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a division of hughes aircraft company

RESEARCH ON GRAVITATIONAL MASS SENSORS

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I. INTRODUCTION AND SUMMARY

A. Purpose and Technical Objectives

The ultimate objective of our effort on gravitational mass sensors is the development of a small, lightweight, rugged sensor to be used on lunar orbiters to measure the mass distribution of the moon and on deep space probes to measure the mass of the asteroids. The basic concepts, the theoretical limitations, and the possible applications have been investigated and are discussed in Section II.

The purpose of the present research program is

1. To develop and refine experimental techniques for the measurement of gravitational and inertial fields, using rotating elastic systems.

2. To develop a more complete understanding of these types of sensors so that accurate predictions of sensor behavior can be made which are based on practical system configurations and measured device sensitivity.

B. Summary of Problem Areas

The major problem area can be summarized in one word — noise. This noise includes background clutter due to external forces and masses other than the one under investigation, external electrical noise and mechanical vibrations, and internal thermal and electronic noise in the sensor and amplifiers. The force of gravitational attraction is very weak, even for large masses, and every effort must be made in sensor design and operation to develop and utilize discrimination techniques that will allow the weak gravitational signal to be picked out from the background clutter and noise.

The problems of background clutter are nearly independent of the particular sensor design. It is felt that the techniques discussed in Section II-D-2, Background Rejection, will suffice for elimination of this source of noise.

The problems of externally and internally generated electrical and mechanical noise have been overcome in previous work on non-rotating gravitational sensors (see Section II-D-1), and the experience gained during this work will aid in the investigation of the very similar problems in rotating sensors. It is expected that each sensor design will have its own versions of these problems and that a major portion of the experimental work will be spent in locating and eliminating or discriminating against these sources of extraneous noise.

One minor problem area which will require special attention in the theoretical portion of the program is the instability and cross-coupling effects that are common to mechanically rotated systems. Typical examples are given in the Appendix. These problems can be avoided by proper choice of sensor configuration and sensor operation based on a thorough theoretical analysis and preliminary experimental studies of each proposed design before extensive experimental work is done.

C. Summary of General Approach

The program has started with parallel efforts consisting of detailed theoretical study and preliminary experimental work. The various possible sensor configurations are being investigated theoretically to determine their suitability as mass sensors under the assumed operating conditions. Various combinations of promising sensor designs and sensor support and drive mechanisms are being constructed and will be operated to verify qualitatively the sensor characteristics, develop signal readout techniques, and search for unexpected sources of instabilities and noise. No attempt will be made to look for gravitational interactions at this stage.

After the preliminary work, one of the sensor configurations will be chosen as the basis for a feasibility model, and a carefully designed version will be constructed. The remainder of the program will be expended in studying the feasibility model both experimentally and theoretically, locating and eliminating the sources of extraneous noise, and determining the sensitivity to gravitational fields. The program objective is a sensor that will detect the presence of a small, nearby moving mass through gravitational interactions.

D. Summary of Work to Date

The work on this program began when the completed contract was received on 26 October 1964. The original paper by Robert L. Forward on the gravitational mass sensor[1] which analyzed a two mass, one spring, radially vibrating sensor was reanalyzed by C. Bell, who found a numerical error in the original analysis. The only effect was to increase the theoretical sensitivity by a factor of 4/3. Bell then analyzed a three mass, two spring, radially vibrating sensor structure (see Appendix). This analysis indicates that radially vibrating sensor structures generally are incapable of measuring the gravitational force gradient because the sensor will fly apart at the necessary rotation speeds; therefore, this class of sensor will be limited in use to measuring the higher order gradients. A theoretical study of a transversely vibrating cruciform sensor structure, which does not have this problem, was started.

R. Morris developed a preamplifier circuit using two field effect transistors and one ordinary transistor (a total of 16 components) for boosting the strain transducer output. The circuit has a gain of 120, an input impedance of 2.5 MΩ, and an equivalent input noise of less than 0.2 μV in a 5% pass band at 190 cps.

An air bearing supported motor is being fabricated by S. D. Howe of the Research and Development Division in Culver City. Delivery is expected on 19 February 1965.

A magnetic suspension and drive of the Beams type is being fabricated by W. H. Dancy of the University of Virginia. Delivery is expected in May 1965.

A cruciform sensor head, with associated vacuum chamber designed by C. Bell, was fabricated for test purposes. The sensor head has four arms that vibrate tangentially to the direction of rotation. See Figs. 3 and 4. The arms are about 1.5 in. long, 0.75 in. wide, and 0.050 in. thick, with a 0.75 in. cube at the end for extra mass. The over-all diameter is about 4.5 in. The sensor alone has a clean resonance at 190 cps with a Q of 170 in air. The four strain transducers, one on each arm, are matched to within 1 dB and have a voltage-strain characteristic of 3.5×10^5 V/unit strain. The associated vacuum chamber, which will be rotated with the sensor head, is 5.5 in. in diameter and 1.5 in. thick. This is rotated in an external frame and supported by mechanical ball bearings.

Readout from the rotating sensor chamber is accomplished with standard slip rings. R. Morris has designed an alternative method of readout which involves inductive coupling between a static coil and a rotating coil containing a ferrite rod.

With the cruciform sensor head installed in its vacuum chamber and the chamber suspended from a wire, the Q of the sensor was found to be about 300; three resonances separated by about two cycles were found, however. These disappeared when the chamber was firmly clamped in a vise. The effect results because the sensor is coupled to the chamber through the fairly stiff supporting rod. Studies of the various ways to eliminate this problem are under way.

The sensor and vacuum chamber were then placed in the frame and the sensor rotated up to speed (6000 rpm) to check its gross characteristics in a rotating system, and to study readout techniques. The slip rings are working quite well at these speeds, and the only effect of the rotation on the sensor seems to be to shift the resonance frequency from 190 to 200 cps. As expected, the vibrational noise due to the mechanical bearings was too large (20 mV) to allow us to observe any gravitational interactions with this unit.

II. THEORY OF GRAVITATIONAL GRADIENT SENSORS

A. Introduction

In order to measure the mass of an object at a distance,[1-6] when both the object and the sensor are in free fall, and in order to determine the attitude of a spacecraft in orbit around the earth without using external referents,[7-18] it is necessary to develop force measuring instruments that will allow us to distinguish between the effects of gravitational forces and inertial reaction forces. At first glance it might be assumed that Einstein's principle of equivalence might preclude such a differentiation, since basically it states that there is no way to distinguish between a gravitational field and an accelerated reference frame. However, the principle of equivalence is valid only for uniform gravitational fields or an infinitesimal region of the reference frame.[19] The principle of equivalence can be applied over a larger region only when the gravitational field is uniform over that region. In reality, however, the gravitational field of a mass is far from uniform. Real gravitational fields have gradients in their vector field patterns. Thus, gradient sensors such as differential accelerometers can distinguish between real gravitational fields and inertial effects due to accelerated reference frames.

For almost all real situations, however, the problem is not one of a fundamental nature; rather, it is a practical one of measuring the very weak gravitational force field in the presence of the much larger inertial force fields. In order to do this, it is necessary to develop a class of sensors capable of using the differences between gravitational and inertial forces in such a way that they ignore the large inertial fields and respond only to the gravitational field.

A detailed analysis of this problem has been carried out, and is presented as an Attachment to this report. The separation of the gravitational effects from the inertial effects is accomplished by using the physical fact that the various force fields differ in their gradient or tensor characteristics and the mathematical fact that a tensor of n^{th} rank, when examined in the rotating reference frame of a sensor, will be found to produce time-varying signals that are at n times the rotational frequency of the sensor. Because of the detailed analysis in the Attachment only a short summary of the theory will be presented here.

B. Characteristics of Gravitational and Inertial Fields

1. Inertial Fields

When a nongravitational force F acts on a vehicle with mass m, it causes a linear acceleration. The linear acceleration of the vehicle creates in the frame of reference of the vehicle a uniform inertial field which has purely vector properties and no spatial gradients:

$$a_f = \frac{F}{m} \quad . \tag{1}$$

If the vehicle using the sensor is rotating at a rate Ω, the rotation sets up a cylindrically symmetric inertial field which increases with increasing distance r from the axis of rotation.

$$a_r = \Omega^2 r \quad . \tag{2}$$

This field has a uniform gradient in the plane of rotation

$$\Gamma_r \approx \Omega^2 \tag{3}$$

but no higher order gradients.

2. Gravitational Fields

When a gravitational field of a mass M at a distance R acts on a vehicle, it sets up a spherically symmetric acceleration field

$$a_g \approx - \frac{GM}{R^2} \quad , \tag{4}$$

which has not only a first order gradient

$$\Gamma_g \approx - \frac{GM}{R^3} \tag{5}$$

but also an unlimited number of higher order gradients

$$T_{ab \ldots n} \approx - \frac{GM}{R^n} \quad . \tag{6}$$

3. Differentiation of Gravitational and Inertial Fields

As is shown in the previous sections and in the Attachment, gravitational and inertial effects have different tensor characteristics. The inertial field created by acceleration is a uniform vector field and has no gradients, while the inertial field created by rotation has a uniform cylindrically symmetric tensor gradient but none of higher order. The gravitational field created by a mass is highly nonuniform, with essentially no limit to the number of higher order gradients. These differences make it theoretically possible to measure independently gravitation, rotation, and acceleration effects; to do so, some form of differential force sensor with tensor response characteristics must be used.

The differential force sensors discussed in the literature[7-18] usually consist of spaced pairs of low level accelerometers with opposed outputs. However, a very good accelerometer is capable of a linearity of only one part in 10^5, and the outputs of two accelerometers cannot be matched to even this degree of accuracy. Thus, it has not been possible to make differential force sensors whose outputs could be combined to cancel out the acceleration terms in order to obtain the rotation and gravitation terms.

The most promising technique is a dynamic one.[1-6, 13, 17] By rotation of specially designed differential force sensors, the static spatial variations of the fields can be transformed into temporal variations in the sensor. Because of the rotational properties of tensors, the various inertial and gravitational effects come out at different frequencies.[3, 4]

The basic concept is that forces are vectors (tensors of first rank), the gradients of forces are tensors of second rank, and higher order gradients are higher rank tensors. In general, the components of a tensor of n^{th} rank, when examined in the rotating reference frame of a sensor, will be found to have time-varying coefficients that are at n times the rotational frequency of the sensor.[3]

C. Rotating Gravitational Gradient Sensors

As was pointed out in the previous section, the gravitational field of a mass creates a pattern of tensions and compressions in space which differs from that of the inertial fields due to rotations or external forces. We propose to develop sensors that transform these different spatial patterns into different frequency components.

The basic gravitational sensor configuration being studied at Hughes[1-6] consists of a mass-spring system with one or more vibrational modes. The system is rotated at some subharmonic of the vibrational mode. If a nonuniform gravitational field is present, the differential forces on the sensor resulting from the gradients of the gravitational field will excite the vibrational modes of the sensor structure. In the schematic of Fig. 1, the gradient of the gravitational field excites vibrations at twice the rotation frequency of the sensor. Similar devices have been proposed by Diesel,[13, 17] Kalmus,[20] and Fitzgerald,[21, 22] although only the device proposed by Diesel was designed to measure the gradient of the force rather than the gravitational force itself. The other proposed devices[20-22] are similar in construction to the Hughes sensors, but in operation are more similar to the weak spring type of static accelerometer or gravimeter. In this class of device, the system is rotated at its "critical speed" so that the restoring forces of the spring are exactly counterbalanced by centrifugal forces. The system is then in unstable equilibrium and a small gravitationally induced force or any other force will cause a large displacement. The Hughes sensors are operated below the "critical speed." The difference between the two types of devices is illustrated by the discussions of Den Hartog.[23]

The basic concept behind the operation of these sensors is an old one in the field of electronics, the concept of chopping. This is used extensively in dc amplifiers, where the low level dc signal is chopped, transformed into an ac signal, and then amplified and measured by phase sensitive detectors. In the gravitational sensors, the chopping of the static gravitational field is accomplished by physically rotating the sensor so that its response to the gravitational field varies with time.

The conversion of a static gravitational interaction into a dynamic gravitational interaction occurs because the rotation of the sensor creates a rotating reference system. From the viewpoint of the sensor, the mass to be measured is somehow whirling around the sensor, attracting it first one way and then the other.

If the sensor has resonant electromechanical modes, these time varying forces will build up vibrations in the modes. The resulting vibrations are no different from those set up by any other force. The theoretical and experimental methods of studying and measuring the vibrations are well known in the field of mechanics.

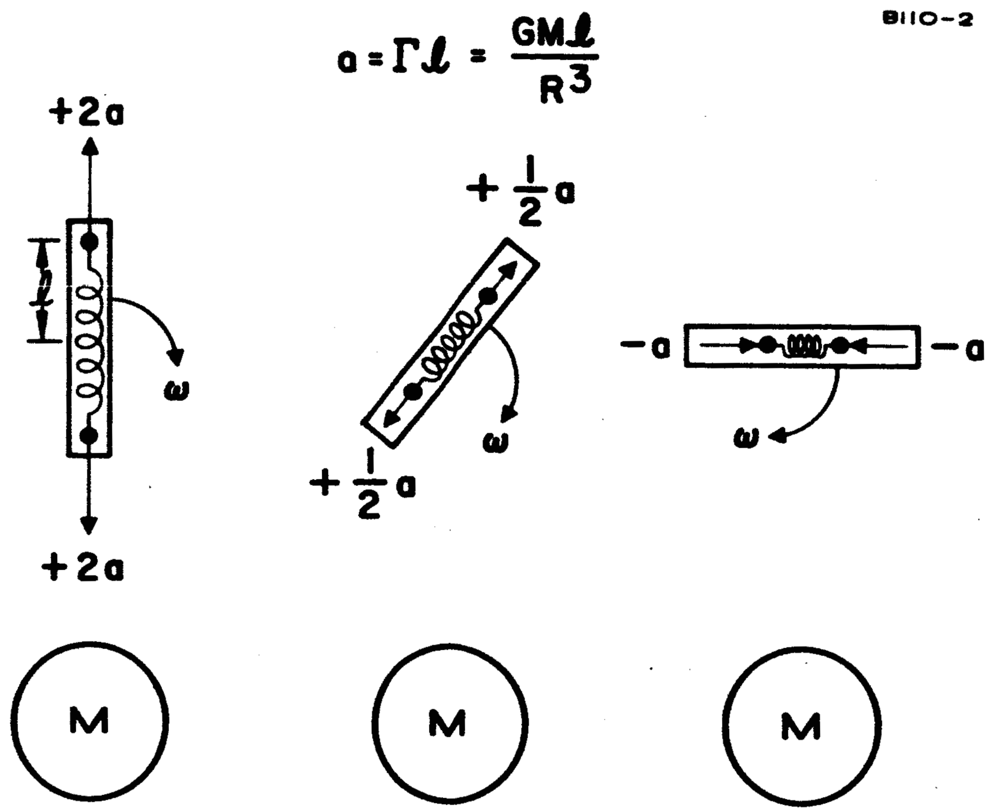


Fig. 1. Response of rotating gradient sensor to gravity gradients.

The sensor configurations that have been identified fall into four general classes (see Fig. 2). One has its effective length along the rotational axis and vibrates perpendicular to the rotational axis (Fig. 2(a)) and three have their effective length rotated in the plane at right angles to the rotational axis, with their vibrations either radially in the rotational plane (Fig. 2(b)), tangentially in the rotational plane (Fig. 2(c)), or at a right angle to the rotational plane (Fig. 2(d)).

The case of Fig. 2(a) is common in the field of mechanics where it is known that a problem of gravitationally driven vibration exists when a drive shaft is rotated at one-half its natural vibrational frequency.[24] For our application, these gravitationally induced vibrations are not a problem, but the desired result. The case of Fig. 2(a) is sensitive to torsional gradients effective along the rotational axis.

The case of Fig. 2(b) has the characteristic that it should measure the difference between the radial gradients in the plane of rotation. A careful analysis (see Appendix) showed, however, that because the same spring is used to generate the centripetal force and the vibrational restoring force, if we attempt to design the sensor to detect the gravitational force gradient of an object, the spring will not have enough strength to resist the centrifugal force. The sensor can be used to measure the higher order gradients, however.

The case of Fig. 2(c) has been analyzed by V. Chobotov.[25] Here again, the gravitationally induced vibrations were considered a problem. This type of sensor measures the difference between the torsional gradients in the plane of rotation.

The case of Fig. 2(d) has the characteristics that it will measure the difference between the torsional gradients perpendicular to the plane of rotation. The coupling of the centrifugal force to the vibrational system is much weaker in this case since a vibrational response does not change the angular momentum about the rotational axis.

The most promising sensor configuration is that shown in Fig. 3. It is a variation of type 2(c) and consists of four masses on the end of four transversely vibrating arms. The gradient of the gravitational field causes differential torques on the arms. As the sensor rotates, the direction of the applied differential torque varies at a frequency which is twice the rotation frequency of the sensor, and a phase which is related to the direction to the exciting mass (see Fig. 4).

Further investigations of this and other sensor configurations will continue.

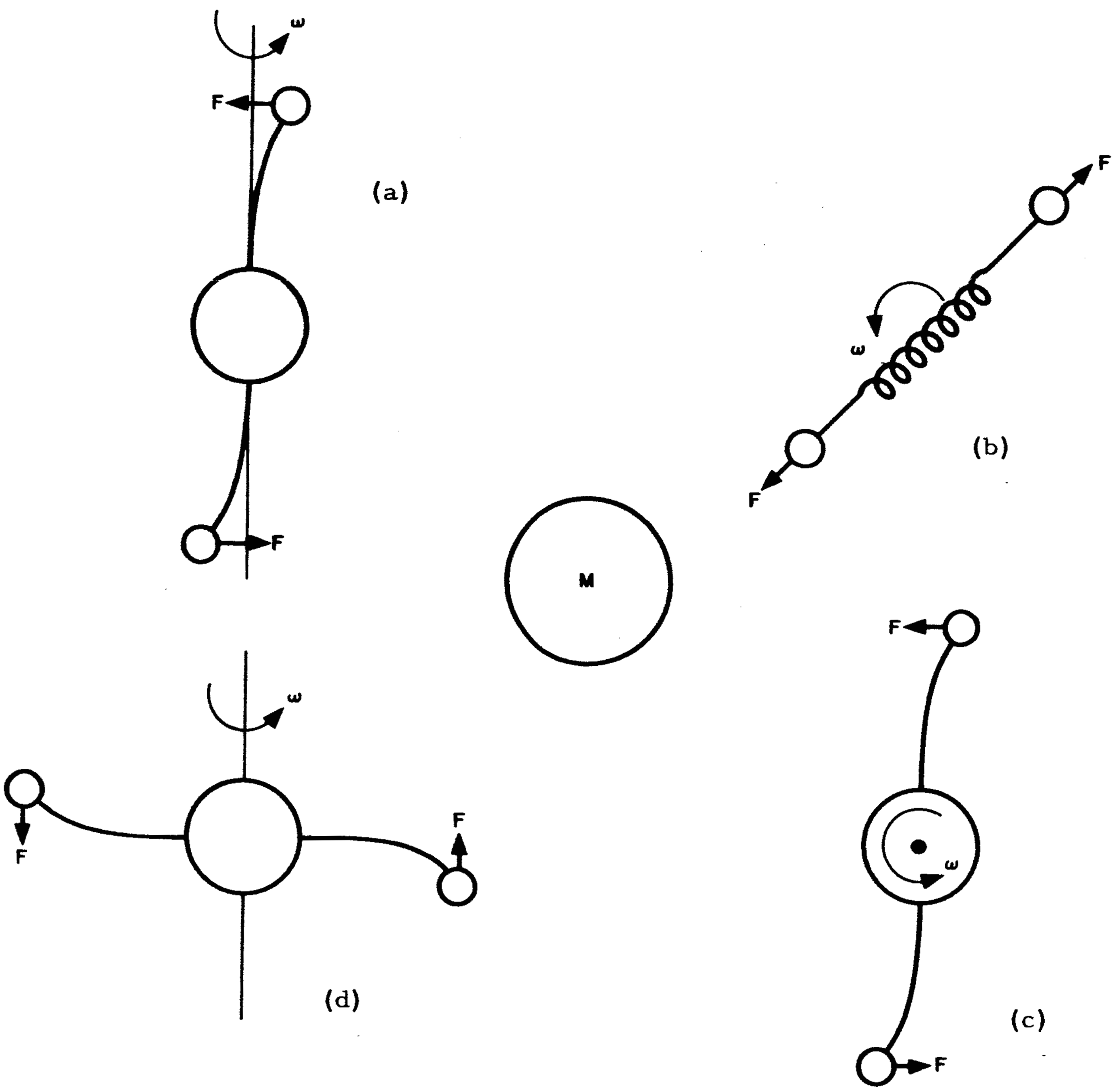


Fig. 2. Possible sensor configurations.

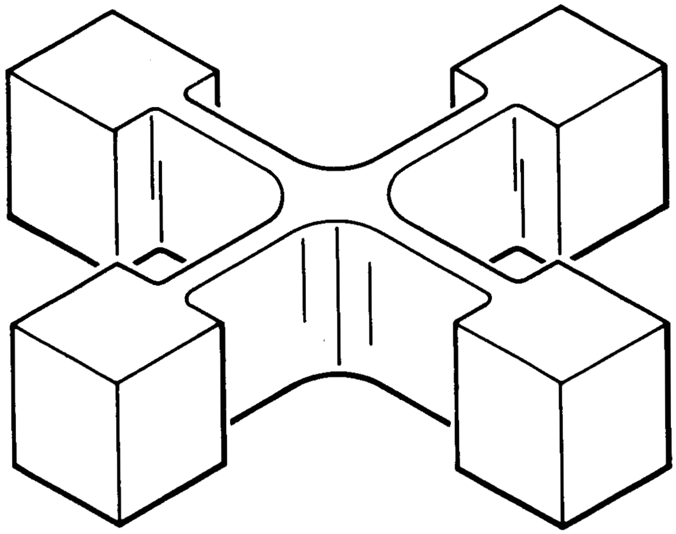


Fig. 3. Cruciform mass sensor.

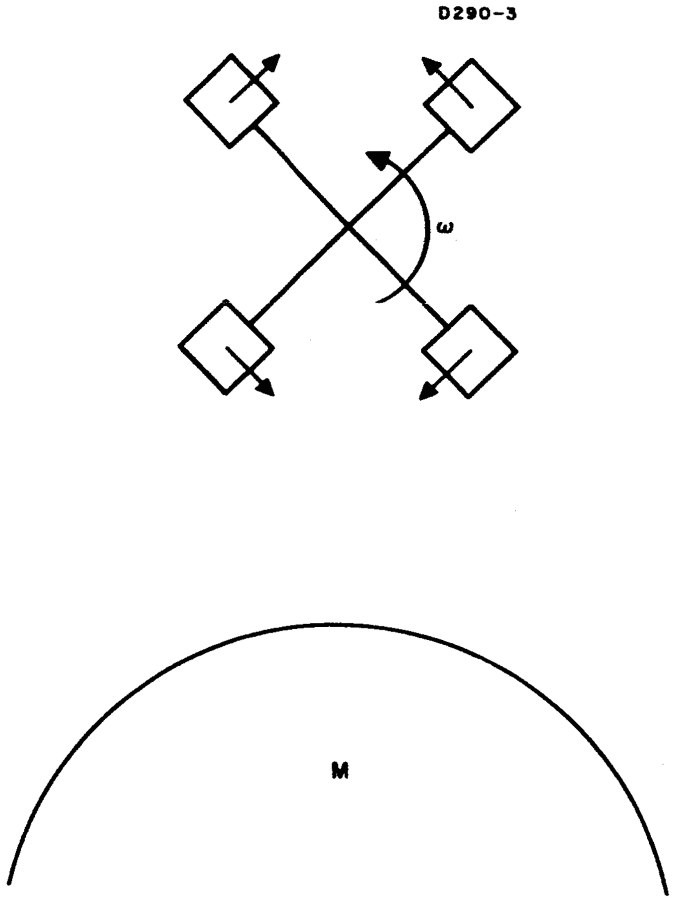


Fig. 4. Phase of 2ω vibrations.

D. Discussion of Noise

1. Thermal Noise

The fundamental sensitivity of any sensor is determined by the thermal noise limitation. In practice, this limit can never be reached, but many systems can approach it very closely. This is especially true of low frequency devices, since the electronics available in this region has been highly developed and will contribute only a few degrees of extra equivalent noise temperature to the physical temperature of the sensor.

Because this basic limit is dependent upon energy considerations, its calculation depends only upon very general parameters of the sensor, such as its temperature, mass, effective length, and time of integration. The results can then be applied to all sensors, regardless of their detailed design. The basic formula states that the signal-to-noise ratio is given by the ratio of the signal energy stored in the sensor to the thermal energy (kT) present in the sensor. In a dynamic case, the signal energy is stored partly in the kinetic energy and partly in the potential energy of the spring.

$$S = KE + PE = \frac{1}{2} m\dot{\xi}^2 + \frac{1}{2}K\xi^2 \qquad (7)$$

where we assume that the amplitude of the spring extension due to the signal forces is given by[1]

$$\xi = \xi_o \sin \omega t \approx \frac{GM \ell \tau}{R^3 \omega} \sin \omega t \quad . \qquad (8)$$

Since the spring constant, mass, and frequency are related by $K = m\omega^2$, the average stored signal energy is just

$$S = \frac{1}{2} m (\omega^2 \overline{\xi}^2) + \frac{1}{2} (m\omega^2) \overline{\xi}^2 = m\omega^2 \overline{\xi}^2 = m\omega^2 \xi_o^2 \overline{\sin^2 \omega t} \ . \qquad (9)$$

Since the average value of $\sin^2 \omega t = 1/2$, we see that the signal-to-noise ratio is

$$\frac{S}{N} = \frac{m\omega^2 \xi_o^2}{2kT} \qquad (10)$$

or the amplitude necessary for a specified signal-to-noise ratio is given by

$$\xi_o \approx \frac{(S/N)^{1/2}}{\omega} \left(\frac{2\,kT}{m}\right)^{1/2} . \qquad (11)$$

This equation, combined with (8), yields the minimum gradient that can be measured for a thermally limited sensor:

$$\Gamma = \frac{GM}{R^3} = \frac{(S/N)^{1/2}}{\ell\ \tau} \left(\frac{2\,kT}{m}\right)^{1/2} \qquad (12)$$

where S/N is the desired signal-to-noise ratio; T, m, and ℓ are the temperature, mass, and length, respectively, of the sensor; M is the mass of the object at distance R; and τ is the integration time.

The major problem to be faced in the design, construction, and operation of the proposed gravitational mass sensors is the identification and elimination of external and internal sources of electronic and mechanical noise so that the sensor is limited only by thermal fluctuations.

Fortunately, the principal investigator has faced these problems previously during the process of construction and operation of a detector for gravitational radiation. This work was done as part of a Ph. D. thesis at the University of Maryland under the direction of Professor Joseph Weber.[26-28] The detector, a nonrotating version of the proposed sensors, uses the longitudinal vibrations in a large aluminum rod to detect time varying gravitational fields. The vibrations in the rod are sensed by piezoelectric strain transducers placed at the points of maximum strain. The detector and its associated electronics were potentially sensitive to acoustic, seismic, and electromagnetic noise sources, but by proper design of acoustic filters and electromagnetic shielding and the use of noise discrimination techniques, the thermal noise in the vibrational mode could be seen. The thermal noise limit for this detector was much lower than that of the proposed sensors because of the much larger mass used for the detector, so that the gravitational mass sensors will need proportionately less shielding.

2. Background Rejection

One difficulty in using these devices will be the spurious background signals generated by masses other than the one under investigation. This will not always be a problem since the sensors measure the various gradients of the gravitational field and are much more sensitive at close range. If the exact position of the object is known, phase coherent detection and correlation between two different sensors could be used to discriminate against the background clutter. Also, partial discrimination can be obtained by orienting the sensor rotation perpendicular to the disturbing mass.

The most important technique for eliminating the background clutter from all the other disturbing masses such as the sun and the using vehicle itself is to operate the sensor in the mode in which it responds preferentially to moving objects. This is done by adjusting the rotational angular frequency slightly below (or above) the proper frequency for vibrational resonance.

$$f_r = \frac{\omega_r}{2\pi} < \frac{f_v}{2} \tag{13}$$

At this rotational rate, all the disturbing masses will be inducing vibrational forces at twice the rotational frequency and if the vibrational mode has a high Q, these forces will lie outside the acceptance bandwidth of the vibrational mode and it will not be excited.

If we now operate the sensor so that there is a constant relative velocity v between it and the object to be measured, the changing line of sight is found to be equivalent to a relative increase in angular rotation. (See Figs. 5 and 6.)

$$\omega_r + \dot{\theta} \approx \omega_r + \frac{v}{b} = 2\pi \left(\frac{f_v}{2}\right) . \tag{14}$$

Thus the detector is rotating with respect to the moving object at the proper angular rate to excite the vibrations in the bending modes, but all other inputs are off resonance.

A technique very similar to this will be valuable for laboratory testing purposes. The sensor to be tested can be rotated at a right angle to the local vertical at a rotational rate that is off resonance. A mass quadrupole test mass can then be suspended from above and spun with its rotational rate and direction chosen so that the combined rotations bring the driving forces into resonance with the vibrational mode.

In reality, of course, the discrimination obtained by this technique is only relative, and a very strong disturbing signal can still be seen even after the discrimination is effected. However, if there is relative motion between the sensor and the object to be measured, the

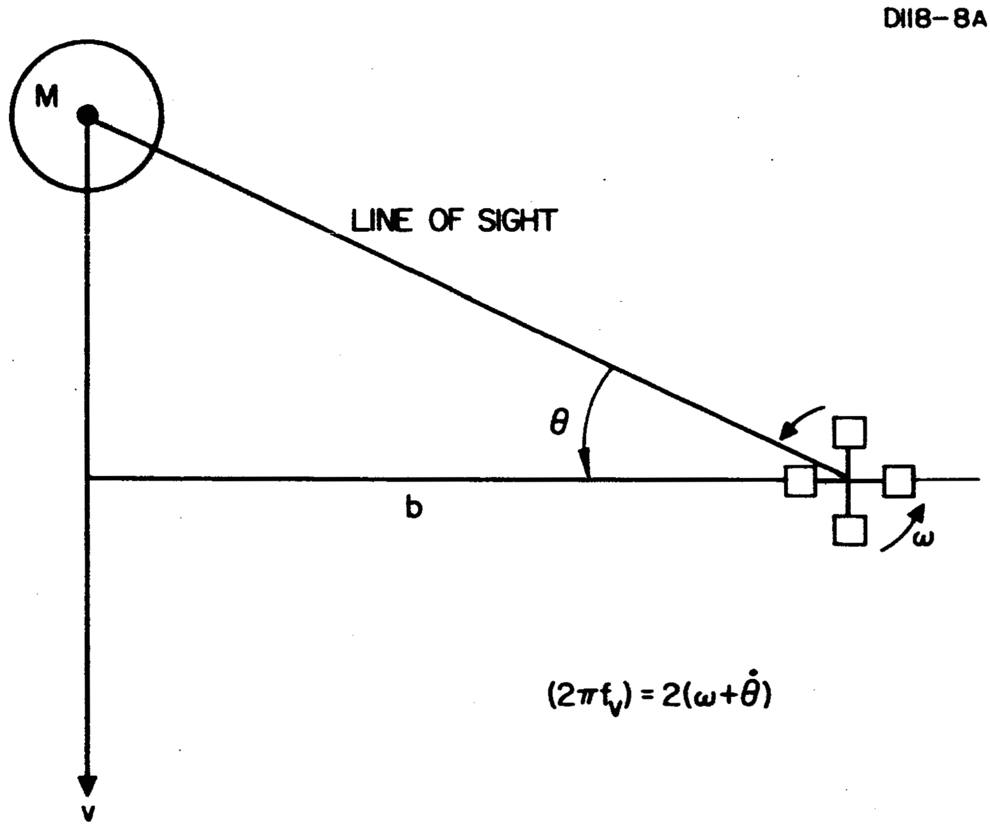


Fig. 5. Relative angular rotation due to relative velocity.

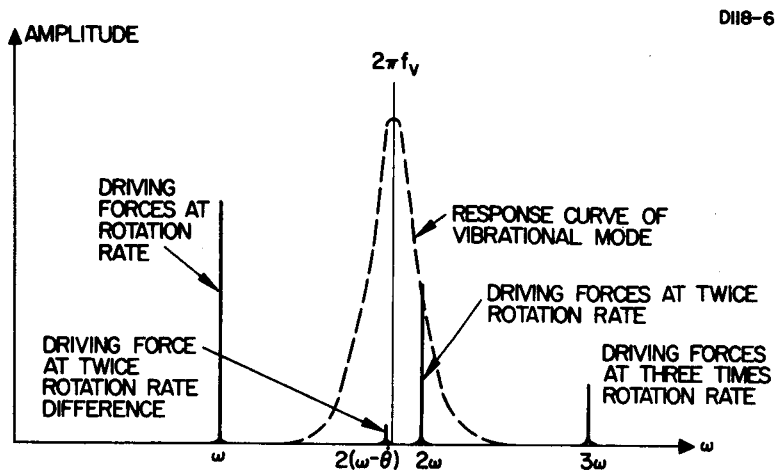


Fig. 6. Response of vibrational mode to driving forces from various sources.

excitation frequency will be different from that of the background clutter, and the signal from the object being measured can be detected by the beat notes that it causes. This can be illustrated by Fig. 7, which shows the response of an electromechanical analog of a sensor. The sensor analog was driven off resonance by a very strong but stable clutter signal. A simulated flyby was then made with a frequency and amplitude swept signal that was 1000 times weaker. The output of the sensor analog was then detected and the beat note of about 0.8 cps between the strong clutter signal and the stored energy from the simulated flyby was plotted. The rise time of 1.5 sec is the duration of the flyby and the fall time of 3.0 sec is the decay time of the vibrational mode. The two passes show that the effect is repeatable, and other experiments showed the expected one-to-one relation between signal strength and beat note amplitude.

There are various ways of using the proposed sensors to obtain further information or better discrimination. For example, the gravitational field pattern varies nonuniformly from one measurement point to another, and a series of measurements at different distances from a mass would allow verification of the range and an unambiguous identification of the gravitationally induced signal. In addition, the primary axes of the gravitational stress pattern are oriented with respect to the line of sight to the mass being investigated, and if the sensor and the mass are in relative motion, the changing line of sight will cause phase shifts in the sensor output.

If an object (e.g., the moon) has a complex mass distribution, its gravitational field will have higher order multipole terms that will vary with relative orientation of the object and the sensor. If the sensor is in orbit around the mass, these higher order multipole terms will also show up as different frequency components, with each order at a different multiple of the orbital frequency.

3. Mechanical Noise

Since the proposed sensors consist of mechanically resonant mass-spring systems, they are potentially susceptible to mechanical noise with frequency components at the frequency of resonance of the sensor. The various types of mechanical noise that could presumably be present are acoustic noise coupling to the sensor through the air, vibrational noise from the bearings or drive coupling through the sensor support, and inertial noise due to poor balance in the sensor support or accelerations and rotations of the vehicle containing the sensor.

In theory, these sources of mechanical noise (except the inertial noise due to certain types of rotation of the using vehicle) should not affect the operation of the sensor, since they do not excite the proper

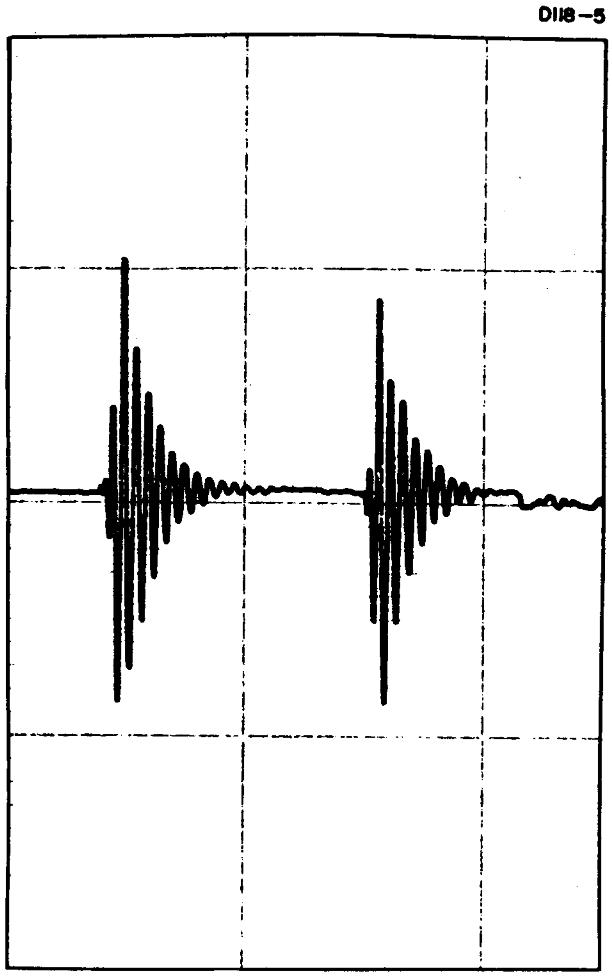


Fig. 7. Simulated flyby with constant background.

mode of oscillation. This is most easily seen by reference to Fig. 8, which shows the response of a cruciform sensor. This applies to all sensors since in general the mechanical noises are forces, forces are vector quantities or first rank tensors, and the sensors are designed to sense only second rank tensors (see Attachment).

In practice, however, these mechanical noises do couple into the sensor and out through the electronics because of asymmetries or nonlinearities in the various mechanical and electrical components. Thus, the mechanical noise problem is a second order one. However, the gravitational interaction which we are seeking is very weak, and a second order sensor coupling to mechanical noise can easily hide the gravitational signal if the noise sources are not carefully eliminated.

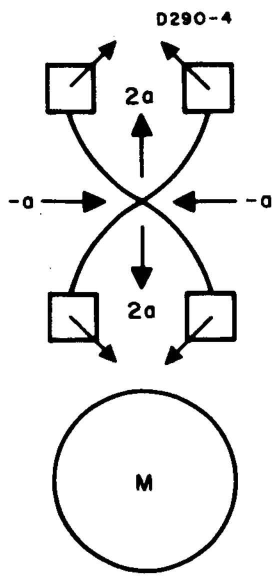
To eliminate the acoustic coupling, the sensor will be operated in a vacuum chamber. Previous experience at the University of Maryland[26-28] has shown that a pressure of a few microns is sufficient for decoupling any acoustic noise.

The vibrational noise from the bearings and drive and the inertial noise due to unbalance of the rotating systems, which couple to the sensor through the sensor support, are major problem areas that are being investigated in the experimental portion of the program. The investigation to eliminate this type of noise has just begun, and will be our major concern during the next quarter. Various different bearings and drives are being purchased, and their noise characteristics will be determined by test. Various sensor supports will be investigated in an effort to find one which transmits the low frequency torques and forces needed for rotation and support, but does not transmit the high frequency torques and vibrations that will excite the sensor resonance.

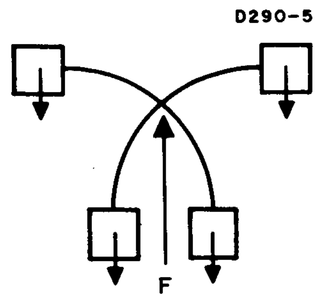
There is no way, even in theory (see Attachment), to eliminate the coupling of the sensor to the inertial noise caused by precession of a single sensor by an external torque. This will not affect the present research program and should not affect any possible applications, since the techniques discussed in Section II-D-2 are applicable to this type of noise.

E. Applications

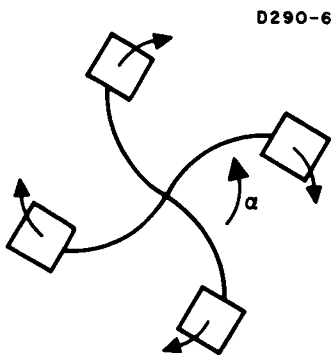
Although the ultimate configuration of a spaceborne gravitational mass sensor will be strongly affected by the results obtained in the research and development phases, the general characteristics such as the mass and size can be estimated now. The sensor proper will weigh about 1/4 lb, will be about 6 in. to 1 ft long, and will be rotating about its midpoint at a rotational speed in the neighborhood of 6000 rpm (100 cps). The package containing the rotating sensor and the electronics will be a flat cylinder about 1 ft in diameter and 2 in. thick, and will weight about 5 to 10 lb.



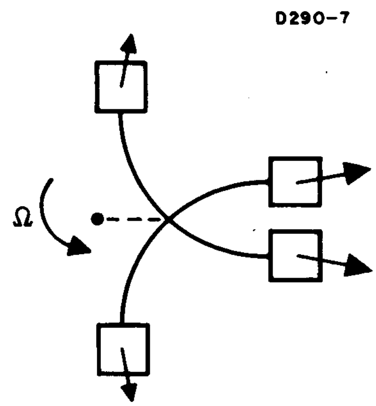
(a).
Response to gravitational force gradient.



(b).
Response to linear acceleration.



(c).
Response to angular acceleration.



(d).
Response to asymmetric rotation.

Fig. 8. Response of cruciform sensor to various gravitational and inertial sources.

The initial applications of a gravitational mass sensor will be to measure the masses of the asteroids and the variations in the gravitational field of the moon. However, since the devices respond in different ways to inertial and gravitational forces, they could also be used as sensors for active attitude control as proposed in Refs. 7-18.

1. Asteroid Mass Measurement

The investigation of the asteroid belt will be primarily directed toward determining the origin of the asteroids. An important parameter in this determination is a measurement of their density. Measurement of the volume can be obtained from photographs during a flyby, and the development of a sensor to measure the mass during a flyby will relieve the necessity for matching orbits with an asteroid in order to determine its mass.

To determine the ultimate limit on this type of application, we shall assume a thermally limited gravitational mass sensor detecting the gravitational force gradient of the asteroid. The usual thermally limited equation from (12) is

$$\frac{GM \ell \tau}{R^3} = (S/N)^{1/2} (2kT/m)^{1/2} \quad , \qquad (15)$$

where S/N is the desired signal-to-noise ratio; T, m, and ℓ are the temperature, mass, and length of the sensor; M is the mass of the asteroid; and τ is the integration time. The integration time, however, is not completely independent of the range, since in a flyby the effective integration time must be less than the time of effective interaction of the sensor and asteroid.

$$\tau < \frac{2R}{v} \qquad (16)$$

where v is the velocity of the probe and R is the miss distance or effective range. Using (16), our maximum range for a given set of operating parameters is given by

$$R^2 = \frac{GM \ell}{v\,(S/N)^{1/2}} \left(\frac{2m}{kT}\right)^{1/2} \quad . \qquad (17)$$

If we assume a room temperature sensor of effective mass 100 g (1/4 lb) and effective length of 1 ft on a probe with a relative veloctiy of 10 km/sec, the range in meters for a 10 to 1 signal-to-noise ratio is

$$R^2 = 4.4 \times 10^{-6} M \qquad (18)$$

where M is in kilograms. Estimates for the number of asteroids of each mass range are available in Allen's Astrophysical Quantities.[29] Data from these tables are summarized in Table I.

TABLE I

Asteroid Measurement Range for Small Flyby Sensor

Number of Asteroids	Radius, km	Mass, kg	Measurement Range, km	Integration Time $\tau = 2R/v$ sec (v = 10 km sec)
6	140	5×10^{19}	15000	3000 (~ 1 hour)
25	70	6×10^{18}	5000	1000
80	44	1.5×10^{18}	2500	500
200	28	4×10^{17}	1300	260
500	18	1×10^{17}	660	130
1250	11	2.5×10^{16}	330	66 (~ 1 min)

2. Gravity Survey of Moon

A detailed gravity survey of the moon will tell us a great deal about its past history and internal structure. A surface survey with gravimeters is obviously time consuming, and a survey using the variations in the orbital parameters of a lunar satellite will smooth out localized features. The proposed gravitational force gradient sensors will read out the gravity difference directly, and thus the geodetic information will be available in real time. The gravitational force gradient of the moon is 1.7×10^{-6} sec^{-2}, which is about the same as that of the earth. (For measurements near the surface of a body, the gradient is proportional to the density.) The magnitude of the variations in this gradient due to the higher harmonics of the mass distribution of the moon are also expected to be as large or larger than those of the earth since the moon's lower gravity allows larger mass imbalances to exist. The largest effects will occur for density variations on the moon's surface that have an effective radius equal to or greater than the altitude of the lunar orbiter. If we assume an altitude of 50 km, objects with characteristic dimensions of 50 km will create gravitational gradients of the order of 10^{-7} to 10^{-8} sec^{-2}. This is a variation of a few percent in the basic gradient of the moon and should be easily measured by a sensor in a lunar orbiter, since the thermal noise limit is over 30 dB down.

Using (8), we see that a gravitational gradient anomaly $\Gamma = G\Delta M/R^3$ at the sensor will cause strains in the sensor of

$$\epsilon = \frac{\xi}{\ell} \approx \frac{GM}{R^3} \frac{\tau}{\omega} \sin \omega t = \Gamma \frac{\tau}{\omega} \sin \omega t \quad . \qquad (19)$$

If we assume that the sensor resonant frequency is 160 cps ($\omega = 10^3$) and the integration time is $\tau = 10$ sec, a gravitational gradient anomaly of 10^{-8} sec^{-2} will cause a strain of 10^{-10} in./in. The strain transducers used on the sensor have a "gauge factor" of about 10^5 volts per unit strain (V/in./in.), so that the electrical signal due to the anomaly will be about 10 μV. Since it is a narrow-band signal, this is easily measured with conventional electronic circuits.

III. EXPERIMENTAL PROGRAM

A. Sensor Design

Because theoretical analysis in the Appendix has shown that there would be limitations on the use of radial type mass sensor design and because design problems of a radial type support structure were quite complex, it was decided to try a cantilever type sensor design (see Fig. 3) for our preliminary studies of sensor support and drive mechanisms. Four arms were chosen in order to counterbalance the rotational torques which are produced by the interaction between the sensor and the sensed mass. These torques will cause angular accelerations at twice the rotation frequency for a two arm sensor, and four times the rotation frequency in a four arm sensor, etc. Each cantilever beam was designed for a natural frequency of 195 cps. Upon receipt of the sensor, barium titanate strain transducers were mounted on each of the four arms and the natural frequency and Q of the system were checked both in a solid mounting in a vise and suspended freely by a wire. In each case there was one natural frequency (190.8 cps) and one value of Q ($\approx$ 173).

The sensor was then mounted in a vacuum chamber (not evacuated) (see Fig. 9). This system was suspended from a wire and again tested. This time multiple natural frequencies occurred in the range of 185 to 190 cps. The magnitude, frequency, and number of these resonant peaks differed, depending on which arm or arms were excited, and which arm or arms were used for readout.

Evacuation of the chamber did not seem to change the multiple peak problem, but the Q of the system rose to about 300. The chamber was then clamped in a vise and retested, and the multiple peaks were replaced by a single resonance at 189.5 cps.

This implies that the sensor torques are coupling to the vacuum chamber through the support rod. One hypothesis indicates that since the vacuum chamber has a mass comparable to the sensor mass, a number of new modes of oscillation are possible at frequencies near the design frequency. However, a preliminary analysis did not confirm this hypothesis. A more detailed analysis is under way, and the sensor support rod is being redesigned to lower the coupling of the sensor to the chamber.

The sensor and vacuum chamber were then placed in the frame (Fig. 10) and driven by air pressure to a rotational speed of about 6000 rpm (100 cps). The unfiltered, unamplified output of the sensor arm at this speed was $\approx$ 100 mV; this was entirely a result of the vibrational noise from the ball bearings. The frequency of this resonance was $\approx$ 200 cps.

M 3615

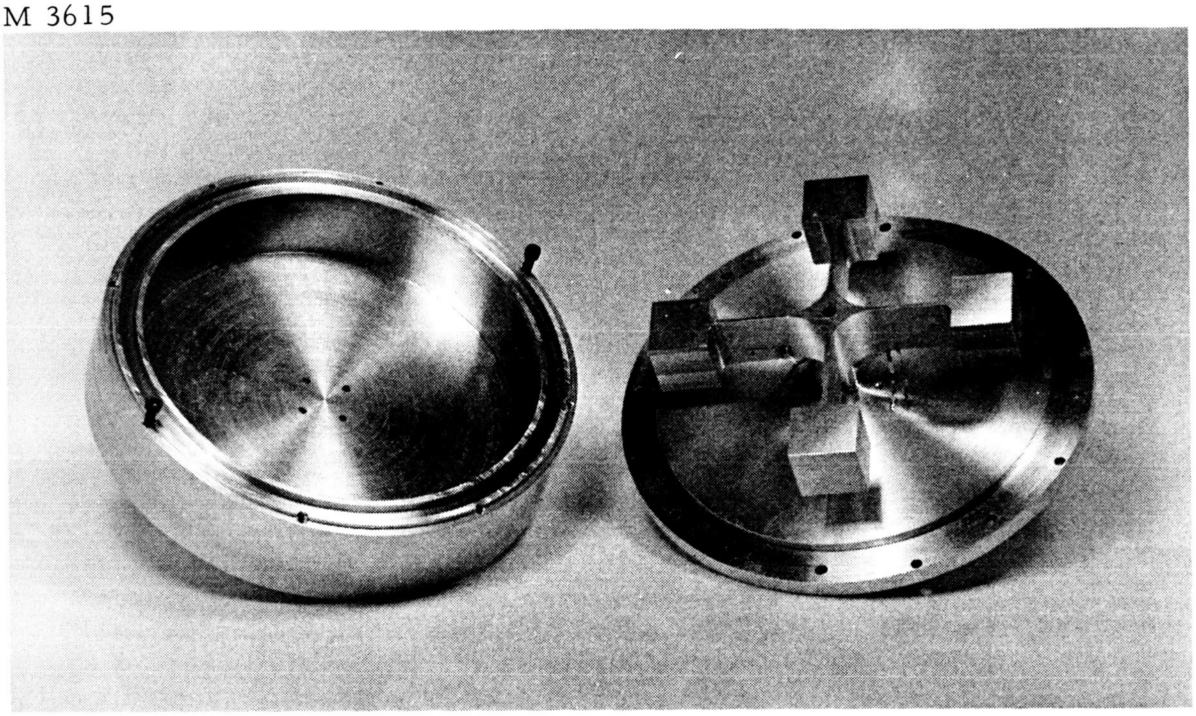
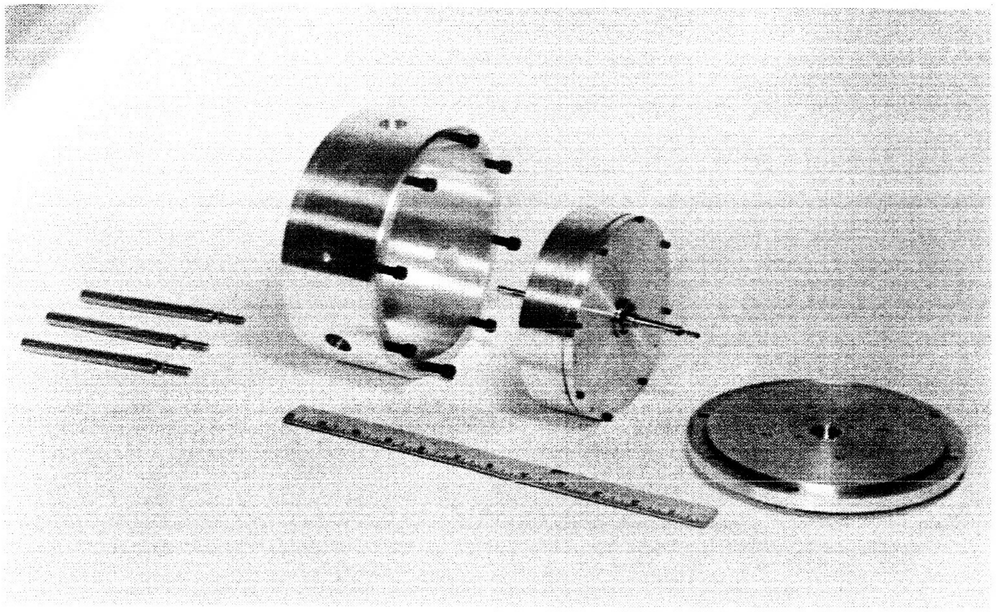


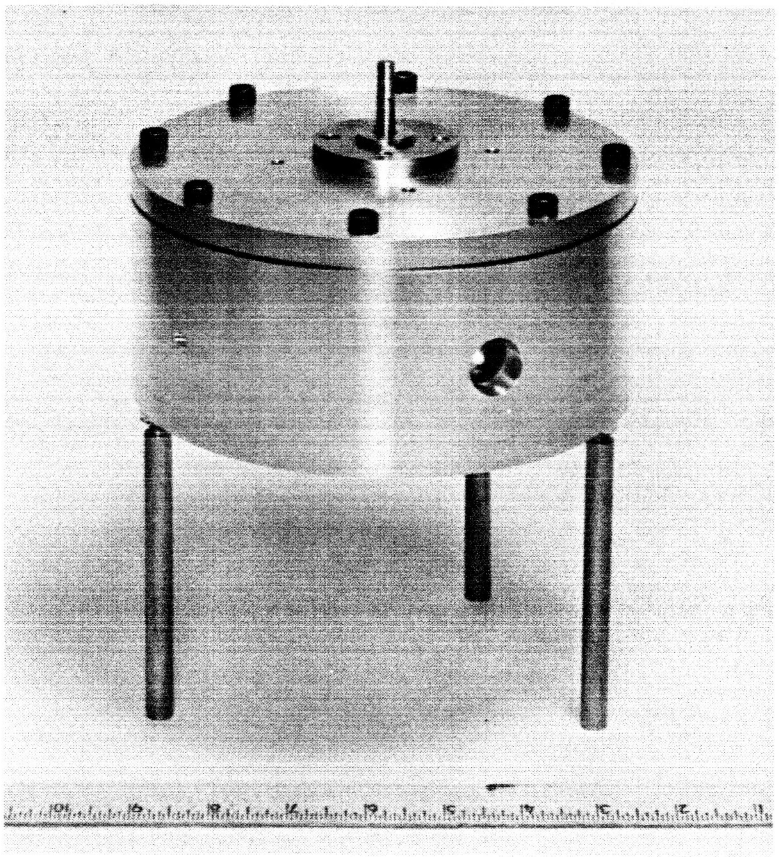
Fig. 9. Cruciform gravitational mass sensor.

M 3677



(a). Exploded view.

M 3676



(b). Assembled test setup.

Fig. 10. Ball bearing motor mount and frame with rotor structure.

The vacuum chamber sensor combination (rotor) was then dynamically balanced to a center of mass location of within 5 μin. of the center of rotation. This balancing brought the magnitude of the above mentioned 200 cps bearing noise down to approximately 20 mV. Further investigations are being conducted to determine ways of reducing the noise still further.

The commercially purchased slip rings are working quite well at these speeds and voltage levels.

B. Sensor Drive and Support

As was pointed out in Section II-D-3, a major problem area is the vibrational and inertial noise introduced by the driving mechanism which rotates the sensor and the vibrational noise introduced by the supporting bearings. Our first model uses standard high quality ball bearings and, as expected, these have a great deal of noise at multiples of the rotation frequency. It is expected that the air bearings and magnetic bearings being fabricated for the program will have substantially less noise.

1. Air Bearing Support and Drive

A low noise air bearing support assembly is being built by the Electromechanical Department under the direction of S. D. Howe.

The device features a hollow central post with air channels, a concentric synchronous motor stator, and an enclosure for the rotating mass sensor/vacuum chamber. The chamber is supported by a cylinder which fits over the central post. The cylinder is also the rotor of the synchronous motor (see Fig. 11).

Signals from the mass sensor are retrieved through slip rings at the top of the sensor chamber.

2. Magnetic Suspension Support and Drive

One of the most promising bearings being investigated in terms of noise free operation is a Beams type magnetic suspension. A large number of these have been constructed by Prof. J. W. Beams and his colleagues at the University of Virginia for various purposes, including a suspension capable of levitating and driving 20 lb ultracentrifuge rotors. To date they have not been able to find any evidence of high frequency vibrational noise in their suspensions. Their magnetically

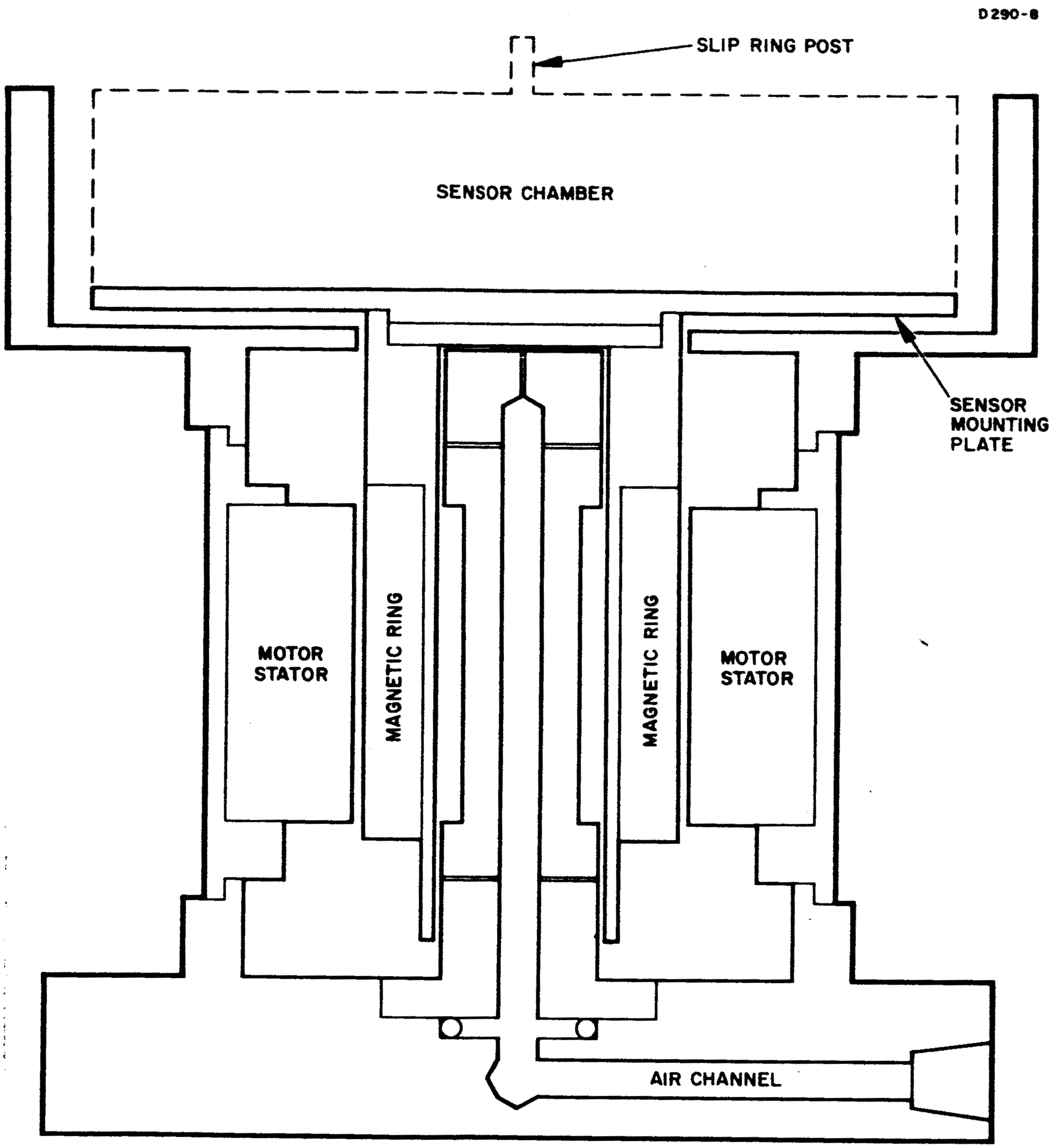


Fig. 11. Sensor air support and synchronous motor assembly.

supported ultracentrifuges are quiet enough to accomplish equilibrium sedimentation analyses on high molecular weight compounds[30] that are not possible with previously available flexible shaft ultracentrifuges because of the vibrational noise generated in their support structure.

Magnetic suspensions are not available commercially, but W. H. Dancy of the Instrument Development Group of the Research Laboratories for the Engineering Sciences at the University of Virginia has agreed to construct a unit for us with the following characteristics.

1. A main supporting solenoid coil capable of magnetically supporting ferrous bodies weighing in excess of 5 lb will be fabricated. The solenoid will be constructed in several sections so that adequate cooling of the coil assembly may be assured.

2. A gas-tight lateral damping assembly will be fitted to the supporting solenoid to facilitate possible operation of the support system in conjunction with a vacuum chamber.

3. A two-phase electric motor will be provided to accelerate the suspended rotor. It is anticipated that the motor will be wound in a four pole configuration. An attempt will be made to operate the motor at the top of the rotor. Should this configuration be unstable, the motor will be placed at the bottom of the rotor and a 3-in.-diameter access hole will be provided in the center of the assembly.

4. All electronic components and power supplies required for stable operation of the magnetic support system will be furnished. The design of the magnetic support system will, in general, follow the design that is discussed in "The Application of High Rotational Speed Techniques to the Study of the Adhesion of Electrodeposits," William H. Dancy, Jr. ASTIA 363-396L.

5. A two-phase power amplifier and the related power supplies having the capability of accelerating the rotor up to angular speeds of at least 1000 revolutions/sec when the rotor is operated in an enclosed chamber maintained at a pressure less than 50 mTorr will be supplied. Higher chamber pressures may be used at lower angular speeds.

6. A photoelectric pickup capable of providing an electrical signal that is proportional to the angular speed of the rotor will be provided.

The complete support system and drive assembly will be assembled and tested in the laboratories of the Research Laboratories for the Engineering Sciences. Hughes Research Laboratories will furnish the rotor for this test. Following these tests, the complete magnetic support assembly will be packed for shipment to the Hughes Research Laboratories.

C. Electronics

The general electronic circuit for the mass sensor is shown in Fig. 12. The parallel connection of four strain transducers (Gulton SC-2) serves two purposes: It reduces the source impedance seen by the preamplifier by 4, and it cancels output of vibrational modes other than the desired one.

The output of the strain transducers (1 to 100 μV at 190 cps) is amplified by a low noise preamplifier (Fig. 13) to a level of 1.5 to 150 mV. The preamplifier is mounted in the rotating sensor vacuum chamber.

Power for the preamplifier and the preamplifier output are presently retrieved from the sensor via slip rings. It is possible that the mechanical and electrical noise produced by the slip rings will require their replacement by more sophisticated means, but the sensor support and rotation system has not been refined to the point where slip ring noise may be seen yet. If the slip rings are too noisy, the battery may be mounted inboard and the signal retrieved through a transformer with a rotary primary. Some preliminary investigation along this line has indicated its practicality.

The preamplifier output in the frequency region desired is next amplified to a level of approximately 1 V by a band-pass amplifier such as the GR 1232-A and fed to a lock-in amplifier (or phase sensitive detector) such as the Princeton Applied Research JB-4.

The sensor is rotated at a synchronous speed (6000 rpm) by an oscillator-amplifier-motor system. The oscillator also supples the phase reference for the lock-in amplifier.

The output of the amplifier is a dc level proportional to the amplitude of the sensor output at the frequency and phase selected by the oscillator and lock-in amplifier.

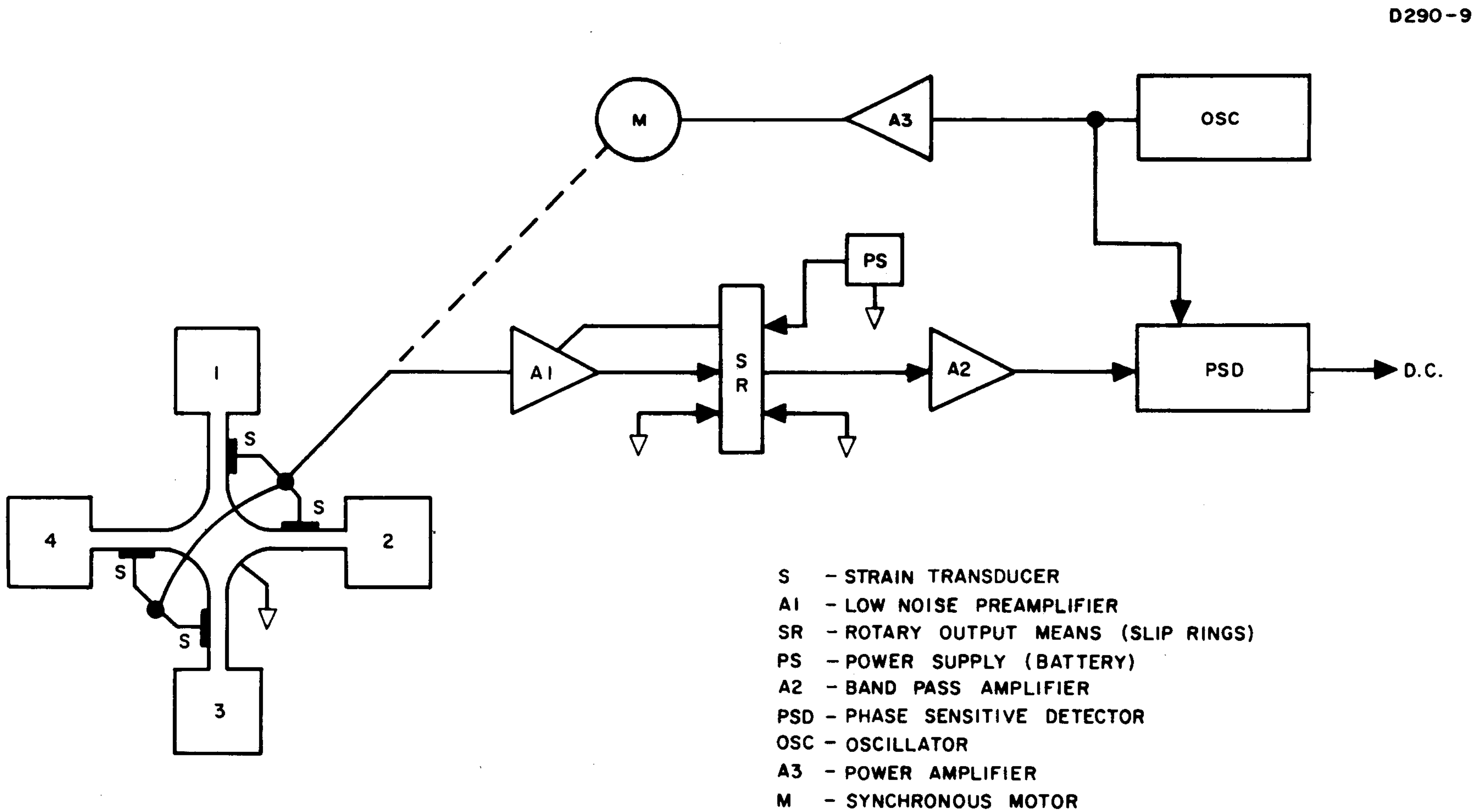


Fig. 12. General mass sensor electronics.

D290-10

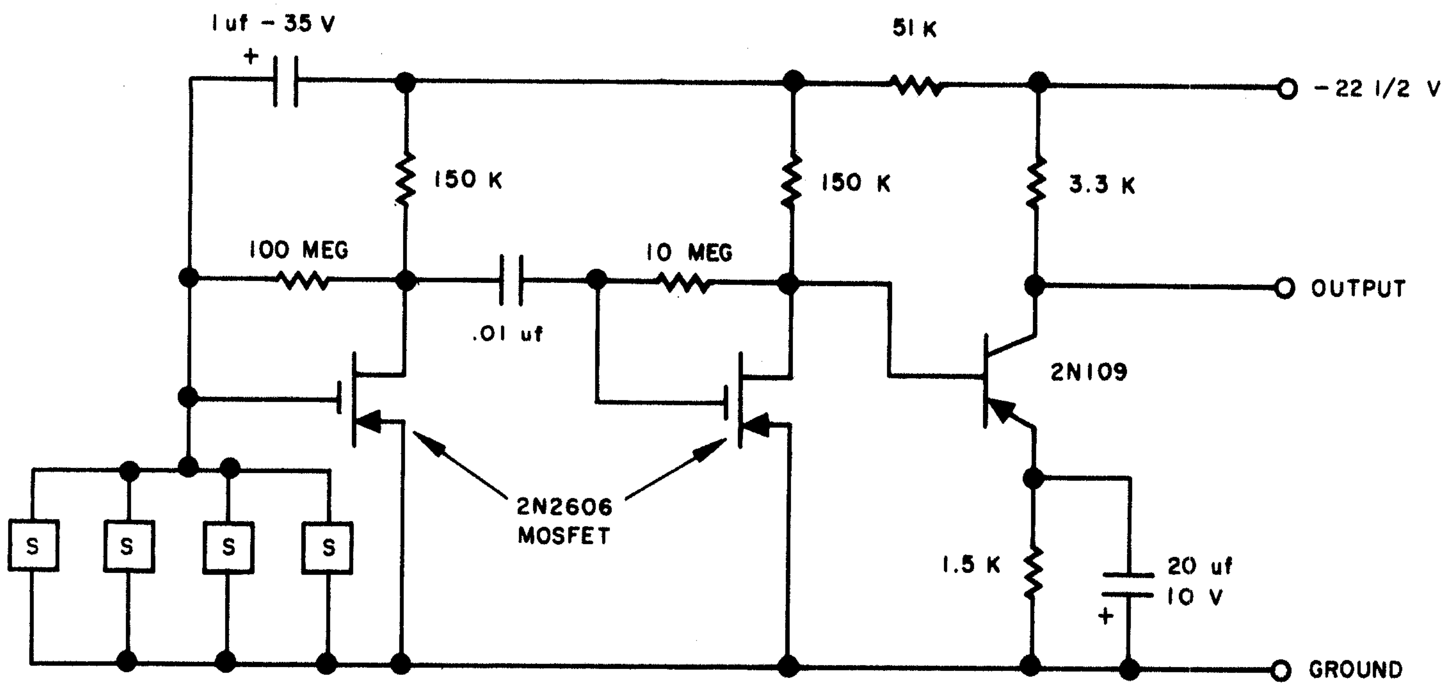


Fig. 13. Preamplifier circuit.

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IV. CONCLUSIONS

We have shown analytically that radially vibrating structures (Fig. 2(b)) are not suitable for use as gravitational mass sensors because of a mechanical instability at a rotation frequency of one-half the vibrational frequency.

We have shown experimentally that transversely vibrating structures (Fig. 2(c)) are stable under rotations at one-half their vibration frequency and therefore have the basic structural integrity necessary for their use as sensors.

Because we have a sensor structure that can be rotated at the necessary rotational speeds, there does not seem to be any other barrier to the construction of a gravitational mass sensor, except for the recognized major problem area of bearing and drive noise.

V. RECOMMENDATIONS

It is recommended that the following investigations be continued:

1. Experimental investigation of noise in bearings and drives

2. Experimental and analytical investigation of the multimode oscillations of a transversely vibrating cruciform sensor structure and the effects of varying the coupling to external structures

3. Preliminary studies of

    a. Other sensor structures (e.g., axially vibrating cruciform rather than transversely vibrating cruciform)

    b. Other modes of operation (e.g., electrical resonance rather than mechanical resonance)

    c. Other methods of readout (e.g., transformer coupling rather than slip rings)

    d. Other types of bearings (e.g., journal bearings).

APPENDIX – THEORY OF RADIALLY VIBRATING GRAVITATIONAL MASS SENSORS

A system such as that shown in Fig. A-1 is assumed for the calculations of the operation of a freely falling radially vibrating gravitational mass sensor. The sensor consists of two equal masses M_1 and M_2 connected by massless springs to a central mass M_3. The sensor is rotating in free fall near an object with mass M_4. The springs have an initial length of ℓ_o; under the centrifugal force caused by the sensor rotation they experience an extension e_a. The sensor system then oscillates about this extended position with amplitude $\xi_a(t)$.

In the analysis, we will assume the following:

1. The sensor masses are constrained to vibrate only in the radial direction.

2. The radial vibrations ξ_a are small compared with the length ℓ_o and the centrifugal extensions e_a, and their effect on the gravitational interaction of the sensor masses with the detected mass is negligible.

3. The self gravitation of the three sensor masses is negligible compared with the centrifugal force.

4. In contrast to previous analyses,[13, 17, 25] we will not assume that the rotational frequency is constant, but instead will assume that the angular momentum of the total system, including the object being measured, remains constant.

Under these assumptions, the equations of motion of the four masses are written. These equations are then combined and manipulated into a useful form in polar coordinates. These equations of motion and another equation expressing the conservation of angular momentum are then solved to obtain a pair of equations describing the radial vibrations of each spring. These equations are examined to determine the response of the sensor to the gravitational gradients produced by the mass M_4.

It shall be assumed first that the central mass of the sensor is equal to the outer sensor masses; variations in the relative magnitude of this mass will then be discussed.

The equations of motion of the masses shown in Fig. A-1 are as follows:

D290-1

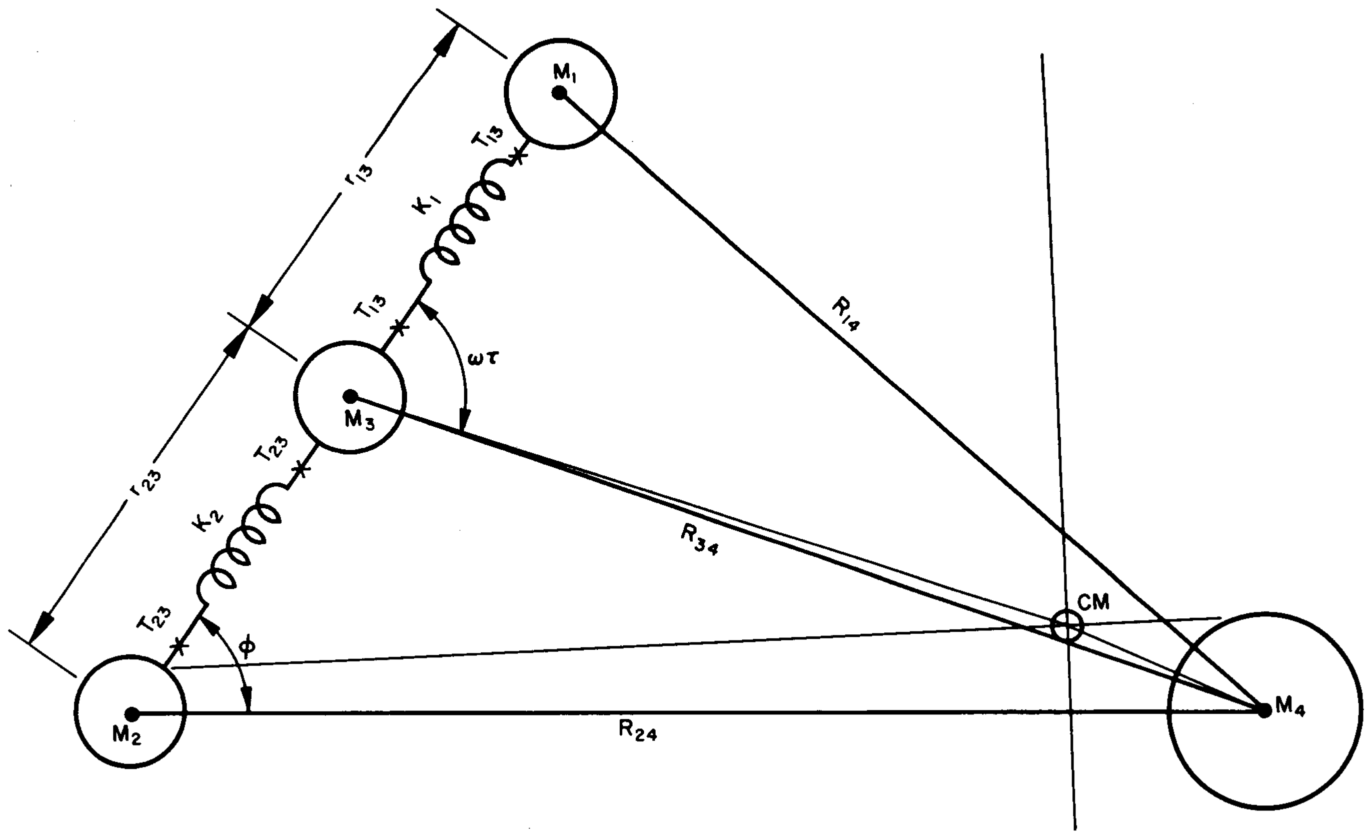

Fig. A-1. Schematic of three mass, two spring, radially vibrating gravitational mass sensor.

For Mass 4

$$M_4 \ddot{x}_4 = - \frac{GM_4 M_1}{(R_{14})^2} \frac{x_4 - x_1}{R_{14}} - \frac{GM_4 M_2}{(R_{24})^2} \frac{x_4 - x_2}{R_{24}} - \frac{GM_4 M_3}{(R_{34})^2} \frac{x_4 - x_3}{R_{34}} \qquad \text{(A-1)}$$

$$M_4 \ddot{y}_4 = - \frac{GM_4 M_1}{(R_{14})^2} \frac{y_4 - y_1}{R_{14}} - \frac{GM_4 M_2}{(R_{24})^2} \frac{y_4 - y_2}{(R_{24})} - \frac{GM_4 M_3}{(R_{34})^2} \frac{y_4 - y_3}{R_{34}} \qquad \text{(A-2)}$$

For Mass 1

$$M_1 \ddot{x}_1 = \frac{GM_1 M_4}{(R_{14})^2} \frac{x_4 - x_1}{R_{14}} + T_{13} \frac{x_3 - x_1}{r_{13}} \qquad \text{(A-3)}$$

$$M_1 \ddot{y}_1 = \frac{GM_1 M_4}{(R_{14})^2} \frac{y_4 - y_1}{R_{14}} + T_{13} \frac{y_3 - y_1}{r_{13}} . \qquad \text{(A-4)}$$

For Mass 2

$$M_2 \ddot{x}_2 = \frac{GM_2 M_4}{(R_{24})^2} \frac{x_4 - x_2}{R_{24}} + T_{23} \frac{x_3 - x_2}{r_{23}} \qquad \text{(A-5)}$$

$$M_2 \ddot{y}_2 = \frac{GM_2 M_4}{(R_{24})^2} \frac{y_4 - y_2}{R_{24}} + T_{23} \frac{y_3 - y_2}{r_{23}} . \qquad \text{(A-6)}$$

For Mass 3

$$M_3 \ddot{x}_3 = \frac{GM_4 M_3}{(R_{43})^2} \frac{x_4 - x_3}{R_{43}} - T_{13} \frac{x_3 - x_1}{r_{13}} - T_{23} \frac{x_3 - x_2}{r_{23}} \tag{A-7}$$

$$M_3 \ddot{y}_3 = \frac{GM_4 M_3}{(R_{43})^2} \frac{y_4 - y_3}{R_{43}} - T_{13} \frac{y_3 - y_1}{r_{13}} - T_{23} \frac{y_3 - y_2}{r_{23}} \tag{A-8}$$

where

$$T_{a3} = k_a(e_a + \xi_a) + D \dot{\xi}_a$$

$$r_{a3} = (\ell_o + e_a) + \xi_a \approx \ell_a + \xi_a \quad .$$

We will assume initially that the three sensor masses are equal

$$M_1 = M_2 = M_3 = m$$

and the two springs are identical so that

$$k_1 = k_2 = k$$

$$e_1 = e_2 = e \quad .$$

In addition, the mass being measured is assumed larger than the sensor

$$M_4 = M >> 3m \quad .$$

Subtracting (A-7) from (A-3) and (A-5), and similarly (A-8) from (A-4) and (A-6), and using the above simplifications, we obtain equations expressing the relative motion of the sensor masses:

$$\ddot{x}_1 - \ddot{x}_3 = GM\left[\frac{x_4 - x_1}{(R_{14})^3} - \frac{x_4 - x_3}{(R_{34})^3}\right] + \frac{2T_{13}}{m}\,\frac{x_3 - x_1}{r_{13}} + \frac{T_{23}}{m}\,\frac{x_3 - x_2}{r_{23}}$$

(A-9)

$$\ddot{x}_2 - \ddot{x}_3 = GM\left[\frac{x_4 - x_2}{(R_{24})^3} - \frac{x_4 - x_3}{(R_{34})^3}\right] + \frac{2T_{23}}{m}\,\frac{x_3 - x_2}{r_{23}} + \frac{T_{13}}{m}\,\frac{x_3 - x_1}{r_{13}}$$

(A-10)

$$\ddot{y}_1 - \ddot{y}_3 = GM\left[\frac{y_4 - y_1}{(R_{14})^3} - \frac{y_4 - y_3}{(R_{34})^3}\right] + \frac{2T_{13}}{m}\,\frac{y_3 - y_1}{r_{13}} + \frac{T_{23}}{m}\,\frac{y_3 - y_2}{r_{23}}$$

(A-11)

$$\ddot{y}_2 - \ddot{y}_3 = GM\left[\frac{y_4 - y_2}{(R_{24})^3} - \frac{y_4 - y_3}{(R_{34})^3}\right] + \frac{2T_{23}}{m}\,\frac{y_3 - y_2}{r_{23}} + \frac{T_{13}}{m}\,\frac{y_3 - y_1}{r_{13}} .$$

(A-12)

The gravitational terms which drive this system of equations are very small compared with the inertial terms and the spring tension terms; therefore, we can assume for the gravitational terms that

$$\omega \equiv \text{constant}$$

$$R_{34} = R \equiv \text{constant}$$

$$R_{14}^2 = R^2 + r_{13}^2 - 2 r_{13} R \cos \omega t$$

$$R_{24}^2 = R^2 + r_{23}^2 + 2 r_{23} R \cos \omega t$$

$$x_4 - x_1 = R - r_{13} \cos \omega t$$

$$x_4 - x_2 = R + r_{23} \cos \omega t$$

$$x_4 - x_3 = R$$

$$y_4 - y_1 = - r_{13} \sin \omega t$$

$$y_4 - y_2 = + r_{23} \sin \omega t$$

$$y_4 - y_3 = 0 \quad .$$

This is not strictly true, but the differences are of order G^2 and are therefore negligible. We cannot make these kinds of assumptions for the other terms, but instead must use the relations

$$x_1 - x_3 = r_{13} \cos \phi \qquad \text{(A-13)}$$

$$\ddot{x}_1 - \ddot{x}_3 = \ddot{r}_{13} \cos \phi - (2\dot{r}_{13}\dot{\phi} + r_{13} \ddot{\phi}) \sin \phi - r_{13} \cos \phi \, \dot{\phi}^2 \qquad \text{(A-14)}$$

$$x_2 - x_3 = - r_{23} \cos \phi \qquad \text{(A-15)}$$

$$\ddot{x}_2 - \ddot{x}_3 = - \ddot{r}_{23} \cos \phi + (2\dot{r}_{23}\dot{\phi} + r_{23}\ddot{\phi}) \sin \phi + r_{23} \cos \phi \, \dot{\phi}^2 \qquad \text{(A-16)}$$

$$y_3 - y_1 = - r_{13} \sin \phi \tag{A-17}$$

$$\ddot{y}_1 - \ddot{y}_3 = \ddot{r}_{13} \sin \phi + (2\dot{r}_{13}\dot{\phi} + r_{13}\ddot{\phi}) \cos \phi - r_{13} \dot{\phi}^2 \sin \phi \tag{A-18}$$

$$y_3 - y_2 = r_{23} \sin \phi \tag{A-19}$$

$$\ddot{y}_2 - \ddot{y}_3 = - \ddot{r}_{23} \sin \phi - (2\dot{r}_{23}\dot{\phi} + r_{13}\ddot{\phi}) \cos \phi + r_{13} \dot{\phi}^2 \sin \phi \tag{A-20}$$

(The equations (A-14), (A-16), (A-18), and (A-20) are correct and therefore the rest of the analysis will differ from the similar but simpler analysis in Ref. 1, which contains an error of 2 in its corresponding eqs. (25) and (27).)

If we multiply (A-9) and (A-10) by cos ϕ and add them to (A-11) and (A-12), respectively, multiplied by sin ϕ and then make the above substitutions into the various terms and simplify, we obtain equations representing the radial motion of the spring mass systems of the sensor driven by the gravitational force terms:

$$\ddot{r}_{13} - r_{13}\dot{\phi}^2 + \frac{2T_{13} - T_{23}}{m} = \frac{GM}{R^2} \left[\frac{\cos \omega t - \frac{r_{13}}{R}}{\left[1 + \left(\frac{r_{13}}{R}\right)^2 - 2 \frac{r_{13}}{R} \cos \omega t\right]^{3/2}} - \cos \omega t \right] \tag{A-21}$$

$$\ddot{r}_{23} - r_{23}\dot{\phi}^2 + \frac{2T_{23} - T_{13}}{m} = - \frac{GM}{R^2} \left[\frac{\cos \omega t + \frac{r_{23}}{R}}{\left[1 + \left(\frac{r_{23}}{R}\right)^2 + 2 \frac{r_{23}}{R} \cos \omega t\right]^{3/2}} - \cos \omega t \right] . \tag{A-22}$$

If we then expand the denominator in the gravitational driving terms and keep the largest factors of (r/R) for each term in cos $n\omega t$ we obtain

$$\ddot{r}_{13} - r_{13}\dot{\phi}^2 + \frac{2T_{13} - T_{23}}{m}$$

$$= \frac{GM\, r_{13}}{R^3}\left[\frac{1}{2} + \frac{9}{8}\left(\frac{r_{13}}{R}\right)\cos\omega t + \frac{3}{2}\cos 2\omega t + \frac{15}{8}\left(\frac{r_{13}}{R}\right)\cos 3\omega t + \frac{35}{16}\left(\frac{r_{13}}{R}\right)^2\cos 4\omega t\right] ; \qquad \text{(A-23)}$$

in a similar manner,

$$\ddot{r}_{23} - r_{23}\dot{\phi}^2 + \frac{2T_{23} - T_{13}}{m}$$

$$= -\frac{GM\, r_{23}}{R^3}\left[-\frac{1}{2} + \frac{9}{8}\left(\frac{r_{23}}{R}\right)\cos\omega t - \frac{3}{2}\cos 2\omega t + \frac{15}{8}\left(\frac{r_{23}}{R}\right)\cos 3\omega t - \frac{35}{16}\left(\frac{r_{23}}{R}\right)^2\cos 4\omega t\right] . \qquad \text{(A-24)}$$

The usual analysis then proceeds with the assumption that the angular velocity of the sensor is a constant and sets $\dot{\phi}^2 = \omega^2$. However, we have found that if we assume that the total angular momentum of both the sensor and the object under measurement remains constant, the conservation of angular momentum principle (acting through the centrifugal force term $(-r\dot{\phi}^2)$) leads to a modification of the effective spring constant and to additional driving terms of the same order of magnitude as the obvious gravitational terms. Therefore, in order to obtain a correct expression for $-r\dot{\phi}^2$, we start from the law of conservation of angular momentum

$$\text{Ang. Momentum} = \Sigma\, m_i\,(\overline{r}_i \times \overline{P}_i) = 2m\ell^2\omega ; \qquad \text{(A-25)}$$

$$\therefore M_1(x_1\dot{y}_1 - y_1\dot{x}_1) + M_2(x_2\dot{y}_2 - y_2\dot{x}_2) + M_3(x_3\dot{y}_3 - y_3\dot{x}_3)$$

$$+ M_4(x_4\dot{y}_4 - y_4\dot{x}_4) = 2m\ell^2\omega .$$

For this part of the problem it can be assumed that $r_{13} = r_{23} = r$; then

$$x_1 = x_3 + r\cos\phi$$
$$x_2 = x_3 - r\cos\phi$$

$$y_1 = y_3 + r\sin\phi$$
$$y_2 = y_3 - r\sin\phi$$

$$\dot{x}_1 = \dot{x}_3 + \dot{r}\cos\phi - r\sin\phi\,\dot{\phi}$$
$$\dot{x}_2 = \dot{x}_3 - \dot{r}\cos\phi + r\sin\phi\,\dot{\phi}$$

$$\dot{y}_1 = \dot{y}_3 + \dot{r}\sin\phi + r\cos\phi\,\dot{\phi}$$
$$\dot{y}_2 = \dot{y}_3 - \dot{r}\sin\phi - r\cos\phi\,\dot{\phi} \quad .$$

From the center of mass location

$$M_4 x_4 + m(x_1 + x_2 + x_3) = 0 \quad , \tag{A-26}$$

but

$$x_1 + x_2 = 2x_3 \tag{A-27}$$

$$\therefore\ x_4 = -\frac{3M}{m} x_3 \quad .$$

Similarly,

$$y_4 = -\frac{3M}{m} y_3$$

$$\dot{x}_4 = -\frac{3M}{m} \dot{x}_3$$

$$\dot{y}_4 = -\frac{3M}{m} \dot{y}_3 \quad .$$

Substituting all this into (A-25) and expanding and collecting terms, we obtain an expression for the instantaneous angular velocity of the sensor:

$$r^2 \dot{\phi} = \ell^2 \omega - \left(\frac{3}{2} + \frac{9m}{2M}\right) \frac{M^2}{9m^2} (x_4 \dot{y}_4 - y_4 \dot{x}_4) \quad . \qquad \text{(A-28)}$$

Equation (A-28) shows that the angular velocity $\dot{\phi}$ of the sensor is a function of the motion of the object being measured. Since the sensor is a rotating mass quadrupole rather than a simple gravitating body, we find that the sensor and the mass under measurement do not fall smoothly toward each other. Superimposed on the expected radial acceleration is an oscillating radial motion as well as an oscillating transverse motion. This can be shown by taking the equations of motion for the mass M_4 (eqs. (A-1) and (A-2)), expanding the denominators of the gravitational terms, collecting terms of the same frequency, dropping the higher order terms, and substituting for higher powers of cos $n\omega t$ to obtain

$$\ddot{x}_4 = - \frac{Gm}{R^2} \left[3 + \frac{9}{2}\left(\frac{r}{R}\right)^2 \cos 2\omega t - \frac{35}{16}\left(\frac{r}{R}\right)^4 \cos 4\omega t\right] ; \qquad \text{(A-29)}$$

in a similar manner,

$$\ddot{y}_4 = - \frac{Gm}{R^2} \left[- 3 \left(\frac{r}{R}\right)^2 \sin 2\omega t - \frac{135}{8}\left(\frac{r}{R}\right)^4 \sin 4\omega t\right] . \qquad \text{(A-30)}$$

These expressions may be integrated directly if we assume that R, r, and ω are constant, the initial velocity of the mass M_4 is zero, and that the initial position is given by

$$x_4(0) = \frac{3mR}{M + 3m}$$

$$y_4(0) = 0 \quad .$$

When we carry out the integration and examine the angular momentum of the mass M_4, we find that

$$(x_4\dot{y}_4 - y_4\dot{x}_4) \approx x_4(0)\dot{y}_4 = - \frac{3mR}{M + 3m}\left(\frac{3Gmr^2}{2\omega R^4}\cos 2\omega t + \frac{35Gmr^4}{32\,\omega R^6}\cos 4\omega t\right) \quad \text{(A-31)}$$

because all other terms are of the order G^2 and are negligible.

Substituting (A-31) into (A-28) and solving for the instantaneous angular velocity $\dot{\phi}$, we obtain

$$\dot{\phi} \approx \frac{\ell^2}{r^2}\omega + \frac{3}{4}\frac{GM}{\omega R^3}\cos 2\omega t + \frac{35}{64}\frac{GM\ell^2}{\omega R^5}\cos 4\omega t \;. \quad \text{(A-32)}$$

We may now use this equation to obtain the desired centrifugal force term

$$- r\dot{\phi}^2 = - \frac{\ell^4\omega^2}{r^3} - \frac{3GM\ell}{2R^3}\cos 2\omega t - \frac{35}{32}\frac{GM\ell^3}{R^5}\cos 4\omega t \quad \text{(A-33)}$$

where in the gravitational terms we have assumed $r \approx \ell$ and have dropped all other terms in G^2 or $G\xi$. We now see that the centrifugal term is not constant, but contains gravitationally driven ac components that are of the same order of magnitude as the gravitational terms in (A-23) and (A-24).

We now use the fact that the radial extension of each spring consists of an initial length ℓ_o and extension due to the dc component of the centrifugal force e and a time varying vibration ξ_1, e.g.,

$$r_{13} = \ell_o + e + \xi_1 = \ell + \xi_1 \;, \quad \text{(A-34)}$$

and the tension in the spring contains not only the reaction to the extension of the spring, but also dissipation, e.g.,

$$T_{13} = k\,(e + \xi_1) + D\dot{\xi}_1 \;. \quad \text{(A-35)}$$

Using (A-33), (A-34), and (A-35) in (A-23) and (A-24), we can obtain equations for the radial motion of the springs which are expressed in terms of the time varying vibrations ξ and the initial centrifugal extension e of the springs.

$$\ddot{\xi}_1 + \frac{D}{m}(2\dot{\xi}_1 - \dot{\xi}_2) + \left(3\omega^2 + \frac{2k}{m}\right)\xi_1 - \frac{k}{m}\xi_2 + \left(\frac{k}{m}e - \ell_o\omega^2 - e\omega^2\right)$$

$$- \frac{9}{8}\frac{GM\ell^2}{R^4}\cos\omega t$$

$$- 3\frac{GM\ell}{R^3}\cos 2\omega t - \frac{15}{8}\frac{GM\ell^2}{R^4}\cos 3\omega t - \frac{105}{32}\frac{GM\ell^3}{R^5}\cos 4\omega t = 0 \qquad \text{(A-36)}$$

and

$$\ddot{\xi}_2 + \frac{D}{m}(2\dot{\xi}_2 - \dot{\xi}_1) + \left(3\omega^2 + \frac{2k}{m}\right)\xi_2 - \frac{k}{m}\xi_1$$

$$+\left(\frac{k}{m}e - \ell_o\omega^2 - e\omega^2\right) + \frac{9}{8}\frac{GM\ell^2}{R^4}\cos\omega t - 3\frac{GM\ell}{R^3}\cos 2\omega t$$

$$+ \frac{15}{8}\frac{GM\ell^2}{R^4}\cos 3\omega t - \frac{105}{32}\frac{GM\ell^3}{R^5}\cos 4\omega t = 0 \;. \qquad \text{(A-37)}$$

The equations are valid only if the constant terms are zero

$$\frac{k}{m}e - \ell_o\omega^2 - e\omega^2 = 0 \quad \text{or} \quad e = \frac{\ell_o\omega^2}{\frac{k}{m} - \omega^2} \qquad \text{(A-38)}$$

and this puts a condition on the strength of the spring usable in the sensor for any given rotational frequency. As long as k/m is appreciably greater than ω^2, the initial extension of the spring due to the centrifugal force is reasonable; if the sensor has a rotational speed greater than the natural static vibrational frequency determined by the spring constant $k/m = (2\pi f)^2$, however, the extension given by (A-38) becomes infinite and the sensor flies apart.

These combined equations of motion ((A-36) and (A-37)) then describe the behavior of the sensor as it is driven by the various gravitational terms at the various frequencies. If we wish to sense the driving force at twice the rotational frequency caused by the gravitational force gradient, we pick a spring constant so that excitations of the form

$$\xi_1 = A \sin 2\omega t \qquad \text{(A-39)}$$

$$\xi_2 = B \sin 2\omega t \qquad \text{(A-40)}$$

will predominate. Since under these conditions only the gravitational driving term at $2\omega t$ will be important, the equations of motion (A-36) and (A-37) become

$$\ddot{\xi}_1 + \frac{D}{m}(2\dot{\xi}_1 - \dot{\xi}_2) + \left(3\omega^2 + \frac{2k}{m}\right)\xi_1 - \frac{k}{m}\xi_2 = \frac{3GM\ell}{R^3}\cos 2\omega t \qquad \text{(A-41)}$$

$$\ddot{\xi}_2 + \frac{D}{m}(2\dot{\xi}_2 - \dot{\xi}_1) + \left(3\omega^2 + \frac{2k}{m}\right)\xi_2 - \frac{k}{m}\xi_1 = \frac{3GM\ell}{R^3}\cos 2\omega t \qquad \text{(A-42)}$$

where we have assumed that (A-38) holds. If we substitute (A-39) and (A-40) into (A-41) and (A-42), we find that there are two possible values for the spring constant that will cause a resonance at the driving frequency 2ω; these are

$$\frac{k}{m} = \omega^2 \qquad \text{(symmetric oscillation)}$$

$$\frac{k}{m} = \frac{1}{3}\omega^2 \qquad \text{(antisymmetric oscillation)} .$$

They are not at $k/m = (2\omega)^2$, as would be expected from a naive analysis, since the centrifugal force term and conservation of momentum conditions create an effective restoring force in addition to the restoring force of the spring.

Unfortunately, however, the two allowable spring constants that would permit resonance at 2ω in a rotating sensor frame of reference are too weak to allow (A-38) to be satisfied, and the sensor will fly apart.

If we wish to sense the driving force at three times the rotational frequency resulting from the gradient of the gravitational force gradient, we find that it is possible to pick the spring constant as either

$$\frac{k}{m} = 2\omega^2 \quad \text{(antisymmetric)}$$

or

$$\frac{k}{m} = 6\omega^2 \quad \text{(symmetric)} \ .$$

These correspond to an initial extension of the spring of (eq. (A-38))

$$e = \frac{\ell_o \omega^2}{\frac{k}{m} - \omega^2} = \frac{\ell_o}{2}$$

and

$$e = \frac{\ell_o}{5} \ ,$$

which are large but not unreasonable for a coil spring. Since the gravitational driving forces at 3ω in (A-36) and (A-37) are antisymmetric, the proper choice for the spring constant is $k/m = 2\omega^2$. At this frequency the resonant vibrations of the springs will have the form

$$\xi_1 = - \xi_2 = \frac{5}{24} \frac{GM\ell^2 Q}{\omega^2 R^4} \sin 3\omega t$$

where

$$Q = \frac{(3\omega)m}{D} \quad .$$

Similarly, to detect the fourth gradient of the gravitational potential which produces symmetric driving forces at 4ω, we choose our spring constant as

$$\frac{k}{m} = 13\,\omega^2 \qquad e = \frac{\ell_o}{12}$$

and the solutions of the equations of motion are

$$\xi_1 = \xi_2 = \frac{35}{128} \frac{GM\ell^3 Q}{\omega^2 R^5} \sin 4\omega t$$

where

$$Q = \frac{(4\omega)m}{D} \quad .$$

It may be noted that if the central sensor mass were to go to zero, only symmetric response would be possible since

$$\xi_1 \equiv \xi_2$$

and we find that only even gradients of the gravitational potential can be measured with a two mass, one spring sensor in free fall. However, as a previous analysis has shown, the sensor cannot be used to measure the second order gradient because of the infinite extension predicted by (A-38).

If the analysis is repeated with the central sensor mass of 10 times the outer sensor masses, we find that for response at ωt

$$\frac{k}{m} = -\,2\,\omega^2$$

$$\frac{k}{m} = -\,\frac{5}{3}\,\omega^2$$

for response at $2\,\omega t$

$$\frac{k}{m} = \omega^2$$

$$\frac{k}{m} = \frac{5}{6}\,\omega^2$$

for response at $3\,\omega t$

$$\frac{k}{m} = 6\omega^2$$

$$\frac{k}{m} = 5\omega^2$$

and for response at $4\,\omega t$

$$\frac{k}{m} = 13\,\omega^2$$

$$\frac{k}{m} = \frac{65}{6}\,\omega^2 \quad .$$

It may now be seen that the response to the frequency component at $2\omega t$ is controlled by $\omega^2 = k/m$, regardless of the size of the central sensor mass; if this value of ω^2 is substituted into (A-38) the value of e becomes infinite (i.e., the spring is stretched beyond its distortion point).

Therefore the $3\,\omega$ or 4ω responses must be observed in this type of sensor in order to obtain gravity gradient data. Thus, in general, it does not seem possible to measure the gravitational force gradient which introduces forces at 2ω with a radially vibrating sensor of this design.

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ATTACHMENT

The following paper, which is tutorial in nature, presents an extended discussion of the tensor properties of gravitational and inertial fields. It is a modified version of the original paper which was released as Hughes Research Laboratories Research Report No. 301 in March 1964. It is planned to publish it in the American Journal of Physics.

The Principle of (Non)Equivalence and Its Application to Gravitational and Inertial Sensors†

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ABSTRACT

It is often thought that the Principle of Equivalence used by Einstein as the starting point for his General Theory of Relativity is valid under all circumstances; thus, it is often said that because of the Principle of Equivalence, there is no way to distinguish between gravitational and inertial effects. This paper emphasizes that the correct definition of the Principle of Equivalence is so restricted mathematically that it has little relationship to the real world of experimental physics. It is demonstrated that in fact it is possible to devise an instrument to measure independently the effects of acceleration, rotation, and gravitation.

INTRODUCTION

The Principle of Equivalence of gravitational and inertial forces was used by Einstein as a philosophical starting point for the development of the General Theory of Relativity. The success of the field equations of general relativity in predicting non-Newtonian gravitational effects, such as the precession of the perihelion of Mercury, has lead to a tendency to elevate the Principle of Equivalence to the level of a physical

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law that is valid under all circumstances. The statement is sometimes heard that because of the Principle of Equivalence there is <u>no</u> way to distinguish between gravitational and inertial effects.[1]

The proper definition of the Principle of Equivalence is[2]:

> "It is always possible at a point in space-time to transform to a (in general accelerated) co-ordinate system such that the effects of gravity will disappear over a differential region in the neighborhood of the point."

This definition states that the Principle of Equivalence is only valid over a differential region. This requirement is made in order that the gravitational field can be assumed to be uniform. The Principle of Equivalence can be applied over a larger region only when the gravitational field is uniform over that region. In reality, however, the gravitational field of a mass is far from uniform and the "uniform gravitational field" turns out to be a convenient mathematical fiction.

Real gravitational fields have gradients in their vector field patterns. Thus, gradient sensors such as differential accelerometers can distinguish between real gravitational fields and inertial effects due to accelerated reference frames. As is shown in a later section, it is even possible to design a sensor which can distinguish between the effects of gravitation, rotation, and acceleration.[3-18]

It must be emphasized that because a gradient sensor can detect the difference between gravitational and inertial effects, this does not mean that the Principle of Equivalence is invalid. By their very nature, gradient sensors must operate over an extended region of space and thus they do not satisfy the assumptions of the definition of the Principle of Equivalence.

PRINCIPLE OF (NON)EQUIVALENCE

In a completely specious manner we now propose a new principle called the "Principle of (Non)Equivalence."

> "It is always possible at a point in space-time to distinguish between a (in general accelerated) coordinate system and the effects of gravity provided a large enough region is taken around the neighborhood of the point."

Of course one can think of mass distributions that nearly invalidate this principle; for example, a flat disc of thickness t, radius R and density ρ has a gravitational field near the center that is almost completely uniform[19]

$$a = -2\pi G \rho t \left[1 - \frac{r}{(R^2 + r^2)^{1/2}} \right] \approx -2\pi G \rho t \qquad (1)$$

where $r \ll R$.

For almost all real experimental situations, however, the problem is not one of a fundamental nature; rather, it is a practical one of measuring the very weak gravitational force field in the presence of the much larger inertial force fields. In order to do this, it is necessary to develop a class of sensors capable of using the differences between gravitational and inertial forces in such a way that they completely ignore the large inertial fields and respond only to the gravitational field.

INERTIAL FIELDS

When nongravitational forces act on a vehicle, they cause both acceleration and rotation. In order to sense these effects, we must measure the reaction of a mass to these inertial force fields.

Acceleration Inertial Field

The linear acceleration of a vehicle of mass m due to an applied force F creates a uniform inertial field in the frame of reference of the vehicle which has purely vector properties and no spatial gradients (see Fig. 1):

$$a_i = \frac{1}{m} F_i = \frac{1}{m} (F_x, F_y, F_z) . \tag{2}$$

The accelerating force field can be detected by any force or acceleration measuring device, such as an accelerometer.

Rotation Inertial Field

If the vehicle using the sensor is rotating, the rotation sets up a cylindrically symmetric inertial field (see Fig. 2).

$$a_j = \Omega^2 d_j = (\Omega^2 x, \Omega^2 y, 0) \tag{3}$$

where Ω is the angular velocity and d is the position vector from the axis of rotation. (For purposes of clarity, we have chosen the rotation axis along the z axis.) This acceleration field not only has a radial gradient resulting from the change in the magnitude of the acceleration vector with a change in radius, but also a tangential gradient due to the change in direction of the acceleration vector with a change in angle.

The resultant acceleration gradient field is a tensor field and is unusual in that it is zero in the direction of the rotation axis and has a value of Ω^2 in the directions at right angles to the rotation axis.

$$R_{ij} = \nabla_i a_j = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} (\Omega^2 x, \Omega^2 y, 0) = \begin{pmatrix} \Omega^2 & 0 & 0 \\ 0 & \Omega^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{4}$$

This gradient is constant and has no higher order gradients.

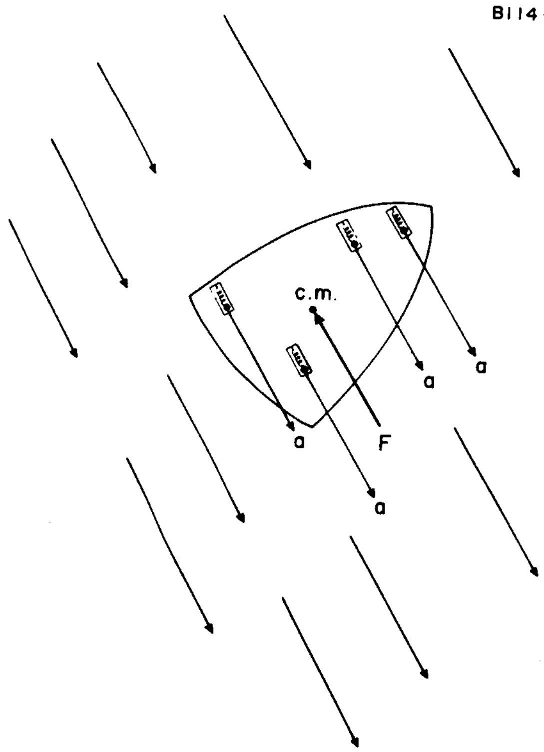

Fig. 1. The uniform inertial reaction acceleration field created by a force on a body.

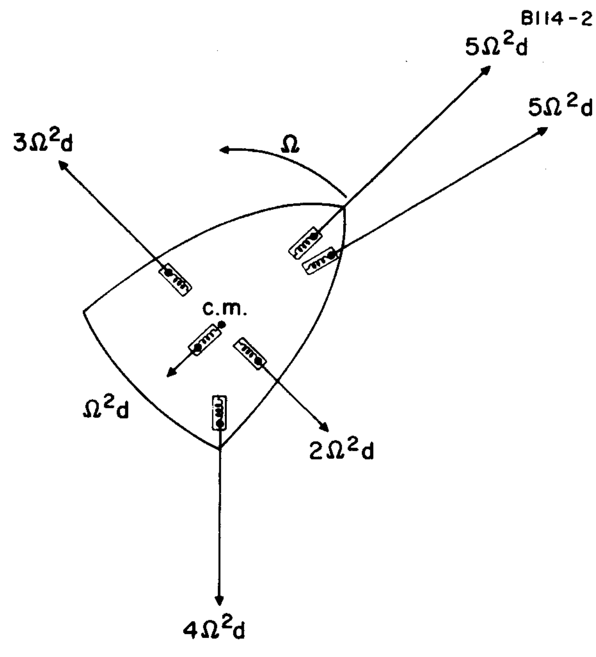

Fig. 2. The radially increasing, cylindrically symmetric, inertial reaction acceleration field created by a rotation of a body.

GRAVITATIONAL FIELD

In order to sense the gravitational field of an object, either the gravitational potential or one of its gradients must be measured.

Gravitational Potential

According to Newton's law of gravitation, a mass M characteristically sets up a field in the space around it, which interacts with other masses. If a small test mass m is placed at a distance R from the first mass, it is found that the system has a potential energy given by

$$\phi = -\frac{GMm}{R} \tag{5}$$

where $G = 6.67 \times 10^{-11}\ m^3/kg\ sec^2$. Strictly speaking, the above formula applies only to a spherically symmetric mass, but the concept can be extended to more complicated distributions of mass by simply adding the contributions of each part of the distribution.

The gravitational potential is not directly measurable since the point of zero reference can be changed arbitrarily. Differences in potential energy can be measured by allowing the masses to attract each other and measuring the change in kinetic energy. The experiments using the Mössbauer effect are an example of this.

Gravitational Force

The gradient of the potential is the gravitational force field. Since the inertial mass and the gravitational mass are the same for all bodies, the gravitational force field is equivalent to a gravitational acceleration field.

$$a_k = \frac{1}{m} F_k = -\frac{1}{m} \nabla_k \phi \tag{6}$$

$$= -\frac{GM}{(x^2 + y^2 + z^2)^{3/2}} (x,\ y,\ z) \text{ in Cartesian coordinates}$$

$$= \left(-\frac{GM}{R^2},\ 0,\ 0\right) \text{ in spherical coordinates.}$$

This accelerating force field can be detected by any force or acceleration measuring device such as an accelerometer or gravity meter, provided the center of mass of the sensing device and the object being investigated are not moving with respect to each other (see Fig. 3).

The Gravitational Force Gradient

If the object under measurement is in free fall with respect to the sensor, the only measurable components of the gravitational field are the gravitational force gradients which are the components of a symmetric tensor.

$$\Gamma_{ij} = \nabla_i \nabla_j \frac{\phi}{m} = \frac{1}{m} \begin{vmatrix} \frac{\partial^2 \phi}{\partial x \, \partial x} & \frac{\partial^2 \phi}{\partial y \, \partial x} & \frac{\partial^2 \phi}{\partial z \, \partial x} \\ \frac{\partial^2 \phi}{\partial x \, \partial y} & \frac{\partial^2 \phi}{\partial y \, \partial y} & \frac{\partial^2 \phi}{\partial z \, \partial y} \\ \frac{\partial^2 \phi}{\partial x \, \partial z} & \frac{\partial^2 \phi}{\partial y \, \partial z} & \frac{\partial^2 \phi}{\partial z \, \partial z} \end{vmatrix} . \qquad (7)$$

The gravitational force gradient is best known to us as the tides on the earth due to the gravitational field of the sun. Since the amplitude of the gravitational force due to the sun varies as the inverse square of the distance from the sun, and since the direction of the force vector varies with angle, we can see from Fig. 4 that the gravitational force due to the sun varies from point to point on the earth. If we look at these force vectors from the viewpoint of the center of mass of the earth, we see that after subtracting out the center of mass motion, we are left with a radial tension and tangential compression. It is important to realize that the effects of the slight angular convergence are of the same order of magnitude as the radial gradient effects. This will always be true and the angular effects must always be included for a correct calculation of gradients.

The gravitational force gradient field of an object can be measured by a gradiometer. The usual form of a gradiometer consists of two accelerometers on the ends of a rod of length ℓ. In this form, the tension,

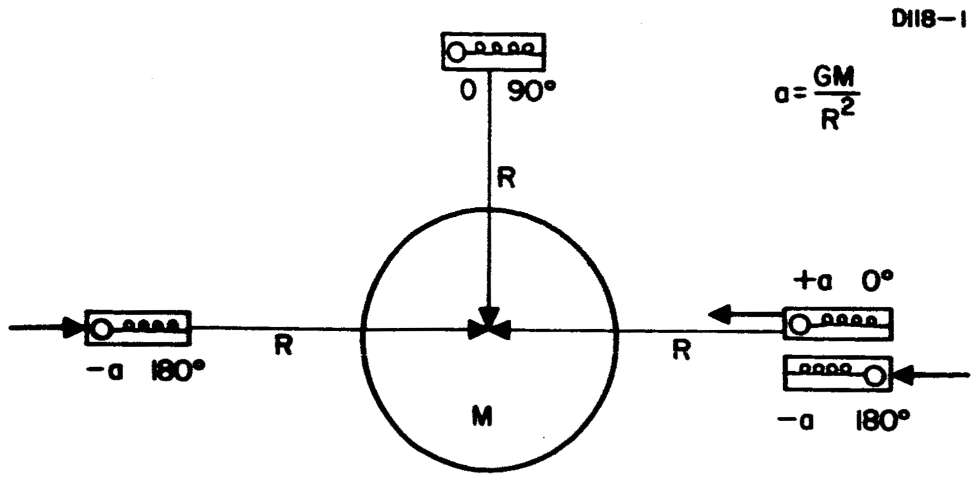

Fig. 3. Accelerometer reading as a function of relative orientation to attracting mass.

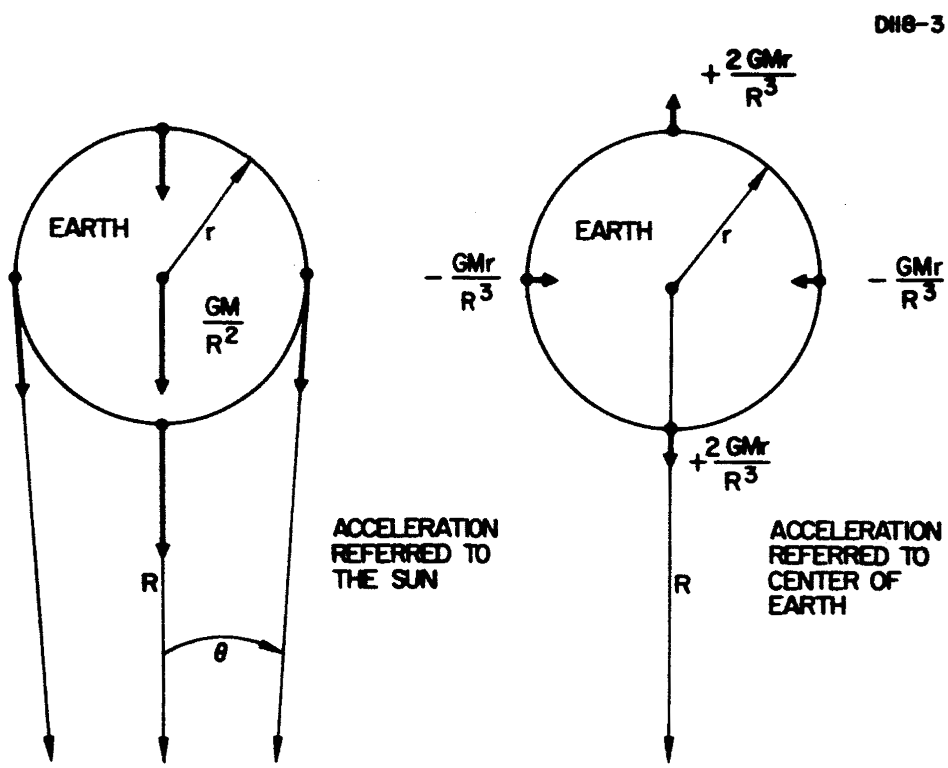

Fig. 4. Radial and azimuthal contributions to the earth's tides.

compression, or torque due to the gradient will cause the accelerometers to indicate an acceleration given by

$$a_i = \sum_{j=1}^{3} \Gamma_{ij} \ell_j \tag{8}$$

where we have assumed that the rod is lying in the $\pm j$ direction and the accelerometers are oriented in the $\pm i$ direction (see Fig. 5).

If we have a simple mass to measure, by proper orientation of the sensor the gravitational force gradient tensor can be simplified to

$$\Gamma_{ij} = \begin{vmatrix} \Gamma_{rr} & 0 & 0 \\ 0 & \Gamma_{tt} & 0 \\ 0 & 0 & \Gamma_{tt} \end{vmatrix} \tag{9}$$

which consists of the radial gravitational force gradient

$$\Gamma_{rr} = +\ 2\ \frac{GM}{R^3} \tag{10}$$

and the tangential gravitational force gradient

$$\Gamma_{tt} = -\ \frac{GM}{R^3}\ . \tag{11}$$

These gradients cause tensions and compressions in a gradiometer oriented either radially or tangentially (see Figs. 5(a) and 5(b)).

If the sensor is located 45° to this elementary orientation, there is a torsional component

$$\Gamma_{xy}\ (\phi = 45^\circ) = \frac{3}{2}\ \frac{GM}{R^3} \tag{12}$$

as well as a reduced tension component.

$$\Gamma_{xx}\ (\phi = 45^\circ) = \frac{1}{2}\ \frac{GM}{R^3} \tag{13}$$

(See Fig. 5(c)).

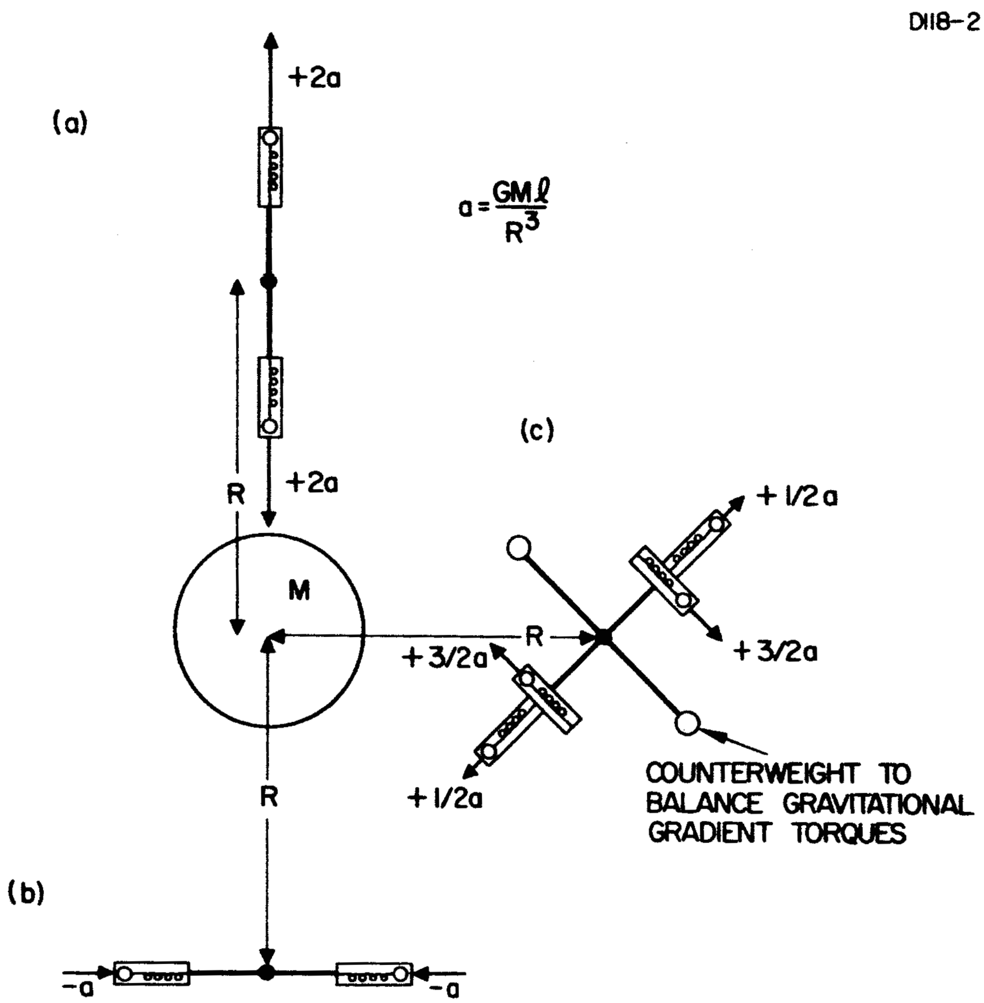

Fig. 5. Types of gravitational gradients and sensor configurations. (a) Case showing radial tension; (b) case showing tangential compression; (c) more general case showing tension and torque.

Higher Order Gradients

Unlike the rotational inertial field which has a uniform force gradient and therefore has no higher order gradients, there is essentially no limit to the number of higher gravitational gradients that can be measured, provided the sensor is close enough and the object under investigation is massive enough that the interaction overcomes the sensor noise. These higher order gradients are complicated tensors of high rank, and sophisticated techniques and sensors may be able to obtain a great deal of information from them. Basically, they have the form:

$$T_{ab\cdots n} = \frac{1}{m} \frac{\partial^n \phi}{\partial x^a \, \partial x^b \ldots \partial x^n} \approx \frac{GM}{R^n} \quad . \tag{14}$$

DIFFERENTIAL INERTIAL SENSOR

In order to demonstrate a portion of the Principle of (Non)Equivalence, let us design a device that can distinguish between the effects of a generalized acceleration and a generalized rotation. Once we have shown that the inertial fields can be separated in the general case, we can justify the use of more simplified forms of inertial fields in analyses that contain the gravitational fields. Since the rotational inertial field has a gradient and the acceleration-al inertial field does not, a differential accelerometer should be able to distinguish between the effects of rotation and acceleration. To be able to detect all the components of the rotation and acceleration we need three sets of orthogonally oriented differential accelerometers (see Fig. 6). Each accelerometer will respond to the component of the acceleration force along its active axis and to the combined centrifugal force due to the rotations about the two axes orthogonal to its active axis. The six accelerometers then give us six different readings

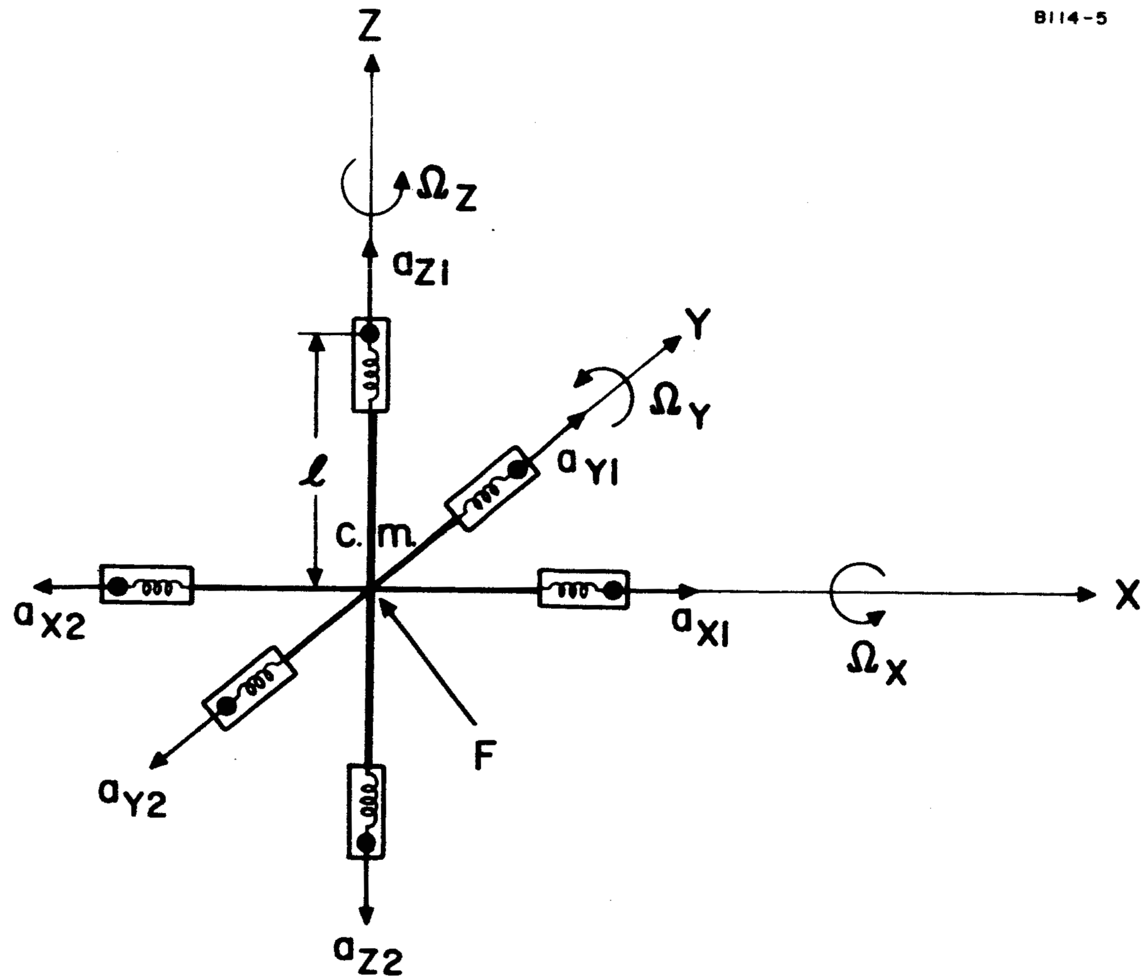

Fig. 6. Differential inertial sensor.

$$a_{x1} = \frac{F_x}{m} + \left(\Omega_y^2 + \Omega_z^2\right)\ell \tag{15}$$

$$a_{x2} = -\frac{F_x}{m} + \left(\Omega_y^2 + \Omega_z^2\right)\ell \tag{16}$$

$$a_{y1} = \frac{F_y}{m} + \left(\Omega_z^2 + \Omega_x^2\right)\ell \tag{17}$$

$$a_{y2} = -\frac{F_y}{m} + \left(\Omega_z^2 + \Omega_x^2\right)\ell \tag{18}$$

$$a_{z1} = \frac{F_z}{m} + \left(\Omega_x^2 + \Omega_y^2\right)\ell \tag{19}$$

$$a_{z2} = -\frac{F_z}{m} + \left(\Omega_x^2 + \Omega_y^2\right)\ell \quad . \tag{20}$$

It is fairly obvious that the three components of the accelerating force can be obtained by subtracting the outputs of the accelerometer pairs

$$\frac{F_i}{m} = \frac{a_{i1} - a_{i2}}{2} \tag{21}$$

where $i = 1, 2, 3$.

It is not so obvious, but still true, that the magnitude of the three components of the rotation at any instant are given by a combination of the sums of the outputs of the accelerometer pairs:

$$\Omega_k^2 = \frac{(a_{i1} + a_{i2}) + (a_{j1} + a_{j2}) - (a_{k1} + a_{k2})}{4\ell} \tag{22}$$

where $i, j, k = 1, 2, 3 \rightarrow 3, 1, 2$.

Thus, three sets of orthogonally oriented differential accelerometers can be used to distinguish acceleration from rotation. This is not so surprising since a blindfolded person is also capable of the same distinction. (He is also capable of telling the rotation sense since his inner canal utilizes the Coriolis effect.) But we know that a blindfolded person is not capable of distinguishing between an accelerated reference frame and a gravitational field. Can we design an instrument that can measure separately the forces due to acceleration, rotation and gravitation?

DIFFERENTIAL GRAVITATIONAL SENSOR

It has long been known in the published literature[3-18] that a differential force measuring device can be used to distinguish between gravitational and inertial effects. These devices are usable in free fall and can determine, without outside reference, whether they are in free fall in deep space or in orbit around a planet. If they are in orbit, they can also determine the plane of the orbit, the direction of the local vertical with respect to the planet, and, if the planet (and its mass) are known, the radial distance to the planet.

We demonstrate the general concept with a simplified, two-dimensional version of the differential inertial sensor in Fig. 6. This uses only two orthogonal pairs of differential accelerometers, and, to keep the analysis simple, we also assume that the gravitating mass, the accelerating force F, and the plane of the sensor rotation Ω are all in the plane of the accelerometers (see Fig. 7).

The acceleration experienced by the four accelerometers is given by

$$a_{x1} = \frac{F_x}{m} + \Omega^2 \ell + \frac{GM}{(R+\ell)^2} \tag{23}$$

$$a_{x2} = -\frac{F_x}{m} + \Omega^2 \ell - \frac{GM}{(R-\ell)^2} \tag{24}$$

$$a_{y1} = \frac{F_y}{m} + \Omega^2 \ell - \frac{GM}{\rho^2}\,\frac{\ell}{\rho} \tag{25}$$

$$a_{y2} = -\frac{F_y}{m} + \Omega^2 \ell - \frac{GM}{\rho^2}\,\frac{\ell}{\rho} \tag{26}$$

where $\rho^2 = (R^2 + \ell^2)^{1/2}$.

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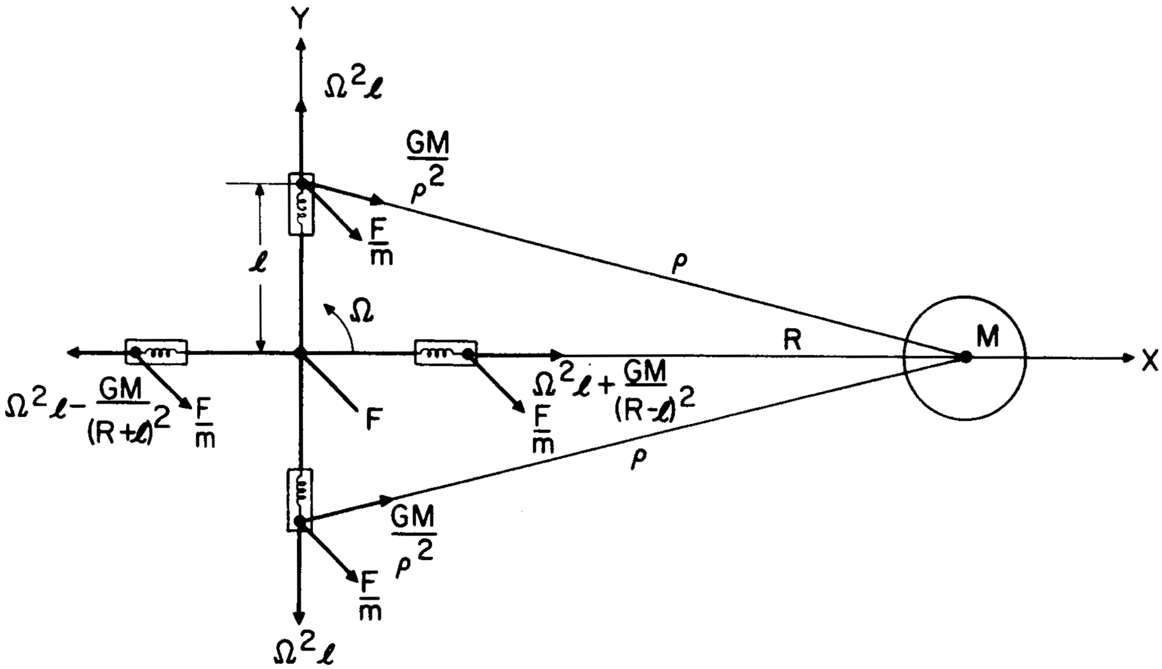

Fig. 7. Differential gravitational sensor.

If we expand the gravitational terms into the gravitational force component and the gravitational gradient component we obtain

$$a_{x1} = \frac{F_x}{m} + \Omega^2 \ell + \frac{GM}{R^2} - \frac{2GM\ell}{R^3} + \dots \qquad (23a)$$

$$a_{x2} = -\frac{F_x}{m} + \Omega^2 \ell - \frac{GM}{R^2} - \frac{2GM\ell}{R^3} + \dots \qquad (24a)$$

$$a_{y1} = \frac{F_y}{m} + \Omega^2 \ell + \frac{GM\ell}{R^3} + \dots \qquad (25a)$$

$$a_{y2} = -\frac{F_y}{m} + \Omega^2 \ell + \frac{GM\ell}{R^3} + \dots \quad . \qquad (26a)$$

If we add the two pairs, we obtain the total of the two gradient fields, the rotational field, and the gravitational field.

$$a_{x1} + a_{x2} = 2\Omega^2 \ell - \frac{4GM\ell}{R^3} \qquad (27)$$

$$a_{y1} + a_{y2} = 2\Omega^2 \ell + \frac{2GM\ell}{R^3} \quad . \qquad (28)$$

Note that whereas the rotational gradient causes a tension in both pairs, the gravitational gradient causes a tension in one pair and a compression in the other. We can therefore obtain the gravitational gradient term by subtracting the two sums:

$$\frac{6GM\ell}{R^3} = (a_{y1} + a_{y2}) - (a_{x1} + a_{x2}) \qquad (29)$$

and the rotational gradient term by adding

$$6\Omega^2 \ell = 2(a_{y1} + a_{y2}) + (a_{x1} + a_{x2}) \quad . \qquad (30)$$

In a similar manner, by taking the difference between the accelerometers, we can obtain the forces on the sensor, including the component of force needed in the x direction to prevent acceleration of the center of mass of the sensor in the gravitational force field (GM/R^2). Of course, if there were no such support force, the sensor would be in free fall, the center of mass would move with the acceleration $a = GM/R^2$, and the accelerations read by the x-direction accelerometers would not contain the GM/R^2 terms.

We have demonstrated the known fact that all types of differential accelerometers are capable of distinguishing between acceleration, rotation, and gravitation. However, the problem has always been an academic one, because none of the differential accelerometers produced to date have the physical ability to measure the very weak gravitational gradient forces in the presence of the unavoidable and much larger acceleration and rotation forces. A very good accelerometer is only capable of a linearity of one part in 10^5, and two accelerometers cannot be matched in output to anywhere near this degree of accuracy. Thus, it has not been possible to make differential accelerometers whose outputs could be added to cancel out the force terms in order to obtain the rotation and gravitation terms.

We propose a new type of differential accelerometer, based on the rotational properties of tensors, which is able to respond only to the gradients of a force field and which can completely ignore a linear acceleration.

ROTATIONAL PROPERTIES OF TENSORS

A scalar is a tensor of zero rank. At any given point in space it can be expressed by a single number which is independent of the coordinate system in which it is measured.

$$A = a \quad . \tag{31}$$

A scalar can exhibit time varying properties by changing its magnitude with time.

$$A = a(t) \quad . \tag{32}$$

A vector is a tensor of first rank. At any given point in space it can be expressed by three numbers which are its components along the axes of some coordinate system defined at that point.

$$A_i = (a_1, a_2, a_3) \quad . \tag{33}$$

A vector cannot only change its magnitude with time, but it also can change its direction by rotation. The rotation is expressed mathematically by a matrix of second rank. For example, if the vector is rotating about the z axis at the angular rate ω, the effect of the rotation $\phi = \omega t$ at any time t is given by

$$S_{ij} = \begin{pmatrix} \cos \omega t & \sin \omega t & 0 \\ -\sin \omega t & \cos \omega t & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad . \tag{34}$$

If a vector is originally lying along the x axis of a coordinate system and either the coordinate system or the vector is rotating about the z axis, the components of the vector expressed in terms of the coordinate axes are given by

$$A_i = \sum_{j=1}^{3} S_{ij} A_j = \begin{pmatrix} \cos \omega t & \sin \omega t & 0 \\ -\sin \omega t & \cos \omega t & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_x \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} a_x \cos \omega t \\ -a_x \sin \omega t \\ 0 \end{pmatrix} . \tag{35}$$

Notice that the new representation of the vector has sinusoidally time-varying components with a frequency that is equal to the relative rotation of the vector and the coordinate system.

What is usually called a tensor is actually a tensor of second rank. At any given point in space it can be expressed by nine numbers which are dependent upon the coordinate system in which they are defined:

$$A_{ij} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \quad . \tag{36}$$

A tensor cannot only change its magnitude with time, but it can also change its direction by rotation. The rotation of a tensor is handled mathematically by the formula

$$A_{ab} = \sum_{i=1}^{3} \sum_{j=1}^{3} S_{ai} S_{bj} A_{ij} \quad . \tag{37}$$

For a general tensor subjected to a rotation about the z axis, the form of the rotated tensor is a fairly complicated one.

$$A_{ab} = \begin{pmatrix} a_{11}\cos^2 \omega t + a_{22}\sin^2 \omega t & a_{12}\cos^2 \omega t - a_{21}\sin^2 \omega t & a_{13}\cos \omega t + a_{23}\sin \omega t \\ +(a_{12}+a_{21})\sin \omega t \cos \omega t & +(a_{22}-a_{11})\sin \omega t \cos \omega t & \\ a_{21}\cos^2 \omega t - a_{12}\sin^2 \omega t & a_{22}\cos^2 \omega t + a_{11}\sin^2 \omega t & a_{23}\cos \omega t - a_{13}\sin \omega t \\ +(a_{22}-a_{11})\sin \omega t \cos \omega t & -(a_{12}+a_{21})\sin \omega t \cos \omega t & \\ a_{31}\cos \omega t + a_{32}\sin \omega t & a_{32}\cos \omega t - a_{31}\sin \omega t & a_{33} \end{pmatrix} . \tag{38}$$

Because of the double application of the rotation matrix we now have products of $\sin \omega t$ and $\cos \omega t$ in some of the components of the rotated tensor. It is well known that these can be reduced to combinations of constant terms and terms in $\cos 2\omega t$ and $\sin 2\omega t$. Thus, in general, a rotating tensor of second rank has sinusoidally time-varying components which have a frequency twice the relative rotational frequency of the tensor and the coordinate system.

The tensors of higher rank rotate by the general rule

$$A_{ab\ldots ef} = \sum_{h=1}^{3} \sum_{i=1}^{3} \cdots \sum_{m=1}^{3} \sum_{n=1}^{3} S_{ah} S_{bi} \cdots S_{em} S_{fn} A_{hi\ldots mn} \tag{39}$$

and, in general, a tensor of n^{th} rank will have time-varying coefficients that are at n times the rotational frequency.

To give some specific physical examples, let us assume that we define a coordinate system with three orthogonal pairs of differential accelerometers. Then the force gradient due to the gravitational field of a mass M at a distance R along the x axis is given by:

$$\Gamma_{ij} = \frac{GM}{R^3}\begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} . \tag{40}$$

Now, if we rotate the mass in the x-y plane around the differential accelerometer (to eliminate temporarily the problem of centrifugal forces), the resulting gradient in the reference frame of the differential accelerometer is

$$\Gamma_{k\ell} = \sum_{i=1}^{3} \sum_{j=1}^{3} S_{ki}\, S_{\ell j}\, \Gamma_{ij} = \frac{GM}{R^3}\begin{pmatrix} \frac{1}{2} + \frac{3}{2}\cos 2\omega t & -\frac{3}{2}\sin 2\omega t & 0 \\ -\frac{3}{2}\sin 2\omega t & \frac{1}{2} - \frac{3}{2}\cos 2\omega t & 0 \\ 0 & 0 & -1 \end{pmatrix} . \tag{41}$$

Force gradients can thus be distinguished by their frequency behavior in a rotating reference frame. The existence of the doubled frequency in the components of a gradient makes sense physically, since a gradient can be represented by a double headed arrow; if it is turned through 180°, the new orientation is identical to the one at 0°. The gradient sensors shown in Fig. 5 also have this property. If, instead of rotating the mass, we had rotated the sensor, the rotation would have created the inertial gradient

$$R_{ij}(z) = \begin{pmatrix} \omega^2 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{42}$$

In this gradient, $a_{11} = a_{22}$; when we substitute this back into the general expression (38) for the form of the rotated tensor, we find that there are no time-varying components to the rotated tensor.

$$R_{ab}(z) \equiv \sum_{i=1}^{3} \sum_{j=1}^{3} S_{ai} S_{bj} R_{ij}(z) = \begin{pmatrix} \omega^2 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} = R_{ij}(z) \quad . \qquad (43)$$

Thus, although the rotation of the sensor about the z axis creates a force gradient, the gradient is constant in the x-y plane and time-varying components do not occur. This means that a differential accelerometer rotating about its z axis can distinguish gravitational gradients from rotational gradients in the x-y plane by the frequency difference between the two gradients.

However, if the differential accelerometer is slowly precessing about some other axis, e.g., at a rate Ω about the y axis, there will exist a rotational gradient given by

$$R_{ij}(y) = \begin{pmatrix} \Omega^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \Omega^2 \end{pmatrix} . \qquad (44)$$

This gradient does not have the degeneracy exhibited by the gradient due to rotation about the z axis in that $a_{11} \neq a_{22}$, therefore it will have time varying components when viewed in the rotating reference frame of the rotating sensor.

The combined effects of the three gradients, one ($R_{ij}(z)$) due to the sensor rotation about the z axis at an angular velocity ω, one ($R_{ij}(y)$) due to the sensor precession about the y axis at an angular velocity Ω, and one (Γ_{ij}) due to a mass M out at a distance R along the x axis is found to have the following components in the rotating reference frame of the sensor:

$$A_{ab} = \begin{pmatrix} \left(\omega^2 + \frac{\Omega^2}{2} + \frac{GM}{2R^3}\right) & -\left(\frac{\Omega^2}{2} + \frac{3GM}{2R^3}\right)\sin 2\omega t & 0 \\ +\left(\frac{\Omega^2}{2} + \frac{3GM}{2R^3}\right)\cos 2\omega t & & \\ -\left(\frac{\Omega^2}{2} + \frac{3GM}{2R^3}\right)\sin 2\omega t & \left(\omega^2 + \frac{\Omega^2}{2} + \frac{GM}{2R^3}\right) & 0 \\ & +\left(\frac{\Omega^2}{2} + \frac{3GM}{2R^3}\right)\cos 2\omega t & \\ 0 & 0 & \Omega^2 - \frac{GM}{R^3} \end{pmatrix} \tag{45}$$

Notice that everywhere a gravitational gradient term appears, there is also a term due to the slow precession Ω. It is theoretically possible to separate the two effects by proper combinations of the time-varying portion of a_{11} or a_{22} and the static portion of a_{33}; however, it is difficult to do so in practice because it is necessary to find a small difference between two large numbers. A very high precession rate is not necessary in order to generate a rotational force gradient that is larger than the earth's gradient. The earth's gradient is

$$\frac{GM}{R^3} = 1.5 \times 10^{-6}\ \text{sec}^{-2} \quad . \tag{46}$$

For a rotational gradient to have the same magnitude as the earth's gradient,

$$\Omega^2 = 1.5 \times 10^{-6} \left(\frac{\text{rad}}{\text{sec}}\right)^2 \quad , \tag{47}$$

a rotational rate of only

$$\Omega = 1.2 \times 10^{-3}\ \text{rad/sec} \tag{48}$$

or 0.012 rpm is required. In order to separate the gravitational effects from all the rotational effects by frequency discrimination, it is therefore necessary to utilize the fact that there are no higher order gradients to the rotational force field, whereas the gravitational force field has an unlimited number of higher order gradients.

To measure these higher order gradients, complicated sensors with multiplicated lengths and differential pairs of differential accelerometers should be used. However, if a first order gradient sensor such as a differential accelerometer with a single length is used, these higher order gradients show up as nonlinear perturbations in the gravitational force gradient field. These perturbations have periodic variations with the orientation of the sensor that will allow them to be singled out.

ROTATING NONUNIFORM GRADIENT SENSOR

As a simplified example of a sensor that can be used to detect either acceleration, rotation, or gravitation by frequency selection, let us assume the configuration shown in Fig. 8. Here we have an accelerometer at the end of a rod of length ℓ rotating around a fixed point at a distance r from the center of mass of the using vehicle.* The rod ℓ is forced to move at a constant angular velocity ω. The vehicle is subject to forces F and has a residual inertial rotation Ω_z ($\ll \omega$) about the z axis and a residual rotation Ω_x ($\ll \omega$) about the x axis. A nearby mass M is in the x-y plane at a distance R from the center of the sensor.

The accelerometer will now respond to all these effects. The inertial linear acceleration reaction of the mass in the accelerometer will be directed opposite to the force F and will have the amplitude $a = F/m$, where m is the mass of the using vehicle. The inertial rotation reaction of the mass in the accelerometer will have three components: one with amplitude $\omega^2\ell$ directed radially outward from the axis of rotation of the sensor, one with amplitude $\Omega_z^2 d = \Omega_z^2 (r^2 + \ell^2 + 2r\ell \cos \omega t)^{1/2}$ radially outward from the z axis through the c.m. of the vehicle, and one of amplitude $\Omega_x^2 \ell \sin \omega t$ directed outward from the x axis. The gravitation

*An accelerometer is being used in this example for clarity of the analysis. A real accelerometer would be saturated by the rotation necessary to accomplish the desired frequency separation. More practical rotating sensors are described in Refs. 9, 14, 15 and 17.

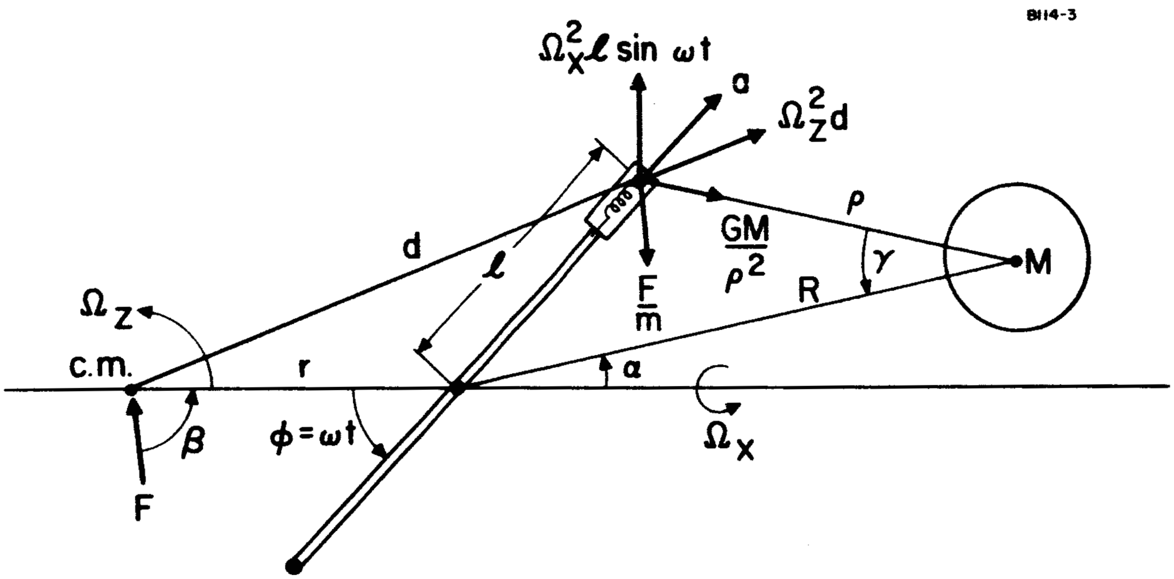

Fig. 8. Effect of gravitational and inertial fields on rotating sensor.

reaction of the mass in the accelerometer will have the amplitude GM/ρ^2 and will be in the direction of the gravitating mass. However, the accelerometer responds only to the components which are along its sensitive axis, and this sensitive axis is rotating due to the rotation of the rod. Thus the rotation of an accelerometer on the end of the rod not only changes the amplitude of the various forces by changing the effective length of the sensor with time, but the rotation also changes the directional response of the sensor to these time-varying amplitudes.

Because of all these forces the acceleration read by the accelerometer at the end of the rod will vary with the angle of rotation in a complicated manner.

$$a = \frac{F}{m} \cos(\omega t + \beta) + \Omega_z^2 d \cos \omega t + \omega^2 \ell + \frac{GM}{\rho^2} \cos(\omega t + \gamma - \alpha) + \Omega_x^2 \ell \sin^2 \omega t \,. \tag{49}$$

The primary effect results from the directional sensitivity of the accelerometer to the various accelerations, but the effects that interest us arise from the change in the gravitational force with angle, distance, and time. If we simplify the inertial terms and keep all the components, and expand the gravitational term, but keep only the highest order terms of each frequency component, we obtain

$$\begin{aligned} a = {} & \omega^2 \ell + \frac{1}{2} \Omega_x^2 \ell + \Omega_z^2 \ell + \frac{GM}{R^2} \left[\frac{1}{2} \frac{\ell}{R} + \cdots \right] \\ & - \frac{F}{m} \cos(\omega t + \beta) - \Omega_z^2 r \cos \omega t + \frac{GM}{R^2} \left[\cos(\omega t - \alpha) + \cdots \right] \\ & - \frac{1}{2} \Omega_x^2 \ell \cos 2\omega t + \frac{GM}{R^2} \left[\frac{3}{2} \frac{\ell}{R} \cos 2(\omega t - \alpha) + \cdots \right] \\ & + \frac{GM}{R^2} \left[\frac{15}{8} \left(\frac{\ell}{R} \right)^2 \cos 3(\omega t - \alpha) + \cdots \right] \\ & + \frac{GM}{R^2} \left[\frac{35}{16} \left(\frac{\ell}{R} \right)^3 \cos 4(\omega t - \alpha) + \cdots \right] \\ & + \cdots \,. \end{aligned} \tag{50}$$

(In free fall, the odd harmonics of the gravitational interaction will drop out.)

Now, if we design the accelerometer at the end of the rotating rod so that it is not highly damped, but instead is tuned to some harmonic of the rotational frequency, the sensor will respond preferentially to one of the frequency components of (50).

If the accelerometer were tuned to $(k/m)^{1/2} = 3\omega$, it would see only the gravitational term even in the presence of the usually much larger rotational and acceleration terms. If it were tuned to $(k/m)^{1/2} = 2\omega$, it would see the terms due to the rotations at right angles to the z axis (assuming that the grvitational gradient is weaker.) Other sensors rotating about the other axes would enable all of the rotational components to be measured without the interference of the usually much larger force term. If it were tuned to $(k/m)^{1/2} = \omega$, it would see the term due to the forces on the using vehicle (assuming that the rotational and gravitational effects are weaker). Again, other sensors rotating about other axes would enable all the components of the force vector to be determined.

SUMMARY

By designing sensors that take force measurements over an extended region of space it is possible to measure independently the effects of acceleration, rotation, and gravitation.

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