NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

Technical Memorandum No. 33-287

Application of Double-Averaged Equations of Satellite Motion to a Lunar Orbiter

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ABSTRACT

Equations of motion for satellites disturbed by both third-body and oblateness effects are presented in their averaged and double-averaged forms. These equations are appropriate for long-term studies of satellite orbits. In particular, they should serve as the starting point for study and classification of orbits of close lunar satellites.

I. INTRODUCTION

This Report presents the results of some of the research done in connection with determining the effects of third-body perturbations on satellite motion. In particular, the material contained herein supplements work already reported in Refs. 1, 2, and 3.

The motion of a satellite with periodic perturbation may be described mathematically in any one of three ways: (1) by the complete equations of motion, (2) by the averaged equations of motion, or (3) by the double-averaged equations of motion.

In the double-averaged equations, the averages are taken with respect to both the satellite period and the third-body period. The resulting differential equations are comparatively simple, lending themselves to analytical and simple numerical interpretation. Williams and Lorell (Ref. 1) used these equations in an analytical interpretation of the motion, while Lorell (Ref. 2) extended the study to a numerical analysis and a classification of the motion. Reference 4 contains an analysis of Earth satellite orbits perturbed by the Sun and Moon, and Lidov, in Ref. 5, presents a general treatment of the subject.

The work discussed in this Report is aimed specifically at artificial satellites of the Moon and is concerned with the applicability of the double-averaged equations. In Part II, the double-averaged equations are derived directly from the complete equations of motion. This derivation and the form of the result should be compared with the single-averaged equations as given in Ref. 3. It is included here for completeness, since it was not given in the referenced papers.

In Part III, the implications for satellite lifetimes are interpreted in terms of specific orbits. Here, the complete equations of motion were integrated, showing both short-term and long-term effects. In addition, the effects of the Sun and the Earth are compared.

II. DERIVATION OF THE DOUBLE-AVERAGED EQUATIONS

The motion of a satellite perturbed only by a distant third body has been studied recently using averaging methods (Refs. 1–5). In Ref. 3, the basic single-averaged equations are derived. (Single average refers to average with respect to the satellite motion only; double average refers to average with respect to both the satellite motion and the third-body motion.)

In Refs. 1–5, the single-averaged equations are presented for the third body in an arbitrary orbit. This model is applicable to general situations (e.g., Moon satellite perturbed by Earth and Sun). The position of the third body with respect to the reference coordinate system is arbitrary. On the other hand, the model for the double-averaged equations assigns the third body to an elliptic orbit in the x-y plane.

The double-averaged equations are of doubtful value for simulation purposes. However, for general study of long-term orbit behaviors, they are very useful. References 1, 2, 4, and 5 include studies and classification of orbits using these equations. References 2 and 5 suggest in addition the inclusion of oblateness effects in such studies, and present the appropriate equations. Performance of these studies, however, remains for the future.

The perturbing function due to a third body is well known, and may be found in any text on celestial mechanics. A form suitable to the present use is given by Clemence and Brouwer (Ref. 6, p. 310):

$$R = \frac{\mu_3 r^2}{r_3^3}\left[\left(-\frac{1}{2} + \frac{3}{2}\cos^2\psi\right) + \frac{r}{r_3}\left(-\frac{3}{2}\cos\psi + \frac{5}{2}\cos^3\psi\right) + \cdots\right] \tag{1}$$

in which the expansion is in powers of r/r_3, where[1]

R = disturbing function

r = distance from satellite to primary

r_3 = distance from disturbing third body to primary

ψ = satellite–primary–third-body angle

In the case of an artificial satellite of the Moon disturbed by the Earth, the ratio r/r_3 is of the order 1/100. Therefore, it is reasonable as a first approximation to consider the truncated disturbing function

$$R = \frac{\mu_3 r^2}{r_3^3}\left(-\frac{1}{2} + \frac{3}{2}\cos^2\psi\right) \tag{2}$$

The derivation of the single-averaged equations is given in Refs. 1 and 2 in matrix form, and will not be repeated here. The derivation of the double-averaged equations follows.

Let a bar denote a single average and a double bar a double average; e.g., R, $\bar{R}$, and $\bar{\bar{R}}$ are the perturbing function and its single and double averages. The procedure is to represent R first in cartesian coordinates and then in Kepler elements, assuming the third body in a Kepler ellipse about the primary. Thus,

$$R = \frac{3}{2}\frac{\mu_3 p^2}{p_3^3}\left(\frac{r}{p}\right)^2\left(\frac{p_3}{r_3}\right)^3\left[-\frac{1}{3} + \left(\frac{x}{r}\frac{x_3}{r_3} + \frac{y}{r}\frac{y_3}{r_3} + \frac{z}{r}\frac{z_3}{r_3}\right)^2\right] \tag{3}$$

[1]The units here are arbitrary. However, for use in conjunction with the equations for oblateness, the unit of length must be the lunar radius (see footnote 5).

where

$$\frac{x}{r} = \cos u \cos \Omega - \cos i \sin u \sin \Omega$$

$$\frac{y}{r} = \cos u \sin \Omega + \cos i \sin u \cos \Omega$$

$$\frac{z}{r} = \sin i \sin u$$

in which

$$u = \omega + f$$

$$\frac{r}{p} = \frac{1}{1 + e \cos f}$$

and

$$p = a(1 - e^2) = \text{semilatus rectum}$$

$$a, e, i, \omega, \Omega = \text{usual Kepler elements}$$

$$f = \text{true anomaly}$$

With subscript 3 on each variable, the notation described by the above equations applies to the third-body motion.

Choose the coordinate axes so that the third-body orbit is in the x-y plane, and the x-axis is in the direction of periapsis of the third-body orbit. Then,

$$\frac{x_3}{r_3} = \cos f_3$$

$$\frac{y_3}{r_3} = \sin f_3 \qquad (4)$$

$$\frac{z_3}{r_3} = 0$$

and

$$R = \frac{3}{2}\frac{\mu_3 p^2}{p_3^3}\left(\frac{r}{p}\right)^2\left(\frac{p_3}{r_3}\right)^3\left[-\frac{1}{3} + \left(\frac{x}{r}\cos f_3 + \frac{y}{r}\sin f_3\right)^2\right] \qquad (5)$$

After expanding and averaging twice,

$$\bar{\bar{R}} = \frac{3}{2}\frac{\mu_3 p^2}{p_3^3}\left[-\frac{1}{3}\overline{\left(\frac{r}{p}\right)^2}\,\overline{\left(\frac{p_3}{r_3}\right)^3} + \overline{\left(\frac{x}{r}\frac{r}{p}\right)^2}\,\overline{\left(\frac{p_3^3}{r_3^3}\cos^2 f_3\right)} + \overline{\left(\frac{y}{r}\frac{r}{p}\right)^2}\,\overline{\left(\frac{p_3^3}{r_3^3}\sin^2 f_3\right)}\right] \qquad (6)$$

The next step is to evaluate the averages. To start with, it is understood that averaging over the satellite variable means an average with respect to the mean anomaly of the satellite orbit, and for the third body, the average is with respect to the mean anomaly of its orbit.

Averages are obtained as follows:

$$\overline{\left(\frac{r}{p}\right)^2} = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{r}{p}\right)^2 dM = \frac{1}{2\pi}\int_0^{2\pi}\frac{1}{(1+e\cos f)^2}\frac{dM}{df}dt = (1-e^2)^{3/2}\frac{1}{2\pi}\int_0^{2\pi}\frac{df}{(1+e\cos f)^4} = \frac{1+\frac{3}{2}e^2}{(1-e^2)^2} \tag{7}$$

where M is the mean anomaly and

$$\frac{dM}{df} = \frac{r^2}{p^2}(1-e^2)^{3/2}$$

$$\overline{\left(\frac{p_3}{r_3}\right)^3} = (1-e_3^2)^{3/2}\frac{1}{2\pi}\int_0^{2\pi}(1+e\cos f_3)\,df_3 = (1-e_3^2)^{3/2} \tag{8}$$

$$\overline{\left(\frac{p_3^3\cos^2 f_3}{r_3^3}\right)} = \overline{\left(\frac{p_3^3\sin^2 f_3}{r_3^3}\right)} = \frac{1}{2}(1-e_3^2)^{3/2} \tag{9}$$

Thus, the last two terms of Eq. (6) may be combined to give

$$\frac{1}{2}(1-e_3^2)^{3/2}\left[\overline{\left(\frac{x}{r}\frac{r}{p}\right)^2} + \overline{\left(\frac{y}{r}\frac{r}{p}\right)^2}\right] = \frac{1}{2}(1-e_3^2)^{3/2}(1-e^2)^{3/2}\frac{1}{2\pi}\int_0^{2\pi}\left(1-\frac{z^2}{r^2}\right)\frac{df}{(1+e\cos f)^4} \tag{10}$$

But,

$$1-\frac{z^2}{r^2} = 1-\sin^2 i\sin^2 u = 1-\sin^2 i(\cos^2\omega + \sin 2\omega\sin f\cos f - \cos 2\omega\cos^2 f) \tag{11}$$

Furthermore,

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{\sin f\cos f}{(1+e\cos f)^4}df = 0 \tag{12}$$

and

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{\cos^2 f}{(1+e\cos f)^4}df = \frac{\frac{1}{2}+2e^2}{(1-e^2)^{7/2}} \tag{13}$$

Hence, the expression in Eq. (10) becomes

$$\frac{(1-e_3^2)^{3/2}}{2(1-e^2)^2}\left[(1-\sin^2 i\cos^2\omega)\left(1+\frac{3}{2}e^2\right) + \sin^2 i\cos 2\omega\left(\frac{1}{2}+2e^2\right)\right] \tag{14}$$

Finally, substituting and combining terms in Eq. (6), the expression for $\overline{\overline{R}}$ is found to be[2]

$$\overline{\overline{R}} = \frac{\mu_3 a^2}{a_3^3(1-e_3^2)^{3/2}}\left[\frac{1}{4}\left(1+\frac{3}{2}e^2\right) - \frac{3}{8}\sin^2 i(1-e^2+5e^2\sin^2\omega)\right] \tag{15}$$

[2] The double bar $(=)$ is omitted from the Kepler elements in these equations because there is no ambiguity; i.e., all variables are averaged variables.

The double-averaged equations of motion are then derived by substituting this disturbing function in the Lagrange planetary equation, giving

$$\frac{da}{dt} = 0 \tag{16}$$

$$\frac{de}{dt} = \frac{15}{8} \frac{\mu_3 e}{n a_3^3} \frac{(1 - e^2)^{1/2}}{(1 - e_3^2)^{3/2}} \sin 2\omega \sin^2 i \tag{17}$$

$$\frac{di}{dt} = -\frac{15}{16} \frac{\mu_3 e^2}{n a_3^3 (1 - e^2)^{1/2} (1 - e_3^2)^{3/2}} \sin 2\omega \sin 2i \tag{18}$$

$$\frac{d\omega}{dt} = \frac{3}{2} \frac{\mu_3 (1 - e^2)^{1/2}}{n a_3^3 (1 - e_3^2)^{3/2}} \left[1 + \frac{5 \sin^2 \omega (e^2 - \sin^2 i)}{2(1 - e^2)} \right] \tag{19}$$

$$\frac{d\Omega}{dt} = -\frac{3}{4} \frac{\mu_3 \cos i}{n a_3^3 (1 - e^2)^{1/2} (1 - e_3^2)^{3/2}} \left[(1 - e^2) \cos^2 \omega + (1 + 4e^2) \sin^2 \omega \right] \tag{20}$$

and if χ is the sixth Kepler element representing time of pericenter passage,

$$\frac{d\chi}{dt} = -\frac{3}{4} \frac{\mu_3}{n a_3^3 (1 - e_3^2)^{3/2}} \left\{ \frac{7}{3} + e^2 - \sin^2 i \left[(1 - e^2) \cos^2 \omega + 2(3 + 2e^2) \sin^2 \omega \right] \right\} \tag{21}$$

III. DOUBLE-AVERAGED EQUATIONS WITH OBLATENESS

To account for both oblateness of the primary and third-body effects, it is necessary to consider the relative orientation between the primary's equator and the third-body orbits. A convenient way to identify the important angles and the relations between them is to set up three right-handed coordinate systems as follows:[3]

1. x, y, z, with the xy-plane the plane of the third-body orbit
2. $\bar{x}$, $\bar{y}$, $\bar{z}$, with the $\overline{xy}$-plane the plane of the satellite orbit, and the x-axis in the direction of periapsis
3. x', y', z', with the $x'y'$-plane the plane of the equator and the x-axis in arbitrary direction (to be specialized later)

Let the coordinate transformations be given in terms of Euler angles:

$$\begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix} = A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = B \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \tag{22}$$

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = C \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{23}$$

[3] The bar (—) notation here is not to be confused with its previous meaning as an average.

Thus,

$$B = AC^{-1} \tag{24}$$

Here the matrices A, B, and C can be written in terms of the respective sets of Euler angles:

$$A = \begin{pmatrix} \cos\omega\cos\Omega - \cos i \sin\omega\sin\Omega & \cos\omega\sin\Omega + \cos i\sin\omega\cos\Omega & \sin i\sin\omega \\ -\sin\omega\cos\Omega - \cos i\cos\omega\sin\Omega & -\sin\omega\sin\Omega + \cos i\cos\omega\cos\Omega & \sin i\cos\omega \\ \sin i\sin\Omega & -\sin i\cos\Omega & \cos i \end{pmatrix} \tag{25}$$

with inverse A^{-1} given by

$$A^{-1} = \begin{pmatrix} \cos\omega\cos\Omega - \cos i\sin\omega\sin\Omega & -\sin\omega\cos\Omega - \cos i\cos\omega\sin\Omega & \sin i\sin\Omega \\ \cos\omega\sin\Omega + \cos i\sin\omega\cos\Omega & -\sin\omega\sin\Omega + \cos i\cos\omega\cos\Omega & -\sin i\cos\Omega \\ \sin i\sin\omega & \sin i\cos\omega & \cos i \end{pmatrix} \tag{26}$$

The matrices B and B^{-1} are of the same form, but with barred angles;[4] i.e., $\bar{\Omega}$, $\bar{\omega}$, $\bar{i}$. The matrices C and C^{-1} are of the same form, but with primed angles; i.e., Ω', ω', i'.

Specifically, the Euler angles Ω, ω, i relate the satellite orbit to the third-body orbit; the angles $\bar{\Omega}$, $\bar{\omega}$, $\bar{i}$ relate the satellite orbit to the equator; and the angles Ω', ω', i' relate the equator to the third-body orbit.

The relation between the three sets of Euler angles is given by Eq. (24). Explicit formulas can be obtained by expansion. Thus, equating the matrix element of the third row, third column gives

$$\cos\bar{i} = \cos i\cos i' + \sin i\cos\Omega\sin i'\cos\Omega' + \sin i\sin\Omega\sin i'\sin\Omega' \tag{27}$$

$$\cos i = \cos\bar{i}\cos i' + \sin\bar{i}\sin\bar{\Omega}\sin i'\sin\omega' + \sin\bar{i}\cos\bar{\Omega}\sin i'\cos\omega' \tag{28}$$

These equations may be simplified by choosing the coordinate axes judiciously within the prescribed frame. Thus, there is no loss of generality in taking the x-axis along the direction to the ascending node of the equator requiring $\Omega' = 0$. Equation (27) then becomes

$$\cos\bar{i} = \cos i\cos i' + \sin i\sin i'\cos\Omega \tag{29}$$

Since ω' represents the position of the reference meridian in the primary, it is useful to leave it in the equations. The angles $\bar{\Omega}$ and $\bar{\omega}$ are then determined by the following four equations:

$$\sin\bar{i}\sin\bar{\Omega} = \sin i\sin\bar{\Omega}\cos\omega' - \sin i\cos\Omega\cos i'\sin\omega' + \cos i\sin i'\sin\omega' \tag{30}$$

$$\sin\bar{i}\cos\bar{\Omega} = \sin i\sin\Omega\sin\omega' + \sin i\cos\Omega\cos i'\cos\omega' - \cos i\sin i'\cos\omega' \tag{31}$$

$$\sin\bar{i}\sin\bar{\omega} = -\sin i'\cos\omega\sin\Omega - \sin i'\cos i\sin\omega\cos\Omega + \cos i'\sin i\sin\omega \tag{32}$$

$$\sin\bar{i}\cos\bar{\omega} = +\sin i'\sin\omega\sin\Omega - \sin i'\cos i\cos\omega\cos\Omega + \cos i'\sin i\cos\omega \tag{33}$$

[4] See footnote 3.

The force function for a nonspherical primary can be expressed as an expansion in spherical harmonics and the result represented as functions of the Kepler elements of the orbit. However, the resulting expression is very unwieldy. Therefore, only the first few terms will be reproduced here. The averaged disturbing function $\bar{R}_{OB}$ thus derived[5] is

$$\bar{R}_{OB} = \mu\left[\frac{(1-e^2)^{3/2}}{4p^3}(1-3\cos^2\bar{i})\,C_{20} - \frac{3(1-e^2)^{3/2}}{4p^3}\sin 2\bar{i}\sin\bar{\Omega}\,C_{21} + \frac{3}{4}\frac{(1-e^2)^{3/2}}{p^3}\sin 2\bar{i}\cos\bar{\Omega}\,S_{21}\right.$$
$$+\frac{3}{2}\frac{(1-e^2)^{3/2}}{p^3}\sin^2\bar{i}\cos 2\bar{\Omega}\,C_{22} + \frac{3}{2}\frac{(1-e^2)^{3/2}}{p^3}\sin^2\bar{i}\sin 2\bar{\Omega}\,S_{22}$$
$$\left.+\frac{3}{2}\frac{e(1-e^2)^{3/2}}{p^4}\sin\bar{\omega}\sin\bar{i}\left(\frac{5}{4}\sin^2\bar{i}-1\right)C_{30} + \cdots\right] \tag{34}$$

in which the angles $\bar{i}$, $\bar{\omega}$, $\bar{\Omega}$ are, of course, referred to the equator coordinate system of the primary.

The complete disturbing function for the double-averaged equations is then obtained by adding Eqs. (12) and (34). The corresponding equations of motion are obtained in the usual manner using the Lagrange equations.

Since the equations of motion are derived thus from a potential function, they have an integral, namely, the corresponding Hamiltonian. Thus,

$$\frac{\mu}{2a} + \bar{R}_{OB} + \bar{\bar{R}} = \text{const} \tag{35}$$

Taking only the C_{20} term of $\bar{R}_{OB}$, the combined disturbing function is

$$\frac{\mu_3 a^2}{a_3^3(1-e_3^2)^{3/2}}\left[\frac{1}{4}\left(1+\frac{3}{2}e^2\right) - \frac{3}{8}\sin^2 i\,(1-e^2+5e^2\sin^2\omega)\right] + \mu C_{20}\frac{1}{4a^3(1-e^2)^{3/2}}\left[1-3(\cos i\cos i' + \sin i\sin i'\cos\Omega)^2\right] \tag{36}$$

and the integral corresponding to Eq. (35) can be expressed in the form

$$(1-e^2)\cos^2 i + 2e^2\left(1-\frac{5}{2}\sin^2 i\sin^2\omega\right) + B\left[1-3(\cos i\cos i' + \sin i\sin i'\cos\Omega)^2\right] = \text{const} \tag{37}$$

where

$$B = \frac{2a_3^3\,\mu C_{20}(1-e_3^2)^{3/2}}{3\mu_3 a^5(1-e^2)^{3/2}} \tag{38}$$

Note that when C_{20} is not considered (i.e., $C_{20}=0$), the differential equations (16 through 21) apply, and these have the two integrals

$$(1-e^2)\cos^2 i = \alpha \tag{39}$$

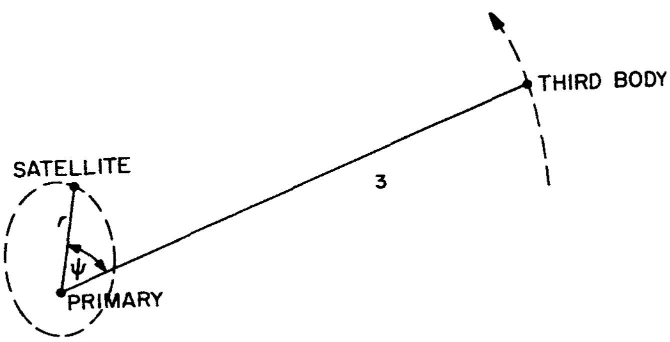
$$e^2\left(1-\frac{5}{2}\sin^2 i\sin^2\omega\right) = \beta \tag{40}$$

compatible with Eq. (37).

[5]Already averaged with respect to the satellite motion. Here, the unit of length is the average lunar radius.

IV. IMPLICATIONS FOR LIFETIME OF A LUNAR SATELLITE

A satellite of the Moon, whose osculating orbit is described by Eqs. (16) to (21), will continue in orbit until it hits the Moon – which cannot occur unless the point of closest approach of the osculating orbit is below the surface of the Moon. The only other possible end to the orbit, namely escape, is not possible, since $da/dt = 0$.

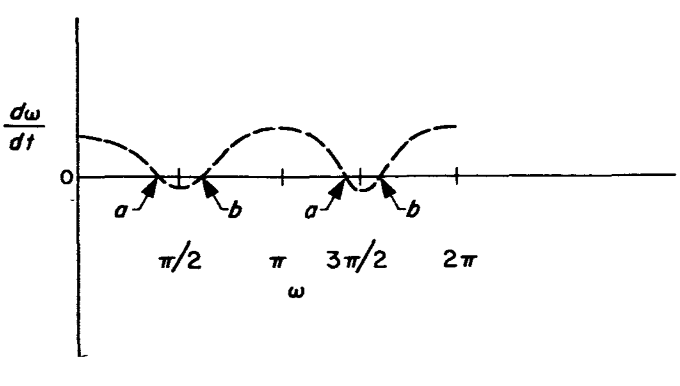


Let $q\,[= a(1-e)]$ be the radius of closest approach of the osculating ellipse, and let $r_{☾}$ be the mean radius of the Moon. Then, $q = r_{☾}$ is a criterion for measuring satellite lifetime. Since $da/dt = 0$, we have

$$\frac{dq}{dt} = -a\frac{de}{dt} \tag{41}$$

and it is apparent that the life of the orbit will be in jeopardy as long as de/dt is positive.

Looking at Eq. (17), we see that the sign of de/dt depends on the quadrant of ω – the first or third quadrant giving a positive sign. If, therefore, there is a situation for which ω remains in the first or third quadrant, then q will steadily decrease, until eventually $q < r_{☾}$ and impact on the lunar surface may occur. To check on the behavior of ω, we look at Eq. (19).



Sketch 2 is based on Eq. (19) and shows that $d\omega/dt$ is negative near $\omega = \pi/2$ and $\omega = 3\pi/2$, provided only that i is close enough to $\pi/2$. Note also that if $i \sim \pi/2$, then di/dt is close to zero (Eq. 18), so that i will remain close to $\pi/2$ for a long interval of time. Points a of the sketch are stable points in the sense that for ω close to a, the rate equation is such that ω tends toward a. On the other hand, the points b are unstable. Thus, in general, ω will be driven toward one of the points a unless outside forces prevail.[6]

But, the points a are in the first and third quadrants, which leads to the unstable condition $dq/dt < 0$ noted above. It follows, therefore, that unstable orbits are possible—and, in fact, all orbits will tend to be unstable if i is close to $\pi/2$. The ultimate lifetime of any particular orbit depends in the long run on whether i decreases to the point where $d\omega/dt$ is never negative before the satellite hits the Moon.

These conclusions were reported by STL (Ref. 4) and were supported by numerical examples for Earth satellites. The studies reported by Williams and Lorell (Ref. 1) and by Lorell (Ref. 2) give a more thorough analysis of the orbit behavior and an orbit classification based on the double-averaged equations. It remains to show the compatibility with the complete (unaveraged) equations.

The problem here is a practical one, because there is no serious question as to the validity of the averaged equations. For an artificial satellite of the Moon, does the short-term behavior reflect any of the characteristics suggested by the averaged equations? How does the relative position of the third body in its orbit enter into the satellite motion?

To answer these questions, the complete equations of motion were integrated numerically for four lunar satellite orbits. The results exhibit both long- and short-period effects and serve as a convincing argument in support of the usefulness of the double-averaged equations. These results are offered only as an example of the possible

[6]The reader is referred to Ref. 2, where a more penetrating analysis is made. Specifically, the top graph in Fig. 10 of Ref. 2 illustrates the case under discussion. Point a would be represented by $\omega \sim 45$ deg and b by $\omega \sim 135$ deg. Then, a line of constant e, say $e = 0.6$, cuts one of the curves at a vertical point, say the curve for $\beta = 0.05$. Passing to a curve to the left, it is apparent that $\dot{\omega} > 0$, while for a curve to the right, $\dot{\omega} < 0$, all at the same value of $e = 0.6$, of course.

orbital motions, and are not intended to prove anything more than that there are situations in which the averaged equations can be informative, even over the short term.

The osculating elements for four lunar satellite orbits are plotted for one period in Figs. 1 through 6. These orbits differ from each other initially only in their inclination, node, and angle to pericenter (see tabulation in Fig. 1). The orbits are highly eccentric ($e = 0.63$), steeply inclined to the ecliptic plane ($i = 90$ deg, except for orbit 80, where $i = 65$ deg), and comparatively close to the Moon ($a = 5438$ km).

For three of the orbits—77, 79, 80—the orbital plane is approximately parallel to the Earth–Moon line. The plane of orbit 78 is perpendicular to the Earth–Moon line.

These orbits were chosen specifically to illustrate the case of instability, in which the point of closest approach to the Moon steadily decreases. In this regard, the following points may be noted:

1. In all cases, ω is in the first or third quadrant.
2. The net variation of ω over the whole period is less than 0.02 deg, except for orbit 78; i.e., the variation per month is of the order of 1.4 deg at this rate (see Fig. 5).
3. The net variation of i is less than 0.03 deg/period (equivalent to 2 deg/month).
4. The variation in a within the period is of the order of 7 km—but the net variation is zero.
5. The net change in q over one period is -14 km, equivalent to 33.6 km/day, or to lunar impact in 8 days if the same rate were to persist.
6. The entire change in q occurs between true anomalies 140 and 210 deg, i.e., in the region of apocenter.

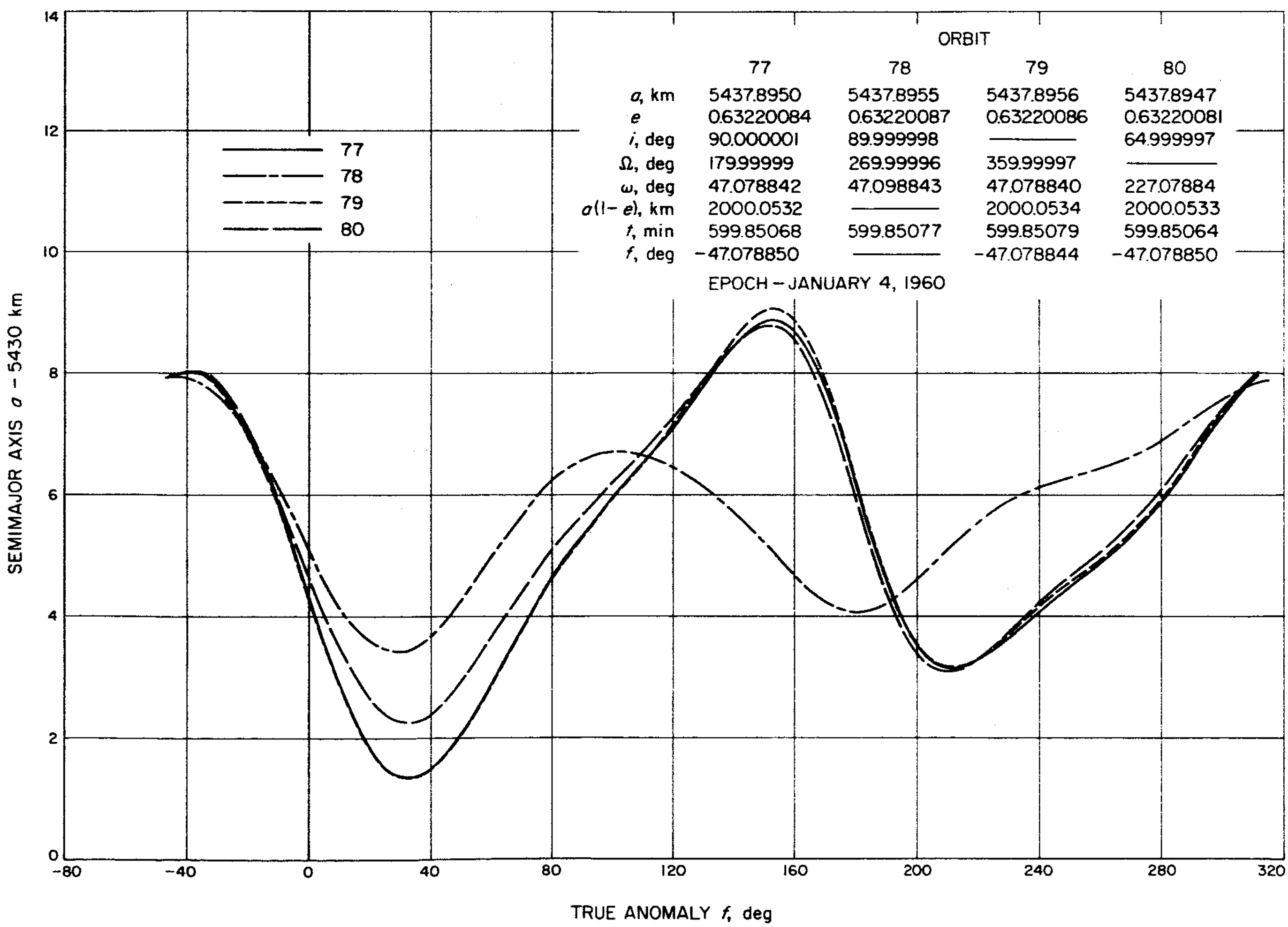


Fig. 1. Semimajor axis

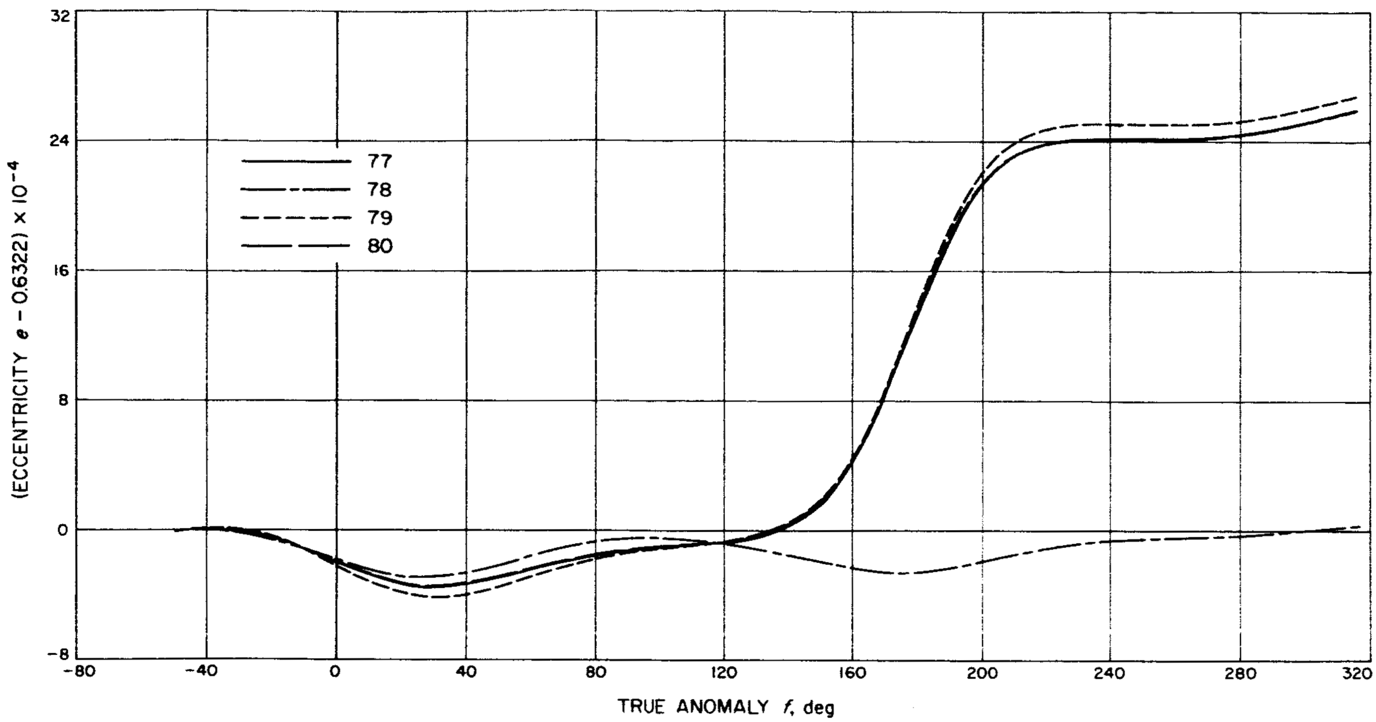


Fig. 2. Eccentricity

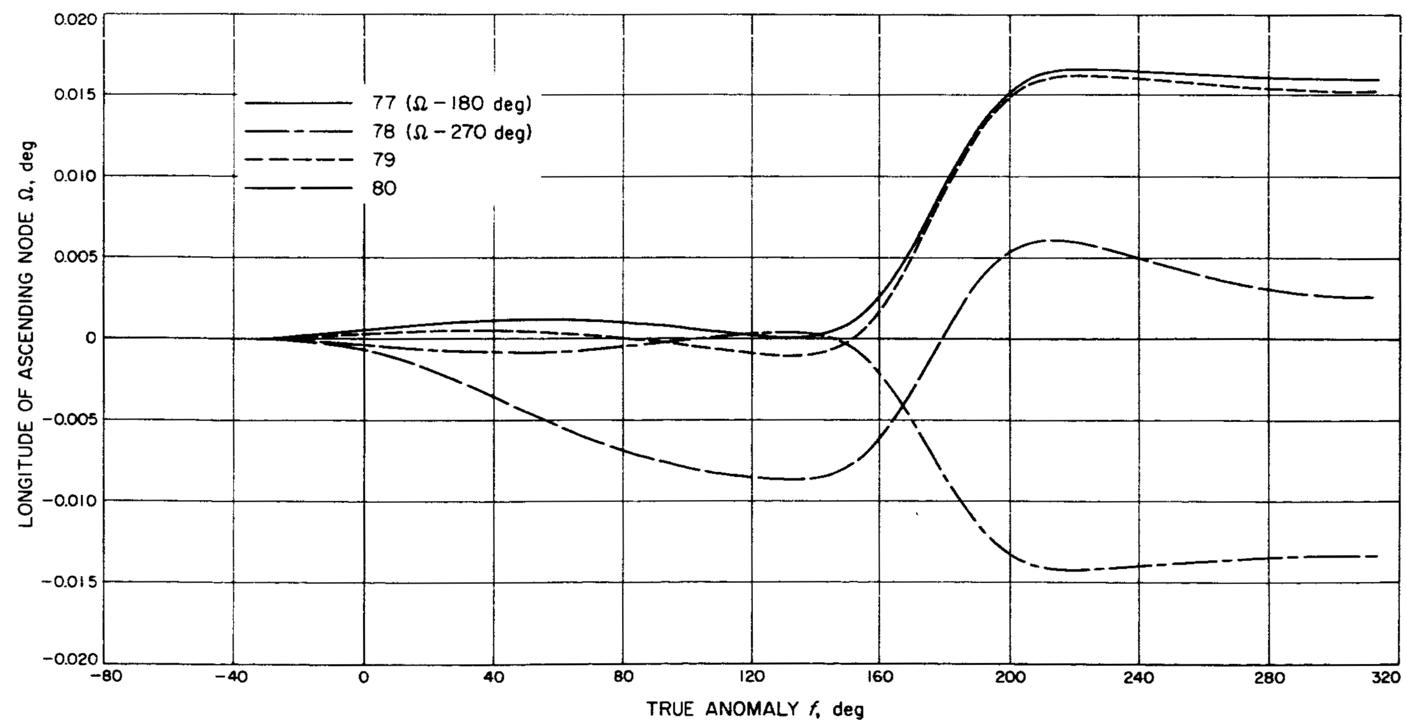


Fig. 3. Longitude of ascending node

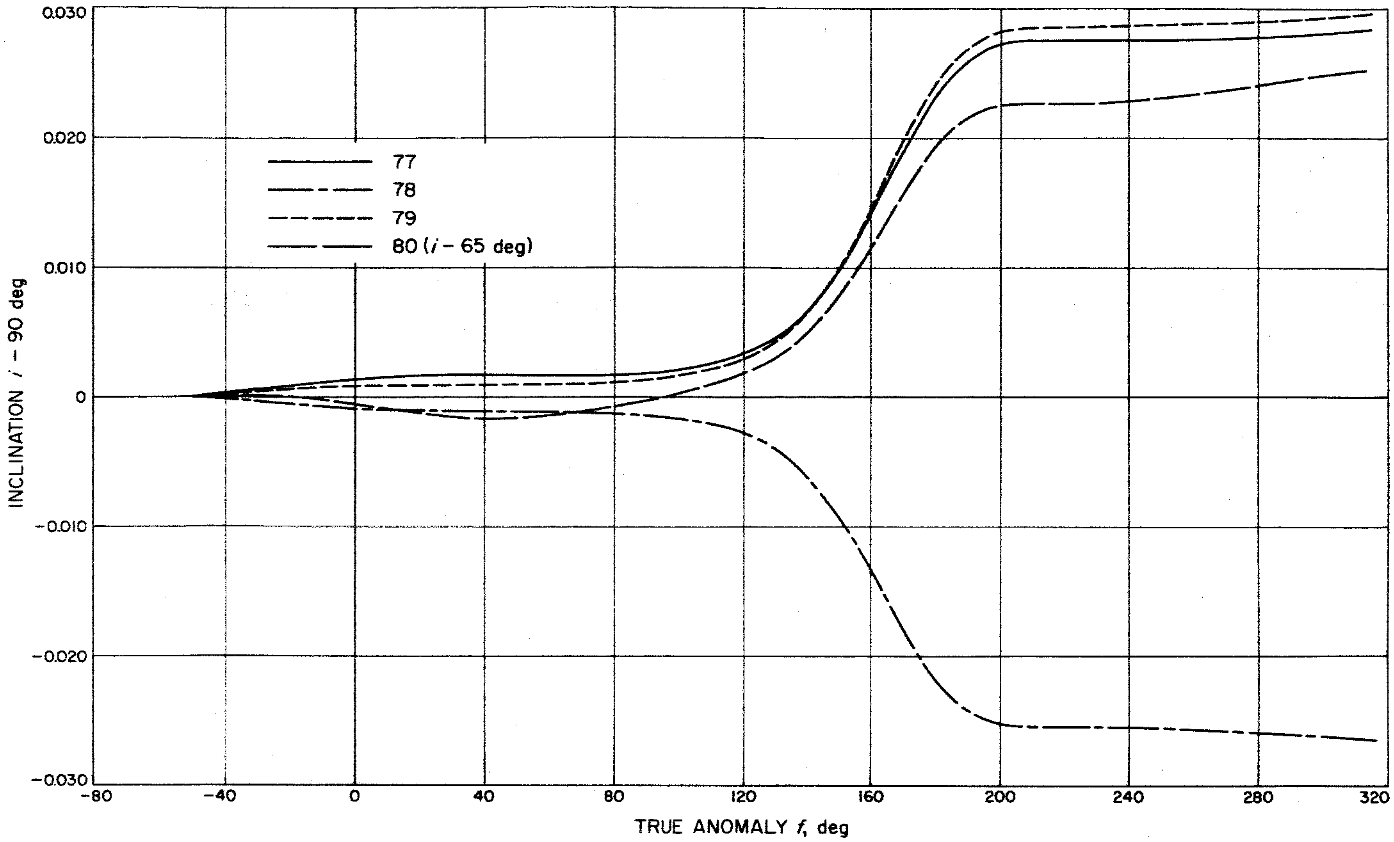


Fig. 4. Inclination

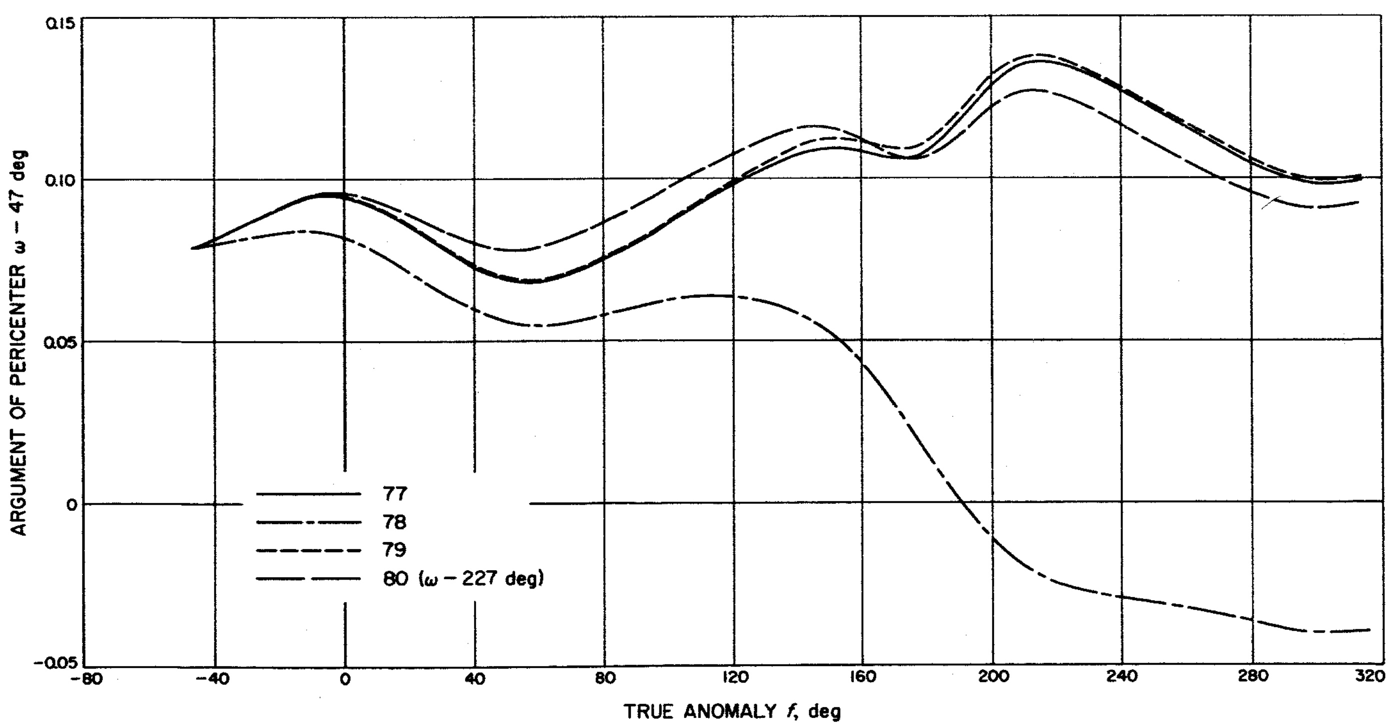


Fig. 5. Argument of pericenter

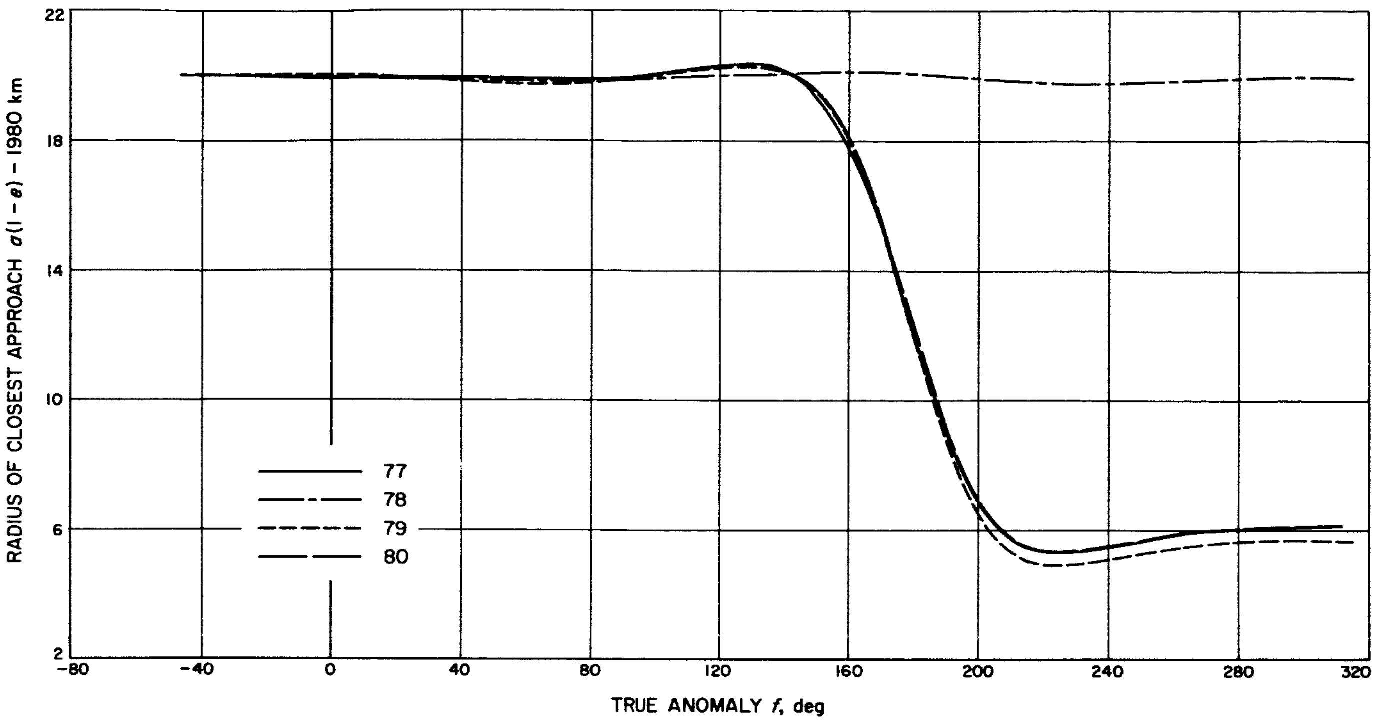


Fig. 6. Radius of closest approach

The residence time in this portion of the trajectory is 5 hr—just half the period.

7. Only orbit 78, whose plane is perpendicular to the Earth–Moon line, behaves differently from the others.

To check the long-term trends, orbit 80 was continued to impact, which occurred after 16 days, 15 hr, 48 min.

A second set of computations was used to evaluate the relative effects of Sun and Earth. In these computations, all the perturbing factors were suppressed, except the Earth in orbit 83 and the Sun in orbit 84. The two orbits were similar to orbits 77 to 80 in all respects, except that initially they were aligned along the Moon–Sun line in order to maximize the effect of the Sun. Because of the relative motion of the Earth around the Moon, the Earth's effect oscillates over a period of half a month.

The results are shown in Fig. 7, in which the radius of closest approach is plotted against time. For proper comparison, the value for the Sun has been multiplied by 170 ($= n_E^2/n_S^2$). In this type of plot, theory predicts that the maximum rates should be equal, which is confirmed by Fig. 7 for t between 6 and 10 days.

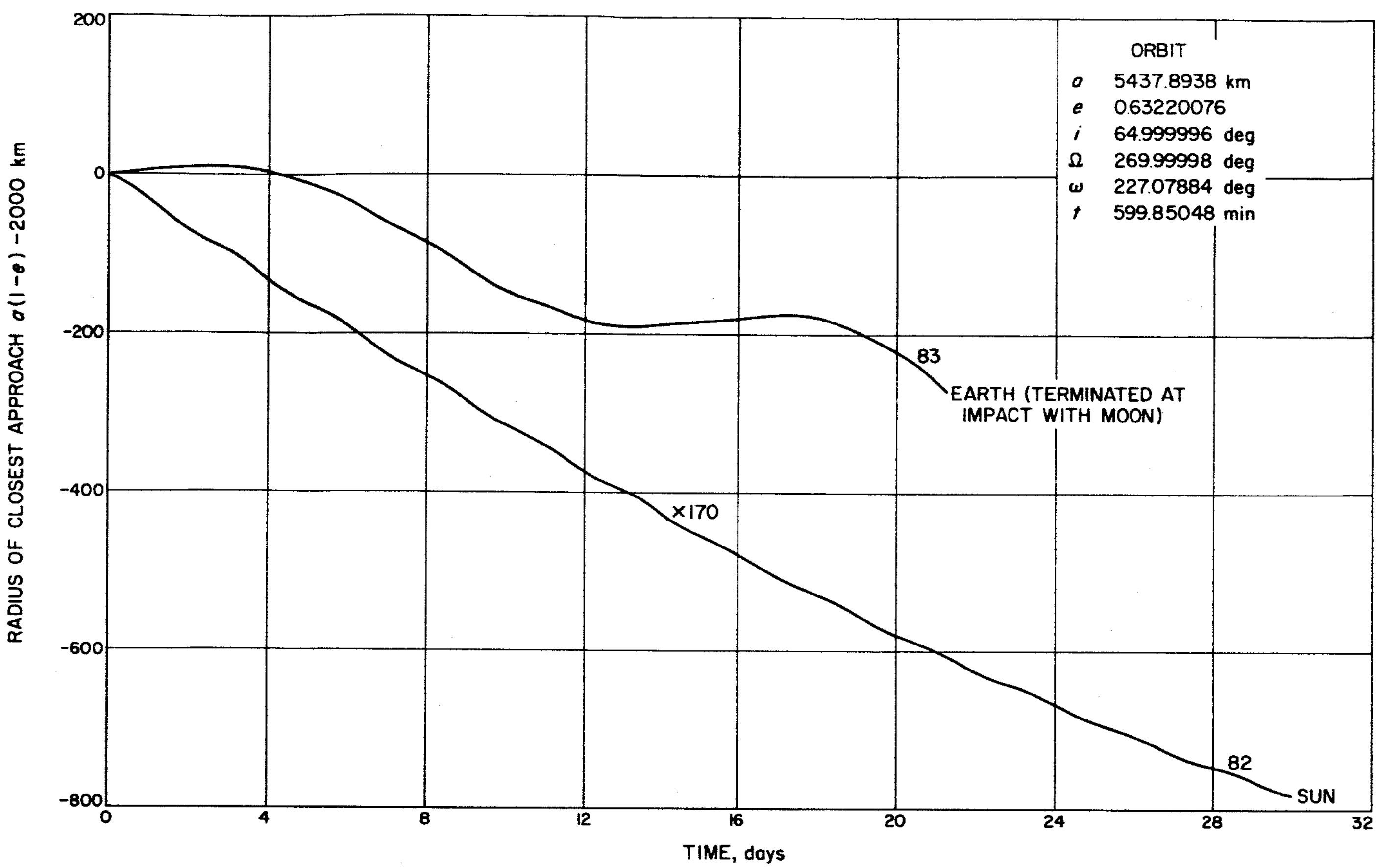


Fig. 7. Effect of Earth and Sun on radius of closest approach

V. CONCLUSIONS

Third-body perturbations of artificial satellites can cause instability, which in some cases may terminate the orbit in a very short time. The conditions for instability are:

1. The inclination i must be appreciable; the closer it is to 90 deg, the more likely the orbit is to be unstable.

2. The eccentricity e must not be zero.

3. The argument of pericenter must be in the first or third quadrant. However, it will normally drift to one of these quadrants if the other conditions for instability are met.

4. The semimajor axis must be large enough so that the dominant perturbing force is the third body rather than oblateness of the primary.

In the case of satellites of the Moon, the perturbations due to the Earth are generally of the order of 170 times those due to the Sun.[7] However, under favorable conditions, as for instance when the satellite orbit plane is parallel to the Moon–Sun line and perpendicular to the Moon–Earth line, the effect of the Sun may exceed that of the Earth for a period of days.

[7]The instantaneous average rates due to Earth and Sun have the ratio 170. However, the ratio of the amplitudes of the periodic effects is $(170)^{1/2} \approx 13$.

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