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On the Impact Induced Stress Waves in Long Bars

S. Utku

M. E. Alper

M. E. Alper, *Manager*
Applied Mechanics Section

JET PROPULSION LABORATORY
CALIFORNIA INSTITUTE OF TECHNOLOGY
PASADENA, CALIFORNIA

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ABSTRACT

The work presented in this report is confined to the initial impactive behavior of uniform prismatic bars of constant pre-impactive velocity. Various nonlinear stress-strain relationships with no strain-rate effect are considered. The fact that the bar is laterally unconfined is taken into account in the equation of conservation of mass. The lateral motion is ignored; however, its effect is discussed during the derivation of Rankine–Hugoniot equations for the impacting bar of strain-hardening material. For the case of no strain hardening, the analysis in an Eulerian reference system is carried out with the method of characteristics until the occurrence of first unloading. Various formulas are derived for the possible iterative computation of stresses, strains, and velocities. The general behavior of energy dissipators is studied by means of a bar of elastoplastic material with infinite strain hardening. It is shown that the bar will be crushed from both ends and, provided that lateral motion is prevented, it is also shown that the shock temperatures are helpful in energy dissipation. This paper indicates the number of measurements required for obtaining the first estimate of mechanical properties of the unknown impacted surface when the properties of the impacting bar are known.

I. INTRODUCTION

Planetary landings have made the impactive behavior of structural elements, individually or as a whole, an increasingly important phenomenon. The objective of this report is to answer some of the basic questions, such as the pressures and velocities of interface, and stress fronts. To answer these questions, a laterally unconfined uniform bar of initial velocity U_0, and mass density ρ_0 is considered. A general $\sigma = \sigma(\varepsilon)$ function is used as the equation of state of the material. The governing equations of the mathematical model are obtained from the conditions of conservation of mass and conservation of momentum. By changing the end conditions, the following cases are simulated: (1) bar impacting a rigid surface at rest at one end and free at the other end, (2) bar impacting a flexible surface at rest at one end and free at the other end, (3) bar impacting a rigid surface at rest at one end and attached to another bar of the same initial speed, and (4) bar of elastoplastic property with infinite strain-hardening impacting a rigid surface at rest at one end and attached to a rigid mass of the same initial speed at the other end.

II. FORMULATION OF THE PROBLEM

For simplicity, a prismatic bar of unit cross-sectional area is considered. Referring to an Eulerian coordinate system (X, T) (Fig. 1), $\rho(X, T)$ the mass density, $U(X, T)$ the particle velocity, and $\underline{\varepsilon}(X, T)$ the strain at a point (X, T) are defined. If ρ_0 is the initial mass density,

$$\rho_0 = \rho(1 + \mu\underline{\varepsilon}) \tag{1}$$

which relates instantaneous mass at a point to the instantaneous strain at the same point. Here, μ is a constant between zero and one indicating the Poisson effect. In terms of Poisson's ratio ν, $\mu = 1 - 2\nu$. Since the amount of mass getting into a control volume in unit time is equal to the time rate of change of the mass in the control volume,

$$\frac{\partial}{\partial X}(\rho U) + \frac{\partial}{\partial T}\rho = 0 \tag{2}$$

Using Eq. (1) in (2), one obtains as the requirement of conservation of mass

$$(1 + \mu\underline{\varepsilon})\frac{\partial U}{\partial X} - \mu U\frac{\partial \underline{\varepsilon}}{\partial X} - \mu\frac{\partial \underline{\varepsilon}}{\partial T} = 0 \tag{3}$$

It is assumed that $\partial\mu/\partial\underline{\varepsilon} = 0$.

The requirement for conservation of momentum leads to

$$\rho a = \frac{\partial \sigma}{\partial X} \tag{4}$$

where $a(X, T)$ is the acceleration, and σ is the stress. For large deformations in the Eulerian coordinates, the acceleration is

$$a = \frac{\partial U}{\partial T} + \frac{\partial U}{\partial X}U \tag{5}$$

Since $\sigma = \sigma(\underline{\varepsilon})$, one can write

$$\frac{\partial \sigma}{\partial X} = \frac{d\sigma}{d\underline{\varepsilon}}\frac{\partial \underline{\varepsilon}}{\partial X} \tag{6}$$

Using Eqs. (5) and (6), and Eq. (1) in (4), one obtains as the requirement of equilibrium

$$U\frac{\partial U}{\partial X} + \frac{\partial U}{\partial T} - (1 + \mu\underline{\varepsilon})\frac{d\sigma}{d\underline{\varepsilon}}\left(\frac{1}{\rho_0}\right)\frac{\partial \underline{\varepsilon}}{\partial X} = 0 \tag{7}$$

Equations (3) and (7) are the governing equations for the unknowns $\underline{\varepsilon}$ and U. Defining a quantity c (which is $1/(\mu)^{1/2}$ times the speed of sound in the undisturbed material) as

$$c = \left[\frac{\left(\frac{d\sigma}{d\underline{\varepsilon}}\right)_{\underline{\varepsilon}=0}}{\mu\rho_0}\right]^{1/2} \tag{8}$$

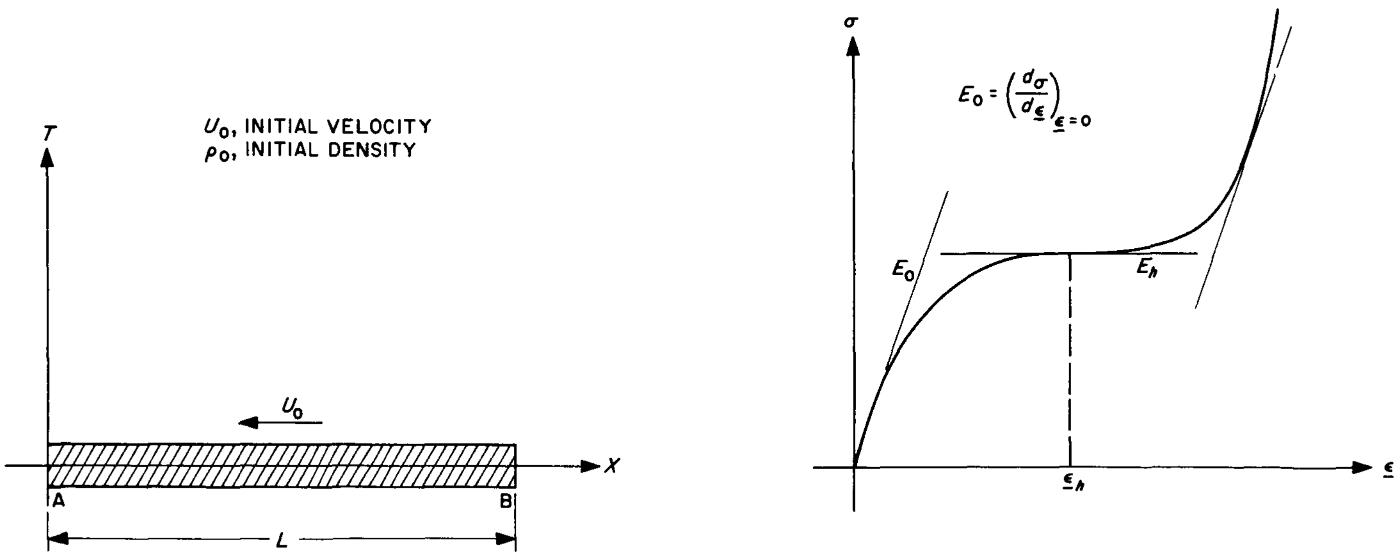


Fig. 1. Impacting bar and nonlinear uniaxial stress-strain relationship

one can nondimensionalize the foregoing equations with the following nondimensional variables

$$x = \frac{X}{L} \tag{9a}$$

$$t = \frac{T}{(L/c)} \tag{9b}$$

$$v = \frac{U}{c} \tag{9c}$$

$$v_0 = \frac{U_0}{c} \tag{9d}$$

$$E = \frac{\left(\frac{d\sigma}{d\underline{\varepsilon}}\right)}{\left(\frac{d\sigma}{d\underline{\varepsilon}}\right)_{\underline{\varepsilon}=0}} \tag{9e}$$

$$\varepsilon = \mu\underline{\varepsilon} \tag{9f}$$

By applying the nondimensional variables to Eqs. (3) and (7),

$$v\frac{\partial v}{\partial x} + \frac{\partial v}{\partial t} - (1+\varepsilon)E\frac{\partial \varepsilon}{\partial x} = 0 \tag{10}$$

$$(1+\varepsilon)\frac{\partial v}{\partial x} - v\frac{\partial \varepsilon}{\partial x} - \frac{\partial \varepsilon}{\partial t} = 0 \tag{11}$$

where Eq. (10) represents the requirement of conservation of momentum, and Eq. (11) represents the requirement of conservation of mass.

The initial conditions for the bar incident to impact are

$$\left.\begin{matrix} v = v_0 \\ \varepsilon = 0 \end{matrix}\right\} \text{at } t = 0 \qquad 0 < x < 1 \tag{12}$$

III. SOLUTION BY METHOD OF CHARACTERISTICS

The loci of disturbance fronts, which exist only if the governing equations are not elliptic, are called characteristic lines. The relationship between v and ε along these lines may be obtained as follows. The total differentials of v and ε in any arbitrary directions are

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial t}dt \tag{13}$$

$$d\varepsilon = \frac{\partial \varepsilon}{\partial x}dx + \frac{\partial \varepsilon}{\partial t}dt \tag{14}$$

Combining Eqs. (10), (11), and (13), (14) in matrix form yields

$$\begin{bmatrix} v & 1 & -(1+\varepsilon)E & 0 \\ 1+\varepsilon & 0 & -v & -1 \\ dx & dt & 0 & 0 \\ 0 & 0 & dx & dt \end{bmatrix} \begin{Bmatrix} \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial t} \\ \frac{\partial \varepsilon}{\partial x} \\ \frac{\partial \varepsilon}{\partial t} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ dv \\ d\varepsilon \end{Bmatrix} \tag{15}$$

If the determinant of the left-hand coefficient matrix can be made to vanish, one has to allow discontinuities in the first derivatives of the unknown quantities across the associated directions for which Eqs. (13) and (14) were written. These directions, if they exist, may be obtained by

$$\begin{vmatrix} v & 1 & -(1+\varepsilon)E & 0 \\ 1+\varepsilon & 0 & -v & -1 \\ dx & dt & 0 & 0 \\ 0 & 0 & dx & dt \end{vmatrix} = 0 \tag{16}$$

which yields

$$\left(\frac{dt}{dx}\right)_\alpha = \frac{1}{v + (1+\varepsilon)(E)^{1/2}} \tag{17}$$

$$\left(\frac{dt}{dx}\right)_\beta = \frac{1}{v - (1+\varepsilon)(E)^{1/2}} \tag{18}$$

as the characteristic directions. The relationship between v and ε in these directions may be obtained by equating the determinant of the left-hand matrix of Eq. (15) to zero, after one of the vectors is replaced by the right-hand vector; this is a necessary condition for a solution to exist, and is shown by

$$\begin{vmatrix} v & 1 & -(1+\varepsilon)E & 0 \\ 1+\varepsilon & 0 & -v & 0 \\ dx & dt & 0 & dv \\ 0 & 0 & dx & d\varepsilon \end{vmatrix} = 0 \qquad (19)$$

which, when Eqs. (17) and (18) are used, yields

$$dv - (E)^{1/2}\, d\varepsilon = 0 \qquad \alpha \text{ direction} \qquad (20)$$

$$dv + (E)^{1/2}\, d\varepsilon = 0 \qquad \beta \text{ direction} \qquad (21)$$

These ordinary differential equations (20 and 21) can be integrated along the characteristics.

A nondimensional quantity θ is defined as

$$\theta = \frac{\int^{\varepsilon} (E)^{1/2}\, d\varepsilon}{\int^{\varepsilon} d\varepsilon} \qquad (22)$$

Note that $E(\varepsilon)$ represents a nondimensional tangent modulus whose value for $\varepsilon = 0$ is unity. The nondimensional quantity θ can be looked upon as an average square-root tangent modulus in the strain range defined by the initial- and steady-state strain.

From Eq. (22), the integration of Eqs. (20) and (21) yields

$$v - \theta\varepsilon = \alpha \qquad (23)$$

$$v + \theta\varepsilon = \beta \qquad (24)$$

or by inversion

$$v = \frac{\alpha + \beta}{2} \qquad (25)$$

$$\varepsilon = \frac{\beta - \alpha}{2\theta} \qquad (26)$$

where α and β are the integration constants. Note that if α and β are known, ε can be solved from Eqs. (22) and (26) by iteration, and v can be directly solved from Eq. (25). With α_i and β_i defined as the initial values of α and β, they can be derived from Eqs. (23) and (24) by using the initial conditions (Eq. 12) as

$$\alpha_i = v_0 \qquad (27)$$

$$\beta_i = v_0 \qquad (28)$$

In region ABC (Fig. 2), a *constant state* exists because all α and β curves starting from line AB have the same α_i and β_i values. When an α (or β) characteristic curve meets a boundary, it will reflect to become a β (or α) characteristic curve; the new value of α (or β) of the reflected wave should be computed from the boundary condition prescribed at that boundary.

A. Bar Impacting a Stationary Rigid Surface at Left End and Free at Right End

1. Impact Without Shock

The boundary conditions for this case are

$$v = 0 \quad \text{at} \quad x = 0 \quad \text{for} \quad t \geqq 0 \qquad (29)$$

$$\varepsilon = 0 \quad \text{at} \quad x = 1 + vt \quad \text{for} \quad t \geqq 0 \qquad (30)$$

Any α_i curve extending up to the right boundary will reflect back as a β curve whose β can be found by using Eq. (30) in Eq. (24). This gives

$$\beta_r = v_0 \qquad (31)$$

Equation (31) implies that the constant state ABC (Fig. 2) should be extended to cover CBD. In this extended region (ABD), the characteristics are straight lines, the slopes of which can be computed from Eqs. (17) and (18) by using Eqs. (12). This gives

$$\left(\frac{dt}{dx}\right)_{\alpha_i} = \frac{1}{1 + v_0} \qquad (32)$$

$$\left(\frac{dt}{dx}\right)_{\beta_i} = \frac{1}{v_0 - 1} \qquad (33)$$

It is known that a constant-state region can be neighbored by either a *simple-state* or another constant-state region separated by a shock front (Ref. 1). Temporarily leaving the case of AD being a shock front, assume that the simple wave and the next constant-state regions are as shown in Fig. 2. In the AEF region, the particle velocity is as indicated by Eq. (29). The β curves initiated from AB, which have β constants equal to v_0, will become α

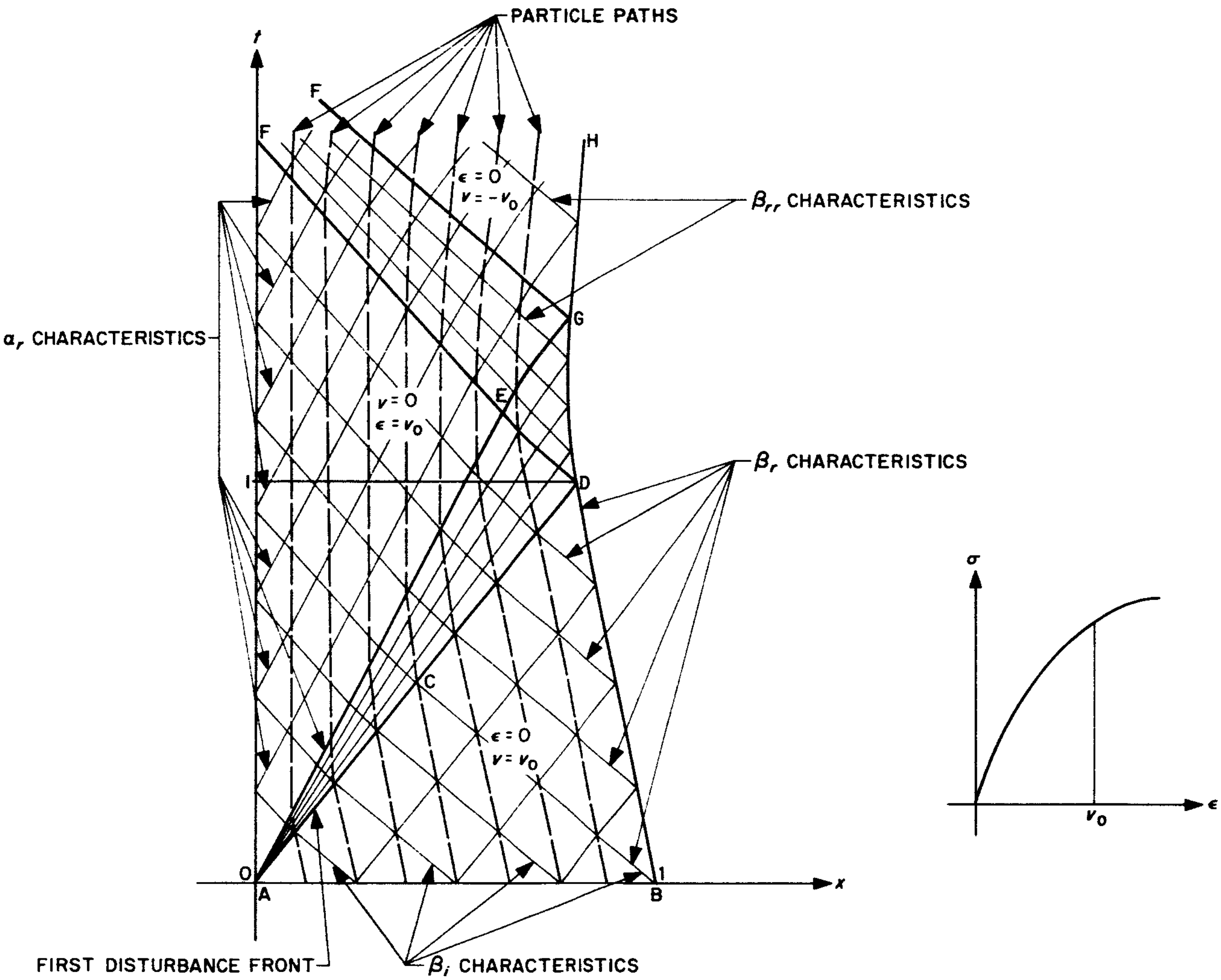


Fig. 2. Bar impacting a stationary rigid surface at left end and free at right end ($\varepsilon_{max} < \underline{\varepsilon}_h$—nonlinear case)

curves after reflection. The α_r constants of the reflected curves can be computed from Eqs. (23) and (29) as

$$\alpha_r = -\theta\varepsilon \tag{34}$$

The strain in AEF can then be obtained from Eqs. (26) and (34), remembering that $\beta_i = v_0$. In this case

$$\varepsilon = \frac{v_0}{\theta} \tag{35}$$

Using Eq. (22), Eq. (35) reduces to

$$\int_0^{\varepsilon} (E)^{1/2}\, d\varepsilon = v_0 \tag{36}$$

For the cases where

$$\int_0^{\infty} (E)^{1/2}\, d\varepsilon = \text{finite} \tag{37}$$

Eq. (36) may not have a solution. This means that infinitely large strains will take place on the contact surface (i.e., the interface will also be a failure front). Assume that Eq. (36) has a real root of ε_r so that

$$\int_0^{\varepsilon_r} (E)^{1/2}\, d\varepsilon = v_0 \tag{38}$$

is satisfied. Having computed ε and v in AEF, the slope of AE can be computed from Eq. (17) as

$$\left(\frac{dt}{dx}\right)_{\alpha_r} = \frac{1}{(1+\varepsilon_r)(E_r)^{1/2}} \tag{39}$$

where $E_r = E(\varepsilon_r)$. By comparing the slopes defined by Eqs. (32) and (39), it can be seen that the necessary condition for a simple wave region to take place is

$$\left(\frac{dt}{dx}\right)_{\alpha_i} < \left(\frac{dt}{dx}\right)_{\alpha_r} \tag{40}$$

or, by use of Eq. (35)

$$(1+\varepsilon_r)(E_r)^{1/2} < 1 + v_0 = 1 + \varepsilon_r\theta_r \tag{40a}$$

which is always satisfied, provided $E \leq 1$ for all strains; this is the actual case for all materials up to strain hardening. If Eq. (40) is not satisfied, there will be no simple wave region ADE. Consequently, AE will become a shock front between two constant-state regions.

If the material is linearly elastic without any yield point, the solution in ADF is

$$E = (E)^{1/2} = \theta = 1\,, \qquad \alpha_r = -v_0\,, \qquad \beta_i = v_0 \tag{41a}$$

$$v = 0 \tag{41b}$$

$$\varepsilon_r = v_0 \tag{41c}$$

$$\left(\frac{dt}{dx}\right)_{\beta} = \frac{1}{v_0 - 1} \tag{41d}$$

$$\left(\frac{dt}{dx}\right)_{\alpha_r} = \left(\frac{dt}{dx}\right)_{\alpha_i} = \frac{1}{v_0 + 1} \tag{41e}$$

which again indicates that there is no simple wave region (points D, E and G in Fig. 2 overlap). In this case AD is a shock front, but there is no energy loss due to heat across the shock (see Section III 2).

Without having the unloading characteristics of the stress-strain curve (Fig. 1), it is not possible to study the conditions in the DFG region where the prevailing physical condition is unloading. For the elastic case (DFH Fig. 3), one can find that

$$\begin{aligned} \beta_{rr} &= -v_0 \\ \alpha_r &= -v_0 \\ \varepsilon &= 0 \\ v &= -v_0 \\ \left(\frac{dt}{dx}\right)_{\alpha_r} &= \frac{1}{1 - v_0} \\ \left(\frac{dt}{dx}\right)_{\beta_{rr}} &= -\frac{1}{1 + v_0} \end{aligned} \tag{42}$$

Figure 3 shows the solution up to $t = 2$. For $t > 2$, the elastic bar will leave the rigid surface with $-v_0$ velocity and with its original length.

In Figs. (2) and (3), the particle paths are indicated by dashed lines. When there is a simple wave region, it is clear that the particles gradually change their velocity vector from the initial value to the at rest value. However, if there is a shock front in place of a simple wave region, the velocity change will be abrupt.

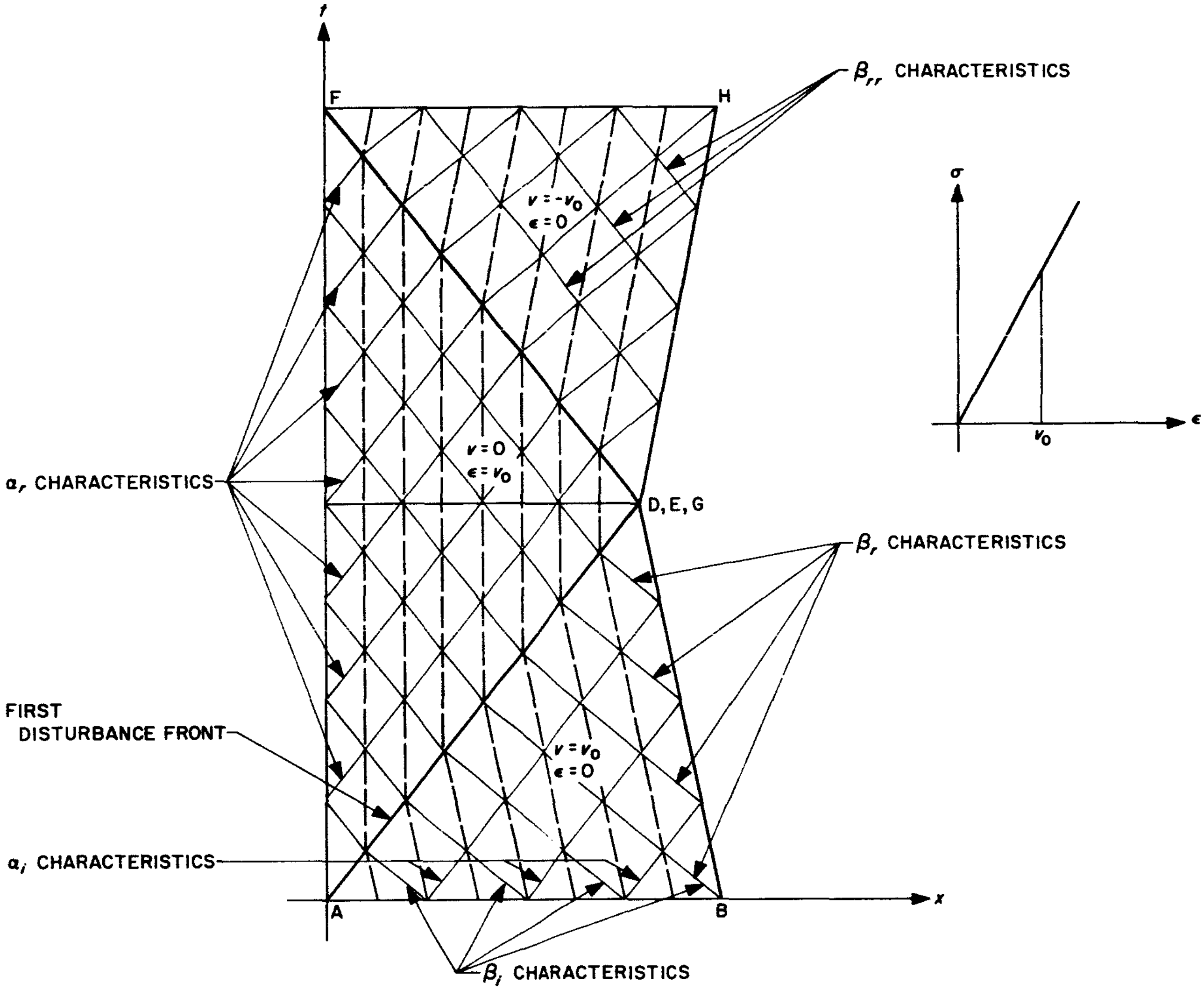


Fig. 3. Elastic bar impacting a stationary rigid surface at left end and free at right end

We will conclude the study of a bar impacting a stationary rigid surface at left end and free at right end by restating the following:

1. The interface of impact remains at rest.

2. A step strain of $\varepsilon_r = v_0/\theta_r$ will take place on the interface and will survive at least for a nondimensional time of 2.

2. Impact With Shock

A *shock front* is defined as the loci of points in the x,t plane where there may be discontinuities in the unknown functions ε and v. If there are discontinuities in the *first derivatives* of the unknown functions (ε and v), the loci of points are called *characteristics*. With these definitions, the α characteristic initiating from the origin (Fig. 2) also is a shock front. However, we prefer to call shock fronts as those fronts across which one has to allow temperature differences for the sake of conservation of energy; this additional constraint shows that the α characteristic passing from the origin in the elastic case is no longer a shock front. We will call this front a *stress front* to emphasize the fact that there are still discontinuities across the front in the unknown functions. The definition of a shock front then becomes the loci of points across which the physical quantities have discontinuities and temperature difference takes place to compensate for the mechanical energy loss.

With the new definition, the case for which Eq. (36) has no real roots or, more generally, the case for which

$$v_0 \geqq \int_0^\infty (E)^{1/2} d\varepsilon \tag{43}$$

holds, the interface will become a failure front which is a shock front. The velocity of the shock in this case is zero. The solution is qualitatively shown in Fig. 4.

Consider the case when the strain, ε_r, in region AEF of Fig. 2 is larger than ε_h, the strain-hardening strain (Fig. 1), where

$$[E(\varepsilon_h)]^{1/2} > [E(\varepsilon_r)]^{1/2}$$

In this case, it can be seen that Eq. 40 is not satisfied, and that the shock will propagate with a velocity H for which

$$(1 + v_0) = \left(\frac{dx}{dt}\right)_{\alpha_i} < H < \left(\frac{dx}{dt}\right)_{\alpha_r} = (1 + \varepsilon_r)(E_r)^{1/2} \tag{44}$$

holds (Fig. 5).

Let us obtain the shock relations using the Rankine–Hugoniot procedure (Ref. 2). Call W (a positive number) the speed of the shock relative to the particles ahead of it. The particle velocity, density, strain, stress, and temperature in front of the shock are U_0^*, ρ_0^*, ε_0^*, σ_0^*, and T_0^*. Respectively, the corresponding quantities behind the shock are U_r, ρ_r, ε_r, σ_r, and T_r. With the aid of Fig. 6, the requirements that the incoming and outgoing flows should leave no mass, no momentum, and no energy in the shock can be written as

mass

$$\rho_0^* W = \rho_r [W + U_0^* - U_r] \tag{45}$$

momentum

$$-\sigma_0^* + \rho_0^* W^2 = -\sigma_r + \rho_r [W + U_0^* - U_r]^2 \tag{46}$$

energy

$$-\frac{\sigma_0^*}{\rho_0^*} + \frac{1}{\rho_0^*}\int_0^{\varepsilon_0^*} \sigma d\underline{\varepsilon} + \underline{c}\, T_0^* + \frac{W^2}{2} = -\frac{\sigma_r}{\rho_r} + \frac{1}{\rho_0^*}\int_0^{\varepsilon_r} \sigma d\underline{\varepsilon} + \underline{c}\, T_r + \frac{(W + U_0^* - U_r)^2}{2} \tag{47}$$

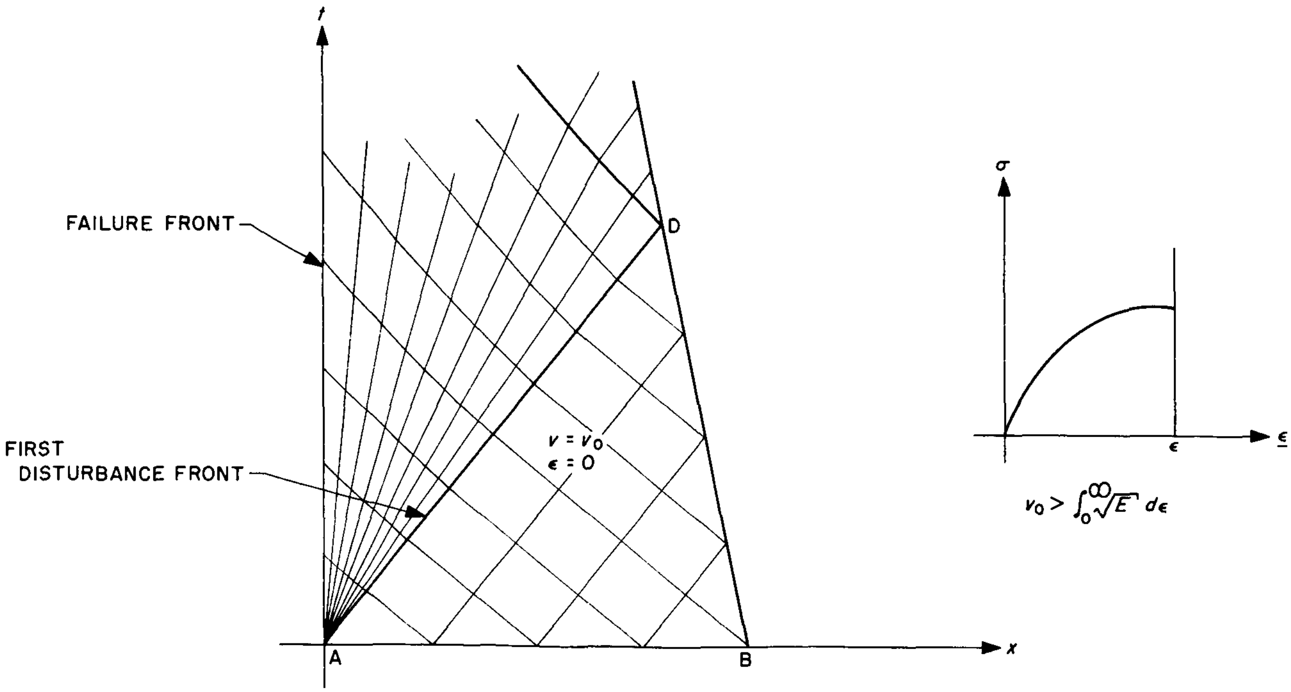


Fig. 4. Bar impacting rigid surface and failing $\left[v_0 > \int_0^\infty (E)^{1/2} d\varepsilon\right]$

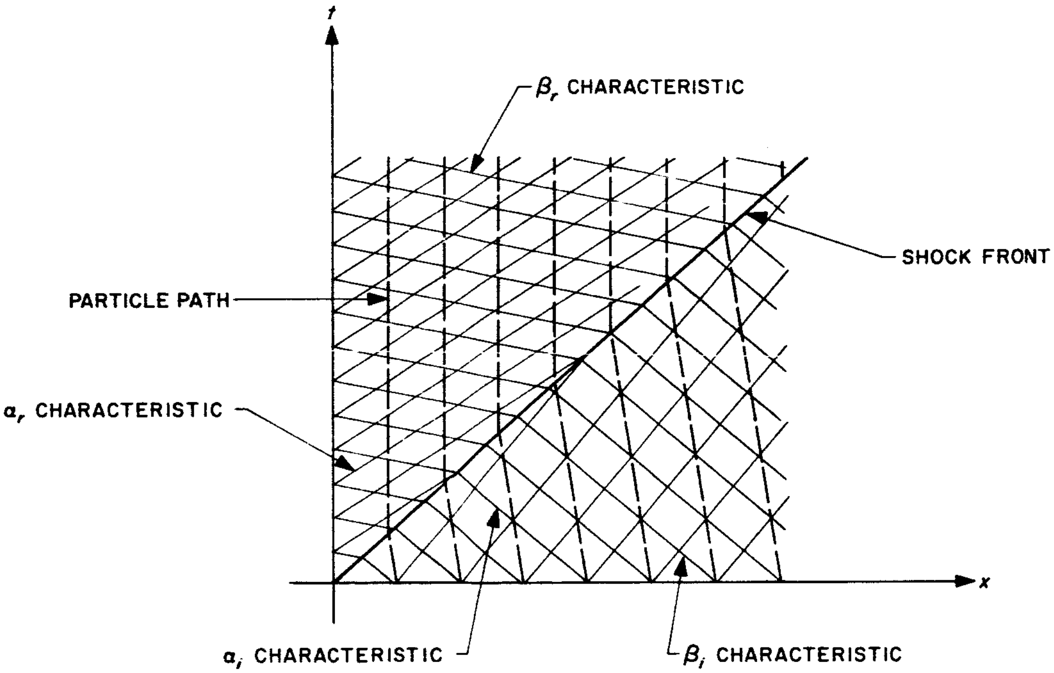


Fig. 5. Right-moving shock

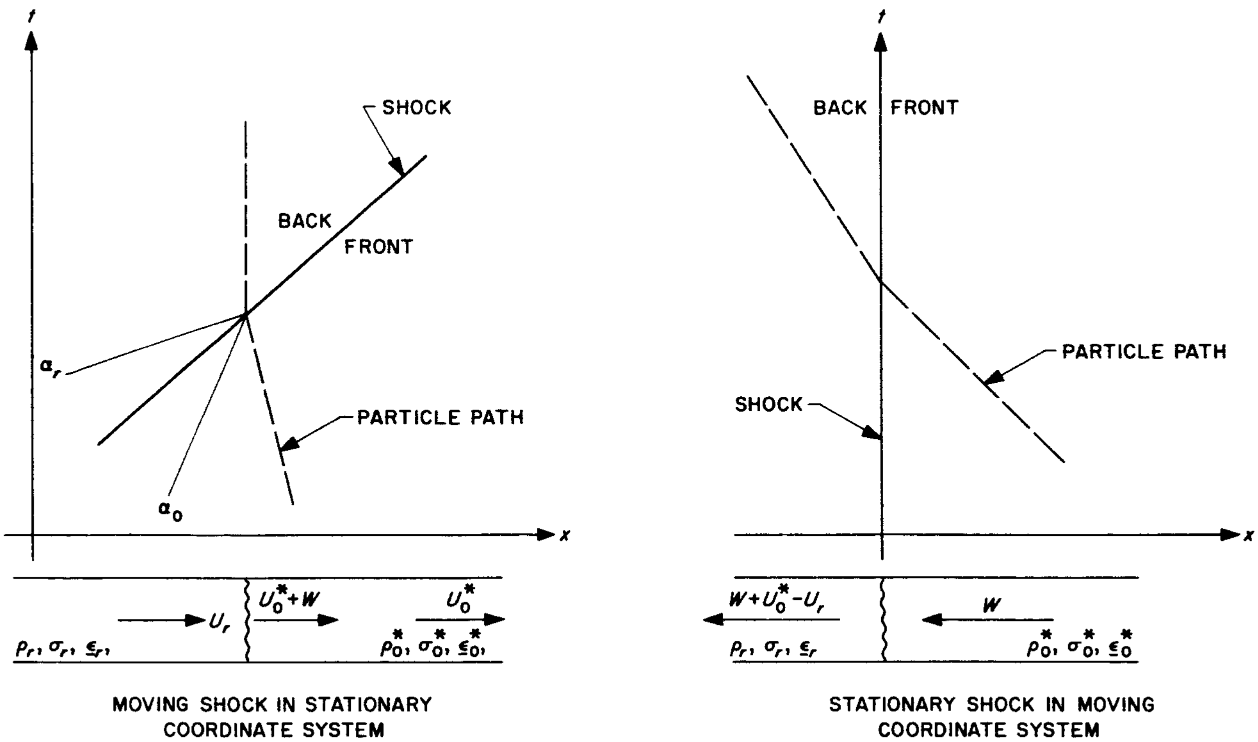


Fig. 6. Representation of shock in moving and stationary coordinate systems

In Eq. 47, the first three terms (on both sides) are pressure, deformation, and thermal energies. The constant $\underline{c}$ represents the specific heat of the bar material.

Using Eqs. (45), (46), and (1)

$$W^2 = \frac{1 + \mu \underline{\varepsilon}_0^*}{\mu \rho_0} \frac{\sigma_r - \sigma_0^*}{(\underline{\varepsilon}_r - \underline{\varepsilon}_0^*)} \tag{48}$$

or

$$W = \frac{1}{\rho_0} \frac{\sigma_r - \sigma_0^*}{U_0^* - U_r} \tag{49}$$

From Eqs. (48) and (49)

$$\underline{\varepsilon}_r - \underline{\varepsilon}_0^* = \frac{U_0^* - U_r}{W\mu}(1 + \mu\varepsilon_0^*) \tag{50}$$

For the cases where there are no simple wave regions, observing that ahead of the shock $\underline{\varepsilon}_0^* = \sigma_0^* = 0$, $U_0^* = U_0$ and behind the shock $U_r = 0$, one obtains

$$W^2 = \frac{1}{\rho_0 \mu} \frac{\sigma_r}{\underline{\varepsilon}_r} \tag{51}$$

or

$$W = \frac{1}{\rho_0} \frac{\sigma_r}{U_0} \tag{52}$$

which requires

$$\underline{\varepsilon}_r = \frac{U_0}{\mu W} \tag{53}$$

For this case, if the material is linearly elastic because of Eqs. (41c), (9d) and (9f)

$$W = c \tag{54}$$

and, the velocity of the front

$$W + U_0 = c + U_0 = c(1 + v_0) \tag{55}$$

which, when compared with Eq. (39), indicates that the shock front is coincident with the α characteristic passing through the origin. Let us see that this is a stress front rather than a shock front. The necessary condition for this is that no heat energy is generated behind the shock. Using Eqs. (49) and (1), Eq. (47) becomes

$$\frac{1}{2}(\underline{\varepsilon}_r - \underline{\varepsilon}_0^*)(\sigma_r + \sigma_0^*)\left(\frac{\mu}{1 + \mu\underline{\varepsilon}_0^*}\right) = \int_{\underline{\varepsilon}_0^*}^{\underline{\varepsilon}_r} \sigma d\underline{\varepsilon} + \rho_0^* \underline{c}(T_r - T_0^*) \tag{56}$$

In Eq. 56, the first term on the right is the increase of deformation energy; the second term is the increase of thermal energy as the material crosses the shock. It is obvious that the first term on the right is equal to the term on the left if the material is linearly elastic, and the bar is laterally confined and initially unstrained ($\mu = 1, \varepsilon_0^* = 0$). For this case

$$T_r = T_0 \tag{57}$$

implying that there is no temperature increase behind the shock. If the bar is elastic and laterally unconfined, Eq. (56) shows a temperature decrease when the bar is initially unstrained. This physically unjustified result is a consequence of the fact that the lateral motion has been ignored in the derivation of Eq. (56).

In Fig. 7, three different types of stress-strain relationships are shown (i.e., nonlinear, linear, and locking); assume that the material of the bar may be any one of these. For the nonlinear case, a simple wave region always develops. For the upper boundary (AE) of the simple wave region (Fig. 2) to fall in the solution domain, it is necessary that Eq. (36) have a real solution. If this is not the case, the rigid boundary will be a failure surface, as well as a contact surface. For the linear case, there is no simple wave region. Boundary regions AE and AD (Fig. 2) are overlapped; this line is a stress front, but it is not considered as a shock front since there is no temperature rise across the front. For the locking case, there is no simple wave region. However, there is a shock front which, in the x, t plane, has a slope of

$$\left(\frac{dt}{dx}\right)_s = \frac{c}{W + U_0} = \left\{ v_0 + \left[\frac{\frac{\sigma_r}{\underline{\varepsilon}_r}}{\left(\frac{d\sigma}{d\underline{\varepsilon}}\right)_{\underline{\varepsilon}=0}} \right]^{1/2} \right\}^{-1} \tag{58}$$

The energy (indicated by the shaded area in Fig. 7c) is provided by a temperature increase across the shock. The shock for this case is shown in Fig. 5. The values of $\underline{\varepsilon}_r$ and σ_r can be found from $\sigma(\underline{\varepsilon})$ curve and

$$\frac{1}{2}\underline{\varepsilon}_r \sigma_r = \frac{1}{2}\rho_0 U_0^2 \tag{59}$$

Equation (59) can be derived from Eqs. (45), (49), and (1), and the conditions $\varepsilon_0^* = \sigma_0^* = 0$, $U_0^* = U_0$. However, since most of the solids do not behave as shown in Fig. 7c, a shock will normally not take place, because of a possible strain hardening, unless the material is laterally confined.

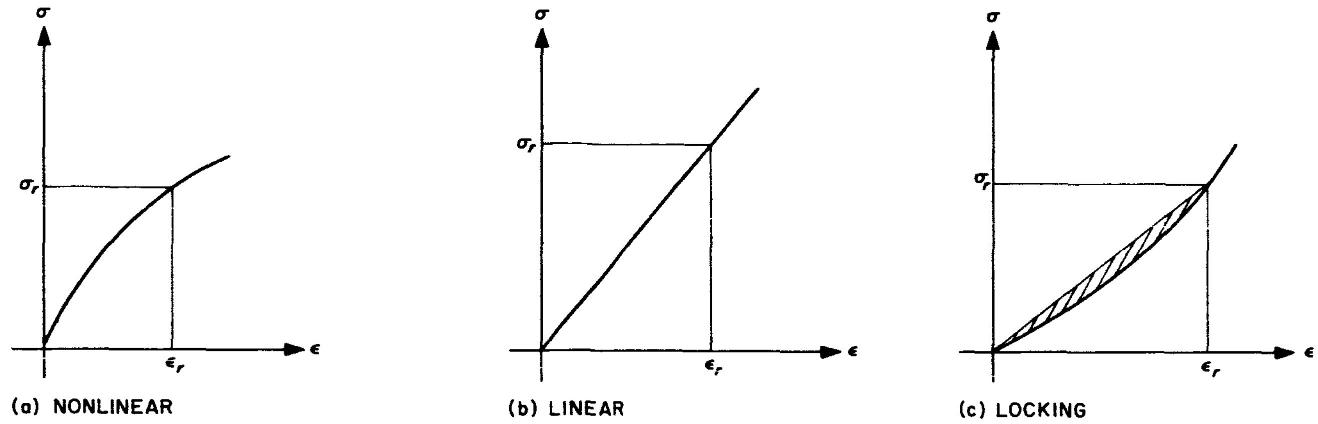


Fig. 7. Various stress-strain relationships

B. Bar Impacting a Flexible Surface at Rest at Left End and Free at Right End

The boundary conditions for this case at the interface, and at $x = 1 + v_0 t$ are, respectively

$$U_f = \bar{U}_f, \qquad \sigma_f = \bar{\sigma}_f \qquad \text{for} \qquad t \geqq 0 \tag{60}$$

$$\varepsilon = 0 \qquad \text{for} \qquad t \geqq 0 \tag{61}$$

where $\bar{U}_f$ and $\bar{\sigma}_f$ are unknown and pertain to the flexible surface (see Fig. 8). When the α_i characteristics meet the free boundary, they become β_r curves that have the same constant value as β_i, i.e.,

$$\beta_r = \beta_i = v_0 \tag{62}$$

In the flexible surface, the initial conditions are

$$\bar{\varepsilon}_i = 0, \qquad \bar{v}_i = 0 \qquad \text{for} \qquad t = 0 \tag{63}$$

(Symbols with bar above represent, in the flexible surface, the counterparts of entities defined for the bar).

From Eqs. (63) and (23)

$$\bar{\alpha}_i = 0 \tag{64}$$

The interface velocity and the interface strain computed from the characteristics of the bar by the use of Eqs. (25), (26), and (62) are

$$v_f = \frac{\alpha_r + \beta_i}{2} = \frac{\alpha_r + v_0}{2} \tag{65}$$

$$\varepsilon_f = \frac{\beta_i - \alpha_r}{2\theta} = \frac{v_0 - \alpha_r}{2\theta} \tag{66}$$

The interface velocity and the interface strain can also be computed from the characteristics in the flexible body by the use of Eqs. (25), (26), and (64) as

$$\bar{v}_f = \frac{\bar{\alpha}_i + \bar{\beta}_r}{2} = \frac{\bar{\beta}_r}{2} \tag{67}$$

$$\bar{\varepsilon}_f = \frac{\bar{\beta}_r - \bar{\alpha}_i}{2\bar{\theta}} = \frac{\bar{\beta}_r}{2\bar{\theta}} \tag{68}$$

Noting that the nondimensionalization in the flexible body is performed by the use of

$$\bar{c} = \left[\frac{\left(\dfrac{d\bar{\sigma}}{d\bar{\varepsilon}} \right)_{\bar{\varepsilon}=0}}{\bar{\rho}_0 \; \bar{\mu}} \right]^{1/2} \tag{69a}$$

$$\bar{x} = \frac{\bar{X}}{L} \tag{69b}$$

$$\bar{t} = \frac{\bar{T}}{L/\bar{c}} \tag{69c}$$

$$\bar{v} = \frac{\bar{U}}{\bar{c}} \tag{69d}$$

$$v_0 = \frac{\bar{U}_0}{\bar{c}} \tag{69e}$$

$$\bar{E}\,(\bar{\varepsilon}) = \frac{\dfrac{d\bar{\sigma}}{d\bar{\varepsilon}}}{\left(\dfrac{d\sigma}{d\bar{\varepsilon}} \right)_{\bar{\varepsilon}=0}} \tag{69f}$$

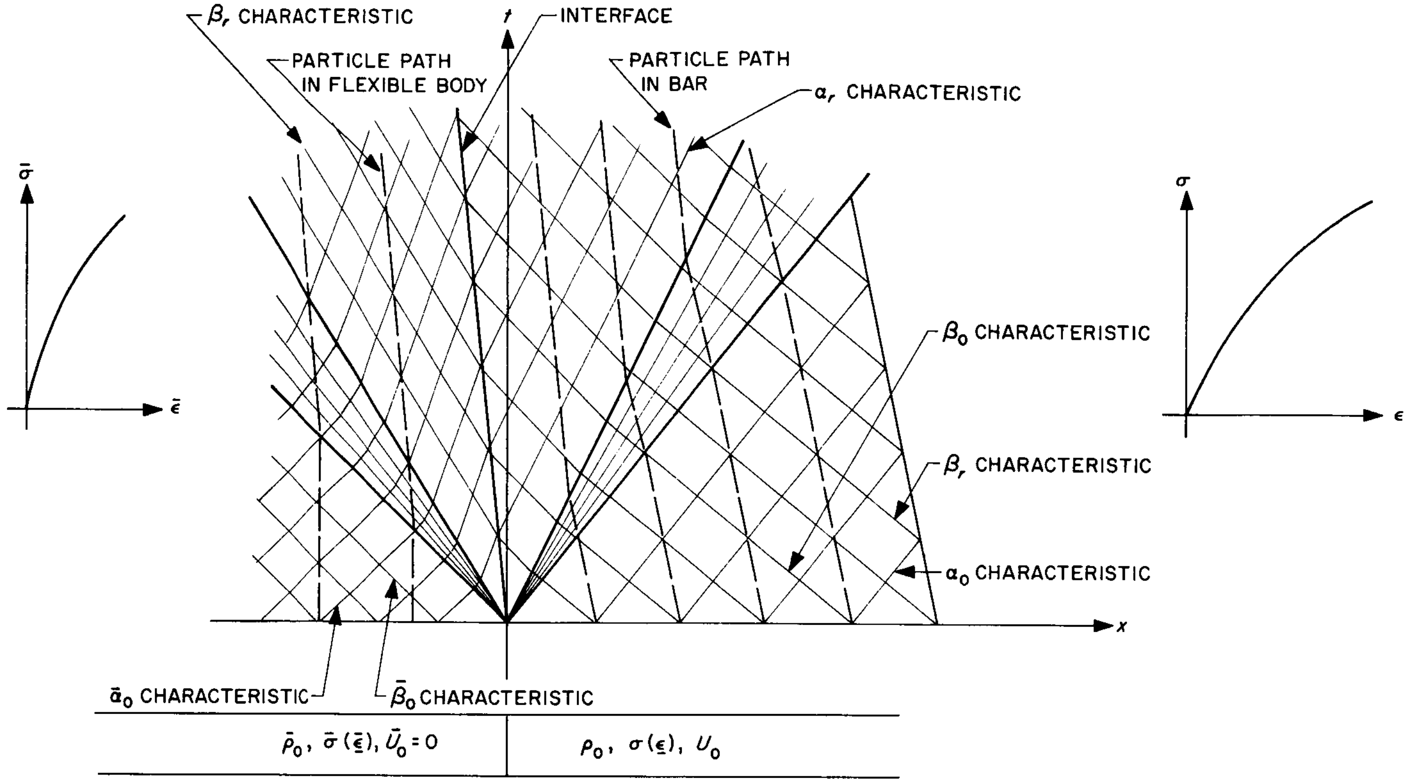


Fig. 8. Bar impacting flexible surface at rest at left end and free at right end (no shock, no failure case)

and defining

$$\gamma = \frac{\sigma(\varepsilon)}{\varepsilon\left(\dfrac{d\sigma}{d\varepsilon}\right)_{\varepsilon=0}} \tag{70}$$

$$\bar{\gamma} = \frac{\bar{\sigma}(\bar{\varepsilon})}{\bar{\varepsilon}\left(\dfrac{d\bar{\sigma}}{d\bar{\varepsilon}}\right)_{\bar{\varepsilon}=0}} \tag{71}$$

one can write from Eqs. (60), (66), and (68)

$$\frac{\bar{\gamma}}{\bar{\mu}}\left(\frac{d\bar{\sigma}}{d\bar{\varepsilon}}\right)_{\bar{\varepsilon}=0}\frac{\bar{\beta}_r}{2\bar{\theta}} = \frac{v_0 - \alpha_r}{2\theta}\,\frac{\gamma}{\mu}\left(\frac{d\sigma}{d\varepsilon}\right)_{\varepsilon=0}$$

or, using Eqs. (69a) and (8)

$$\bar{\gamma}\,\bar{c}^2\,\bar{\rho}_0\,\frac{\bar{\beta}_r}{\bar{\theta}} = \frac{v_0 - \alpha_r}{\theta}\,\gamma\,c^2\,\rho_0 \tag{72}$$

and from Eqs. (65) and (67)

$$\frac{\bar{\beta}_r}{2}\bar{c} = \frac{\alpha_r + v_0}{2}c$$

or

$$\bar{\beta}_r\bar{c} = (\alpha_r + v_0)\,c \tag{73}$$

Defining the nondimensional quantities e and f as

$$e = \frac{\bar{c}\,\bar{\rho}_0}{c\,\rho_0} = \left[\frac{\left(\dfrac{d\bar{\sigma}}{d\bar{\varepsilon}}\right)_{\bar{\varepsilon}=0}\bar{\rho}_0\mu}{\left(\dfrac{d\sigma}{d\varepsilon}\right)_{\varepsilon=0}\rho_0\bar{\mu}}\right]^{1/2} \tag{74}$$

$$f = \frac{\bar{\gamma}}{\gamma}\,\frac{\theta}{\bar{\theta}} \tag{75}$$

and using Eqs. (74) and (75) in obtaining α_r and $\bar{\beta}_r$, from Eqs. (72) and (73), one can write

$$\alpha_r = v_0\frac{1 - ef}{1 + ef} \tag{76}$$

which, when substituted into Eqs. (65) and (66) gives

$$v_f = v_0 \frac{1}{1 + ef} \tag{77}$$

$$\varepsilon_f = \frac{v_0}{\theta} \frac{ef}{1 + ef} \tag{78}$$

It should be noted that v_f (Eq. 77) is nondimensionalized with c (f and θ are positive nondimensional quantities). The average tangent modulus is θ, and f is a function of θ's and γ's. The γ's are a measure of the nonlinearity in the stress-strain relationships. For linearly elastic materials, $\gamma = \theta = \bar{\gamma} = \bar{\theta} = 1$ for which Eqs. (77) and (78) become

$$v_f = \frac{v_0}{1 + e} \tag{79}$$

$$\varepsilon_f = v_0 \frac{e}{1 + e} \tag{80}$$

Note that e (Eqs. 79 and 80) is a nondimensional, positive quantity that can take values between zero and infinity. The case of the bar impacting an infinitely soft material is represented by $e = 0$. For $e = \infty$, Eqs. (77) and (78) are reduced to Eqs. (29) and (35), respectively. Equation (78) is very useful in the computation of the initial contact strains, and, hence, the constant stresses.

If at least one of the materials at the contact surface cannot provide the ε_f of Eq. (78), a failure front will be established at the contact surface. However, because of the lateral confinement of the impacted material, the strain-hardening phenomenon may take place; this may cause a shock front in the flexible body (Fig. 9) for which

$$\left(\frac{d\bar{t}}{d\bar{x}}\right)_s = -\left[\frac{\left(\frac{\bar{\sigma}_f}{\bar{\varepsilon}_f}\right)}{\left(\frac{d\bar{\sigma}}{d\bar{\varepsilon}}\right)_{\bar{\varepsilon}=0}}\right]^{-1/2} \tag{81}$$

In Fig. 10, various possible cases of a bar impacting a flexible body are shown qualitatively. Any one of these cases can be easily studied analytically.

C. Bar Impacting a Rigid Surface at Rest at Left End, and Attached to Another Bar of the Same Initial Velocity at the Right End

The boundary conditions for this case are

$$v = 0 \quad \text{at} \quad x = 0 \quad \text{for} \quad t \geqq 0 \tag{82}$$

$$v = v_f = \bar{v}_f \frac{c}{\bar{c}} \qquad \sigma = \bar{\sigma}_f \quad \text{at} \quad x = 1 + v_f t \quad \text{for} \quad t \geqq 0 \tag{83}$$

where v_f and σ_f are unknown.

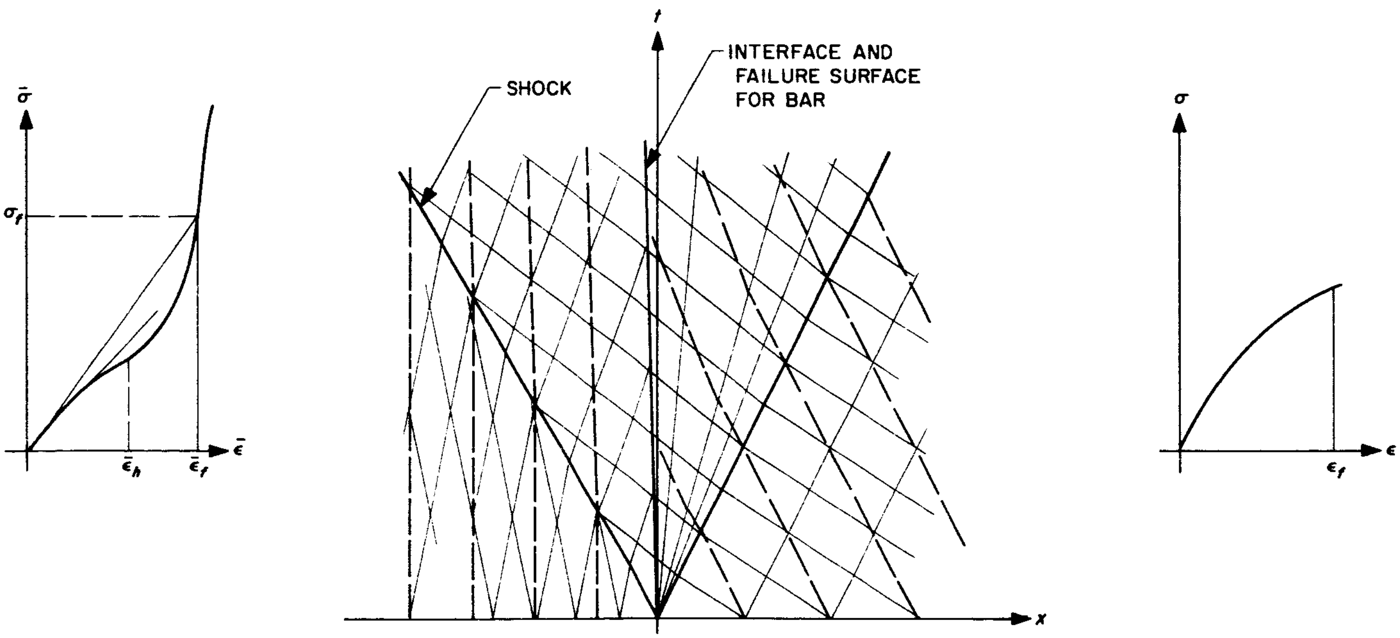
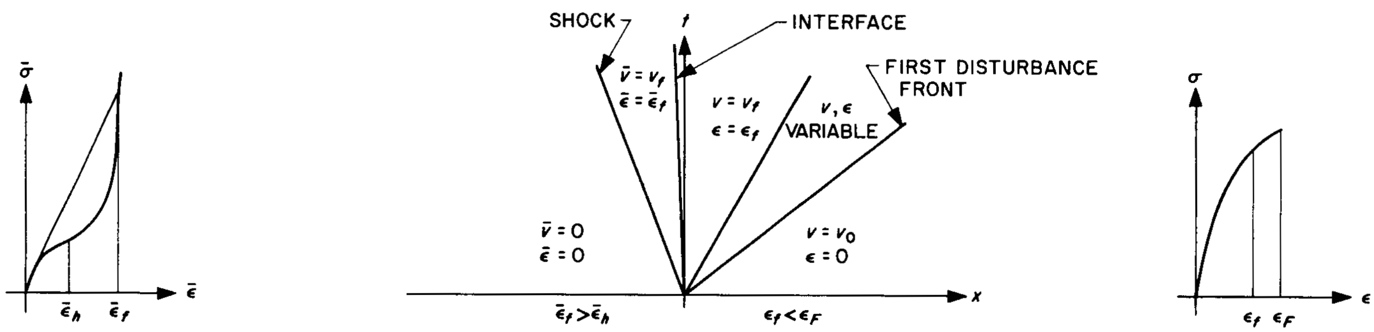


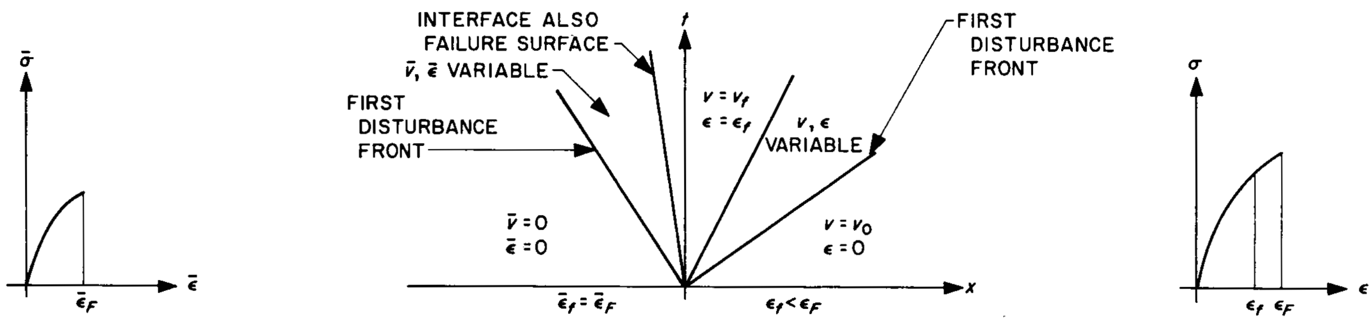
Fig. 9. Impacting bar fails on the interface, creating a shock wave in the impacted body ($v_0 > \varepsilon_f$, $\bar{\varepsilon}_f > \bar{\varepsilon}_h$)

		IMPACTED BODY		
		NO SHOCK	SHOCK	FAILURE
IMPACTING BAR	NO SHOCK	(8).	(10b)	(10c)
	SHOCK	(10d)	(10e)	(10f)
	FAILURE	(10g)	(9)	(10h)

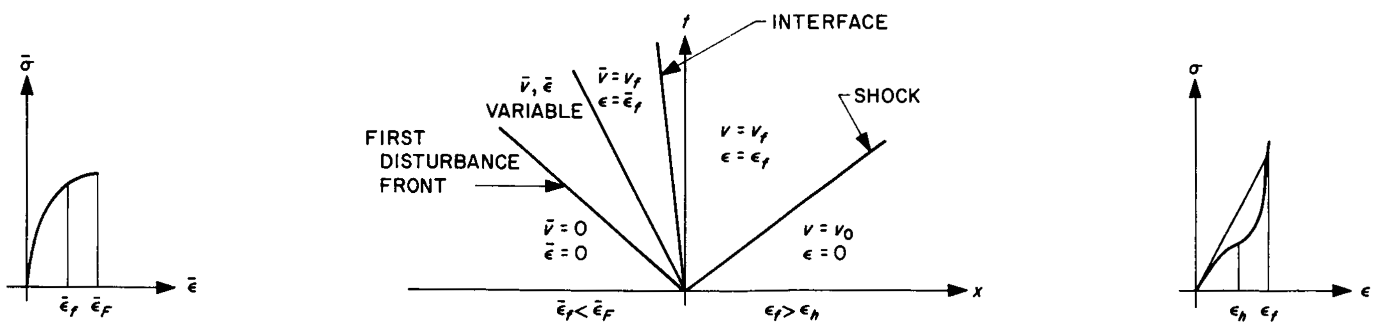
(a) TABLE FOR FIGURE NUMBERS OF VARIOUS CASES



(b) NO SHOCK IN BAR, SHOCK IN IMPACTED BODY

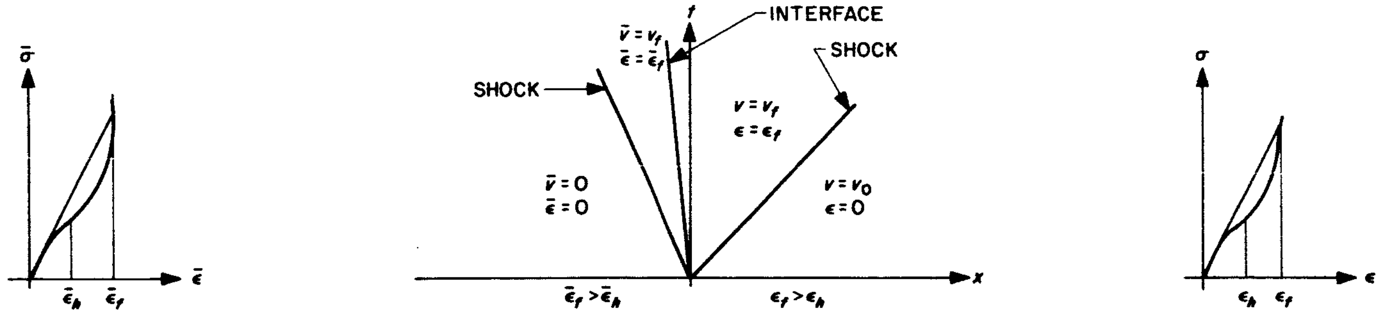


(c) NO SHOCK IN BAR, FAILURE IN IMPACTED BODY

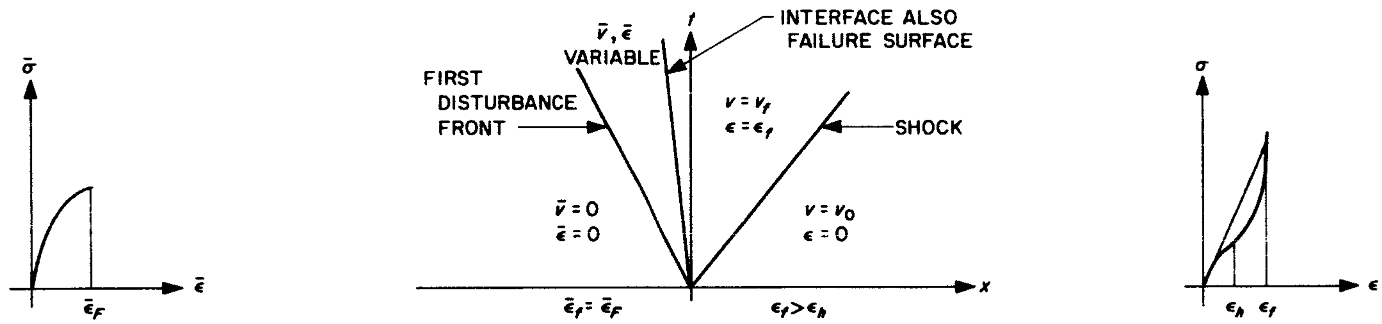


(d) SHOCK IN BAR, NO SHOCK IN IMPACTED BODY

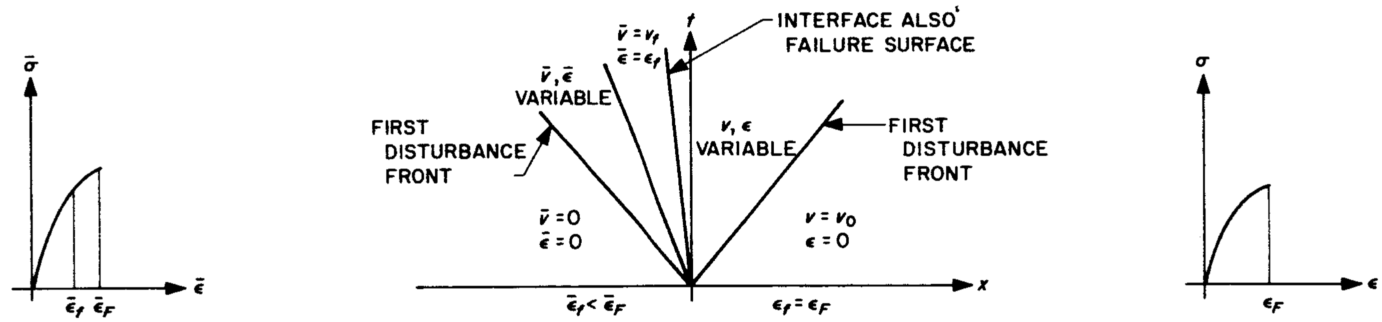
Fig. 10. Various cases of bar impacting a flexible body



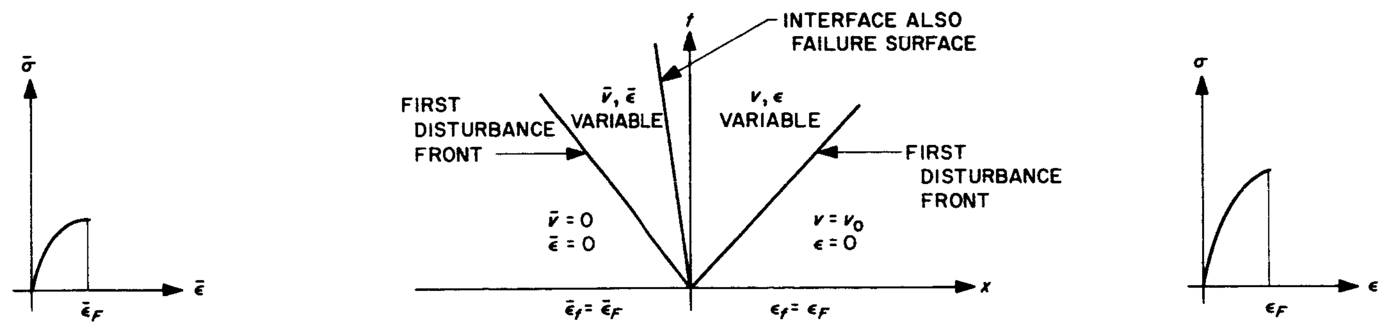
(e) SHOCK IN BAR, SHOCK IN IMPACTED BODY



(f) SHOCK IN BAR, FAILURE IN IMPACTED BODY



(g) FAILURE IN BAR, NO SHOCK IN IMPACTED BODY



(h) FAILURE IN BAR, FAILURE IN IMPACTED BODY

Fig. 10. Various cases of bar impacting a flexible body (Cont'd)

(Symbols with bar above represent, in the bar to the right, the counterpart of entities defined for the bar to the left.)

Let us first study the case of linearly elastic bars (Fig. 11). When the β_i curve meets the boundary on the left, it becomes an α_r curve that has a constant value $-v_0$ (from Eq. 34 with $\theta = 1$). The initial conditions in the neighboring bar to the right are

$$\left.\begin{aligned} \bar{\varepsilon}_i &= 0 \\ \bar{U}_i &= U_0 \end{aligned}\right\} \quad \text{for} \quad t = 0 \tag{84}$$

Then, from Eq. (24)

$$\bar{\beta}_i = \frac{v_0 c}{\bar{c}} \tag{85}$$

Equation (85) also is the value of the $\bar{\beta}_r$ characteristics coming from the free boundary at the right, similar to Eq. (31). Note that the quantities for the bar at the right are assumed nondimensionalized with Eqs. (69). The interface velocity and the interface strain computed from the characteristics of the bar to the left by the use of Eqs. (25) and (26) are

$$v_f = \frac{\alpha_i + \beta_r}{2} = \frac{v_0 + \beta_r}{2} \tag{86}$$

$$\varepsilon_f = \frac{\beta_r - \alpha_i}{2} = \frac{\beta_r - v_0}{2} \tag{87}$$

The same quantities can be computed from the bar on the right by the use of the same equations along the interface. These equations are

$$\bar{v}_f = \frac{\bar{\alpha}_r + \bar{\beta}_i}{2} = \frac{\bar{\alpha}_r + v_0 \dfrac{c}{\bar{c}}}{2} \tag{88}$$

$$\varepsilon_f = \frac{\bar{\beta}_i - \bar{\alpha}_r}{2} = \frac{v_0 \dfrac{c}{\bar{c}} - \bar{\alpha}_r}{2} \tag{89}$$

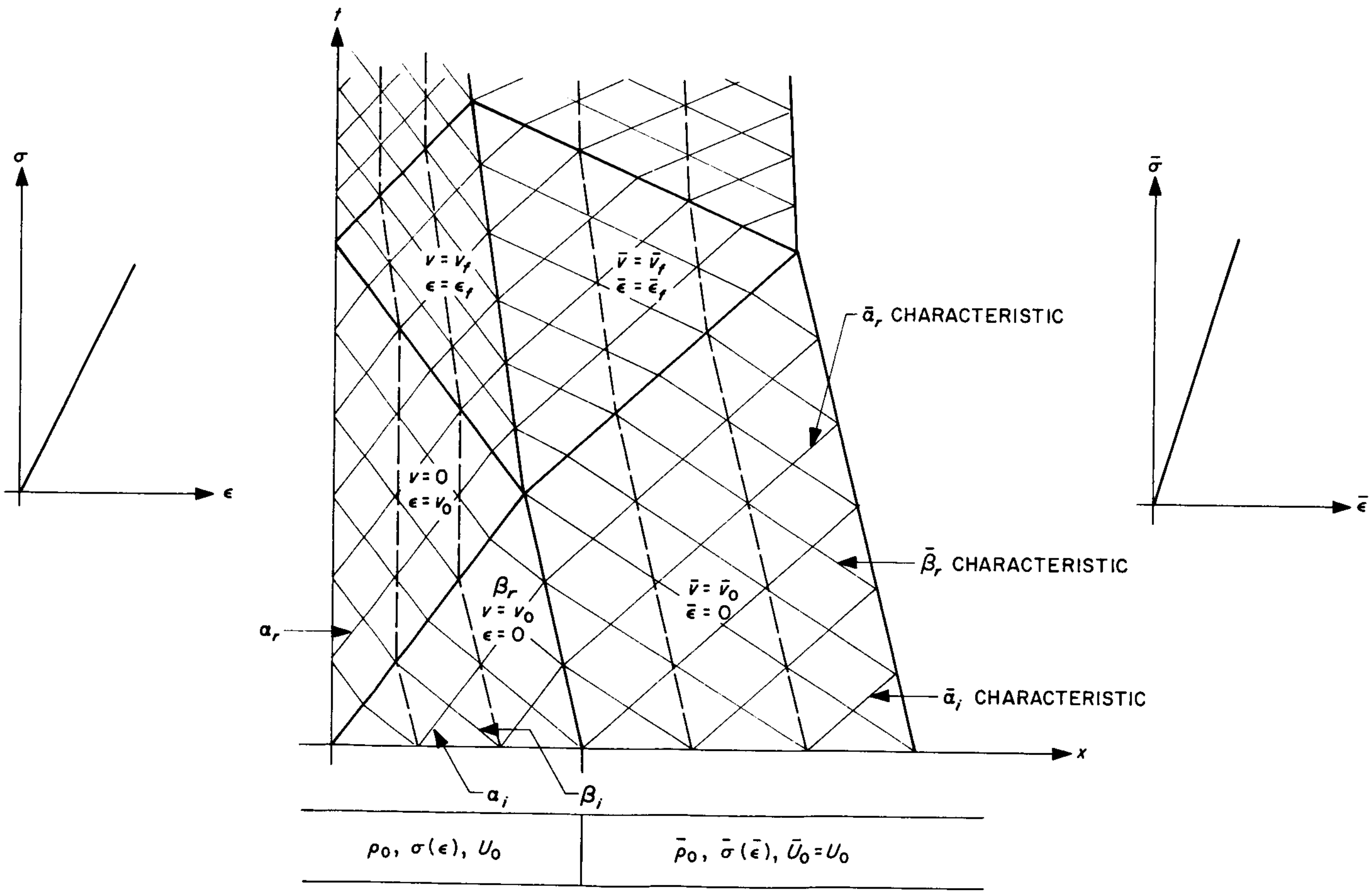


Fig. 11. Elastic bar impacting a rigid surface at rest at left end, and attached to another elastic bar of the same initial velocity at right end

Using Eqs. (69) through (71), and Eqs. (86) through (89), one writes from Eqs. (83) with $\bar{\theta} = \theta = \gamma = \bar{\gamma} = \bar{E} = E = 1$

$$\left(\frac{d\sigma}{d\varepsilon}\right)_{\varepsilon=0} \frac{\beta_r + v_0}{2\mu} = \left(\frac{d\bar{\sigma}}{d\bar{\varepsilon}}\right)_{\bar{\varepsilon}=0} \frac{v_0 \dfrac{c}{\bar{c}} - \bar{\alpha}_r}{2\bar{\mu}}$$

or

$$\left(\frac{d\sigma}{d\varepsilon}\right)_{\varepsilon=0} \frac{\beta_r + v_0}{\mu} = \left(\frac{d\bar{\sigma}}{d\bar{\varepsilon}}\right)_{\bar{\varepsilon}=0} \frac{\dfrac{c}{\bar{c}} v_0 - \bar{\alpha}_r}{\bar{\mu}} \qquad (90)$$

and

$$c\,\frac{-v_0 + \beta_r}{2} = \bar{c}\,\frac{\bar{\alpha}_r + v_0 \dfrac{c}{\bar{c}}}{2}$$

or

$$c\,(-v_0 + \beta_r) = \bar{c}\left(\bar{\alpha}_r + \frac{c}{\bar{c}} v_0\right) \qquad (91)$$

Using Eq. (74), one can solve for α_r and β_r from Eqs. (90) and (91)

$$\bar{\alpha}_r = v_0 \frac{c}{\bar{c}} \frac{e - 3}{e + 1} \qquad (92)$$

$$\beta_r = v_0 \frac{3e - 1}{e + 1} \qquad (93)$$

where e is as defined in Eq. (74).

Using Eq. (93) in Eqs. (86) and (87), the equations

$$\varepsilon_f = v_0 \frac{2e}{1 + e} \qquad (94)$$

and

$$v_f = v_0 \frac{e - 1}{e + 1} \qquad (95)$$

are obtained. Note that e is a measure of the relative rigidities of the bars. The case of $e = 0$ corresponds to a free end, and $e = \infty$ corresponds to a rigid end. When both bars are identical, $e = 1$.

The case of nonlinear material properties is complicated because of the presence of the incoming simple wave and the outgoing simple wave at the interface (see Fig. 12). However, some time later, the interface will be between two constant states, one in each bar. From Eqs. (34) and (35), the incoming α still has the value $-v_0$. Using Eqs. (25) and (26), Eqs. (86) through (89) can be written as

$$v_f = \frac{\beta_r - v_0}{2} \qquad (96)$$

$$\varepsilon_f = \frac{\beta_r + v_0}{2\theta} \qquad (97)$$

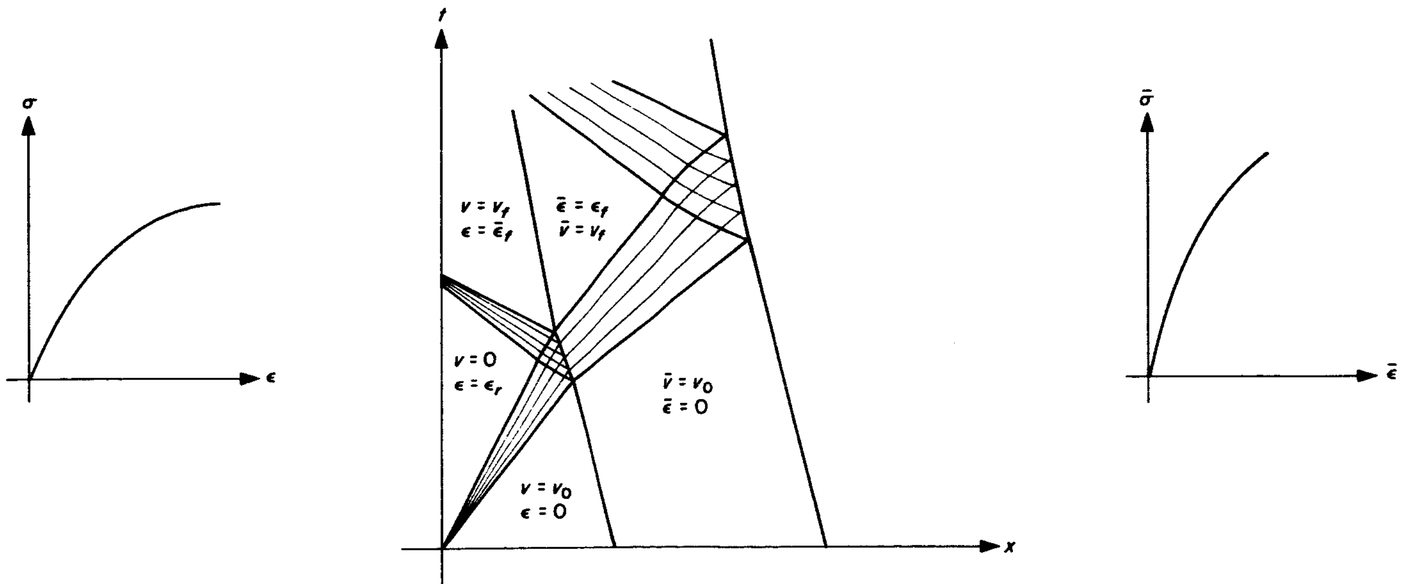


Fig. 12. Qualitative representation of the case of Fig. 11 when the bar materials are nonlinear

$$v_f = \frac{\bar{\alpha}_r + v_0 \frac{c}{\bar{c}}}{2} \tag{98}$$

$$\varepsilon_f = \frac{-\bar{\alpha}_r + v_0 \frac{c}{\bar{c}}}{2\bar{\theta}} \tag{99}$$

Equations (96) through (99), and (69) through (71) lead to the following from Eq. (83)

$$\gamma \left(\frac{d\sigma}{d\varepsilon}\right)_{\varepsilon=0} \frac{\beta_r + v_0}{2\mu\theta} = \bar{\gamma} \left(\frac{d\bar{\sigma}}{d\bar{\varepsilon}}\right)_{\bar{\varepsilon}=0} \frac{-\alpha_r + v_0 \frac{c}{\bar{c}}}{2\bar{\theta}\bar{\mu}} \tag{100}$$

and

$$c\frac{\beta_r - v_0}{2} = \bar{c}\frac{\bar{\alpha}_r + v_0 \frac{c}{\bar{c}}}{2} \tag{101}$$

Solving for $\bar{\alpha}_r$ and β_r yields

$$\bar{\alpha}_r = v_0 \frac{c}{\bar{c}} \frac{ef - 3}{ef + 1} \tag{102}$$

$$\beta_r = v_0 \frac{3ef - 1}{ef + 1} \tag{103}$$

where e and f are as defined by Eqs. (74) and (75).

Using Eq. (103) in Eqs. (96) and (97) leads to

$$v_f = v_0 \frac{ef - 1}{ef + 1} \tag{104}$$

$$\varepsilon_f = \frac{v_0}{\theta} \frac{2ef}{ef + 1} \tag{105}$$

The strain induced in the neighboring bar can be easily computed by the use of Eq. (83). If the σ_f/ε_f ratio is larger than the initial tangent modulus, shock fronts will take place. This case will not be treated in this work.

D. An Elastoplastic Bar of Infinite Strain-Hardening Impacting a Rigid Surface at Rest at Left End, and Attached to a Rigid Mass M of the Same Initial Velocity at Right End

The stress-strain diagram for this case is shown in Fig. 13. The presence of the yield condition (because of Eq. 36) will permit only the propagation of strains smaller than ε_y. However, to deal with every phase of the stress-strain law of Fig. 13, assume

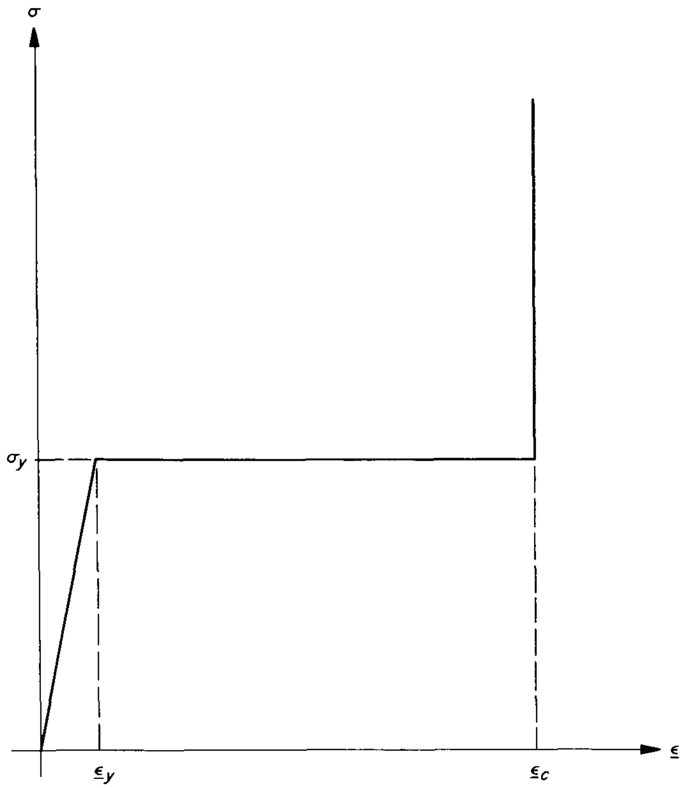


Fig. 13. Stress-strain relation of elastoplastic material with infinite strain hardening

$$v_0 > \varepsilon_y \tag{106}$$

The phenomenon of $v_0 > \varepsilon_y$ is described in Fig. 14. First, a strain front of magnitude ε_y (line AD, Fig. 14) will sweep across the bar with a velocity

$$v_1 = \frac{1}{(dt/dx)_\alpha} = 1 + v_0 \tag{107}$$

causing the particles behind the front to slow down by an amount ε_y. When particles of constant velocity $v_0 - \varepsilon_y$ meet the boundary at rest, they will stack on it with $\varepsilon = \varepsilon_c$, creating a shock front of constant speed (line AF, Fig. 14). On the other hand, when the strain front of magnitude ε_y meets the rigid end at the right, it will tend to reflect back; however, the reflected wave falling in the plastic range will not be able to propagate. At this instant, some portion of the bar to the left will come to rest with a strain ε_c with the remaining portion still moving at a velocity $v_0 - \varepsilon_y$. The mass M on the right still has a velocity of v_0 that is just beginning to decelerate because

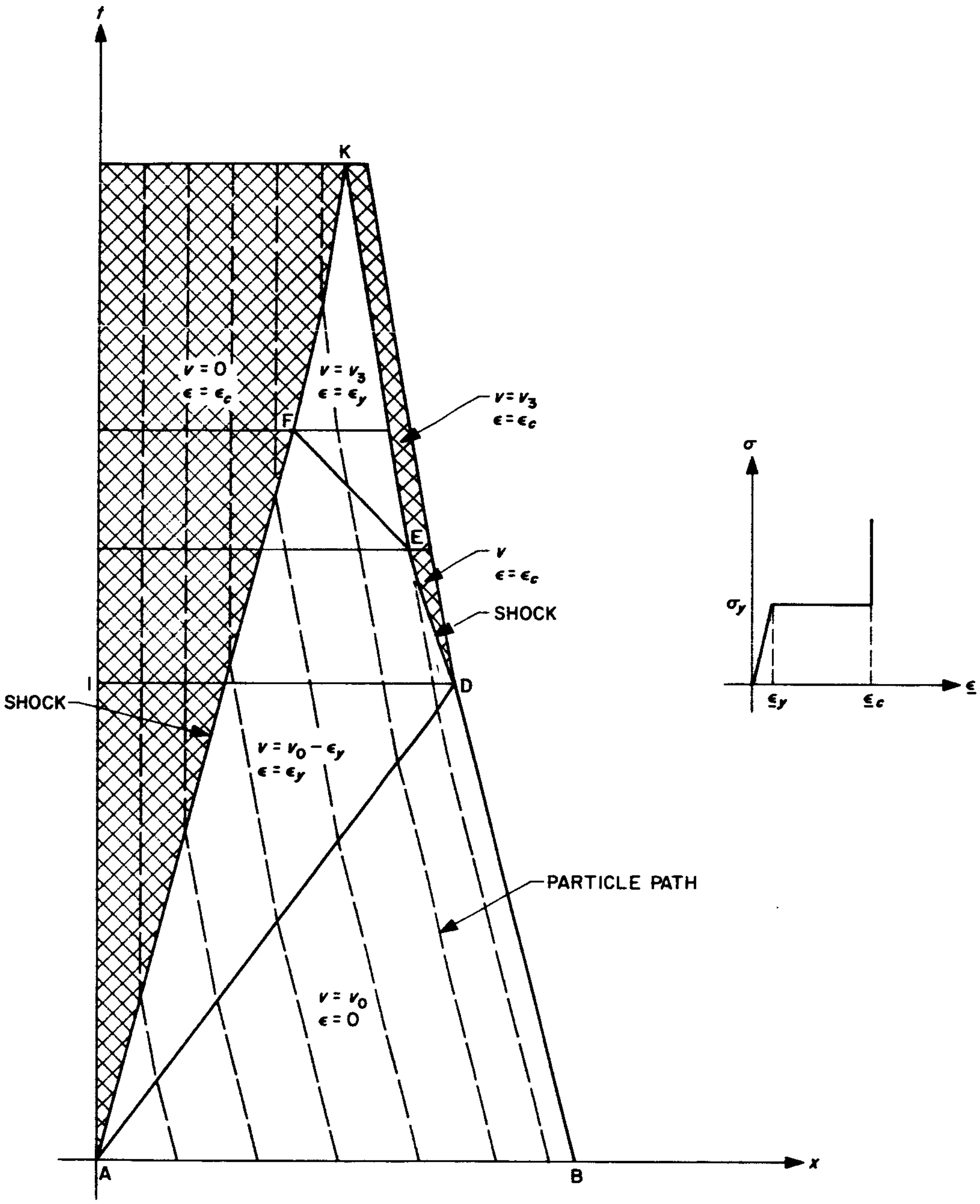


Fig. 14. Elastoplastic bar of infinite strain-hardening impacting a rigid surface at rest at left, and attached to a rigid mass *M* of the same initial velocity at right end

of a force $\sigma_c > \sigma_y$. This means that some stacking of particles with $\varepsilon = \varepsilon_c$ (i.e., another shock) will take place until the mass M and the particles stacked on it are brought to a velocity of $v_0 - \varepsilon_y$. The speed of this shock will gradually decrease and disappear when unloading starts (curved line DE, Fig. 14). Further deceleration of M will tend to unload the particles of velocity $v_0 - \varepsilon_y$ and strain ε_y. Assuming that the unloading takes place on a line parallel to the initial tangent modulus in the stress-strain plane, this front of imminent unloading (line EF, Fig. 14) will have a propagation speed of

$$v_2 = \frac{1}{(dt/dx)_\beta} = v_0 - 1 \qquad (108)$$

The particles behind this front will now follow the motion of the mass M. Until this front meets the shock front at the left end of the bar, all particles of velocity $v_0 - \varepsilon_y$ will continue to stack up at the left end. After the unloading front, the particle paths will be curved, hence, the speed of the shock front at the left will start to decrease (curved line FK, Fig. 14). These paths can be computed from the deceleration of mass M, and the mass of the remaining portion of the bar in motion under constant force σ_y. If, at the time when the end with the rigid mass M meets the stacked particles at point K, mass M still has some kinetic energy, it will be consumed by generation of a strong shock with very high temperatures.

The constant speed of the shock front, at the left up to point F (Fig. 14), can be computed from Eqs. (53) and (8) with $\varepsilon_r = \varepsilon_c$ and $U_0 = c(v_0 - \varepsilon_y)$ as

$$\frac{W}{c} = \frac{v_0 - \varepsilon_y}{\varepsilon_c}$$

or

$$\frac{dx}{dt} = \frac{W}{c} + v_0 - \varepsilon_y = \frac{v_0 - \varepsilon_y}{\varepsilon_c} + v_0 - \varepsilon_y \qquad (109)$$

As a first approximation, assume that the mass of the bar is negligible in comparison with M. Then the speed of mass M up to point D is v_0. Beyond point D, up to point K, this velocity is

$$v_3 = v_0 - \frac{\sigma_y}{M}(t - 1) \qquad (110)$$

assuming $\sigma_c \approx \sigma_y$ along DE. In Eq. (110), σ_y/M is assumed nondimensionalized with c^2/L. The particle velocity above line EF is also v_3. The particle velocity below line AD is v_0, and the particle velocity between lines AD and EF is $v_0 - \varepsilon_y$. The speed of the shock at the left after point F is now available from Eq. (109) by replacing $v_0 - \varepsilon_y$ with v_3

$$\frac{dx}{dt} = \frac{v_3}{\varepsilon_c} + v_3 \qquad (111)$$

At this stage, it would be well to compute the maximum axial velocity of the bar for which the system would be stopped exactly at point K (Fig. 14) under zero gravity field. Let δ be defined as

$$\delta = \frac{\text{mass of the rigid body } M}{\text{mass of the bar}} \qquad (113)$$

Writing the requirement that the maximum deformation energy plus the increase in thermal energy ΔE_t in the bar is just equal to the initial kinetic energy of the system, in terms of the dimensional quantities

$$\Delta E_t + L\left[\sigma_y\left(\varepsilon_c - \frac{\varepsilon_y}{2}\right)\right] = \frac{1}{2}\rho_0 L(1 + \delta)U_0^2$$

Assuming $\varepsilon_y/2$ as being small compared to ε_c, and neglecting ΔE_t, the following nondimensional equation is derived

$$v_0 = \left[\frac{2\varepsilon_c\varepsilon_y}{1 + \delta}\mu\right]^{1/2} \qquad (114)$$

where v_0 is the approximate velocity required to stop the system before appreciable shock temperatures are induced.

IV. SUMMARY OF RESULTS

The work presented in this report has been on the initial impactive behavior of uniform prismatic bars of constant pre-impactive velocity. Taking into account the fact that the bar was free to deform laterally and the material was nonlinear, the govering equations of the one-dimensional problem were obtained through the use of Eulerian coordinates. After nondimensionalization, the solution for various end conditions was derived by the method of characteristics. It was shown that strain hardening of the bar material may cause the shock phenomenon to occur. The Rankine–Hugoniot equations were used to study the impact-induced shocks of the bar. It was determined that the lateral motion of the bar particles should be considered to correctly compute the shock temperatures.

The impactive behavior of an energy dissipator—an elastoplastic material with infinite strain hardening—was studied in a one-dimensional model. As experimentally observed, the energy dissipator will be crushed from both ends if the far end of the dissipator is attached to a rigid body such as a capsule; this crushing is a function of the yield strain of the dissipator—the smaller the yield strain, the smaller the crushed length at the far end. If lateral motion is prevented, the shock temperatures are helpful in energy dissipation.

The equations given for impact-induced stresses, strains and velocities are so arranged that an iteration scheme may be readily applied to solve them. For example, the iteration may be started by assuming a linear material (i.e., $\theta = \gamma = 1$) and solve for the first approximations of impact-induced strains. Then, these approximations may be used to obtain better estimates of θ and γ, and so on.

The equations given for the impact-induced strains also are helpful in indicating the number of parameters involved in the determination of impact-induced strains and stresses. The number of parameters indicate the number of independent measurements required for obtaining the first estimates of the mechanical properties of an unknown impacted surface when the properties of the impacting bar are known.

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