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MINIMAX STUDIES

y W. A. Glasser, K. D. Graham, and C. A. Harvey

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ABSTRACT

Two brief studies of the effect of non-zero initial conditions on the performance according to the minimax criterion and on the selection of minimax controllers from a given set of controllers are reported. The results of two studies of extremal bounded amplitude, bounded rate inputs to linear systems are also reported.

The first study of the effect of non-zero initial conditions considers one flight condition for a vehicle of the Saturn V Type with first order gimbal dynamics. The control configuration has pitch rate, lagged pitch attitude and normal acceleration feedbacks. Each of the optimal controllers had one positive pole, one negative pole and a stable complex pair of poles. The positive pole is small and its magnitude decreases with increasing magnitude of initial conditions.

The second study of the effect of non-zero initial conditions considers two flight conditions for "Model Vehicle Number 2 for Advanced Control Studies" with no gimbal dynamics. The control configuration has pitch attitude, pitch rate and lateral velocity feedbacks. The optimal gains are found to be monotone functions of the magnitude of initial conditions. Further, the stability of the optimal system tends to increase with increasing magnitude of initial conditions.

The first study of extremal inputs is restricted to an oscillator. The theoretical development indicates the relation between several sets of necessary conditions and one sufficient condition. One set of necessary conditions is shown to be sufficient and from these conditions general explicit formulas for extremal inputs are derived.

The last study pertains to the development of computational algorithms for extremal inputs for general linear stationary systems. Two algorithms are presented, and an example of computer results obtained from one algorithm is given.

FOREWORD

This document partially comprises the final report prepared by Honeywell, Incorporated for George C. Marshall Space Flight Center, Huntsville, Alabama, 35812 under Contract NAS 8-11206.

The application of optimal (minimax) control theory to a piecewise constant approximation of a large launch booster for the first 84 seconds of flight is presented in NASA CR-546. A linear piecewise constant controller is determined which minimizes the maximum of several cost items.

The work on this contract was supervised by Mr. C. R. Stone and Dr. E. R. Rang. Section 2 was prepared by Mr. W. A. Glasser. Section 3 was prepared by Mr. K. D. Graham. Sections 4 and 5 were prepared by Dr. C. A. Harvey. Dr. J. Y. S. Luh contributed to the results of Section 4. The linear programming formulation presented in Section 5 was developed by Dr. P. Treuenfels.

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INTRODUCTION

The need to design controllers for large launch boosters provides the motivation for the minimax studies. Four areas of investigation are discussed. The first two studies are concerned with the effect of non-zero initial conditions on the selection of minimax controllers. The remaining studies are aimed at theoretical developments which are necessary for the inclusion of a bound on the time rate of change of the disturbance in the minimax problem statement. The inclusion of such a constraint would yield a closer approximation to disturbances which are encountered in practice.

NON-ZERO INITIAL CONDITIONS

The purpose for studies in Sections 2 and 3 was to examine the effect of non-zero initial conditions on control cost (performance index) and on selection of minimax controllers for large launch boosters. In both sections, rigid vehicles with linear controllers and bounded amplitude winds are assumed.

Saturn V Study

The vehicle in Section 2 is a typical Saturn V booster with first order gimbal dynamics for a ten-second flight condition characterized by maximum dynamic pressure and Mach number of about 1.7. Gains for a good controller for this vehicle with zero initial conditions were known from work on NASA Contract NASw-563 (Honeywell MPG Report 1541-TR 14). This controller had pitch rate, lagged pitch attitude and normal acceleration feedbacks. Four values of initial conditions were chosen on each state variable (pitch attitude, pitch rate, lateral velocity, and gimbal angle). The gain grid chosen represented 54 different controllers and contained the controller for zero-initial conditions. The wind velocity had a magnitude of 75 meters per second.

It was found that a set of only four controllers minimized the control cost for all of the 16 initial conditions. In particular, the controller for zero initial conditions was also best for small values of initial conditions on pitch attitude, pitch rate, gimbal angle, and all values of lateral velocity considered.

The four best controllers all had one positive and one negative real pole, and a stable complex pair of poles. The positive pole was small and its magnitude decreased with increasing magnitude of initial conditions. The closed loop natural frequency and the damping ratio of the complex pair decreased with increasing amplitude of initial conditions.

The ranges of values of the positive pole, natural frequency, and damping ratio of the four best controllers are as follows:

$$0.001738 \leq \text{real pole} \leq 0.005408$$

$$0.780 \text{ cps} \leq f \leq 0.898 \text{ cps}$$

$$0.201 \leq \zeta \leq 0.063$$

Model Vehicle Number 2 Study

Data for the vehicle in Section 3 is taken from the data package "Model Vehicle Number 2 for Advanced Control Studies" and perfect gimbal dynamics were assumed. A cost item corresponding to bending moment was included in this study. Two flight conditions were considered: (1) one was sixteen seconds long near Mach 0.55 with dynamic pressure about one-third of maximum; and

(2) the second was eight seconds long at Mach 1 with about eight-tenths maximum dynamic pressure. The controllers had pitch attitude, pitch rate, and lateral velocity feedback gains with the wind disturbance introduced in such a manner that the gains could easily be converted to equivalent ones for controllers with pitch attitude, pitch rate, and either normal acceleration or attack angle feedback signals.

Good controllers for zero initial conditions were known for both flight conditions from Honeywell Report 12003-FTR1. Each had relatively high gains and all real poles with one of them positive. The positive pole was small for the first flight condition (real pole at 0.00055) and large for the second one (real pole at 0.30809). These particular flight conditions were selected because it was expected that the influence of initial conditions would be comparatively large with higher controller gains, and particularly so with the controller having the large positive pole.

Three iterations of cost computations were performed. A total of 125 controllers was included in each gain grid. Three values of non-zero initial conditions were chosen for pitch attitude, pitch rate, and lateral velocity, with a range of four to one between the minimum and maximum values in each case. The wind velocity was 59 meters per second in the first flight condition and 75 meters per second in the second one.

The results for the first flight condition are generally summarized as follows:

- All gains are monotone non-decreasing/non-increasing with the magnitude of any initial conditions
- All minimax controllers have a negative real and a stable complex pair of closed loop poles
- The real pole is much closer to the origin than the complex pair and its distance from the origin decreases with increasing magnitude of initial conditions

- The damping ratio and natural frequency of the complex pair increase slightly with increasing magnitude of initial conditions.
- The range of values of the real pole Z_o is $-0.0011 \leq z_o \leq -0.00018$.
- The range of values of ζ and ω_n of the complex poles is $0.64 \leq \zeta \leq 0.87$ and $0.071 \leq \omega_n$ cps ≤ 0.094.
- It is possible to select one fixed gain controller which gives good performance for each initial condition.
- One initial condition, the maximum value of the initial condition on lateral velocity considered, must be excepted for several of the above conclusions. However, this initial condition appears to be larger than need be considered, so its exception is not serious.

The conclusions for the second flight condition are similar:

- All gains are monotone non-decreasing/non-increasing with the magnitude of initial conditions on the state variables.
- All minimax controllers have real poles and one of them is positive.
- The distance of the positive poles from the origin decreases with increasing magnitude of initial conditions ($0.041 \leq z_o \leq 0.3291$).
- It is possible to select one fixed gain controller which gives good performance for each initial condition.
- One initial condition, the maximum of the initial condition on pitch attitude considered, must be excepted for the third and fourth conclusions.

THEORETICAL DEVELOPMENTS

The purpose of section 4 is to present theoretical developments applicable to the minimax control problem with a bound on the time derivative of the disturbance. The extremal inputs may be thought of as worst disturbances to the system and the desired result of this study is a means of characterizing such inputs. The discussion is restricted to an oscillator so that explicit results are achieved. The oscillator is general, however, in the sense that extremal inputs may have an arbitrary number of segments on which the input is at its extreme. The theoretical development presented for the oscillator can be generalized. The results of such a generalization are presented in section 5. The discussion of the theory associated with the oscillator indicates the relationship between the necessary conditions for extremal inputs obtained by Gamkrelidze, Bryson, Denham and Dreyfus, and Russell and Schmaedeke, and the sufficient conditions obtained by Russell. The necessary conditions of Russell and Schmaedeke are shown to be sufficient. These conditions are used to determine general explicit formulas for extremal inputs.

COMPUTATIONAL ALGORITHMS

The purpose of section 5 is to develop computational algorithms which may be used to determine extremal inputs for general linear stationary systems. Necessary and sufficient conditions for extremal inputs are presented. A computational algorithm is formulated based on these conditions. Also a linear programming formulation of an approximation to the problem is given. Results of a computer program developed from this last formulation are presented for an example.

SECTION 2

MINIMUM WIND EFFECT CONTROL OF A SATURN V LAUNCH VEHICLE WITH NON-ZERO INITIAL CONDITIONS

A natural criterion for launch booster control is the minimum wind effect criterion. Small errors in the system response do not degrade performance. Hence, there is no reason for saying that control performance is not optimum if the controller permits small errors. The major concern is the maximum value that error components attain over the entire launch trajectory. Hence, a desirable control criterion is one that rates controllers (in terms of a performance index) according to their capabilities for holding the maximum normalized error component to a minimum over the launch interval.

The synthesis of such a controller presents a formidable task. First, it must be assumed that the launch vehicle can be adequately described over the portion of the launch trajectory of interest by a set of linear, constant-coefficient differential equations. A second and less restrictive assumption is that the controller is linear fixed-gain. Further, it is assumed that the wind disturbance is bounded by a known maximum speed. Under these assumptions, a minimum wind effect controller is synthesized for a Saturn V launch vehicle with non-zero initial conditions.

Given the launch vehicle data for the maximum dynamic pressure flight condition, a controller is synthesized which minimizes maximum weighted error components over a fixed time interval with worst disturbances within a given class of bounded amplitude disturbances and a specified vehicle initial condition. The resulting controller is a linear, fixed-gain feedback controller whose optimal gains are a function of vehicle initial conditions.

The linear representation of the longitudinal rigid-body body equations of motion of a Saturn V launch booster is chosen to illustrate the synthesis technique of a minimum wind-effect controller for a linear stationary system with non-zero initial conditions and amplitude bounded disturbances. The vehicle data is that for the maximum dynamic pressure flight condition.

The synthesis procedure selects the controller gains such that a specified performance index will be minimized for a given disturbance and vehicle initial condition. This results in the need to integrate a system of first-order, piecewise linear, autonomous, ordinary differential equations. The computation may be readily accomplished with the use of either an analog or digital computer.

The numerical results indicate that the optimal gains (which minimize the performance index) are a function of the vehicle initial conditions. Furthermore, the vehicle has an unstable closed-loop pole for certain optimal gains. Having determined the optimal gains for a given vehicle initial condition, a linear fixed-gain controller which minimizes the performance index for the specified initial condition is determined.

SYSTEM REPRESENTATION

To illustrate the synthesis of a minimum wind effect controller for a linear stationary system with non-zero initial conditions and amplitude bounded disturbances, a linear representation of the longitudinal rigid body equations of motion of a Saturn V launch vehicle is considered (reference 1). The assumed equations of motion are:

$$\begin{aligned} \ddot{\phi} &= -C_1\alpha - C_2\beta \\ \ddot{z} &= \gamma_1\alpha + \gamma_2\phi + \gamma_3\beta \qquad (1) \\ \alpha &= \phi + (v_w - \dot{z})/v \end{aligned}$$

The control equation is:

$$\tau\dot{\beta} + \beta = K_3 \{\dot{\phi} + K_2\phi + [K_1\gamma_1 - C_1\tau - C_1K_1 (C_M - C_G)]\alpha +$$

$$+ [K_1\gamma_3 - C_2\tau - C_2K_1 (C_M - C_G)] \beta\}$$

Introducing $x_1 = \phi$ (attitude angle); $x_2 = \dot{\phi}$ (attitude rate); $x_3 = \dot{z}$ (displacement rate of center of gravity); and $x_4 = \beta$ (gimbal motor defection angle) yields the following set of closed loop equations:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -C_1 & 0 & C_1/v & -C_2 \\ \gamma_1 + \gamma_2 & 0 & -\gamma_1/v & \gamma_3 \\ k_1 & k_2 & k_3 & k_4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ -C_1/v \\ \gamma_1/v \\ -k_3 \end{bmatrix} g^{(t)} \qquad (2)$$

where:

$$k_1 = (K_3/\tau) \{K_2 + K_1\gamma_1 - C_1 [\tau + K_1 (C_M - C_G)]\}$$

$$k_2 = K_3/\tau \qquad (3)$$

$$k_3 = (-K_3/\tau v) \{K_1\gamma_1 - C_1 [\tau + K_1 (C_M - C_G)]\}$$

$$k_4 = \{K_3 [K_1\gamma_3 - C_2\tau - C_2K_1(C_M - C_G)] - 1\}/\tau$$

Equivalently (2) may be written as:

$$\dot{x} = A_Q X + C_R \, g(t)$$

The open loop set of equations is:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -C_1 & 0 & C_1/v & -C_2 \\ \gamma_1 + \gamma_2 & 0 & -\gamma_1/v & \gamma_3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 0 \\ -C_1/v \\ \gamma_1/v \\ 0 \end{bmatrix} g(t) \quad (4)$$

where:

$$u = k_1 x_1 + k_2 x_2 + k_3 x_3 + k_4 x_4 - k_3 \, g(t)$$

The four real parameters k_1, k_2, k_3, and k_4 may be thought of as pseudo-gains. However, these parameters must be constrained so that the solutions (of the defining equations for k_1, k_2, k_3, k_4) for the gains, time constant and accelerometer location are physically realizable. In order for the time constant τ to be real it is necessary that:

$$[(k_4 C_1 + v k_3 C_2)(C_M - C_G) - k_4 \gamma_1 - v k_3 \gamma_3]^2$$

$$- 4k_2 (C_1 \gamma_3 - C_2 \gamma_1) [C_1 (C_M - C_G) - \gamma_1] \geq 0 \quad (5)$$

Denoting by b_8 and b_9 the minimum and maximum values respectively of $C_M - C_G$ such that C_M represents an accelerometer location on the vehicle, it is possible to express the constraint on the k's in the form:

$$E(k) \cap [b_8, b_9] \neq \phi$$

where:

$$E(k) = \{\lambda: [(k_4C_1 + vk_3C_2)\lambda - k_4\gamma_1 - vk_3\gamma_3]^2$$

$$\geq 4k_2 (C_1\gamma_3 - C_2\gamma_1)(C_1\lambda - \gamma_1)\}$$

This constraint is just a mathematical way of stating that the k's must be chosen so that there is some accelerometer location on the vehicle for which the corresponding value of τ is real. For a set of acceptable gains, the control law is given by:

$$u = k_1\phi + k_2\dot{\phi} + k_3\dot{z} + k_4\beta - k_3 g(t) \qquad (6)$$

The data used represents that for a typical Saturn class launch booster. Units for the data are meters, radians, and seconds. $T = 10$, $C_1 = -0.2165$, $C_2 = 1.1381$, $\gamma_1 + \gamma_2 = 27.66$, $\gamma_1/v = 0.0133$, $\gamma_3 = 17.65$, $v = 507$.

The control criterion (performance index) is defined as:

$$C(u) = \max_{0 \leq i \leq S} C_i(u)$$

where:

$$C_i(u) = \max_{0 \leq t \leq T} \max_{g \in G} | d^i \cdot x(t, u, g) |$$

for each u in the class U, with x(t, u, g) denoting the solution of (2) and d^i representing a non-zero constant weighting vector for i = 1, 2, ... , S where S is a positive integer. A controller is said to be optimal in case it is an element of U which minimizes C(u).

For the present problem, it is possible to write the performance index (reference 2) as:

$$C_i(u) = \max_{t \in [0,\ T]} \{ |\lambda_i(t)| + \mu_i(t) \} \tag{7}$$

The functions $\lambda_i(t)$ and $\mu_i(t)$ may be obtained as solutions of sets of piece-wise linear autonomous differential equations.

The term $\lambda_i(t)$ may be expressed as:

$$\lambda_i(t) = d^i \cdot e^{A_Q t} x^\circ \tag{8}$$

which can be obtained from the solution of the linear system $\dot{x} = A_Q x$ with $x(0) = x^\circ$. For the example being considered, this results in the system of equations given by:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -C_1 & 0 & C_1/v & -C_2 \\ \gamma_1+\gamma_2 & 0 & -\gamma_1/v & \gamma_3 \\ k_1 & k_2 & k_3 & k_4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} , \quad x^\circ = \begin{bmatrix} \phi_o \\ \dot{\phi}_o \\ \dot{z}_o \\ \beta_o \end{bmatrix} \tag{9}$$

Forming the inner product $d^i \cdot e^{A_Q t} x_o$ yields:

$$\lambda_1 = d_1^1 \phi + d_2^1 \dot{\phi} + d_3^1 \dot{z} + d_4^1 \beta$$

$$\lambda_2 = d_1^2 \phi + d_2^2 \dot{\phi} + d_3^2 \dot{z} + d_4^2 \beta$$

$$\lambda_3 = d_1^3 \phi + d_2^3 \dot{\phi} + d_3^3 \dot{z} + d_4^3 \beta \qquad (10)$$

$$\lambda_4 = d_1^4 \phi + d_2^4 \dot{\phi} + d_3^4 \dot{z} + d_4^4 \beta$$

If the d_j^i are chosen such that:

$$\begin{cases} d_j^i = 0 \text{ if } j \neq i \\ d_j^i = d^i \text{ if } j = i \end{cases}$$

Then the system of equations given by (10) reduces to:

$$\phi = \lambda_1 / d^1$$

$$\dot{\phi} = \lambda_2 / d^2$$

$$\dot{z} = \lambda_3 / d^3 \qquad (11)$$

$$\beta = \lambda_4 / d^4$$

If $\phi_o = \dot{\phi}_o = \dot{z}_o = \beta_o = 0$, the solution of (9) is identically zero. Consequently, the $|\lambda_i|$'s indicate the contribution to the cost created when the initial conditions for the vehicle are non-zero with no disturbance present.

The second function $\mu_i(t)$ may be expressed as:

$$\mu_i = \int_0^t d^i \cdot e^{A_Q(t-\tau)} C_R \gamma_i(\tau) d\tau \qquad (12)$$

with $\gamma_i(t)$ given by:

$$\gamma_i(t) = (V_w)_{max} \operatorname{sgn} [d^i \cdot e^{A_Q(t-\tau)}] \qquad (13)$$

which represents a worst disturbance condition. Substitution of expression (13) into (12) yields:

$$\mu_i = (V_w)_{max} \int_0^t |d^i \cdot e^{A_Q\tau} C_R| d\tau \qquad (14)$$

To simplify, notation $(V_w)_{max}$ will be replaced by V_w in all that follows. The term $e^{A_Q\tau}C_R$ may be determined by solving the set of differential equations:

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -C_1 & 0 & C_1/v & -C_2 \\ \gamma_1+\gamma_2 & 0 & -\gamma_1/v & \gamma_3 \\ k_1 & k_2 & k_3 & k_4 \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \\ Z_4 \end{bmatrix}, \quad z(0) = \begin{bmatrix} 0 \\ -C_1/v \\ \gamma_1/v \\ -k_3 \end{bmatrix} \qquad (15)$$

The μ's may be thought of as representing the costs induced by the disturbance for zero vehicle initial conditions.

ANALOG COMPUTATION

The computation necessary to determine the performance index, C(u), is readily accomplished using an analog computer. For purposes of scaling, the system of equations given by (4) was rewritten as:

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \\ \dot{y}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -C_1 & 0 & C_1 & -C_2 \\ \frac{(\gamma_1+\gamma_2)}{v} & 0 & \frac{-\gamma_1}{v} & \frac{\gamma_3}{v} \\ k_1 & k_2 & k_3 v & k_4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} + \begin{bmatrix} 0 \\ -C_1 \\ \gamma_1/v \\ -k_3 v \end{bmatrix} \frac{g(t)}{v} \tag{16}$$

where:

$$y_1 = \phi, \; y_2 = \dot{\phi}, \; y_3 = \dot{z}/v, \text{ and } y_4 = \beta.$$

This was necessitated by the small numerical value of k_3 (approximately 0.00049) which was optimal. The product $k_3 v$ is approximately equal to -0.249 to which the analog potentiometers may be readily adjusted. Accordingly, equations (9) and (15) are modified. Since the disturbance is normalized with respect to the vehicle velocity, v, the expression for μ_i now becomes:

$$\mu_i = \frac{V_w}{v} \int_0^t \left| d^i \cdot e^{A_Q \tau} C_R \right| d\tau \tag{17}$$

The expressions $e^{A_Q t} x^o$ and $e^{A_Q t} C_R$ are each evaluated using four integrators and the necessary summing and inverting circuits. The inner products are easily formed since the following set of weighting vectors, d^i, is used for this example:

$$d^1 = \begin{bmatrix} 1/2.35 \\ 0 \\ 0 \\ 0 \end{bmatrix}; \quad d^2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}; \quad d^3 = \begin{bmatrix} 0 \\ 0 \\ 1/523 \\ 0 \end{bmatrix}; \quad d^4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/1.33 \end{bmatrix}$$

The weighting vectors were selected by determining the maximum value of the transient response of each of the parameters for a similar launch vehicle to a disturbance input and then normalizing such that $d^1\phi_{max} = d^2\dot{\phi}_{max} = d^3\dot{z}_{max} = d^4\beta_{max}$.

The absolute values were formed using two diodes and two summing amplifiers. A complete wiring diagram for the analog computer is shown in Figure 1.

The computer is scaled such that ten volts equals one degree or one degree per second. Since the performance index as determined from the analog computation is in degrees as opposed to radians for the digital computation, one must convert degrees to radians or vice versa for comparison purposes.

The primary use of the analog computer was that of observing the system response for a given initial condition and corresponding optimal gain set. Furthermore, it gave a convenient check of the digital computation results.

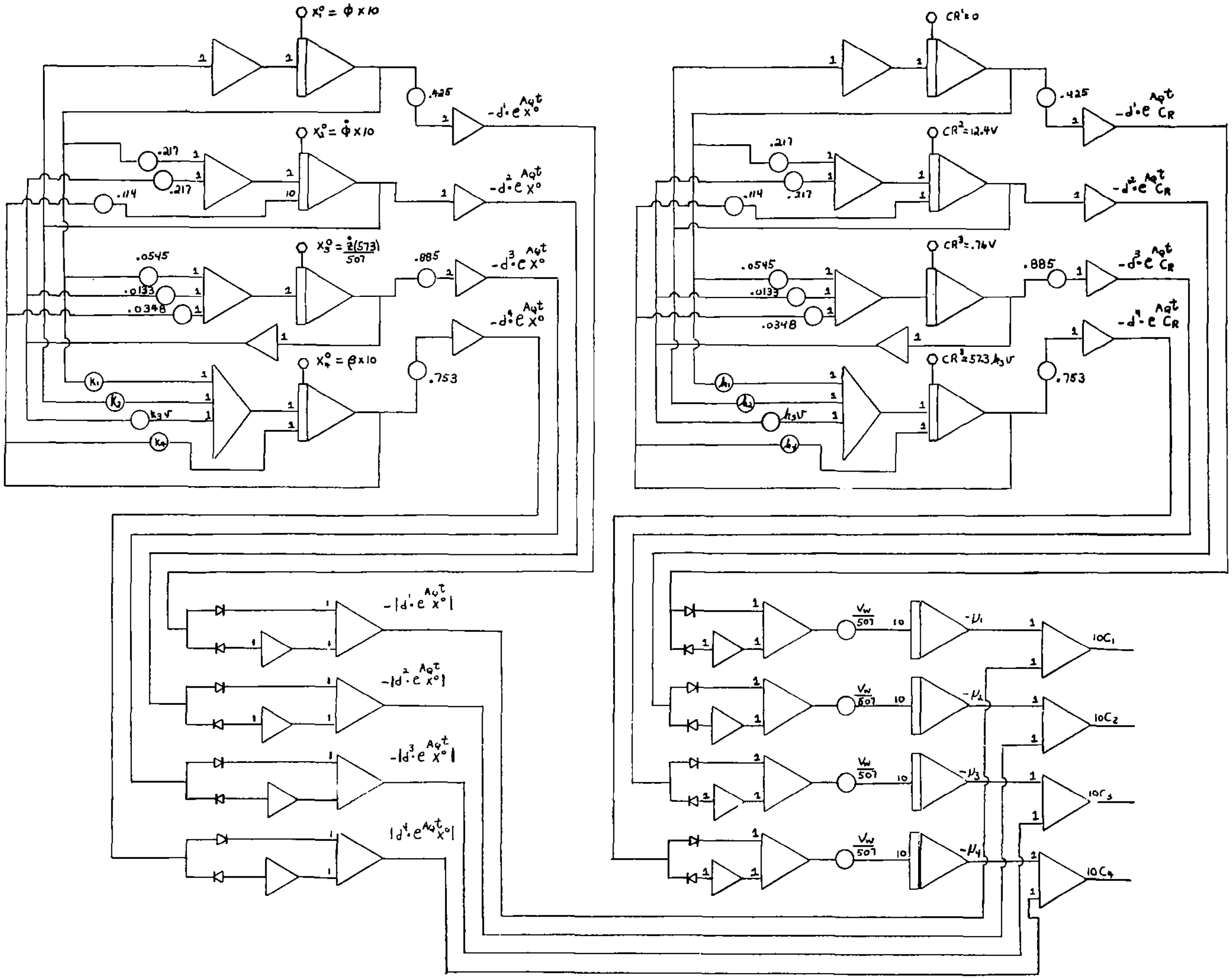

Figure 1. Analog Computation of the Performance Index C(U) for Nonzero Initial Conditions

DIGITAL COMPUTATION

A program was written for the Honeywell H-1800 digital computer to evaluate C(u) by solving the systems of equations described by (9), (12), and (15). Initially it was assumed that the set of minimizing gains for non-zero initial conditions would be close to the optimal gains for zero initial conditions. Consequently, the refined grid of Example 3 of Reference 1 was chosen to minimize C(u) for non-zero initial conditions. This gain grid consisted of the following set of gains (Gain Grid II):

$$k_1 = 0.401765,\ 0.602647,\ 0.802647$$

$$k_2 = 0.471405,\ 0.942809,\ 1.414214$$

$$k_3 = -0.00049105,\ -0.00036828$$

$$k_4 = -0.353553,\ -0.707107,\ -1.060066$$

To more closely observe the dependence of the gains on the vehicle initial conditions, another gain grid refinement was made (Gain Grid III):

$$k_1 = 0.401765,\ 0.502207,\ 0.602647$$

$$k_2 = 1.178511,\ 1.414214,\ 1.649916$$

$$k_3 = -0.00049105,\ -0.00042977$$

$$k_4 = -0.530330,\ -0.707107,\ -0.883586$$

All possible combinations of gains were taken resulting in a total of 54 gain sets. These were conveniently numbered 1 through 54 and consequently any reference to a particular gain set number is only significant with respect to the manner in which the combinations were ordered.

The performance index was minimized for the following set of non-zero initial conditions:

ϕ_o = 1, 2, 3, and 4 deg

$\dot{\phi}_o$ = 0.5, 1.0, 1.5 and 2.0 deg/sec

$\dot{z}_o$ = 1, 2, 3, and 4 m/sec

β_o = 0.5, 1.0, 1.5, and 2.0 deg

SIMULATION RESULTS

The numerical results presented herein will be those obtained using Gain Grid III and a disturbance magnitude, V_w of 75 m/sec. Table 1 identifies the gains which minimized C(u) for the set of initial conditions used. Table 2 lists the initial conditions and the corresponding optimal gains. Table 3 shows the location of the controlled vehicle poles. Approximately 16 minutes of digital computer time was required to determine the optimal gains for the set of initial conditions considered. For small values of ϕ_o, $\dot{\phi}_o$, β and all values of $\dot{z}_o$ considered, the optimal gain set is equal to the optimal gain set for zero initial conditions.

The change in gains with a change of initial conditions is shown in Figure 2 which indicates that the gains are functions of the initial conditions.

If the initial conditions are sufficiently small, then $\mu_i >> | \lambda_i |$ and the optimal gain set will be equal to the optimal gains for zero initial conditions. Also if V_w is sufficiently large, the optimal gains will be independent of initial conditions. Initially the k's were chosen such that $|k_i| \leq M_i$. For all the optimal gain sets, the gain k_3 is at its maximum value. Consequently a smaller value of C(u) may have been obtained had the bounds on k_3 been increased in magnitude.

Table 1. Identification of Optimal Gains

	k_1	k_2	k_3	k_4
Gain 8	0.401765	1.414214	-0.00049105	-0.707107
Gain 14	0.401765	1.649916	-0.00049105	-0.707107
Gain 31	0.502207	1.649916	-0.00049105	-0.530330
Gain 49	0.602647	1.649916	-0.00049105	-0.530330

	C(u) in Radians for $X^\circ = 0$
Gain 8	0.121686
Gain 14	0.124523
Gain 31	0.134310
Gain 49	0.142547

V_w = 75 m/sec and T = 10 sec

Table 2. Optimal Gain and Minimum Cost for Given Set of Initial Conditions

$X_1^{\cdot} = \phi_o$	$X_2^{\cdot} = \dot{\phi}_o$	$X_3^{\cdot} = \dot{z}$	$X_4^{\cdot} = \beta$	Minimizing Gain	Cost C(u) in Radians
1.0°				8	0.164166
2.0°				49	0.194999
3.0°				49	0.221071
4.0°				49	0.250300
	0.5°/sec			8	0.133654
	1.0°/sec			8	0.145514
	1.5°/sec			31	0.150867
	2.0°/sec			31	0.158254
		1 m/sec		8	0.137765
		2 m/sec		8	0.158381
		3 m/sec		8	0.169858
		4 m/sec		8	0.185952
			0.5°	8	0.137249
			1.0°	14	0.152562
			1.5°	31	0.164756
			2.0°	31	0.174840

Min C(u) = 0.121686 for $X^o = 0$ and V_w = 75 m/sec
Corresponding Gain Set is Gain 8.

Table 3. Location of Controlled Vehicle Poles

Root		Real Part	Imaginary Part	
Gain 8	1	0.005408	0.000000	
	2	-0.273208	0.000000	
	3	-0.226303	1.113617	f = 0.898 cps
	4	-0.226303	-1.113617	ζ = 0.201
Gain 14	1	0.005308	0.000000	
	2	-0.229022	0.000000	
	3	-0.248347	1.227910	f = 0.815 cps
	4	-0.248347	-1.227910	ζ = 0.159
Gain 31	1	0.002385	0.000000	
	2	-0.315827	0.000000	
	3	-0.115094	1.261882	f = 0.791 cps
	4	-0.115094	-1.261882	ζ = 0.092
Gain 49	1	0.001738	0.000000	
	2	-0.380567	0.000000	
	3	-0.080662	1.268127	f = 0.780 cps
	4	-0.080662	-1.268127	ζ = 0.063

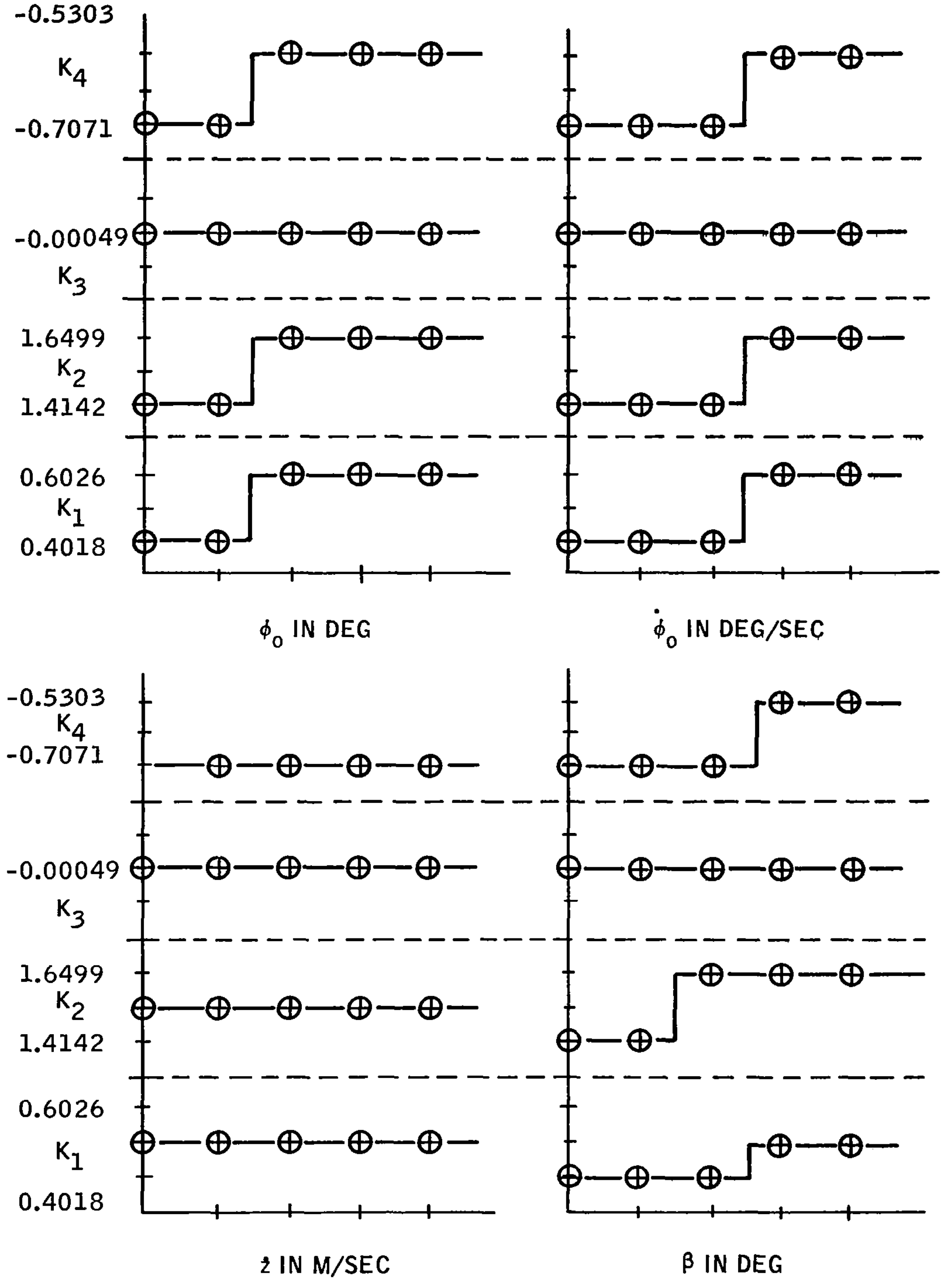

Figure 2. Change in Gains with a Change in Initial Conditions

The responses of the optimally controlled system to a worst disturbance of maximum amplitude of 75 m/sec with the initial conditions previously given are shown in Figures 3 through 11. Observation of the analog traces indicates that the vehicle parameter $\dot{z}$ is the major contributor to the increase in the performance index for non-zero initial conditions over zero initial conditions. In fact, without exception

$$C(u) = \max_{1 \leq i \leq 4} C_i(u) = C_3$$

where:

$$C_3 = \max_{t \in [0,\ 10]} \{\mu_3 + |d^3 \dot{z}|\}$$

The analog traces of the λ's are proportional to the transient response of the vehicle with non-zero initial conditions as given by (8). For the problem at hand:

$$\phi = 2.35\lambda_1$$

$$\dot{\phi} = \lambda_2$$

$$\dot{z} = 575\lambda_3$$

$$\beta = 1.33\lambda_4$$

Observation of the C_i traces for large initial conditions (i. e., Figure 4) shows that the max $\{\mu_i + |d^i x_1|\}$ may occur before T equals 10 seconds because of the oscillatory component of the performance index.

The closed loop pole positions are presented in Table 3. One of the two real roots is unstable except for the optimal gain corresponding to ϕ_o equals 4 degrees. Over the range of ϕ_o, the frequency and damping ratio of the

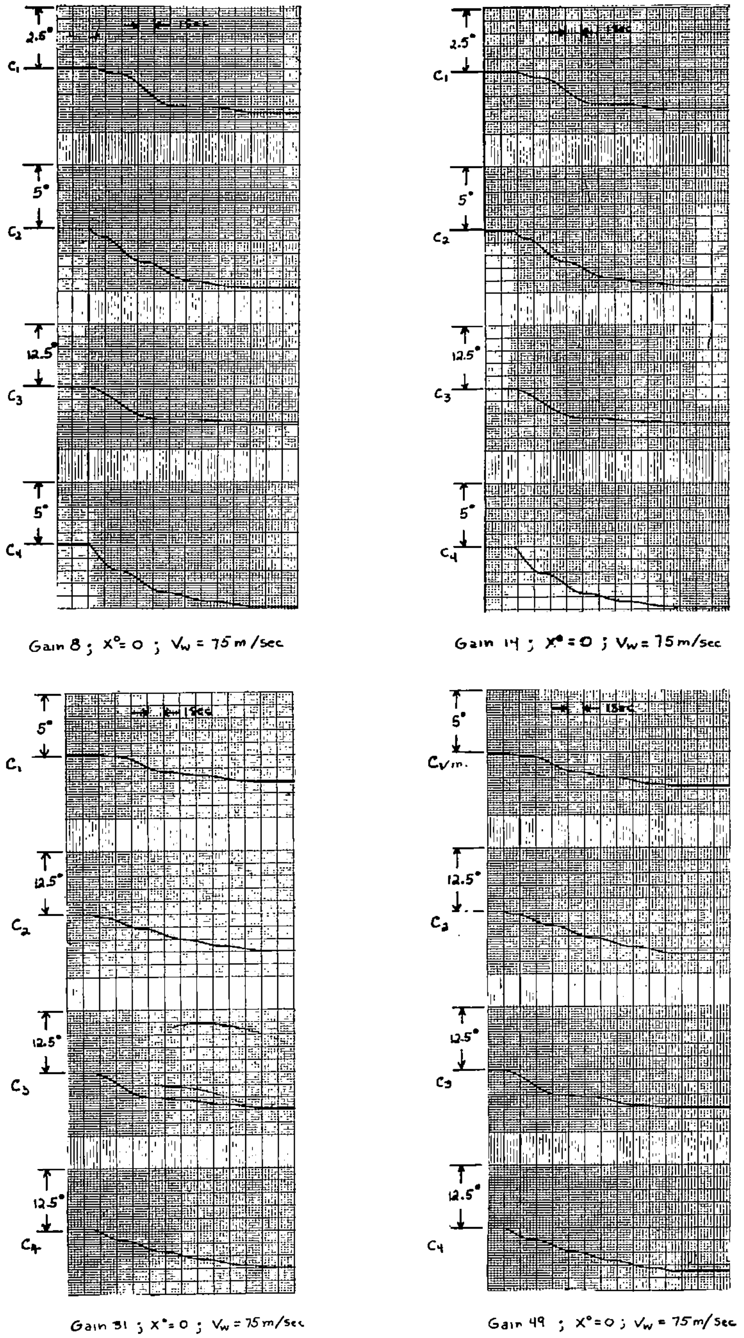

Figure 3. Worst Disturbance Responses of Optimally Controlled System with Zero Initial Conditions

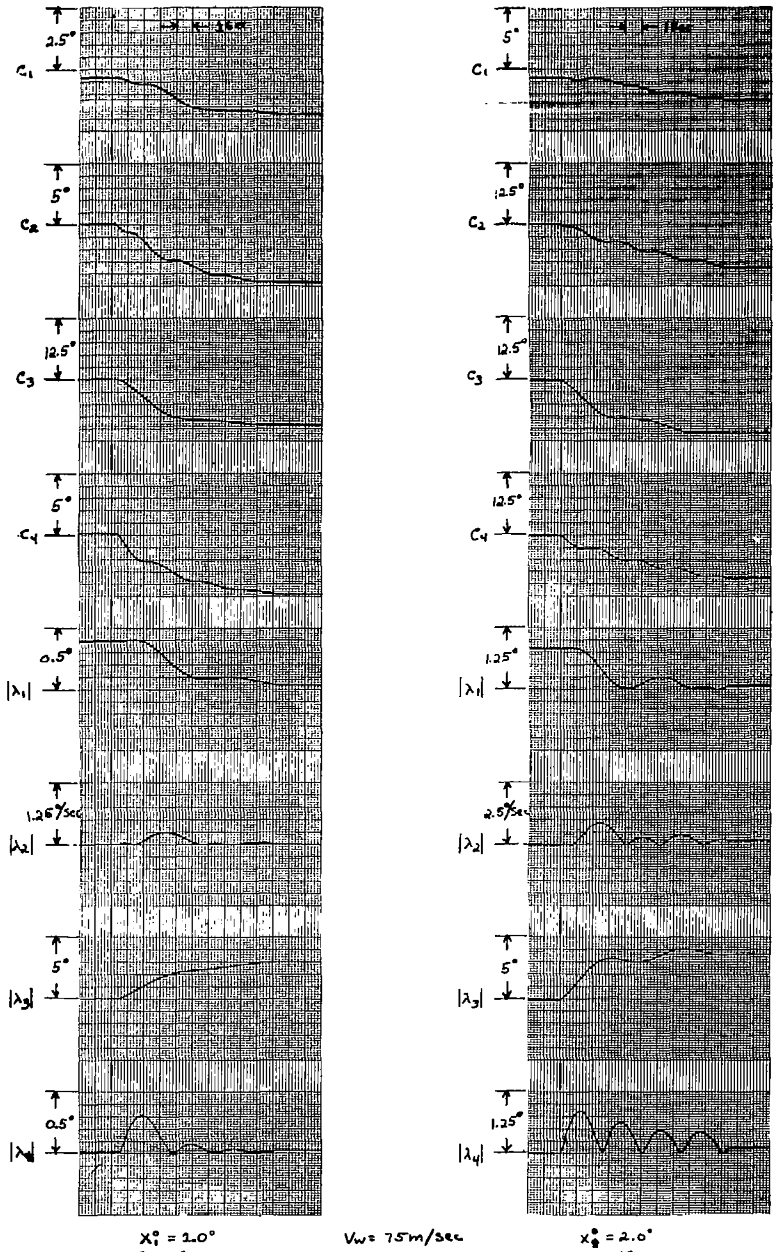

Figure 4. Worst Disturbance Responses of Optimally Controlled System with ϕ = 1 and 2 degrees

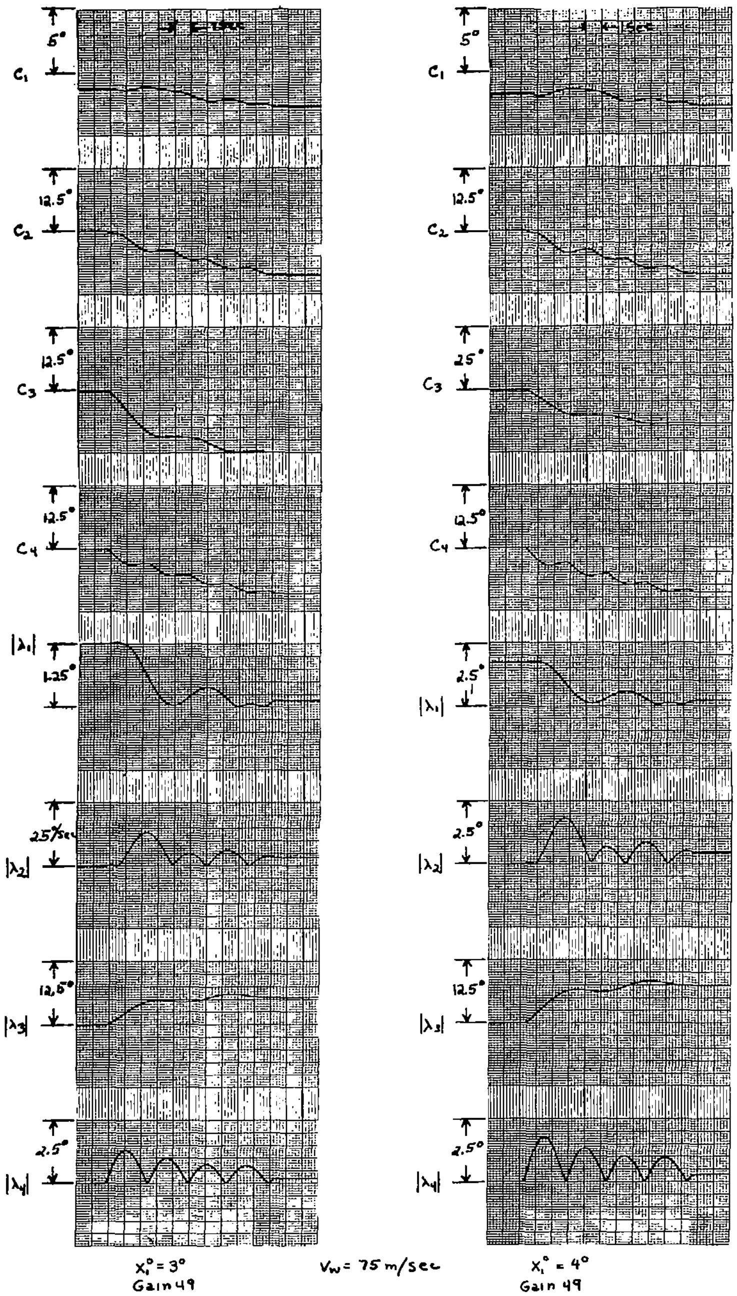

Figure 5. Worst Disturbance Responses of Optimally Controlled System with ϕ = 3 and 4 degrees

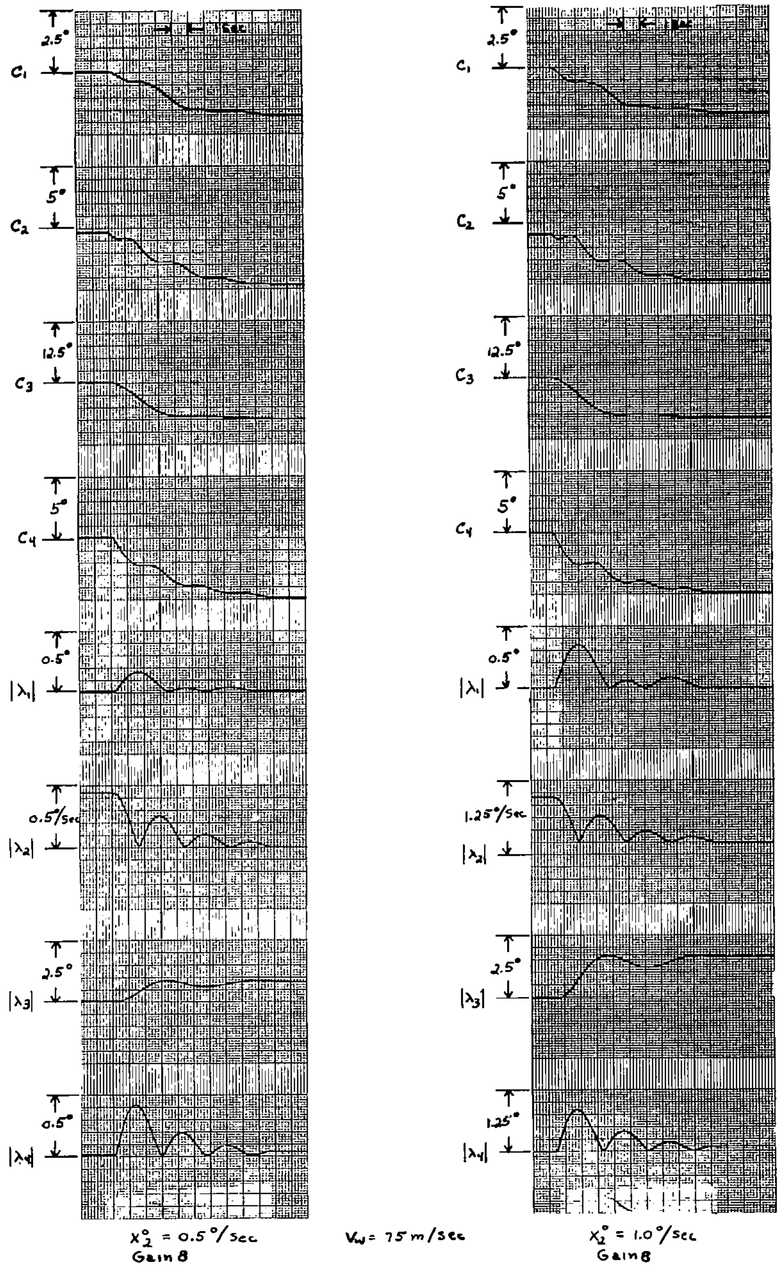

Figure 6. Worst Disturbance Responses of Optimally Controlled System with $\dot{\phi} = 0.5^\circ$/sec and 1.0°/sec

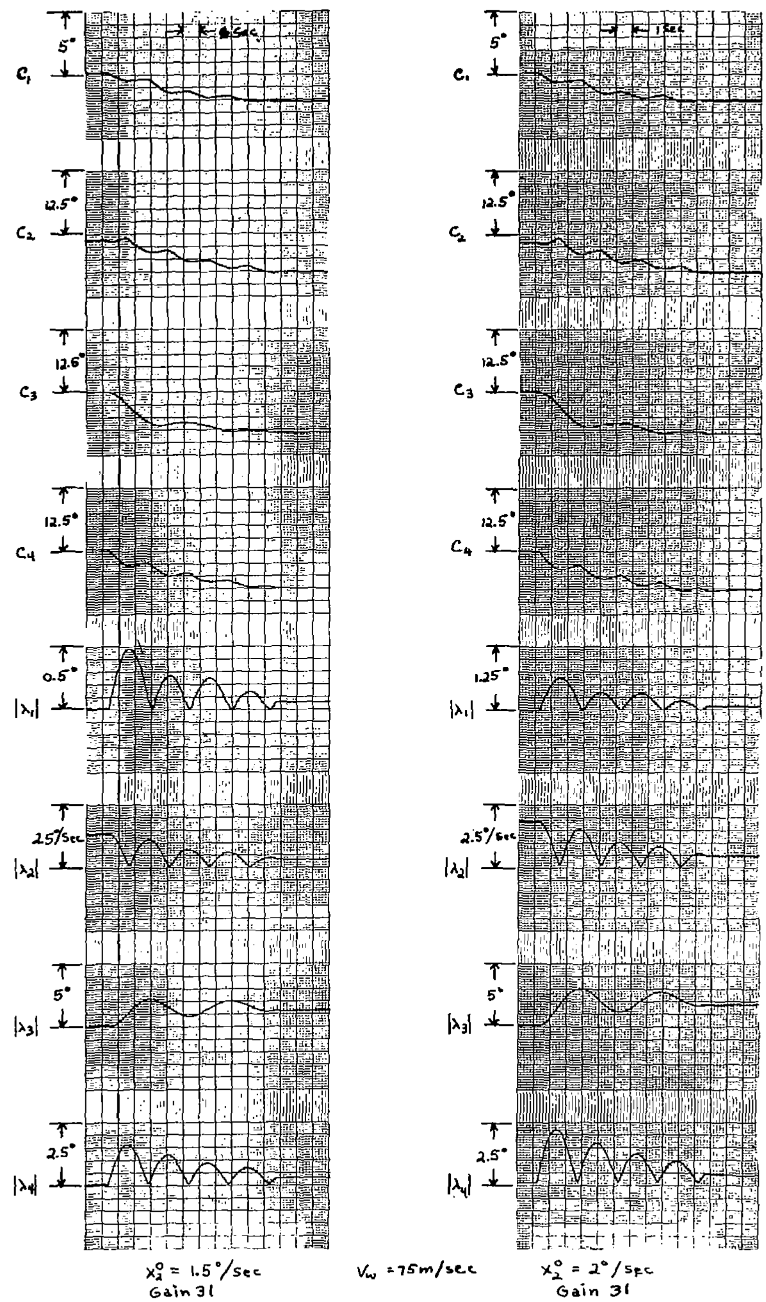

Figure 7. Worst Disturbance Response of Optimally Controlled System with $\dot{\phi} = 1.5^o/sec$ and $\dot{\phi} = 2^o/sec$

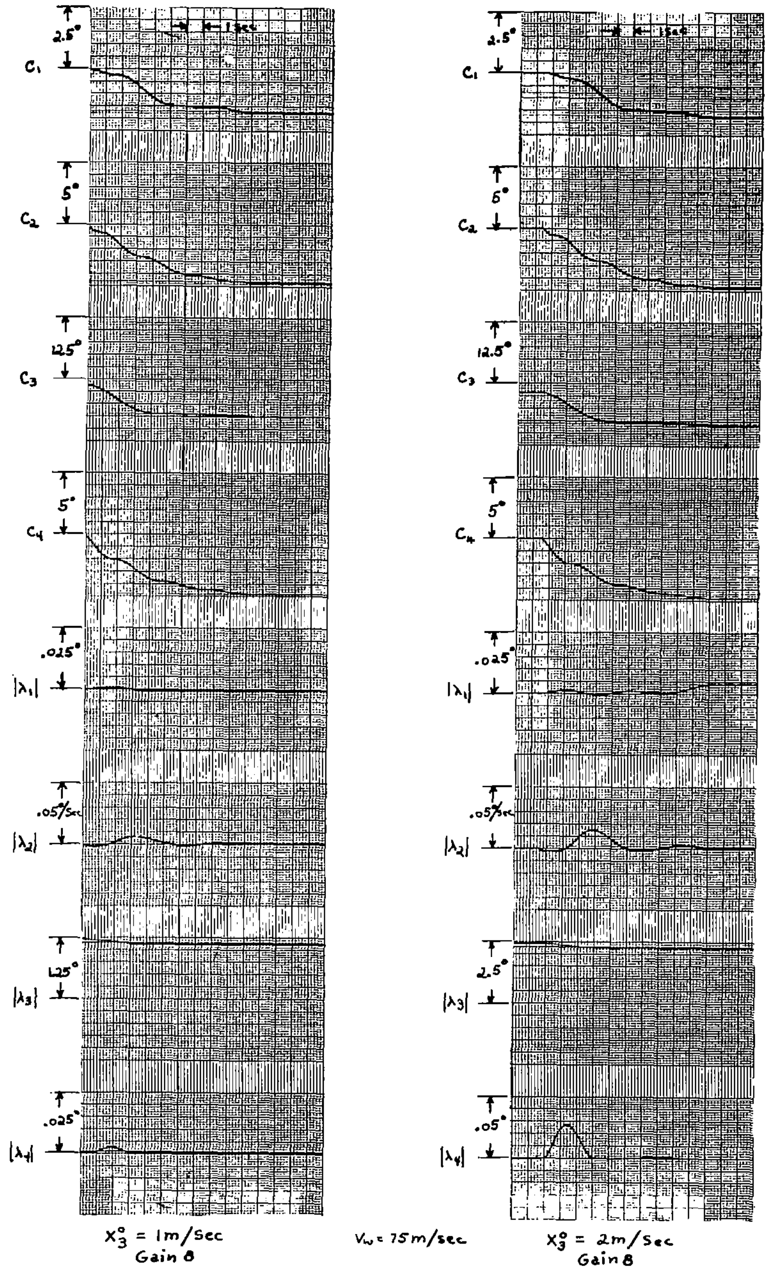

Figure 8. Worst Disturbance Responses of Optimally Controlled System with $\dot{z}$ = 1 m/sec and 2 m/sec

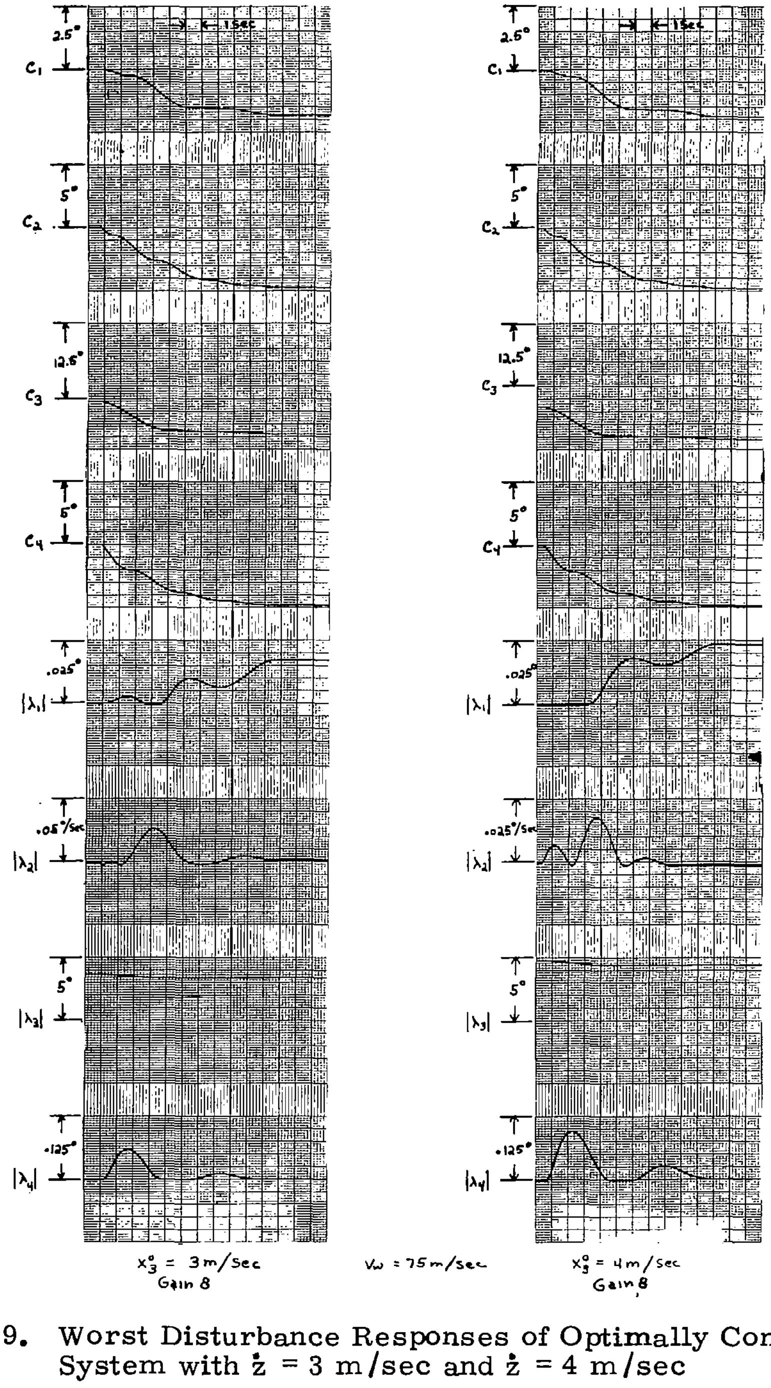

Figure 9. Worst Disturbance Responses of Optimally Controlled System with $\dot{z}$ = 3 m/sec and $\dot{z}$ = 4 m/sec

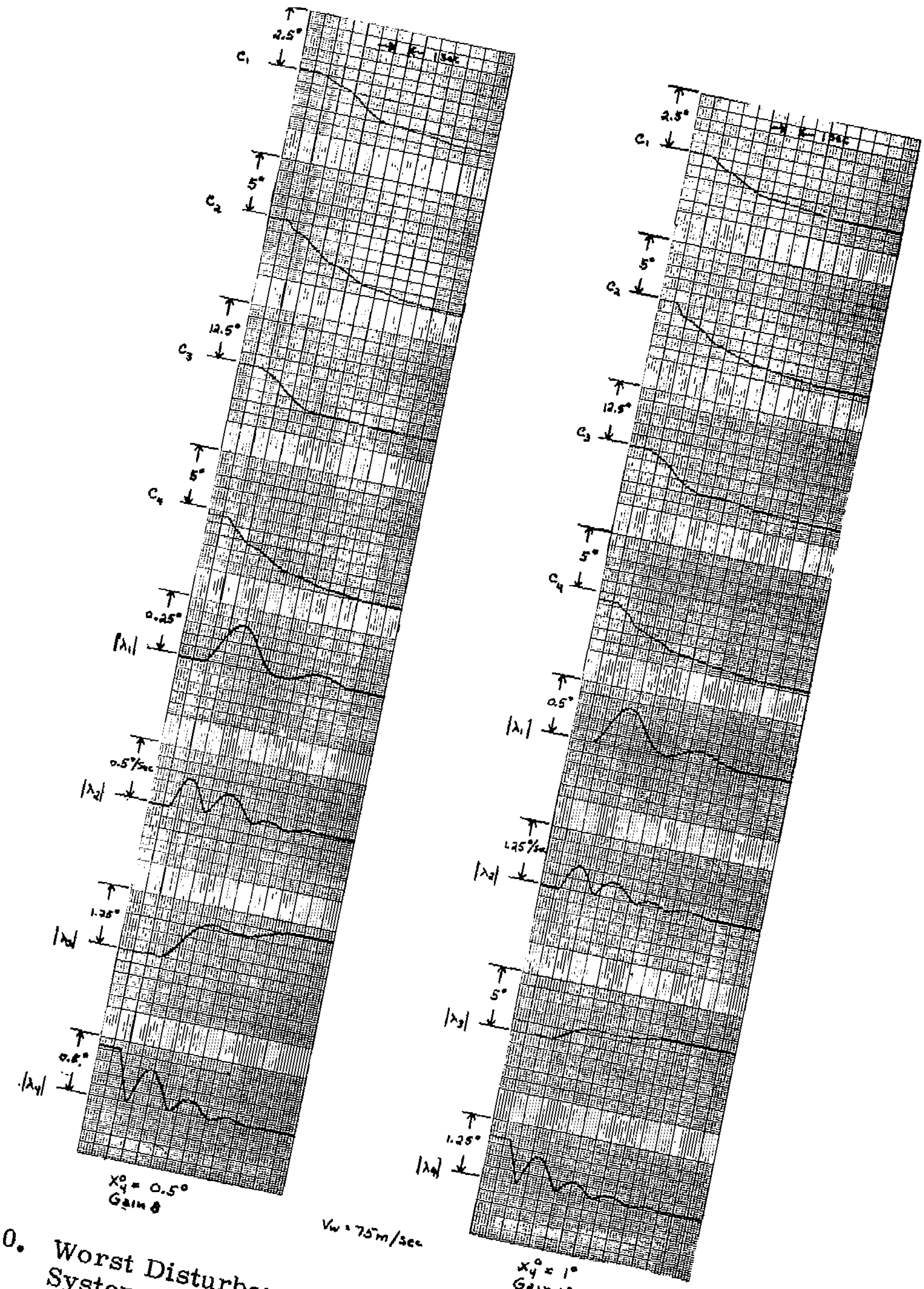

Figure 10. Worst Disturbance Responses of Optimally Controlled System with $\beta = 0.5$ and 1 degree

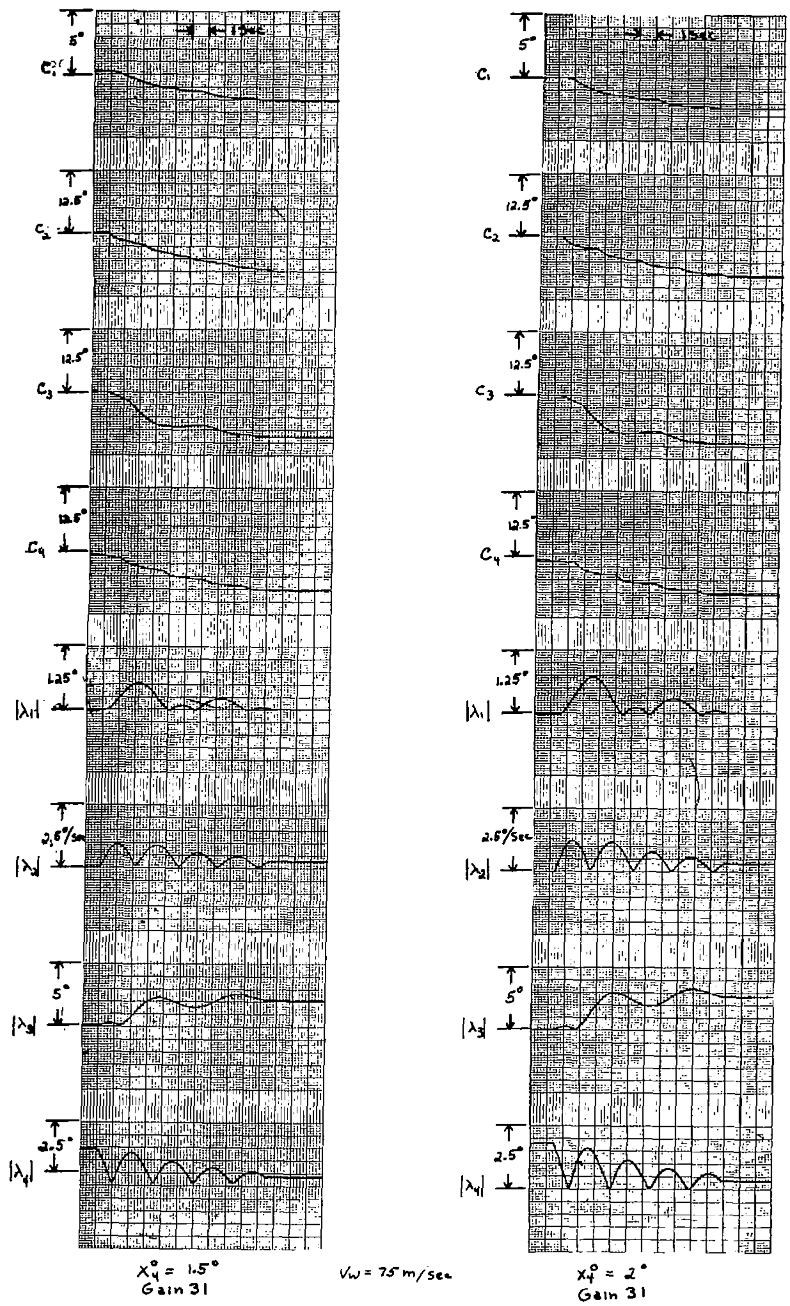

Figure 11. Worst Disturbance Responses of Optimally Controlled System with β = 1.5 and 2 degrees

complex pair of roots decreased by 13.2 percent and 78 percent respectively. For the range of $\dot{\phi}_o$ and β_o the frequency and damping ratio decreased by 11.9 percent and 54 percent respectively. The change in frequency and damping ratio is also observable from the analog traces.

CONCLUSIONS

The synthesis technique developed for minimum wind effect control of a linear stationary system with non-zero initial conditions and amplitude bounded disturbances was found to be feasible. A linear feedback controller for the launch vehicle was determined in a systematic fashion and provided adequate control of the vehicle. This approach to launch vehicle controller synthesis has much merit in terms of development time and cost.

REFERENCES

1. Harvey, C. A., "Minimum Disturbance Effects Control of Linear Systems with Linear Controllers", MH MPG Report 1541-TR14.

2. Harvey, C. A., "Minimax Control of Linear Stationary Systems with Non-Zero Initial Conditions and Amplitude Bounded Disturbances", Appendix, Honeywell Report 12003-FTR1.

SECTION 3

THE EFFECT OF NON-ZERO INITIAL CONDITIONS ON SELECTION OF MINIMAX CONTROLLERS

Results of the effect of the magnitude of non-zero initial conditions on selection of minimax controllers for a piecewise constant approximation of a large launch booster are given for two time intervals (flight conditions) which occur during the first 84 seconds of flight.

This section is a supplement to Honeywell Report 12003-FTR1. That report contains the results of applying optimal control theory to selection of linear, fixed-gain controllers for each interval with zero initial conditions on each interval. It also contains a description of the mathematical approximation of the launch booster and the various flight conditions.

Good zero-initial-condition controllers for intervals I_3 and I_5 were selected from Report 12003-FTR1 as a starting point. The techniques and computations described in Report 12003-FTR1 were used to arrive at the results given in this report.

PROBLEM SUMMARY

Choice of Intervals

Interval I_3 was chosen for study because its good zero-initial-condition controller (hereafter called (X^o = 0)-controllers) were of comparatively high gain and responses (hence also cost items) were expected to be more sensitive to gain change and initial conditions (hereafter called I. C.) than in an interval with lower gain controllers. Interval I_5 was chosen because it contained the event of Mach 1 and because the good (X^o = 0)-controllers had one large positive eigenvalue.

Choice of Initial Conditions (I. C. 's) and Gains

It was decided to study the effect of I. C. 's on each state variable individually to eliminate the possibility of I. C. 's on two or more state variables cancelling their individual effects.

It is known from the results presented in Appendix A of Report 12003-FTR1 that non-zero I. C. 's on the state variables increase the cost of control for a given controller. From results in Appendix A of Report 12003-PR6, it is known that the best controller of a given set of controllers is dependent on the magnitude of the I. C. 's.

Two problems involved in extending previous results and techniques to non-zero I. C. 's are:

A) The I. C. 's at the start of I_3 or I_5 should have magnitudes which are typical of an actual response of a reasonably well-controlled vehicle subjected to typical (not maximal) disturbances in the earlier portions of a flight; and

B) The gain grid should have increments consistent with the size of the I. C. 's.

The details of picking values for I. C. 's with the properties described in (A) will be described in MAGNITUDE OF INITIAL CONDITIONS. The problem described in (B) can be clarified by an example. As stated above, it is known that the best controller of the set represented by a given gain grid depends on the magnitude of the I. C. 's.

Assume that the grid contains its best ($X^o = 0$) - controller somewhere near the midpoint of the grid, and that typical non-zero I. C. 's (as described in (A) are used. If the gain grid is too coarse, the same controller of that set will remain the best controller for the chosen I. C. 's. If the gain grid is too fine, the best controllers with non-zero I. C. 's will be on the boundary of the gain grid. In either of these extreme cases, very little is learned about how much the optimal gains depend on the I. C. 's.

One way to have quantitative results would be to have:

- Several values for the I. C. on a given state variable which covered a typical range.
- A gain grid with increments such that to each different value of a given I. C. would correspond a different best controller in the grid;
- The best controller systematically related to the magnitude of the I. C.

The following results presented in Section 3 substantially have these properties.

RESULTS OF MINIMAX COMPUTATIONS

Initial gain grids were chosen which contained the controllers in I_3 and I_5 specified in the right half of Figure 12, Report 12003-FTR1. The grid sizes had increments of about ten percent except that the K_1 increment in I_5 was about 30 percent. In only three iterations of minimax computations for each interval, the gain grids and sets of I. C. 's listed in Tables 3-1 and 3-6 were attained. The results will be considered from the points of view of control costs, eigenvalues (stability), and controller gains. It will be seen that the non-zero I. C. 's lead to controllers of slightly higher cost, and more stability. Both of these results were anticipated.

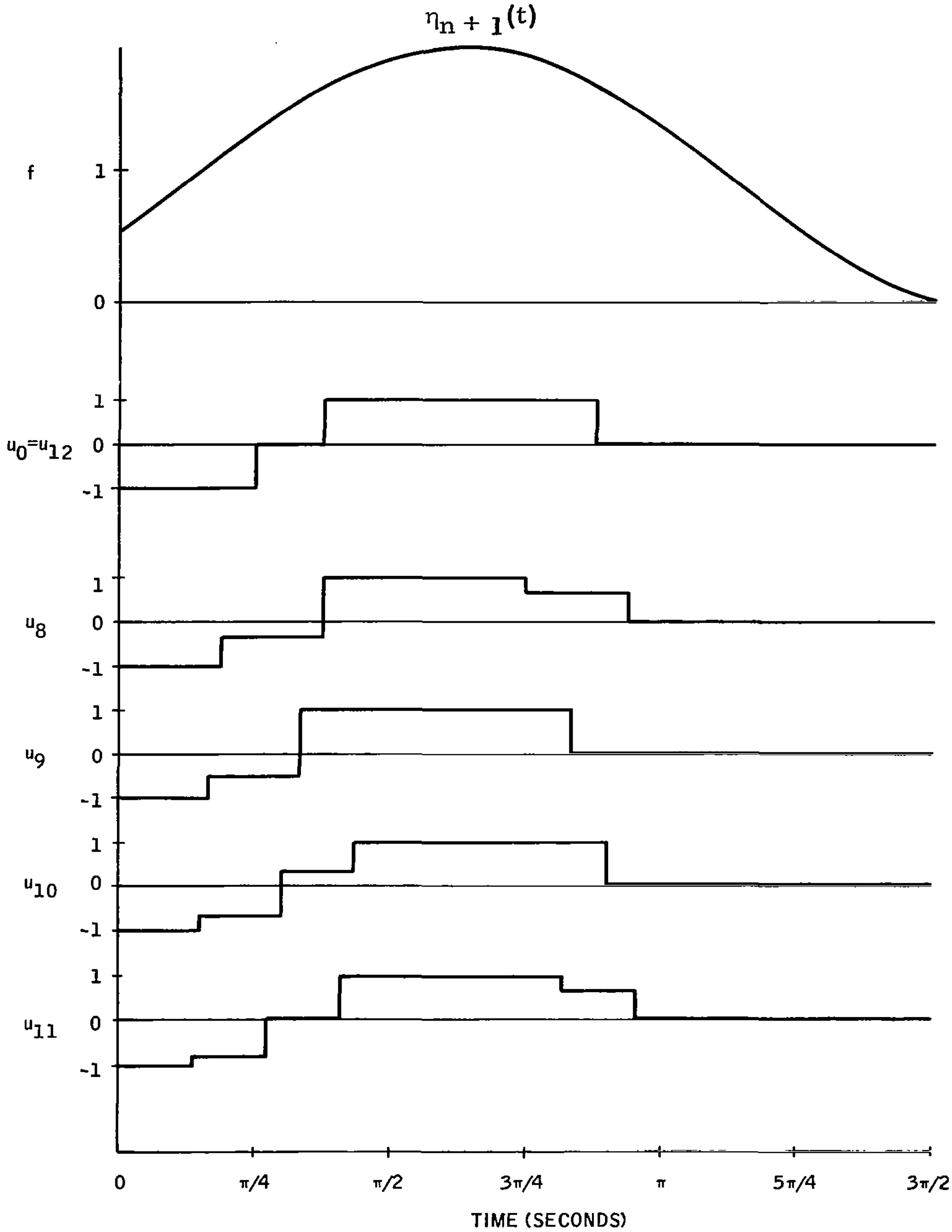

Figure 12. Extremal Input and Approximations Obtained

Table 3-1. Gain Grid and Initial Conditions for Interval I_3

	Gain Index				
Gain	1	2	3	4	5
K_1	0.50	0.55	0.60	0.65	0.70
K_2	1.4	1.6	1.8	2.0	2.2
K_3	-0.13	-0.12	-0.11	-0.10	-0.09

Initial Conditions				
	Amount	$\phi(0)$ rad	$\dot{\phi}(0)$ rad/sec	$\dot{Z}(0)$ m/sec
$X^o = 0$		0	0	0
ϕ deg	0.1	0.001745	0	0
	0.2	0.003490	0	0
	0.4	0.006980	0	0
$\dot{\phi}$ deg/sec	0.02	0	0.000349	0
	0.04	0	0.000698	0
	0.08	0	0.001396	0
$\dot{z}$ m/sec	0.172285	0	0	0.172285
	0.344570	0	0	0.344570
	0.698140	0	0	0.698140

Identification of Controllers

For convenient reference to controllers in the following discussion, a given controller in Tables 3-1 and 3-6 will be identified by the index values of its three gains, rather than by the gain values themselves. For example, in interval I_3 (Table 3-1), the controller with gains $(K_1, K_2, K_3) = (0.50, 1.4, -0.09)$ will be designated as K(115). This happened to be the best controller for zero I. C. It will be noted that all three of its values lie on the boundary of the control box (hence, this controller is at a corner of the box), rather than on the interior as was postulated on page 37. This presents no problem since it was established in arriving at the gain grid for I_3 that the best controllers for non-zero I. C.'s would lie toward the interior of the gain box. Similar comments apply for I_5 (Table 3-6), where the best ($X^o = 0$)-controller (which was K(154)) was on a face of the gain box.

Costs, Gain Changes, and Closed Loop Poles

Results of minimax computations for the various I. C.'s are given for I_3 and I_5 in Tables 3-2 and 3-7 respectively. Each table gives the costs and closed-loop poles of the three best controllers in the grids for ten I. C.'s. The results for each interval will be discussed separately and supplementary tables and graphs will be given to illustrate various conclusions.

Interval I_3

Some facts and general conclusions from Table 3-2 are as follows:

- Only 13 different controllers (out of 30 possible) are represented in Table 3-2. Of these, only eight are needed to provide the two lowest cost controllers for ten I. C.'s.

Table 3-2. Three Best Controllers, Their Costs, and Closed Loop Poles for Ten Initial Conditions in Interval I_3

I. C.	Controllers and Costs $C(K_l, K_m, K_n)$ = C(l, m, n.)	Closed Loop Poles		ζ and ω_n of Complex Poles	
		Real	Complex Pair	ζ	ω_n
$X^\circ = 0$	$C_5(115) = 0.04347$	-0.00110	-0.2800 ±i · 0.3278	0.6496	0.4311
	$C_5(254) = 0.04424$	-0.00018	-0.4367 ±i · 0.1381	0.9535	0.4580
	$C_5(154) = 0.04446$	0.00111	-0.4374 ±i · 0.1146	0.9997	0.4375
$\|\phi\|$ = 0.1 deg = 0.001745	$C_3(115) = 0.04363$	-0.00110	-0.2800 ±i · 0.3278	0.6496	0.4311
	$C_5(244) = 0.04473$	-0.00018	-0.3977 ±i · 0.2250	0.8707	0.4570
	$C_5(234) = 0.04530$	-0.00018	-0.3587 ±i · 0.2813	0.7869	0.4559
$\|\phi\|$ 0.2 deg = 0.00349	$C_5(234) = 0.04538$	-0.00018	-0.3587 ±i · 0.2813	0.7869	0.4559
	$C_3(115) = 0.04598$	-0.00110	-0.2800 ±i · 0.3278	0.6496	0.4311
	$C_5(324) = 0.04606$	-0.00127	-0.3192 ±i · 0.3518	0.6719	0.4750
$\|\phi\|$ = 0.4 deg = 0.00698	$C_3(224) = 0.04662$	-0.00018	-0.3197 ±i · 0.3235	0.7030	0.4548
	$C_5(214) = 0.04711$	-0.00018	-0.2807 ±i · 0.3565	0.6187	0.4537
	$C_5(314) = 0.04712$	-0.00128	-0.2802 ±i · 0.3824	0.5910	0.4741
$\|\dot{\phi}\|$ = 0.02 $\frac{deg}{sec}$ = 0.000349 sec^{-1}	$C_5(115) = 0.04350$	-0.00110	-0.2800 ±i · 0.3278	0.6496	0.4311
	$C_5(244) = 0.04468$	-0.00018	-0.3977 ±i · 0.2250	0.8704	0.4570
	$C_5(144) = 0.04484$	0.00111	-0.3984 ±i · 0.1779	0.9131	0.4363
$\|\dot{\phi}\|$ = 0.04 $\frac{deg}{sec}$ = 0.000698 sec^{-1}	$C_5(115) = 0.04358$	-0.00110	-0.2800 ±i · 0.3278	0.6496	0.4311
	$C_5(244) = 0.04470$	-0.00018	-0.3977 ±i · 0.2250	0.8704	0.4570
	$C_5(234) = 0.04528$	-0.00018	-0.3587 ±i · 0.2813	0.7869	0.4559
$\|\dot{\phi}\|$ = 0.08 $\frac{deg}{sec}$ = 0.001396 sec^{-1}	$C_5(244) = 0.04475$	-0.00018	-0.3977 ±i · 0.2250	0.8704	0.4570
	$C_5(234) = 0.04533$	-0.00018	-0.3587 ±i · 0.2813	0.7869	0.4559
	$C_3(115) = 0.04548$	-0.00110	-0.2800 ±i · 0.3278	0.6496	0.4311
$\|\dot{z}\|$ = 0.1723 m/sec	$C_3(244) = 0.04485$	-0.00018	-0.3977 ±i · 0.2250	0.8704	0.4570
	$C_5(234) = 0.04535$	-0.00018	-0.3587 ±i · 0.2813	0.7869	0.4559
	$C_3(115) = 0.04555$	-0.00110	-0.2800 ±i · 0.3278	0.6496	0.4311
$\|\dot{z}\|$ = 0.3446 m/sec	$C_3(234) = 0.04609$	-0.00018	-0.3587 ±i · 0.2813	0.7869	0.4559
	$C_5(224) = 0.04621$	-0.00018	-0.3197 ±i · 0.3235	0.7030	0.4548
	$C_5(214) = 0.04716$	-0.00018	-0.2807 ±i · 0.3565	0.6187	0.4537
$\|\dot{z}\|$ = 0.6891 m/sec	$C_3(542) = 0.05167$	0.00018	-0.3984 ±i · 0.3300	0.7701	0.5173
	$C_5(532) = 0.05209$	0.00018	-0.3594 ±i · 0.3707	0.6960	0.5163
	$C_3(552) = 0.05212$	0.00017	-0.4374 ±i · 0.2780	0.8439	0.5183

- For eight of ten initial conditions, the three best controllers are asymptotically stable.

- Two of the three best ($X^o = 0$)-controllers were asymptotically stable. In earlier gain grids, the ($X^o = 0$)-controllers always had one positive closed loop pole; e.g., see Figure 12 of Report 12003-FTR1. One iteration of grid mapping resulted in the grid (Table 1) for which results are shown here.

The increasing of I.C.'s leads to systematic changes in the minimax controller gains. The simplest illustration is shown by looking at the progression of best controllers as I.C.'s on the state variable $\dot{\phi}$ progress from 0 to the maximum value considered. Table 3 shows the collection.

Table 3-3. Best Controllers for I C.'s on $\dot{\phi}$

$\lvert\dot{\phi}^o\rvert$ $\frac{deg}{sec}$	Best Controller and Cost	Closed Loop Poles		ζ and W_n of Complex Poles	
		Real	Complex Pair	ζ	ω_n
0	$C_5(115) = .04347$	-.00110	-.2800 ± i.3278	.6496	.4311
.02	$C_5(115) = .04350$	"	"	"	"
.04	$C_5(115) = .04358$	"	"	"	"
.08	$C_5(244) = .04475$	-.00018	-.3977 ± i.2250	.8707	.4570

It is seen that as $\lvert\dot{\phi}\rvert$ increases, the gain index (hence the gain) is:

- Monotone increasing * on K_1
- Monotone increasing * on K_2
- Monotone decreasing * on K_3

With minor modification, the same result is true for I. C.'s on ϕ and $\dot{Z}$. The modification in the case of I. C.'s on $\dot{Z}$ is that one should choose the second best controller for the smallest non-zero I. C. in order for all gains to be monotone with $|\dot{Z}^o|$. The cost penalty paid for this substitution is only about one percent. Table 4 illustrates the gain changes as $|\dot{Z}^o|$ increases.

Table 3-4. Best Controllers for I. C.'s on $\dot{Z}$

$\lvert\dot{Z}^o\rvert \frac{m}{sec}$	Best Controller and Cost	Closed Loop Poles		ζ and ω_n of Complex Poles	
		Real	Complex Pair	ζ	ω_n
0	$C_5(115) = .04347$	-.00110	-.2800 ± i.3278	.6496	.4311
.1723	**$C_3(234) = .04535$	-.00018	-.3587 ± i.2813	.7869	.4559
.3446	$C_3(234) = .04609$	"	"	"	"
.6891	$C_3(542) = .05167$	.00018	-.3984 ± i.3300	.7701	.5173

The same situation is true for I. C.'s on ϕ.

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* More accurate terminology is monotone non-increasing/non-decreasing. The simplification used above is common.

** Second best controller for this I. C.

Table 3-5. Best Controllers for I. C.'s on ϕ

$\lvert\phi^\circ\rvert$ deg	Best Controller and Cost	Closed Loop Poles		ζ and ω_n of Complex Pair	
		Real	Complex	ζ	ω_n
0	$C_5(115) = 0.04347$	-0.00110	$-0.2800 \pm i\ 0.3278$	0.6496	0.4311
0.1	$C_5(115) = 0.04363$	-0.00110	$-0.2800 \pm i\ 0.3278$	0.6496	0.4311
0.2	*$C_5(115) = 0.04538$	-0.00110	$-0.2800 \pm i\ 0.3278$	0.6496	0.4311
0.4	$C_5(224) = 0.04662$	-0.00018	$-0.3197 \pm i\ 0.3235$	0.7030	0.4548

* Second best controller for this I. C.

Comparison of Tables 3-3, 3-4, and 3-5 shows that all minimax gains are monotone with increasing absolute values of I. C.'s, and also that each minimax gain is monotone in the same direction with every I. C.; i.e.,

- K_1 is monotone increasing with $\lvert$I. C.$\rvert$,
- K_2 is monotone increasing with $\lvert$I. C.$\rvert$,
- K_3 is monotone decreasing with $\lvert$I. C.$\rvert$.

The total increases of costs over the ranges of I. C.'s are about 2.9 percent in Table 3-3, 18 percent in Table 3-4, (8.1 percent in Table 3-4 if the largest value of $\dot{Z}^\circ$ is omitted) and 7.4 percent in Table 3-5.

As far as closed loop poles are concerned the indication from Table 3-3, 3-4, and 3-5 is that the real pole moves from -0.00110 to the right as $\lvert$I. C.$\rvert$ increases. It remains negative except for the largest value of $\lvert\dot{Z}^\circ\rvert$. The behavior of the complex poles is most easily interpreted from the damping ratio ζ and natural frequency ω_n. It is seen that, with the largest value of $\lvert\dot{Z}^\circ\rvert$ again excepted, both ζ and ω_n increase with $\lvert$I. C.$\rvert$.

The uniform behavior of gains, real poles, damping ratio, and natural frequency with increases in magnitude of any of the initial conditions, and the small variation in control costs with initial conditions suggest the possibility of picking one fixed-gain controller for I_3 which is good for a large number of non-zero I. C. 's. A first candidate might be the (X° = o) controller itself (K(115)), since it occurs more often in Table 3-2 than any other (seven times). But it turns out that this controller gives some rather high costs for the largest values of $|\phi^\circ|$ and $|\dot{Z}^\circ|$. A better compromise controller is K(234) (which occurs six times in Table 3-2). For every I. C. considered, the costs for K(234) exceed those of K(115) for X° = o by less than 26 percent, less than 12 percent if the largest value of $|\dot{Z}^\circ|$ is excepted, and less than 6 percent if the largest values of both $|\dot{Z}^\circ|$ and $|\phi^\circ|$ are excepted.

Thus, for interval I_3, it has been found that, for the gain grid considered:

(1) All gains are monotone non-decreasing/non-increasing with the magnitude of I. C. on any state variable;

(2) All minimax controllers have a negative real and a stable complex pair of closed loop poles;

(3) The real pole is much closer to the origin than the complex pair and its distance from the origin decreases with increasing magnitude of I. C. 's ($-0.0011 \leq Z_o \leq -0.00018$);

(4) The damping ratio and natural frequency of the complex pair increase slightly with increasing magnitude of I. C. 's ($0.64 \leq \zeta \leq 0.87$ and $0.43 \leq \omega_n$ rad/sec ≤ 0.57);

(5) It is possible to select one fixed gain controller which gives good performance for each I. C.;

(6) One I. C., the maximum value of $|\dot{Z}^\circ|$ considered, must be excepted for conclusions (2), (3), (4), and (5). It is shown in MAGNITUDE OF INITIAL CONDITIONS that this exception is probably not serious

Since the only exceptions concerned $\dot{Z}^\circ$, we may also conclude that the selection of minimax controllers for bounded winds is not particularly sensitive to I. C. 's on ϕ and $\dot{\phi}$ in interval I_3. This bears the qualification that a single compromise controller which is to be used for all I. C. 's will give better performance if the effect of I. C. 's on selection of minimax controllers has been considered. This qualification is apparently not a severe one, since in this example where good ($X^\circ = o$) controllers were known to begin with, only three iterations of minimax computations were required to arrive at the results presented.

Interval I_5

The selection of minimax controllers in interval I_5 was more strongly influenced by I. C. 's than was the case in interval I_3. This is shown by the fact that 22 different controllers are represented in Table 3-7, while only 13 occurred in the corresponding table for interval I_3 (Table 3-2). Nevertheless it will be seen that the conclusions are very similar for both intervals. Table 3-8 is extracted from Table 3-7, and serves to illustrate the conclusions.

Inspection of Table 3-8 shows that, again, all gain indices (hence gains) are monotone with increasing $|I. C.|$. On all three state variables, as $|I. C.|$ increases,

(1) K_1 is monotone increasing,

(2) K_2 is monotone decreasing, and

(3) K_3 is monotone decreasing.

Again, the cost penalty for substituting second best controllers in two spots was well below one percent.

Table 3-6. Gain Grid and Initial Conditions for Interval I_5

Gain \ Gain Index	1	2	3	4	5
K_1	-1.2	-0.9	-0.6	-0.3	0
K_2	1.7	2.0	2.3	2.6	2.9
K_3	-0.070	-0.065	-0.060	-0.055	-0.050

Initial Conditions

	Amount	ϕ (o) rad	$\dot{\phi}$ (o) rad/sec	$\dot{Z}$ (o) m/sec
X° = o		0	0	0
ϕ deg	0.2	0.00349	0	0
	0.4	0.00698	0	0
	0.8	0.01396	0	0
$\dot{\phi}$ deg/sec	0.04	0	0.000698	0
	0.08	0	0.001396	0
	0.16	0	0.002792	0
$\dot{Z}$ m/sec	0.308865	0	0	0.308865
	0.617730	0	0	0.617730
	1.23546	0	0	1.23546

Table 3-7. Three Best Controllers, Their Costs, and Closed Loop Poles for Ten Initial Conditions in Interval I_5

I_5	Controllers and Costs $C(K_\ell, K_m, K_n) = C(\ell mn)$	Closed Loop Poles			Comments
$X^\circ \equiv 0$	C_3 (154) = 0.03909	0.3291	-0.0186	-1.5457	
	C_5 (235) = 0.03914	0.3021	-0.0191	-1.2661	
	C_3 (155) = 0.03925	0.3288	-0.0183	-1.5457	
ϕ = 0.2 deg =	C_5 (513) = 0.04352	0.0399	-0.0570	-0.7142	
0.00349 rad	C_3 (514) = 0.04389	0.0379	-0.0548	-0.7144	
	C_3 (414) = 0.04400	0.1559	-0.0259	-0.8613	
ϕ = 0.4 deg =	$C_{3,5}$ (512) = 0.04496	0.0418	-0.0592	-0.7141	C_3 (512) = C_5 (512)
0.00698 rad	C_3 (521) = 0.04568	0.0399	-0.0568	-0.8407	
	C_5 (511) = 0.04612	0.0436	-0.0613	-0.7140	
ϕ = 0.8 deg =	C_3 (511) = 0.05366	0.0436	-0.0613	-0.7140	
0.01396 rad	C_3 (512) = 0.05499	0.0418	-0.0592	-0.7141	
	C_3 (521) = 0.05570	0.0399	-0.0568	-0.8407	
$\dot{\phi} = 0.04 \frac{deg}{sec}$ =	C_3 (245) = 0.04071	0.2796	-0.0191	-1.3696	Choose C_5 (244) to
0.000698 sec^{-1}	C_5 (244) = 0.04073	0.2798	-0.0194	-1.3696	make gains monotone
	C_5 (335) = 0.04076	0.2184	-0.0206	-1.1809	with $\dot{\phi}$
$\dot{\phi} = 0.08 \frac{deg}{sec}$ =	C_5 (344) = 0.04182	0.2009	-0.0211	-1.2891	
0.001396 sec^{-1}	C_5 (415) = 0.04186	0.1552	-0.0249	-0.8613	
	C_3 (325) = 0.04189	0.2392	-0.0207	-1.0757	
$\dot{\phi} = 0.16 \frac{deg}{sec}$ =	C_5 (433) = 0.04340	0.1267	-0.0263	-1.0837	
0.002792 sec^{-1}	C_5 (414) = 0.04346	0.1559	-0.0259	-0.8613	
	C_3 (424) = 0.04347	0.1394	-0.0257	-0.9710	
$\dot{Z}$ = 0.308865	C_5 (244) = 0.04067	0.2798	-0.0194	-1.3696	
$\frac{m}{sec}$	C_5 (315) = 0.04067	0.2636	-0.0207	-0.9740	
	C_3 (335) = 0.04072	0.2184	-0.0206	-1.1809	
$\dot{Z}$ = 0.61773	C_3 (415) = 0.04198	0.1552	-0.0249	-0.8613	Choose C_3 (324) to
$\frac{m}{sec}$	C_3 (324) = 0.04206	0.2396	-0.0212	-1.0757	make gains monotone
	C_3 (334) = 0.04216	0.2188	-0.0211	-1.1809	with $\dot{Z}^\circ$
$\dot{Z}$ = 1.23546	C_5 (512) = 0.04482	0.0418	-0.0592	-0.7141	
$\frac{m}{sec}$	C_3 (422) = 0.04497	0.1409	-0.0274	-0.9710	
	C_3 (413) = 04530	0.1566	-0.0268	-0.8612	

Table 3-8. Extract of Table 3-7

I.C.			Best Controller and Cost	Closed Loop Poles		
$\lvert\phi^{\circ}\rvert$ deg	$\lvert\dot{\phi}^{\circ}\rvert$ deg/sec	$\lvert\dot{Z}^{\circ}\rvert$ m/sec				
0	0	0	C_3 (154) = 0.03909	0.3291	-0.0186	-1.5457
0.2	0	0	C_5 (513) = 0.04352	0.0399	-0.0570	-0.7142
0.4	0	0	C_3 (512) = 0.04496	0.0418	-0.0592	-0.7141
0.8	0	0	C_3 (511) = 0.05366	0.0436	-0.0613	-0.7140
0	0	0	C_3 (154) = 0.03909	0.3291	-0.0186	-1.5457
0	0.04	0	*C_5 (244) = 0.04073	0.2798	-0.0194	-1.3696
0	0.08	0	C_5 (344) = 0.04182	0.2009	-0.0211	-1.2891
0	0.16	0	C_5 (433) = 0.04340	0.1267	-0.0263	-1.0837
0	0	0	C_3 (154) = 0.03909	0.3291	-0.0186	-1.5457
0	0	0.308865	C_5 (244) = 0.04067	0.2798	-0.0194	-1.3696
0	0	0.61773	*C_3 (324) = 0.04026	0.2396	-0.0212	-1.0757
0	0	1.23546	C_5 (512) = 0.04482	0.0418	-0.0592	-0.7141

*Second Best Controller for this I. C.

The locations of all closed loop poles for the minimax controller in Table 3-8 follow a definite pattern. In I. C. 's on $\dot{\phi}$ and $\dot{Z}$, a given pole continues to move in the same direction as |I. C.| increases. In particular, the positive pole moves toward the origin as |I. C.| increases. For I. C. 's on ϕ, even the first non-zero I. C. results in minimax controllers whose poles are substantially different from those of the (X° = 0)-controller. Of most interest is the positive pole, which is only about one-tenth as far from the origin for non-zero I. C. 's as when $X^\circ = 0$. And when $\phi^\circ \neq 0$, all pole locations are quite similar.

The total cost increases over the ranges of the I. C. 's in Table 3-8 are 37 percent on $|\phi^\circ|$ (15 percent if the largest value of $|\phi^\circ|$ is omitted), 11 percent on $|\dot{\phi}^\circ|$, and 14. 7 percent on $|\dot{Z}^\circ|$. The most popular controller in Table 3-7 is K(512). It is therefore a candidate to be considered as a compromise controller for all the I. C. 's. It looks surprisingly good: the costs for K(512) exceed the (X° = 0) -cost of K(154) by about 41 percent; but if the largest value of $|\phi^\circ|$ is excepted, the cost excess is only 15 percent. These values are about the same as the cost increases for the minimax controllers in the grid over the ranges of the I. C. 's.

Thus for interval I_5 it has been found that, for the gain grid considered:

(1) All gains are monotone non-decreasing/non-increasing with the magnitude of I. C. 's on the state variables;

(2) All minimax controllers have real poles and one of them is positive;

(3) The distance of the positive pole from the origin decreases with increasing magnitude of I. C. 's ($0.0418 \leq Z_o \leq 0.3291$)

(4) It is possible to select one fixed gain controller which gives good performance for each initial condition.

(5) One initial condition, the maximum of $|\phi^\circ|$ considered, must be excepted for conclusions (3) and (4).

MAGNITUDE OF INITIAL CONDITIONS

It remains to be shown that values of I. C. 's chosen are, in some sense, "reasonable". The initial values of I. C. 's were chosen from previously computed cost data for $X^\circ = 0$ controllers.

The natural output of the minimax computations is a set of cost items, each of which is proportional to the maximum amplitude which a cost variable can achieve with a bang-bang wind disturbance. Furthermore, the switching times are not in general the same for two different cost variables. Thus, a set of cost items (for a given controller) does not represent the terminal values of a response, but rather the maximum values the cost variables can achieve at any time during a set of responses of a given time duration.

Maximal amplitudes of individual cost variables were readily available but response data was not, so it was decided to choose initial values of state variables* for I_3 amd I_5 equal to one-third their maximum amplitudes with zero initial conditions for I_2 and I_4, respectively. The rationale was that maximal responses correspond loosely to "three-sigma" responses and that one third of these amounts would be more "typical" of state variable amplitudes at the end of I_2 and I_4, hence at the beginning of I_3 and I_5.

A second estimate of suitable I. C. values for ϕ and $\dot{Z}$ was made from response curves of a similar vehicle subjected to five different synthetic wind profiles. These data were supplied by personnel at the George C. Marshall Space Flight Center. The cross-wind velocity in each profile built up at a certain rate from zero to a specified maximum value which occurred at times t = 48, 56, 64, 72, and 80 seconds respectively. A gust with an altitude depth of.3 km was superimposed on each profile at the instant it reached its specified maximum value. This maximum value corresponded to an attack angle due to wind α_w ($\equiv \frac{Vm}{V}$) of about 10. 3 degrees.

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* The state variables are a subset of the cost variables

The left half of Table 3-9 shows the response amplitude of ϕ and $\dot{Z}$ from the M. S. F. C. data at t = 36 and t = 60 (the beginning times for I_3 and I_5, respectively). The middle section shows the values obtained from minimax cost information, and the right section shows the values actually used.

The top half of the table shows that, for I_3, the range of chosen initial values for ϕ' is about right. However, the largest initial value for $\dot{Z}$ substantially exceeds that estimated either from responses or cost data. This fact has the effect of strengthening previous conclusions for interval I_3, since it was this largest value of $|\dot{Z}^\circ|$ which required the several exceptions to that list of conclusions.

In interval I_5 (bottom half of Table 3-9), the chosen range of I. C.'s on $\dot{Z}$ is ample. The chosen range of I. C.'s on ϕ is suitable when compared with cost data, but is not large enough to encompass the attitude responses from the two wind profiles peaking at 48 and 56 seconds. This fact weakens the previous conclusion concerning the possibility of using a single fixed-gain controller for all I. C. in interval I_5, since the maximum value of $|\phi^\circ|$ used was already excluded from that conclusion. It therefore seems likely that some technique such as gain modification with the amplitude of ϕ would be advisable in interval I_5. There also exists the possibility that further grid mapping would lead to a more desirable grid from which to choose controllers.

CONCLUSIONS

In general, the results are at least as good as anticipated, especially in interval I_3. Starting in that interval with an unstable but good $(X^\circ = 0)$-controller, only three iterations of grid mapping yielded a gain grid in which the best controllers for all I. C.'s (both zero and non-zero) were stable and the changes in gain were monotone with increasing values of I. C.'s on each state variable. A bonus result was that the gains changed in the same direction

Table 3-9. Relative Magnitudes of I. C.'s

Internal	From Responses to Wind Profiles					From Cost Data (1/3 max amplitude)		Values Chosen	
	Profile Peaks at t	ϕ deg	$\dot{z}$ $\frac{m}{sec}$	α_w deg	α_w deg	$\lvert\phi\rvert$ deg	$\lvert\dot{z}\rvert$ $\frac{m}{sec}$	$\lvert\phi^\circ\rvert$ deg	$\lvert\dot{z}^\circ\rvert$ $\frac{m}{sec}$
	t	t = 36	t = 36	t = 36	t = 52	t = 36	t = 36	t = 36	t = 36
I_3	48	-0.4	0.2	3.2	10.2	0.222	0.36	0.1	0,172285
	56	0	0	0	6.1			0.2	0.344570
	64	0	0	0	3.3			0.4	0.698140
	72	0	0	0	0				
	80	0	0	0	0				
I_5		t = 60	t = 60	t = 60	t = 68	t = 60	t = 60	t = 60	t = 60
	48	*	*	*	*			0.2	0.308865
	56	-1.8	0.7	10.3	*	0.65	0.824	0.4	0.617730
	64	-1	0.8	5.7	10.3			0.8	1.23546
	72	-0.7	0.2	3.4	5.5				
	80	0	0	0	2				
	I_3 corresponds to $36 \leq t < 56$; I_5 corresponds to $60 \leq t < 68$				* Not shown in data			$\lvert\alpha_w\rvert$ disturbance always 10.4 deg	

with I. C. 's on every state variable. This suggested the possibility of a compromise controller in the grid which would give good performance for all I. C. 's, and one such controller was found.

The same conclusions are applicable to interval I_5 except that the controller started with was unstable and the minimax controllers for non-zero I. C. 's were still unstable, but much less so.

The technique of choosing minimax controllers using zero I. C. 's, and then using these as a starting point for considering non-zero I. C. 's proved to be efficient.

SECTION 4

EXTREMAL BOUNDED AMPLITUDE, BOUNDED-RATE INPUTS FOR A HARMONIC OSCILLATOR

The problem considered is the determination of extremal inputs to a forced harmonic oscillator described by equation (1).

$$\ddot{y} + y = v(t) \tag{1}$$

The system is assumed to be initially at rest, i.e., $y(0) = \dot{y}(0) = v(0) = 0$. The input, $v(t)$, is admissible if it satisfies the following constraints for $t \geq 0$:

- $v(t)$ is continuous, with piece-wise continuous derivative $\dot{v}(t)$
- $|v(t)| \leq 1$
- $|\dot{v}(t)| \leq k/\pi,\ k > 0$

An extremal input is an admissible input defined on the interval $[0, T]$ that maximizes $[y(T) \cos \theta + \dot{y}(T) \sin \theta] \cos \phi + v(T) \sin \phi$ for some T, θ and ϕ where $T > 0$, $0 \leq \theta < 2\pi$ and $|\phi| < \frac{\pi}{2}$. Extremal inputs may be characterized as follows. The response of the system (1) to an input $v(t)$ forms a trajectory in a three dimensional Euclidean space with coordinates of a point on the trajectory given by $y(t)$, $\dot{y}(t)$ and $v(t)$. Let the set of attainability at $t = T$ be the set of all endpoints (points with coordinates $y(T)$, $\dot{y}(T)$, $v(T)$) of trajectories corresponding to admissible inputs on the interval $0 \leq t \leq T$. Such a set is closed and bounded. An extremal input on the interval $0 \leq t \leq T$ is an admissible input to which corresponds a trajectory with an endpoint that is a boundary point of the set of attainability. Hence, an equivalent definition of an extremal input is an input that is a time optimal regulation input with the time reversed.

The problem can easily be formulated as one with a phase constraint. Let $u(t) = \dot{v}(t)$ and introduce the vector x(t) with components $x_1(t) = y(t)$, $x_2(t) = \dot{y}(t)$, $x_3(t) = v(t)$. Then equation (1), with initial conditions specified and admissible inputs, may be represented by

$$\dot{x} = Ax + bu, \quad x(0) = 0, \quad |u| \leq k/\pi, \text{ where:} \tag{2}$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The phase constraint is then $|x_3| \leq 1$. A discussion of the development of necessary and sufficient conditions for extremal inputs in problems of this type is given in reference 1, pp. 1-2. For this particular problem these conditions say, essentially, that extremal inputs are given by:

$$u(t) = (k/\pi)\operatorname{sgn}[\psi(t)b] \tag{3}$$

where sgn(0) = 0 and $\psi(t)$ is a piecewise continuous solution of an adjoint equation with a piecewise continuous right-hand side. The discontinuities are allowed only at values of t which are endpoints of maximal intervals in which the corresponding response has $|x_3| = 1$. The points of discontinuity of ψ can be further restricted to occur only at right-hand endpoints of such intervals. A more detailed statement of these conditions will be given in the section on APPLICABLE THEORY. Also it will be shown that allowing at most one discontinuity in ψ at t = T, the extremal input with respect to $\psi(T)\,x(T)$ is given by (3). Thus, since the response x(T) depends only on the input on the open interval (0, T), extremal inputs correspond to continuous solutions of piecewise continuous adjoint equations.

This problem is chosen for two reasons. The first is that it presents a case in which the number of segments or arcs of extremal responses which lie on the phase constraint can be made arbitrarily large. This makes it possible to determine that such segments are interrelated. The second is that an explicit solution can be obtained.

In the DERIVATION OF EXTREMALS the theory will be used to treat a particular case. Ranges of values of k and θ will be chosen and the extremal inputs will be derived for all values of the parameters T and ψ.

Formulas for extremal inputs are given in EXPLICIT REPRESENTATION OF EXTREMALS for the parameters k, T, θ and ϕ with ranges $k \geq 2.5$, $T > 0$, $0 \leq \theta < \pi$ and $|\phi| < \frac{\pi}{2}$.

The results are then summarized in the CONCLUSIONS.

APPLICABLE THEORY

The necessary conditions for the present problem will be based on general necessary conditions given by Gamkrelidze, references 2 and 3, and improved by Bryson, Benham and Dreyfus, reference 4. Then the necessary conditions given by Russell and Schmaedeke, reference 5 will be cited. Comparison of these necessary conditions with sufficient conditions obtained by Russell, reference 1, show that the necessary conditions are also sufficient.

The notation in reference 3 will be followed with the exception that the components x_o and ψ_o will not be included in the vectors x and ψ. Thus $f(x, u) = Ax + bu$. The phase constraint is represented by requiring x to lie in the region G represented by:

$$G = \{x: g(x) = (x_3)^2 - 1 \leq 0\}. \tag{4}$$

Then $p(x, u) = 2x_3 u$ and $\frac{\partial p}{\partial x} = (0, 0, 2u)$. $H(\psi, x, u) = \psi f(x, u)$ and $m(\psi, x) = \psi Ax$, $M(\psi, x) = \max_{u \in U} H(\psi, x, u)$, where $U = \{u: |u| \leq k/\pi\}$.

Theorem 25 of reference 2 may be stated as follows (taking note of theorems 1, 22 and 24 of reference 2):

Suppose that x(t) is an optimal trajectory of equation (2), corresponding to the optimal control u(t), and that x(t) lies entirely in G for $0 \leq t \leq T$ and contains a finite number of junction points. Also suppose that each of its sections which lies on the boundary of G is regular. Let $0 < \tau_1 \leq \tau_2 < \cdots < \tau_{2q-1} \leq \tau_{2q} \leq T$ denote the junction points. Then there exists a piecewise continuous vector $\psi(t) = (\psi_1(t), \psi_2(t), \psi_3(t))$ and a piecewise continuous, piecewise smooth scalar-valued function $\lambda(t)$ such that:

$$\frac{dx}{dt} = \frac{\partial H(\psi, x, u)}{\partial \psi} = f(x, u) = Ax + bu \tag{5}$$

$$\frac{d\psi}{dt} = -\frac{\partial H(\psi, x, u)}{\partial x} + \chi(t)\,\lambda(t)\,\frac{\partial p(x, u)}{\partial x} = -\psi A + \chi(t)\lambda(t)\,(0, 0, 2u) \tag{6}$$

$$H(\psi(t), x(t), u(t)) = M(\psi(t), x(t))\,[1 - \chi(t)] + \chi(t)m(\psi(t), x(t)) \tag{7}$$

where $\chi(t)$ is equal to zero when $g(x(t)) < 0$ and is equal to one when $g(x(t)) = 0$, $\lambda(t) = \frac{1}{2}\,\psi(t)b\,\mathrm{sgn}\,[x_3(t)]$. The vector $\psi(t)$ is zero nowhere on $[0, T]$. On $\tau_{2i-1} \leq t \leq \tau_{2i}$, $g(x(t)) = 0$ and $\psi^+(\tau_{2i-1})$ is tangent to the boundary $g(x) = 0$ at $x(\tau_{2i-1})$ and $d\lambda(t)/dt \leq 0$ for $\tau_{2i-1} < t < \tau_{2i}$, $i = 1, 2, \ldots, q$. Furthermore, at junction points the following jump condition is satisfied:

$$\text{either } \psi^+(\tau_i) = \psi^-(\tau_i) + \mu_i \,\mathrm{grad}\, g(x(\tau_i)), \tag{8}$$

$$\text{or } \quad \psi^-(\tau_i) + \mu_i \,\mathrm{grad}\, g(x(\tau_i)) = 0, \; \mu_i \neq 0, \tag{9}$$

where μ_i is a real number.

In reference 4 is is shown that the jump condition (8) and (9) may be replaced by:

$$\psi^-(\tau_{2i-1}) = \psi^+(\tau_{2i-1}) \quad (10)$$

$$\psi^-(\tau_{2i}) = \psi^+(\tau_{2i}) + \mu_i \text{grad } g(x(\tau_{2i})) \quad (11)$$

ıus ψ can be defined so that it is continuous at τ_{2i-1}, $i=1, 2, \ldots, q$.

summary of these results for the problem being considered is that if x(t) ıd u(t) are optimum then: (Introducing the notation $\tau_o = 0$ and $\tau_{2q+1} = T$, $\tau_{2q} < T$).

quation (5) is satisfied ($dx/dt = Ax + bu$) and there exists a piecewise contin- ›us $\psi(t)$ such that:

$$\frac{d\psi}{dt} = -\psi A + \frac{1}{2}\chi(t)\,\psi_3(t)\,\text{sgn}[x_3(t)]\,(0, 0, 2u) \quad (12)$$

here $\chi(t) = 0$ if $\tau_{2i} < t < \tau_{2i+1}$, and $\chi(t) = 1$, if $\tau_{2i+1} \leq t \leq \tau_{2i}$, $i=0, 1, 2, \ldots, q$. urthermore $u(t) = 0$ and $g(x(t)) = 0$, if $\chi(t) = 1$ and $u(t) = (k/\pi)\text{sgn}[\psi_3(t)]$ and $(x(t)) < 0$, if $\chi(t) = 0$. The first two components of $\psi(t)$ are continuous and the ıird is continuous except possibly at τ_{2i}, $i=1, 2, \ldots, q$. At these points $_3^-(\tau_{2i}) = \psi_3^+(\tau_{2i}) + 2\mu_i x_3(\tau_{2i})$. Since $\chi(t)u(t) = 0$ for each $t \in [0, T]$, equation 2) may be written as:

$$d\psi/dt = -\psi A \quad (13)$$

ote that this formulation gives an adjoint equation with a continuous right- and side and that u(t) differs from $(k/\pi)\text{sgn}[\psi(t)b]$ when $\chi(t) = 1$, since if $(t) = 1$, $u(t) = 0$ and $\psi_3(t) \neq 0$. A slightly different formulation can be made hich will change these results. In the above formulation the constraint was djoined by setting:

$$h(\psi, x, u) = H(\psi, x, u) - \chi(t)\lambda p(x, u) \quad (14)$$

where $p(x, u) = dg[x(t)]/dt$. Adding the constraint by setting:

$$h(\psi, x, u) = H(\psi, x, u) - \chi(t)\zeta g(x) \quad (15$$

yields an equivalent formulation, reference 6.

If the formulation indicated by (15) were used then $\psi(t)$ would satisfy:

$$d\psi/dt = -\psi A + \chi\zeta \text{grad } g(x) = -\psi A + \chi\zeta(0, 0, 2x_3) \quad (16$$

where $\zeta = \frac{1}{2}\psi_2 \text{sgn} x_3$. Also u(t) would satisfy:

$$u(t) = (k/\pi)\text{sgn}[\psi_3(t)], \quad 0 < t < T \quad (17$$

where sgn(0) = 0.

The necessary conditions given in reference 5 are also applicable to this problem. They give further information regarding extremal inputs. These results may be summarized in the following definitions and theorems from reference 5 (stated for the present problem in terms of notation given above).

Definition 1

The input u_o is extremal if there exists a non-trivial (continuous) solution ψ of (13) such that $\int_o^T \psi_3(t)\, u_o(t)\, dt = \max_u \int_o^T \psi_3(t) u(t) dt$. The maximization is taken over all u with $|u(t)| \leq k/\pi$ and $\left| \int_o^t u(\tau) d\tau \right| \leq 1$ for $0 < t < T$.

Definition 2

Let u(t) be an admissible input on the interval $[0, T]$. An interval of type B for the input u is a maximal closed subinterval of the interval $[0, T]$ on which $|v(t)| = 1$. $\left(v(t) = \int_0^t u(\tau)d\tau\right)$.

Definition 3

An interval of type P_1 for u(t) is a maximal closed subinterval of $[0, T]$ in the interior of which $|v(t)| < 1$.

Definition 4

An interval of type P_2 for u(t) is a maximal subinterval of $[0, T]$ whereon $|v(t)| \neq 1$ and $|u(t)| = k/\pi$ and sgn $[u(t)]$ is constant.

Theorem 1

Let u(t) be an extremal input and assume $|v(t)| = 1$ on a subinterval of $[0, T]$. Then, on that subinterval:

$$v(t)\,[d\psi_3(t)/dt] \leq 0. \tag{18}$$

Theorem 2

Let u(t) be an extremal input. Then $|u(t)| = k/\pi$, almost everywhere, on an interval of type P_1 for u(t).

Theorem 3

Let u(t) be an extremal input and suppose there exists an interval I of type P_1 for u(t) with one endpoint, say t*, which is an interior point of [0, T]. Then for $t \in I$:

$$u(t) = (k/\pi)\,\text{sgn}[\psi_3(t) - \psi_3(t^*)]. \qquad (19$$

Theorem 4

Let u(t) be an extremal input and suppose that the entire interval [0, T] is of type P_1 for u(t). Then there exists a constant c such that $u(t) = (k/\pi)\text{sgn}[\psi_3(t) - c]$ for $0 < t < T$. If there are at least two intervals of type P_2 for u(t) contained in [0, T], then the constant c is equal to $\psi_3(\tau)$ where τ is any endpoint of an interval of type P_2 which is in the interior of [0, T].

In view of equation (13) the inequality (18) is equivalent to:

$$v(t)\psi_2(t) \geq 0. \qquad (20)$$

Also since $v(0) = x_3(0) = 0$, the point 0 is an endpoint of an interval of type P_1 for any extremal input, i.e., there exists a $\tau_1 \leq T$ such that $[0, \tau_1]$ is an interval of type P_1. If $\tau_1 = T$ then from theorem 4 it is clear that a continuous p(t) exists, namely $p(t) = \psi_3(t) - c$, such that $u(t) = (k/\pi)\text{sgn}p(t)$. If $\tau_1 < T$ there exists an integer $N \geq 1$ and a sequence $\tau_1 \leq \tau_2 < \tau_3 \leq \cdots \leq \tau_{2N} \leq T$ such that $[\tau_{2i-1}, \tau_{2i}]$ is an interval of type B for $i = 1, 2, \ldots, N$. $[\tau_{2i}, \tau_{2i+1}]$ is an interval of type P_1 for $i = 1, 2, \ldots, N-1$ and if $\tau_{2N} < T$ then $[\tau_{2N}, T]$ is an interval of type P_1. In this case defining p(t) to be equal to 0 on intervals of type B and to be equal to $\psi_3(t) - \psi_3(t^*)$ on each interval of type P_1, where $t^* \in (0, T)$ is an endpoint of the interval of type P_1 yields a continuous function p(t) such that $u(t) = (k/\pi)\text{sgn}[p(t)]$. Note that from Theorem 3 if $[\tau_{2i}, \tau_{2i+1}] \subset (0, T)$

then $\psi_3(\tau_{2i}) = \psi_3(\tau_{2i+1})$. Also note that the vector $\psi(t)$ with components $\psi_1(t)$, $\psi_2(t)$, $p(t)$ satisfies equation (16). These necessary conditions can be summarized as follows: if $u(t)$ is an extremal control there exists a continuous solution $\psi(t)$ of equation (16) such that $u(t)$ is given by equation (17) and on each interval where $|v(t)| = 1$ the inequality (20) is satisfied.

Extremal inputs defined by Definition 1 are interpreted geometrically in the statement and proof of Lemma 1 of reference 5. It is shown that if a non-zero vector $\psi(t)$ is chosen to satisfy equation (13) over $0 \leq t \leq T$, then the response $x_e(t)$ to the corresponding extremal input has the property that $\psi(T)x_e(T) \geq \psi(T)x(T)$, where $x(t)$ is any response corresponding to an admissible input on $[0, T]$. This property can be maintained when $\psi(t)$ is taken as a solution of equation (16) if a discontinuity in $\psi_3(t)$ is allowed at $t = T$ and $\psi(t)$ is continuous on $[0, T]$. This is accomplished by setting $\psi_3^-(T)$ equal to the value of $p(T)$. Furthermore if the corresponding $v(T)$ is less than one in magnitude no discontinuity is required. If the corresponding $|v(T)|$ equals one a discontinuity may appear but the following inequality must be satisfied:

$$v(T)\ [\psi_3(T) - \Psi_3^-(T)] \geq 0. \tag{21}$$

A sufficient condition for an input to be extremal which is applicable to this problem is given in reference 1. If $|v(T)| < 1$ and the time scale is reversed Theorem 1 of reference 1 is applicable. Consider, then, the system

$$\dot{y} = Cy + d\,\omega, \quad |\omega| \leq k/\pi, \quad g(y) \leq 0, \tag{22}$$

where $C = -A$, $d = -b$, $y(t) = x(T-t)$, $\omega(t) = u(T-t)$ and in particular $y(T) = 0$ and $g[y(0)] < 0$. Theorem 1 of reference 1 states:

Let $\omega(t)$ be defined on $[0, T]$ and assume $\omega(t)$ transfers y from $y(0)$ to 0 in $[0, T]$ with $g(y) \leq 0$, and let $\eta(t)$ be a covariant vector function defined and continuous on $[0, T]$ with the possible exception of points $t_1, t_2, \ldots, t_{r-1}$ where r is an odd integer and $0 = t_o < t_1 \leq t_2 < \ldots \leq t_r = T$. If k is odd,

$g(\xi(t)) < 1$, and if k is even $g(\xi(t)) = 1$, for $t \in I_k = [t_{k-1}, t_k]$ where $\xi(t)$ is the response to u(t). Let $\nu_1, \nu_2, \ldots, \nu_{r-1}$ be non-negative real numbers and $\zeta(t)$ be a function defined on $[0, T]$, non-negative on each I_{2k}. Also let $||\eta(T)|| \neq 0$ and

$$\dot{\eta} = -\eta C + \chi(t)\zeta(t)\nabla g[\xi(t)], \quad (\nabla = \text{grad}) \tag{23}$$

where $\chi(t) = 0$, if $t \in I_{2k+1}$, and $\chi(t) = 1$, if $t \in I_{2k}$, for $k = 1, 2, \ldots, r-1$,

$$\eta^+(t_k) - \eta^-(t_k) = \nu_k \nabla g[\xi(t_k)]. \tag{24}$$

Let $H(\omega, t) = \eta d\omega$ for all $|\omega| \leq k/\pi$ when $t \neq t_k$, $k = 1, 2, \ldots, r-1$.

If $H(\omega(t), t) = \max_{|\omega| \leq k/\pi} H(\omega, t)$ for almost all $t \in [0, T]$, then $\omega(t)$ is an optimal controller.

Now let $\psi(t) = -\eta(T-t)$ and $\tau_k = T - t_k$, $k = 0, 1, \ldots, r$. Then:

$$\dot{\psi} = -\psi A + \chi(t)\ \zeta(t)\nabla g[x(t)] \tag{25}$$

$$\psi^+(\tau_k) - \psi^-(\tau_k) = \nu_k \nabla g[x(t_k)] \tag{26}$$

$$H(\omega, T-t) = \eta(T-t)d\omega = -\psi(t)\ (-b)\omega = \psi(t)b\omega \tag{27}$$

Thus, if $H[\omega(T-t), T-t] = \max_{|\omega| \leq k/\pi} [\psi(t)b\omega]$ where $\psi(t)$ satisfies (25) and (26), then $\omega(T-t)$ is optimal, i.e., $\bar{u}(t)$ is extremal. Hence the necessary conditions given above are also sufficient when $|v(T)| < 1$, since in this ease each $\nu_k = 0$ and $\zeta(t) = \frac{1}{2}\psi_2(t)\operatorname{sgn}[x_3(t)] \geq 0$ on each I_{2k} which follows from (20).

In the case when $|v(T)| = 1$ the theorem of Russell could be modified by requiring (24) to hold only for $k = 2, \ldots, r-1$ since in this case $t_1 = t_o = 0$. This establishes that, also in this case, the inputs determined from the necessary condition are extremal.

These results are summarized as follows: a necessary and sufficient condition for u(t) to be an extremal input is that there exist a vector $\psi(t)$, continuous on (0, T), that satisfies equation (16), such that u(t) is given by equation (17), and on each interval in which $|v(t)| = 1$, the inequality (20) is satisfied. Furthermore, $\psi^-(T)$ is an external normal to the set of attainability at the point x(T), and if $\psi(T)$ is an external normal to the set of attainability at x(T), then $\psi(T) = \psi^-(T)$ when $|v(T)| < 1$, and inequality (21) is satisfied, if $|v(T)| = 1$.

Now let $\psi(T) = (\cos\theta\cos\phi,\ \sin\theta\cos\phi,\ \sin\phi)$. Then it is easily shown that $\psi_2(t) = \cos\phi\sin(\theta+T-t)$ and

$$\psi_3(t)\sec\phi = \begin{cases} \tan\phi, & \text{for } t = T, \\ \cos\theta - \cos(\theta+T-t) + \tan\theta + \delta(\tan\phi_o - \tan\phi), & \text{for } \tau_{2N} \le t < T, \quad \text{if } \tau_{2N} < T \\ 0, & \text{for } t = \tau_{2N}, \quad \text{if } \tau_{2N} < T \\ 0, & \text{for } \tau_{2N-1} \le t < \tau_{2N}, \quad \text{if } \tau_{2N-1} < T \\ \cos(\theta+T-\tau_{2N-1}) - \cos(\theta+T-t), & \text{for } \tau_{2N-2} \le t \le \tau_{2N-1}, \quad \text{if } \tau_{2N-1} < T \\ \cos\theta - \cos(\theta+T-t) + \tan\phi + \delta(\tan\phi_o - \tan\phi) & \text{for } \tau_{2N-2} \le t < T, \quad \text{if } \tau_{2N-1} = T \\ 0, & \text{for } \tau_{2i-1} \le t \le \tau_{2i}, \quad i = 1, 2, \ldots, N-1 \\ \cos(\theta+T-\tau_{2i-1}) - \cos(\theta+T-t), & \text{for } \tau_{2i-2} \le t \le \tau_{2i-1}, \quad i = 1, 2, \ldots, N-1 \end{cases} \quad (28)$$

where $0 = \tau_o < \tau_1 \le \tau_2 < \ldots < \tau_{2N-1} \le \tau_{2N} \le T$ and δ is equal to zero or one. If $|v(T)| < 1$, then $\delta = 0$, and if $|v(T)| = 1$, then $\delta = 1$ and $v(T)(\tan\phi - \tan\phi_o) \ge 0$. Thus, u(t) is an extremal input corresponding* to $\psi(T)$ if and only if

*This means $\psi(T)$ is an external normal to the set of attainability at x(T) where x(T) is the response to u(t).

$$u(t) = (k/\pi)\,\text{sgn}[\psi_3(t)\sec\phi], \quad 0 < t < T \tag{29}$$

where $\psi_3(t)\sec\phi$, satisfies (28) on $0 \leq t \leq T$, and

$$|v(t)| < 1, \quad \tau_{2i-2} < t < \tau_{2i-1};\ i=1, 2, \ldots, N, \tag{30}$$

$$(\tau_{2i} - \tau_{2i-1})\, v(t) \sin(\theta+T-t) \geq 0, \quad \tau_{2i-1} \leq t \leq \tau_{2i};\ i=1, 2, \ldots, N, \tag{31}$$

$$|v(\tau_{2i-1})| = 1, \quad i=1, 2, \ldots, N, \tag{32}$$

$$\cos(\theta+T-\tau_{2i-2}) = \cos(\theta + T-\tau_{2i-1});\ i=2, 3, \ldots, N-1, \tag{33}$$

$$\cos\theta-\cos(\theta+T-\tau_{2N}) + \tan\phi+ \delta(\tan\phi_o - \tan\phi) = 0, \text{ if } \tau_{2N-1} < T, \tag{34a}$$

$$\cos\theta - \cos(\theta+T-\tau_{2N-2}) + \tan\phi + \delta(\tan\phi_o - \tan\phi) = 0, \text{ if } \tau_{2N-1} = T, \tag{34}$$

$$(T-\tau_{2N-1})\,[\cos(\theta+T-\tau_{2N-2}) - \cos(\theta+T-\tau_{2N-1})] = 0, \tag{35}$$

$$\delta\, v(T)\,(\tan\phi - \tan\phi_o) \geq 0. \tag{36}$$

Equations (32) through (35) are 2N equations in the unknown parameters, N, τ_i, i=1, 2, . . . , 2N, ϕ_o and δ. The relations (28) through (31) are constraints which a solution must satisfy. The simplicity of this problem permits explicit solution of these constrained equations. The nature of τ_1 is determined in the next paragraph.

Consider equation (32) with i=1, namely, $|v(\tau_1)| = 1$. Equations (28) and (29) yield $v(\tau_1) = (k/\pi)\int_0^{\tau_1} \text{sgn}[\cos(\theta+T-\tau_1)-\cos(\theta+T-t)]dt$. The zeros of $\cos(\theta+T-\tau_1)-\cos(\theta+T-t)$ occur at $t = \theta + T \pm (\theta + T - \tau_1 + 2m\pi)$ for m an integer. Since there is at most one zero in the interval $(\tau_1 - 2\pi, \tau_1)$, and k is greater than two, τ_1 must be less than 2π. Hence, in the interval $(0,\tau_1)$, u(t) can change sign at most once. Therefore, either $\tau_1 = \pi/k$, or there exists an integer m_o such that $0 < 2(\theta+T+m_o\pi)-\tau_1 < (\pi/k) < \tau_1$ and $\tau_1 - 2[2(\theta+T+m_o\pi)-\tau_1] = \pi/k$. That is, if $\tau_1 \neq \pi/k$ then there exists an integer m_o such that

$$\tau_1 = \frac{4}{3}(\theta+T+m_o\pi) + (\pi/3k) \tag{37}$$

where m_o satisfies

$$\pi/2k < \theta + T + m_o\pi < 2\pi/k. \tag{38}$$

Furthermore, if $\tau_1 = \pi/k$ there can be no zero in $(0, \frac{\pi}{k})$. Hence, there exists an integer m_o such that

$$2(\theta + T + m_o\pi) - \tau_1 \leq 0 < \frac{\pi}{k} = \tau_1 \leq 2(\theta + T + m_o\pi) + 2\pi - \tau_1.$$

This is equivalent to

$$(\pi/k) - \pi \leq \theta + T + m_o\pi \leq (\pi/2k). \tag{39}$$

For i=2, 3, . . . , N-1, τ_{2i-2} and τ_{2i-1} may be determined as follows. The general solution of equation (33) for τ_{2i-2} is

$$\tau_{2i-2} = \theta + T \pm (\theta + T - \tau_{2i-1} + 2m\pi)$$

where m is an integer. Equations (28) and (29) yield $|v(\tau_{2i-2})| = 1$ and $u(t) = (k/\pi)\,\mathrm{sgn}[\cos(\theta + T - \tau_{2i-1}) - \cos[\theta + T - t]$ for $\tau_{2i-2} < t < \tau_{2i-1}$. There is at most one sign change of u(t) in $\tau_{2i-1} - 2\pi < t < \tau_{2i-1}$. The maximum length that an interval of constant sign for u(t) can have is $2\pi/k$ which is less than π when k is greater than 2. Thus $\tau_{2i-1} - \tau_{2i-2}$ must be less than 2π and is greater than zero by definition. Therefore,

$$\tau_{2i-2} = 2(\theta + T) - \tau_{2i-1} + 2m_i\pi, \tag{40}$$

where m_i is such that $\tau_{2i-1} - 2\pi < 2(\theta + T) - \tau_{2i-1} + 2m_i\pi < \tau_{2i-1}$, i.e.,

$$\tau_{2i-1} - \pi < \theta + T + m_i\pi < \tau_{2i-1}. \tag{41}$$

Thus, u(t) is of constant sign in $(\tau_{2i-2}, \tau_{2i-1})$ and hence from (34) , $\tau_{2i-1} = \tau_{2i-2} + (2\pi/k)$. This result, together with equation (41), yields the following:

$$\tau_{2i-2} = \theta + T + m_i\pi - (\pi/k) \tag{42}$$

$$\tau_{2i-1} = \theta + T + m_i\pi + (\pi/k). \tag{43}$$

The integers m_i can be readily determined from the relations $\tau_2 \geq \tau_1 > \tau_2 - \pi$ and $m_{i+1} = m_i + 1$ for i = 2, 3, . . . , N-1. In case $\tau_1 \neq \pi/k$ it is easily shown that $m_2 = m_o + 1$. If $\tau_1 = \pi/k$ then $m_2 = m_o + 2$ if $(\pi/k) - \pi \leq \theta + T + m_o\pi < (2\pi/k) - \pi$, and $m_2 = m_o + 1$ if $(2\pi/k) - \pi \leq \theta + T + m_o\pi \leq (\pi/2k)$.

Note that if $\tau_{2N-1} < T$, then equation (35) implies that equations (42) and (43) hold also for i = N.

Now consider equations (34a) and (34b). Suppose $\delta = 1$, i.e. $|v(T)| = 1$. If $\tau_{2N} = T > \tau_{2N-1}$, $\phi_o = 0$. If $T = \tau_{2N-1}$ then $\tan \phi_o = \cos(\theta + T - \tau_{2N}) - \cos\theta$. ϕ_o can also be determined from limiting cases when $\delta = 0$. Note that when $\phi = \phi_o$ the value of δ is immaterial.

Suppose now that $\delta = 0$. Equation (34) can be written as

$$\cos(\theta + T - \tau_{2N}) = \rho, \tag{44}$$

where

$$\rho = \rho(\theta, \phi) = \cos\theta + \tan\phi. \qquad (45)$$

It is clear from equation (45) that ρ must be restricted by the relation $|\rho| \leq 1$.

Thus $\tau_{2N} = \theta + T \pm [\cos^{-1}\rho + 2m\pi]$ where $0 \leq \cos^{-1}\rho \leq \pi$. Since $|v(T)| < 1$, $\tau_{2N} < T$. It is readily shown that $T - \tau_{2N} < 2\pi$. Hence, $T - \tau_{2N}$ is equal to $2\pi - \theta \pm \cos^{-1}\rho$ or $-\theta + \cos^{-1}\rho$. The three cases: $\theta > \cos^{-1}\rho$, $\theta = \cos^{-1}\rho$ and $\theta < \cos^{-1}\rho$, will be considered separately. Before proceeding with the analysis of each case let it be noted that $v(\tau_{2N})\,\mathrm{sgn}[u(\tau_{2N}^{+})] = -1$ and that $\mathrm{sgn}\, u(\tau_{2N}^{+}) = \mathrm{sgn}\, d\psi_3^{+}/dt\,|_{\tau_{2N}} = \mathrm{sgn}[-\sin(\theta + T - \tau_{2N})]$.

In the first case, $\theta > \cos^{-1}\rho$, $T - \tau_{2N} = 2\pi - \theta \pm \cos^{-1}\theta$. Suppose $\tau_{2N} = T + \theta - 2\pi - \cos^{-1}\rho$. Then $u(t)$ changes sign at $T + \theta - 2\pi + \cos^{-1}\rho$ and $\mathrm{sgn}\, u(\tau_{2N}^{+}) = \mathrm{sgn}[-\sin(2\pi + \cos^{-1}\rho)] = -1$. Therefore $v(\tau_{2N}) = +1$ and

$$u(t) = \begin{cases} -(k/\pi) & \tau_{2N} < t < T + \theta - 2\pi + \cos^{-1}\rho \\ +(k/\pi) & T + \theta - 2\pi + \cos^{-1}\rho < t < T \end{cases}$$

The constraint $|v(t)| < 1$, $\tau_{2N} < t < T$ imposes the conditions $T-(T+\theta-2\pi+\cos^{-1}\rho) < T + \theta - 2\pi + \cos^{-1}\rho - \tau_{2N} < 2\pi/k$, which simplifies to

$$(2\pi-\theta)/3 < \cos^{-1}\rho < \pi/k. \qquad (46)$$

Now if $\tau_{2N} = T + \theta - 2\pi + \cos^{-1}\rho$, u(t) is constant on (τ_{2N}, T) and sgn $u(\tau_{2N}^{+}) = +1$. Thus $v(\tau_{2N}) = -1$ and $u(t) = k/\pi$ for $\tau_{2N} < t < T$. The constraint $T - \tau_{2N} < 2\pi/k$ is equivalent to

$$\cos^{-1}\rho > 2\pi - \theta - (2\pi/k) \tag{47}$$

Notice that ϕ_o is determined by $|v(T)| = 1$. This corresponds to $\cos^{-1}\rho = (2\pi-\theta)/3$ from (46) or $\cos^{-1}\rho = 2\pi - \theta - (2\pi/k)$ from (47).

In case $\theta = \cos^{-1}\rho$, $\tau_{2N} = T + \theta - 2\pi + \cos^{-1}\rho$ and the results are the same as when $\theta > \cos^{-1}\rho$ and $\tau_{2N} = T + \theta - 2\pi + \cos^{-1}\rho$.

In the final case, $\theta < \cos^{-1}\rho$, $\tau_{2N} = T + \theta - \cos^{-1}\rho$ or $T + \theta - 2\pi + \cos^{-1}\rho$. If $\tau_{2N} = T + \theta - \cos^{-1}\rho$, $u(t) = -k/\pi$ for $\tau_{2N} < t < T$ and $v(\tau_{2N}) = +1$. The constraint $|v(T)| < 1$ imposes the condition

$$\cos^{-1}\rho < \theta + (2\pi/k) \tag{48}$$

If $\tau_{2N} = T + \theta - 2\pi + \cos^{-1}\rho$, $v(\tau_{2N}) = -1$ and

$$u(t) = \begin{cases} k/\pi, & \tau_{2N} < t < T + \theta - \cos^{-1}\rho \\ -k/\pi, & T + \theta - \cos^{-1}\rho < t < T \end{cases}$$

The constraint $|v(t)| < 1$ for $\tau_{2N} < t < T$ yields the condition

$$\pi - (\pi/k) < \cos^{-1}\rho < (2\pi + \theta)/3 \tag{49}$$

In this case ϕ_o, as determined by $|v(T)| = 1$, is given by $\cos^{-1}\rho = \theta + 2\pi/k$ from (48) and $\cos^{-1}\rho = (2\pi + \theta)/3$ from (49).

It is concluded from the preceding analysis that the parameters $\{\tau_i, \phi_o, \delta, N\}$ are not arbitrary but are functions of T, θ, ϕ and k. One important consideration that has been neglected for the most part in the analysis is the required relation between $v(\tau_{2i}) = v(\tau_{2i-1})$ and $u(\tau_{2i}^{+})$, namely that $v(\tau_{2i})\, u(\tau_{2i}^{+}) < 0$. Imposing this constraint will remove all the ambiguities that remain but requires a detailed case by case analysis. An example will be considered next to demonstrate the complications involved in deriving the totality of extremals.

DERIVATION OF EXTREMALS

Before proceeding with the derivations for a sample case it can be noted that the formula to be used for τ_{2N} depends on the value of θ. That is: (assuming $k \geqslant 2$) the interval (46) is of positive length if $\theta \geq 2\pi - (3\pi/k)$ the interval determined by (47) and $\theta \geq \cos^{-1}\rho$ has positive length if $\theta > \pi - (\pi/k)$ and the interval (49) is of positive length if $\theta > \pi - (3\pi/k)$. Hence the nature of extremals will depend on the relation of θ to the values $2\pi - (3\pi/k)$, $\pi - (\pi/k)$ and $\pi - (3\pi/k)$. Other such break points in θ arise from considering the special cases when $N \leq 1$. The distribution of the break points in θ relative to the interval $[0, \pi]$ and the ordering of all break points depend on the value of k. The values 2, 2.5, and 3 are critical values of k which determine the number and ordering of break points in θ. For example, if $k \geq 3$ the break points in θ for τ_{2N} are $\pi - (3\pi/k)$ and $\pi - (\pi/k)$, whereas if $k < 3$ they are $\pi - (\pi/k)$ and $2\pi - (3\pi/k)$. The other break points in θ are $\pi - (\pi/2k)$ and $m\pi - (5\pi/2k)$ where $m = 2$ if $2 < k < 2.5$ and $m = 1$ if $k > 2.5$.

Now consider the case of $k > 3$ and $\pi - (3\pi/k) \leq \theta < \pi - (5\pi/2k)$. In this case as in all cases, if $T \leq \pi/k$ the constraint $|v(t)| \leq 1$, $0 \leq t \leq T$, is always satisfied and N and δ are both zero.

The possible values for τ_{2N} when $T > \pi/k$ can be determined as follows. The inequalities $\pi - (5\pi/2k) < \pi - (\pi/k)$ and $\pi \leq 2\pi - (3\pi/k)$ are satisfied when $k \geq 3$. Thus, with θ restricted to $[\pi - (3\pi/k), \pi - (5\pi/2k)]$, the inequality $\theta \geq \cos^{-1}\rho$ implies $\cos^{-1}\rho < 2\pi - \theta - (2\pi/k)$ and $\pi/k < (2\pi - \theta)/3$, i.e., neither (46) nor (47) is ever satisfied. Hence $\tau_{2N} = t$ when $\theta \geq \cos^{-1}\rho$ (or equivalently $\phi \geq 0$) and $\phi_o = 0$. When $\phi < 0$ and $|v(T)| < 1$:

$$\tau_{2N} = T + \theta - \cos^{-1}\rho \text{ if } v(\tau_{2N}) = +1 \tag{50}$$

$$\tau_{2N} = T + \theta - 2\pi + \cos^{-1}\rho \text{ if } v(\tau_{2N}) = -1, \tag{51}$$

subject, of course, to $\tau_{2N} \geq \tau_{2N-1}$. The limiting cases when $|v(T)| = 1$ are $\tau_{2N} = T - (2\pi/k)$, which corresponds to $\cos^{-1}\rho = \theta + (2\pi/k)$ in (50), and $\tau_{2N} = T - 4(\pi - \theta)/3$, which corresponds to $\cos^{-1}\rho = (2\pi + \theta)/3$ in (51).

When $\pi/k < T \leq \pi - \theta + (\pi/2k)$ (38) indicates $\tau_1 = \pi/k$ and it is easily deduced that if $\tau_1 = \pi/k$, $v(\tau_1) = +1$. It is also easy to see that N is at most 1 if $|v(T)| < 1$. Thus, if $N = 1$, $\tau_{2N} = \tau_2 = T + \theta - \cos^{-1}\rho$ from (50) when $|v(T)| < 1$. Furthermore, as $v(T)$ ranges from +1 to -1, τ_2 ranges from T to $T - (2\pi/k)$. Hence, if $T \geq 3\pi/k$, $\tau_2 \geq \tau_1$. But if $T < 3\pi/k$, $T - (2\pi/k) < \pi/k$ so that $\tau_2 = \tau_1$ for some value of ϕ. As ϕ is decreased from this particular value $N = 0$ until $v(T) = -1$ at which point $N = 1$ and $\tau_1 = \tau_2 = T$. Thus, the following results are obtained.

For $\pi/k < T < 3\pi/k$:

$\phi \geq 0$	implies $N = \delta = 1$, $\phi_o = 0$, $\tau_1 = \pi/k$, $\tau_2 = T$
$0 > \phi > \phi_1$	implies $N = 1$, $\delta = 0$, $\tau_1 = \pi/k$, $\tau_2 = T + \theta - \cos^{-1}\rho$

$\phi_1 > \phi > \phi_4$ implies $N = 0$

$\phi_4 \geq \phi$ implies $N = \delta = 1, \; \phi_o = \phi_4, \; \tau_1 = \tau_2 = T,$

where ϕ_1 and ϕ_4 satisfy:

$$\tan \phi_1 = -\cos\theta + \cos(T + \theta - \pi/k)$$

$$\tan \phi_4 = -\cos\theta + \cos(\theta + T/2 + \pi/2k)$$

For $3\pi/k \leq T \leq \pi - \theta + (\pi/2k)$:

$\phi \geq 0$ implies $N = \delta = 1, \; \phi_o = 0, \; \tau_1 = \pi/k, \; \tau_2 = T$

$0 > \phi > \phi_8$ implies $N = 1, \; \delta = 0, \; \tau_1 = \pi/k, \; \tau_2 = T + \theta - \cos^{-1}\rho$

$\phi_8 \geq \phi$ implies $N = 2, \; \delta = 1, \; \phi_o = \phi_8, \; \tau_1 = \pi/k, \; \tau_2 = T - (2\pi/k),$

$$\tau_3 = \tau_4 = T$$

where $\tan\phi_8 = -\cos\theta + \cos(\theta + 2\pi/k)$.

When $\pi - \theta + (\pi/2k) < T < \pi - \theta + (\pi/k)$ it is possible that τ_1 is given by (37) with $m_o = -1$. From (39) it is clear that τ_1 is given by (37) if $\pi - \theta + (\pi/2k) < T < \pi - \theta + (\pi/k)$ and if $\pi - \theta + (\pi/k) \leq T < \pi - \theta + (2\pi/k)$, then τ_1 is either given by (37) or is equal to π/k. As shown above $\tau_{2N} = T$ and $\phi_o = 0$ and $N = \delta = 1$ if $\phi \geq 0$. Then τ_1 is given by (37). With τ_1 given by (37), $v(\tau_1) = +1$ so that τ_{2N} is given by (50), i.e., $\tau_1 = T + \theta - \cos^{-1}\rho$. Now $T - (2\pi/k) \geq 4(\theta+T-\pi)/3+(\pi/3k)$ is equivalent to $T \leq 4(\pi - \theta) - (7\pi/k)$. Thus the following result is obtained for $\pi - \theta + (\pi/2k) < T \leq 4(\pi-\theta) - (7\pi/k)$:

$\phi \geq 0$: $N = \delta = 1, \; \phi_o = 0, \; \tau_1 = [(\pi/k) + 4(\theta + T - \pi)]/3, \; \tau_2 = T$

$0 > \phi > \phi_8$: $N = 1, \; \delta = 0, \; \tau_1 = [(\pi/k) + 4(\theta + T - \pi)]/3, \; \tau_2 = \theta + T - \cos^{-1}\rho$

$\phi_8 \geq \phi$: $N = 2$, $\delta = 1$, $\phi_o = \phi_8$, $\tau_1 = [\pi/k + 4(\theta + T - \pi)]/3$, $\tau_2 = T -$

$(2\pi/k)$, $\tau_3 = \tau_4 = T$.

If $4(\pi - \theta) - (7\pi/k) < T < \pi - \theta + 2\pi/k$ the above result for $\phi \geq 0$ holds. For $0 > \phi \geq \phi_2$ where $\tan\phi_2 = -\cos\theta + \cos([T + (\pi/k) + 2\pi + \theta]/3)$; $\theta + T - \cos^{-1}\rho \geq [(\pi/k) + 4(\theta + T - \pi)]/3$. Hence for this range of ϕ; $N = 1$, $\delta = 0$, $\tau_1 = [(\pi/k) + 4(\theta + T - \pi)]/3$ and $\tau_2 = \theta + T - \cos^{-1}\rho$. Then for $\phi_2 > \phi > \phi^*$, for some ϕ^*, $N = 0$ and $u(t)$ is given by:

$$u(t) = \begin{cases} -(k/\pi), & \text{for } 0 < t < \theta + T - 2\pi + \cos^{-1}\rho \\ (k/\pi), & \text{for } \theta + T - 2\pi + \cos^{-1}\rho < t < \theta + T - \cos^{-1}\rho \\ -(k/\pi), & \text{for } \theta + T - \cos^{-1}\rho < t < T \end{cases}$$

The value of ϕ^* is the maximum of the values ϕ_1 and ϕ_6 where $\theta + T - 2\pi + \cos^{-1}\rho(\theta, \phi_1) = \pi/k$ and $T = (\pi/k) + 4[\pi - \cos^{-1}\rho(\theta, \phi_6)]$. It can be shown that $\phi_6 > \phi_1$ when $4(\pi - \theta) - (7\pi/k) < T < [4(\pi - \theta)/3] + (\pi/k)$, and $\phi_1 \geq \phi_6$ if $[4(\pi - \theta)/3] + \pi/k \leq T < \pi - \theta + (2\pi/k)$. Thus, the following result is obtained for $4(\pi - \theta) - (7\pi/k) < T < [4(\pi - \theta)/3] + (\pi/k)$:

$\phi \geq 0$: $N = \delta = 1$, $\phi_o = 0$, $\tau_1 = [(\pi/k) + 4(\theta + T - \pi)]/3$, $\tau_2 = T$

$0 > \phi \geq \phi_2$: $N = 1$, $\delta = 0$, $\tau_1 = [(\pi/k) + 4(\theta + T - \pi)]/3$, $\tau_2 = \theta + T - \cos^{-1}\rho$

$\phi_2 > \phi > \phi_6$: $N = 0$

$\phi_6 \geq \phi$: $N = \delta = 1$, $\phi_o = \phi_6$, $\tau_1 = \tau_2 = T$.

In the case, $[4(\pi - \theta)/3] + (\pi/k) \leq T < \pi - \theta + (2\pi/k)$ and $\phi_1 \geq \phi$, $\tau_1 = \pi/k$ and $v(\tau_1) = -1$. Thus $\tau_{2N} = \theta + T - 2\pi + \cos^{-1}\rho$ as long as $|v(T)| < 1$ and the limiting case occurs if $\theta + T - 2\pi + \cos^{-1}\rho = T + [4(\theta - \pi)/3]$. Thus for this range of T the following result is obtained:

$\phi \geq 0$: $N = \delta = 1$, $\phi_o = 0$, $\tau_1 = [(\pi/k) + 4(\theta + T - \pi)]/3$, $\tau_2 = T$

$0 > \phi \geq \phi_2$: $N = 1$, $\delta = 0$, $\tau_1 = [(\pi/k) + 4(\theta + T - \pi)]/3$, $\tau_2 = \theta + T - \cos^{-1}\rho$

$\phi_2 > \phi > \phi_1$: $N = 0$

$\phi_1 \geq \phi > \phi_5$: $N = 1$, $\delta = 0$, $\tau_1 = \pi/k$, $\tau_2 = \theta + T - 2\pi + \cos^{-1}\rho$

$\phi_5 \geq \phi$: $N = 2$, $\delta = 1$, $\phi_o = \phi_5, \tau_1 = \pi/k$, $\tau_2 = T + [4(\theta - \pi)/3]$, $\tau_3 = \tau_4 = T$.

where $\tan\phi_5 = -\cos\theta + \cos[(2\pi + \theta)/3]$.

Now consider T in the interval $m\pi - \theta + (2\pi/k) \leq T \leq (m+1)\pi - \theta + (\pi/2k)$, where m is any integer greater than zero. Since (38) is not satisfied, $\tau_1 = \pi/k$ whenever $\tau_2 \geq (\pi/k)$. It is seen that $\theta + T - m\pi - (\pi/k) \geq \pi/k$ and that $\theta+T-\pi + (\pi/k) < T < \theta+T-(\pi/k)$ for the ranges of θ and k considered. Thus setting $\tau_{2i} = \theta + T - \pi - (m-i)\pi - \pi/k$ and $\tau_{2i+1} = \tau_{2i} + (2\pi/k)$, $\tau_2 \geq \tau_1$ and $\tau_{2m+1} < T < \tau_{2m+2}$. Hence for $\phi \geq 0$, $N = m+1$, $\delta = 1$, $\phi_o = 0$, $\tau_1 = \pi/k$, $\tau_{2N} = T$ and the intermediate τ_i are as given above. For $0 > \phi \geq \phi_3$ the only changes are that $\delta = 0$ and $\tau_{2N} = \theta + T - \cos^{-1}\rho$, where $\tan\phi_3 = -\cos\theta + \cos(\pi - \pi/k)$. For $\phi_3 > \phi > \phi_5$, N is reduced from m+1 to m with no change in τ_i if $i < 2m$ and $\tau_{2m} = \theta + T - 2\pi + \cos^{-1}\rho$. For $\phi_5 \geq \phi$, $\delta = 1$, $\phi_o = \phi_5$, $N = m+1$, $\tau_{2m} = T + 4(\theta - \pi)/3$, $\tau_{2m+1} = \tau_{2m+2} = T$ and the remaining τ_i are unchanged.

Consider the interval $m\pi - \theta + (\pi/2k) < T < m\pi - \theta + (2\pi/k)$ with $m > 1$. In this case $\theta + T - (m-1)\pi - (\pi/k) > [4(\theta + T - m\pi)+(\pi/k)]/3$. Hence the results are similar to those just obtained except that $\tau_1 = [4(\theta + T - m\pi)+(\pi/k)]/3$, i.e.,

$\phi \geq 0$: $N = m$, $\delta = 1$, $\phi_o = 0$, $\tau_1 = [4(\theta + T - m\pi)+(\pi/k)]/3$, $\tau_{2i} = \theta + T - (m-i)\pi - (\pi/k)$

$\tau_{2i+1}=\tau_{2i} + (2\pi/k)$, $i=1, 2, \ldots, N - 1$, $\tau_{2N} = T$.

$0 > \phi \geq \phi_3$: Only changes from $\phi \geq 0$ are $\delta = 0$, $\tau_{2N} = \theta + T - \cos^{-1}\rho$

$\phi_3 > \phi > \phi_5$: Only changes from $\phi \geq 0$ are $\delta = 0$, $N = m-1$, $\tau_{2N} = \theta + T -$

$2\pi + \cos^{-1}\rho$

$\phi_5 \geq \phi$: $N = m$, $\delta = 1$, $\phi_o = \phi_5$, $\tau_{2m-2} = T + [4(\theta - \pi)/3]$, $\tau_{2m-1} = \tau_{2m} = T$ and τ_1 through τ_{2m-3} are the same as when $\phi \geq 0$.

This completes the derivation of the extremals for all $T \geq 0$, $k \geq 3$, $|\phi| < \pi/2$, and $\pi - (3\pi/2k). < \theta < \pi - (5\pi/k)$.

EXPLICIT REPRESENTATION OF EXTREMALS

The formulas for the extremals will be given in a tabular form for $T > 0$, $k \geq 2.5$, $|\phi| < \pi/2$ and $0 \leq \theta < \pi$. To simplify the table the extremals will not be given for the break points in T and θ. For any particular case of interest where T and θ are break points the extremal could be readily determined from consideration of the neighboring intervals.

The following functions of k, θ, ϕ and T are introduced to simplify future expressions and notation of the dependence on the parameters will be suppressed.

$$h_1 = \pi/k$$

$$h_2 = (2\pi + \theta)/3$$

$$h_3 = (4\pi + \theta)/3$$

$$h_4 = \theta + T - \cos^{-1}(\tan\phi + \cos\theta)$$

$$h_5 = [4(\theta + T - \pi) + (\pi/k)]/3$$

$$h_6 = \theta + T - 2\pi + \cos^{-1}(\tan\phi + \cos\theta)$$

$$h_7 = T + [4(\theta - \pi)/3]$$

$$h_8 = \theta + T - \pi - (\pi/k)$$

Let $g(k) = 0$ if $2.5 \leq k < 3$ and $g(k) = 1$ if $k \geq 3$ and define

$$\theta_o = 0$$

$$\theta_1(k) = \pi - [5 + g(k)](h_1/2)$$

$$\theta_2(k) = \pi - [2 + 3g(k)](h_1/2)$$

$$\theta_3(k) = \pi - [1 + g(k)](h_1/2)$$

$$\theta_4(k) = [2 - g(k)]\pi - [6 - 5g(k)](h_1/2)$$

$$\theta_5 = \pi$$

Then let $j(\theta, k)$ be defined implicitly by $\theta_{j(\theta,k)}(k) < \theta < \theta_{j(\theta,k)+1}(k)$ and the dependence on θ and k will be suppressed from now on. Also introduce the functions $q_i(j) = \delta_{ij}$ for i, j = 0, 1, 2, 3, 4 where $\delta_{ij} = 0$ if $i \neq j$ and $\delta_{ii} = 1$. The dependence on j will be suppressed in the future.

Further, introduce the functions $\phi_i(T, \theta, k)$, defined implicitly by $|\phi_i| < \pi/2$ and $\tan\phi_i + \cos\theta = \rho_i$, for i = 1, 2, . . . , 11. The ρ_i are defined as follows:

$$\rho_1 = \cos(T + \theta - h_1)$$

$$\rho_2 = \cos[h_2 + (T + h_1)/3]$$

$$\rho_3 = \cos(\pi - h_1)$$

$$\rho_4 = \cos[\theta + (T + h_1)/2]$$

$$\rho_5 = \cos(h_2)$$

$$\rho_6 = \cos[\pi - (t - h_1)/4]$$

$$\rho_7 = -\rho_6$$

$$\rho_8 = \cos(\theta + 2h_1)$$

$$\rho_9 = \cos(h_3)$$

$$\rho_{10} = \cos[h_3 + (T + h_1)/3]$$

$$\rho_{11} = -\rho_3$$

The break points in T, i.e., endpoints of intervals in T in which the form of the extremals remains unchanged, are given below:

$$T_o = h_1$$

$$T_1 = 3h_1(q_o + g\, q_1) + [4(\pi - \theta) - h_1](1-g)q_1/3 + [\pi - \theta + (h_1/2)]$$
$$(q_2 + gq_3) + \{h_1 + [4(\pi - \theta)/3]\}[q_3(1-g) + q_4]$$

$$T_2 = [\pi - \theta + (h_1/2)](q_o + q_1) + [4(\pi - \theta) - h_1][(1-g)q_2 + gq_3] +$$
$$\{h_1 + [4(\pi - \theta)/3]\}gq_2 + 3h_1[(1-g)q_3 + q_4]$$

$$T_3 = [4(\pi - \theta) - 7h_1][(1-g)q_o + gq_1] + (\pi - \theta + 2h_1)g(q_o + q_2) +$$
$$\{h_1 + [4(\pi - \theta)/3]\}(1-g)q_1 + 3h_1[(1-g)q_2 + gq_3] +$$
$$[2\pi - \theta + (h_1/2)][(1-g)q_3 + q_4]$$

$$T_4 = \{h_1 + [4(\pi-\theta)/3]\}[(1-g)q_o + gq_1] + [2\pi - \theta + (h_1/2)][g(q_o + q_3)+q_2] +(\pi-\theta+2h_1)(1-g)q_1 + (2\pi-\theta+2h_1)[(1-g)q_3 + gq_4] + [4(2\pi - \theta)-7h_1](1-g)q_4$$

$$T_5 = (\pi-\theta+2h_1)[(1-g)q_o+gq_1]+(2\pi-\theta+2h_1)[g(q_o+q_3+q_4)+q_2] + [2\pi-\theta+(h_1/2)](1-g)(q_1+q_3) + \pi(1-g)q_3 + \{h_1+[4(2\pi-\theta)/3]\}(1-g)q_4$$

$$T_{2n} = n\pi-\theta+(h_1/2)+(3h_1/2)[q_4+(1-g)(q_1+q_3)]-\pi[q_1+(1-g)(q_o+q_4)], \quad n \geq 3$$

$$T_{2n+1} = n\pi-\theta+2h_1-(3h_1/2)[q_4+(1-g)(q_1+q_3)] +\pi[g(q_4-q_1)+(1-g)(q_3-q_o)], \quad n \geq 3$$

The significant break points in ϕ depend on T, θ, and k in a complicated way. But throughout any given θ interval and T interval they are either 0 or one of the ϕ_i, i=1, 2, . . . , 11, defined above. Thus, for $T_{m-1} < T < T_m$, let $\phi^i(T, \theta, k)$, i=1, 2. 3, 4, be defined by:

$$\begin{bmatrix} \phi^1 \\ \phi^2 \\ \phi^3 \\ \phi^4 \end{bmatrix} = A_m \begin{bmatrix} q_o \\ q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix}, \quad m = 1, 2, 3, \ldots,$$

where:

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & \phi_4(1-g) & \phi_4 \\ 0 & 0 & 0 & \phi_4(1-g) & \phi_4 \\ \phi_1 & \phi_1 & \phi_1 & [\phi_4(1-g)+\phi_1 g] & \phi_4 \\ \phi_4 & \phi_4 & [\phi_4(1-g)+\phi_6 g] & \phi_6 & \phi_6 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0 & 0 & 0 & \phi_4(1-g) & \phi_4 \\ 0 & 0 & 0 & \phi_4(1-g) & \phi_4 \\ \phi_8 & [\phi_1(1-g)+\phi_8 g] & \phi_2 & [\phi_1(1-g)+\phi_2 g] & \phi_1 \\ \phi_8 & [\phi_6(1-g)+\phi_8 g] & \phi_6 & [\phi_5(1-g)+\phi_6 g] & \phi_5 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 0 & 0 & \phi_4(1-g) & [\phi_8(1-g)+\phi_4 g] & \phi_8 \\ 0 & 0 & [\phi_4(1-g)+\phi_2 g] & [\phi_8(1-g)+\phi_4 g] & \phi_8 \\ \phi_8 & [\phi_2(1-g)+\phi_8 g] & \phi_1 & [\phi_8(1-g)+\phi_1 g] & \phi_8 \\ \phi_8 & [\phi_6(1-g)+\phi_8 g] & \phi_5 & \phi_5 & \phi_5 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} 0 & 0 & \phi_8(1-g) & \phi_8 & \phi_8 \\ 0 & \phi_2(1-g) & \phi_8(1-g) & \phi_8 & \phi_8 \\ [\phi_2(1-g)+\phi_8 g] & [\phi_1(1-g)+\phi_2 g] & [\phi_8(1-g)+\phi_3 g] & \phi_8 & \phi_8 \\ [\phi_6(1-g)+\phi_8 g] & [\phi_5(1-g)+\phi_6 g] & \phi_5 & \phi_5 & \phi_5 \end{bmatrix}$$

$$A_5 = \begin{bmatrix} 0 & 0 & \phi_8(1-g) & \phi_8 & [\phi_7(1-g)+\phi_8 g] \\ \phi_2(1-g) & \phi_2 g & \phi_8(1-g) & \phi_8 & [\phi_7(1-g)+\phi_8 g] \\ \phi_1(1-g) & [\phi_3(1-g)+\phi_1 g] & [\phi_8(1-g)+\phi_3 g] & \phi_8 & [\phi_{10}(1-g)+\phi_8 g] \\ [\phi_5(1-g)+\phi_8 g] & \phi_5 & \phi_5 & \phi_5 & \phi_5 \end{bmatrix}$$

$$A_6 = \begin{bmatrix} 0 & 0 & \phi_8(1-g) & \phi_8 & \phi_9(1-g)+\phi_8 g \\ 0 & 0 & \phi_8(1-g) & \phi_8 & \phi_1(1-g)+\phi_8 g \\ \phi_3(1-g) & \phi_3 & [\phi_8(1-g)+\phi_3 g] & \phi_8 & \phi_{10}(1-g)+\phi_8 g \\ [\phi_5(1-g)+\phi_8 g] & \phi_5 & \phi_5 & \phi_5 & \phi_5 \end{bmatrix}$$

$$A_m = \begin{bmatrix} 0 & 0 & \phi_8(1-g) & \phi_8 & [\phi_9(1-g)+\phi_8 g] \\ 0 & 0 & \phi_8(1-g) & \phi_8 & [\phi_9(1-g)+\phi_8 g] \\ \phi_3(1-g) & \phi_3 & [\phi_8(1-g)+\phi_3 g] & \phi_8 & [\phi_{11}(1-g)+\phi_8 g] \\ [\phi_5(1-g)+\phi_8 g] & \phi_5 & \phi_5 & \phi_5 & \phi_5 \end{bmatrix}, \quad m \geq 7$$

Let $\phi^o = \pi/2$ and $\phi^5 = -\pi/2$ and introduce the functions r_i, $i=0, 1, 2, 3, 4$, where $r_i = 1$ if $\phi^i < \phi < \phi^{i+1}$ and $r_i = 0$ otherwise. Extremals for $\phi = \phi^i$ will not be defined but they can be easily determined by considering the neighboring intervals.

Now the expressions for N, δ, ϕ_o and τ_i for $i=1, 2, \ldots, 2N$ can be given. For $T < T_o$ it is clear that $\delta = 0$ and $N = 0$. When $T > T_o$, $\delta = r_o + r_4$ and $\phi_o = r_o\phi^1 + r_4\phi^4$. In this case N and τ_i are functions of all the parameters T, θ, ϕ and k. The dependence on T will be shown implicitly by expressing the dependence on m(T) where $T_{m(T)-1} < T < T_{m(T)}$. All other dependence will be suppressed, e.g. N = N(m). Furthermore N(m) dominates $\tau_i(m)$, i.e., $\tau_i(m)$ given by the formulas for $i > N(m)$ are meaningless.

$$N(1) = 1 - r_3$$

$$\tau_1(1) = (1 - r_3 - r_4) \{h_1 + T - h_1) [q_4 + (1-g)q_3]\} + Tr_4$$

$$\tau_2(1) = r_2(h_4-T) [1 - q_4 - (1-g)q_3] + T (1-r_3)$$

$$N(2) = 1 + r_4 - (r_3 + r_4) [q_1(1-g) + q_2 + q_3 g] - r_2[q_3(1-g) + q_4]$$

$$\tau_1(2) = h_1 \{q_o+q_1+(r_3+r_4)[q_4+(q_3-q_1)(1-g)]\} + h_5(1-r_3-r_4)(q_2+q_3 g) +$$
$$T\{r_4[q_1+q_2+(q_3-q_1)g] + (r_o+r_1)[q_3(1-g)+q_4]\}$$

$$\tau_2(2) = T(1-r_2-r_3) + (Tr_2 + h_4 r_3 - 2h_1 r_4) (q_o+q_1 g) + h_4 r_2 [q_1(1-g)+$$
$$q_2+q_3 g] + [h_1 r_3 + (h_1-T)r_4][q_3(1-g)+q_4]$$

$$\tau_3(2) = \tau_4(2) = Tr_4[q_o + q_1 g + q_3(1-g) + q_4]$$

$$N(3) = 1+r_4-(r_3+r_4)q_1(1-g)-r_2(q_2+q_3 g) + (1-r_3-r_4)[q_3(1-g)+q_4]$$

$$\tau_1(3) = h_5[q_o-q_1(r_3+r_4)(1-g)+q_2(r_o+r_1)g+q_1] + T\{r_4 q_1(1-g)+(r_o+r_1)$$
$$[q_2(1-g)+q_3 g]\} + h_1 \{(r_3+r_4)[q_2(1-g)+q_3 g] + q_3(1-g)+q_4\}$$

$$\tau_2(3) = [T(1-r_3)+h_4 r_3-2h_1 r_4](q_o+q_1 g) + [T(1-r_2 -r_3)+h_4 r_2]q_1(1-g)+$$
$$(h_4-T)r_1 q_2 g + [(T-2h_1) (1-r_3 -r_4)+h_6 r_3+h_7 r_4](1-q_o-q_1) + 2h_1$$
$$(1-r_3 -r_4) (q_2+q_3 g)$$

$$\tau_3(3) = \tau_4(3) = Tr_4[1-q_1(1-g)] + T(1-r_3 -r_4)[q_3(1-g)+q_4]$$

$$N(4) = 2 - r_3 - [1 - (1+r_3+r_4)g]q_o - [1 + (r_2 - r_3 - r_4) (1-g)]q_1$$

$$\tau_1(4) = h_5+(T-h_5)r_4[q_o(1-g)+q_1 g] + (h_1-h_5)[(r_2+r_3)q_1(1-g)+q_2+$$
$$(q_o+q_3)g]-(4\pi/3) [q_3(1-g)+q_4]$$

$$\tau_2(4) = [T(1-r_2)+h_4r_2][q_o(1-g)+q_1g]+(Tr_o+h_4r_1)q_1(1-g)+h_8[q_o+q_2(1-r_3-r_4)]g + (h_6r_3+h_7r_4)(1-q_o-q_1g)+(T-2h_1)(1-r_3-r_4)(1-q_o-q_1-q_2g)$$

$$\tau_3(4) = Tr_4[q_1(1-g)+q_2g]+(h_8+2h_1)[q_o+q_2(1-r_3-r_4)]g+T(1-r_3)(1-q_o-q_1-q_2g)$$

$$\tau_4(4)=\tau_3(4)(1-g)+[T(1-r_3)+(h_4r_3-2h_1r_4)q_o+(h_4-T)r_2q_2]g$$

$$\tau_5(4) = \tau_6(4) = Tr_4q_og$$

$$N(5)=2-r_3-(1+r_2-r_3-r_4)[(q_o+q_4)(1-g)+q_1g]+(r_3+r_4)q_og+q_3(1-g)+q_4g$$

$$\tau_1(5) = [h_5(r_o+r_1)+h_1(r_3+r_4)][q_o(1-g)+q_1g]+h_1[(q_1+q_3)(1-g)+q_4g]+[h_5-(4\pi/3)][q_2+(q_o+q_3)g] + \{T(r_o+r_1)+[h_5-(4\pi/3)](r_3+r_4)\}q_4(1-g)$$

$$\tau_2(5) = [Tr_o+h_4r_1][q_o(1-g)+q_1g]+[h_6r_3+h_7r_4][1-q_3+(q_3-q_o-q_4)g] + T(r_o+r_1)q_4g+h_8\{(1-r_3-r_4)[q_1(1-g)+q_2g]+q_og\}+(h_8-\pi)[q_3(1-g)+q_4g]+(T-2h_1)(1-r_3-r_4)[q_2(1-g)+q_3g]$$

$$\tau_3(5)=T+(h_8+2h_1-T)\{1-r_3-r_4)[q_1(1-g)+q_2g]+q_og\}+(h_8-\pi+2h_1-T)[q_3(1-g)+q_4g]$$

$$\tau_4(5)=T+(h_4-T)[r_2q_1(1-g)+(r_3q_o+r_2q_2)g]-2h_1r_4q_og-[2h_1(1-r_3-r_4)+(T-h_6)r_3+(T-h_7)r_4][q_3(1-g)+q_4g]$$

$$\tau_5(5)=\tau_6(5)=Tr_4q_og+T(1-r_3)[q_3(1-g)+q_4g]$$

$$N(6) = 3-r_3-q_o(1-g)-q_1+(r_3+r_4)q_og+(r_o-r_2+r_3+r_4-2)q_4(1-g)$$

$$\tau_1(6)=h_1\{[q_o+q_2+(r_o+r_1)q_4](1-g)+(1-q_4)g\}+h_5q_1(1-g)+[h_5-(4\pi/3)]\{[q_3+(r_3+r_4)q_4](1-g)+q_4g\}-(4\pi/3)[q_3(1-g)+q_4g]$$

$\tau_2(6)=h_8\{(1-r_3-r_4)[q_o(1-g)+q_1]+q_2(r_3+r_4)g\}+(h_6r_3+h_7r_4)[(q_o+q_4)(1-g)+q_1]+(h_8-\pi)\{q_2(1-g)+q_3+[q_o+q_2(1-r_3-r_4)+q_4]g\}+\{[h_7-(4\pi/3)]r_o+(h_4-2\pi)r_1\}q_4(1-g)$

$\tau_3(6)=(h_8+2h_1)\{(1-r_3-r_4)[q_o(1-g)+q_1]+q_2+q_3+(q_o+q_4)g\}-\pi\{[q_o+q_2(1-r_3-r_4)+q_4]g+q_3+q_2(1-g)\}+T\{[r_4q_o+(r_o+r_4)q_4](1-g)+r_4q_1\}$

$\tau_4(6)=[T(1-r_2-r_3)+h_4r_2][q_o(1-g)+q_1]+(T-2h_1)(1-r_3-r_4)[q_2(1-g)+q_3 q_4g]+(h_6r_3+h_7r_4)(q_2+q_3+q_4g)+T(r_o+r_4)q_4(1-g)+h_8[q_o+q_2(1-r_3-r_4)]g$

$\tau_5(6)=T(1-r_3)[q_2(1-g)+q_3+q_4g]+Tr_4q_2g+(h_8+2h_1)[q_o+q_2(1-r_3-r_4)]g$

$\tau_6(6)=T(1-r_3)[q_2+q_3+q_4g]+(h_4-T)r_2q_2g+[T+(h_4-T)r_3-2h_1r_4]q_og$

$\tau_7(6)=\tau_8(6)=Tr_4q_og$

General expressions can be written for the parameters N and τ_i for $m \geq 7$. It is convenient to introduce the integer M(m) defined as follows: M is equal to N, if $\tau_{2N-1} < T$, and otherwise M is equal to N-1. Explicitly,

$$M = N-r_4[q_o+q_1+q_4(1-g)+q_2g]-(1-r_3)[q_2(1-g)+q_3+q_4g]-(r_o+r_1)q_4(1-g),$$

and $\tau_{2N-1} = \tau_{2N} = T$ if $M = N-1$. With M defined in this way, the range of the index i in the following expressions is $1 \leq i \leq M-1$.

For $n \geq 3$;

$N(2n+1)=n-r_3-q_o[(1-g)-(r_3+r_4)g]+[q_3-(1+r_2-r_3-r_4)q_4](1-g)-(q_1-q_4)g$

$\tau_1(2n+1)=[h_5-(n-2)(4\pi/3)][q_o(1-g)+q_1g]+[h_5-(n-1)(4\pi/3)][q_2+(q_o+q_3)g]+h_1[(q_1+q_3)(1-g)+q_4]$

$$\tau_{2i}(2n+1)=h_8+(i-n+1)\pi+\pi[(q_o-q_3)(1-g)+(q_1-q_4)g],\ \tau_{2i+1}(2n+1)=\tau_{2i}(2n+1)+2h_1$$

$$N(2n+2)=N(2n+1)+q_o+q_2+(q_1+q_3)g$$

$$\tau_1(2n+2)=h_1[q_o+q_2+(q_1+q_3)g]+[h_5-(n-1)(4\pi/3)](q_1+q_4)(1-g)+[h_5-n(4\pi/3)][q_3(1-g)+q_4g]$$

$$\tau_{2i}(2n+2)=\tau_{2i}(2n+1)-\pi[q_o+q_2+(q_1+q_3)g]$$

$$\tau_{2i+1}(2n+2)=\tau_{2i}(2n+2)+2h_1$$

Also, for $n \geq 7$

$$\tau_{2M}(n) = [T(r_o+r_1)+h_4r_2][q_o+q_1+(q_2-q_o)g]+[T(1-r_3)+h_4r_3-2h_1r_4]q_og + (h_6r_3+h_7r_4)(1-q_og)+(T-2h_1)(1-r_3-r_4)[q_2+q_3+(q_4-q_2)g] + \{[h_7-(4\pi/3)](r_o+r_1)+(h_4-2\pi)r_2\}\ q_4(1-g)$$

The above formulas give a complete explicit representation of the extremal inputs for the problem considered.

CONCLUSIONS

The necessary conditions for an extremal input given in references 2, 3, and 4 for problems with bounded phase coordinates were discussed with respect to the problem at hand. These conditions give the result that an extremal input is proportional to a signum function in which the argument of the signum function is an adjoint solution. These conditions allow discontinuities in the adjoint solution at certain junction points. The necessary conditions of reference 5, interpreted for this problem show that these discontinuities are not required. One discontinuity is required if it is desired that the adjoint represent any external normal to the set of attainability. This discontinuity only occurs when the external normal is not unique. It is shown that when the necessary conditions of reference 5 are satisfied the sufficient conditions of reference 1 are satisfied.

These conditions imply that u(t) is an extremal input if and only if it can be represented by equations (28) and (29) subject to the constraints (30) through (35). In this representation the junction points and the discontinuity are introduced as parameters. The constraints (30) through (35) are determining equations and inequalities for these parameters. An example shows how these constraints are used to determine these parameters.

The results are presented in tabular form for extremal inputs giving explicit formulas for the necessary parameters.

Responses having an arbitrarily large number of arcs which lie on the phase constraint boundary can be obtained from extremal inputs if the time interval $[0, T]$ is sufficiently long. In such a case the input tends to be in resonance with the oscillator over an intermediate segment of the time interval. During an initial segment the input seeks to get into the proper phase relationship with the oscillator. A final segment of the interval is spent attaining the proper terminal value for the input.

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SECTION 5
EXTREMAL BOUNDED-AMPLITUDE BOUNDED RATE INPUTS

This section considers the problem of determining extremal bounded-amplitude bounded-rate inputs to linear stationary systems. The discussion will be restricted to the case of a scalar input.

Consider a system represented by the vector differential equation:

$$\dot{x}(t) = Ax(t) + bw(t) \tag{1}$$

Here it is assumed that w(t) is a scalar input which is a continuous function of time with a piece-wise continuous derivative $\dot{w}(t)$. It is also assumed that $|w(t)|$ and $|\dot{w}(t)|$ satisfy the constraints (2) and (3).

$$|w(t)| \leq k \tag{2}$$

$$|\dot{w}(t)| \leq 1 \tag{3}$$

The vector, x(t), is an n-vector representing the state (or response) of the system and A and b are constant n x n and n x 1 matrices, respectively. The system is assumed to be initially at rest, i. e. x (0) = w (0) = 0. The input, w(t), can be adjoined to the state of the system by introducing $x_{n+1}(t) = w(t)$ and for convenience set $\dot{w}(t) = v(t)$. Then the system is represented by:

$$\dot{x} = \begin{bmatrix} A & b \\ o & o \end{bmatrix} x + \begin{bmatrix} o \\ 1 \end{bmatrix} v, \quad x(0) = 0 \tag{4}$$

where $\dot{x}$ represents the n + 1-vector and the constraints are:

$$|X_{n+1}(t)| \leq k \tag{5}$$

$$|v(t)| \leq 1 \tag{6}$$

'or any T > 0 the set of attainability, K(T), is defined as the set of all x(T) 'hich correspond to solutions of (4) subject to (5) and (6) for $0 \leq t \leq T$. 'his set is a closed, bounded, convex set in Euclidean n+1 dimensional space. nputs which give rise to boundary points of K(T) are defined to be extremal nputs. Since K(T) is convex, it has the property that at any boundary point, ay $x_B(T)$, there is a hyperplane, π, containing the boundary point which supports :(T), i.e. There is a vector ψ normal to π such that $\psi \cdot [x_B(T) - x(T)] \geq 0$ or any x(T) in K(T). Such a vector, ψ, will be called an external normal to :(T) at $x_B(T)$.

'hus, it is possible to interpret extremal inputs in the following way. On a ime interval, [0, T] an extremal input gives rise to a response x(T) that is uch that the projection of x(T) on some vector ψ is a maximum. Hence, if v(t) in (1) is considered as a disturbance and if the significant effects of the listurbance can be described as linear combinations of the state of the system hen the worst disturbances are extremal inputs. The results obtained for the problem of determining extremal inputs provide a means for evaluating the performance index for a controller in the minimax problem with bounded-amplitude, bounded-rate disturbances. A brief description of such a problem is as follows:

Consider a system described by

$$\dot{x} = \hat{A}x + \hat{b}\,u + \hat{c}\,w, \quad x(0) = 0, \; w(0) = 0$$

with u, the scalar control, a linear combination of the components of x and w where w is a scalar disturbance that satisfies $|w(t)| \leq k$ and $|\dot{w}(t)| \leq 1$. The closed loop system may be written in the form of equation (4). For a fixed time, T, the performance index of a controller u is given by $C_i(u) = \max_{1 \leq i \leq s} C_i(u)$ with $C_i(u) = \max |d(i)\; x(T)|$ where the maximum is taken with respect to all allowable v's and a given set of row vectors, d(i), i=1, 2, . . . , s.

With this problem in mind it is desired to develop an efficient computational method to determine the maximum of the scalar product of a given vector, d, and x(T) subject to the constraints (5) and (6).

In the following discussion the necessary and sufficient conditions for extremal inputs will be given. Based on these conditions two computational algorithms will be formulated. Then the results of a computer program derived from one of these formulations will be discussed.

NECESSARY AND SUFFICIENT CONDITIONS

The necessary and sufficient conditions state that extremal inputs are signum functions of appropriate adjoint solutions. The adjoint solutions are continuous on the open interval (0, T) with a possible discontinuity at T. Determining equations are derived for the junction times (times when the corresponding extremal response enter or leave the phase constraint) and the discontinuity if it occurs.

A discussion of the development of these conditions is given in the section on extremal inputs for the harmonic oscillator. The results presented there are written in terms of that specific problem. The generalization of those results for problems of the type considered here is as follows:

$\eta(T)\, x(T) = \int_o^T \eta_{n+1}(t)\, v(t)\, dt$ where $\eta(t)$ is a row vector satisfying

$$\dot{\eta} = -\eta \begin{bmatrix} A & b \\ o & o \end{bmatrix} \tag{7}$$

hus a precise definition of an extremal input, v_o, is that there exists a non-
ivial (continuous) solution $\eta(t)$ of equation (7) such that $\int_0^T \eta_{n+1}(t)\, v_o(t)\,dt =$

ax $\int_0^T \eta_{n+1}(t)\, v(t)dt$ where the maximization is taken over all v(t) which

atisfy (5) and (6) for $0 < t < T$ with $x_{n+1}(0) = 0$. A necessary and sufficient
ondition for v(t) to be an extremal input is that:

i) There exist a set of junction points $0 < \tau_1 \leq \tau_2 < \ldots < \tau_{2q-1} \leq \tau_{2q} \leq T$ and setting $\tau_o = 0$ such that $g[x(t)] < 0$ for $\tau_{2i} < t < \tau_{2i+1}$ and $g[x(t)] = 0$ for $\tau_{2i+1} \leq t \leq \tau_{2i}$, $i = 0, 1, \ldots, q-1$ where $g[x(t)] = [x_{n+1}(t)]^2 - k^2$ and

ii) there exists a (row) vector $\psi(t)$ continuous on $(0, T)$ such that

$$v(t) = \text{sgn}\,[\psi_{n+1}(t)] \quad 0 < t < T \tag{8}$$

where $\psi(t)$ satisfies:

$$\dot{\psi}(t) = \psi(t) \begin{bmatrix} A & b \\ o & o \end{bmatrix} + \chi(t)\, \zeta(t) \text{ grad } g\,[x(t)] \tag{9}$$

where $\zeta(t) = 1/2\, \psi(t) \begin{bmatrix} b \\ o \end{bmatrix} \text{sgn}\,[x_{n+1}(t)]$ and $\chi(t) = 1$ of $g[x(t)] = 0$

and $\chi(t) = 0$ if $g[x(t)] < 0$ and

iii) on each interval where $g[x(t)] = 0$ the following inequality is satisfied

$$\psi(t) \begin{bmatrix} b \\ o \end{bmatrix} x_{n+1}(t) \geq 0 \tag{10}$$

Furthermore, $\psi(T^-)$ is an external normal to the set of attainability at the point $x(T)$, and if $\psi(T)$ is an external normal to the set of attainability at $x(T)$ then each of the first n components of $\psi(T)$ is equal to the corresponding component of $\psi(T^-)$ and

$$\psi_{n+1}(T) = \psi_{n+1}(T^-) \quad \text{if } g[x(t)] < 0 \tag{11}$$

and

$$[\psi_{n+1}(T) - \psi_{n+1}(T^-)]x_{n+1}(T) \geq 0 \quad \text{if } g[x(T)] = 0 \tag{12}$$

Now suppose $\eta(T)$ is a given vector and $v(T)$, $0 < t < T$, is the corresponding extremal input. Then there is a $\psi(t)$ satisfying (9) through (12) such that $v(t)$ and $\psi(t)$ are related by equation (8). The functions $\eta(t)$ and $\psi(t)$ are related in the following way if $\psi(T)$ is chosen properly.

$$\psi(T) = \eta(T) \tag{13}$$

$$\psi_i(t) = \eta_i(t), \quad 0 \leq t \leq T, \quad i = 1, 2, \ldots, n \tag{14}$$

$$\psi_{n+1}(t) = \eta_{n+1}(t) - \eta_{n+1}(\tau_{2i+1}), \quad \tau_{2i-2} \leq t \leq \tau_{2i-1}, \quad i = 1, 2, \ldots, q-1 \tag{15}$$

$$\psi_{n+1}(t) = 0 \qquad , \tau_{2i-1} \leq t \leq \tau_{2i}, \quad i = 1, 2, \ldots, q-1 \tag{16}$$

$$\psi_{n+1}(t) = \eta_{n+1}(t) + \delta[\psi_{n+1}(T^-) - \eta_{n+1}(T)], \quad \tau_{2q-2} \leq t \leq T \text{ if } \tau_{2q-1} = T \tag{17}$$

$$\psi_{n+1}(t) = \eta_{n+1}(t) - \eta_{n+1}(\tau_{2q-1}) \qquad , \tau_{2q-2} \leq t \leq \tau_{2q-1} \text{ if } \tau_{2q-1} < T \tag{18}$$

$$\psi_{n+1}(t) = 0. \qquad , \tau_{2q-1} \leq t < \tau_{2q} \text{ if } \tau_{2q-1} < T \tag{19}$$

$$\psi_{n+1}(t) = 0 \qquad , t = \tau_{2q} \text{ if } \tau_{2q} < T \tag{20}$$

$$\psi_{n+1}(t) = \eta_{n+1}(t) + \delta\,[\psi_{n+1}(T^-) - \eta_{n+1}(T)],\ \tau_{2q} \leq t < T \text{ if } \tau_{2q} < T \tag{21}$$

In equations (17) and (21) δ is equal to zero if $g[x(T)] < 0$ and is equal to one if $g[x(T)] = 0$.

Thus, if $\eta(T)$ is given, the corresponding extremal input $v(t)$ can be determined by finding $\psi(t)$. From equations (13) through (21) it is seen that $\psi(t)$ can be found from $\eta(t)$ if quantities q, δ, $\psi_{n+1}(T^-) - \eta_{n+1}(T)$ and τ_i, $i = 1, 2, \ldots, 2q$ are known. These quantities are not arbitrary and certain determining equations and inequalities exist.

Recalling that $v(t) = \text{sgn}[\psi_{n+1}(t)]$ from equation (8) and that by definition $g[x(t)] < 0$ for $\tau_{2i-2} < t < \tau_{2i-1}$, $i = 1, 2, \ldots, q$, it is evident that the τ_i are constrained by the implicit inequalities:

$$\left| w(\tau_{2i-2}) + \int_{\tau_{2i-2}}^{t} \text{sgn}\,[\eta(\tau) - \eta(\tau_{2i-1})]\, d\tau \right| < k,\ \tau_{2i-2} < t < \tau_{2i-i} \tag{22}$$

Also the τ_i are constrained from (10) taking note of (14) by:

$$(\tau_{2i} - \tau_{2i-1})\, w(t)\, \eta(t) \begin{bmatrix} b \\ \\ o \end{bmatrix} \geq 0,\ \tau_{2i-1} \leq t \leq \tau_{2i},\ i = 1, 2, \ldots, q \tag{23}$$

where of course $w(t) \equiv w(\tau_{2i})$ and $|w(\tau_{2i})| = k$. From (11), (12), and (13) follows the constraint:

$$\delta\, w(T) [\eta_{n+1}(T) - \psi_{n+1}(T^-)] \geq 0. \tag{24}$$

Now for determining equations there are the obvious equations

$$|w(\tau_{2i-1})| = k, \quad i = 1, 2, \ldots, q. \tag{25}$$

From the continuity of $\psi(t)$ and equations (15) and (16) it follows that:

$$\eta(\tau_{2i}) = \eta(\tau_{2i+1}), \quad i = 1, 2, \ldots, q-2. \tag{26}$$

Also from the continuity of $\psi(t)$ and equations (16) and (17) one can obtain:

$$\eta_{n+1}(\tau_{2q-2}) + \delta [\psi_{n+1}(T^-) - \eta_{n+1}(T)] = 0 \text{ if } \tau_{2q} = T \tag{27}$$

If $\tau_{2q} < T$ one can obtain from (20),(21) and the continuity of $\psi(t)$ at τ_{2q} that $\eta_{n+1}(\tau_{2q}) + \delta [\psi_{n+1}(T^-) - \eta_{n+1}(T)] = 0$. If $\tau_{2q} = T > \tau_{2q-1}$, then $\delta = 1$ and $\psi_{n+1}(T^-) = 0$ so that again $\eta_{n+1}(\tau_{2q}) + \delta [\psi_{n+1}(T^-) - \eta_{n+1}(T)] = 0$.

Hence one obtains the equation:

$$(T - \tau_{2q-1}) \{\eta_{n+1}(\tau_{2q}) + \delta [\psi_{n+1}(T^-) - \eta_{n+1}(T)]\} = 0. \tag{28}$$

If $\tau_{2q-1} < T$ the continuity of $\psi(t)$ at τ_{2q-1} along with equations (16) and (18) implies $\eta(\tau_{2q-2}) = \eta(\tau_{2q-1})$. Hence, one can write the equation:

$$(T - \tau_{2q-1}) [\eta(\tau_{2q-2}) - \eta(\tau_{2q-1})] = 0 \tag{29}$$

The parameter δ is the following function of $T - \tau_{2N}$:

$$\delta = \begin{cases} 0 & \text{if} \quad T - \tau_{2N} > 0 \\ 1 & \text{if} \quad T - \tau_{2N} = 0 \end{cases} \tag{30}$$

Equations (25) through (30) are determining equations for, q, τ_i, δ, and $\psi_{n+1}(T^-) - \eta_{n+1}(T)$ subject to the constraints (22) through (24).

FORMULATION OF A COMPUTATIONAL ALGORITHM

The first computational algorithm is formulated as a nonlinear programming problem. The second algorithm is based on a finite sum approximation of the integrals involved in the first formulation.

Equation (7) can be solved explicitly by determining the eigenvalues and eigenvectors of the matrix A. Hence $\eta_{n+1}(t)$, $0 < t \leq T$ can be assumed known explicitly as a function of t. A value for q can be chosen and then the integral from 0 to T of the product $\eta_{n+1}(t)\, v(t)$ may be obtained as a function of τ_i, i = 1, 2, ,,, 2q, δ, and $\psi_{n+1}(T^-) - \eta_{n+1}(T)$ by making use of equations (8) and (13) through (21). That is,

$$\int_0^T \eta_{n+1}(t)\, v(t)\, dt = J\,[\tau_i,\ \delta,\ \psi_{n+1}(T^-) - \eta_{n+1}(T)\,]. \tag{31}$$

The parameters, τ_i, δ, and $\psi_{n+1}(T^-) - \eta_{n+1}(T)$ are to be constrained by the inequalities

$$\tau_o = 0 < \tau_1 \leq \tau_2 < \ldots < \tau_{2q-1} \leq \tau_{2q} \leq T \tag{32}$$

and the constraints (22) through (30). Then one can maximize J subject to the constraints just cited. This is a mathematical programming problem. If q were not chosen properly no solution would exist and q could be varied until a solution exists.

There are major difficulties present in this method. To obtain J as a function of the parameters shown it is necessary to be able to determine explicitly the zeros of v(t). This is equivalent to solving transcendental equations explicitly which is not generally possible. Also to obtain explicit constraints from (22) through (30) the zeros of v(t) must be known explicitly. Another major problem may be that it is not easy to determine whether a solution exists for a particular value of q.

Because of the difficulties involved in the above method, an approximation to the problem was made which leads to a linear programming problem. In essence, the computational problem is to maximize the functional:

$$I(v) = \int_o^T \eta_{n+1}(t)\, v(t)\, dt \qquad (33)$$

over the function space of all piecewise continuous functions v(t) defined for $0 < t < T$ and satisfying the constraints (5) and (6) for all t in the interval $0 \leq t \leq T$. Basic to the approach is a finite-sum approximation of the integrals, which may be thought of as sampling'' the integrands at a finite number of points. Let m such sampling points, $s_1, s_2, \ldots, s_m$ and m +1 auxiliary points $r_o, r_1, \ldots, r_m$ be chosen to satisfy:

$$0 = r_o < s_1 < r_1 < s_2 < \ldots < s_m < r_m = T \qquad (34)$$

To approximate the integral, $\int_o^t g(s)\,ds$, for a given t, p is chosen so that $|t - r_p| \leq |t - r_i|$ and the integral is approximated by:

$$\sum_{i=1}^{p} g(s_i)\,(r_i - r_{i-1})$$

where g denotes an arbitrary piecewise continuous function.

It is desirable that the points of discontinuity of g are included in the auxiliary points r_i. The points of discontinuity of the integrands of interest are the points of discontinuity of v(t). These points are unknown and hence only estimates of them can be used.

The approximate problem to be solved is to maximize

$$S \equiv \sum_{i=1}^{m} \eta_{n+1}(s_i)\; v(s_i)\; (r_i - r_{i-1}) \tag{35}$$

subjects to the constraints:

$$|v(s_i)| < 1 \quad \text{for } i = 1, 2, \ldots, m \tag{36}$$

$$\left|\sum_{i=1}^{j} v(s_i)\,(r_i - r_{i-1})\right| \leq k \quad \text{for } j = 1, 2, \ldots, m \tag{37}$$

The linear programming formulation of this problem is, Given the points s_i and r_i and the function η_{n+1}, maximize the linear form:

$$L = \sum_{i=1}^{m} \eta_{n+1}(s_i)\,(r_i - r_{i-1})\; x_i \tag{38}$$

in the m unknowns x_i, subject to the 3m linear inequalities:

$$x_j \leq 2 \tag{39}$$

$$\sum_{i=1}^{j} (r_i - r_{i-1})\, x_i \leq k + r_j \tag{40}$$

$$\sum_{i=1}^{j} \left[- (r_i - r_{i-1})\, x_i \right] \leq k - r_j. \tag{41}$$

For this linear programming formulation non-negative unknowns $x_i = v(s_i) + 1$ have been introduced.

A computer program has been written to solve this linear program with a maximum of 14 unknowns with equally spaced sampling and auxiliary times. Results were obtained from this program for an example and they are presented below.

COMPUTER RESULTS

The example chosen is one for which the exact solution is known. The following problem is considered. The system is: $\ddot{x} + x = u,\ \dot{u} = v,\ |v| \leq 1,\ |u| \leq \pi/k$ with $x(0) = \dot{x}(0) = u(0) = 0$

The vector $\eta(T)$ is $[(\pi/k) \cos\theta,\ (\pi/k) \sin\theta,\ (\pi/k) \tan\phi]$. The function $\eta_{n+1}(t)$ is

$$\eta_{n+1}(t) = \cos\theta - \cos(\theta + T - t) + (\pi/k) \tan\phi.$$

Values of k, θ, ϕ, and T are 4, $\pi/8$, 0 and $3\pi/2$ respectively.

The exact solution to this problem is

$$v(t) = \begin{cases} -1 & \text{for } 0 < t < \pi/4 \\ 0 & \text{for } \pi/4 < t < 3\pi/8 \\ +1 & \text{for } 3\pi/8 < t < 7\pi/8 \\ 0 & \text{for } 7\pi/8 < t < 3\pi/2. \end{cases} \tag{42}$$

The value of the integral with this extremal input is $(\pi/4)\cos(\pi/8) + 2\sin(\pi/4)$ which is approximately 2.1398. Approximate solutions to the problem were found by using the computer program with the number of unknowns, m, equal to 8, 9, 10, 11, and 12. The resulting inputs along with a graph of $\eta_{n+1}(t)$ are shown in Figure 12. The input obtained with m = 12 is the same (to nine significant figures) as the input given by (42). This happened because the allowed breaks in v(t) for m = 12 include all the break points of the exact solution. The values obtained for the approximated integral are:

2.1001	for m = 8,
2.1120	for m = 9,
2.1082	for m = 10,
2.1150	for m = 11,
2.1490	for m = 12.

These results indicate that the approximation can be made adequate if enough unknowns are introduced. However, for the present program, the computation time increases rapidly with an increase in the number of unknowns. For example, the computation time necessary to obtain the solutions in this case are:

15 seconds	for m = 8
34 seconds	for m = 9
58 seconds	for m = 10
91 seconds	for m = 11
132 seconds	for m = 12.

Thus for more complicated problems this method may be too costly to use.

SUMMARY

Solutions to the problem of determining extremal bounded amplitude, bounded rate inputs to linear stationary systems are presented. Necessary and sufficient conditions for extremal inputs are given and a set of determining equations are derived from these conditions. Certain constraints are also derived that must be satisfied along with the determining equations.

A computational algorithm is formulated which exhibits what appear to be major computational difficulties. A linear programming formulation of an approximation to the problem is given along with computer results for an example. An adequate approximation can be made with this formulation. But the computation time may become excessive for an adequate approximation.

NASA-Langley, 1966 CR-547

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