A t t e n t i o n: Code SC, NASA
Washington, D.C. 20546 U S A

Innsbruck, June 30, 1966

F I N A L R E P O R T

GPO PRICE	$
CFSTI PRICE(S)	$
Hard copy (HC)	$1.00
Microfiche (MF)	.50

ff 653 July 65

Covering the Period 1 July 1965 - 31 July 1966

RESEARCH GRANT NGR 52 - 046 - 001

Lie Series Solutions of Partial Differential Equations *

FACILITY FORM 602

N66 30760	
(ACCESSION NUMBER)	(THRU)
11	1
(PAGES)	(CODE)
CR-76314	19
(NASA CR OR TMX OR AD NUMBER)	(CATEGORY)

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* This research has been supported by N A S A (National Aeronautics and Space Administration)

I. Scientific Work Done in the Period of 1 July 1965 to 31 July 1966

The scientific program was concerned with the application of Lie series to the solution of linear ordinary differential equations of second order.

Lie series were invented by GROEBNER, Head of the Department of Mathematics at the University of Innsbruck, to solve special problems in algebraic geometry. It was found, however, that these series, named by GROEBNER after S. LIE, were very useful to solve differential equations. A lot of theoretical development and physical applications of Lie series have already been published.

A series of the following kind:

$$\sum_{\nu}^{\infty}\frac{t^{\nu}}{\nu!}D^{\nu}f(z) = f(z) + tDf(z) + \frac{t^2D^2f(z)}{2!} + \ldots\ldots\ldots$$

is called Lie series; $f(z)$ is any function which depends on the complex variables $z_1,z_2,\ldots\ldots,z_n$ and is holomorphic in the neighborhood of the point z_o. D is a linear differential operator defined by

$$D = \vartheta_1(z)\frac{\partial}{\partial z_1} + \vartheta_2(z)\frac{\partial}{\partial z_2} + \ldots\ldots\ldots + \vartheta_n(z)\frac{\partial}{\partial z_n}$$

the coefficients $\vartheta_i(z)$ represent functions of the complex variables $z_1,z_2,\ldots\ldots\ldots,z_n$, which are all assumed to be holomorphic in the neighborhood of the point z_o. The convergence of the Lie series was proved by GROEBNER, using the method of CAUCHY's majorants.

Lie series can be used to solve not only ordinary (linear and nonlinear) but also partial differential equations and are from this point of view a quite general method to solve differential equations.

If we consider an ordinary differential equation of the order n we can write it in the form:

$$Z^{(n)}(t) = \vartheta_1(t, Z, Z', Z'', \ldots\ldots\ldots, Z^{(n-1)})$$

or

$$\begin{aligned} Z_1' &= \vartheta_1(t, Z_1, Z_2, \ldots\ldots\ldots, Z_n) \\ Z_2' &= \vartheta_2(t, Z_1, Z_2, \ldots\ldots\ldots, Z_n) \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ Z_n' &= \vartheta_n(t, Z_1, Z_2, \ldots\ldots\ldots, Z_n) \end{aligned} \qquad (1)$$

As the ϑ_i are analytic functions we may expand the functions $\vartheta_i (i=1,2,\ldots,n)$ in a power series:

$$\vartheta_i = \sum_{\nu}^{\infty} \frac{t^\nu}{\nu!} D^\nu \vartheta_{io} \qquad (2)$$

the index "o" indicates that after applying the operator D ν- times, $t, Z_1, Z_2, \ldots\ldots, Z_n$ have to be replaced by the initial values $t_o, z_1, z_2, \ldots\ldots\ldots, z_n$.

Further the equation is valid:

$$Z_i' = \frac{d}{dt} Z_i(t) = \frac{d}{dt} \sum_{\nu}^{\infty} \frac{t^\nu}{\nu!} D^\nu z_i = \sum_{\nu}^{\infty} \frac{t^\nu}{\nu!} D^\nu \vartheta_{io} \qquad (3)$$

Eq.(2) and Eq.(3) show that the Lie series $Z(t) = \sum_{\nu}^{\infty} \frac{t^\nu}{\nu!} D^\nu z_{io}$ solve Eq. (1).

During our research period we investigated the general ordinary linear homogeneous and inhomogeneous differential equation of second order. The solutions were presented by two alternative forms, one of them still contains D-operators, the other one consists of well-known functions plus an integral term. These general solutions were applied to special problems in physics and the differential equations, which result from the separation of the Helmholtz equation.

As numerically examples we have computed the Mathieu functions and the Weber functions.

<u>Scientific Reports</u>

Report No 1 : The Representation of Mathieu functions Using Lie series, by F. CAP and D. FLORIANI. This report deals with theoretical investigations of Mathieu functions.

Report No 2: On the Solution of the General Homogeneous Linear Differential Equation of Second Order, Using Lie Series, by A.SCHETT and J. WEIL. - The solution which is presented by Lie series for regular domains, can be evaluated numerically by recurrence formula.

Report No 3: A Compact Method for Solving the Homogeneous Linear Differential Equation of Second Order, Using Lie Series. Matrix formalism is here used and the solution is presented by known functions plus an integral term. It is essentially a seminar lecture held by Prof. GROEBNER, Head of the Department of Mathematics.

Report No 4: On the Solution of the General Inhomogeneous Linear Differential Equation of Second Order, Using Lie Series, by A. SCHETT and J.WEIL. The solution is presented by two alternative forms, one of which can be evaluated by recurrence formulas, whereas the other can be evaluated by means of iterative methods.

Report No 5: On the Numerical Calculation of the Mathieu Functions, By D. FLORIANI. This report deals with theoretical and numerical investigations of the Mathieu functions, using Lie series.

Report No 6: The Numerical Calculation of Weber Parabolic Cylinder Functions, by A.SCHETT and J. WEIL. The recurrence formula method is used to compute the Weber functions.

Report No 7: Lie Series Solution of the Equations Resulting from a Separation of the Helmholtz Equation in Special Coordinate Systems (PartI), by A. SCHETT and J.WEIL. The general representations of the solutions of Rep. No 2 and Rep. No 3 are applied to special problems.

Report No 8: Lie Series Solution of the Equations Resulting from a Separation of the Helmholtz Equation in Special Coordinate Systems (PartII), by A.SCHETT and J.WEIL. The representations of the genral solutions of Rep. No 2 and Rep. no 3 are applied to special problems.

Publications

1. Theoretical and Numerical Investigations of Mathieu Functions, Using Lie Series, by F. CAP and D.FLORIANI (to be published)
2. On the Solution of the General Homogeneous Linear Differential Equation of Second Order, Using Lie Series, by A.SCHETT and J.WEIL (Under press, Acta Physica Austriaca)
3. On the Solution of the General Inhomogeneous Linear

Differential Equation of Second Order, Using Lie Series, by A.SCHETT and J.WEIL (Under press, Monatshefte für Mathematik)

4. The Numerical Computation of Weber Parabolic Cylinder Function, by A.SCHETT and J.WEIL. (To be published in ZAMM).
5. Lie Series Solution of the Equations Resulting from a Separation of the Helmholtz Equation in Special Coordinate Systems, by A.SCHETT and J.WEIL (Under press, Acta Physica Austriaca)
6. Monograph entitled: Solution of Ordinary Differential Equations by Means of Lie Series, by F.CAP, [illegible], [illegible], A.SCHETT and J.WEIL, in publication by N A S A.

II. Personal Data and a Short Survey of the Work Done by the Collaborators of the Team.

a) Personal Data: The team consists of:

Professor Dr. Ferdinand CAP (Principal Investigator)
D. FLORIANI from 15 July 1965 to 30 June 1966
Dr. Alois SCHETT from 1 October 1965 to 31 July 1966
Dr. Jürgen WEIL from 1 October 1965 to 31 July 1966

Professor Dr. Ferdinand CAP

Being the principal investigator he was responsible for the coordination and general supervision of the work, comprising general investigations of the applicability of Lie series to various problems and a lot of discussions with the collaborators, usually held once a week.

Dietmar FLORIANI

After generally dealing with the separation of the Helmholtz equation in the 11 coordinate systems he turned to the Mathieu equation, which he investigated theoretically and numerically.

Dr. Alois SCHETT

was initially concerned with investigations on the use of Laplace transformation for solving ordinary differential equations then he turned in collaboration with Dr. WEIL to the application of Lie series to solve the homogeneous and inhomogeneous ordinary differential equation of second order. Furthermore he applied - in collaboration with Dr. WEIL - the general solution by means of Lie series, to special problems in physics and differential equations resulting from the separation of the Helmholtz equation. Finally he computed the Weber functions using Lie series representation.

<u>Dr. Jürgen WEIL</u>

was initially concerned with considerations on the use of Laplace transformation and attended a course on computing by our ZUSE 23 V. Then he collaborated with Dr. SCHETT. Finally he was responsible for a proper English in all our reports and publications.