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Axisymmetric Thermal Stresses in Sandwich Shells of Revolution with Application to Shallow, Spherical Shells

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ABSTRACT

The problem of computing the thermal stresses in a sandwich shell of revolution with a weak core is approached by computing separately a particular solution of the equations which involve the temperature distribution, and then superposing a solution for an edge-loaded shell to satisfy the boundary conditions. The component solutions for a shallow spherical shell are included in the Report. In order to retain the effects of transverse normal and shear strain, the governing equations of equilibrium and the stress–strain relations are obtained by applying Reissner's variational principle. The particular solution for the heated shell is obtained by an order of magnitude argument and is applicable for slowly varying temperature distributions only. The solutions for the edge-loaded shell indicate that the cross influence coefficients (rotation due to a force, etc.) can be obtained from those of an equivalent isotropic shell, but that the direct influence coefficients (displacement due to a force, etc.) are substantially affected by the effect of transverse shear strain. The effect of transverse normal strain is appreciable only in the particular solution.

I. INTRODUCTION

As the requirements for lightweight structure in air and spacecraft demand a consideration of sandwich construction, it is of increasing interest to be able to predict the thermal as well as the mechanical stresses induced in such structures. It is the purpose of this Report to present the equations describing a thin sandwich shell of revolution under such a combined loading; as well as a solution of such equations for the case of a shallow spherical shell.

E. Reissner (Ref. 1) derives the complete set of equations for the mechanical stresses in an unheated shell, using the principle of minimum complementary energy wherein the equations of equilibrium are introduced as restraints. As a principal contribution, it is noted that the effects of transverse normal strain and finite shear strain could be of great importance.

The combined thermoelastic equations are given by Grigolyuk and Kiryukhin (Ref. 2). In that paper, the stress–displacement relations are obtained directly from Hooke's Law, while the equations of equilibrium follow from an application of the principle of minimum potential energy. However, as these equations are specifically derived for a sandwich shell with a stiff core, the effects of transverse normal strain and finite shear strain are omitted.

As the purpose of this Report is the formulation of equations describing a sandwich shell with a weak core, it was decided to proceed along the lines of Ref. 3 and use the Reissner variational theorem (Ref. 4) to develop the required equations into which the thermoelastic stress–strain law could be introduced. These equations are developed in Section II for axisymmetric deflection of a general shell of revolution acted upon by surface pressure, edge forces and moments and heated such that the temperature has a linear variation with the thickness coordinate. In the sections which follow, the solution is presented for a complete shallow spherical shell restrained at the edge.

The method of solution follows that described in Ref. 5 for the isotropic shell. The complete solution satisfying the equations of equilibrium and some prescribed boundary conditions is formulated as the sum of a particular solution satisfying the thermoelastic equations and a solution for an edge-loaded unheated shell. The latter solution contains the edge shear force and meridional bending moment as parameters. These parameters are determined by the requirement that the composite solution must satisfy the two boundary conditions at the edge.

II. GENERAL EQUATIONS

The equations describing the stresses and displacements of a thin sandwich shell of revolution are derived below using the Reissner variational theorem (Ref. 4). With the location of a point in the shell being given in terms of the three parameters ϕ, θ, ζ as shown in Fig. 1, the orthogonal line elements in a surface ζ = constant are

$$ds_\phi = R_1 (1 + \zeta/R_1)\, d\phi \qquad ds_\theta = R_2 \sin\phi\, (1 + \zeta/R_2)\, d\theta$$

the general volume element is

$$dV = R_1 R_2 \sin\phi\, (1 + \zeta/R_1)(1 + \zeta/R_2)\, d\phi\, d\theta\, d\zeta$$

and the appropriate form of the variational theorem is

$$\delta \int_V F\, dV = \delta \int_S (p_\phi U_\phi + p_\zeta W)\, dS$$

where S is the surface on which the stresses p_ϕ, p_ζ are prescribed,

$$F = \sigma_{\phi\phi}\epsilon_\phi + \sigma_{\theta\theta}\epsilon_\theta + \sigma_{\zeta\zeta}\epsilon_\zeta + \tau_{\phi\zeta}\gamma_{\phi\zeta} - f$$

and

$$\epsilon_\phi = \frac{\partial f}{\partial \sigma_{\phi\phi}} \qquad \epsilon_\theta = \frac{\partial f}{\partial \sigma_{\theta\theta}} \qquad \epsilon_\zeta = \frac{\partial f}{\partial \sigma_{\zeta\zeta}} \qquad \gamma_{\phi\zeta} = \frac{\partial f}{\partial \tau_{\phi\zeta}}$$

If the face material is isotropic, the stress–strain law is

$$\epsilon_\phi = \frac{1}{E}\left[\sigma_{\phi\phi} - \nu(\sigma_{\theta\theta} + \sigma_{\zeta\zeta})\right] + \alpha\Delta T$$

$$\epsilon_\theta = \frac{1}{E}\left[\sigma_{\theta\theta} - \nu(\sigma_{\phi\phi} + \sigma_{\zeta\zeta})\right] + \alpha\Delta T$$

$$\epsilon_\zeta = \frac{1}{E}\left[\sigma_{\zeta\zeta} - \nu(\sigma_{\phi\phi} + \sigma_{\theta\theta})\right] + \alpha\Delta T$$

$$\gamma_{\phi\zeta} = \frac{2(1+\nu)}{E}\tau_{\phi\zeta}$$

so that

$$f = \frac{1}{2E}\left\{\sigma_{\phi\phi}^2 + \sigma_{\theta\theta}^2 + \sigma_{\zeta\zeta}^2 - 2\nu\left(\sigma_{\phi\phi}\sigma_{\theta\theta} + \sigma_{\phi\phi}\sigma_{\zeta\zeta} + \sigma_{\theta\theta}\sigma_{\zeta\zeta}\right) + 2\left(1+\nu\right)\tau_{\phi\zeta}^2\right\} + \alpha\Delta T\left(\sigma_{\phi\phi} + \sigma_{\theta\theta} + \sigma_{\zeta\zeta}\right)$$

If the core material is transverse orthotropic, the stress–strain law is

$$\epsilon_\phi = \frac{1}{E_c}\left(\sigma_{\phi\phi} - \nu_c\,\sigma_{\theta\theta}\right) - \frac{\nu_\zeta}{E_\zeta}\sigma_{\zeta\zeta} + \alpha_c\Delta T$$

$$\epsilon_\theta = \frac{1}{E_c}\left(\sigma_{\theta\theta} - \nu_c\,\sigma_{\phi\phi}\right) - \frac{\nu_\zeta}{E_\zeta}\sigma_{\zeta\zeta} + \alpha_c\Delta T$$

$$\epsilon_\zeta = \frac{1}{E_\zeta}\left[\sigma_{\zeta\zeta} - \nu_\zeta\left(\sigma_{\phi\phi} + \sigma_{\theta\theta}\right)\right] + \alpha_\zeta\Delta T$$

$$\gamma_{\phi\zeta} = \frac{1}{G_\zeta}\tau_{\phi\zeta}$$

so that

$$f = \left(\sigma_{\phi\phi}^2 + \sigma_{\theta\theta}^2 - 2\nu_c\,\sigma_{\phi\phi}\,\sigma_{\theta\theta}\right)/2E_c + \left[\sigma_{\zeta\zeta}^2 - 2\nu_\zeta\,\sigma_{\zeta\zeta}\left(\sigma_{\phi\phi} + \sigma_{\theta\theta}\right)\right]/2E_\zeta + \tau_{\phi\zeta}^2/2G_\zeta + \alpha_c\,T\left(\sigma_{\phi\phi} + \sigma_{\theta\theta}\right) + \alpha_\zeta\Delta T\,\sigma_{\zeta\zeta}$$

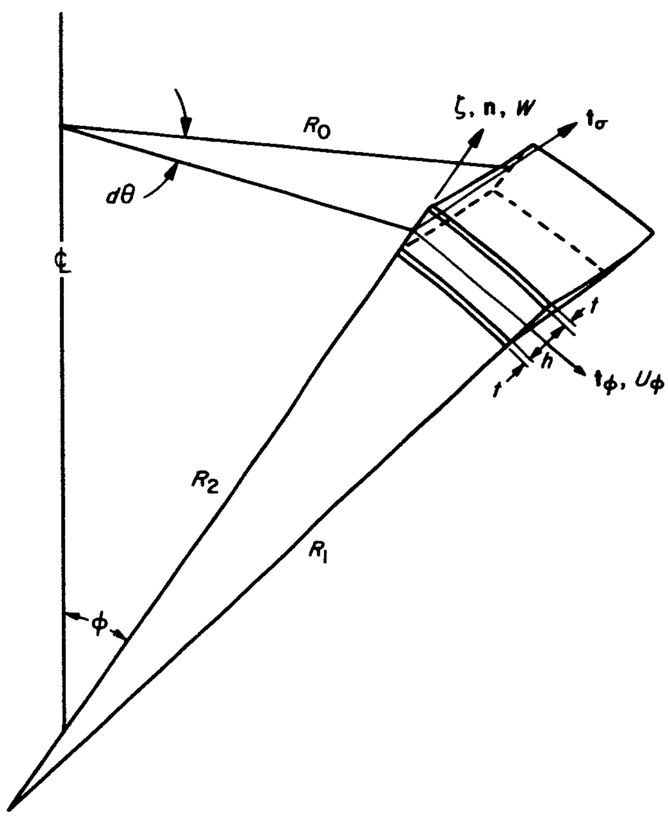


Fig. 1. The geometry of an element of the shell

If we now make a soft core hypothesis that the contribution to the function F of the normal stresses $\sigma_{\phi\phi}$, $\sigma_{\theta\theta}$ of the core is negligible in comparison to that of the face material, i.e., we require

$$\frac{h}{t}\frac{(\sigma_{\phi\phi})_{\text{core}}}{(\sigma_{\phi\phi})_{\text{face}}} \ll 1 \qquad \frac{h\alpha_c}{t\alpha}\frac{(\sigma_{\phi\phi})_{\text{core}}}{(\sigma_{\phi\phi})_{\text{face}}} \ll 1$$

$$\frac{\nu_\zeta E}{\nu E_\zeta}\frac{h}{t}\frac{(\sigma_{\phi\phi})_{\text{core}}}{(\sigma_{\phi\phi})_{\text{face}}} \ll 1 \qquad \frac{E}{E_c}\frac{h}{t}\frac{(\sigma_{\phi\phi})^2_{\text{core}}}{(\sigma_{\phi\phi})^2_{\text{face}}} \ll 1$$

then, the form of the function F for the core becomes

$$F^{(c)} = \sigma_{\zeta\zeta}\,\epsilon_\zeta + \tau_{\phi\zeta}\,\gamma_{\phi\zeta} - (\sigma^2_{\zeta\zeta}/2E_\zeta + \tau^2_{\phi\zeta}/2G_\zeta) - \alpha_\zeta \Delta T\,\sigma_{\zeta\zeta}$$

Following Ref. (3), let the displacements in the core be taken in the form

$$U_\phi(\phi,\zeta) = U(\phi) + \frac{2\zeta}{h}\beta(\phi)$$

$$W(\phi,\zeta) = w(\phi) + \frac{2\zeta}{h}\epsilon_m(\phi) + \left(\frac{2\zeta}{h}\right)^2 \omega_m(\phi)$$

In this way, we introduce a transverse normal strain that has a linear variation through the thickness.

If the thickness of the face material is small in comparison with that of the core, it is reasonable to expect that the displacements in the face are independent of the thickness variable ζ and are continuous with the displacements of the core. Hence, take

$$\left.\begin{aligned} U_\phi &= U^{(+)}_\phi = U + \beta \\ W &= W^{(+)} = w + \epsilon_m + \omega_m \end{aligned}\right\} \text{ for the upper face}$$

$$\left.\begin{aligned} U_\phi &= U^{(-)}_\phi = U - \beta \\ W &= W^{(-)} = w - \epsilon_m + \omega_m \end{aligned}\right\} \text{ for the lower face}$$

With the appropriate strain components in the form

$$\epsilon_\phi = \frac{1}{R_1(1+\zeta/R_1)}\left(\frac{\partial U_\phi}{\partial \phi} + W\right)$$

$$\epsilon_\sigma = \frac{1/R_2}{1+\zeta/R_2}(U_\phi \cot\phi + W)$$

$$\epsilon_\zeta = \frac{\partial W}{\partial \zeta}$$

$$\gamma_{\phi\zeta} = \frac{1/R_1}{1+\zeta/R_1}\left(\frac{\partial W}{\partial \phi} - U_\phi\right) + \frac{\partial U_\phi}{\partial \zeta}$$

it follows that

$$\epsilon^{(+)}_\phi = \frac{1/R_1}{1+\zeta/R_1}(U' + \beta' + w + \epsilon_m + \omega_m)$$

$$\epsilon^{(+)}_\theta = \frac{1/R_2}{1+\zeta/R_2}[(U+\beta)\cot\phi + w + \epsilon_m + \omega_m]$$

$$\epsilon^{(+)}_\zeta = 0 = \epsilon^{(-)}_\zeta$$

$$\gamma^{(+)}_{\phi\zeta} = \frac{1/R_1}{1+\zeta/R_1}(w' + \epsilon'_m + \omega'_m - U - \beta)$$

$$\epsilon_{\phi}^{(-)} = \frac{1/R_1}{1 + \zeta/R_1}(U' - \beta' + w - \epsilon_m + \omega_m)$$

$$\epsilon_{\theta}^{(-)} = \frac{1/R_2}{1 + \zeta/R_2}[(U - \beta)\cot\phi + w - \epsilon_m + \omega_m]$$

$$\gamma_{\phi\zeta}^{(-)} = \frac{1/R_1}{1 + \zeta/R_1}(w' - \epsilon_m' + \omega_m' - U + \beta)$$

$$\epsilon_{\zeta}^{(c)} = \frac{2}{h}\left(\epsilon_m + \frac{4\zeta}{h}\omega_m\right)$$

$$\gamma_{\phi\zeta}^{(c)} = \frac{1/R_1}{1 + \zeta/R_1}\left[w' - U + \frac{2R_1}{h}\beta + \left(\frac{2\zeta}{h}\right)\epsilon_m' + \left(\frac{2\zeta}{h}\right)^2\omega_m'\right]$$

where ( )′ denotes differentiation with respect to (ϕ).

Finally, let us take the following temperature distribution

$$\Delta T(\phi,\zeta) = T_0(\phi) + T_1(\phi)\,\zeta \qquad \left(-\frac{h}{2} \leqslant \zeta \leqslant \frac{h}{2}\right)$$

$$\Delta T = T^{(+)} = T_0(\phi) + T_1(\phi)\frac{h}{2}$$

$$\Delta T = T^{(-)} = T_0(\phi) - T_1(\phi)\frac{h}{2}$$

and stress distribution

$$\left.\begin{aligned} \sigma_{\zeta\zeta} &= \frac{q(\phi)}{2}(1 + 2\zeta/h) + \frac{\left(\frac{3S}{2h} + \frac{T\zeta}{h^2}\right)\left[1 - \left(\frac{2\zeta}{h}\right)^2\right]}{(1 + \zeta/R_1)(1 + \zeta/R_2)} \\ \tau_{\phi\zeta} &= \frac{\tau_m(\phi)}{1 + \zeta/R_2} \end{aligned}\right\} \left(-\frac{h}{2} \leqslant \zeta \leqslant \frac{h}{2}\right)$$

$$\left.\begin{aligned} \sigma_{\zeta\zeta} &= q(\phi) \\ \sigma_{\phi\phi} &= \sigma_{\phi\phi}^{(+)}(\phi) \\ \sigma_{\theta\theta} &= \sigma_{\theta\theta}^{(+)}(\phi) \\ \tau_{\phi\zeta} &= \tau_m(\phi)\left(1 + \frac{h/2 - \zeta}{t}\right)\Big/(1 + \zeta/R_2) \end{aligned}\right\} \text{for the upper face}$$

$$\left.\begin{aligned} \sigma_{\zeta\zeta} &= 0 \\ \sigma_{\phi\phi} &= \sigma_{\phi\phi}^{(-)}(\phi) \\ \sigma_{\theta\theta} &= \sigma_{\theta\theta}^{(-)}(\phi) \\ \tau_{\phi\zeta} &= \tau_m(\phi)\left(1 + \frac{h/2 + \zeta}{t}\right)\Big/(1 + \zeta/R_2) \end{aligned}\right\} \text{for the lower face}$$

as being a reasonable form consistent with the assumed form of the strain distribution. In this way, the average transverse normal strain in the core (ϵ_m) is compatible with the stress resultant (S) and the linearly varying component of the transverse normal strain (ω_m) is compatible with the stress resultant (T). The stress resultants in the core are essentially those suggested in Ref. (3).

Thus, accounting for a surface loading as well as edge loading on the edges $\phi = \phi_1, \phi_2$ $(\phi_2 > \phi_1)$, the final form of the variational theorem is as follows:

$$\delta \int_{V^{(+)}} F^{(+)}\, dV^{(+)} + \delta \int_{V^{(c)}} F^{(c)}\, dV^{(c)} + \delta \int_{V^{(-)}} F^{(-)}\, dV^{(-)} =$$

$$\int_{\theta=0}^{2\pi} \int_{\phi_1}^{\phi_2} R_1 R_2 \sin\phi \left(1 + \frac{h/2+t}{R_1}\right)\left(1 + \frac{h/2+t}{R_2}\right) q(\phi)\, (\delta w + \delta\epsilon_m + \delta\omega_m)\, d\phi$$

$$+ \int_{\theta=0}^{2\pi} \int_{\zeta=-(h/2+t)}^{(h/2+t)} \left\{ R_2 \sin\phi \left(1 + \frac{\zeta}{R_2}\right) d\theta\, d\zeta\, (\sigma_{\phi\phi}\, \delta U_\phi + \tau_{\phi\zeta}\, \delta W) \right\} \Bigg|_{\phi_1}^{\phi_2}$$

On substituting the assumed form for the stresses and displacements and carrying out the indicated integration over ζ and subsequent variation, we obtain the following set of equations as the Euler equations of the variational problem:

$$\sigma_{\phi\phi}^{(+)} - \nu\left(\sigma_{\theta\theta}^{(+)} + q\right) + \alpha E T^{(+)} = \frac{E}{R_1}\left(U' + \beta' + w + \epsilon_m + \omega_m\right) \frac{1 + (h+t)/2R_2}{1 + \dfrac{h+t}{2}\dfrac{R_1+R_2}{R_1 R_2} + \dfrac{h^2 + 2ht + 4t^2/3}{4R_1R_2}}$$

$$\sigma_{\theta\theta}^{(+)} - \nu\left(\sigma_{\phi\phi}^{(+)} + q\right) + \alpha E T^{(+)} = \frac{E}{R_2}\left[(U + \beta)\cot\phi + w + \epsilon_m + \omega_m\right] \frac{1 + (h+t)/2R_1}{1 + \dfrac{h+t}{2}\dfrac{R_1+R_2}{R_1 R_2} + \dfrac{h^2 + 2ht + 4t^2/3}{4R_1 R_2}}$$

$$\sigma_{\phi\phi}^{(-)} - \nu\, \sigma_{\theta\theta}^{(-)} + \alpha E T^{(-)} = \frac{E}{R_1}\left(U' - \beta' + w - \epsilon_m + \omega_m\right) \frac{1 - (h+t)/2R_2}{1 - \dfrac{h+t}{2}\dfrac{R_1+R_2}{R_1 R_2} + \dfrac{h^2 + 2ht + 4t^2/3}{4R_1 R_2}}$$

$$\sigma_{\theta\theta}^{(-)} - \nu\, \sigma_{\phi\phi}^{(-)} + \alpha E T^{(-)} = \frac{E}{R_2}\left[(U - \beta)\cot\phi + w - \epsilon_m + \omega_m\right] \frac{1 - (h+t)/2R_1}{1 - \dfrac{h+t}{2}\dfrac{R_1+R_2}{R_1 R_2} + \dfrac{h^2 + 2ht + 4t^2/3}{4R_1 R_2}}$$

$$2\epsilon_m/h = q/2E_\zeta + \frac{6S/5}{E_\zeta h}\left[1 + \frac{h^2}{28}\left(\frac{1}{R_1^2} + \frac{1}{R_1 R_2} + \frac{1}{R_2^2}\right)\right] - \frac{T}{35E_\zeta}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) + \alpha_\zeta T_0$$

$$\frac{4\omega_m}{15h} = \frac{q}{30E_\zeta} - \frac{S}{35E_\zeta}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) + \frac{\alpha_\zeta h T_1}{30} + \frac{2T}{105E_\zeta h}\left[1 + \frac{h^2}{12}\left(\frac{1}{R_1^2} + \frac{1}{R_1 R_2} + \frac{1}{R_2^2}\right)\right]$$

$$\frac{\tau_m h}{G_\zeta}\left[1 + \frac{h^2}{12R_2}\left(\frac{1}{R_2} - \frac{1}{R_1}\right) + \frac{4(1+\nu)\, t\, G_\zeta}{3Eh}\left(1 + \frac{h^2 + ht + 2t^2/5}{4R_2}\frac{R_1 - R_2}{R_1 R_2}\right)\right]$$

$$= 2\beta + \frac{h+t}{R_1}\left(w' - U + \omega_m' \frac{t + h/3}{t+h}\right)$$

$$\tau_m \frac{h+t}{R_1} + \frac{1}{R_1 R_2 \sin\phi}\frac{d}{d\phi}\left\{R_2 \sin\phi \left[\left(1 + \frac{h+t}{2R_2}\right) t\, \sigma_{\phi\phi}^{(+)}\right.\right.$$

$$\left.\left. + \left(1 - \frac{h+t}{2R_2}\right) t\sigma_{\phi\phi}^{(-)}\right]\right\} - \frac{\cot\phi}{R_2}\left[\left(1 + \frac{h+t}{2R_1}\right) t\, \sigma_{\phi\phi}^{(+)} + \left(1 - \frac{h+t}{2R_2}\right) t\, \sigma_{\theta\theta}^{(-)}\right] = 0$$

$$\frac{1}{R_1 R_2 \sin\phi}\frac{d}{d\phi}\left\{R_2 \sin\phi\left[\left(1-\frac{h+t}{2R_2}\right)t\,\sigma_{\phi\phi}^{(-)}-\left(1+\frac{h+t}{2R_2}\right)t\,\sigma_{\phi\phi}^{(+)}\right]\right\}$$
$$+\frac{\cot\phi}{R_2}\left[\left(1+\frac{h+t}{2R_1}\right)t\,\sigma_{\theta\theta}^{(+)}-\left(1-\frac{h+t}{2R_1}\right)t\,\sigma_{\theta\theta}^{(-)}\right]+2\tau_m=0$$

$$\frac{h+t}{R_1 R_2 \sin\phi}\frac{d}{d\phi}(R_2\sin\phi\,\tau_m)-\frac{t}{R_1}\left[\left(1+\frac{h+t}{2R_2}\right)\sigma_{\phi\phi}^{(+)}+\left(1-\frac{h+t}{2R_2}\right)\sigma_{\phi\phi}^{(-)}\right]$$
$$-\frac{t}{R_2}\left[\left(1+\frac{h+t}{2R_1}\right)\sigma_{\theta\theta}^{(+)}+\left(1-\frac{h+t}{2R_1}\right)\sigma_{\theta\theta}^{(-)}\right]=-q^*$$

$$2S/h+q\left(1+\frac{h}{6}\frac{R_1+R_2}{R_1R_2}+\frac{h^2/12}{R_1R_2}\right)+\frac{t}{R_1}\left[\left(1+\frac{h+t}{2R_2}\right)\sigma_{\phi\phi}^{(+)}\right.$$
$$\left.-\left(1-\frac{h+t}{2R_2}\right)\sigma_{\phi\phi}^{(-)}\right]+\frac{t}{R_2}\left[\left(1+\frac{h+t}{2R_1}\right)\sigma_{\theta\theta}^{(+)}-\left(1-\frac{h+t}{2R_1}\right)\sigma_{\theta\theta}^{(-)}\right]=q^*$$

$$4T/15\,h+\frac{2q}{3}\left(1+\frac{h}{2}\frac{R_1+R_2}{R_1R_2}+\frac{3h^2/20}{R_1R_2}\right)-\frac{t+h/3}{R_1R_2\sin\phi}\frac{d}{d\phi}(\tau_m R_2\sin\phi)$$
$$+\frac{t}{R_1}\left[\left(1+\frac{h+t}{2R_2}\right)\sigma_{\phi\phi}^{(+)}+\left(1-\frac{h+t}{2R_2}\right)\sigma_{\phi\phi}^{(-)}\right]+\frac{t}{R_2}\left[\left(1+\frac{h+t}{2R_1}\right)\sigma_{\theta\theta}^{(+)}+\left(1-\frac{h+t}{2R_1}\right)\sigma_{\theta\theta}^{(-)}\right]=q^*$$

where

$$q^*=q\left(1+\frac{h+2t}{2R_1}\right)\left(1+\frac{h+2t}{2R_2}\right)$$

III. AN EDGE-LOADED, COLD, SPHERICAL SHELL

For the special case of an unheated spherical shell loaded at the edges only, the general equations can be rewritten in the following form

$$\epsilon_\phi^{(0)} + \omega_m/a = \frac{1}{2Et}(N_{\phi\phi} - \nu N_{\theta\theta})$$

$$\epsilon_\theta^{(0)} + \omega_m/a = \frac{1}{2Et}(N_{\theta\theta} - \nu N_{\phi\phi})$$

$$\frac{Q_\phi}{G_\zeta}\frac{h}{h+t}\left(1 + \frac{2t\,G_\zeta}{3h\,G}\right) = 2\beta + \frac{h+t}{a}(w' - U) + \frac{h+3t}{3a}\omega_m'$$

$$\beta' + \epsilon_m = \frac{a}{Et(h+t)}(M^*_{\phi\phi} - \nu M^*_{\theta\theta})$$

$$\beta \cot\phi + \epsilon_m = \frac{a}{Et(h+t)}(M^*_{\theta\theta} - \nu M^*_{\phi\phi})$$

$$\epsilon_m E_\zeta = \frac{3}{5}S - Th/35\,a$$

$$\omega_m E_\zeta = \frac{1}{14}(T - 3Sh/a)$$

$$S = -\frac{h/a}{h+t}(M^*_{\phi\phi} + M^*_{\theta\theta})$$

$$T = -\frac{5h}{2a}\frac{h}{h+t}(N_{\phi\phi} + N_{\theta\theta})$$

$$\frac{dM^*_{\phi\phi}}{d\phi} + (M^*_{\phi\phi} - M^*_{\theta\theta})\cot\phi = aQ_\phi$$

$$N_{\phi\phi} = Q_\phi \cot\phi$$

$$N_{\theta\theta} = \frac{dQ_\phi}{d\phi}$$

where

$$Q_\phi = (h+t)\,\tau_m$$

$$N_{\phi\phi} = t\left[\left(1 + \frac{h+t}{2a}\right)\sigma_{\phi\phi}^{(+)} + \left(1 - \frac{h+t}{2a}\right)\sigma_{\phi\phi}^{(-)}\right]$$

$$N_{\theta\theta} = t\left[\left(1 + \frac{h+t}{2a}\right)\sigma_{\theta\theta}^{(+)} + \left(1 - \frac{h+t}{2a}\right)\sigma_{\theta\theta}^{(-)}\right]$$

$$M^*_{\phi\phi} = t\frac{h+t}{2}\left[\left(1 + \frac{h+t}{2a}\right)\sigma_{\phi\phi}^{(+)} - \left(1 - \frac{h+t}{2a}\right)\sigma_{\phi\phi}^{(-)}\right]$$

$$M^*_{\theta\theta} = t\frac{h+t}{2}\left[\left(1 + \frac{h+t}{2a}\right)\sigma_{\theta\theta}^{(+)} - \left(1 - \frac{h+t}{2a}\right)\sigma_{\theta\theta}^{(-)}\right]$$

$$\epsilon_\phi^{(0)} = \frac{1}{a}(U' + w)$$

$$\epsilon_\theta^{(0)} = \frac{1}{a}(U\cot\phi + w)$$

The quantities $Q_\phi, N_{\phi\phi}, N_{\theta\theta}$ are true stress resultants as

$$(Q_\phi, N_{\phi\phi}, N_{\theta\theta}) = \int_{-\left(\frac{h}{2}+t\right)}^{\left(\frac{h}{2}+t\right)} (\tau_{\theta\zeta},\ \sigma_{\phi\phi},\ \sigma_{\theta\theta})(1 + \zeta/a)\,d\zeta$$

but the quantities $M^*_{\phi\phi}, M^*_{\theta\theta}$ are not, as

$$M_{\phi\phi} = \int_{-\left(\frac{h}{2}+t\right)}^{\left(\frac{h}{2}+t\right)} \zeta\,\sigma_{\phi\phi}(1 + \zeta/a)\,d\zeta$$

$$= t\frac{h+t}{2}\left[\sigma_{\phi\phi}^{(+)}\left(1 + \frac{h+t}{2a} + \frac{t^2/6a}{h+t}\right) - \sigma_{\phi\phi}^{(-)}\left(1 - \frac{h+t}{2a} - \frac{t^2/6a}{h+t}\right)\right]$$

$$M_{\theta\theta} = \int_{-\left(\frac{h}{2}+t\right)}^{\left(\frac{h}{2}+t\right)} \zeta\,\sigma_{\theta\theta}(1 + \zeta/a)\,d\zeta$$

$$= t\frac{h+t}{2}\left[\sigma_{\theta\theta}^{(+)}\left(1 + \frac{h+t}{2a} + \frac{t^2/6a}{h+t}\right) - \sigma_{\theta\theta}^{(-)}\left(1 - \frac{h+t}{2a} - \frac{t^2/6a}{h+t}\right)\right]$$

As we expect the operation of differentiation with respect to ϕ to change the resulting order of magnitude of the term differentiated, i.e.,

$$\frac{dQ_\phi}{d_\phi} = Q_\phi O(a/\delta) \qquad a/\delta \gg 1$$

where δ is the meridional distance over which changes in the variable are appreciable, it follows that

$$Q_\phi = \frac{M^*_{\phi\phi}}{a} O\left(\frac{a}{\delta}\right) \qquad M^*_{\phi\phi} = \frac{Eth}{a} \beta O\left(\frac{a}{\delta}\right)$$

$$\beta = \frac{h}{a} w O\left(\frac{a}{\delta}\right) \qquad w = \frac{a}{Et} N_{\theta\theta} O(1)$$

$$N_{\theta\theta} = Q_\phi O\left(\frac{a}{\delta}\right)$$

from what are expected to be the dominant terms. These relations are consistent only if

$$\delta^4 = a^2 h^2 O(1)$$

so that

$$N_{\theta\theta} = \frac{M^*_{\phi\phi}}{a} O\left(\frac{a}{h}\right) \qquad S = \frac{M^*_{\phi\phi}}{a} O(1)$$

$$T = \frac{h}{a} N_{\theta\theta} O(1)$$

$$E_\zeta \epsilon_m = \frac{M^*_{\phi\phi}}{a} O(1) \qquad 14\, E_\zeta \omega_m = \frac{h}{a} N_{\theta\theta} O(1)$$

If we limit our analysis to shells such that

$$\frac{E}{E_\zeta} \frac{th}{a^2} \ll 1$$

the contribution of ω_m to the stress–strain relations is negligible as well as the contribution of ϵ_m to the moment–curvature relations. Thus, the stress–strain and moment–curvature relations can be simplified to read

$$\epsilon^{(0)}_\phi = \frac{1}{2Et} (N_{\phi\phi} - \nu N_{\theta\theta})$$

$$\epsilon^{(0)}_\theta = \frac{1}{2Et} (N_{\theta\theta} - \nu N_{\phi\phi})$$

$$M^*_{\phi\phi} = \frac{Et(h+t)}{a(1-\nu^2)} (\beta' + \nu \beta \cot \phi)$$

$$M^*_{\theta\theta} = \frac{Et(h+t)}{a(1-\nu^2)} (\nu \beta' + \beta \cot \phi)$$

$$\frac{Q_\phi}{G_\zeta} \frac{h}{(h+t)^2} \left(1 + \frac{2tG_\zeta}{3hG}\right) = \frac{2\beta}{h+t} + \frac{1}{a}(w' - U)$$

Following the analysis for the classical isotropic shell, it is convenient to reduce this set of equations to a pair relating the dependent variables β, Q_ϕ. With the following identity

$$\frac{dw}{d_\phi} - U = a\left[\frac{d\epsilon^{(0)}_\theta}{d_\phi} + (\epsilon^{(0)}_\theta - \epsilon^{(0)}_\phi) \cot \phi\right]$$

it follows that

$$\frac{1}{a}\left(\frac{dw}{d_\phi} - U\right) = \frac{1}{2Et}\left[L(Q_\phi) + \nu Q_\phi\right]$$

$$= \frac{Q_\phi}{G_\zeta} \frac{h}{(h+t)^2}\left(1 + \frac{2tG_\zeta}{3hG}\right) - \frac{2\beta}{h+t}$$

where

$$L(\;) = \frac{d^2(\;)}{d\phi^2} + \cot_\phi \frac{d(\;)}{d\phi} - \cot^2 \phi (\;)$$

Further, the equation of moment equilibrium becomes

$$\frac{(1-\nu^2) a^2 Q_\phi}{Et(h+t)} = L(\beta) - \nu\beta$$

Finally, in terms of the parameters

$$\rho^4 = \frac{(1-\nu^2) a^2}{(h+t)^2} - \left(\frac{\nu}{2}\right)^2 \qquad \kappa = \left(1 + \frac{2tG_\zeta}{3hG}\right) \frac{E}{G_\zeta} \frac{2th}{(h+t)^2}$$

the equation for determining both Q_ϕ, β becomes

$$\left[L - \kappa/2 - \sqrt{(\kappa/2)^2 - \nu\kappa - 4\rho^4}\right]\left[L - \kappa/2 + \sqrt{(\kappa/2)^2 - \nu\kappa - 4\rho^4}\right] (Q_\phi, \beta) = 0$$

At this stage, it is evident that the above equations are analogous to those of Ref. (6) which describe the bending of a shell of transverse isotropic material wherein only the correction for finite shear strain is retained. In the following sections, the results of Ref. (6) will be applied to the calculation of the influence coefficients for shallow, spherical shells.

IV. SHALLOW SPHERICAL SHELLS

In this section, the equations developed for an edge-loaded, cold spherical shell are specialized for shallow opening angles. In this case it is justifiable to replace $\cot \phi$:

$$\cot \phi \approx 1/\phi$$

and so obtain

$$L(\ ) \approx \frac{d^2(\ )}{d\phi^2} + \frac{1}{\phi}\frac{d(\ )}{d\phi} - \frac{(\ )}{\phi^2}$$

As shown in Ref. 6, the form of the solution of the bending equations depends on the relation of the parameter κ to the critical value given by

$$\kappa^* = 4\rho^2 \left[\sqrt{1 + (\nu/2\rho^2)^2} + \nu/2\rho^2\right]$$

For $\kappa < \kappa^*$, the quantities of interest are

$$Q_\phi/2Et = C_1 U_1 + C_2 V_1 + C_3 u_1 + C_4 v_1$$

$$\begin{aligned} -\frac{2\beta}{h+t} = \ & C_1 \left[(\nu - \kappa/2) U_1 + |k|^2 V_1 \sin 2\theta\right] \\ & + C_2 \left[(\nu - \kappa/2) V_1 - |k|^2 U_1 \sin 2\theta\right] \\ & + C_3 \left[(\nu - \kappa/2) u_1 + |k|^2 v_1 \sin 2\theta\right] \\ & + C_4 \left[(\nu - \kappa/2) v_1 - |k|^2 u_1 \sin 2\theta\right] \end{aligned}$$

$$\begin{aligned} -\frac{a}{h+t}\frac{d\beta}{d\phi} = \ & C_1 \left\{(\nu - \kappa/2)\left[-\frac{U_1}{\phi} + |k|(U_0 \cos\theta - V_0 \sin\theta)\right] + |k|^2 \sin 2\theta \left[-\frac{V_1}{\phi} + |k|(V_0\cos\theta + U_0 \sin\theta)\right]\right\} \\ & + C_2 \left\{(\nu - \kappa/2)\left[-\frac{V_1}{\phi} + |k|(V_0 \cos\theta + U_0 \sin\theta)\right] - |k|^2 \sin 2\theta \left[-\frac{U_1}{\phi} + |k|(U_0\cos\theta - V_0 \sin\theta)\right]\right\} \\ & + C_3 \left\{(\nu - \kappa/2)\left[-\frac{u_1}{\phi} + |k|(u_0 \cos\theta - v_0 \sin\theta)\right] + |k|^2 \sin 2\theta \left[-\frac{v_1}{\phi} + |k|(u_0\sin\theta + v_0 \cos\theta)\right]\right\} \\ & + C_4 \left\{(\nu - \kappa/2)\left[-\frac{v_1}{\phi} + |k|(u_0 \sin\theta + v_0 \cos\theta)\right] - |k|^2 \sin 2\theta \left[-\frac{u_1}{\phi} + |k|(u_0\cos\theta - v_0 \sin\theta)\right]\right\} \end{aligned}$$

$$\begin{aligned} N_{\theta\theta}/2Et = \ & C_1\left[-\frac{U_1}{\phi} + |k|(U_0\cos\theta - V_0\sin\theta)\right] + C_2\left[-\frac{V_1}{\phi} + |k|(V_0\cos\theta + U_0\sin\theta)\right] \\ & + C_3\left[-\frac{u_1}{\phi} + |k|(u_0\cos\theta - v_0\sin\theta)\right] + C_4\left[-\frac{v_1}{\phi} + |k|(u_0\sin\theta + v_0\cos\theta)\right] \end{aligned}$$

where

$$k^2 = -\kappa/2 + i\sqrt{4\rho^4 + \nu\kappa - (\kappa/2)^2} = |k|^2 e^{2i\theta}$$

and

$$|k|^2 \cos 2\theta = -\kappa/2 \qquad |k|^2 \sin 2\theta = \sqrt{4\rho^4 + \nu\kappa - (\kappa/2)^2}$$

The functions $U_0, V_0, \cdots, v_1$ are the real and imaginary parts of the Bessel functions of complex argument defined by

$$U_0 + iV_0 = J_0(k\phi) \qquad u_0 + iv_0 = Y_0(k\phi)$$

$$U_1 + iV_1 = J_1(k\phi) \qquad u_1 + iv_1 = Y_1(k\phi)$$

For $\kappa = \kappa^*$, we have

$$Q\phi/2Et = C_1^* I_1(\eta) + C_2^* \eta I_0(\eta) + C_3^* K_1(\eta) + C_4^* \eta K_0(\eta)$$

$$\frac{dQ\phi}{d\phi}/2Et = \sqrt{\frac{\kappa^*}{2}} \left\{ - C_1^* \left(\frac{I_1}{\eta} - I_0\right) + C_2^* (I_0 + \eta I_1) - C_3^* \left(\frac{K_1}{\eta} + K_0\right) + C_4^* (K_0 - \eta K_1) \right\}$$

$$\frac{2\beta}{h+t} = \left(\frac{\kappa^*}{2} - \nu\right)(C_1^* I_1 + C_2^* \eta I_0 + C_3^* K_1 + C_4^* \eta K_0) + \kappa^* (- C_2^* I_1 + C_4^* K_1)$$

$$\frac{2}{h+t}\frac{d\beta}{d\phi} = \sqrt{\frac{\kappa^*}{2}} \left\{ \left(\frac{\kappa^*}{2} - \nu\right) \left[C_1^* \left(I_0 - \frac{I_1}{\eta}\right) + C_2^* (I_0 + \eta I_1) - C_3^* \left(K_0 + \frac{K_1}{\eta}\right) + C_4^* (K_0 - \eta K_1) \right] \right.$$
$$\left. - \kappa^* \left[C_2^* \left(I_0 - \frac{I_1}{\eta}\right) + C_4^* \left(K_0 + \frac{K_1}{\eta}\right) \right] \right\}$$

where $\eta = \phi \sqrt{\frac{\kappa^*}{2}}$ and I_0, I_1, K_0, K_1

are the modified Bessel functions of the first and second kinds respectively.

For $\kappa > \kappa^*$, we have

$$Q\phi/2Et = C_1^+ I_1 (l_1 \phi) + C_2^* K_1 (l_1 \phi) + C_3^+ I_1 (l_2 \phi) + C_4^* K_1 (l_2 \phi)$$

$$\frac{dQ_\phi}{d\phi} \Big/ 2Et = l_1 \left\{ C_1^+ \left[I_0 (l_1\phi) - \frac{I_1(l_1\phi)}{l_1\phi} \right] - C_2^+ \left[K_0(l_1\phi) + \frac{K_1(l_1\phi)}{l_1\phi} \right] \right\}$$
$$+ l_2 \left\{ C_3^+ \left[I_0(l_2\phi) - \frac{I_1(l_2\phi)}{l_2\phi} \right] - C_4^+ \left[K_0(l_2\phi) + \frac{K_1(l_2\phi)}{l_2\phi} \right] \right\}$$

$$\frac{2\beta}{h+t} = (\kappa - \nu - l_1^2) [C_1^+ I_1(l_1\phi) + C_2^+ K_1(l_1\phi)]$$
$$+ (\kappa - \nu - l_2^2) [C_3^+ I_1(l_2\phi) + C_1^+ K_1(l_2\phi)]$$

$$\frac{2}{h+t}\frac{d\beta}{d\phi} = l_1 (\kappa - \nu - l_1^2) \left\{ C_1^+ \left[I_0(l_1\phi) - \frac{I_2(l_1\phi)}{l_1\phi} \right] \right.$$
$$\left. - C_2^+ \left[K_0(l_1\phi) + \frac{K_1(l_1\phi)}{l_1\phi} \right] \right\}$$
$$+ l_2(\kappa - \nu - l_2^2) \left\{ C_3^+ \left[I_0(l_2\phi) - \frac{I_1(l_2\phi)}{l_2\phi} \right] \right.$$
$$\left. - C_4^+ \left[K_0(l_2\phi) + \frac{K_1(l_2\phi)}{l_2\phi} \right] \right\}$$

where

$$l_1^2 = \kappa/2 + \sqrt{(\kappa/2)^2 - \nu\kappa - 4\rho^4}$$
$$l_2^2 = \kappa/2 - \sqrt{(\kappa/2)^2 - \nu\kappa - 4\rho^4}$$

The variation of the factors $|k|$, l_1, l_2 with κ for a thin shell is presented in Fig. 2.

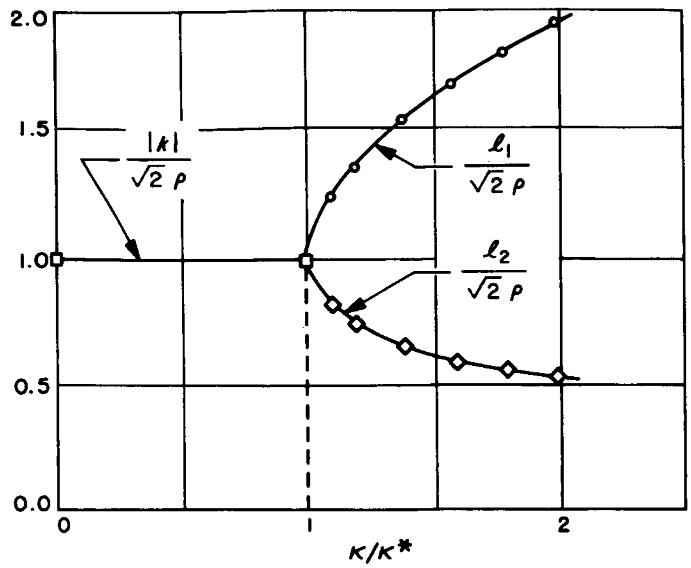


Fig. 2. Coefficients of the independent variable

V. THE INFLUENCE COEFFICIENTS

The particular concern of this section is to obtain the displacements at an edge due to forces and moments applied at that edge. In particular, let us consider a shell closed at the apex and loaded at the outer edge $\phi = \phi_b$ (Fig. 3). The constants C_1, C_2, $\cdots$ are then chosen so that the solution is regular at the apex and satisfies the boundary conditions that

$$H \equiv (Q\phi \sin\phi + N\phi \cos\phi)\big|_{\phi=\phi b} = \left.\frac{Q\phi}{\sin\phi}\right|_{\phi=\phi b} \approx \left.\frac{Q\phi}{\phi}\right|_{\phi=\phi b}$$

$$M = M^*_{\phi\phi}\,(\phi_b)$$

The equation for H simplifies as, for loading with no vertical resultant, we have

$$N_\phi = Q_\phi \cot\phi$$

The horizontal displacement at the edge follows for each particular case of κ, and is given in general by

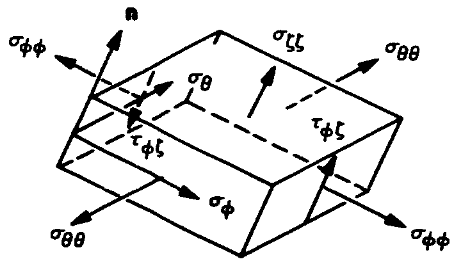
$$\delta_h(\phi = \phi_b) = [W(\zeta = 0)\sin\phi + U_\phi(\zeta = 0)\cos\phi]\big|_{\phi_b}$$

$$= a\,\epsilon^{(0)}_\theta\,(\phi_b)\sin\phi_b$$

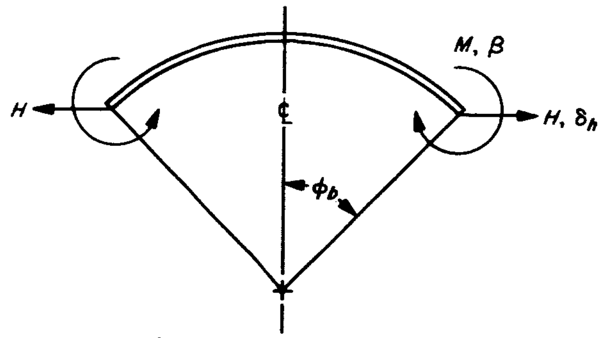
$$\approx \frac{a}{2Et}\left(\phi\,\frac{dQ\phi}{d\phi} - \nu\,Q_\phi\right)\Bigg|_{\phi_b}$$

It should be noted that $M^*_{\phi\phi}$ is not the true bending moment, but that the difference is negligible provided that

$$\frac{t^2/6a}{h+t} \ll 1$$



(a) AN ELEMENT OF SHELL



(b) EDGE LOADING OF A COMPLETE SHELL

Fig. 3. Sign convention

In particular, for $\kappa < \kappa^*$, we take $C_3, C_4 = 0$ and choose C_1, C_2 to satisfy

$$H/2Et = \frac{1}{\phi_b}[C_1 U_1 + C_2 V_1]$$

$$\begin{aligned}
-\frac{2(1-\nu^2)Ma}{Et(h+t)^2} = C_1 \Big\{ &(\nu - \kappa/2)\left[-\frac{1-\nu}{\phi}U_1 + |k|(U_0\cos\theta - V_0\sin\theta)\right] \\
&+ |k|^2 \sin 2\theta\left[-\frac{1-\nu}{\phi}V_1 + |k|(V_0\cos\theta + U_0\sin\theta)\right]\Big\} \\
+ C_2 \Big\{ &(\nu - \kappa/2)\left[-\frac{1-\nu}{\phi}V_1 + |k|(V_0\cos\theta + U_0\sin\theta)\right] \\
&+ |k|^2 \sin 2\theta\left[\frac{1-\nu}{\phi}U_1 - |k|(U_0\cos\theta - V_0\sin\theta)\right]\Big\}
\end{aligned}$$

For $\kappa = \kappa^*$, we take $C_3^*, C_4^* = 0$, and determine C_1^*, C_2^* to satisfy

$$H/2Et = \frac{1}{\phi_b}\left[C_1^* I_1(\eta) + C_2^* \eta I_0(\eta)\right]\Big|_{\phi_b}$$

$$\begin{aligned}
\frac{2(1-\nu^2)Ma}{Et(h+t)^2} = \sqrt{\frac{\kappa^*}{2}}\Big\{&\left(\frac{\kappa^*}{2} - \nu\right)\left[C_1^*\left(I_0 - \frac{1-\nu}{\eta}I_1\right)\right. \\
&\left. + \eta C_2^*\left(I_1 + \frac{1-\nu}{\eta}I_0\right)\right] - \kappa^* C_2^*\left(I_0 - \frac{1-\nu}{\eta}I_1\right)\Big\}\Big|_{\phi_b}
\end{aligned}$$

Finally, for $\kappa > \kappa^*$, we take C_2^+, $C_4^+ = 0$ and determine C_1^+, C_3^+ to satisfy

$$H/2Et = \frac{1}{\phi_b}\left[C_1^+ I_1(l_1\phi) + C_3^+ I_1(l_2\phi)\right]\Big|_{\phi=\phi_b}$$

$$\frac{2(1-\nu^2)Ma}{Et(h+t)^2} = l_1(\kappa - \nu - l_1^2)C_1^+\left[I_0(l_1\phi) - \frac{1-\nu}{l_1\phi}I_1(l_1\phi)\right]\Big|_{\phi_b}$$

$$+ l_2(\kappa - \nu - l_2^2)C_3^+\left[I_0(l_2\phi) - \frac{1-\nu}{l_2\phi}I_1(l_2\phi)\right]\Big|_{\phi_b}$$

In order to evaluate the effect of the shear parameter κ on the influence coefficients, calculations were performed taking

$$\rho^2 = 100 \qquad \nu = 0.30 \qquad \phi_b = 20.26°$$

$$\theta = 50°, 60°, 90°$$

The values of the Bessel functions of complex argument were obtained from Ref. 7. The results of the calculations are presented in Table 1. The corresponding distribution of the stress resultants is presented in Figs. 4–9.

Table 1. Influence coefficients ($\rho^2 = 100$; $\nu = 0.3$; $\phi_b = 20.26°$)

κ/κ^*	$\frac{4Et\beta}{(h+t)H}$	$\frac{Et(h+t)\beta}{(1-\nu^2)Ma}$	$\frac{2Et\delta_h}{aH}$	$\frac{Et(h+t)^2\rho_h}{(1-\nu^2)Ma^2} \times 10^{-2}$
0.1737	−72.21	0.1128	2.441	−0.3612
0.5000	−71.37	0.1266	2.777	−0.3574
1.000	−69.64	0.1434	3.227	−0.3493

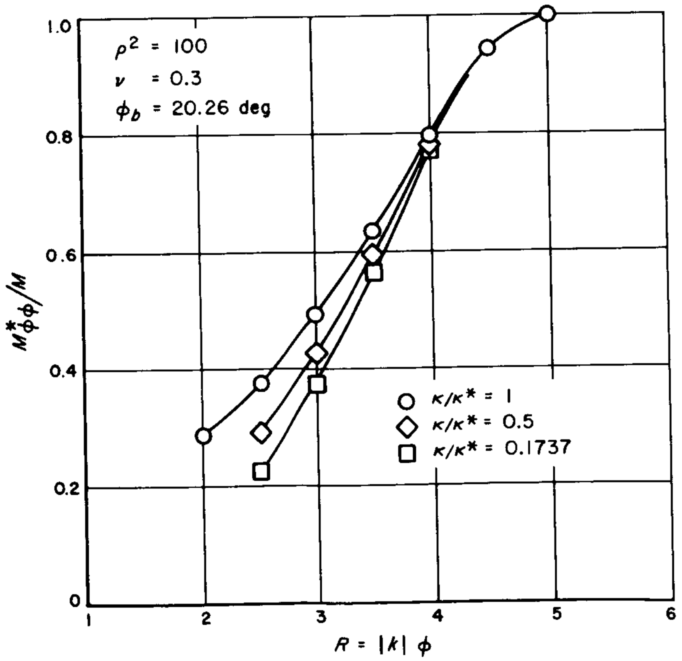


Fig. 4. Bending moment distribution, applied moment

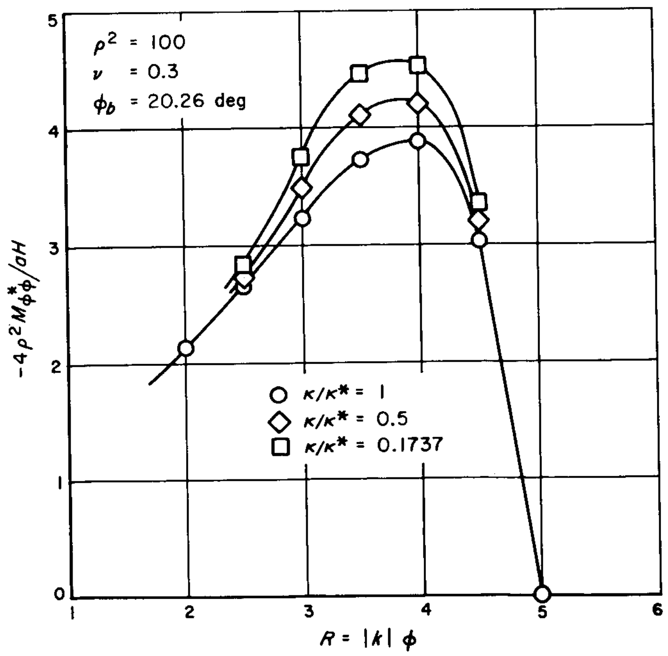


Fig. 5. Bending moment distribution, applied force

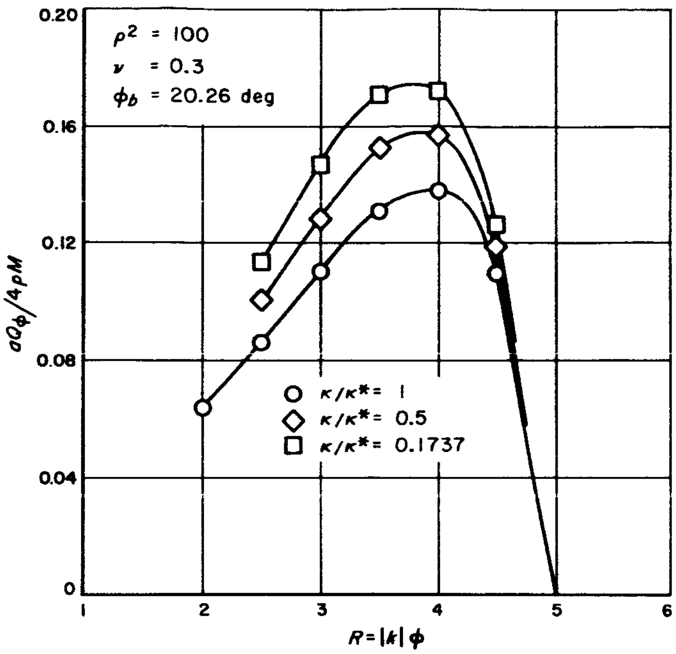


Fig. 6. Transverse shear distribution, applied moment

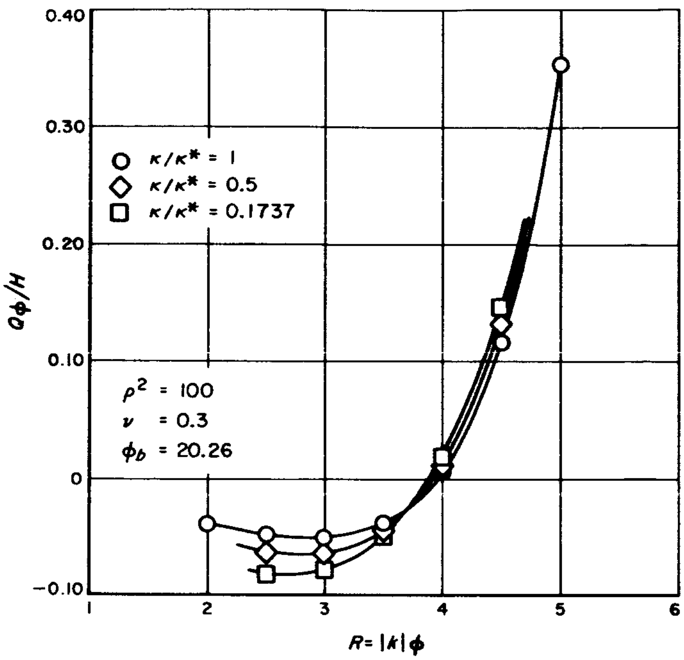


Fig. 7. Transverse shear distribution, applied force

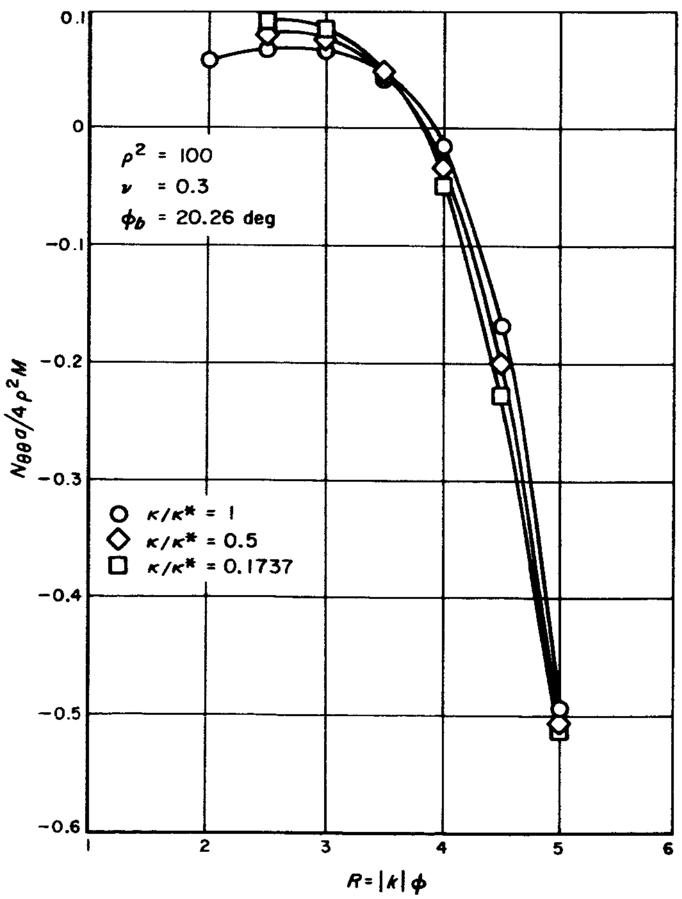


Fig. 8. Circumferential unit force, applied moment

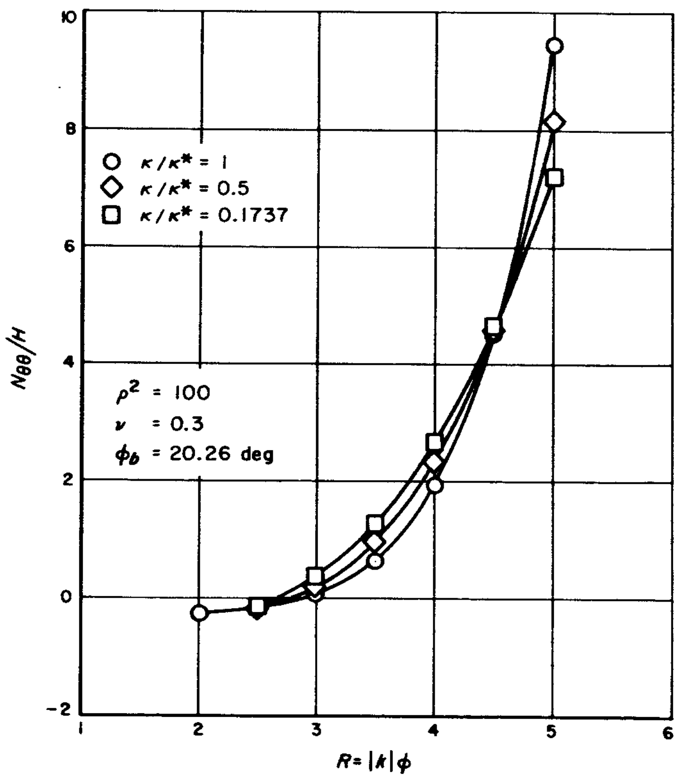


Fig. 9. Circumferential unit force, applied force

VI. HEATING OF AN UNRESTRAINED SHELL

The analysis of the stresses and deflections produced in a restrained shell will proceed as suggested in Ref. 5. If the temperature distribution is sufficiently smooth, a reasonably accurate solution for a heated unrestrained shell can be deduced from an order of magnitude analysis of the equations. The complete solution is then formulated as the superposition of this thermal solution on a solution of an edge-loaded cold shell wherein the edge loading is chosen so as to satisfy the boundary conditions of the original problem. It is convenient to study separately the effects of middle surface heating and a temperature gradient.

VII. MIDDLE SURFACE HEATING ONLY

The effect of middle surface heating can be investigated separately by setting $T_1=0$, $T^{(+)}=T^{(-)}=T_0$. As there is no surface loading, we find $\sigma^{(+)}_{\zeta\zeta}, \sigma^{(-)}_{\zeta\zeta}=0$. Anticipating that the dominant factors in the development of the solution are the effective middle surface strains

$$\epsilon^{(0)}_\phi = \frac{1}{a}(U' + w) \qquad \epsilon^{(0)}_\theta = \frac{1}{a}(U\cot\phi + w)$$

it is convenient to write the Euler equations in the following form: the stress–strain equations

$$\epsilon^{(0)}_\phi = \alpha T_0 + \frac{1}{2Et}(N_{\phi\phi} - \nu N_{\theta\theta}) - \omega_m/a$$

$$\epsilon^{(0)}_\theta = \alpha T_0 + \frac{1}{2Et}(N_{\theta\theta} - \nu N_{\phi\phi}) - \omega_m/a$$

$$\frac{2\beta}{h+t} = -\frac{w' - U}{a} - \frac{1}{3}\frac{h+3t}{h+t}\frac{\omega'_m}{a} + \frac{\kappa}{\kappa^*}\frac{2\rho^2}{Et}Q_\phi \qquad (\kappa \approx 4\rho^2)$$

$$\epsilon_m = \frac{1}{2}\alpha_\zeta h T_0 + \frac{3S}{5E_\zeta} - \frac{Th}{35E_\zeta a}$$

$$\omega_m = -\frac{3Sh}{14E_\zeta a} + \frac{T}{14E_\zeta}$$

the moment–curvature equations

$$M^*_{\phi\phi} = M_T + \frac{Et(h+t)}{a(1-\nu^2)}[(\beta' + \nu\beta\cot\phi) + (1+\nu)\epsilon_m]$$

$$M^*_{\theta\theta} = M_T + \frac{Et(h+t)}{a(1-\nu^2)}[\beta\cot\phi + \nu\beta' + (1+\nu)\epsilon_m]$$

and the equations of equilibrium

$$S = -\frac{h/a}{h+t}(M^*_{\phi\phi} + M^*_{\theta\theta})$$

$$T = -\frac{5}{2}\frac{h^2/a}{h+t}(N_{\phi\phi} + N_{\theta\theta})$$

$$N_{\phi\phi} = Q_\phi\cot\phi \qquad N_{\theta\theta} = \frac{dQ_\phi}{d_\phi}$$

$$\frac{dM^*_{\phi\phi}}{d_\phi} + (M^*_{\phi\phi} - M^*_{\theta\theta})\cot\phi = aQ_\phi$$

where

$$M_T = -\frac{\alpha Et}{2a}\frac{(h+t)^2}{1-\nu}T_0$$

For the particular case $T_0 = \text{Const}$, a satisfactory solution can be seen to be given by

$$\beta, Q_\phi, N_\phi, N_\theta, T = 0$$

$$\epsilon_\phi^{(0)}, \epsilon_\theta^{(0)} = \alpha T_0$$

$$M^*_{\phi\phi}, M^*_{\theta\theta} = \frac{Et\,(h+t)}{a\,(1-\nu)}\frac{hT_0}{2}\left(\alpha_\zeta - \alpha\frac{h+t}{h}\right)$$

$$S = \frac{h^2}{a^2}\frac{EtT_0}{1-\nu}\left(\alpha\frac{h+t}{h} - \alpha_\zeta\right)$$

$$\epsilon_m = \alpha_\zeta\, hT_0/2 \qquad \omega_m = -\frac{3h}{14E_\zeta a}S$$

For the general case, if we make the assumption that the temperature distribution is slowly varying, i.e., the deflections and stresses vary significantly over distances comparable with the radius a, it follows that the process of differentiation with respect to ϕ does not change the order of magnitude of the term differentiated. Thus, it follows from the above system of equations that

$$Q_\phi = \frac{Et\,(h+t)}{a^2}\beta\, O(1)$$

and

$$\left.\begin{matrix} M^*_{\phi\phi}/a, M^*_{\theta\theta}/a, N_{\phi\phi}, N_{\theta\theta} \\ S, aT/h, E_\zeta\, a\, \omega_m/h \end{matrix}\right\} = \frac{M_T}{a}O(1)$$

Further, we find that a satisfactory solution is given by

$$\epsilon_\phi^{(0)}, \epsilon_\theta^{(0)} = \alpha T_0$$

$$\frac{2\beta}{h+t} = -\alpha\frac{dT_0}{d\phi}$$

Once β is determined, the remaining variables follow directly.

As an example, consider the case where

$$T_0 = T_0^*\cos\phi \qquad M_T^* = -\frac{\alpha Et}{2a}\frac{(h+t)^2}{1-\nu}T_0^*$$

With $\dfrac{2\beta}{h+t} = \alpha\, T_0^*\sin\phi$, it follows that

$$\epsilon_m = \frac{1}{2}\alpha_\zeta\, hT_0^*\cos\phi$$

$$M^*_{\phi\phi}, M^*_{\theta\theta} = \frac{\alpha_\zeta\, Et}{2\,(1-\nu)}\frac{h\,(h+t)}{a}T_0^*\cos\phi$$

$$Q_\phi = -\frac{\alpha_\zeta\, Et}{2\,(1-\nu)}\frac{h\,(h+t)}{a^2}T_0^*\sin\phi$$

$$N_{\phi\phi}, N_{\theta\theta} = -\frac{\alpha_\zeta\, Et}{2\,(1-\nu)}\frac{h\,(h+t)}{a^2}T_0^*\cos\phi$$

VIII. A TEMPERATURE GRADIENT ONLY

The effect of a temperature gradient can be investigated separately by setting $T_0 = 0$, $T^{(+)} = -T^{(-)} = T_1 h/2$. With

$$M_T = -\alpha\, Et\, T_1 \frac{h(h+t)}{2(1-\nu)}$$

the Euler equations take the following form: the stress–strain equations

$$\epsilon_\phi^{(0)} = (N_{\phi\phi} - \nu N_{\theta\theta})/2Et + \alpha h T_1 \frac{h+t}{4a} - \omega_m/a$$

$$\epsilon_\theta^{(0)} = (N_{\theta\theta} - \nu N_{\phi\phi})/2Et + \alpha h T_1 \frac{h+t}{4a} - \omega_m/a$$

$$\frac{2\beta}{h+t} = -(w' - U)/a - \frac{1}{3}\frac{h+3t}{h+t}\frac{\omega_m'}{a} + \frac{\kappa}{\kappa^*}\frac{2\rho^2}{Et} Q_\phi$$

$$E_\zeta \epsilon_m = \frac{3}{5} S - \frac{h}{35a} T$$

$$\omega_m = \alpha_\zeta h^2 T_1/8 + \frac{1}{14E_\zeta}(T - 3Sh/a)$$

and the moment–curvature equations

$$M^*_{\phi\phi} = M_T + \frac{Et(h+t)}{(1-\nu^2)a}[\beta' + \nu\beta\cot\phi + (1+\nu)\epsilon_m]$$

$$M^*_{\theta\theta} = M_T + \frac{Et(h+t)}{(1-\nu^2)a}[\beta\cot\phi + \nu\beta' + (1+\nu)\epsilon_m]$$

The equations of equilibrium are unchanged.

For the special case T_1 = Const, a satisfactory solution can be seen to be given by

$$\beta, Q_\phi, N_{\phi\phi}, N_{\theta\theta}, T = 0$$

$$M^*_{\phi\phi}, M^*_{\theta\theta} = M_T$$

$$S = -\frac{2h/a}{h+t} M_T$$

$$\epsilon_\phi^{(0)}, \epsilon_\theta^{(0)} = \frac{h^2 T_1}{4a}\left(\alpha\frac{h+t}{h} - \frac{\alpha_\zeta}{2}\right)$$

For the general case, following the argument given in the previous section for a sufficiently smooth temperature variation, it follows that the bending moments are the dominant factors in the development of the solution. Thus,

$$\frac{Et\beta}{h+t} = Q_\phi O(1), \qquad \text{if } \kappa\rho^2/\kappa^* \leqslant O(1)$$

$$= Q_\phi \kappa\rho^2/\kappa^* O(1), \qquad \text{if } \kappa\rho^2/\kappa^* > O(1)$$

and

$$\left.\begin{matrix} M^*_{\phi\phi}/a, M^*_{\theta\theta}/a, N_{\phi\phi}, N_{\theta\theta} \\ S, aT/h, Et\left(\frac{\omega_m}{a}, \epsilon_\phi^{(0)}, \epsilon_\theta^{(0)}\right)\end{matrix}\right\} = \frac{M_T}{a} O(1)$$

Further, we find that a satisfactory solution is given by

$$M^*_{\phi\phi}, M^*_{\theta\theta} = M_T \qquad Q_\phi = \frac{1}{a}\frac{dM_T}{d\phi}$$

Once Q_ϕ is determined, the remaining variables follow directly.

As an example, consider the case

$$T_1 = T^*_1 \cos\phi \qquad M^*_T = -\alpha E t T^*_1 \frac{h(h+t)}{2(1-\nu)}$$

With $Q_\phi = -\dfrac{M^*_T}{a}\sin\phi$, it follows that

$$N_{\phi\phi}, N_{\theta\theta} = -\frac{M^*_T}{a}\cos\phi$$

$$S = -\frac{2h/a}{h+t} M^*_T \cos\phi \qquad T = \frac{5h^2/a^2}{h+t} M^*_T \cos\phi$$

$$\omega_m = \frac{\alpha_\zeta}{8} h^2 T^*_1 \cos\phi$$

$$\epsilon^{(0)}_\phi, \epsilon^{(0)}_\theta = \frac{h^2 T^*_1}{2a}\cos\phi\left(\alpha\frac{h+t}{h} - \frac{\alpha_\zeta}{4}\right)$$

$$\frac{\beta}{h+t} = \frac{h^2 T^*_1 \sin\phi}{4a}\left[\alpha\frac{h+t}{h}\left(1 + 2\rho^2\frac{\kappa/\kappa^*}{1-\nu}\right) - \frac{\alpha_\zeta}{4}\left(1 - \frac{1}{3}\frac{h+3t}{h+t}\right)\right]$$

IX. EXAMPLES

Let us combine the previous analyses to obtain the solution for a restrained shell that is heated to a uniform temperature T_0. The type of restraint will correspond to fixing a vertical element of the core centerline, i.e., requiring

$$\delta_h, \beta = 0 \qquad \text{at } \phi = \phi_b$$

If $\delta_h^{(H)}$, $\beta^{(H)}$, $\delta_h^{(M)}$, $\beta^{(M)}$, $\delta_h^{(T)}$, $\beta^{(T)}$ are taken to be the horizontal displacement and rotation at the edge due to: the induced shear force, the induced bending moment and the uniform temperature, respectively, then the values of the induced force and moment H, M can be determined from the requirements that

$$\delta_h^{(H)} + \delta_h^{(H)} + \delta_h^{(T)} = 0$$

$$\beta^{(H)} + \beta^{(M)} + \beta^{(T)} = 0$$

As $\beta^{(T)} = 0$, $\delta_h^{(T)} = \alpha a T_0 \sin \phi_b$

it follows from these equations that

$$H = \frac{-\alpha a T_0 \sin \phi_b}{\dfrac{\delta_h^{(H)}}{H} - \dfrac{\delta_h^{(M)}}{M}\dfrac{\beta^{(H)}/H}{\beta^{(M)}/M}} \qquad M = -H\frac{\beta^{(H)}/H}{\beta^{(M)}/M}$$

Using the influence coefficients in Table 1, we find for $\kappa = \kappa^*$ that

$$H/\alpha EtT_0 = -0.2911 \quad M/\alpha EtaT_0 = -0.3534 \times 10^{-2}$$

For a shell with the same kind of restraint, that is heated with a uniform temperature gradient T_1, a similar analysis leads to

$$H = -0.07277 \times \frac{EtT_1h^2}{a} \times \left(\alpha\frac{h+t}{h} - \frac{\alpha_\zeta}{2}\right)$$

$$M = -0.8834 \times 10^{-3} \times EtT_1 h^2 \times \left(\alpha\frac{h+t}{h} - \frac{\alpha_\zeta}{2}\right)$$

for $\kappa = \kappa^*$, as $\beta^{(T)} = 0$

and

$$\delta_h^{(T)} = \frac{T_1h^2}{4}\left(\alpha\frac{h+t}{h} - \frac{\alpha_\zeta}{2}\right)\sin\phi_b$$

X. DISCUSSION

In the first place, let us note that the deflections of the unheated shallow spherical shell presented above show the same correspondence with those of the isotropic case that is familiar in calculations wherein shear strain is neglected. Specifically, if one compares the influence coefficients given in Table 1 with those of Ref. 8, there will be a correspondence provided that we replace the thickness of the isotropic shell h_{iso} by $\sqrt{3}\,(h+t)$ and the Young's modulus of the isotropic shell (E_{iso}) by $(Eh)_{iso} = 2Et$. In particular, following Ref. 8 and noting that the quantity $(2\beta/h+t)$ corresponds to the rotation (V), one finds using the asymptotic forms that

$$\frac{2Et\delta_h}{aH} = 2\rho\phi_b^2 = 2.50$$

$$\frac{4Et\beta}{H(h+t)} = -2\rho^2\phi_b = -70.7$$

$$\frac{Et\delta}{M\rho^4} = -\phi_b/\rho^2 = -0.3536 \times 10^{-2}$$

$$\frac{Et(h+t)\beta}{(1-\nu^2)Ma} = \frac{1}{\rho} = 0.100$$

Thus, it is apparent that the effect of shearing strain, as measured by the parameter κ/κ^*, is most pronounced in the direct deflections, i.e., the horizontal displacement δ_h due to an applied shear force H and the rotation β due to an applied moment M. The cross deflections, i.e., the horizontal displacement due to an applied moment and the rotation due to an applied shear force, appear to show little dependence on this parameter.

As far as the particular solutions for the heated shell are concerned, there appear to exist some similarities with the solution for the isotropic shell given in Ref. (5). For middle surface heating only, the middle surface strains ($\epsilon_\phi^{(0)}$, $\epsilon_\theta^{(0)}$) and the rotation (β) are identical with those of the isotropic shell. However, here the similarity ends, as the remaining quantities are strongly dependent on the transverse material properties.

For the case of a temperature gradient only, the middle surface strain and the rotation as well as the remaining quantities depend on the transverse material properties and hence the solution is not similar to that of the isotropic shell.

Further, it should be noted that the transverse normal strain is important in defining the bending moments for middle surface heating only, and the effect of finite shear strain is important in defining the edge rotation for a temperature gradient only. Thus, it can be concluded that the thermal stress solution for a sandwich shell with a soft core is strongly dependent on the effects of transverse normal strain and finite shear-strain.

Regarding the limitations of the above theory, let us review the assumptions that were made in developing the above partial solution. As was noted in the analysis of the unheated shell, the absence of the effect of transverse normal strain (as measured by ϵ_m, ω_m) depends on the existence of an edge zone wherein the bending effects predominate. Specifically this requires that $\rho \gg 1$, and that the range of (κ) be such that the coefficients ($|k|$, l_1, l_2) in the governing equation for bending do not change their order of magnitude. It follows from Fig. 3 that, for $\kappa/\kappa^* < 2$, these coefficients maintain the required order of magnitude. Apparently, for a greater value of the shear parameter (corresponding to a softer core material), one of the solutions would be so slowly varying as to invalidate the requirement that the edge zone be much narrower than the radius of the shell.

Further, additional terms were neglected on the assumption that the core modulus in the transverse direction (E_ζ) was not too small, and the core and face material thicknesses were small enough so that

$$\frac{E}{E_\zeta}\frac{th}{a^2} \ll 1$$

Finally, for the assumed form of the stresses and displacements to be realistic, i.e., to be able to assume that these quantities in the face material are independent of the transverse coordinate (ζ), the thickness of the face material relative to the core must be small enough so that

$$t/h < 1$$

In summary, the reduction of the original set of equations for an unheated spherical shell to the more simple form that was actually solved requires that

$$\frac{E}{E_\zeta}\frac{th}{a^2} \ll 1 \qquad \rho \gg 1$$

$$\frac{t}{h} < 1 \qquad \kappa/\kappa^* \leqslant 2$$

On the other hand, the analysis leading to the particular solution for the heated shell requires that there be no edge zone, i.e., the temperature variation with (ϕ) must be small enough that all displacement and stress variables vary appreciably only over distances comparable with the radius of the shell. The solution which is presented as being satisfactory shows the same basic assumption with regard to thickness as that for the unheated shell, i.e., $\rho \gg 1$, but requires also that α, α_ζ be such that

$$\alpha/\alpha_\zeta \gg (h/a)^2 \qquad \alpha_\zeta/\alpha \gg \frac{E}{E_\zeta}\left(\frac{th}{a^2}\right)$$

Finally, let us determine the criteria necessary for substantiating the soft-core hypothesis that the meridional and circumferential stresses in the core are much less than their counterparts in the face material. Following the stress–strain law for the core material, we find

$$\sigma_{\phi\phi\,(\text{core})} = \frac{1}{1-\nu_c-2\nu_\zeta^2\dfrac{E_c}{E_\zeta}}\left\{\nu_\zeta E_c\,\epsilon_\zeta - (\alpha_c+\nu_\zeta\alpha_\zeta)E_c\Delta T + \frac{E_c}{1+\nu_c}\left[\left(1-\nu_\zeta^2\frac{E_c}{E_\zeta}\right)\epsilon_\phi + \left(\nu_c+\nu_\zeta^2\frac{E_c}{E_\zeta}\right)\epsilon_\theta\right]\right\}$$

where the strains (ϵ_ζ, ϵ_ϕ, ϵ_θ) are the total strain, i.e., the superposition of the contributions from the edge-loaded unheated shell and the particular solution for the heated shell. The stress in the face material can be written as

$$\sigma_{\phi\phi}^{(+)} = \frac{1/t}{1 + \dfrac{h+t}{2a}} \left\{ \left(\frac{1}{2} N_{\phi\phi} + \frac{M^*_{\phi\phi}}{h+t} \right) \bigg|_H + \left(\frac{1}{2} N_{\phi\phi} + \frac{M^*_{\phi\phi}}{h+t} \right) \bigg|_M + \left(\frac{1}{2} N_{\phi\phi} + \frac{M^*_{\phi\phi}}{h+t} \right) \bigg|_T \right\}$$

where the superposition over the solution of the unheated shell and over that of the heated shell is indicated. However, following the analysis for the unheated shell, one notes that

$$\frac{N_{\phi\phi}}{M/h} \leqslant O(1) \qquad \frac{M^*_{\phi\phi}/h}{H} = O(1)$$

$$E_\zeta \, \epsilon_\zeta (\zeta = 0) = \frac{M}{ah} O(1)$$

so that

$$\sigma_{\phi\phi}^{(+)} = \frac{1/t}{1 + \dfrac{h+t}{2a}} \left\{ H\,O(1) + \frac{M}{h} O(1) + \left(\frac{1}{2} N_{\phi\phi} + \frac{M^*_{\phi\phi}}{h+t} \right) \bigg|_T \right\}$$

Thus, for an evaluation of the two meridional stresses, one must compute the value of (H, M) corresponding to the boundary conditions for an appropriate temperature distribution.

For a restrained shell, following the method of analysis described in Section IX, the equations for determining H, M are of the form

$$H\,O(1) + \frac{M}{h} O(1) = Et\,\epsilon_\theta^{(0)} |_T \sin\phi_b$$

$$H\,O(1) + \frac{M}{h} O(1) = \frac{Et}{\rho h} \beta \,|_T$$

where $\epsilon_\theta^{(0)} |_T$, $\beta|_T$ are the thermal strain and rotation respectively given by the particular solution. Thus, we note that H, M/h are of the same order of magnitude, and that this magnitude depends on the terms $\epsilon_\theta^{(0)} |_T$, $\beta|_T$.

Due to middle surface heating, we have

$$\epsilon_\theta^{(0)} |_T = \alpha T_0 \qquad\qquad \frac{2\beta}{h+t} \bigg|_T = -\alpha \frac{dT_0}{d\phi}$$

so that

$$H = \alpha\,Et\,T_0 \sin\phi_b \, O(1)$$

Also, for the general case,

$$N_{\phi\phi} |_T = \frac{M_T}{a} O(1) \qquad\qquad M^*_{\phi\phi} |_T = M_T\,O(1)$$

$$\epsilon_\zeta(\zeta=0)|_T = \alpha_\zeta T_0$$

where

$$M_T = -\frac{\alpha\,Et}{2a} \frac{(h+t)^2}{1-\nu} T_0$$

Thus, the meridional stresses for middle surface heating become

$$\sigma^{(+)}_{\phi\phi} = \alpha\, E T_0 \sin\phi_b\, O(1)$$

$$\sigma_{\phi\phi}(\zeta=0) = \frac{E_c\, T_0}{1 - \nu_c - 2\nu_\zeta^2 \dfrac{E_c}{E_\zeta}} \left[\alpha\, O(1) - \alpha_c\right]$$

Due to a temperature gradient only, we have

$$Et\, \epsilon^{(0)}_\theta \big|_T = \frac{M_T}{a}\, O(1)$$

$$\left.\frac{Et\,\beta}{h+t}\right|_T = \frac{1}{a}\frac{dM_T}{d\phi}\frac{\kappa\rho^2}{\kappa^*} \qquad (\text{if } \kappa\rho^2/\kappa^* \geqslant O(1)$$

so that

$$M/h = \frac{\kappa}{\kappa^*}\,\alpha\, Et\, h\, T_1 \frac{O(1)}{\rho}$$

Also, for the general case,

$$N_{\phi\phi}\big|_T = \frac{M_T}{a}\, O(1) \qquad\qquad M^*_{\phi\phi} = M_T$$

$$E_\zeta\, \epsilon_\zeta(\zeta=0)\big|_T = \frac{M_T}{ah}\, O(1)$$

where

$$M_T = -\,\alpha Et\, T_1 \frac{h(h+t)}{2(1-\nu)}$$

Thus, the meridional stresses for a temperature gradient only become

$$\sigma^{(+)}_{\phi\phi} = \alpha Eh\, T_1\, O(1)$$

$$\sigma_{\phi\phi}(\zeta=0) = \frac{\alpha E_c\, T_1\, h^2/a}{1 - \nu_c - 2\nu_\zeta^2 \dfrac{E_c}{E_\zeta}} \left\{ O(1) + \frac{\kappa\rho}{\kappa^*}\, O(1) + \nu_\zeta \frac{Et}{E_\zeta h}\, O(1) \right\}$$

Finally the requirements of the soft core hypothesis can be formulated in terms of the ratio of the meridional stresses. For middle surface heating only

$$\frac{\sigma_{\phi\phi}(\zeta=0)}{\sigma^{(+)}_{\phi\phi}} \propto E_c/E$$

if α, α_c are the same order of magnitude; and, for a temperature gradient only

$$\frac{\sigma_{\phi\phi}(\zeta=0)}{\sigma^{(+)}_{\phi\phi}} \propto \frac{E_c}{E}\frac{h}{a} \left\{ O(1) + \frac{\kappa\rho}{\kappa^*}\, O(1) + \nu_\zeta \frac{Et}{E_\zeta h}\, O(1) \right\}$$

Thus, as in the case for the unheated shell, the soft core hypothesis can be applied essentially whenever $E_c/E \ll 1$.

NOMENCLATURE

E	Young's modulus of face material
E_c	Young's modulus of core in surface ζ − const
E_ζ	Young's modulus of core in transverse direction
G_ζ	transverse shear modulus of core
h	thickness of core
K_0, K_1	modified Bessel functions
k^2	$-\kappa/2 + i\sqrt{4\rho^4 + \nu\kappa - (\kappa/2)^2}$
l_1^2, l_2^2	$\kappa/2 \pm \sqrt{(\kappa/2)^2 - \nu\kappa - 4\rho^4}$
$M^*_{\phi\phi}, M^*_{\theta\theta}$	unit moments (for $t/a \ll 1$)
$N_{\phi\phi}, N_{\theta\theta}$	unit forces
q	surface pressure
Q_ϕ	unit transverse shear force
R_1, R_2	principal radii of curvature of surface
S, T	components of $\sigma_{\zeta\zeta}$
t	thickness of face material
T_0, T_1	components of temperature distribution
U	component of meridional displacement
U_ϕ	meridional displacement
U_0, U_1, V_0, V_1	components of Bessel functions of complex argument
w	component of transverse displacement
α	coefficient of thermal expansion of face
α_c	coefficient of thermal expansion of core in a surface ζ − const
α_ζ	coefficient of thermal expansion of core in transverse direction
β	component of meridional deflection
δ_h	horizontal displacement of middle surface
$\epsilon_\phi, \epsilon_\theta, \epsilon_\zeta, \gamma_{\phi\zeta}$	strain components
ϵ_m, ω_m	components of transverse deflection
ζ	transverse coordinate
η	$\phi\sqrt{\kappa^*/2}$
θ	$\lvert k \rvert^2 \cos 2\theta = -\kappa/2;\ \lvert k \rvert^2 \sin 2\theta = \sqrt{4\rho^4 + \nu\kappa - (\kappa/2)^2}$
κ	$\left(1 + \frac{2tG_\zeta}{3hG}\right)\frac{E}{G_\zeta}\frac{2th}{(h+t)^2}$
κ^*	$4\rho^2\left[\sqrt{1+(\nu/2\rho^2)^2} + (\nu/2\rho^2)\right]$
ν	Poisson's ratio of face material
ν_c	Poisson's ratio of core in a surface ζ − const
ν_ζ	Poisson's ratio of core in transverse direction
ρ^4	$\frac{(1-\nu^2)a^2}{(h+t)^2} - \left(\frac{\nu}{2}\right)^2$
$\sigma_{\phi\phi}, \sigma_{\theta\theta}, \sigma_{\zeta\zeta}, \tau_{\phi\zeta}$	stress components
τ_m	component of transverse shear stress
ϕ	polar angle; $\frac{d}{d\phi} \equiv (\ )'$

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