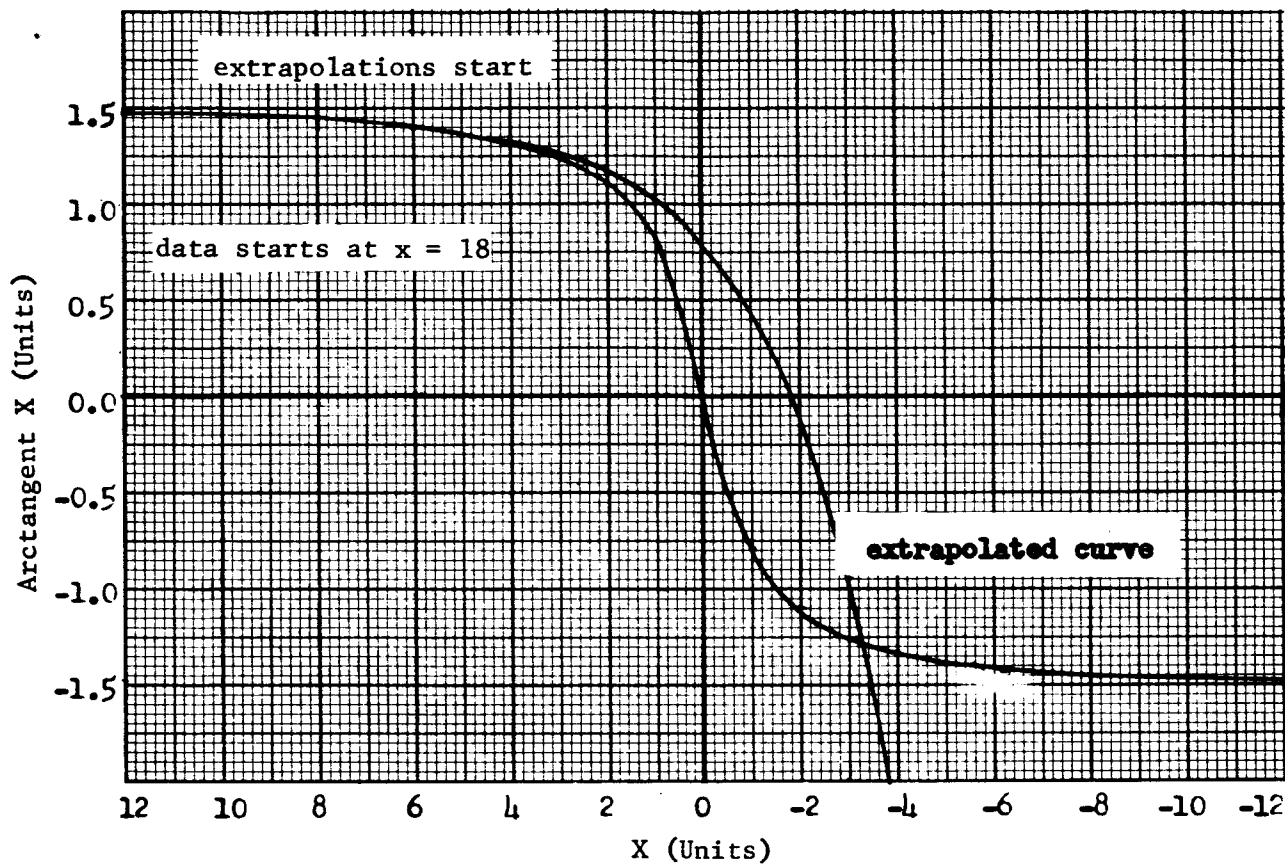
Figure 15-a.   Arctangent X as a Function of X With Extrapolations from Data Increments of 1.0.
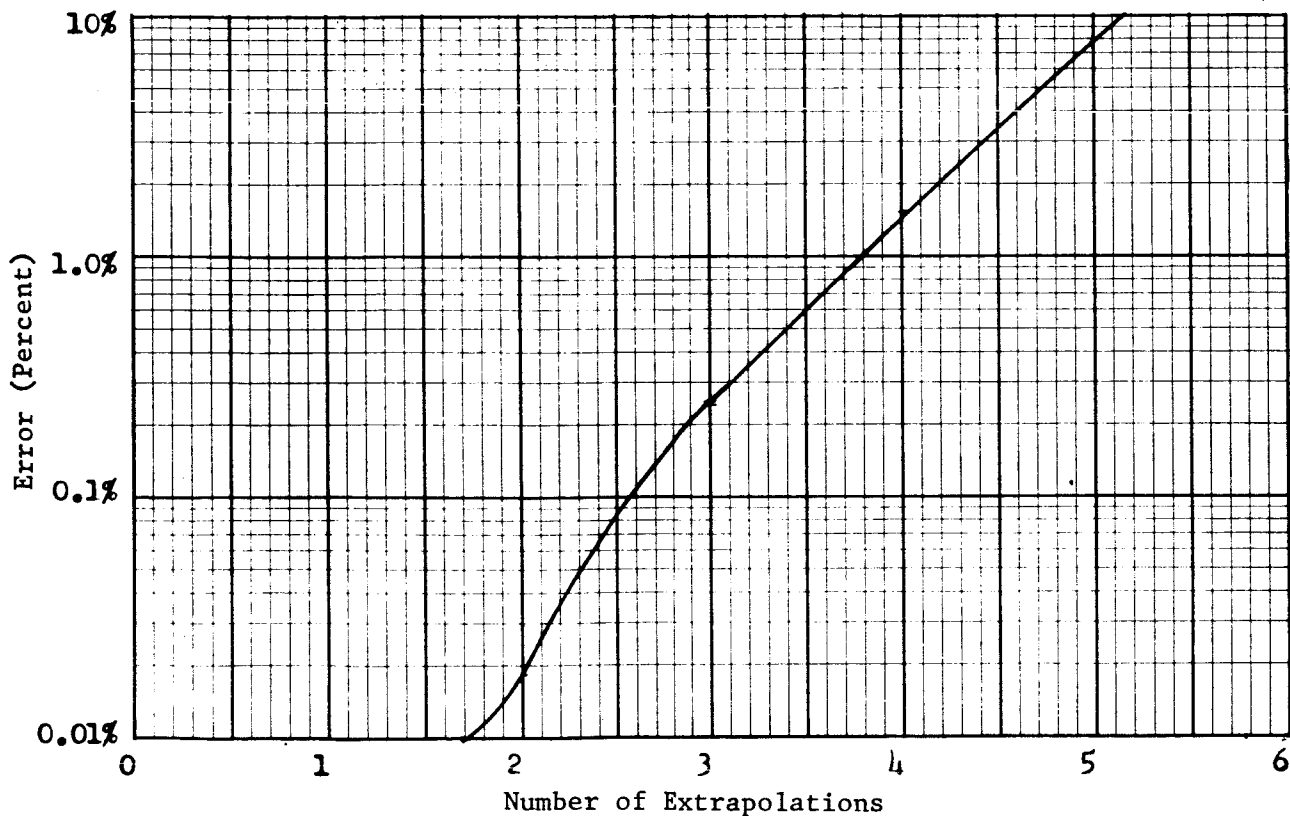
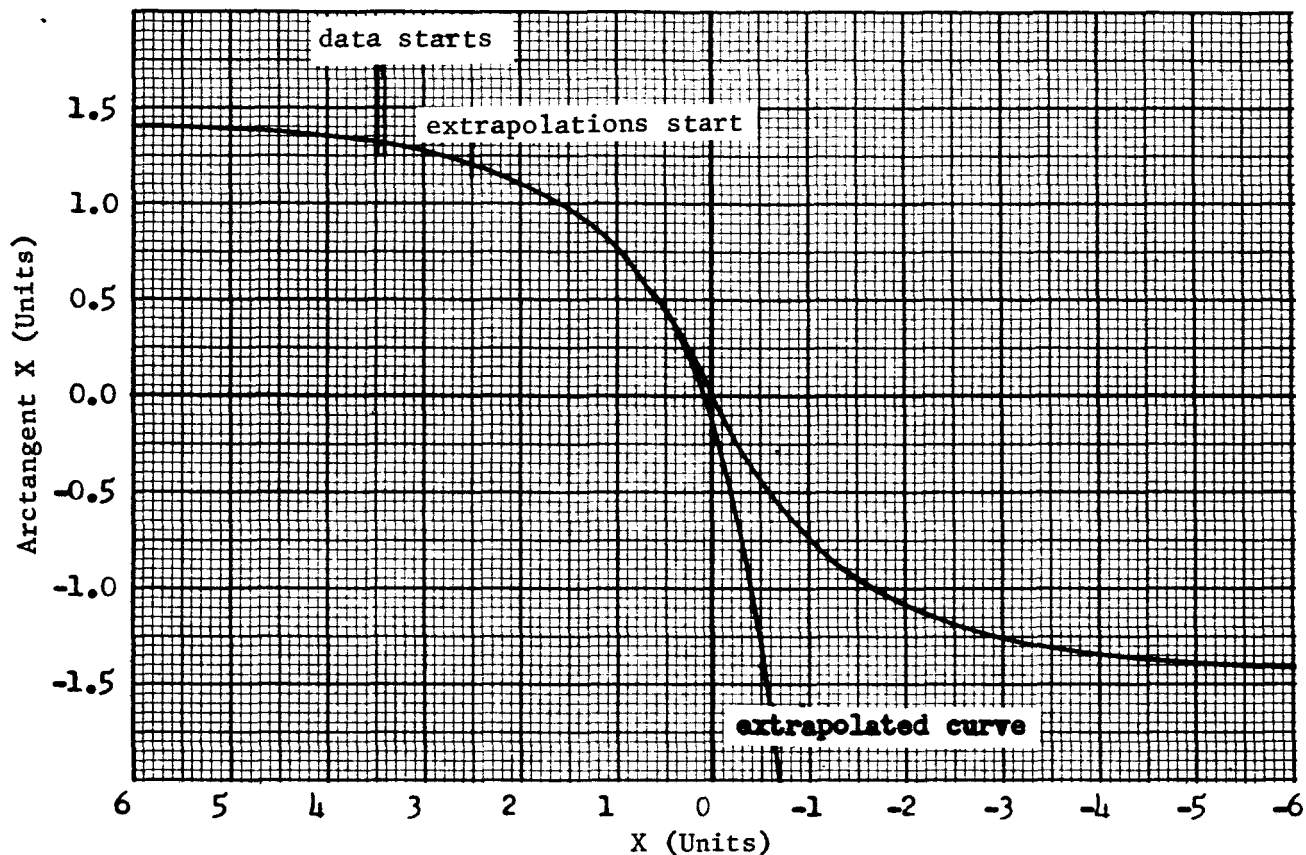Figure 15-b.   Percent Error Versus Number of Extrapolations.
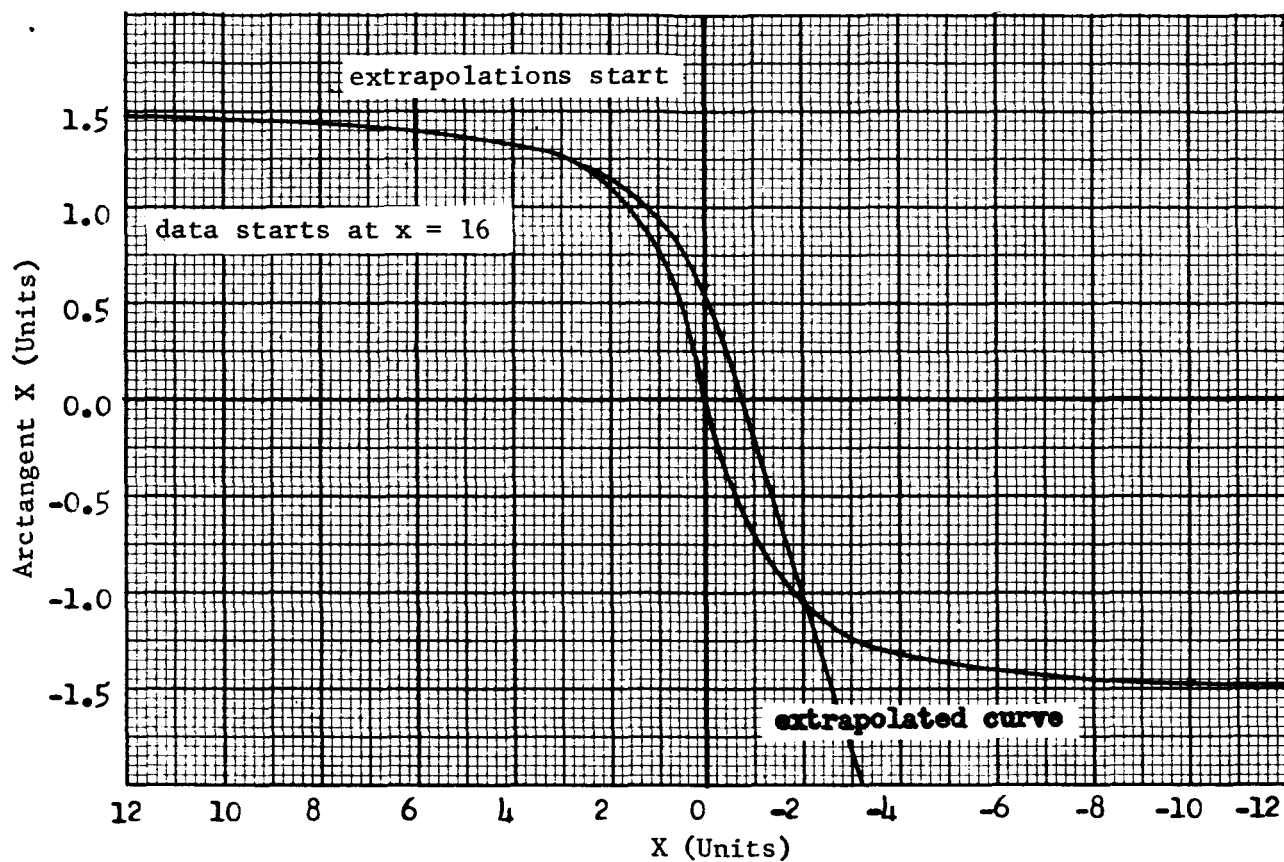
Figure 16-a. Arctangent X as a Function of X With Extrapolations from Data Increments of 1.0.
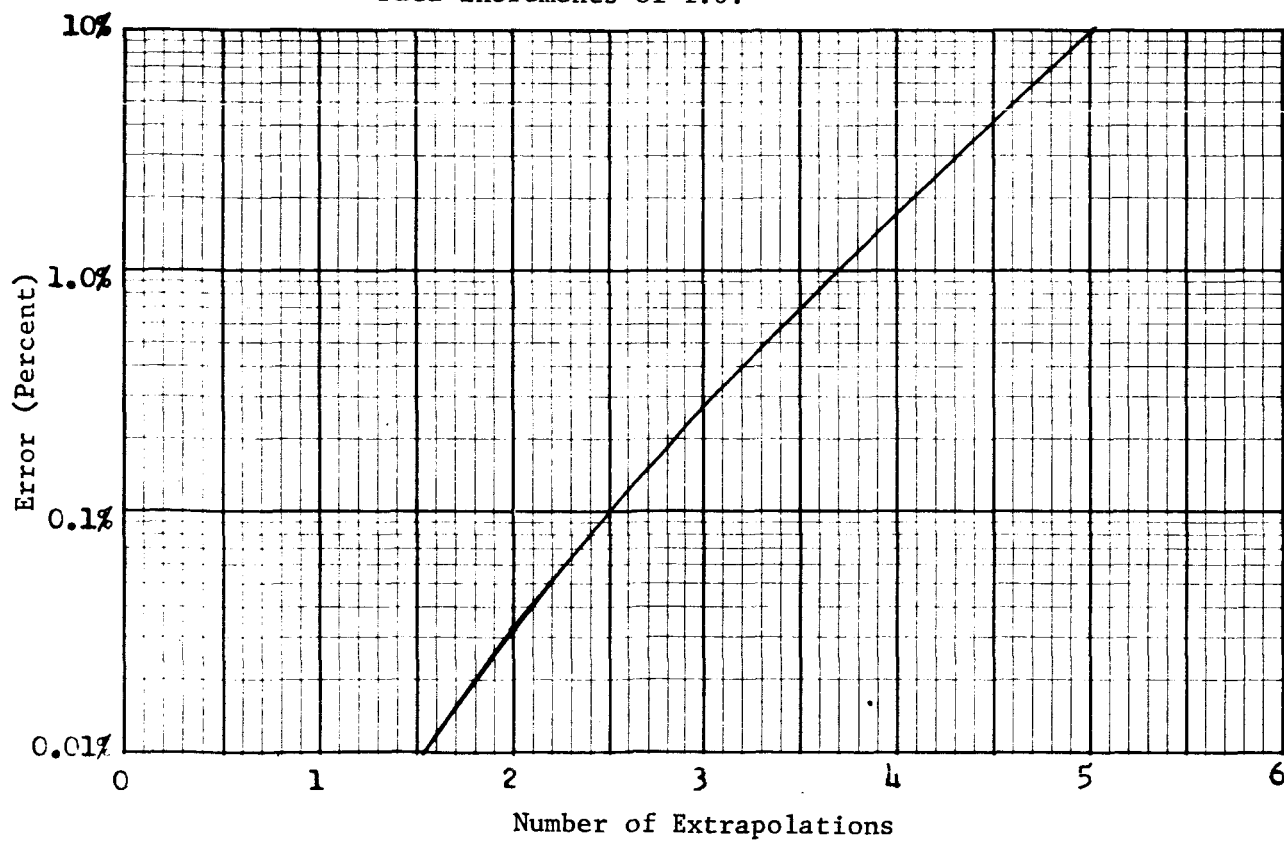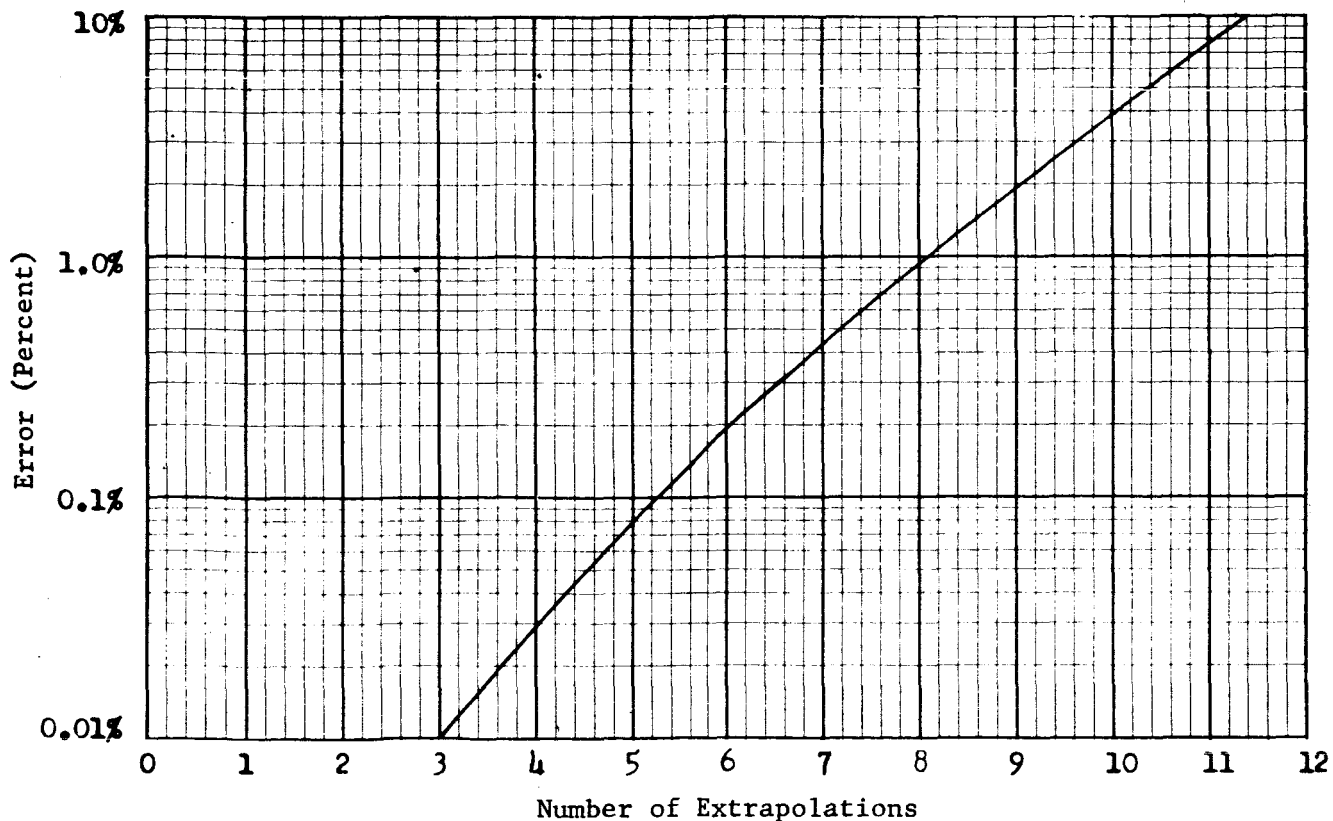

Figure 16-b. Percent Error Versus Number of Extrapolations.
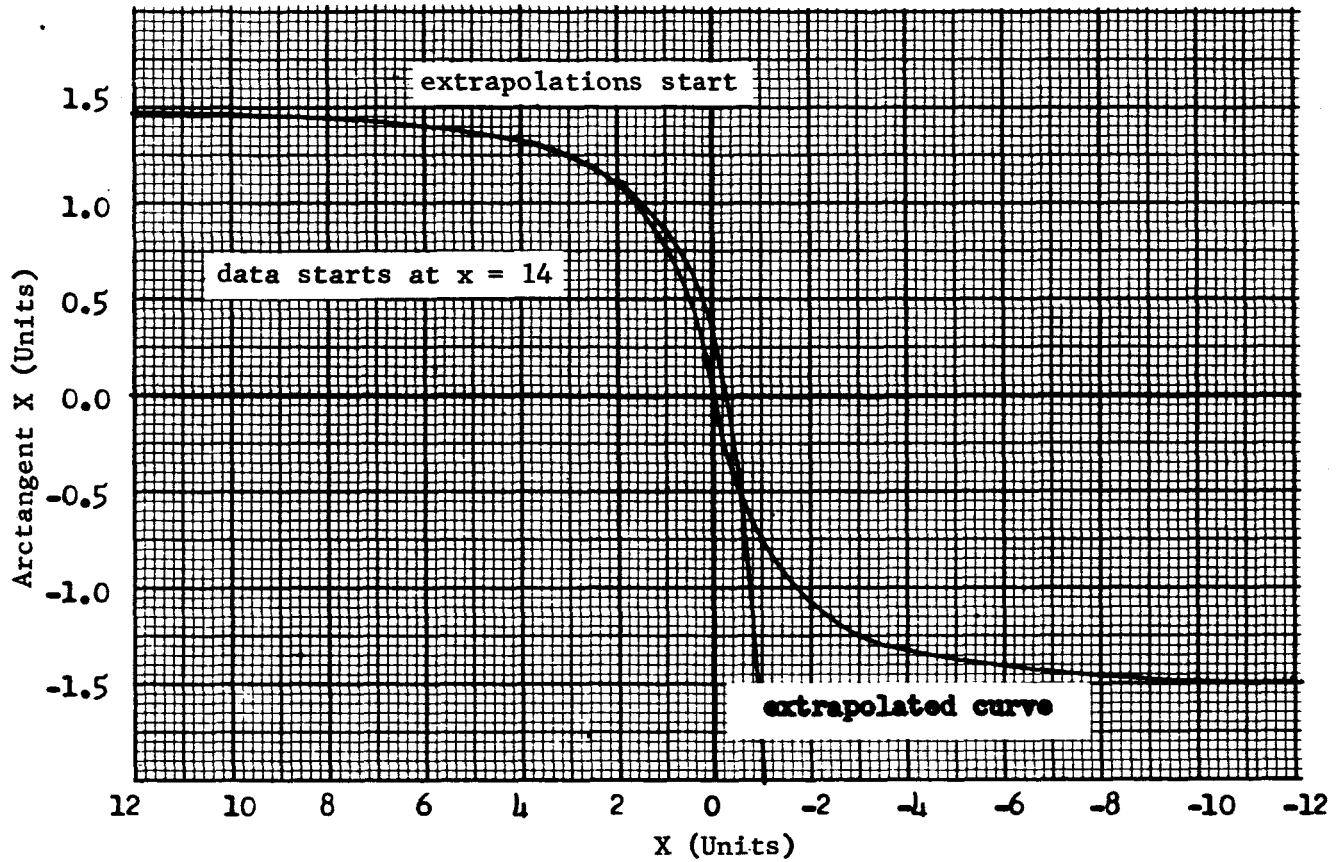
Figure 17-a. Arctangent X as a Function of X With Extrapolations from Data Increments of 1.0.
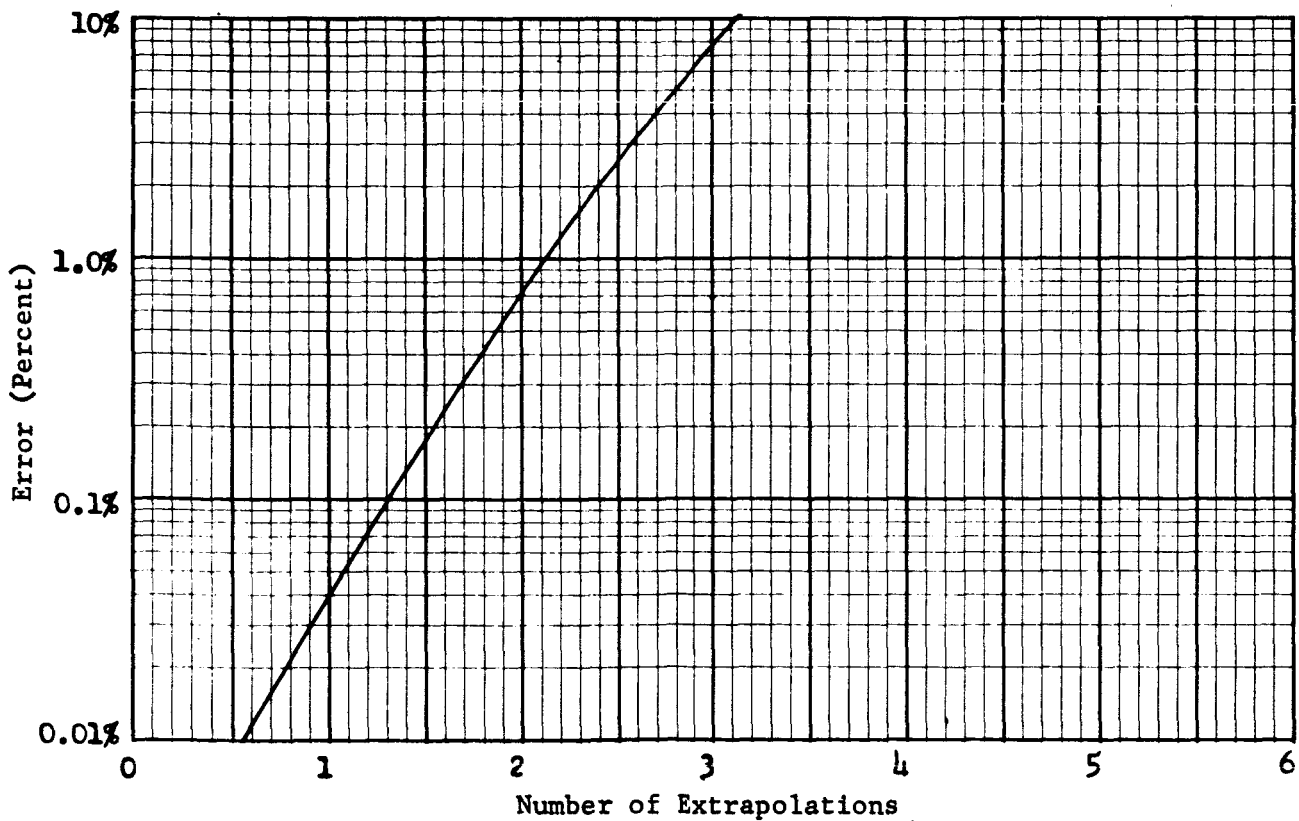
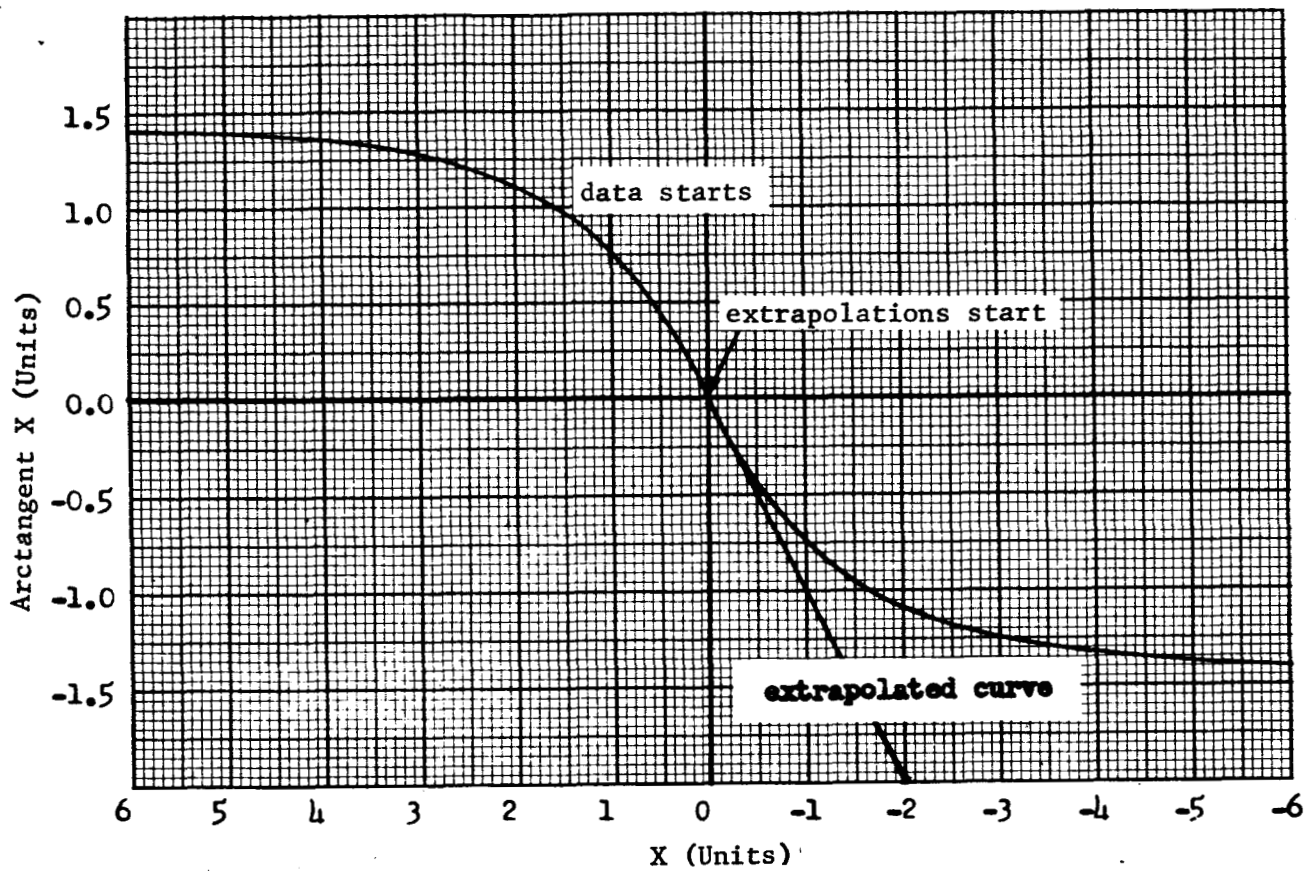Figure 17-b. Percent Error Versus Number of Extrapolations.
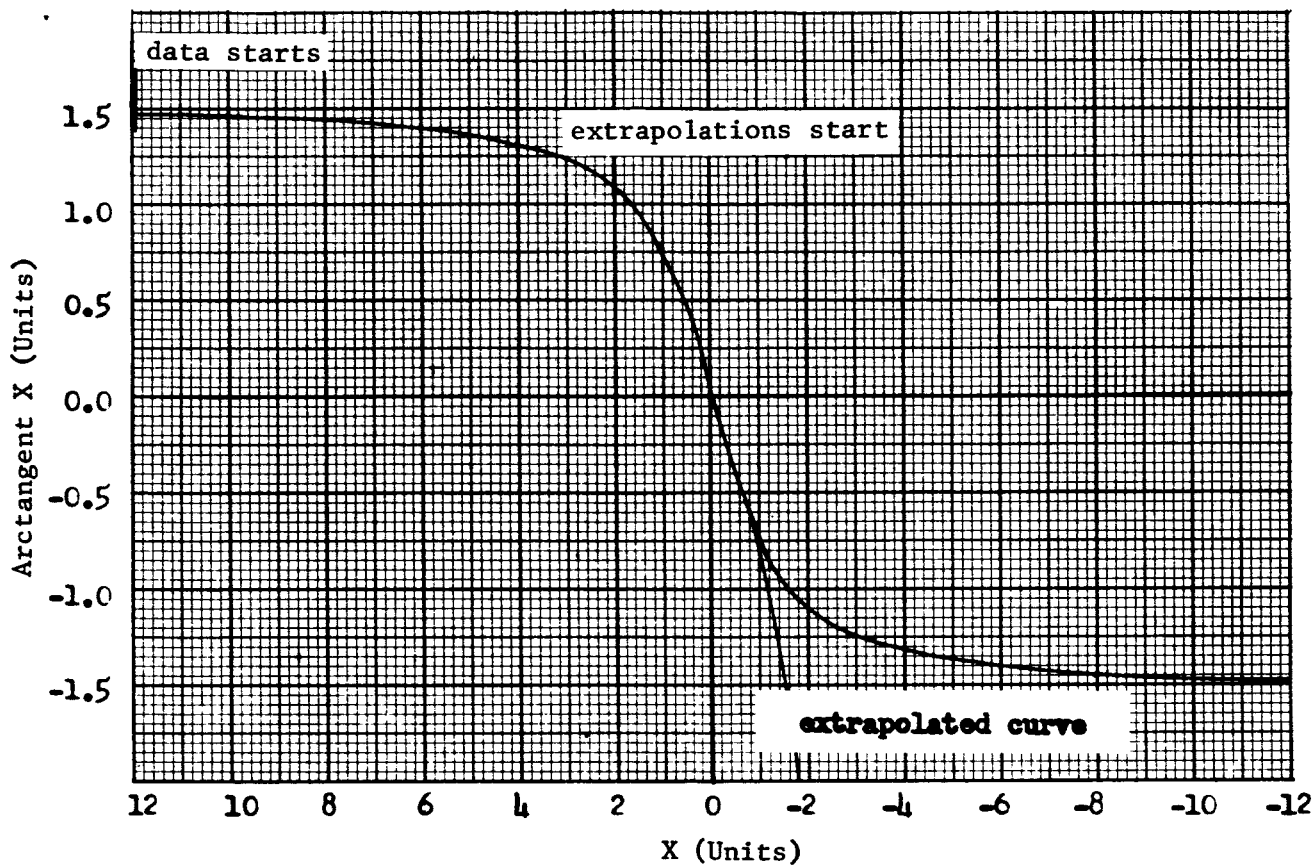
Figure 18-a. Arctangent X as a Function of X With Extrapolations from Data Increments of 1.0.
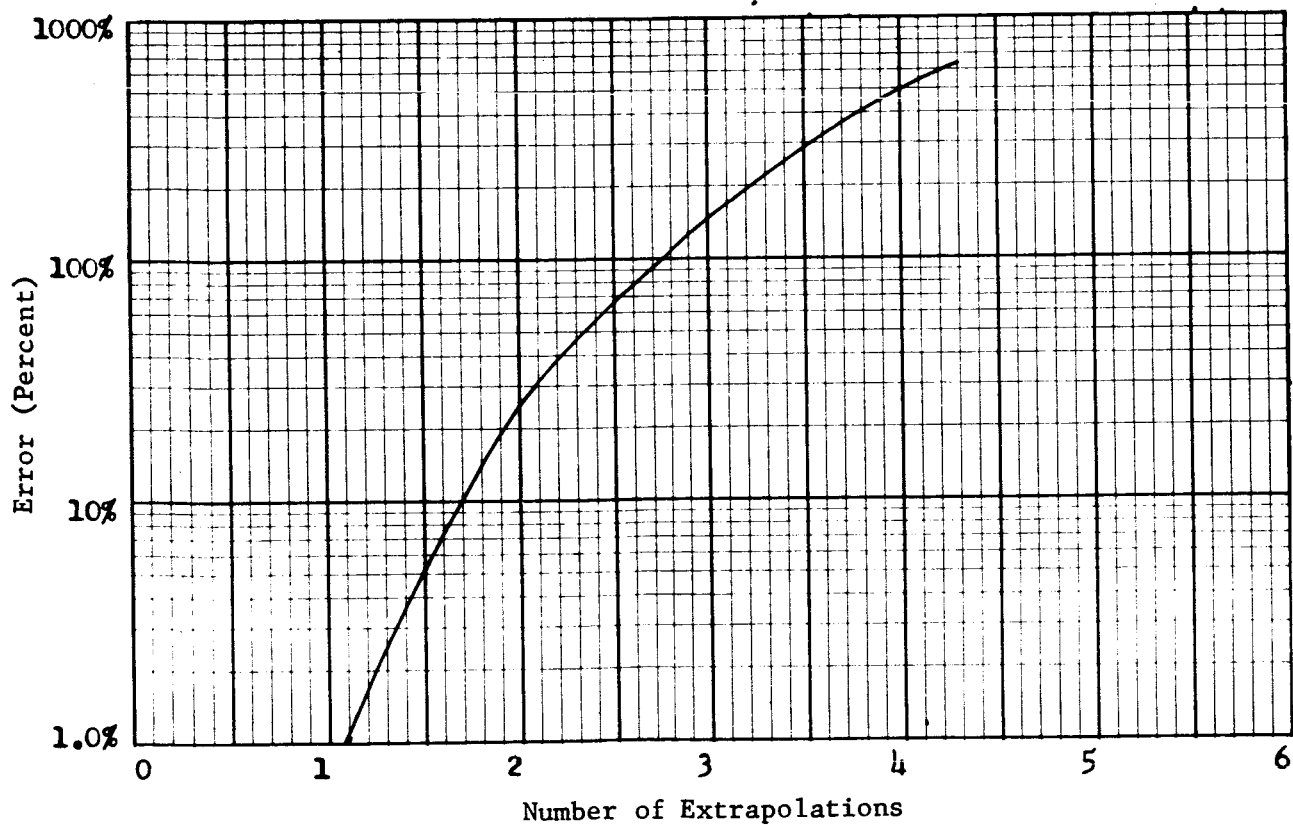
Figure 18-b. Percent Error Versus Number of Extrapolations.

64
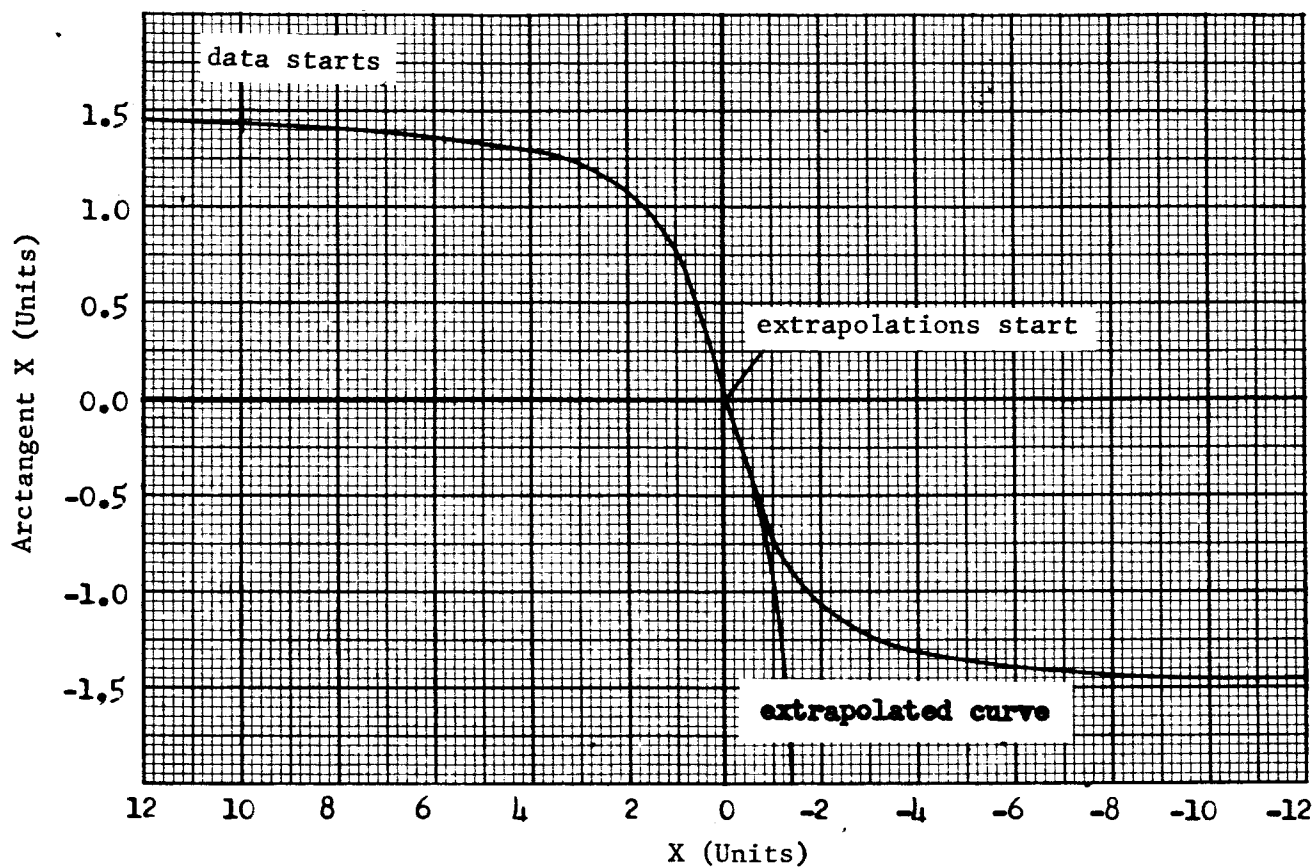
Figure 19-a. Arctangent X as a Function of X With Extrapolations from Data Increments of 1.0.
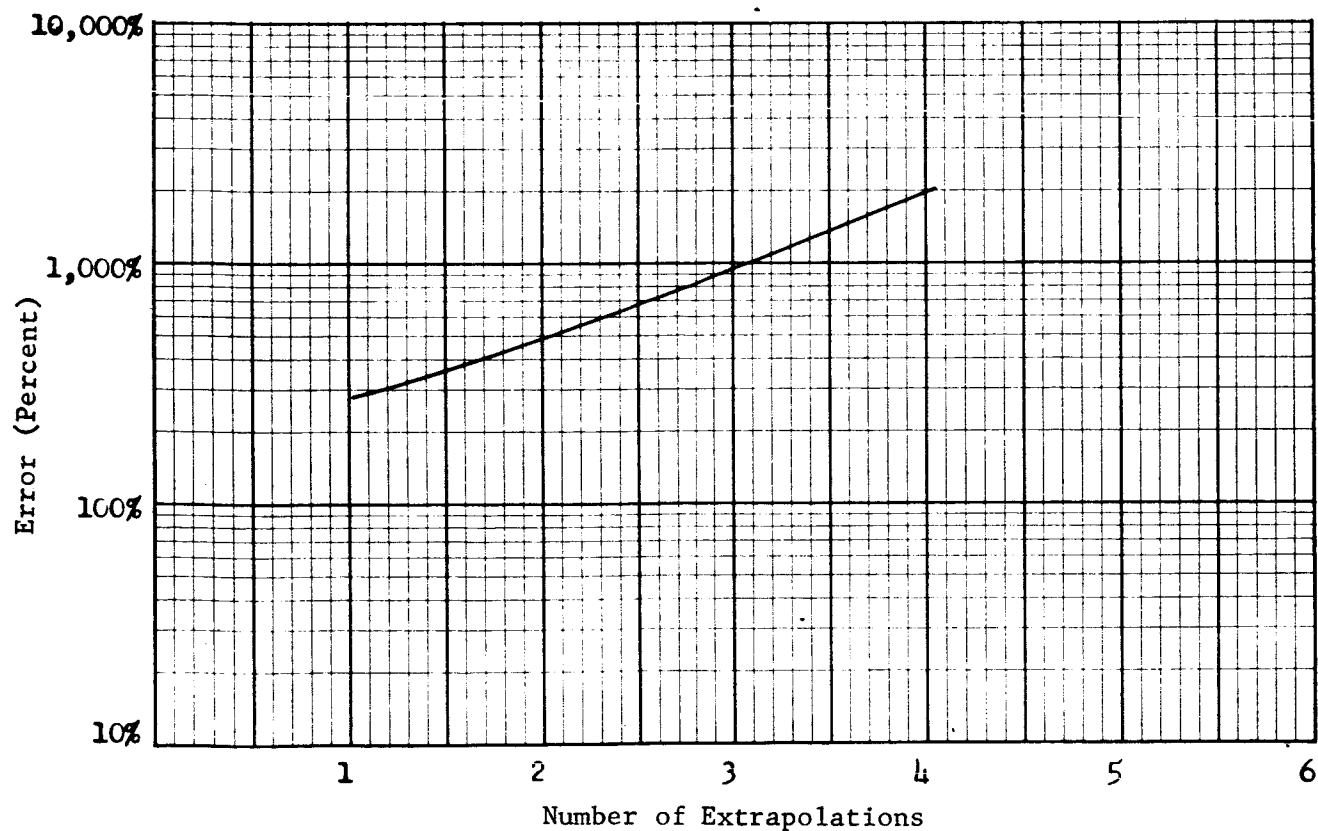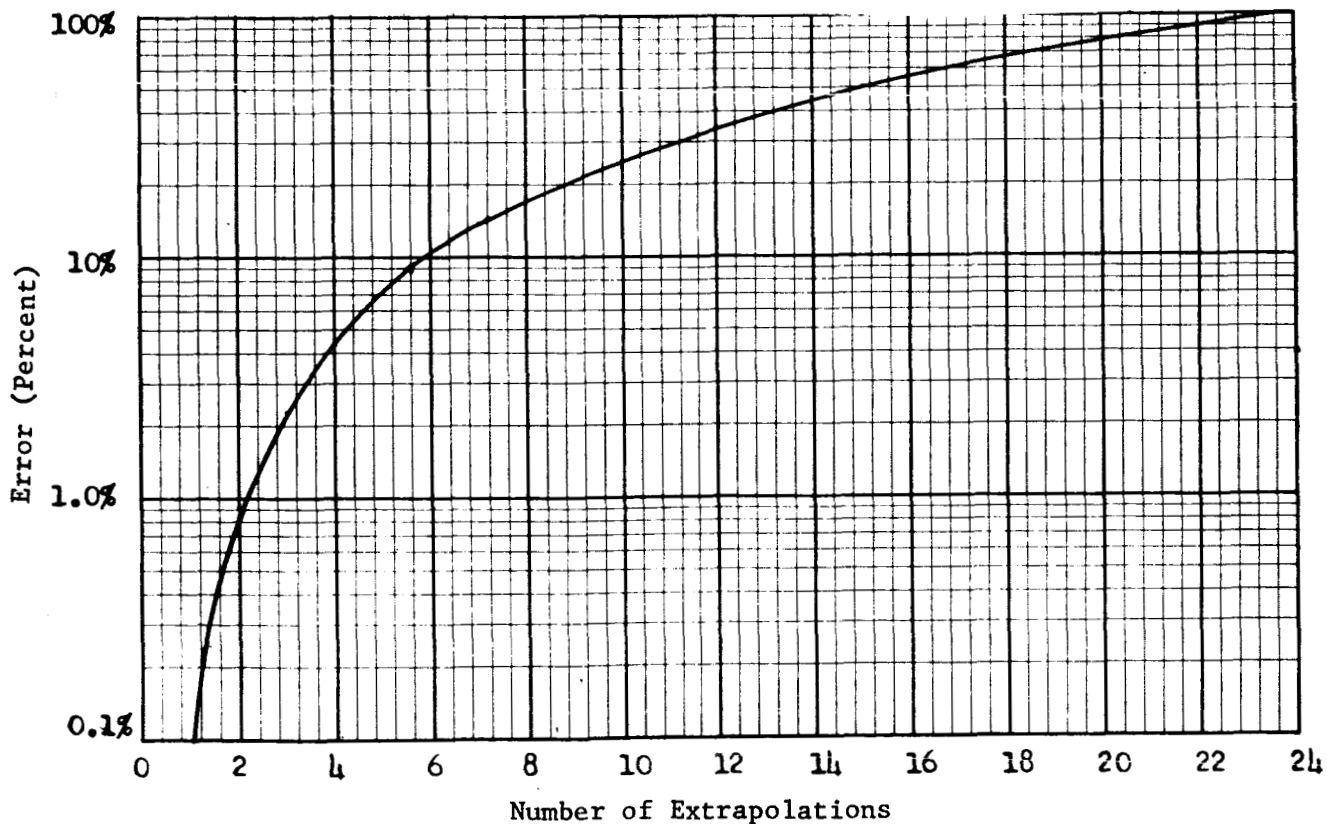

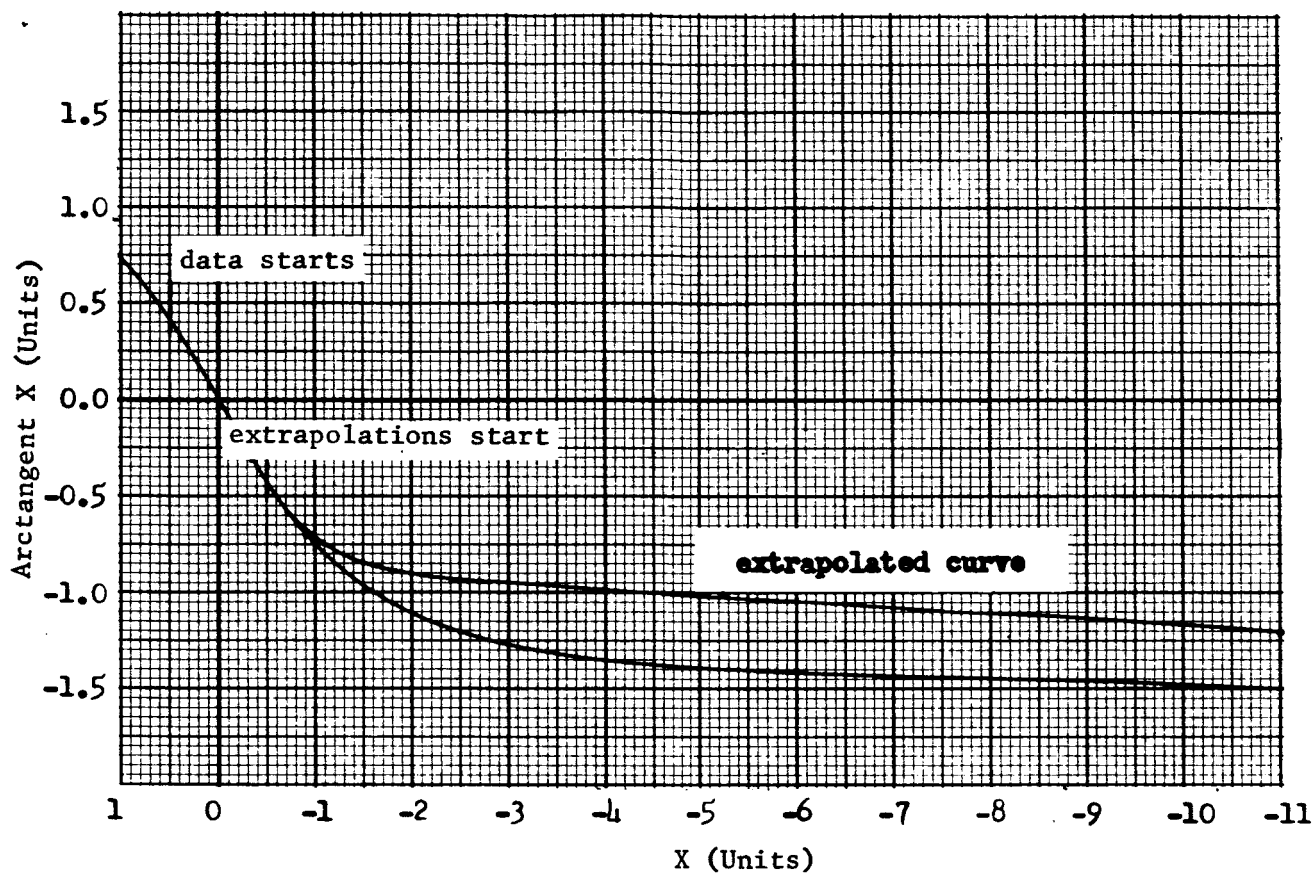Figure 19-b. Percent Error Versus Number of Extrapolations.
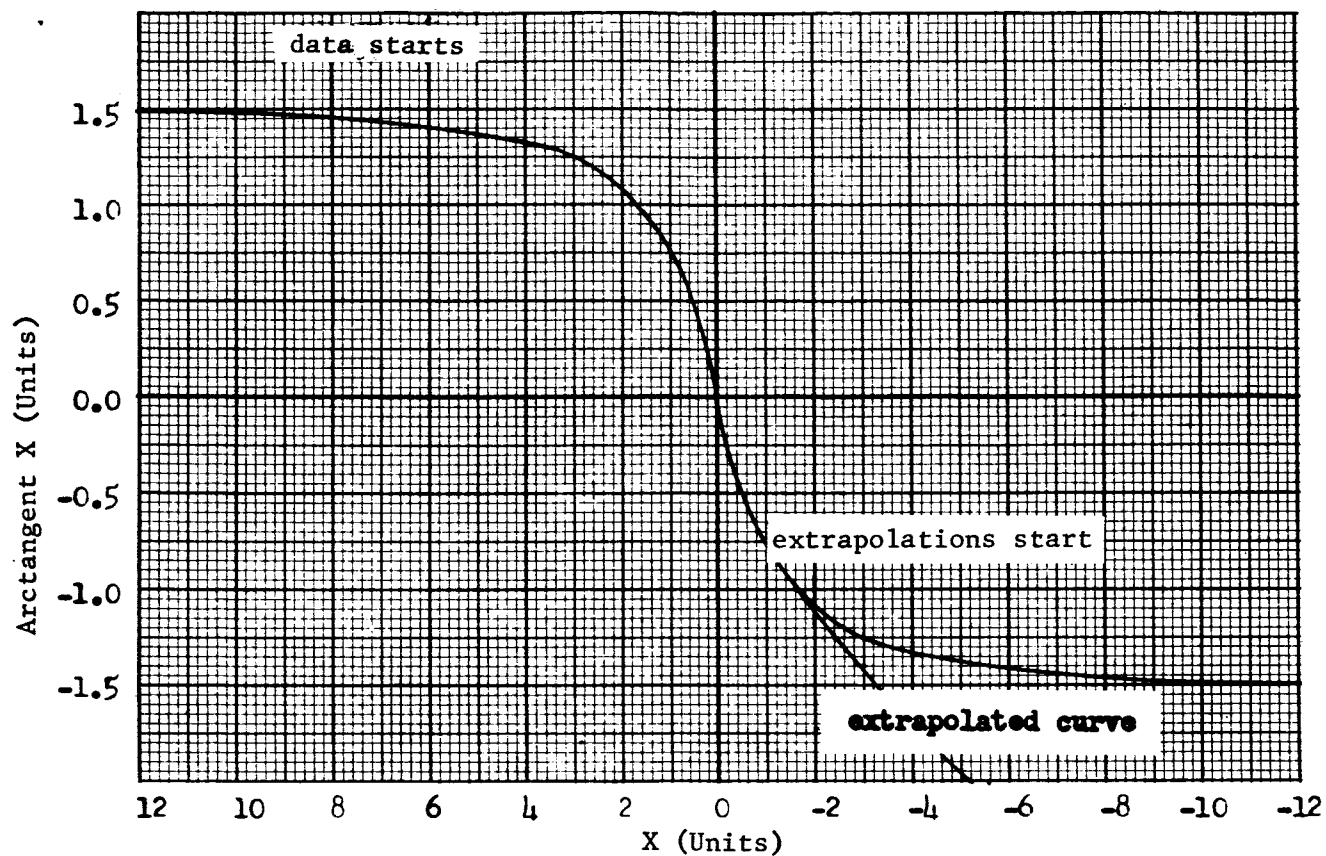
Figure 20-a. Arctangent X as a Function of X With Extrapolations from Data Increments of 1.0.
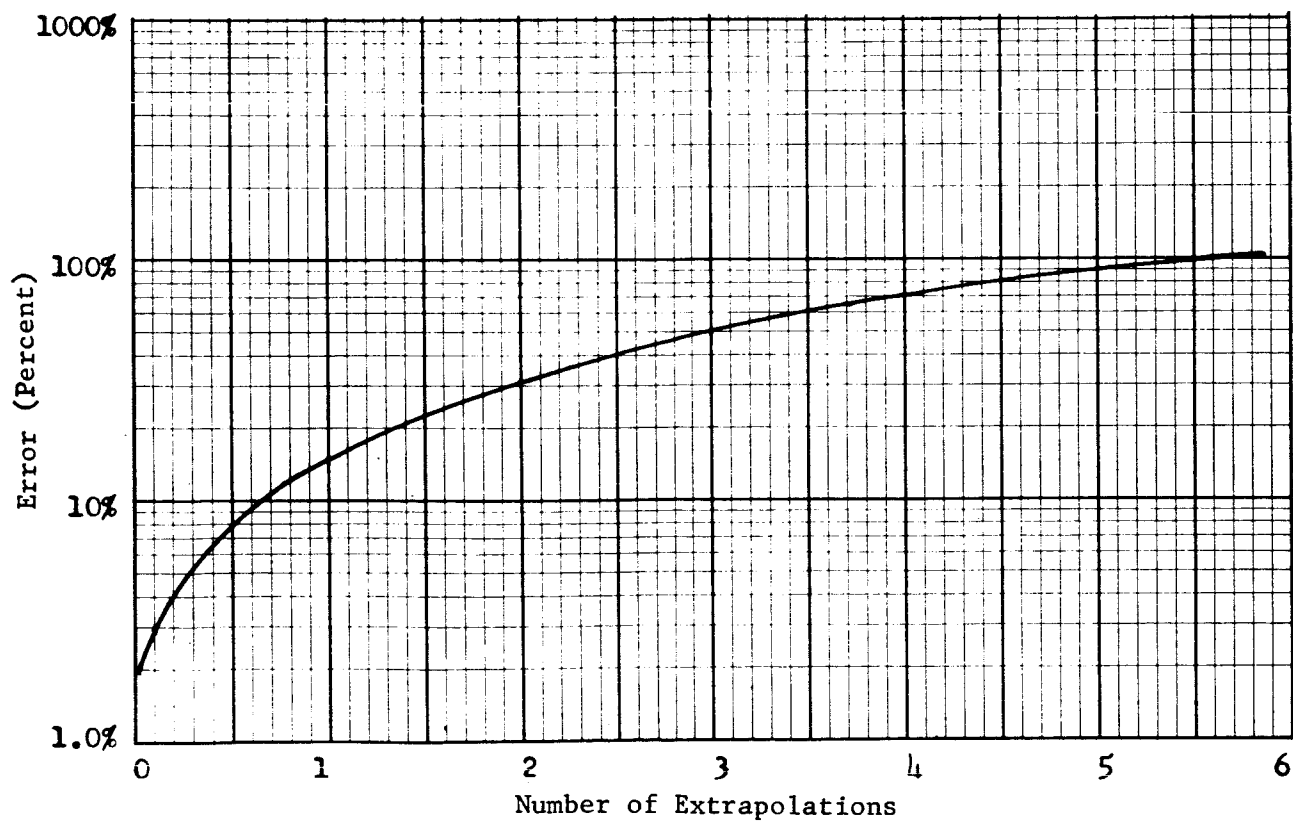
Figure 20-b. Percent Error Versus Number of Extrapolations.
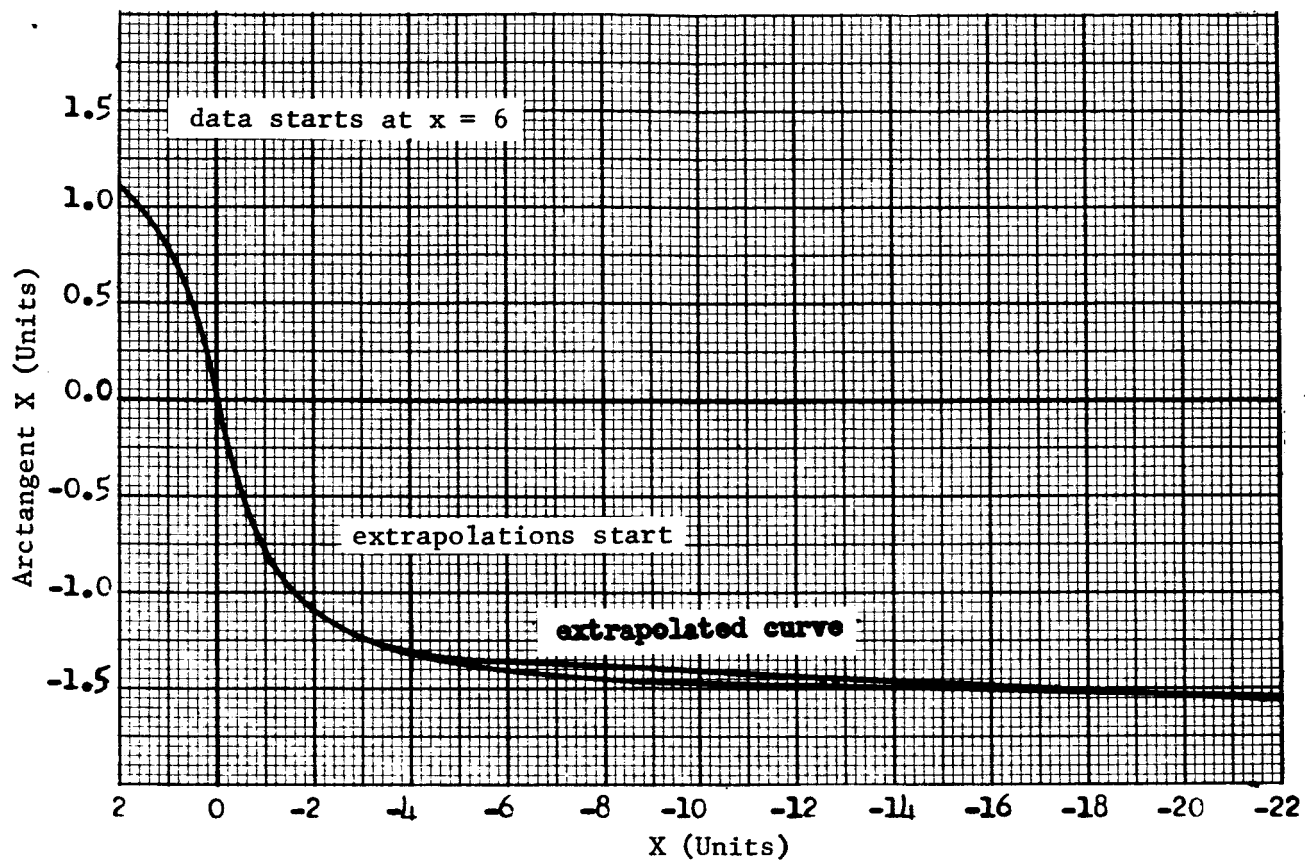
Figure 21-a. Arctangent X as a Function of X With Extrapolations from
Data Increments of 1.0.
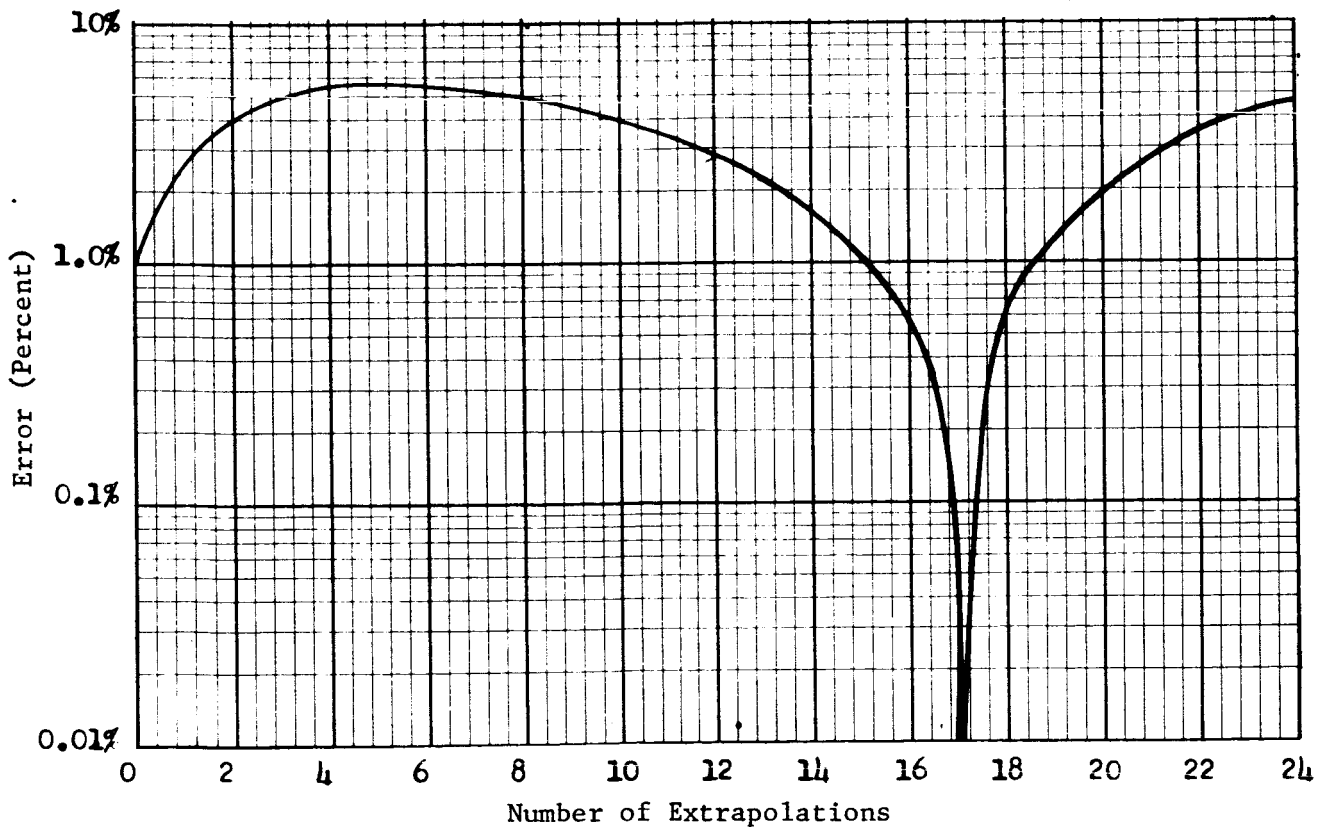
Figure 21-b. Percent Error Versus Number of Extrapolations.
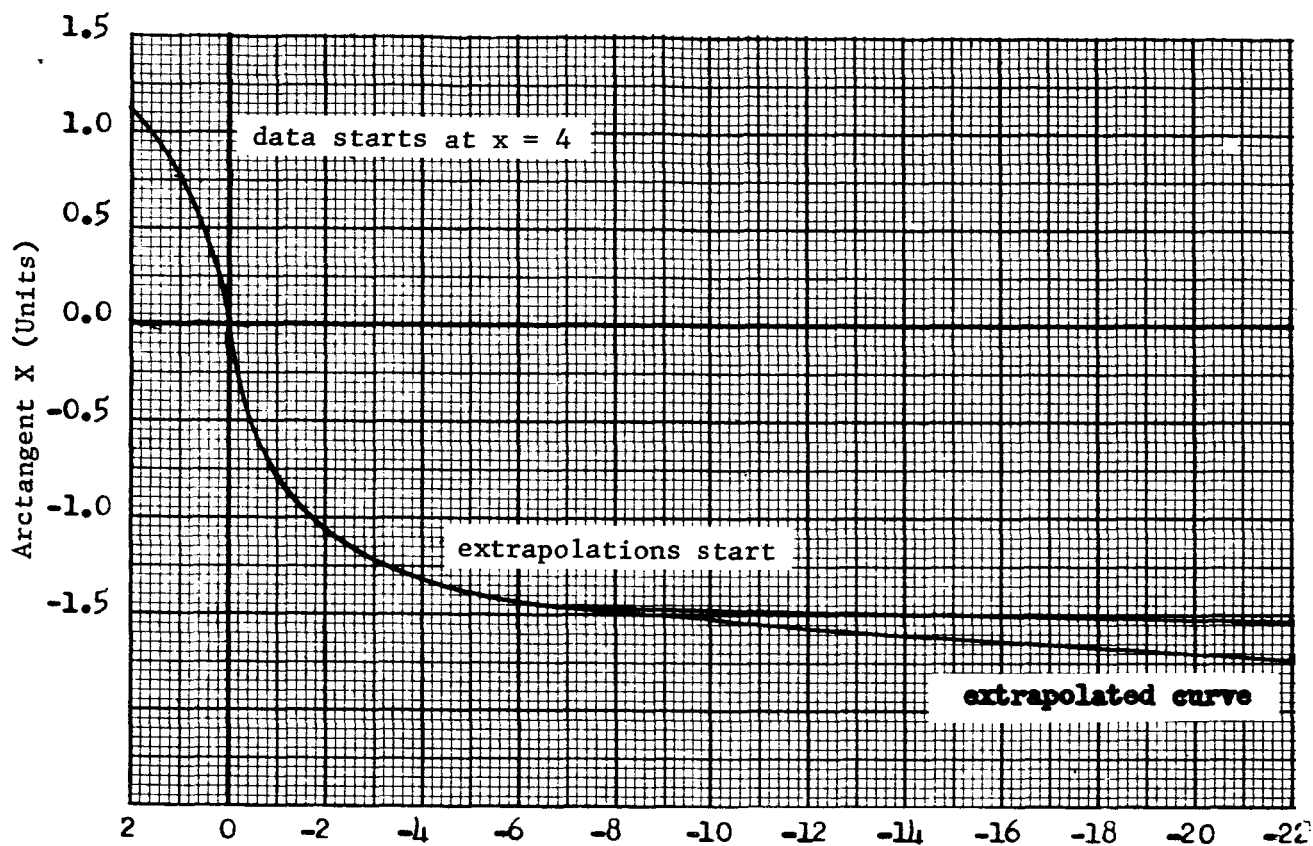
Figure 22-a.   Arctangent X as a Function of X With Extrapolations from
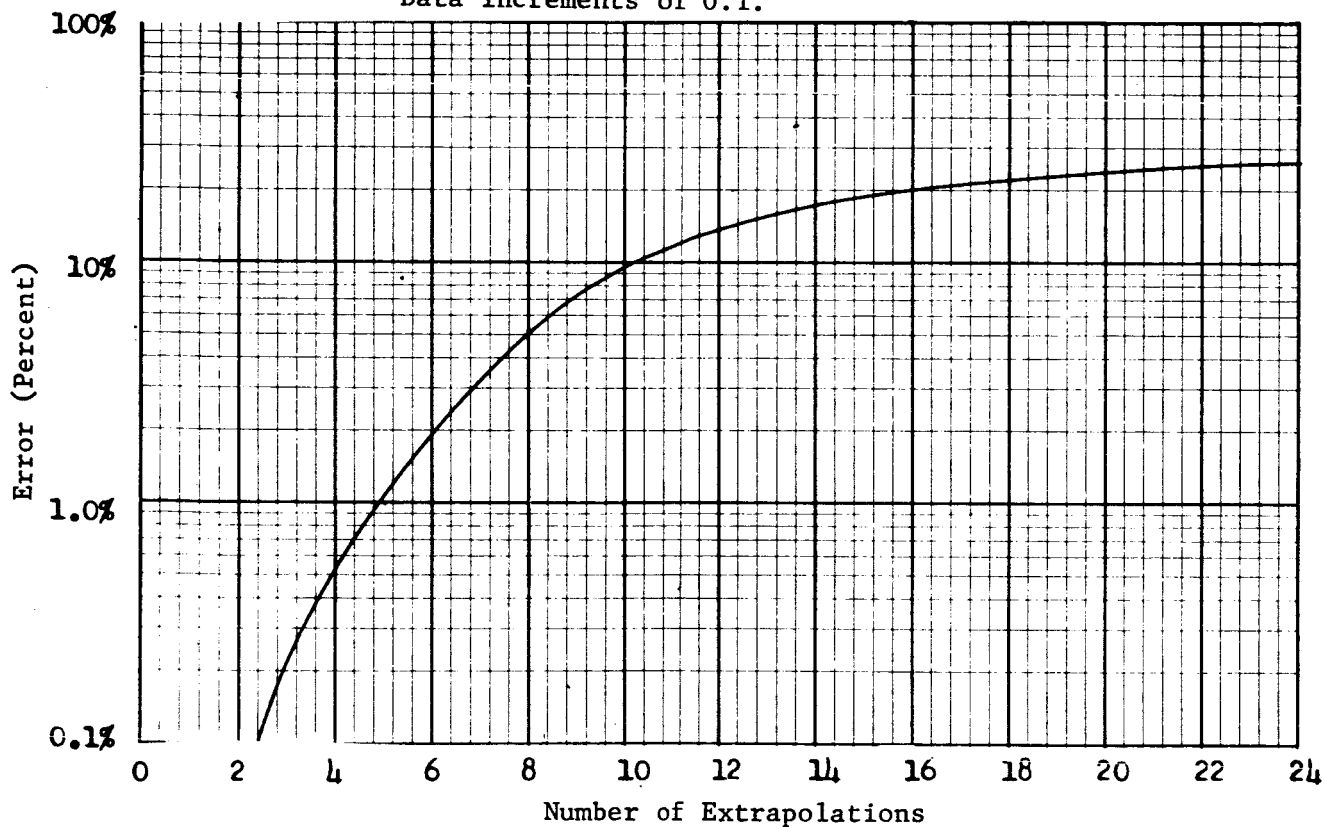Data Increments of 1.0.
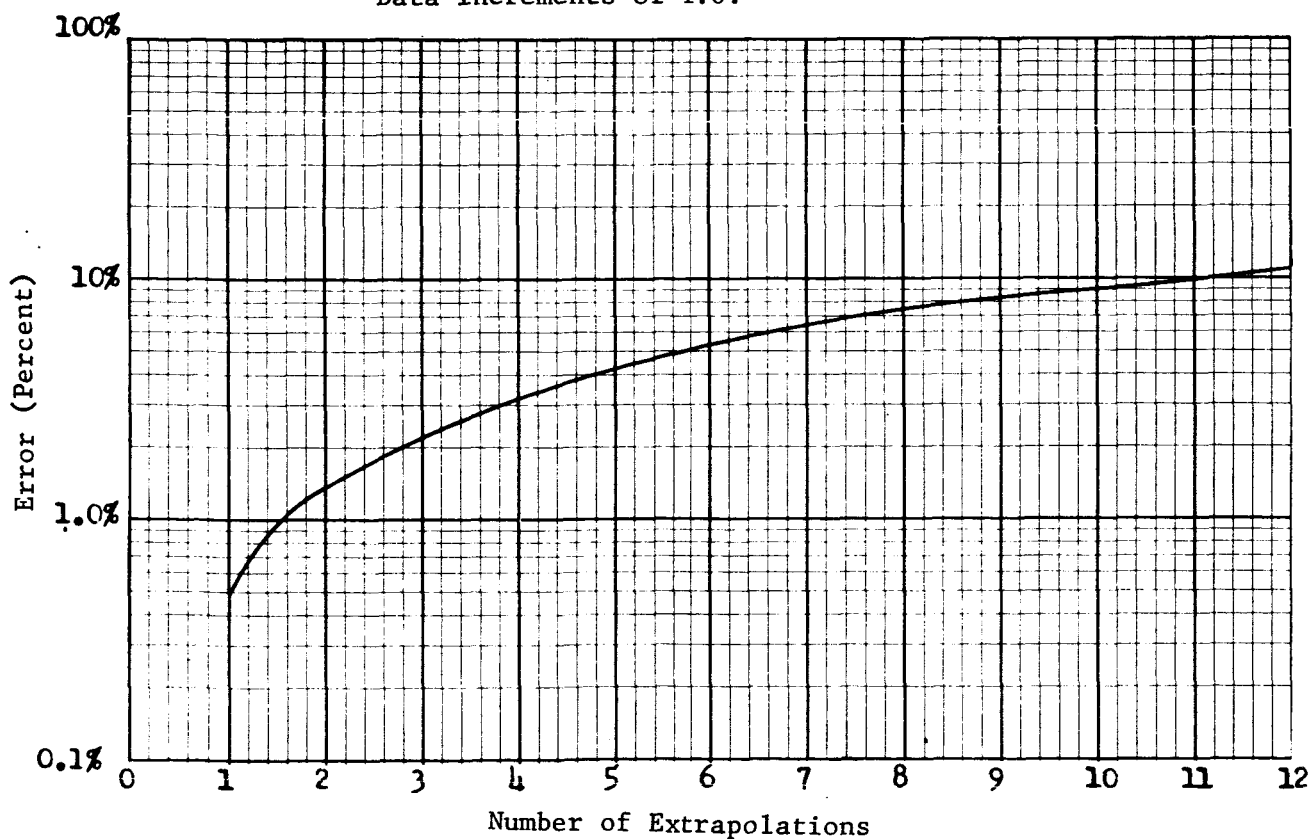
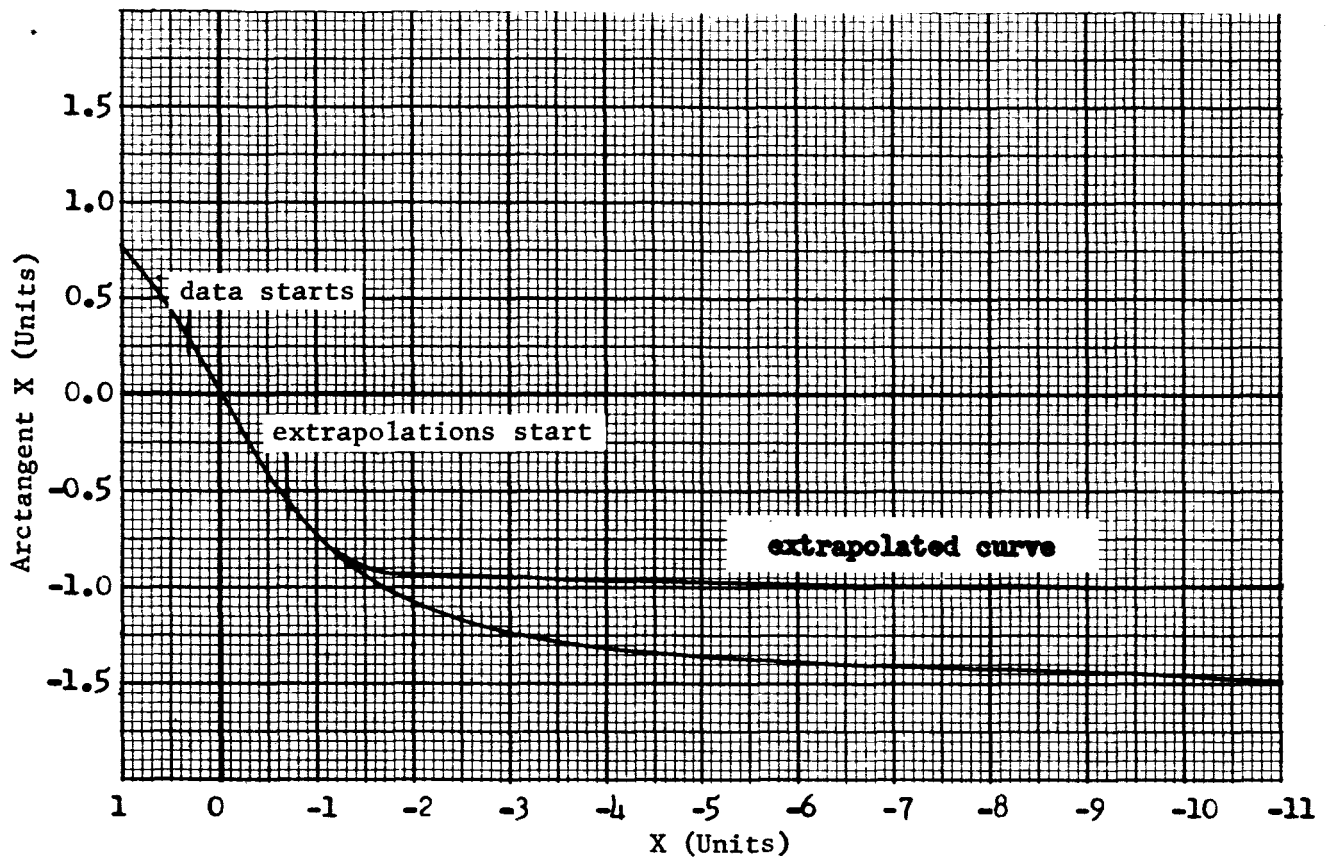Figure 22-b.   Percent Error Versus Number of Extrapolations.
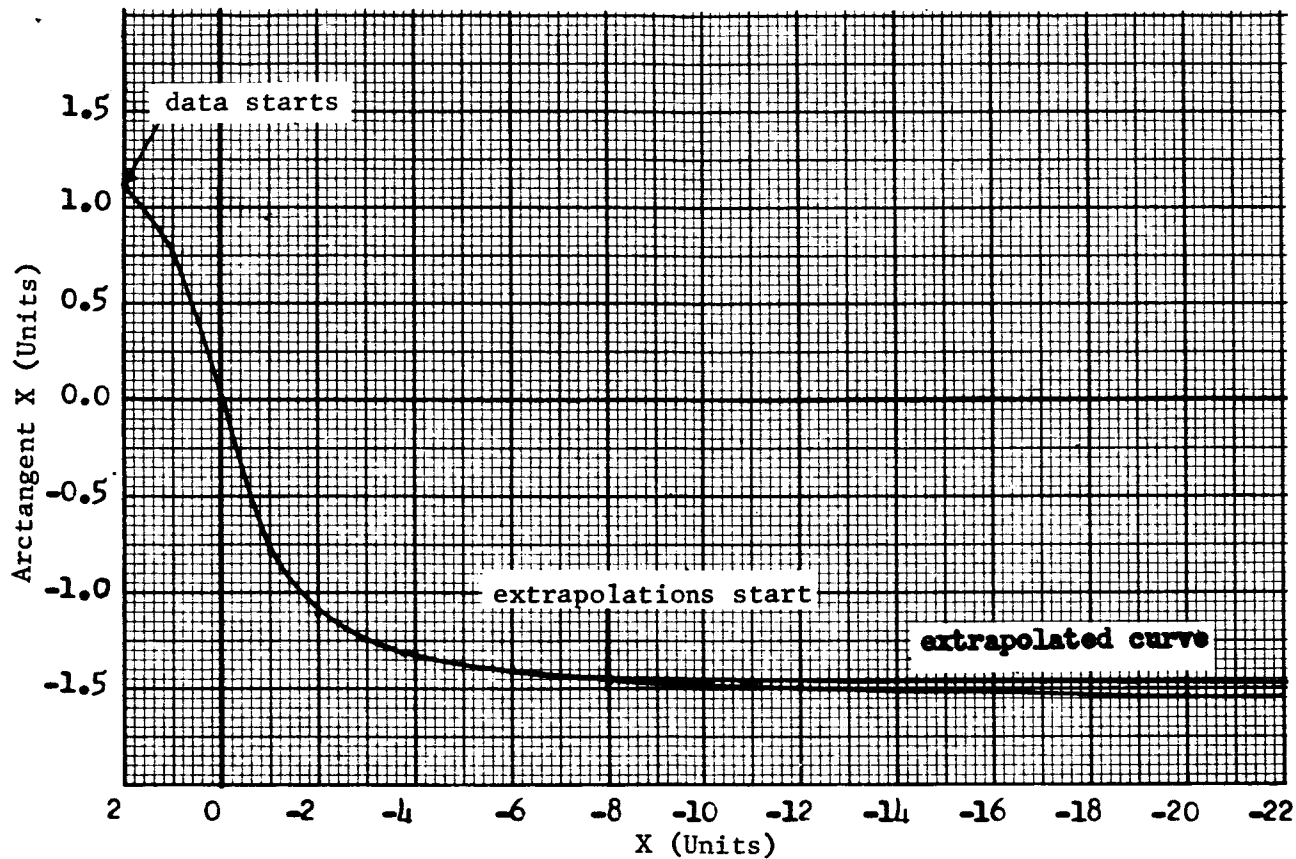
Figure 23-a. Arctangent X as a Function of X With Extrapolations from Data Increments of 1.0.
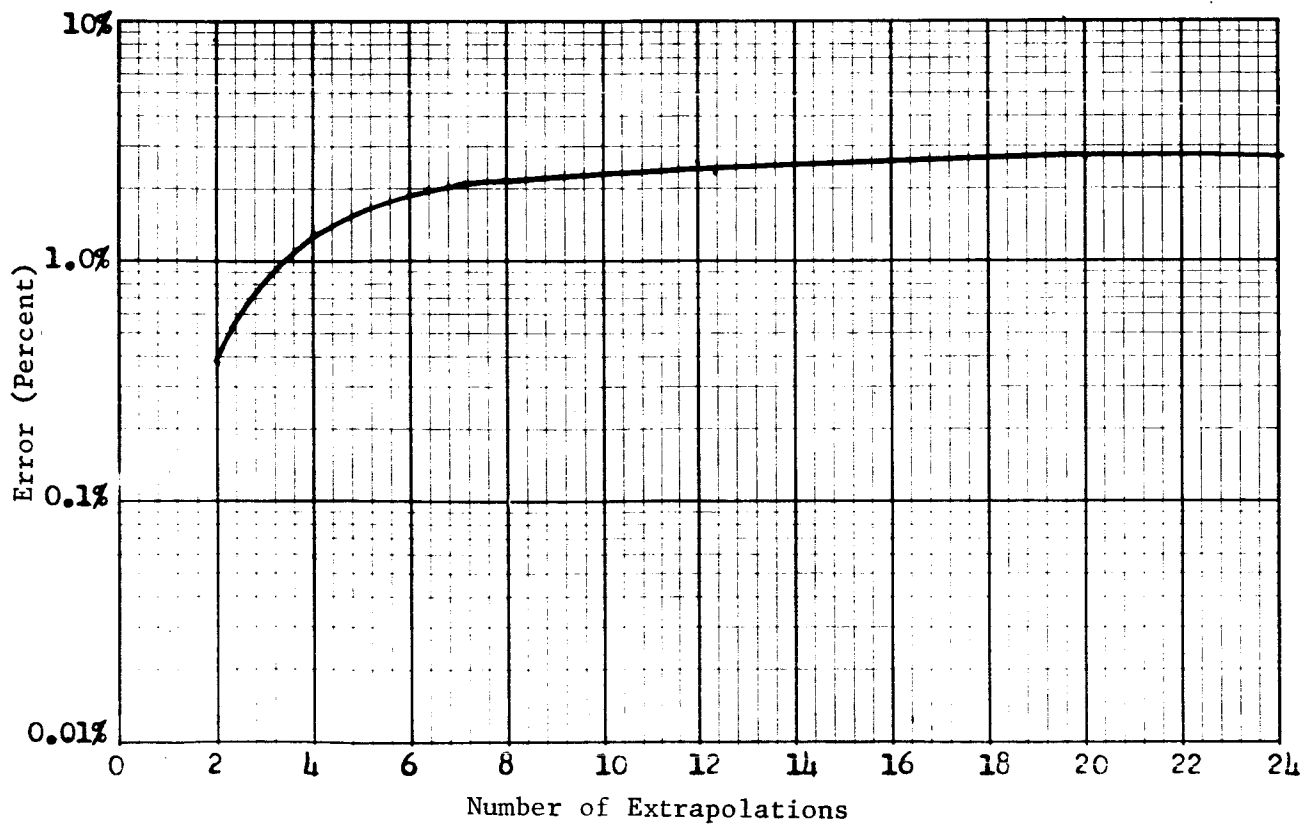
Figure 23-b. Percent Error Versus Number of Extrapolations.
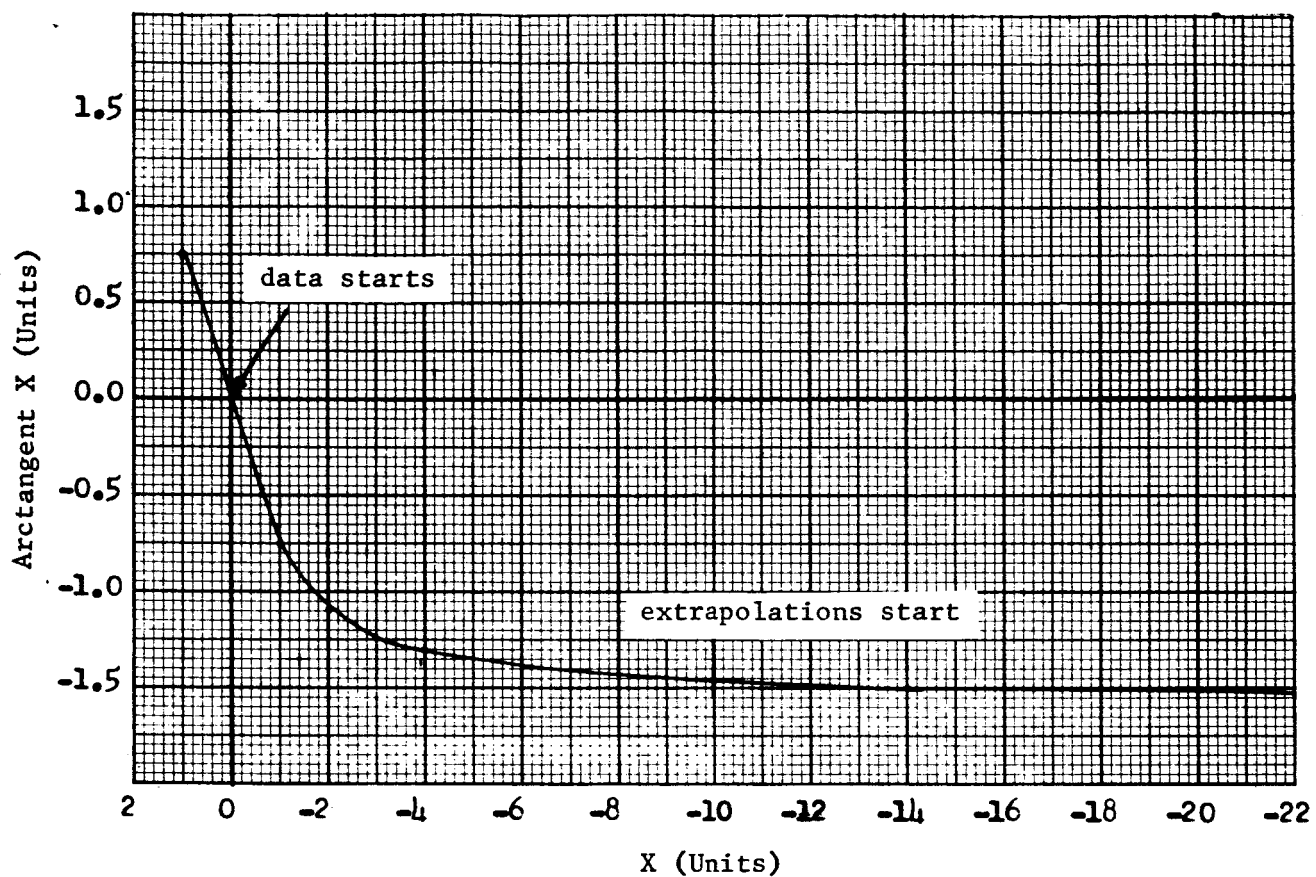
Figure 24-a. Arctangent X as a Function of X With Extrapolations from Data Increments of 1.0.
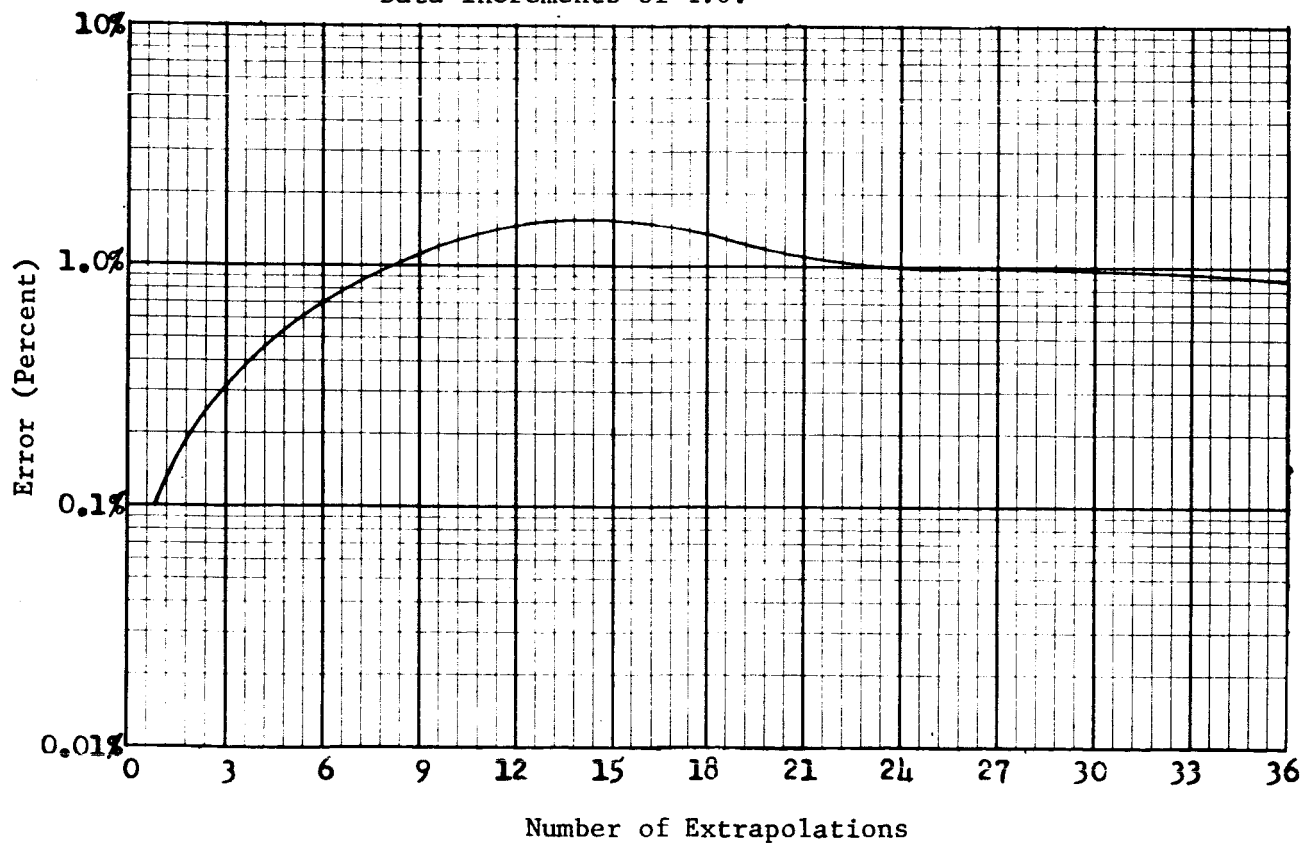
Figure 24-b. Percent Error Versus Number of Extrapolations.
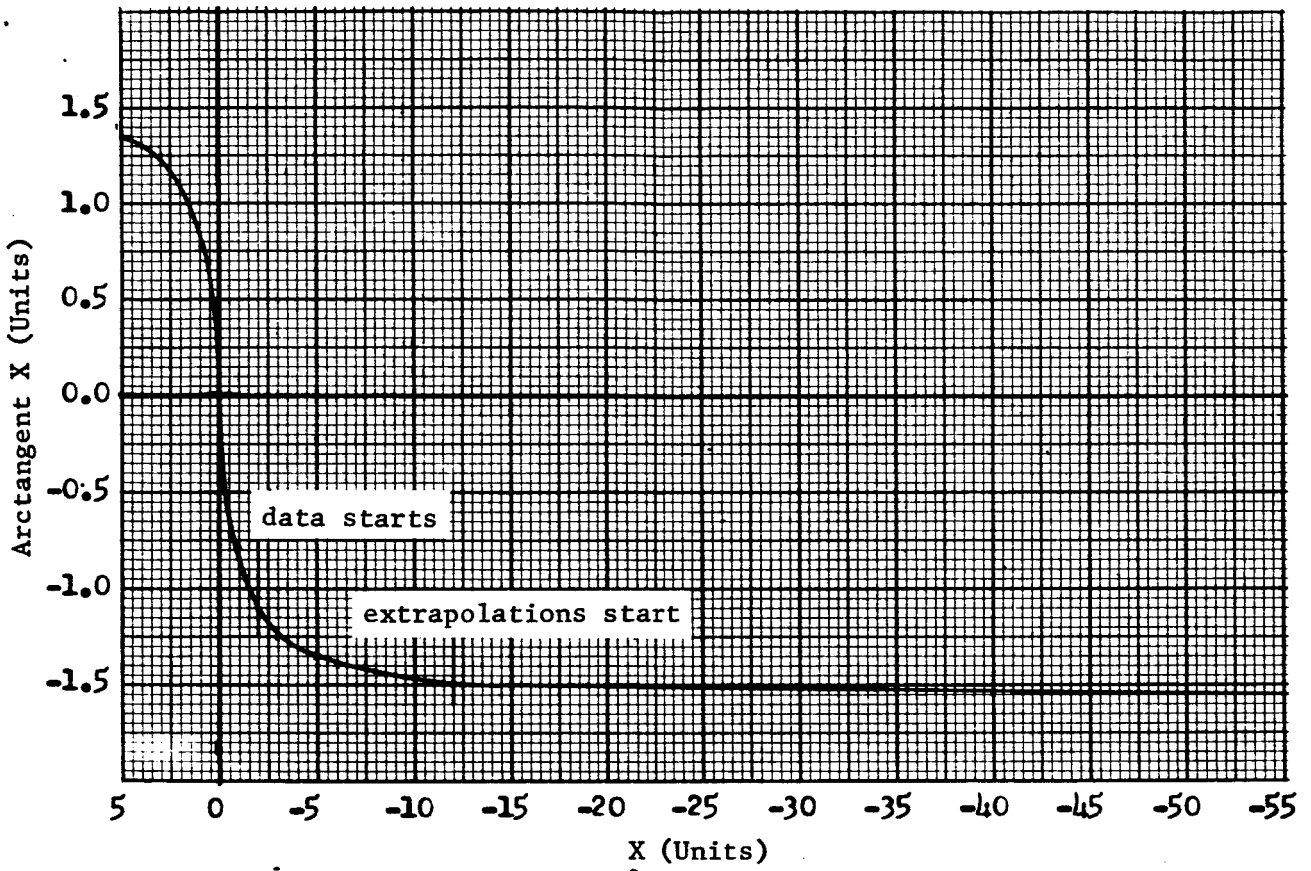
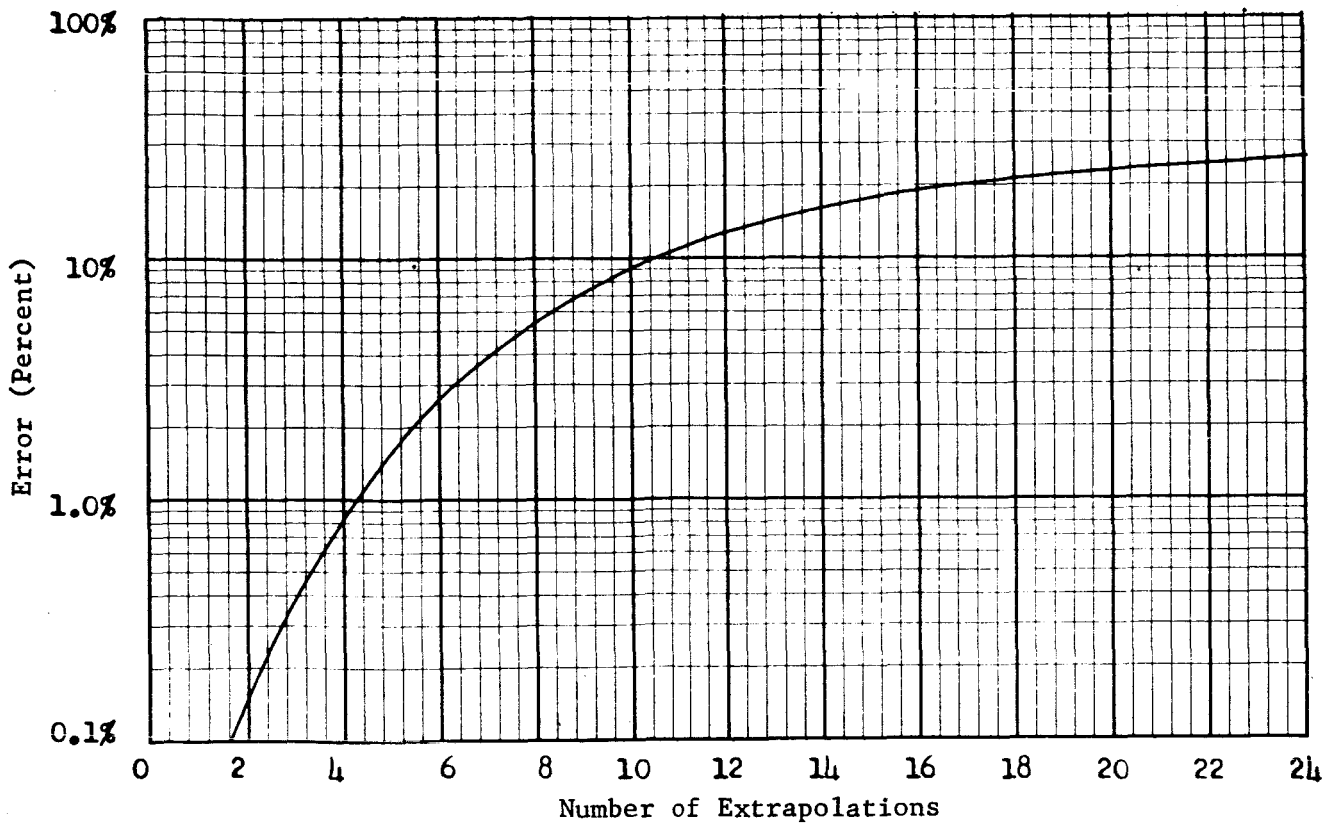Figure 25-a. Arctangent X as a Function of X With Extrapolations from Data Increments of 1.0.
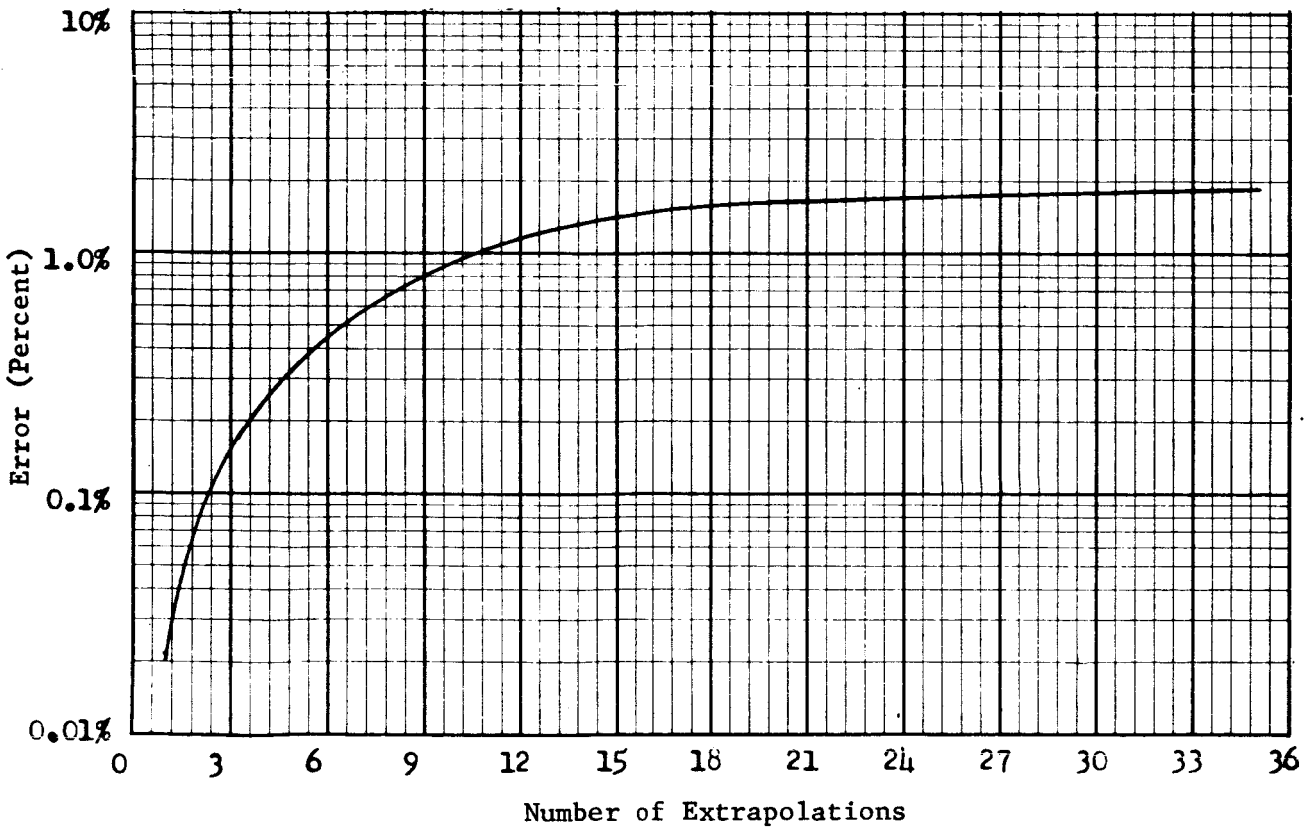
Figure 25-b. Percent Error Versus Number of Extrapolations.
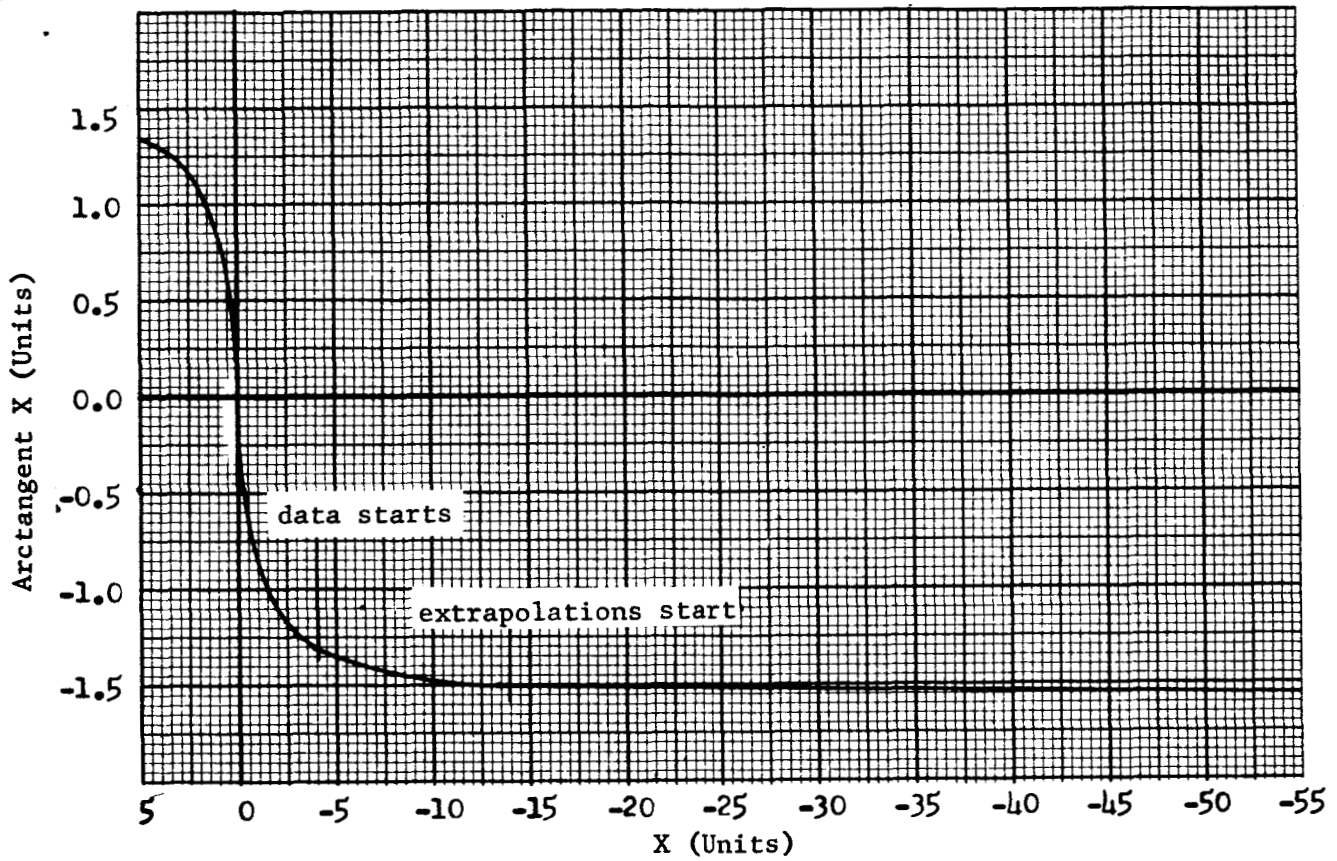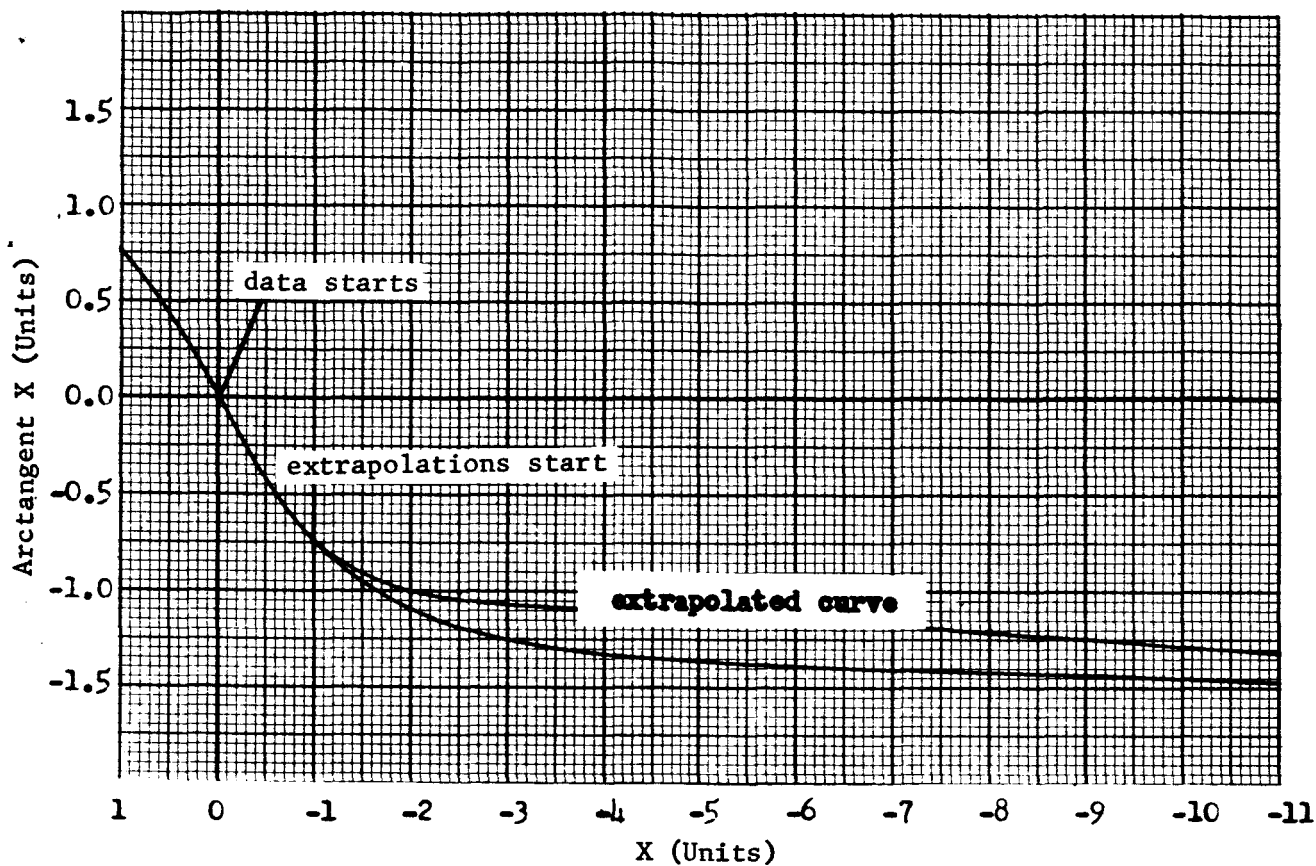
Figure 26-a. Arctangent X as a Function of X With Extrapolations from
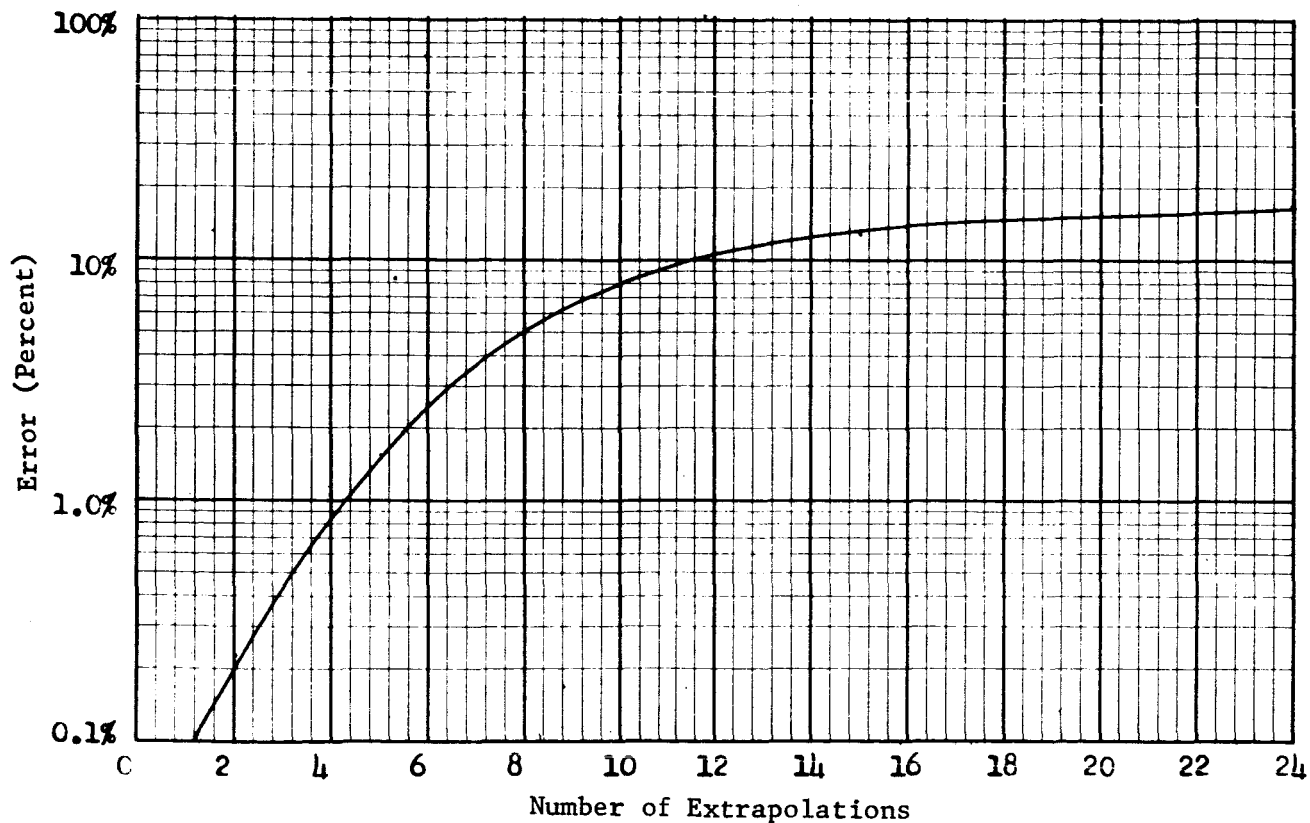Data Increments of 1.0.
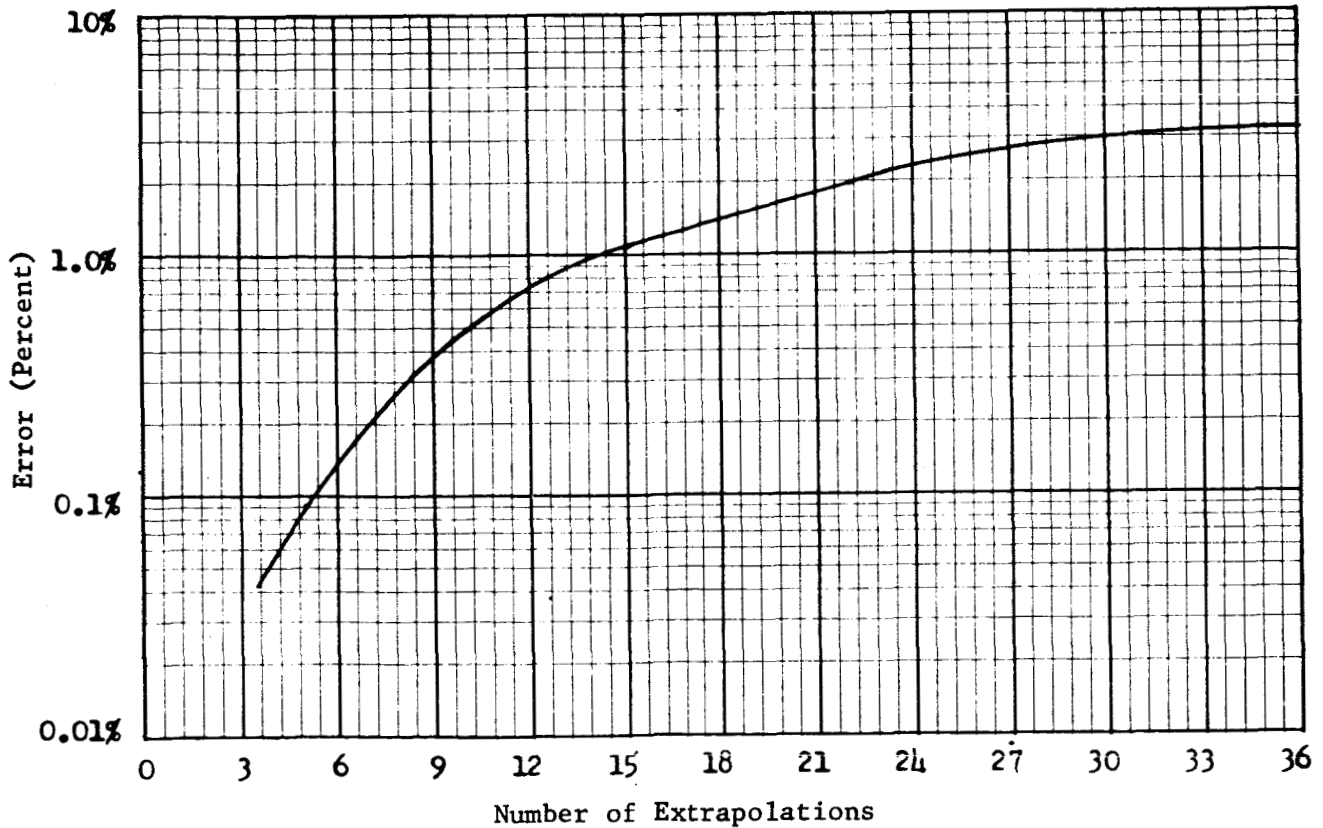
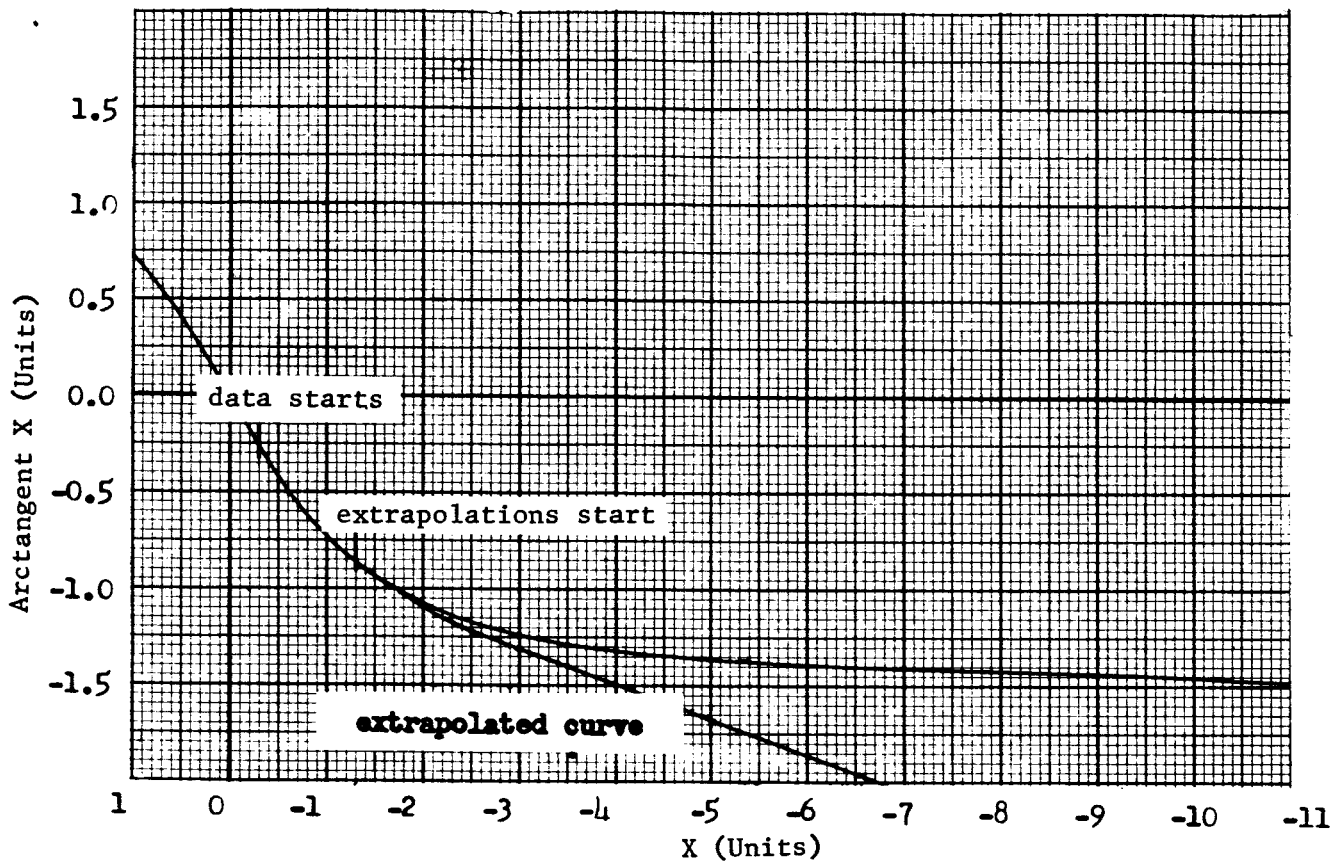Figure 26-b. Percent Error Versus Number of Extrapolations.

Figure 27-a. Arctangent X as a Function of X With Extrapolations from Data Increments of 1.0.

Figure 27-b. Percent Error Versus Number of Extrapolations.

Figure 28-a. Arctangent X as a Function of X With Extrapolations from Data Increments of 1.0.
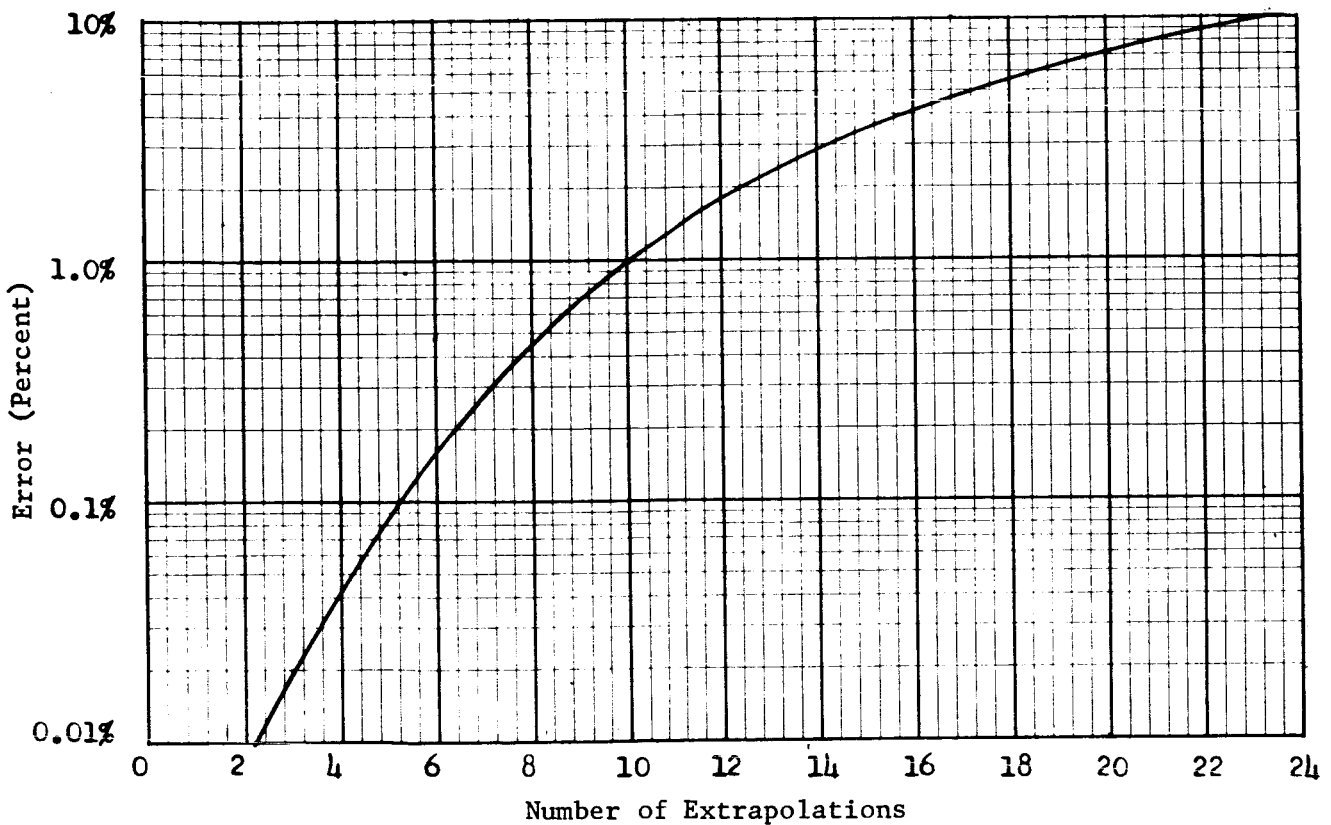

Figure 28-b. Percent Error Versus Number of Extrapolations.

Figure 29-a. Arctangent X as a Function of X With Extrapolations from
Data Increments of 1.0.

Figure 29-b. Percent Error Versus Number of Extrapolations.

data starts
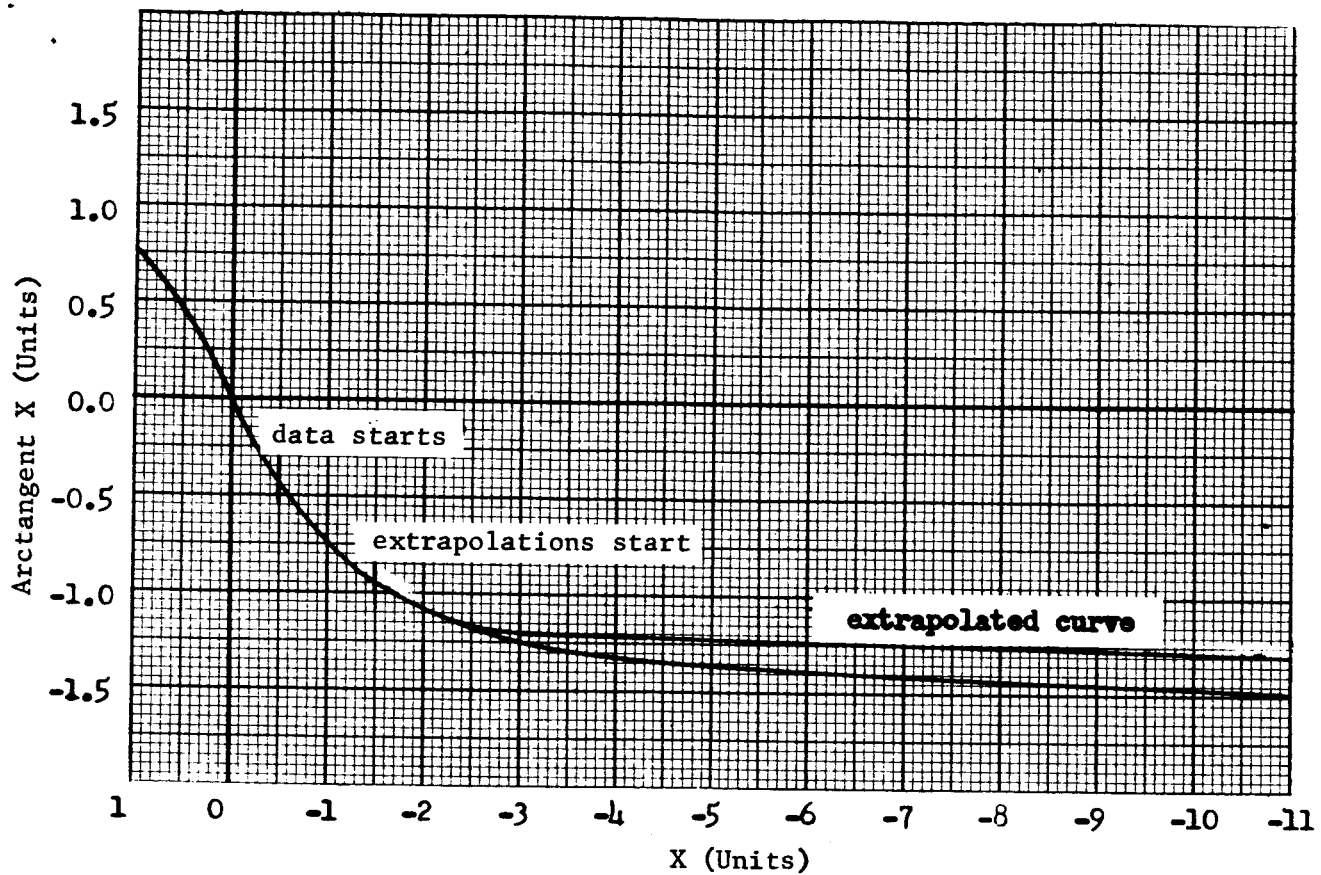
extrapolations start

extrapolated curve

Figure 30-a. Arctangent X as a Function of X With Extrapolations from
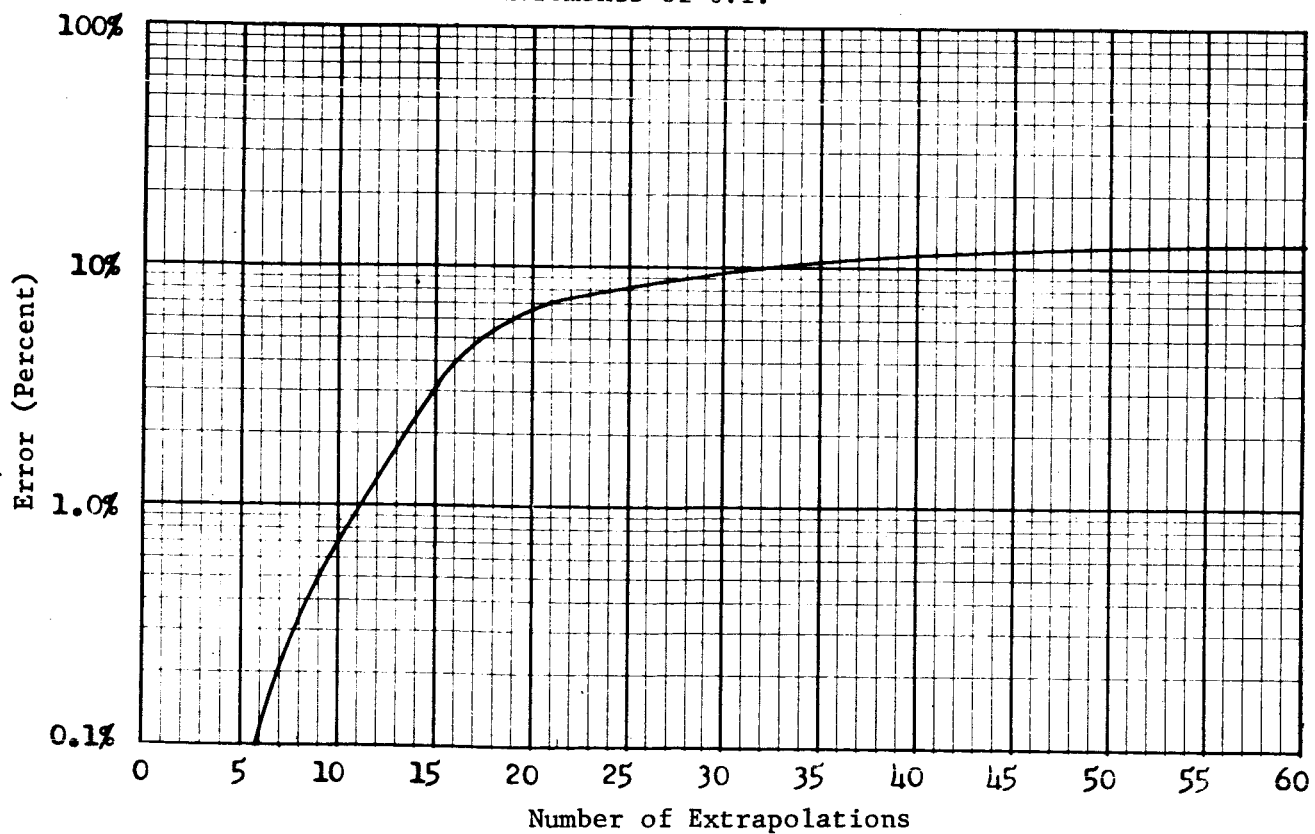Data Increments of 0.1.

Figure 30-b. Percent Error vs Number of Extrapolations.

Figure 31-a. Arctangent X as a Function of X With Extrapolations from Data Increments of 0.1.

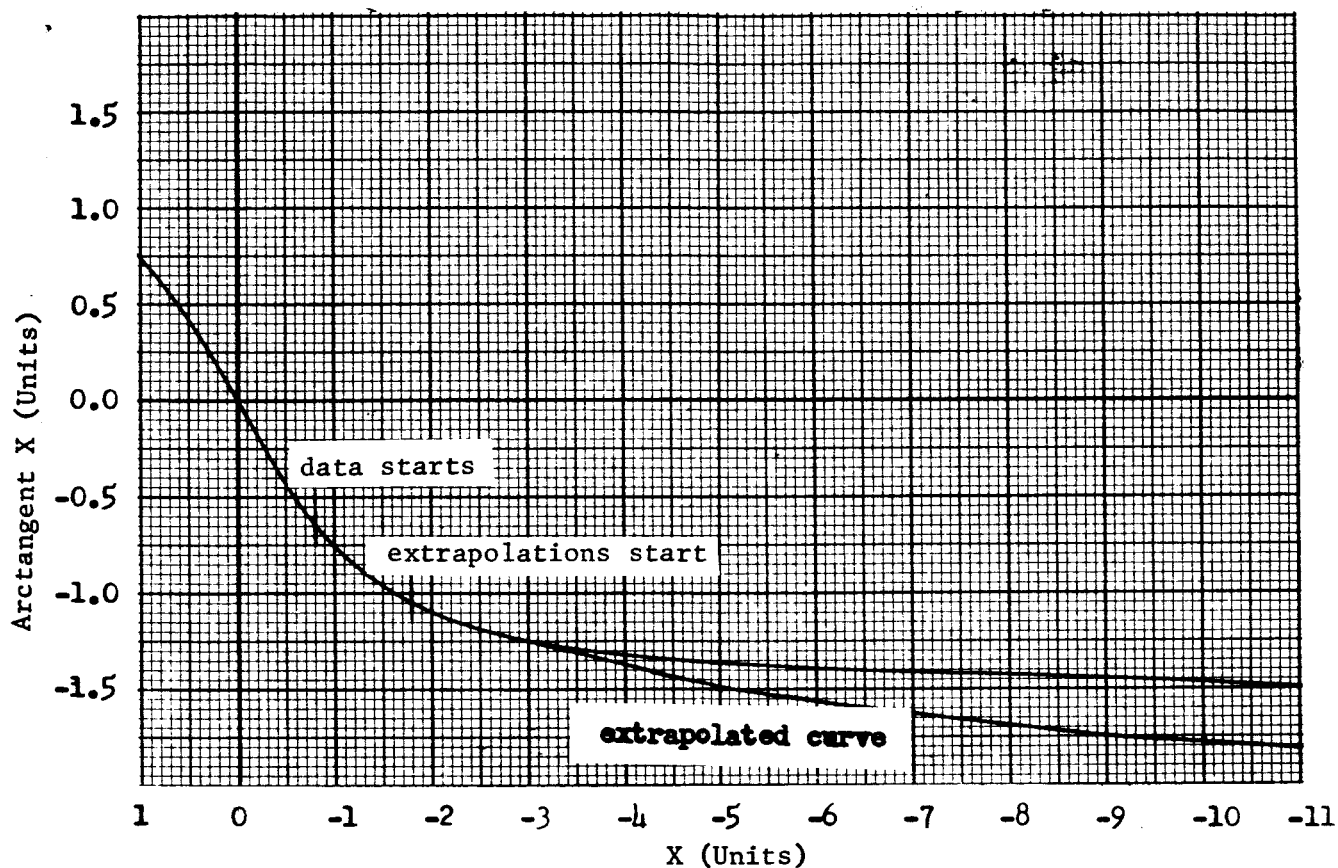Figure 31-b. Percent Error vs Number of Extrapolations.

Figure 32-a. Arctangent X as a Function of X With Extrapolations from Data Increments of 0.1.

Figure 32-b. Percent Error vs Number of Extrapolations.

Figure 33-a. Arctangent X as a Function of X With Extrapolations from
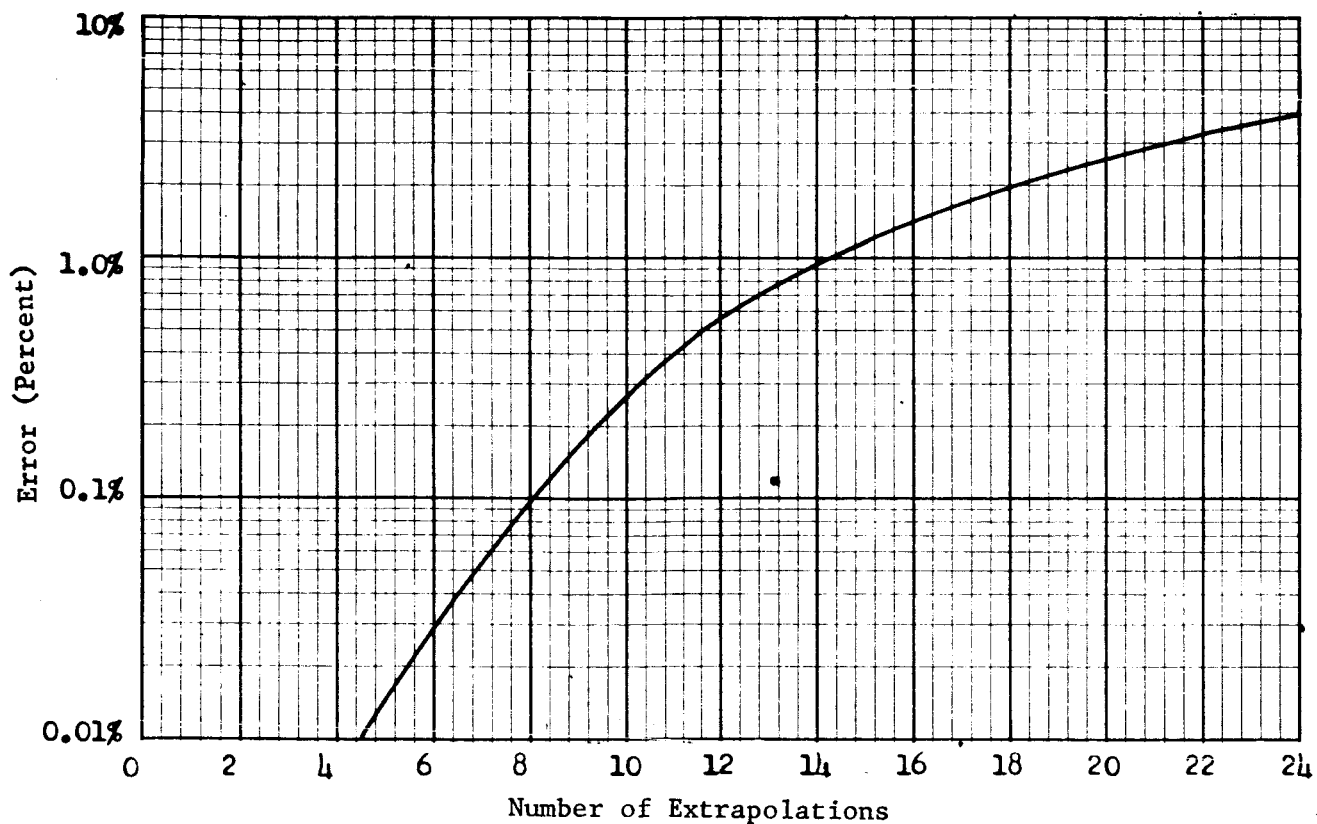Data Increments of 0.1.

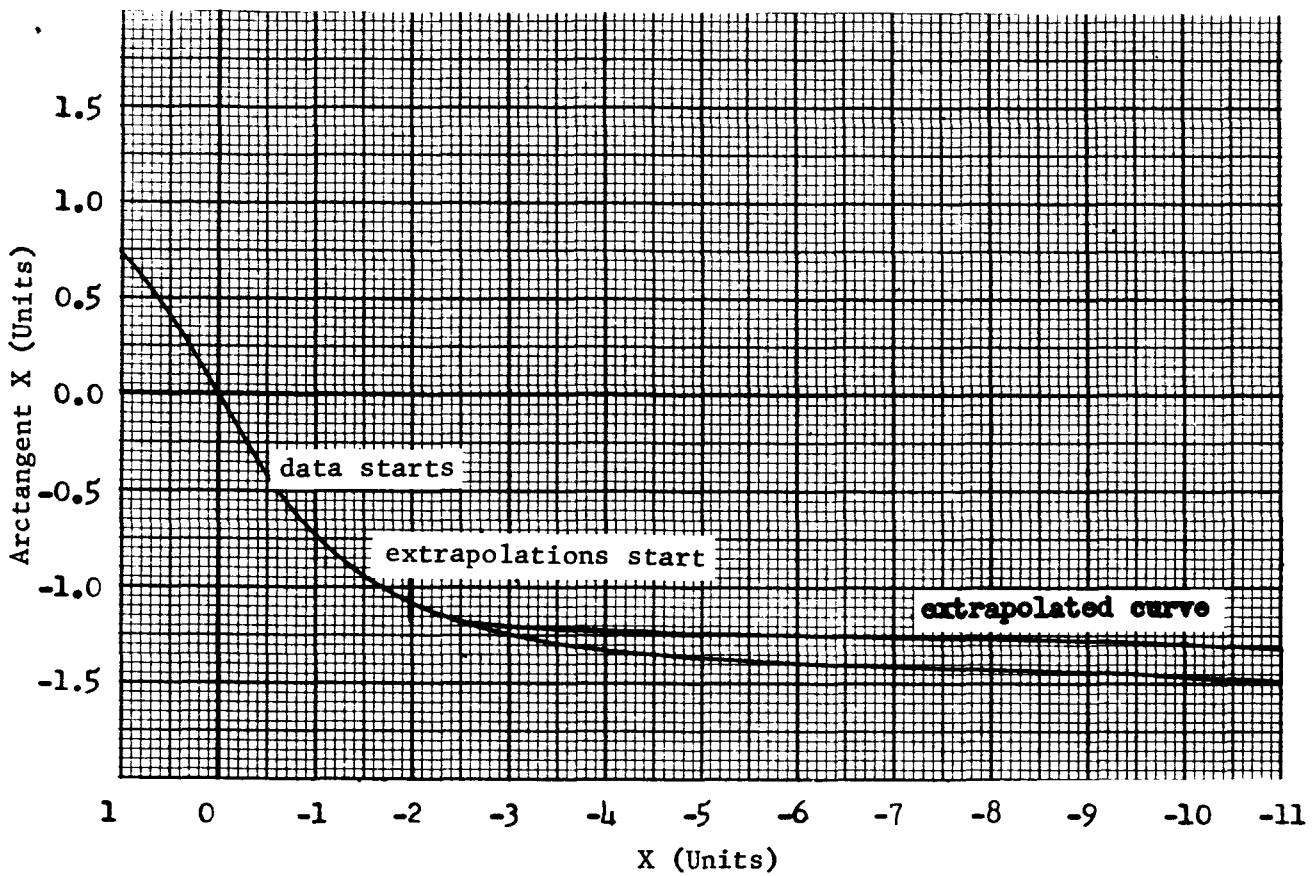Figure 33-b. Percent Error vs Number of Extrapolations.

Figure 34-a.   Arctangent X as a Function of X With Extrapolations from Data Increments of 0.1.

Figure 34-b.   Percent Error vs Number of Extrapolations.

Figure 35-a. Arctangent X as a Function of X With Extrapolations from Data Increments of 0.1.

Figure 35-b. Percent Error vs Number of Extrapolations.

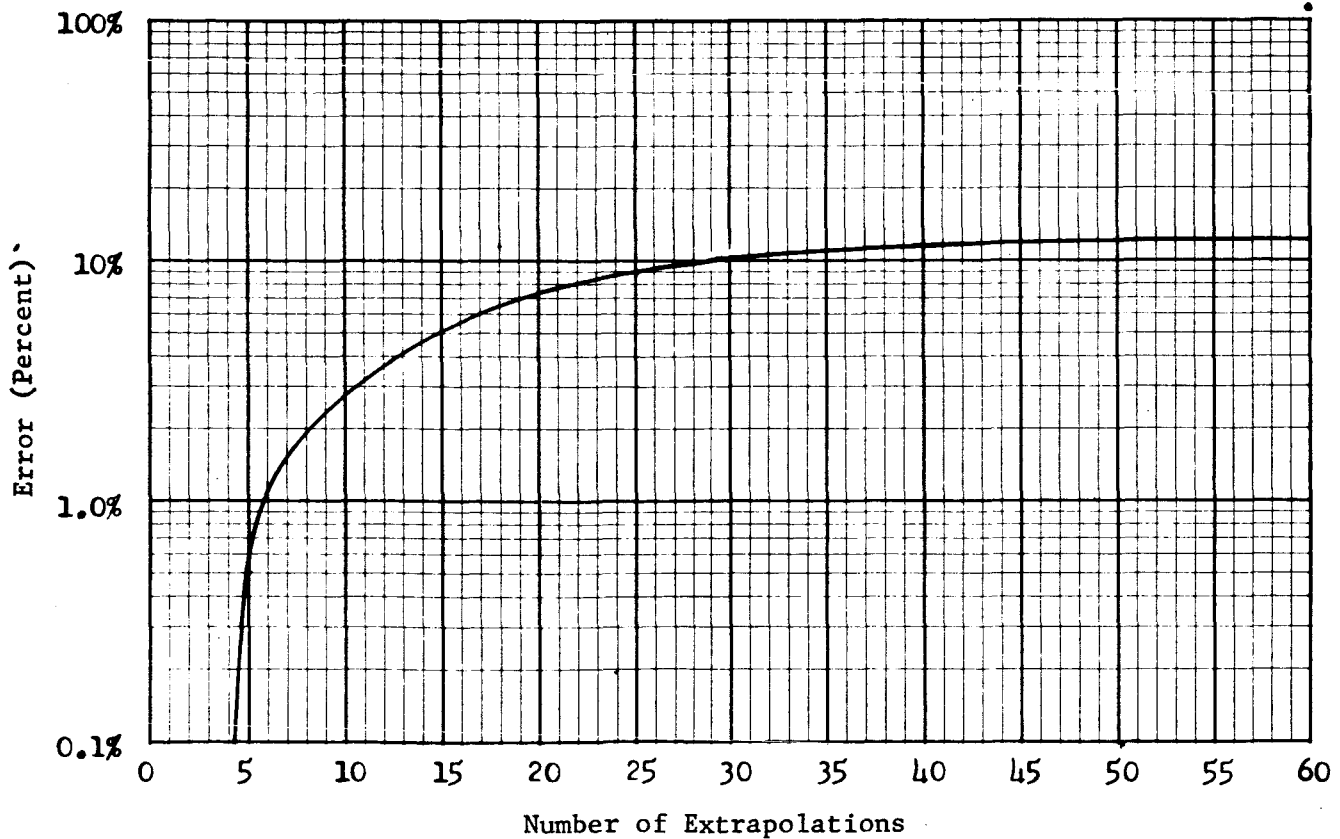Figure 36-a. Arctangent X as a Function of X With Extrapolations from Data Increments of 0.1.

Figure 36-b. Percent Error vs Number of Extrapolations.

Figure 37-a. Arctangent X as a Function of X With Extrapolations from Data Increments of 0.1.

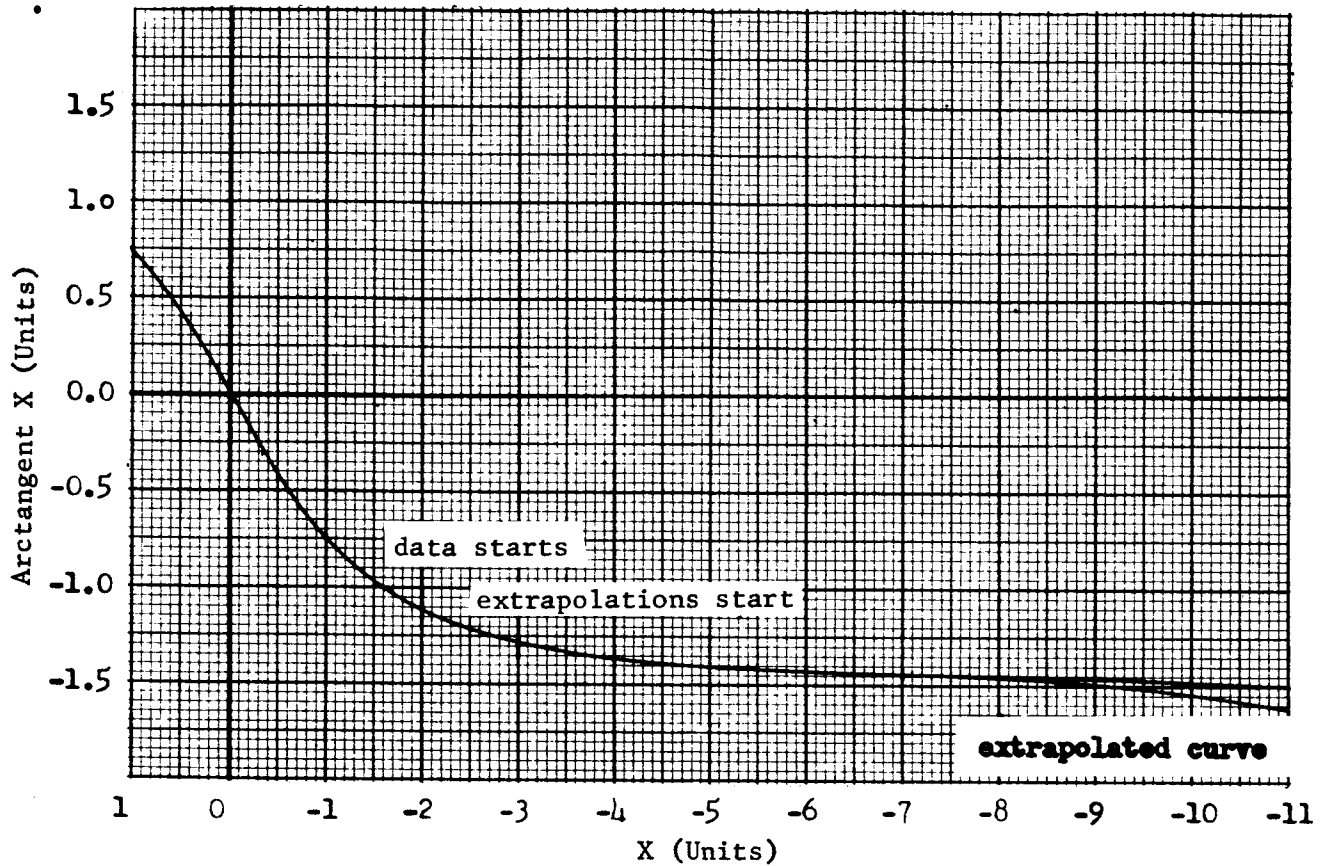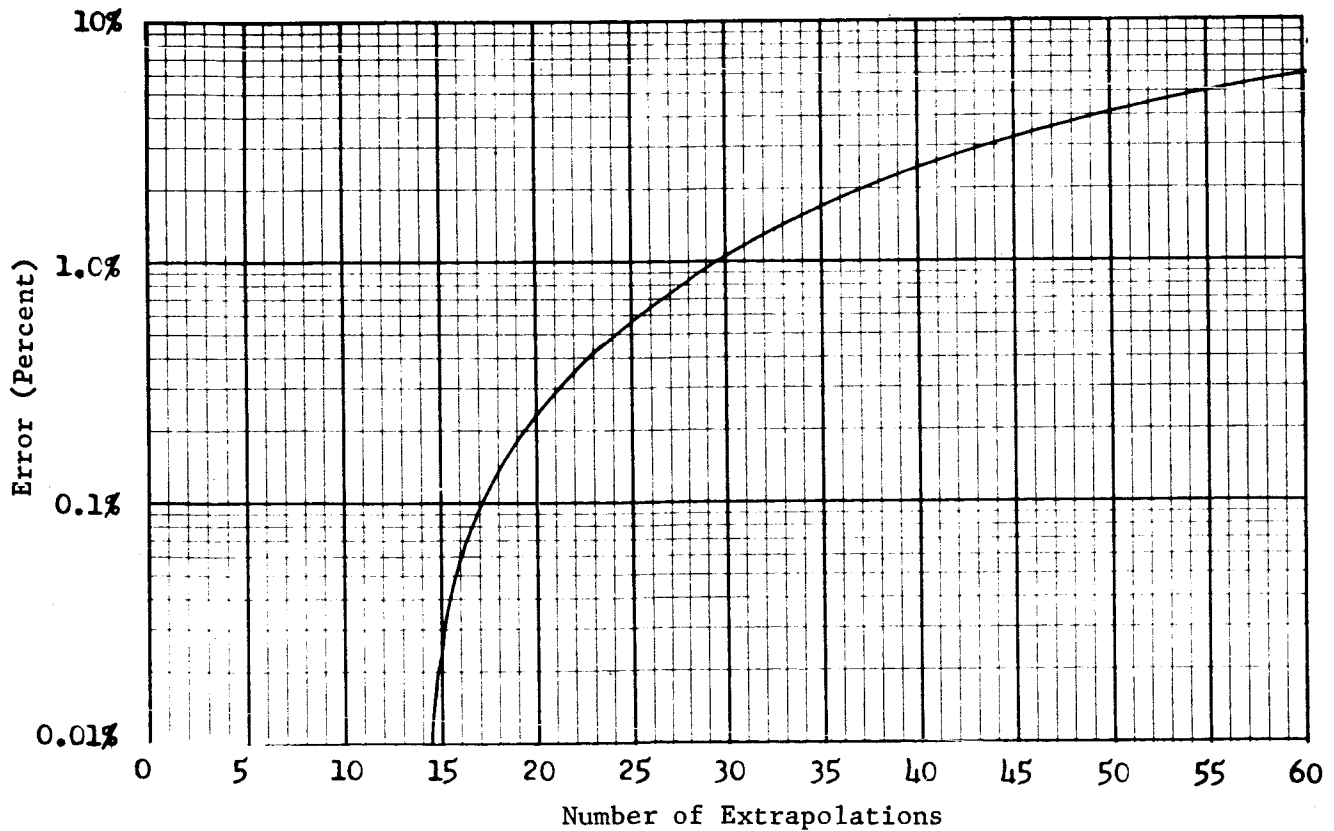Figure 37-b. Percent Error vs Number of Extrapolations.

Figure 38-a. Arctangent X as a Function of X With Extrapolations from Data Increments of 0.1.

Figure 38-b. Percent Error vs Number of Extrapolations.

Figure 39-a. Arctangent X as a Function of X With Extrapolations from Data Increments of 0.1.

Figure 39-b. Percent Error vs Number of Extrapolations.

Figure 40-a. Arctangent X as a Function of X With Extrapolations from Data Increments of 0.1.

Figure 40-b. Percent Error vs Number of Extrapolations.

Figure 41-a.   Arctangent X as a Function of X With Extrapolations from
Data Increments of 0.1.

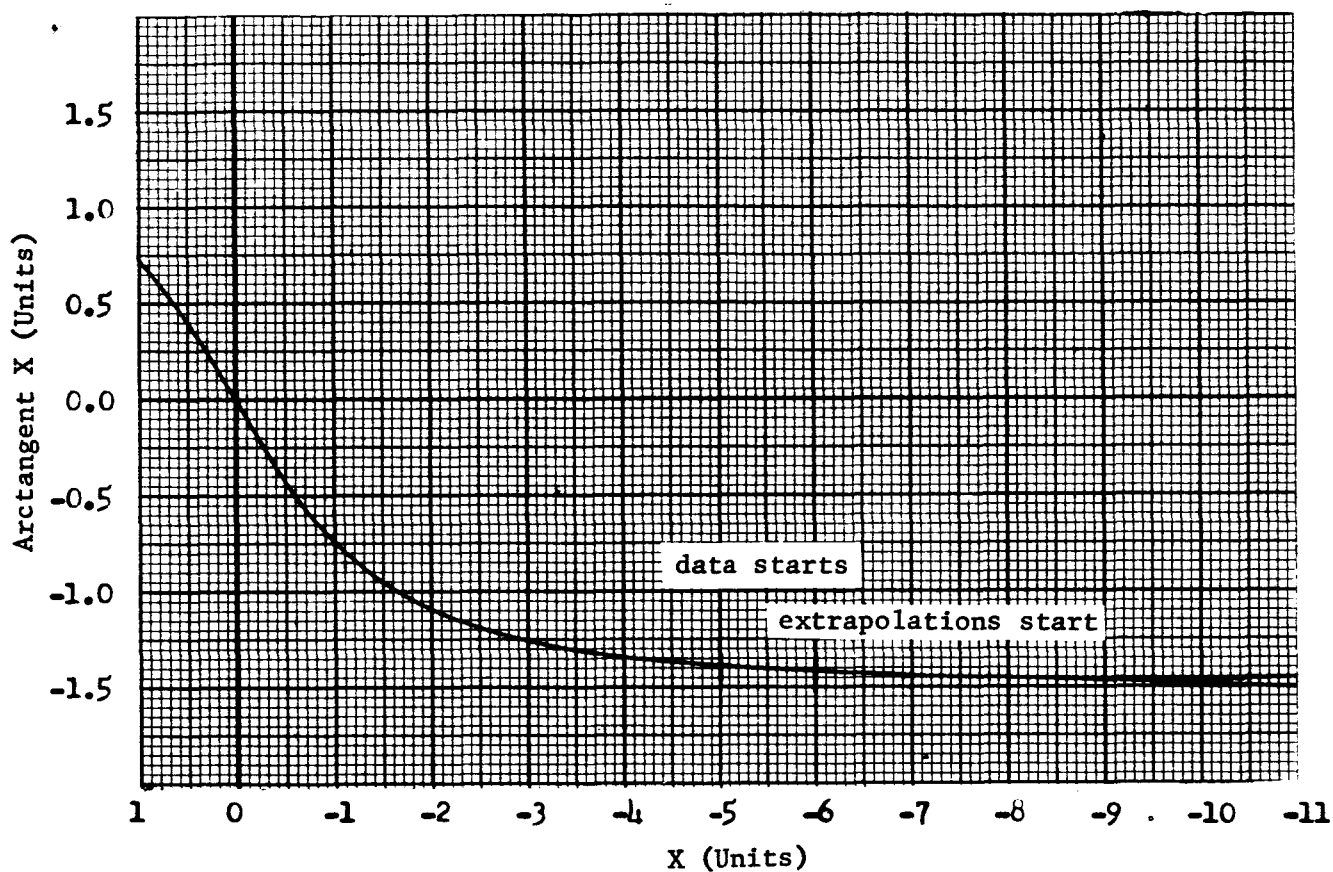
Figure 41-b.   Percent Error vs Number of Extrapolations.

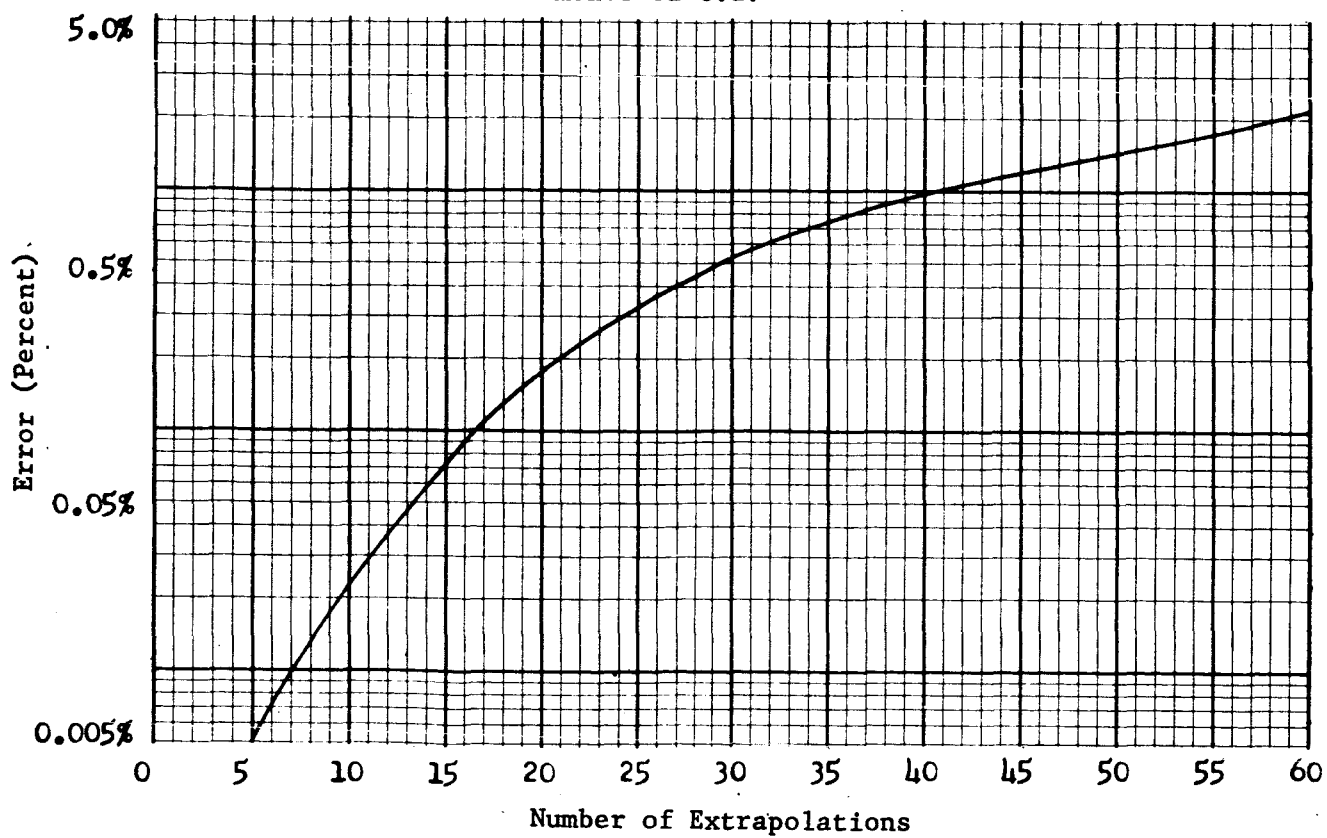Figure 42-a. Arctangent X as a Function of X With Extrapolations from Data Increments of 0.1.

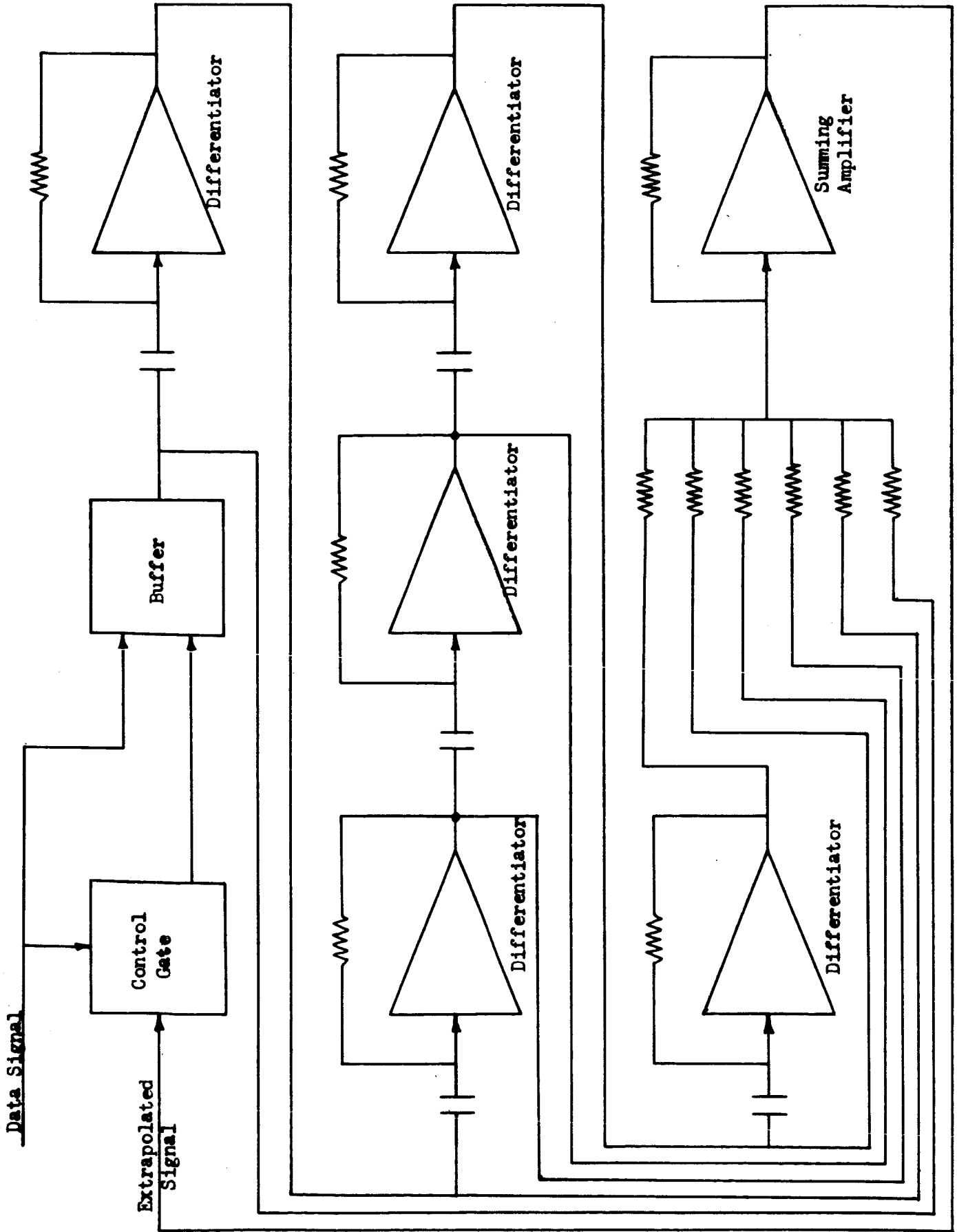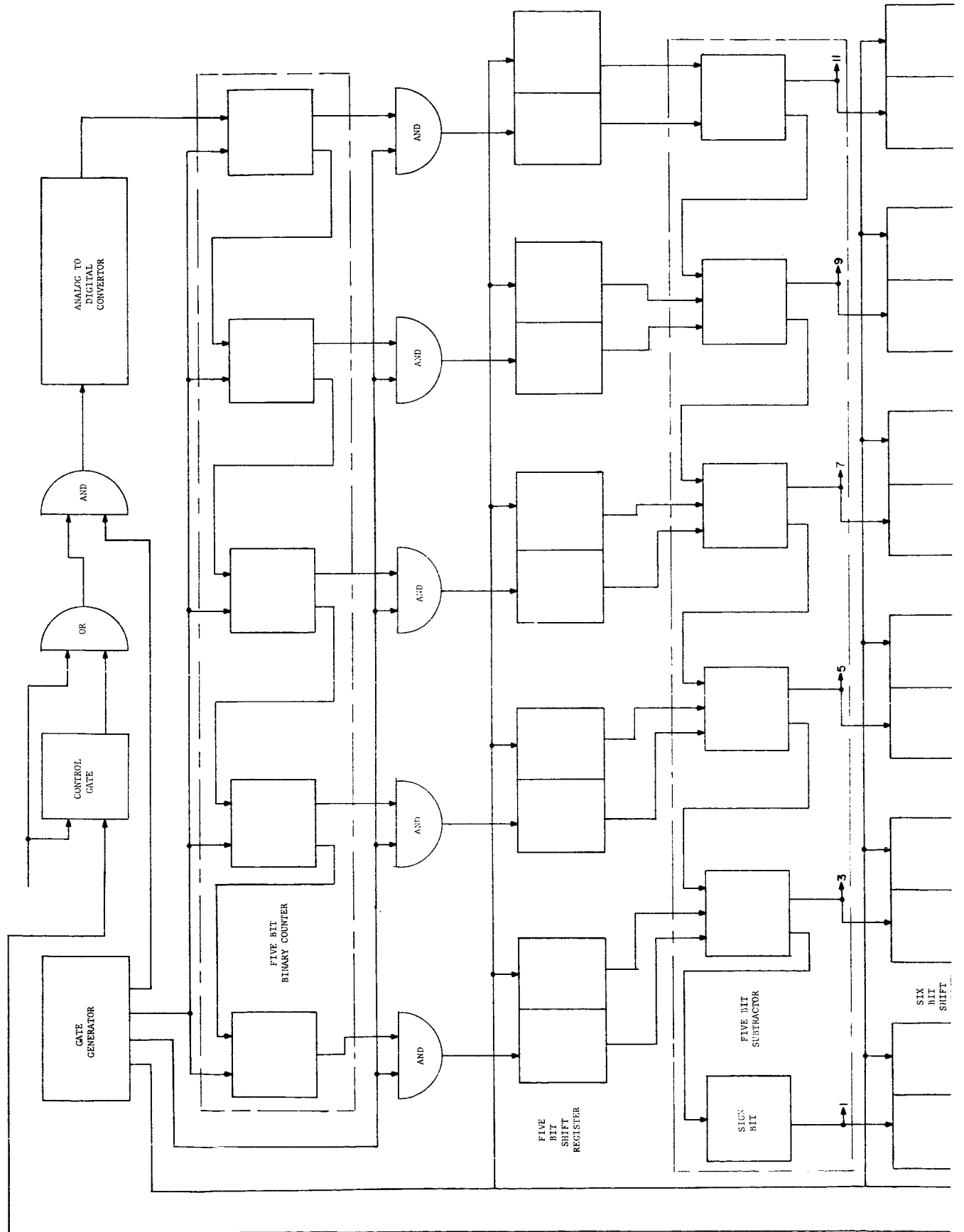Figure 42-b. Percent Error vs Number of Extrapolations.

Figure 43.   Block Diagram Of Analog System To Perform Extrapolations.

ANALOG TO DIGITAL CONVERTOR

GATE GENERATOR

CONTROL GATE

OR

AND

AND

FIVE BIT BINARY COUNTER

AND

AND

AND

AND

AND

FIVE BIT SHIFT REGISTER

FIVE BIT SUBTRACTOR

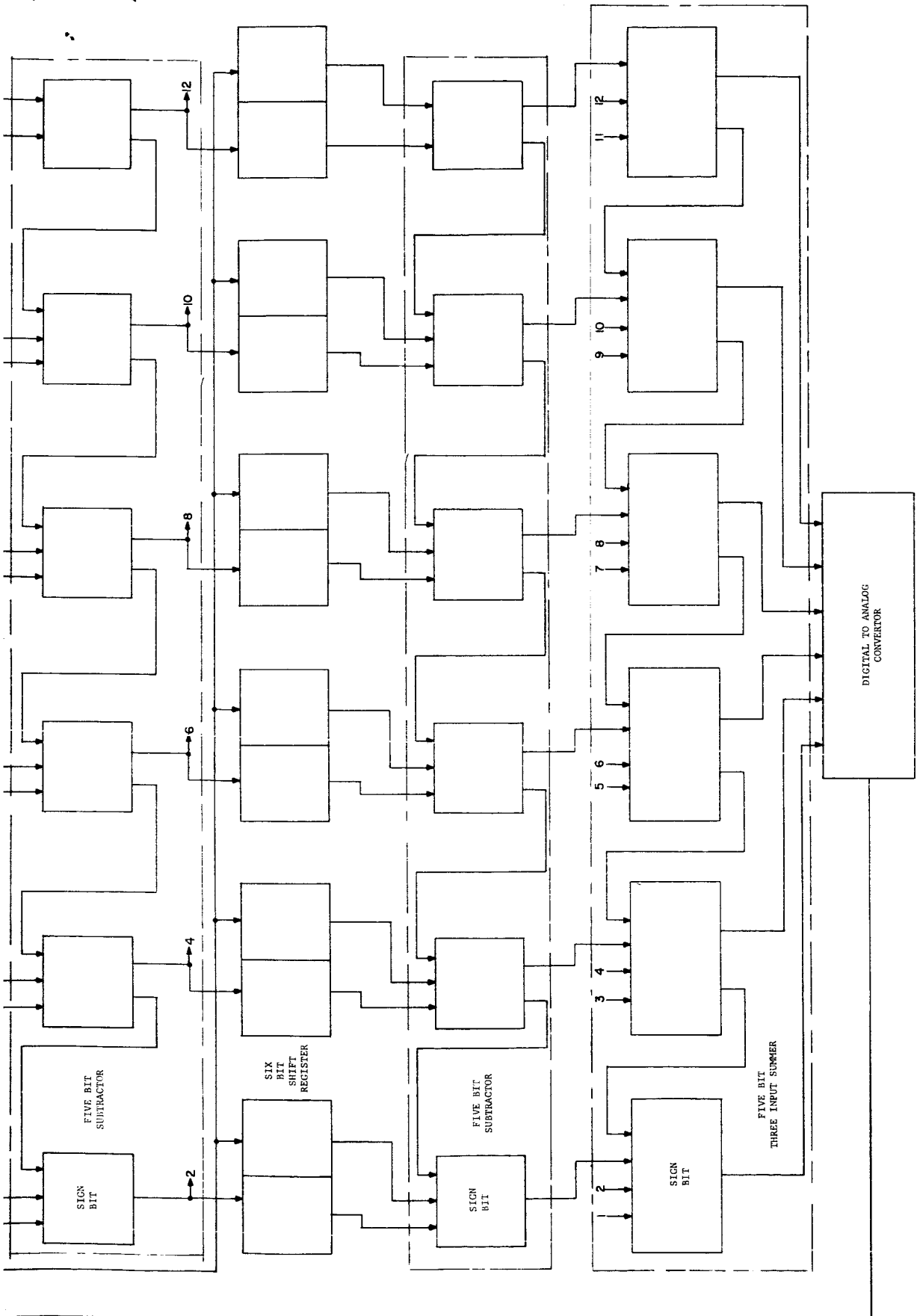SIGN BIT

SIX BIT SHIFT

11

9

7

5

3

1

89-1

FIGURE 44. BLOCK DIAGRAM OF DIGITAL SYSTEM TO PERFORM EXTRAPOLATIONS.

89.2