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## Application of Formal Power Series to Some Classical Problems of Mechanics



by

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## Introduction

One of the oldest problems in the history of mechanics is to find functions of the coordinates which satisfy a simpler (lower order) system of differential equations. In particular, to find functions of the coordinates which satisfy a first order differential equation. The energy integral and the integrals of angular momentum are examples of such functions: they are algebraic functions of the coordinates and their derivatives which satisfy, the simplest of all first order differential equations, $Y' = 0$.

In 1887 Bruns [1] proved that in the three body problem the only algebraic functions of the coordinates and their derivatives which satisfy the equation $Y' = 0$ are those generated by the known integrals. These results were later extended to the n-body problem $(n > 2)$ by Painlevé [2] and to the restricted three body problem, for small values of a parameter $\mu$, which appears in the problem, by Poincaré [3, vol. 1 ch. v].

In this paper we extend these negative results of Bruns and Poincaré. One consequence of our results, for the n-body problem, is that no algebraic function of the coordinates and their derivatives is an exponential. Similar results are obtained for a large class of problems in mechanics.

We use the tools and ideas of differential algebra developed by Ritt [4]. The following is a summary of definitions, notations and results that we use frequently.

A differential field, differential ring, differential ideal is a field, ring, ideal which is closed under a given derivation. In this paper the characteristic of the field is zero and the derivation is the derivative with respect to the time $t$. By $K\langle x\rangle$, $K\{x\}$ we mean the differential field, differential ring respectively of $K$ and the elements $x = (x_1, \ldots, x_n)$. $K(x)$, $K[x]$ is used for field and ring adjunction, respectively.

Let $K$ be a differential field and let $S$ be a set of differential polynomials $P(X) \in K\{X\}$. The perfect (radical) differential ideal $I$ generated by $S$, in the differential ring $K\{X\}$, is the intersection of a finite number of prime differential ideals $\pi_1, \ldots, \pi_\ell$. Each $\pi_i$ has an irreducible manifold of zeros and the manifold of zeros of $S$ is the union of the manifolds of $\pi_i$, $i = 1, \ldots, \ell$. A zero $x$ of $\pi_i$ is called generic if every differential polynomial $P(X) \in K\{X\}$, which vanishes for $X = x$, belongs to $\pi_i$. Every prime differential ideal has a generic zero. Let $y$ be any zero of $S$, then the degree of transcendency of $K\langle y\rangle$ over $K$ is called the order of $y$. The order of a prime differential ideal $\pi$ is the order of its generic zero. We shall call the order of $S$ the $\max.(\text{order } \pi_i)_{1\leq i\leq \ell}$. If $S$ is a system of differential polynomials defining a mechanical system of $n$ degrees of freedom, then order of $S$ is $2n$ which agrees with the usual definition of order of $S$.

The problem that concerns us may now be restated as follows: Let $x$ be a solution of max. order of a mechanical system $S$, what are the differential subfields of $K\langle x\rangle$ which are of order one over $K$? For the n-body problem, the theorems of Bruns and Painlevé give the subfield of constants of $K\langle x\rangle$, where $K = Q(M_1, \ldots, M_n)$, $Q$ stands for the reals and $M_{i, i=1,\ldots n}$ are the masses.

These results are possible to obtain without finding the decomposition of $S$ into its prime components, because all the irreducible manifolds of $S$, of maximal order, are isomorphic (Theorem three).

Theorems one and two prepare the background for the application of formal power series, in an arbitrary parameter, to our problem. In [4] Ritt proved that if a is any zero of the highest or lowest degree terms $P^*$ of a single differential polynomial $P(X)$, then there exists a formal fractional power series of the form

$$x = ac^j + \sum_{i=j+1}^{\infty} a_i c^{k_i} \qquad j = \begin{cases} -1 & \text{if } P^* \text{ is highest degree} \\ +1 & \text{if } P^* \text{ is lowest degree} \end{cases}$$

such that $P(x) = 0$. That this theorem does not hold if $S$ consists of more than one differential polynomial can be shown by the following example which arises in problems of coupled springs:

$$(S) \qquad \begin{aligned} X_1'' &= \alpha_1 X_1 + \beta_1 X_2 + \lambda_1 (X_2 - X_1)^3 \\ X_2'' &= \alpha_2 X_1 + \beta_2 X_2 + \lambda_2 (X_2 - X_1)^3 \end{aligned}$$

$X_1 = X_2 = a \neq 0$ is a solution of the highest degree terms $(X_2 - X_1)^3 = 0$. This solution cannot be extended to a formal fractional power series solution of $S$ except in the case where $\alpha_1 + \beta_1 = \alpha_2 + \beta_2$. However, any solution of the lowest degree terms i.e. of

$$\begin{aligned} X_1'' &= \alpha_1 X_1 + \beta_1 X_2 \\ X_2'' &= \alpha_2 X_1 + \beta_2 X_2 \end{aligned}$$

can be extended to a formal power series solution of $S$. The fact that the highest order terms appear to the first degree, among the highest degree terms or lowest degree terms of the system, makes it possible to find a formal power series solution in an arbitrary parameter, where the first term of the series is a solution of a linear system of the same order. This is, precisely what occurs in a large class of mechanics problems which includes practically all the problems arising in celestial mechanics.

## I. Formal Power Series for Algebraic Systems

Let $K$ be a field. Let

$$x_i = \sum_{j = s_i}^{\infty} x_{ij} c^j \quad ; \quad i = 1, \ldots, n; \; s_i \text{ integers, and}$$

$$(x_{ij})_{\substack{1 \leq i \leq n \\ s_i \leq j \leq \infty}} \text{ are algebraically independent over } K,$$

$c$ transcendental over the field $K(x_{ij})$. Any element $u \in K(x) = K(x_1, \ldots, x_n)$ has the form

$$u = \sum_{k = \ell}^{\infty} u_k c^k \,, \quad \ell \text{ an integer and}$$

$u_k \in K(x_{ij})$ for $\ell \leq k < \infty$.

Lemma 1. Let $u = \sum_{k = \ell}^{\infty} u_k \, c^k \in K[x]$. Then

$$\text{(A)} \quad \frac{\partial u_k}{\partial x_{i,s_i+r}} = \frac{\partial u_{k-r}}{\partial x_{i,s_i}} \qquad \text{for all } r \geq 0\,,$$

where we set $\dfrac{\partial u_k}{\partial x_{i,s_i}} = 0$ if $k < \ell$.

Proof: Let $S$ be the set of all elements $u \in K[x]$ for which (A) holds. $S$ is not empty, since $x_i \in S$, $i = 1, \ldots, n$. Also, let (A) hold for $u, v \in K[x]$ and let

$$u = \sum_{k = \ell_1}^{\infty} u_k c^k \;, \quad v = \sum_{k = \ell_2}^{\infty} v_k c^k, \quad \text{then } w = u + v =$$

$$= \sum_{k = \ell_1}^{\infty} (u_k + v_k) c^k \,.$$ Without loss of generality we assumed

$\ell_2 \geq \ell_1$ and set $v_k = 0$ for $k < \ell_2$.

Now, $$\frac{\partial w_k}{\partial x_{i,s_i+r}} = \frac{\partial u_k}{\partial x_{i,s_i+r}} + \frac{\partial v_k}{\partial x_{i,s_i+r}}, \qquad r \geq 0$$

$$= \frac{\partial u_{k-r}}{\partial x_{i,s_i}} + \frac{\partial v_{k-r}}{\partial x_{i,s_i}}$$

$$= \frac{\partial w_{k-r}}{\partial x_{i,s_i}}$$

So that (A) holds for $u + v$. Similarly, let $w = uv =$

$$= \sum_{k = \ell_1+\ell_2}^{\infty} w_k c^k = \sum_{k = \ell_1+\ell_2}^{\infty} \left( \sum_{j = \ell_1}^{k-\ell_2} u_j v_{k-j} \right) c^k$$

$$\frac{\partial w_k}{\partial x_{i,s_i+r}} = \sum_{j = \ell_1}^{k-\ell_2} v_{k-j} \frac{\partial u_j}{\partial x_{i,s_i+r}} + u_j \frac{\partial v_{k-j}}{\partial x_{i,s_i+r}}$$

$$= \sum_{j = \ell_1}^{k-\ell_2} v_{k-j} \frac{\partial u_{j-r}}{\partial x_{i,s_i}} + u_j \frac{\partial v_{k-j-r}}{\partial x_{i,s_i}}$$

$$= \sum_{j = \ell_1+r}^{k-\ell_2} v_{k-j} \frac{\partial u_{j-r}}{\partial x_{i,s_i}} + \sum_{j = \ell_1}^{k-r-\ell_2} u_j \frac{\partial v_{k-j-r}}{\partial x_{i,s_i}}$$

$$= \sum_{j = \ell_1}^{k-r-\ell_2} v_{k-r-j} \frac{\partial u_j}{\partial x_{i,s_i}} + u_j \frac{\partial v_{k-r-j}}{\partial x_{i,s_i}}$$

$$= \frac{\partial w_{k-r}}{\partial x_{i,s_i}}$$

So (A) holds for $uv$ and hence $S$ is a ring containing $x_1, \ldots, x_n$ and $S = K[x]$.

Lemma 2. Let $u_k$ be as in Lemma 1, then $u_k \in K(x_{ij})_{\substack{1\le i\le n \\ s_i\le j\le s_i+k-\ell}}$ for all $u \in K(x)$.

Proof. For any $u \in K[x]$, by Lemma 1, $\dfrac{\partial u_k}{\partial x_{i,s_i+r}} = 0$ for all $r > k - \ell$. Let $v = \dfrac{1}{u} = \sum_{k=-\ell}^{\infty} v_k c^k = c^{-\ell} \sum_{j=0}^{\infty} w_j c^j$ where $w_j \in K(u_\ell, \ldots, u_{\ell+j}) \in K(x_{im})_{\substack{1\le i\le n \\ s_i\le m\le s_i+j+\ell-\ell \,=\, s_i+j}}$

Since $w_{j+\ell} = v_j$, we have $v_j \in K(x_{im})_{\substack{1\le i\le n \\ s_i\le m\le s_i+j+\ell \,=\, s_i+j-(-\ell)}}$

Hence Lemma 2 holds for all $u \in K(x)$.

Theorem 1. Let $P(X, Y) \in K[X, Y]$.
let $r$, $s$ be integers such that

$P(Xc^s, Yc^r) = P^*(X, Y)c^\ell$ + higher power of $c$, where $P^* \neq kY$; $k \in K$, and the discriminant of $P^*$, considered as a polynomial in $Y$ with coefficients in $K[X]$, is not zero. Then there exists

$$x_i = \sum_{k=s}^{\infty} x_{ik} c^k, \quad i = 1, \ldots, n; \qquad y = \sum_{k=r}^{\infty} y_k c^k$$

such that $P(x, y) = 0$ and the set $(x_{ik})_{\substack{1\le i\le n \\ s\le k<\infty}}$ are algebraically independent while $y_k$ is algebraic over $(x_{im})_{\substack{1\le i\le n \\ s\le m\le k+s-r}}$.

Proof. Let $(x_{ik})$ be a set of algebraically independent elements over $K$ in some field extension of $K$ and let

$Y = \overline{Y} = \sum_{k=r}^{\infty} \overline{Y}_k c^k$, then

$P(x, \overline{Y}) = \sum_{k=\ell}^{\infty} p_k c^k$ . By Lemma 1

$$\frac{\partial p_j}{\partial \overline{Y}_k} = \frac{\partial p_j}{\partial \overline{Y}_{r+(k-r)}} = \frac{\partial p_{j-k+r}}{\partial \overline{Y}_r} \quad \text{for} \quad j \geq \ell + k - r$$

so that $\frac{\partial p_j}{\partial \overline{Y}_k} = \frac{\partial p_j}{\partial \overline{Y}_r}$ if $j = \ell + k - r$

and $\frac{\partial p_j}{\partial \overline{Y}_k} = 0$ if $j < \ell + k - r$. Hence $\overline{Y}_k$ first appears in $p_{\ell+k-r}$. Also, $\frac{\partial p_j}{\partial x_{i,k}} = 0$, if $j < s + k - r$. Now, let $y_r$ be a zero of $p_\ell(x_{is}, \overline{Y}_r)$, $y_r \neq 0$, (such zeros exist because $P^*$ is not a power of Y). Since

$$\left(\frac{\partial p_\ell}{\partial \overline{Y}_r}\right)_{\substack{x_i = x_{ir} \\ \overline{Y}_r = y_r}} \neq 0 \quad \text{we have} \quad \frac{\partial p_{\ell+k-r}}{\partial \overline{Y}_k} \neq 0$$

Let $y_k$ be a zero of

$$p_{\ell+k-r} \in K[x_{im}, y]_{\substack{1 \leq i \leq n \\ s \leq m \leq k+s-r \\ r \leq j \leq k}} ,$$

then $X = x$, $Y = y = \sum_{i=r}^{\infty} y_i c^i$ is a zero of $P(X, Y)$ with the stated properties.

Def. A polynomial $P(X, Y)$ satisfying the conditions of Theorem 1 will be called non-singular.

## II. Algebraic Mechanical Systems

A mechanical system of n-degrees of freedom is given by a system of second order differential equations of the form:

$$X'' = F(X, X')$$

where $X, X', X''$ are n-vectors and the derivatives are with respect to the time - t. If the system is algebraic, the functions $(F_i) = F$ are algebraic functions of $X, X'$. More precisely, the system is of the form:

$$U_i \equiv Q_i(X, X', Y)\, X_i'' + P_i(X, X', Y) = 0$$

$$\text{(S)} \qquad A_j(X, X', Y_j) = 0 \qquad \begin{array}{l} i = 1, \ldots, n \\ j = 1, \ldots, m \end{array}$$

$Y = (Y_1, \ldots, Y_m)$ and

$P_i, Q_i, A_j \in K[X, X', Y]$, where $K$ is some differential field of functions of $t$.

Let $A_j(Xc^s, X'c^s, Y_j c^{r_j}) = A_j^*(X, X', Y_j)c^{\ell_j} + \ldots +$

and let $A_j^*$ be non-singular, $j = 1, \ldots, m$. Then, by Theorem 1, there exist formal power series solution of the form:

$$X_i = x_i = \sum_{j=s}^{\infty} x_{ij} c^j \qquad i = 1, \ldots, n$$

$$X_i' = x_i' = \sum_{j=s}^{\infty} x_{ij}' c^j$$

$$Y_j = y_j = \sum_{k=r_j}^{\infty} y_{jk} c^k \qquad j = 1, \ldots, m$$

such that $A_j(x, x', y_j) = 0$. Since the $(x_{ij}), (x_{ij}')$ are indeterminates, we may set $x_{ij}'$ to the time-derivative of $x_{ij}$. Then

$$U_i(x, x', x'', y) = \sum_{j=f_i}^{\infty} u_{ij} c^j = 0, \qquad i = 1, \ldots, n$$

$$u_{ij} \in K[(x_{\ell\mu}), (x_{\ell\mu}'), (x_{\ell\mu}''), (y_{g\nu})]_{\substack{1 \le \ell \le n \\ s \le \mu \le s + j - f_i \\ 1 \le g \le m \\ r_g \le \nu \le r_g + j - f_i}}$$

and the $(y_{g\nu})$ are algebraic over $(x_{\ell\alpha}), (x_{\ell\alpha}')$ $1 \le \ell \le n$ ;

$$s \le \alpha \le \nu + s - r_g \le r_g + j - f_i + s - r_g = j - f_i + s$$

Thus the $u_{ij}$ are rational integral in $(x_{\ell\mu}'')$, algebraic over $(x_{\ell\mu}), (x_{\ell\mu}')$ $1 \le \ell \le n$ , $s \le \mu \le s + j - f_i$. If we let degree of $y_j = r_j/s$ ; $j = 1, \ldots, m$ then every element of $K\{x, y\}$ has highest and lowest degree terms. $(x_{is})$; $1 \le i \le n$ is of order $2n$ if and only if $u_{if_i}$ is of order 2; that is if $x_i''$ appears among the lowest degree terms of $U_i$ when $s \ge 0$, or among the highest degree terms when $s < 0$.

Therefore, the relative degree of the terms, in which $x_i''$ appears, is determined by the degree assigned to the $y_j$. It may, of course, happen that the system $A_j = 0$ ; $j = 1, \ldots, m$ has more than one formal power series solution and $x_i''$ appears among the lowest (highest) degree terms for one determination of the $y_i$ but not for the other. It may, also, happen that $x_i''$ appears among the lowest degree terms for one determination of the $y_i$ and among the highest degree terms for another determination of $y_j$ , $j = 1, \ldots, m$, as the following example illustrates.

Example:

$$(S) \quad \begin{aligned} YX_1'' + \lambda_1 X_1 + \lambda_2 X_2^2 &= 0 \\ YX_2'' + \lambda_2 X_2 + \lambda_4 X_1^2 &= 0 \end{aligned} \qquad \begin{aligned} X^2 = (X_1^2 + X_2^2)^2 + 1 \\ \lambda_i \in K \; ; \quad i = 1, \ldots, 4 \end{aligned}$$

We may let $X_i = x_i = \sum_{j=1}^{\infty} x_{ij} c^j$ , $Y = y = 1 + \sum_{j=1}^{\infty} y_j c^j$

the degree of $y$ is zero, and $x_{11}, x_{21}$ is a solution of the linear system

$$X_1'' + \lambda_1 X_1 = 0 \quad , \quad X_2'' + \lambda_3 X_1 = 0$$

or we may set $X_i = x_i = \sum_{j=-1}^{\infty} x_{ij} c^j \quad Y = y = \sum_{j=-2}^{\infty} y_j c^j$

the degree of $y$ is two and $x_{11}, x_{21}$ is a solution of the linear system

$$X_1'' = 0 \qquad X_2'' = 0 \quad .$$

Theorem 2. Let (S) be an algebraic mechanical system. Let $A_j(X, X', Y_j) = 0$; $j = 1, \ldots, m$ have a formal power series solution

$$X_i = x_i = \sum_{k=s}^{\infty} x_{ik} c^k \quad ; \qquad i = 1, \ldots, n,$$

$$X_i' = x_i' = \sum_{k=s}^{\infty} x_{ik}' c^k$$

$$Y_j = y_j = \sum_{k=r_j}^{\infty} y_{jk} c^k \quad ; \qquad j = 1, \ldots, m,$$

as described in Theorem 1.

Let $$U_i = \sum_{k=f_i}^{\infty} u_{ik} c^k \qquad \text{where}$$

$$u_{if_i} = q_i(x_{\ell s}, x_{\ell s}', y_{jr_j})[x_{is}'' + L_i(x_{\ell s}, x_{\ell s}')] \quad \substack{1 \leq \ell \leq n \\ 1 \leq j \leq m}$$

and $L_i$ is linear homogeneous in $(x_{\ell s})$, $(x_{\ell s}')$ with coefficients in some algebraic extension of $K$. Then the system (S) has a formal power series solution where the $x_{ik}$ are generic solutions of

$$x_{ik}'' + L_i(x_{\ell k}, x_{\ell k}') = p_i \quad ; \qquad i = 1, \ldots, n$$

and the $p_i$ belong to an algebraic extension of $K\langle(x_{\ell \mu})\rangle$ ; $1 \leq \ell \leq n$ , $s \leq \mu < k$ .

Proof. By Theorem 1 the $x_{ik}$, $x_{ik}'$ are indeterminates. Hence we may specialize $x_{is}$, $x_{is}'$ to be generic solutions of the homogeneous linear system

$$X_i'' + L_i(X, X') = 0 \qquad i = 1, \ldots, n$$

We may extend this specialization to $y_{jr_j}$ $\quad j = 1, \ldots, n$

Since $q_i(x_{\ell s}, x_{\ell s}', y_{jr_j}) \neq 0$ and

$$u_{i,f_i+k} = q_i(x_{\ell s}, x_{\ell s}', y_{jr_j})[x_{i,s+k}'' + L_i(x_{\ell,s+k}, x_{\ell,s+k}')] + A_i$$

where $A_i \in K[x_{\ell\mu}, x_{\ell\mu}', y_{j\nu}]$ ; $1 \leq \ell \leq n, \quad x \leq \mu < s + k,$
$y \leq j \leq m, \quad r_j \leq \nu < r_j + k$

we may successively specialize $x_{i,s+k}$ to be a generic solution of the linear system

$$x_{i,s+k}'' + L_i(x_{\ell,s+k}, x_{s+k}') = - A_{i/q_i} = p_i$$

and then extend the specialization to $y_{j,r_j+k}$; This is possible, for by Theorem 1 $y_{j,r_j+k}$ is algebraic over

$K(x_{\ell\mu}, x_{\ell\mu}')$ $\qquad \ell = 1, \ldots, n$
$\qquad \mu = s, s+1, \ldots, s + k - 1$

and the set $x_{\ell\mu}, x_{\ell\mu}'$ are algebraically independent over $K$. This proves our assertion.

Theorem 3. Let (S) be an algebraic mechanical system with $n$ degrees of freedom. Let $K(X, X', Y)$ be a field extension of degree $r$ over $K(X, X')$. Then (S) has $r$ irreducible manifolds of order $2n$. Any two such manifolds are isomorphic.

Proof. Since $(X, X')$ is a transcendence base for $K(X, X', Y)$ over $K$ we may extend any derivation of $K$ by defining $X'$ to be the derivative of $X$ and $(X')' = \frac{P_i(X, X', Y)}{Q_i(X, X', Y)}$, so that (S) has manifolds of order 2n. Since $K(X, X', Y)$ has r distinct isomorphism in some differential extension field of $K\langle X, X'\rangle$ it follows that (S) has r distinct irreducible manifolds of order 2n. Now, let $M_1$, $M_2$ be any two such manifolds and let $(x, y)$, $(\overline{x}, \overline{y})$ be a generic point of $M_1$, $M_2$ respectively. The isomorphism

$$\sigma: \begin{aligned} x &\longrightarrow \overline{x} \\ x' &\longrightarrow \overline{x}' \end{aligned}$$

can be extended to an isomorphism of $K(x, x', y)$ onto $K(\overline{x}, \overline{x}', \overline{y})$ by letting $\overline{y} = \sigma(y)$. This clearly defines an isomorphism of $K\langle x, y\rangle$ onto $K\langle\overline{x}, \overline{y}\rangle$ ; for the successive derivatives of $\overline{x}, \overline{y}$ are given by the same equations as those given for $x, y$ with the substitution of $\overline{x}, \overline{y}$ for $x, y$.

Remark 1. The statement that (S) has r irreducible manifolds of order 2n means that the field $K(x, x')$ belongs to a differential field $T$ such that $T$ contains r differential subfields $T_1, \ldots, T_r$; $\bigcap_{i=1}^{r} T_i = K(x, x')$ and each $T_i$ is generated by a solution of the system (S) with

$$X = x \qquad X' = x'$$

Remark 2. For the n-body problem, this theorem implies that it is immaterial whether the forces between the bodies are attractive or repulsive; the algebraic properties of the general solution are the same.

**Theorem 4. Let (S) be an algebraic mechanical system** with n degrees of freedom. Let $A_j(X, X', Y_j)$ $j = 1, \ldots, m$ be non-singular. Let $U_i = \sum_{k = f_i}^{\infty} u_{ik}c^k$ (as in Theorem 2) and let $u_{if_i} = q_i(x_{\ell s}, x'_{\ell s}, y_{jr_j})[x''_i + L_i(x_{\ell s}, x'_{\ell s})]$ . Let $\theta \in K < \xi, \eta >$ where $x = \xi$, $y = \eta$ is a solution of order 2n of (S). Let $\theta$ be a solution of a differential equation $B(\Theta) = 0$ where $B(\Theta) \in K\{\Theta\}$. Then $\theta^*(x_{\ell s}, y_{jr_j})$, $1 \leq \ell \leq n$; $1 \leq j \leq m$, is a solution of $B^*(\Theta) = 0$, where $\theta(\xi c^s, \eta_j c^{r_j}) = \theta^*(\xi, \ )c$ + higher powers of c and $B^*$ is the lowest (highest) degree terms of B if $\alpha \geq 0$ $(\alpha < 0)$.

Proof. Since the $A_j$; $j = 1, \ldots, n$, are non-singular, by Theorem 1, $X = x$, $Y = y$ is a solution of the algebraic system $A_j = 0$ and by Theorem 2 $X = x$, $Y = y$ is a solution of the system (S) with $K\langle(x_{\ell s})\rangle$ of order 2n over K. Therefore $K < x, y >$ is of order 2n. By Theorem 3 there exists an isomorphism

$$\sigma: \begin{array}{l} x \longleftrightarrow \xi \\ y \longleftrightarrow \eta \end{array}$$

Therefore, $\theta(x, y)$ is a solution of $B(\Theta) = 0$. Now, $B(\theta(x, y)) = B^*(\theta^*(x_{\ell s}, y_{jr_j}))c^{\beta}$ + higher powers of c.

Hence $\theta^*$ is a solution of $B^* = 0$.

III. Applications

1. The n-Body Problem

$$(S)\qquad X_i'' = \sum_{j \neq i} \frac{M_j(X_j - X_i)}{R_{ij}^3} \qquad i = 1, \ldots, n$$

$$Y_i'' = \sum_{j \neq i} \frac{M_j(Y_j - Y_i)}{R_{ij}^3}$$

$$Z_i'' = \sum \frac{M_j(Z_j - Z_i)}{R_{ij}^3}$$

$$R_{ij}^2 = (X_i - X_j)^2 + (Y_i - Y_j)^2 + (Z_i - Z_j)^2$$

This is a system with $3n$ degrees of freedom. The differential field $K$ is $C(M_1, \ldots, M_n)$; $C$ is the field of complex numbers and the $(M_i)$ are $n$ transcendental constants.

This system, clearly, satisfies the conditions of Theorem 1, 2. Therefore:

$$X_i = x_i = \sum_{k=-1}^{\infty} x_{ik} c^k$$

$$Y_i = y_i = \sum_{k=-1}^{\infty} y_{ik} c^k \qquad R_{ij} = \sum_{k=-1}^{\infty} r_{ijk} c^k$$

$$Z_i = z_i = \sum_{k=-0}^{k=-1} z_{ik} c^k$$

is a solution of (S), where $(x_{i,-1})$, $(y_{i,-1})$, $(z_{i,-1})$ is a generic solution of the homogeneous linear system

$$X_i'' = 0,\ Y_i'' = 0,\ Z_i'' = 0; \qquad i = 1, \ldots, n.$$

The field of constants of $K<(x_{i,-1}), (y_{i,-1}), (z_{i,-1})>$ is $D = K(< (x'_{i,-1}), (y'_{i,-1}), (z'_{i,-1}), (u_k u'_\ell - u'_k u_\ell)>$, where $u_k$ spans the set $\{(x_{i,-1}), (y_{i,-1}), (z_{i,-1})\}$.

Let $X = \xi$, $Y = \eta$, $Z = \zeta$ be any solution of order $6n$ of the n-body problem. Let $\theta$ be a polynomial in $\xi'$, $\eta'$, $\zeta'$, with coefficients in the field $K(\xi, \eta, \zeta, (r_{ij}))$ such that $\theta' = 0$. By Theorem 4, $\theta^*((x_{i,-1}), (y_{i,-1}), (z_{i,-1})) \in D$. Hence the highest degree terms of $\theta$ are polynomials in $\xi'$, $\eta'$, $\zeta'$, $(u_k u'_\ell - u'_k u_\ell)$ with coefficients in $K$. Thus, the highest degree terms of the energy integral is a polynomial in $\xi'$, $\eta'$, $\zeta'$, and the integrals of angular momentum are of the form $u_k u'_\ell - u'_k u_\ell$.

Theorem 5. Let $X = \xi$, $Y = \eta$. $Z = \zeta$ be any solution (real or complex), of order $6n$, of the n-body problem. Let $\theta \in K<\xi, \eta, \zeta>$ and let $\theta$ be a solution of $P(\Theta) = 0$; $P(\Theta) \in K\{\Theta\}$, P or order 1. Then $P^*(\Theta)$ (the highest degree terms of $P(\Theta)$ of degree if $\theta > 0$, the lowest degree terms of $P(\Theta)$ if degree $\theta < 0$) is divisible by $\Theta'$.

Proof. By theorem 4, $\theta^*(x_{i,-1}), (y_{i,-1}), (z_{i,-1})$ is a solution of $P^*(\Theta) = 0$ which is homogeneous and of order 1. If $P^*(\Theta)$ is not divisibel by $\Theta'$, every zero of, $P^*(\Theta) = \Theta' - k_i\Theta$ ($k_i$ $\epsilon$ to an algebraic extension of $K$), is an exponential, but $K<(x_{i,-1}), (y_{i,-1}), (z_{i,-1})>$ has no exponential elements for $K<(x_{i,-1}), (y_{i,-1}), (z_{i,-1})> = D<t>$. Hence $P^*(\Theta)$ is divisible by $\Theta'$.

Cor. Let $\xi$, $\eta$, $\zeta$, $\theta$ be as in Theorem 5. Then $\theta$ cannot be an exponential.

2. Restricted Three-Body Problem.

$$X'' - 2Y' - X = -\frac{(X+\mu)(1-\mu)}{R_1^3} - \frac{(X+\mu-1)\mu}{R_2^3}$$

(S) $$Y'' + 2X' - Y = -\frac{Y(1-\mu)}{R_1^3} - \frac{Y\mu}{R_2^3}$$

$$R_1^2 = (X+\mu)^2 + Y^2 \ , \ R_2^2 = (X+\mu-1)^2 + Y^2$$

Let $K = C(\mu)$ where $C$ is the field of complex numbers and $\mu$ is a transcendental constant over $C$.

It follows from Theorems 2, 3 that (S) has a solution

$$X = x = \sum_{k=-1}^{\infty} x_k c^k \quad , \quad Y = y = \sum_{k=-1}^{\infty} y_k c^k$$

$$R_1 = \sum_{k=-1}^{\infty} r_{1k} c^k \qquad R_2 = \sum_{k=-1}^{\infty} r_{2k} c^k$$

$x_{-1}$, $y_{-1}$ is a generic solution of the homogeneous linear system:

$$X'' - 2Y' - X = 0$$
$$Y'' + 2X' - Y = 0$$

$$r^2_{1,-1} = x^2_{1,-1} + y^2_{1,-1} = r^2_{2,-1}$$

Note that if $X = \xi$, $Y = \eta$, $R_1 = \rho_1$, $R_2 = \rho_2$ is any solution of (S) then

$$K\langle \xi, \eta, \rho_1, \rho_2 \rangle = K\langle \xi, \eta \rangle$$

Lemma 3. Let $X = a$, $Y = b$, $R = \rho$ be a generic solution of the system:

$$X'' - 2Y' - X = 0 \qquad R^2 = X^2 + Y^2$$
$$Y'' + 2X' - Y = 0$$

Then the field of constants of $K\langle a, b, \rho\rangle = K(C_1, C_2)$ where

$$C_1 = a'^2 + b'^2 - (a^2 + b^2)$$

$$C_2 = ab' - a'b + a^2 + b^2$$

Proof. $C_1' = 2(a'a'' + b'b'' - aa' - bb')$

$$= 2(a'[2b' + a] + b'[b - 2a'] - aa' - bb') = 0$$

$$C_2' = ab'' - a''b + 2(aa' + bb')$$

$$= a[b - 2a'] - [2b' + a]b + 2(aa' + bb') = 0$$

Let $u = a' - b$ then

$$u' = a'' - b' = 2b' + a - b' = b' + a$$

and
$$u^2 + u'^2 = (a'^2 + b'^2) + (a^2 + b^2) + 2(ab' - a'b)$$
$$= C_1 + 2C_2$$

Also,
$$a'u - au' = a'(a' - b) - a(b' + a)$$
$$= a'^2 - a^2 - (a'b + ab')$$
$$= u^2 - C_2$$

Therefore, $$\left(\frac{a}{u}\right)' = \frac{u^2 - C_2}{u^2}$$

Also, $u^2 + u'^2 = C_1 + 2C_2$ implies

$$2u'(u + u'') = 0 \quad \text{but} \quad u' = a + b' \neq 0$$

since $a, b$ is generic, so that $u'' = - u$

let $$v = \frac{a}{u} - \frac{C_2}{C_1 + 2C_2}\frac{u'}{u}$$ then

$$v' = \frac{u^2 - C_2}{u^2} + \frac{C_2}{C_1 + 2C_2} + \frac{C_2' \, u'^2}{(C_1 + 2C_2)\, u^2}$$
$$= 1 - \frac{C_2}{u^2} + \frac{C_2}{C_1 + 2C_2} + \frac{C_2(C_1 + 2C_2 - u^2)}{(C_1 + 2C_2)u^2}$$
$$= 1$$

Since $K\langle a, b\rangle = K(C_1, C_2')\langle u, v\rangle$,

for $$a = \left(v + \frac{C_2 \, u'}{(C_1 + 2C_2)u}\right) u$$

$$b = a' - u \,,$$

and $K(C_1, C_2) < u, v >$ is isomorphic to

$K(C_1, C_2) < \sqrt{C_1 + 2C_2}$ sine $t, t >$, we have field of constants of $K< a, b >$ equals field of constants of

$K(C_1, C_2) < \sqrt{C_1 + 2C_2}$ sine $t, t >$ equals $K(C_1, C_2)$.

Now let $\alpha \in K < a. b, \rho >$ ; $\rho^2 = a^2 + b^2$ and let $\alpha' = 0$.

Let $\alpha = \gamma + \delta\rho$ ; $\gamma, \delta \in K < a, b >$

$$\alpha' = \gamma' + \delta'\rho + \delta\rho' = 0, \quad \gamma'\rho + \delta'(a^2 + b^2) + \delta(aa' + bb') = 0$$

$$\therefore \quad \gamma' = 0 \quad \text{and} \quad (\delta^2\rho^2)' = 0 \quad \text{but} \quad \delta^2\rho^2 \in K <a, b>$$

$$\therefore \quad \delta^2\rho^2 \in K(C_1, C_2') = K(a'^2 + b'^2 - a^2 - b^2, ab' - a'b + a^2 + b^2)$$

It is clear that a rational function in the polynomials $a'^2 + b'^2 - a^2 - b^2, ab' - a'b + a^2 + b^2$ cannot be divisible by $a^2 + b^2$. Therefore $\delta = 0$ and $\alpha = \nu \in K(C_1, C_2)$.

Theorem 6. Let $X = \xi$, $Y = \eta$, $R_1 = \rho_1$, $R_2 = \rho_2$ be any (real or complex) solution, of order four, of the restricted three body problem. Let $\theta \in K(\xi, \eta, \rho_1, \rho_2)[\xi', \eta']$ and let $\theta' = 0$. Then $\theta^*$ is a polynomial in $P_1 = \xi'^2 + \eta'^2 - \xi^2 - \eta^2$, $P_2 = \xi\eta' - \xi'\eta + \xi^2 + \eta^2$.

Proof. There is no loss in generality in assuming degree of $\theta \geq 0$, since degree of $\frac{1}{\theta}$ = - degree of $\theta$ and $(\frac{1}{\theta})' = 0$. By theorem 4, $\theta^*(a, b) = 0$, so that

$$\theta^*(a, b) \in K(a'^2 + b'^2 - a^2 - b^2, ab' - a'b + a^2 + b^2) .$$

But $\theta$ is a polynomial in $\xi' \; \eta'$ so that $\theta^*(a, b)$ is a polynomial in

$$P_1(a, b) = a'^2 + b'^2 - a^2 - b^2,$$

$$P_2(a, b) = ab' - a' b + a^2 + b^2$$

Therefore $\theta^*(\xi, \eta)$ is a polynomial in $P_1(\xi, \eta)$, $P_2(\xi, \eta)$.

Theorem 7. Let $X = \xi$, $Y = \eta$, $R_1 = \rho_1$, $R_2 = \rho_2$ be any real solution, of order four, of the restricted three-body problem. Let $\theta \in K < \xi, \eta >$ ; $K = C(\mu)$ where $C$ is the field of real numbers. Then $\theta$ is not a real exponential.

Proof. Same argument as for the n-body problem. For, the differential field $K < a, b >$ does not contain any real exponentials.

## References


[1] H. Bruns, Säachs, Sitzber. 13 (1887), pp. 1-39, 55-82.

[2] P. Painlevé, Comptes Rendus 124 (1897), pp. 173-196
P. Painlevé, Budd. Astr. 15 (1898), pp. 81-93.

[3] H. Poincaré, Les Methodes Nouvelles de la Mécanique Céleste. Dover (1957).

[4] J. F. Ritt, Differential Algebra, New York, Amer. Math. Soc. Colloquium Publications, Vol. 33 (1950).

[5] A. Wintner, The Analytical Foundations of Celestial Mechanics, Princeton, N. J., Princeton University Press (1941).