Relativistic Bremsstrahlung in a Plasma

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ABSTRACT

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The problem of a test particle moving through a plasma, causing in the process the emission of radiation by the plasma electrons, is treated for relativistic conditions. Expressions for longitudinal and transverse electric field components as well as for the radiation spectrum are derived. It is found that relativistic effects are only quantitative in nature and do not alter the general behavior of the spectrum. In particular, no new resonances are found. If the plasma ensemble is non-relativistic and interacts with a single relativistic test particle, the location of the resonance is the same as in the case of a non-relativistic test particle.

I. INTRODUCTION

There is a number of treatments of bremsstrahlung emission from plasmas available in the literature. Bohm and Pines[1] used the kinetic approach, whereas later Majumdar[2] who also discussed the influence of magnetic fields[3], and Cohen[4] started from the hydrodynamic equations. A recent summary was given by Gerwein and Guernsey[5]. Altschuler[6], and Dupree and Tidman[7] computed the resonance emission in the neighborhood of the plasma frequency.

It is the purpose of this report to investigate the same problem in the relativistic frame work, in particular, the influence of relativistic speeds on the behavior of the resonance location near the plasma frequency. We base our analysis on Altschuler's formalism, since it appears to be the simplest and most straight-forward approach. We will thus obtain expressions for the radiation emitted by a single relativistic test particle in the presence of a plasma. We assume that no external magnetic field is present.

We first review the derivation of the hydrodynamic equations under relativistic conditions (Section II). We then derive the electric field distributions and their spectrum (Sections III and IV). The analysis of the radiation spectrum is given in Section V. Finally, in Section VI the results of the averaging procedure over collision parameters are obtained.

II. THE BASIC EQUATIONS OF A RELATIVISTIC FLUID

We denote the spatial coordinates by x_1, x_2, x_3, and let $x_4 = ict$. The summation convention is used, Latin subscripts take the values 1, 2, 3, 4; Greek subscripts the values 1, 2, 3.

Since in a fluid the number of particles must be conserved (disregarding ionization and recombination processes), we have

$$\partial n/\partial t + \partial(nv_\alpha)/\partial x_\alpha = 0, \qquad (1)$$

where n is the particle density, v_α the fluid velocity vector. The covariant formulation of Eq. (1) can be written as

$$\partial(\mu^\circ V_i)/\partial x_i = 0, \; \mu^\circ = n^\circ m_o. \qquad (2)$$

μ° is the proper density of proper mass, n° is the proper number density. m_o is the rest mass of each particle, and V_i is the usual relativistic four-velocity vector defined as

$$V_i = [\; v_\alpha/\sqrt{(1-v^2/c^2)}, \;\; ic/\sqrt{(1-v^2/c^2)}\;]. \qquad (3)$$

Furthermore,

$$n^\circ = n\,\sqrt{(1-v^2/c^2)}. \qquad (4)$$

The set of equations describing conservation of momentum and energy in a relativistic fluid in the presence of an electromagnetic field reads in usual notation

$$\partial(T^{(1)}_{ik} + T^{(2)}_{ik})/\partial x_k = 0, \qquad (5)$$

where[8 9]

$$T^{(1)}_{ik} = [\mu^\circ + \mu^\circ\varepsilon^\circ/c^2 + p/c^2]v_i v_k \; + p\delta_{ik}\,, \qquad (6)$$

and

$$T^{(2)}_{\alpha\beta} = (1/8\pi)\;[E^2 + B^2] - (1/4\pi)\;[E_\alpha E_\beta + B_\alpha B_\beta], \qquad (7)$$

$$T^{(2)}_{\alpha 4} = (i/4\pi)\ (\vec{E}\times\vec{B})_\alpha = T^{(2)}_{4\alpha}\ ,\ T^{(2)}_{44} = -\ (1/8\pi)\ (E^2 + B^2). \tag{8}$$

Here, p is the pressure, ε^o is the internal energy per unit proper mass in a reference frame which moves with the local velocity of the fluid, $\vec{E}$ and $\vec{B}$ are electric and magnetic field vectors.

For the total energy and momentum tensor of the electromagnetic field as given by Eqs. (7) and (8), it follows[10]

$$\partial T^{(2)}_{ik}/\partial x_k = -f_i\ , \tag{9}$$

where f_i is the Lorentz force density written as four-dimensional vector, viz.,

$$f_i = F_{ij}\ J_j/c, \tag{10}$$

with

$$F_{ij} = \partial A_i/\partial x_j - \partial A_j/\partial x_i,\ J_j = (J_\alpha,\ icen^o),\ A_i = (A_\alpha,\ i\phi). \tag{11}$$

J_α and en^o are current density vector and charge density. A_α and ϕ are vector and scalar potentials.

Combining Eqs. (5) and (9), we obtain

$$\partial T^{(1)}_{ik}/\partial x_k = f_i\ . \tag{12}$$

The four equations represented by the tensor equation (12) describe conservation of energy and momentum in a relativistic plasma in the presence of an electromagnetic field. They are invariant with respect to Lorentz transformations. The four equations can be separated in the following two sets of equations:

$$\partial(L\gamma^2 v_\alpha v_\beta)/\partial x_\alpha + \partial(L\gamma^2 v_\beta)/\partial t = -\ \partial p/\partial x_\beta - en[E_\beta + (\vec{v}\times\vec{\beta})_\beta/c], \tag{13}$$

$$\partial(L\gamma^2 v_\alpha)/\partial x_\alpha + \partial(L\gamma^2)/\partial t = \partial p/c^2\partial t - en(E_\alpha v_\alpha)/c^2 \; , \tag{14}$$

where

$$L = \mu^\circ[1 + (\Gamma/\Gamma-1)\;(p/\mu^\circ c^2)], \; \gamma = 1/\sqrt{(1-v^2/c^2)}, \; \varepsilon^\circ = p/(\Gamma-1)\mu^\circ. \tag{15}$$

Γ is the adiabatic coefficient.

To linearize Eqs. (13) and (14), we define

$$v_\alpha = 0 + v_\alpha, \; p = p_o + P, \; n = n_o + N, \; E_\alpha = 0 + E_\alpha \;, \; B_\alpha = 0 + B_\alpha \;, \tag{16}$$

where p_o and n_o denote the uniform pressure and density distribution of the system in equilibrium. Together they describe the relativistic thermal motions in the plasma. Using Eqs. (15) and (16) in Eq.(13), we obtain

$$m_o a^2 n_o\;(\partial v_\beta/\partial t) = - e n_o E_\beta - \partial P/\partial x_\beta, \tag{17}$$

where

$$a^2 = 1 + (\Gamma/\Gamma-1)\;(V^2/c^2), \; V^2 = kT/m_o. \tag{18}$$

V is the thermal plasma speed. The parameter a is due to the relativistic temperature of the plasma. For a non-relativistic ensemble, a = 1. It is worth noting that definition (18) does not hold in an external magnetic field that would introduce additional terms.

Added to the equations of motion are the linearized Maxwell Equations for the field vectors $\vec{E}$ and $\vec{B}$, including a uniformly moving test particle:

$$\nabla\times\vec{E} = - \partial\vec{B}/c\partial t, \tag{19}$$

$$\nabla\times\vec{B} = \partial\vec{E}/c\partial t - 4\pi e n_o\vec{v}/c + 4\pi q\;(\vec{u}/c)\;\delta(\vec{r}-\vec{u}t), \tag{20}$$

$$\nabla\times\vec{E} = -4\pi eN + 4\pi q\ \delta(\vec{r}-\vec{u}t), \tag{21}$$

$$\nabla\times\vec{B} = 0, \tag{22}$$

$$\delta(\vec{r}-\vec{u}t) = \delta(x_1-ut)\cdot\delta(x_2)\cdot\delta(x_3)\ . \tag{23}$$

U is the uniform speed of the test particle moving parallel to the x_1-axis. The test particle is represented by the two terms containing δ-functions on the right of Eqs. (20) and (21). Their invariance properties are discussed by Landau and Lifshitz[11]. Eqs. (11), (17), (19) to (22) are the basic equations of our problem. However, Eqs. (1) and (22) are not independent, but can be obtained from Eqs. (19) to (21). Thus, the set of basic equations in the absence of initial plasma motions reduces to (17), (19), (20), and (21).

The relativistic changes are twofold: if the mean speed of the ensemble is relativistic, $a^2 \neq 1$, and may go as high as 5 for $v \rightarrow c$ and $\Gamma \rightarrow 4/3$. We have carried a^2 along to document the influence of relativistic energies of the ensemble on the structure of the equations, although classical radiation theory breaks down in this domain as can be seen, for instance, from the behavior of the bremsstrahlung cross sections for temperatures of the order 1 keV and above; c.f. Oster[12].

Of major importance for our purposes is the case where the ensemble is non-relativistic, i.e., $a^2 \approx 1$, but the test particle has a relativistic speed $u \rightarrow c$. In comparison with Altschuler's treatment, terms of order u^2/c^2 may not be neglected any longer and in particular the transverse electric field derived in the next Section must be included in the spectrum calculations.

III. SPATIAL ELECTRIC FIELD DISTRIBUTIONS

Electric field and charge density distributions are computed with the aid of the well-known methods of Fourier transforms[6]:

$$\vec{F}(\vec{r},t) = (2\pi)^{-2} \iiiint_{-\infty}^{+\infty} \vec{F}_k(\vec{k},\omega) \exp[i(\vec{k}\cdot\vec{r}-\omega t)]\, d\vec{k}\, d\omega, \tag{24}$$

$$\delta(\vec{r}-\vec{u}t) = (2\pi)^{-2} \iiiint_{-\infty}^{+\infty} \delta(\omega-\vec{k}\cdot\vec{u}) \exp[i(\vec{k}\cdot\vec{r}-\omega t)]\, d\vec{k}\, d\omega \quad , \tag{25}$$

where $d\vec{k} = dk_x dk_y dk_z$. The vector $\vec{k}$ is the wave vector, ω is the angular frequency.

Applying the Fourier transformations (24) and (25) on Eqs. (17), (19), (20), and (21) results in

$$i\omega a^2\vec{v} = e\vec{E}/m + v^2 ikN/n_o, \tag{26}$$

$$\vec{k}\times\vec{E} = \omega\vec{B}/c, \tag{27}$$

$$i\vec{k}\times\vec{B} = - i\omega\vec{E}/c - 4\pi e n_o\vec{v}/c + 2q\vec{u}\ \delta(\omega-\vec{k}\cdot\vec{u})/c \ , \tag{28}$$

$$i\vec{k}\cdot\vec{E} = - 4\pi eN + 2q\ \delta(\omega-\vec{k}\cdot\vec{u}) \ . \tag{29}$$

It is readily seen that aside from the appearance of a^2 in Eq.(26) the relativistic case is formally identical to the non-relativistic case.

These four vector equations constitute ten linear equations. In writing these equations we assume that the wave vector $\vec{k}$ is parallel to the x_1-axis. We now split the electric field into a longitudinal component parallel to the propagation vector $\vec{k}$, and a transverse field component (E_2) perpendicular to $\vec{k}$. We then find

$$\vec{E}_1(\vec{k},\omega) = 2qi \frac{\vec{k}\,[(\vec{k}\cdot\vec{u})\omega - k^2v^2]}{k^2[\omega_p^2 + v^2k^2 - \omega^2a^2]}\,\delta(\omega-\vec{k}\cdot\vec{u})\ , \tag{30}$$

$$\vec{E}_2(\vec{k},\omega) = 2qi \frac{a^2\omega[k^2\vec{u} - \vec{k}(\vec{k}\cdot\vec{u})]}{k^2[\omega_p^2 + a^2(k^2c^2-\omega^2)]}\,\delta(\omega-\vec{k}\cdot\vec{u}), \tag{31}$$

$$N(\vec{k},\omega) = (q/2\pi e)\,\frac{\omega_p^2 + (\vec{k}\cdot\vec{u})\omega - \omega^2a^2}{\omega_p^2 + k^2v^2 - \omega^2a^2}\,\delta(\omega-\vec{k}\cdot\vec{u}), \tag{32}$$

where

$$\omega_p = 4\pi e^2 n_o/m_o \tag{33}$$

represents the plasma frequency as in the non-relativistic case. Note, however, that m_o is the rest mass. The vectors $\vec{E}_1(\vec{k},\omega)$ and $\vec{E}_2(\vec{k},\omega)$ denote the Fourier components of the longitudinal and transverse electric field components. The corresponding spatial components can be written down by using Eqs. (24) and (25), viz.,

$$\vec{E}_1(\vec{r},t) = (q/4\pi^3) \iiint_{-\infty}^{+\infty} \frac{i\vec{k}[(\vec{k}\cdot\vec{u})^2 - k^2v^2]}{k^2[\omega_p^2 + k^2v^2 - a^2(\vec{k}\cdot\vec{u})^2]}\,\exp[i\vec{k}\cdot(\vec{r}-\vec{u}t)]\,d\vec{k}, \tag{34}$$

$$\vec{E}_2(\vec{r},t)\ (qa^2/4\pi^2) \iiint_{-\infty}^{+\infty} \frac{i(\vec{k}\cdot\vec{u})[k^2\vec{u} - \vec{k}(\vec{k}\cdot\vec{u})]}{k^2[\omega_p^2 + a^2k^2c^2 - a^2(\vec{k}\cdot\vec{u})^2]} \times$$

$$\times \quad \exp[i\vec{k}\cdot(\vec{r}-\vec{u}t)\}d\vec{k}\ , \tag{35}$$

$$N(\vec{r},t) = (q/8\pi^2 e) \iiint_{-\infty}^{+\infty} \frac{\omega_p^2 + (\vec{k}\cdot\vec{u})^2 - (\vec{k}\cdot\vec{u})^2a^2}{\omega_p^2 + k^2v^2 - a^2(\vec{k}\cdot\vec{u})^2}\,\exp[i\vec{k}\cdot(\vec{r}-\vec{u}t)\}d\vec{k}. \tag{36}$$

In deriving these equations we have integrated over frequency.

Eqs. (34) - (36) replace the corresponding non-relativistic set. Again, we find that for $a^2 = 1$ the only difference is the requirement to retain the transverse field E_2 from Eq.(35).

IV. SPECTRUM COMPONENTS

In the last Section longitudinal and transverse field components were derived. Using these expressions, we now obtain the spectrum components needed to computed the emission coefficients.

It follows from the theory of Fourier transforms that we can write for Eq.(24)

$$\vec{F}(\vec{r},\omega) = 1/(2\pi)^{-3/2} \iiint \vec{F}(\vec{k},\omega)\, \exp(i\vec{k}\cdot\vec{r})\, d\vec{k} \,. \tag{37}$$

Then, from Eqs. (30) and (31), we get

$$\vec{E}_1(\vec{r},\omega) = 2qi/(2\pi)^{3/2} \iiint \frac{\vec{k}[\omega(\vec{k}\cdot\vec{u}) - k^2v^2]}{k^2[\omega_p{}^2 + v^2k^2 - \omega^2a^2]} \times$$

$$\times \quad \exp(i\vec{k}\cdot\vec{r})\, \delta(\omega-\vec{k}\cdot\vec{u})\, d\vec{k} \,, \tag{38}$$

$$\vec{E}_2(\vec{r},\omega) = 2qia^2\omega/(2\pi)^{3/2} \iiint \frac{[k^2\vec{u} - \vec{k}(\vec{k}\cdot\vec{u})]}{k^2[\omega_p{}^2 + a^2(k^2c^2-\omega^2)]} \times$$

$$\times \quad \exp(i\vec{k}\cdot\vec{r})\, \delta(\omega-\vec{k}\cdot\vec{u})\, d\vec{k}. \tag{39}$$

Eqs. (38) and (39) are written for a test particle moving parallel to the x_1-axis. Let $\hat{x}_o$, $\hat{y}_o$, $\hat{z}_o$ denote the unit vectors along the three axes of a Cartesian coordinate system. Then, Eqs. (38) and (39) give

$$\vec{E}_1(\vec{r},\omega) = 2qi/u(2\pi)^{3/2} \iint \frac{[\omega\hat{x}_o/u + k_y\hat{y}_o + k_z\hat{z}_o]}{[\omega^2/u^2 + k_y{}^2 + k_z{}^2]} \times$$

$$\times \frac{[\omega^2(u^2-v^2) - (k_y{}^2+k_z{}^2)u^2v^2]/u^2}{[\omega_p{}^2 - \omega^2(a^2u^2 - v^2)/u^2 + (k_y{}^2 + k_z{}^2)v^2]} \exp[i(\omega x/u + k_y y + k_z z)]dk_y dk_z \quad (40)$$

$$\vec{E}_2(\vec{r},\omega) = 2qa^2 i\omega/u(2\pi)^{3/2} \times$$

$$\times \iint \frac{[(\omega^2/u^2 + k_y{}^2 + k_z{}^2)u\hat{x}_o - (\hat{x}_o\omega/u + k_y\hat{y}_o + k_z\hat{z}_o)\omega]}{[\omega^2/u^2 + k_y{}^2 + k_z{}^2]\,[\omega_p{}^2 + a^2c^2(k_y{}^2+k_z{}^2) + a^2\omega^2(c^2-u^2)/u^2} \times$$

$$\times \quad \exp[i(\omega x/u + k_y y + k_z z)]\, dk_y\, dk_z \quad . \quad (41)$$

In these equations, k_x, k_y, k_z are the three components of the wave vector $\vec{k}$, x, y, z are the three components of the vector $\vec{r}$. In deriving Eqs. (40) and (41) we have used the identity

$$\delta(\omega-k_x u) = u^{-1}\,\delta(k_x-\omega/u) \quad . \quad (42)$$

We now take our electron to be fixed in the plasma, i.e., we make the straight-line approximation[10]. Let the electron thus be at the point (o,b,0). Then, we can write for Eq.(40):

$$E_{1x}\,(b,\omega) = -\,[2qi\omega/u^2(2\pi)^{3/2}]\; I_1 \;, \quad (43)$$

$$E_{1y}\,(b,\omega) = -\,[2q/u(2\pi)^{3/2}]\;\partial I_1/\partial b, \quad (44)$$

$$E_{1z}\,(b,\omega) = 0 \;, \quad (45)$$

whereas for the transverse components from Eq.(41) we get

$$E_{2x}(b,\omega) = -[2qa^2 i\omega/(2\pi)^{3/2}]\{I_2 - (\omega^2/u^2)I_3\} \quad , \tag{46}$$

$$E_{2y}(b,\omega) = -[2qa^2\omega^2/u(2\pi)^{3/2}]\,\partial I_3/\partial b, \tag{47}$$

$$E_{2z}(b,\omega) = 0. \tag{48}$$

The integrals I_1, I_2, I_3 are defined by

$$I_1 = \iint_{-\infty}^{+\infty} [1 - \{\omega_p^2 - \omega^2(a^2-1)\}/\{\omega_p^2 - (\omega^2/u^2)(a^2u^2-v^2) + (k_y^2+k_z^2)\}] \times$$

$$\times \exp(ibk_y)\,[\omega^2/u^2 + k_y^2 + k_z^2]^{-1}\,dk_y\,dk_z, \tag{49}$$

$$I_2 = \iint_{-\infty}^{+\infty} [\omega_p^2 + a^2c^2(k_y^2+k_z^2) + a^2\omega^2(c^2-u^2)/u^2]\exp(ibk_y)\,dk_y\,dk_z, \tag{50}$$

$$I_3 = \iint_{-\infty}^{+\infty} \{[\omega^2/u^2+k_y^2+k_z^2]\,[\omega_p^2+a^2c^2(k_y^2+k_z^2)+a^2\omega^2(c^2-u^2)/u^2]\}^{-1} \times$$

$$\times \exp(ibk_y)\,dk_y\,dk_z\,. \tag{51}$$

b is the impact parameter.

The integrals (49) - (51) are discussed in the Appendix. Their solutions are grouped in the following according to the cases $au < v$ and $au > v$.

CASE (1): $au < v$

In this case the integrals (43) to (45) can be solved by contour integration; c.f. Appendix. On solving we obtain

$$E_{1x}(b,\omega) = -\sqrt{(2/\pi)}\,[qi\omega/u^2(\omega^2-\omega_{p1}^2)]\,\{\omega^2 K_o(b\omega/u) - \omega_{p1}^2 K_o(b\lambda_1)\} \quad , \tag{52}$$

$$E_{1y}(b,\omega) = \sqrt{(2/\pi)}\ [q/u(\omega^2-\omega_{pl}^2)]\ \{(\omega^3/u)\ K_1(b\omega/u) - \omega_{pl}^2\lambda_1\ K_1(b\lambda_1)\quad , \qquad (53)$$

where

$$\lambda_1^2 = \omega_p^2/v^2 + \omega^2[(1/u^2) - (a^2/v^2)] \qquad (54)$$

and

$$\omega_{pl}^2 = \omega_p^2 - \omega^2(a^2-1). \qquad (55)$$

K_o and K_1 are modified Bessel functions of the second kind. At $\omega^2 = \omega_p^2/a^2$, $E_{1x}(b,\omega)$ and $E_{iy}(b,\omega)$ become indeterminate, but are continuous according to L'Hospital's rule. For a non-relativistic ensemble. Eqs. (52) and (53) reduce to Altschuler's equations[6], since no relativistic terms are left.

Eqs. (52) and (53) are valid for frequencies that satisfy the conditions $\lambda_1^2 > 0$, or

$$[\omega^2/\omega_p^2]\ (a^2 - v^2/u^2) < 1\ . \qquad (56)$$

This inequality is always satisfied for au < v. Hence, it holds for the whole spectrum. If au > v, it is valid if

$$\omega < \omega_p/(a^2 - v^2/u^2)^{1/2}\ , \qquad (57)$$

i.e., only for frequencies below the resonance to be derived presently.

CASE (2): au > v

In this case, the solutions of Eqs. (43) and (44) are:

$$E_{1x}(b,\omega) = -\sqrt{(2/\pi)}\ [q\omega/u^2(\omega^2-\omega_{pl}^{\ 2})] \quad \times$$

$$\times \quad (\pi/2)\omega_{pl}^{\ 2}J_o(b\zeta) + i[\omega^2 K_o(b\omega/u) + (\pi/2)\omega_{pl}^{\ 2}Y_o(b\zeta)] \quad , \qquad (58)$$

$$E_{1y}(b,\omega) = \sqrt{(2/\pi)}\ [q/u(\omega^2-\omega_{pl}^{\ 2})] \quad \times$$

$$\times \quad (\omega^3/u)K_o(b\omega/u) + (\pi/2)\omega_{pl}^{\ 2}\zeta Y_1(b\zeta) - i(\pi/2)\omega_{pl}^{\ 2}\zeta J_1(b\zeta) \quad , \qquad (59)$$

where

$$\zeta^2 = \omega^2[a^2/v^2 - 1/u^2] - \omega_p^{\ 2}/v^2 \ . \qquad (60)$$

J_o, J_1, and Y_o, Y_1 are Bessel functions of first and second kind. Eqs. (59) and (58) are valid for $\zeta^2 > 0$. If we define a resonance frequency ω_r by

$$\omega_r = \omega_p/\sqrt{(a^2 - v^2/u^2)}, \qquad (61)$$

then we see that Eqs. (51) and (52) are valid for $\omega < \omega_r$, and Eqs. (58) and (59) are valid for frequencies $\omega \geq \omega_r$. As compared with the case $a = 1$, a relativistic mean speed thus merely reduces the resonance frequency by an insignificant amount.

The solution for the transverse component, Eqs. (46) and (47), is independent of the relative size of au and v. It reads

$$E_{2x}(b,\omega) = -\sqrt{(2/\pi)}\ [qa^2 i\omega/(\omega^2-\omega_{pl}^{\ 2})]\{(\omega^2/u^2)K_o(b\omega/u) - \lambda_2^{\ 2}K_o(b\lambda_2)\}, \qquad (62)$$

$$E_{2y}(b,\omega) = \sqrt{(2/\pi)}\ [qa^2 i\omega^2/u(\omega^2-\omega_{pl}^{\ 2})]\{\lambda_2 K_1(b\lambda_2) - (\omega/u)K_1(b\omega/u) \quad , \qquad (63)$$

where

$$\lambda_2^{\ 2} = \omega^2[1/u^2 - 1/c^2] + \omega_p^{\ 2}/c^2a^2 = \omega^2/u^2\gamma^2 + \omega_p^{\ 2}/c^2a^2 \ . \qquad (64)$$

Again it is easy to show that Eqs. (62) and (63) are continuous at $\omega = \omega_p$, and at $\omega = \omega_p/a$. Note that the transverse field was neglected in Altschuler's non-relativistic limit.

V. THE RADIATION SPECTRUM

We are now ready to compute the radiation emitted by the plasma. The conceptual approach, as outlined by Altschuler[6], is to consider an arbitrary electron interacting with the combined field of the (positive) test particle and its shielding cloud, as expressed by the fields which we derived in the previous Sections.

The acceleration of the electron at position $\vec{r}$ and time t due to the electric field $\vec{E}(\vec{r},t)$ is

$$\vec{a}(b,\omega) = -(e/m)\,\vec{E}(b,\omega). \tag{65}$$

Then, the spectrum function $Q_\omega(b,\omega)$ follows from Larmor's formula

$$Q_\omega(b,\omega) = (4e^2/3c^3)[\vec{a}^*(b,\omega)\cdot\vec{a}(b,\omega)], \tag{66}$$

where starred quantities are complex conjugates. Eq.(66) was also used by Altschuler in his non-relativistic derivation. Since we have assumed that the particle is in that system of reference in which it is at rest at a given moment, Eq.(66) still holds in our relativistic context.

Eq.(66) can be justified in still another way. In the reference frame in which the electron is initially at rest, and the test particle moves by with a speed $u \sim c$, the corresponding motion is non-relativistic throughout the interaction. This means that in this frame of reference the radiation process can be treated non-relativistically; c.f. Jackson, p.514[10]. Thus, relativistic effects will enter through the transverse component of the electric field and, possibly, relativistic thermal motions.

From Eqs. (65) and (66) we obtain

$$Q_\omega(b,\omega) = (4e^4/3c^3m^2)\ [\vec{E}_1(b,\omega)\cdot\vec{E}_1{}^*(b,\omega) + \vec{E}_2(b,\omega)\cdot\vec{E}_2{}^*(b,\omega)], \qquad (67)$$

where we have substituted for the electric field its longitudinal (E_1) and transverse components.

As one expects, the spectrum function depends on whether $u < (v/a)$ (subsonic case), or $u > v/a$ (supersonic case). We now discuss these two cases separately.

(1) SUBSONIC TEST PARTICLE ($u < v/a$)

To obtain the spectrum function in this case, we substitute Eqs. (52), (53), (63) and (64) into Eq.(67) and find

$$Q_\omega(b,\omega) = (8e^2/3\pi c)\ (qe/mc^2)^2[c^2/u^2(\omega^2-\omega_{pl}{}^2)^2] \quad \times$$

$$\times\ \{(\omega^2/u^2)\{\omega^2K_o(b\omega/u) - \omega_{pl}{}^2K_o(b\lambda_1)\}^2 + \{(\omega^3/u)K_1(b\omega/u) -$$

$$-\ \omega_{pl}{}^2\lambda_1K_1(b\lambda_1)\}^2 + a^4\omega^2u^2\{(\omega^2/u^2)\ K_o(b\omega/u) - \lambda_2{}^2K_o(b\lambda_2)\}^2 +$$

$$+\ \omega^4a^4\{\lambda_2K_1(b\lambda_2) - (\omega/u)\ K_1(b\omega/u)\}^2\} \ . \qquad (68)$$

The first two brackets on the right-hand side denote the terms due to the longitudinal electric field distribution, the last two brackets are due to the transverse electric field. Again, we have two relativistic effects present in Eq.(68): one is given by the appearance of the factor a that describes relativistic thermal motions, whereas the second one refers to the test particle and manifests itself by the fact that the argument of the Bessel functions derived from the transverse electric field depends on $\gamma = 1/\sqrt{(1-u^2/c^2)}$; c.f. Eq.(64).

It is convenient to express Eq.(68) in terms of the following dimensionless parameters:

$$\Omega = \omega/\omega_p; \quad \eta = u/v; \quad \beta = u/c; \quad \lambda = b/(v/\omega_p) = b/\lambda_d, \tag{69}$$

where λ_d is the Debye-length. Then, Eq.(68) takes on the form

$$Q_\Omega(\lambda,\Omega) = (8e^2/3c\pi)\ (qe/mc^2)^2\ [c^2\omega_p{}^2/v^4\eta^4(a^2\Omega^2-1)^2] \quad \times$$
$$\times \quad \{\Omega^2\{\Omega^2 K_o(\lambda\Omega/\eta) - \xi_2{}^2 K_o(\lambda\xi_1/\eta)\}^2 + \{\Omega^3 K_1(\lambda\Omega/\eta) - \xi_2{}^2\xi_1 K_1(\lambda\xi_1/\eta)\}^2 +$$
$$+ \ a^4\Omega^2\{\Omega^2 K_o(\lambda\Omega/\eta) - \xi_4{}^4 K_o(\lambda\xi_4/\eta)\}^2 + \Omega^4 a^4\{\xi_4 K_1(\lambda\xi_4/\eta) - \Omega K_1(\lambda\Omega/\eta)\}^2\} \ , \tag{70}$$

where

$$\xi_1 = \sqrt{[\eta^2 + \Omega^2(1-a^2\eta^2)]}, \tag{71}$$

$$\xi_2 = \sqrt{[1 - \Omega^2(a^2-1)]}, \tag{72}$$

$$\xi_4 = \sqrt{[\Omega^2(1-\beta^2) + \beta^2/a^2]}. \tag{73}$$

(2) SUPERSONIC TEST PARTICLE ($u > v/a$)

Int his case, we insert the supersonic solutions of the longitudinal electric field, that is, Eqs. (58) and (59), into Eq.(67), and obtain the spectrum function

$$Q_\omega(b,\omega) = (8e^2/3\pi c)\ (qe/mc^2)^2[c^2/u^2(\omega^2-\omega_{pl}{}^2)] \quad \times$$

$$\times \quad \{(\pi^2/4)\ (\omega^2\omega_{pl}{}^4/u^2)\{J_o(b\zeta)\}^2 + (\omega^2/u^2)\{\omega^2K_o(b\omega/u) + (\pi\omega_{pl}{}^2/2)Y_o(b\zeta)\}^2 +$$

$$+ \quad \{(\omega^3/u)K_1(b\omega/u) + (\pi\omega_{pl}{}^2/2)\zeta Y_1(b\zeta)\}^2 + (\pi^2\omega_{pl}{}^4/4)\zeta^2\{J_1(b\zeta)\}^2 \quad +$$

$$+ \ \omega^2u^2a^4\{(\omega^2/u^2)K_o(b\omega/u) - \lambda_2{}^2K_o(b\lambda_2)\}^2 +$$

$$+ \quad \omega^4\ a^4\{\lambda_2K_1(b\lambda_2) - (\omega/u)K_1(b\omega/u)\}^2\} \ . \qquad (74)$$

Again, the last two brackets on the right are due to the transverse electric field distribution and contain the relativistic effects of the test particle's motion. All other terms are derived from the longitudinal field distribution.

Rewriting Eq.(74) in terms of dimensionless parameters results in

$$Q_\Omega(\lambda,\Omega) = (8e^2/3\pi c)\ (qe/mc^2)^2[c^2\omega_p{}^2/v^4\eta^4(a^2\Omega^2-1)^2] \quad \times$$

$$\times \quad \{(\pi^2\Omega^2/4)\{J_o(\lambda\xi_3/\eta)\}^2 + \ \Omega^2\{\Omega^2K_o(\lambda\Omega/\eta) + (\pi/2)\xi_2{}^2Y_o(\lambda\xi_3/\eta)\}^2 +$$

$$+ \quad \{\Omega^3\ K_1(\lambda\Omega/\eta) + (\pi/2)\xi_2{}^2\ \xi_3Y_1(\lambda\xi_3/\eta)\}^2 + (\pi^2/4)\xi_2{}^4\xi_3{}^2\{J_1(\lambda\xi_3/\eta)\}^2 +$$

$$+ \quad a^4\Omega^2\{\Omega^2K_o(\lambda\Omega/\eta) - \xi_4{}^2K_o(\lambda\xi_4/\eta)\}^2 +$$

$$+ \quad a^4\Omega^4\{\xi_4K_1(\lambda\xi_4/\eta) - \Omega K_1(\lambda\Omega/\eta)\}^2\} \quad , \qquad (75)$$

where

$$\xi_3 = \sqrt{[\Omega^2(a^2\eta^2-1) - \eta^2]} \ . \qquad (76)$$

The other terms are already defined.

The two spectrum functions (70) and (75) are the generalized form of Altschuler's[6] equations. For non-relativistic situations, i.e., for $a = 1$

and $\beta \ll 1$ or $\gamma = 1$, they reduce to Altschuler's corresponding equations. If we further assume $\omega_p = 0$, that is, in the absence of plasma, both Eqs. (70) and (75) give the same spectrum function, viz.,

$$Q_\Omega(\lambda,\Omega) = (8e^2/\,3\pi c)\ (qe/mc^2)^2(c^2\omega^2/u^4)[\{K_0(\lambda\Omega/\eta)\}^2 + \{K_1(\lambda\Omega/\eta)\}^2]. \qquad (77)$$

Eq.(77) agrees term by term with Oster's result[11].

For applications, the collision parameter must be integrated out. We thus discuss the various subcases after introducing this "radiation cross section" in the next Section.

VI. RELATIVISTIC BREMSSTRAHLUNG CROSS SECTIONS

So far, we have derived the radiation spectrum for one fixed position of the radiating electron with respect to the test particle. The average radiation is found from the radiation cross section $\chi(\omega)$ as defined by [Jackson[10]]

$$\chi(\omega) = 2\int_{b(\min)}^{b(\max)} bQ_\omega(b,\omega)\,db, \qquad (78)$$

where b(min) and b(max) are minimum and maximum impact parameters. The physical dimension of $\chi(\omega)$ is erg cm^2 / sec^{-1}.

We will obtain the radiation cross section for the following cases: (1) in the absence of plasma, (2) in the presence of plasma, subsonic case, (3) in the presence of plasma, supersonic case. In each of these cases, the limits of high and low frequency will be discussed.

(1) IN THE ABSENCE OF PLASMA

The only relativistic effect is the relativistic motion of the ion. Both Eqs. (68) and (74) give for $a = 1$ and $\omega_p = 0$

$$Q_\omega(b,\omega) = (8e^4/3c^3m^2)\ (q^2\omega^2/u^4)\ [\{K_o(b\omega/u)\}^2 + \{K_1(b\omega/u\ )\}^2 +$$

$$+ \quad \{K_o(b\omega/u) - \gamma^{-2}K_o(b\omega/u\gamma)\}^2 + \{\gamma^{-1}K_1(b\omega/\gamma u) - K_1(b\omega/u)\}^2], \qquad (79)$$

where

$$\gamma = 1/\sqrt{(1 - u^2/c^2)} \qquad (80)$$

as before. The first two terms on the right which are due to the longitudinal field distribution do not contain relativistic quantities. The remainder of the two brackets contains the influence of the transverse electric field and, thus, the relativistic parameter γ from the motion of the ion. In the non-relativistic limit $\gamma = 1$, Eq.(79) reduces to (77).

Upon multiplying both sides by $2\pi b db$ and integrating we get

$$\chi(\omega) = (16e^2/3c)\ (qe/mc^2)^2\ (e^2\omega^2/u^4) \quad \times$$

$$\times \quad [(1-1/\gamma^2)\ (b_o{}^2/2\gamma^2)\{K_o{}^2(b_o\omega/\gamma u) - K_1{}^2(b_o\omega/\gamma u)\} +$$

$$+ \quad (b_o\gamma u/\omega\gamma^2)K_o(b_o\omega/\gamma u)K_1(b_o\omega/\gamma u) +$$

$$+ \quad (2b_o u/\omega)K_o(b_o\omega/u)K_1(b_o\omega/u) - (2b_o u/\gamma\omega)K_o(\omega b_o/u)K_1(b_o\omega/\gamma u)], \qquad (81)$$

where we have set b(min) equal to the 90^o-deflection parameter b_o, as is customary in the straight-line approximation [Oster[12]]. Since $b(max) \to \infty$, we have $\chi(\omega) \to 0$ at the upper boundary. Eq.(81) is the general form of the radiation cross section in the absence of plasma.

Our straight-line approximation is most accurate at low frequencies, in particular, at radio frequencies. For small arguments the modified Bessel-functions $K_o(x)$ and $K_1(x)$ can be approximated by

$$K_o(x) \propto \ln(2/\gamma_2 x), \; K_1(x) \propto 1/x; \quad x << 1, \tag{82}$$

where $\gamma_2 = 1.756...$ is Euler's constant.

Inserting (82) into Eq.(81) leads to

$$\chi(\omega) = (16e^2/3c)\;(qe/mc^2)^2\;(e^2/u^2) \times$$

$$\times \;\{\ln(2\gamma u/b_o\omega\gamma_2) - \tfrac{1}{2}(1-1/\gamma^2) + \tfrac{1}{2}(1-1/\gamma^2)\;(\omega^2 b_o^2/\gamma^2 u^2)[\ln(2\gamma u/b_o\omega\gamma_2)]^2\}. \tag{83}$$

Equation (83) denotes the radiation cross section for low frequencies, to be precise, for $\omega << \gamma u/b_o$. Using the definition of γ in Eq.(83), we get

$$\chi(\omega) = A\;[(c^2/u^2)\ln x - 1/2 + (\ln x)^2/2x^2], \tag{84}$$

where

$$A = (16e^2/3c)\;(qe/mc^2)^2, \quad x = 1.123\; u/b_o\omega\gamma_2\;. \tag{85}$$

Since Eq.(83) holds for $x >> 1$, where x is defined by Eq.(85), we see that the third term on the right-hand side of Eq.(84) is small for all practical purposes. This behavior of $\chi(\omega)$ vs. $\omega b_o/\gamma u$ agrees with Jackson's result (p.524)[10].

In the non-relativistic case, where $\gamma = 1$, finally, Eq.(83) gives

$$\chi(\omega)=(16e^2/3c)\;(qe/mc^2)^2\;(c^2/u^2)\;\ln(2u/b_o\omega\gamma_2) \tag{86}$$

and agrees with Jackson's formula (p.511)[10] by taking $b_o = \hbar/mu$.

For high frequencies ($\omega >> \gamma\, u/b_o$), Eq.(81) gives

$$\chi(\omega >> \gamma u/b_o) = (16\pi e^2/3c)\ (qe/mc^2)^2\ (c^2/u^2)\ \times$$
$$\times\ [(3/2)\exp(-2b_o\omega/u) - \gamma^{-1/2}\exp[-2b_o(\omega/u)(1+1/\gamma)], \qquad (87)$$

where we have used the following asymptotic values for $K_o(x)$ and $K_1(x)$:

$$K_o(x) \simeq K_1(x) \simeq (\pi/2x)\ e^{-x}, \quad x >> 1. \qquad (88)$$

For non-relativistic situations, formula (87) reduces to

$$\chi(\omega) = (8e^2\pi/3c)\ (qe/mc^2)^2\ (c^2/u^2)\ \exp[-2b_o\omega/u], \qquad (89)$$

which agrees with Altschuler's result[6].

The discussion shows that our results reduce in the absence of plasma to the expressions that are available in the literature.

(2) IN THE PRESENCE OF PLASMA: SUBSONIC CASE

We have obtained two different spectrum functions (68) and (74) depending on the thermal speed and the velocity of the test particle. We first discuss the subsonic case where $u < v/a$.

To obtain the radiation cross section, we integrate Eq.(68) over all impact parameters and find

$$\chi(\omega) = (16e^2/3c)\ (qe/mc^2)^2\ [c^2/u^2(\omega^2-\omega_{pl}{}^2)^2] \times$$
$$\times \{(1 + a^4)\ (\omega^5 b_o/u)\ K_o(b_o\omega/u)\ K_1(b_o\omega/u) +$$
$$+ (\omega_{pl}{}^4 b_o{}^2/2)\ (\omega^2/u^2 - \lambda_1{}^2)\ [\{K_1(b_o\lambda_1)\}^2 - \{K_o(b_o\lambda_1)\}^2] +$$
$$+ \omega_{pl}{}^4\lambda_1 b_o\ K_o(b\lambda_1)\ K_1(b\lambda_1) - (2\omega^3\omega_{pl}{}^2 b_o/u)\ K_o(b_o\lambda_1)\ K_1(b_o\omega/u) +$$
$$+ a^4\omega^2u^2(b_o{}^2/2)\lambda_2{}^2\ (\omega^2/u^2-\lambda_2{}^2)\ [\{K_o(b\lambda_2)\}^2 - \{K_1(b\lambda_2)\}^2] +$$
$$+ a^4\omega^4 b_o\lambda_2\ K_o(b\lambda_2)\ K_1(b_o\lambda_2) -$$
$$- 2a^4\omega^4\lambda_2 b_o\ K_o(b_o\omega/u)\ K_1(b_o\lambda_2)\}\ . \qquad (90)$$

The details are the same as in deriving Eq.(81). Expression (90) represents the radiation cross section in a plasma when the test particle's speed is smaller than the thermal speed of the electrons in the system. The relativistic effects due to the test particle are included in the parameter λ_2 which is defined (c.f. Eq.(64)) as

$$\lambda_2 = [\omega^2/\gamma^2 u + \omega_p{}^2/a^2c^2]^{1/2}\ . \qquad (91)$$

Again one may retrieve the general non-relativistic case by using Eqs.(54), (55), (80), and (91). It may suffice to quote the low-frequency limit (82) which we obtain for $\omega \to 0$ from Eq.(90):

$$\chi(\omega\to 0;\ u<v/a) = (16e^2/3c)\ (qe/mc^2)^2\ (c^2\omega_p{}^2/u^2v^2) \times$$

$$\times [(b_o{}^2/2)\{ K_o^2(b\omega_p/v) - K_1{}^2(b_o\omega/v)\} + (vb_o/\omega_p)\ K_o(b_o\omega_p/v)\ K_1(b_o\omega_p/v)].\ (92)$$

Since $b_o \ll v/\omega_p$, i.e., the minimum impact parameter is much smaller than the Debye length, Eq.(92) gives

$$\chi(\omega\to 0) = (16e^2/3c)\ (qe/mc^2)^2\ (c^2/u^2) \times$$

$$\times\ [\ln(2v/\gamma_2 b_o\omega_p) - 1/2 + (b_o{}^2\omega_p{}^2/v^2\{\ln(2v/b_o\omega_p\gamma_2)\}^2]. \qquad (93)$$

As before,

$$v/b_o\omega_p >> \ln(2v/b_o\omega_p\gamma_2), \tag{94}$$

and we can neglect the third term on the right. Thus, finally,

$$\chi(\omega\to 0) = (16e^2/3c)\ (qe/mc^2)^2\ (c^2/u^2)\ [\ln(2v/\gamma_2 b_o\omega_p) - 1/2]. \tag{95}$$

This expression agrees completely with the low-frequency approximation given by Oster[12] for a spherically symmetric, shielded potential of the test particle. We see that the relativistic situation leads to the same result as the non-relativistic one.

(3) IN THE PRESENCE OF PLASMA: SUPERSONIC CASE

The case of a supersonic test particle can be treated in a manner completely analogous to the last subsection. The elementary, but lengthy calculation results in the expression

$$\begin{aligned}
\chi(\omega,\ u>v/a) &= (16e^2/3c)\ (qe/mc^2)^2\ [c^2/u^2(\omega^2-\omega_{pl}{}^2)^2] \times \\
&\times[(\pi^2\omega^2\omega_{pl}{}^4 b^2/8u^2)\ \{J_o{}^2(b\zeta) + J_1{}^2(b\zeta) + Y_o{}^2(b\zeta) + Y_1{}^2(b\zeta)\} - \\
&- (\omega^5 b/u)\ K_o(b\omega/u)\ K_1(b\omega/u) + (\pi^2\omega_{pl}{}^4\zeta^2 b^2/8) \times \\
&\times \{J_o{}^2(b\zeta) + J_1{}^2(b\zeta) - (2/b\zeta)J_o(b\zeta)J_1(b\zeta) + Y_o{}^2(b\zeta) + Y_1{}^2(b\zeta) - \\
&- (2/b\zeta)Y_o(b\zeta)Y_1(b\zeta)\} + (\pi\omega^4\omega_{pl}{}^2 b/u^2[\omega^2/u^2+\zeta^2]) \times \\
&\times \{-(\omega/u)Y_o(b\zeta)K_1(\omega b/u) + \zeta K_o(b\omega/u)Y_1(b\zeta)\} - (\omega^3\omega_{pl}{}^2\zeta/u)\ (b/[\omega^2/u^2+\zeta^2]) \times \\
&\times \{(\omega/u)K_o(b\omega/u)Y_1(b\zeta) + Y_o(b\zeta)K_1(b\omega/u)\} - (a^4\omega^5 b/u)K_o(b\omega/u)K_1(b\omega/u) + \\
&+ \omega^2u^2\lambda_2{}^2(b^2/2)a^4\ \{K_o{}^2(b\lambda_2) - K_1{}^2(b\lambda_2)\} + \omega^4a^4\lambda_2{}^2\{K_1{}^2(b\lambda_2) - \\
&- K_o{}^2(b\lambda_2) - (2/b\lambda_2)K_o(b\lambda_2)K_1(b\lambda_2)\} + \\
&+ 2a^4\omega^4\lambda_2 b\cdot K_o(b\omega/u)\ K_1(b\lambda_2)]_{b_o}^{b(max)}\quad ,
\end{aligned} \tag{96}$$

where the parameters are defined as before. In particular,

$$\zeta^2=\omega^2[a^2/v^2 - 1/u^2] - \omega_p^{\,2}/v^2 , \tag{97}$$

which does not contain any effect of the relativistic motion of the test particle, and hence leads to the same resonance behavior as the non-relativistic case, although the amplitude and the width of the resonance are, of course, different. A calculation of these details can be readily carried out with the aid of Eq.(96), but requires extensive numerical work.

The major result is the similarity of the resonance behavior of the relativistic and of the non-relativistic case. In particular, no new resonances appear due to the relativistic motion of the test particle, and the location of the only resonance that does occur in a non-relativistic ensemble is independent of the magnitude of the relative speed between test particle and radiating electrons.

I am indebted to Prof. L. Oster for suggesting this problem to me and for his critical reading of the manuscript.

APPENDIX

We write the integral I_1 from Eq.(49) as sum of two components:

$$I_1 = I_{11} - [\omega_p^2 - \omega^2(a^2-1)]\, I_{12}, \tag{A1}$$

where

$$I_{11} = \iint_{-\infty}^{+\infty} \exp(ibk_y)\, [(\omega^2/u^2) + k_y^2 + k_z^2]^{-1}\, dk_y\, dk_z, \tag{A2}$$

and

$$I_{12} = \iint_{-\infty}^{+\infty} \exp(ibk_y)\, [(\omega^2/u^2) + k_y^2 + k_z^2]^{-1} \times$$

$$\times\ [\omega_p^2 + \omega^2(1/u^2 - a^2/v^2)v^2 + k_y^2 + k_z^2]^{-1}\, dk_y\, dk_z\ . \tag{A3}$$

The integral I_{11} can easily be solved [Altschuler[6]] and becomes

$$I_{11} = 2\pi\, K_o(b\omega/u). \tag{A4}$$

The integral I_{12} is similar to Altschuler's integral I. The only difference is the definition of

$$\lambda_1^2 = (\omega_p^2/v^2) + \omega^2(1/u^2 - a^2/v^2). \tag{A5}$$

Thus, the integral reads [c.f. Altschuler[6]; Appendix A]

$$I_{12} = (2\pi/a^2[\omega^2-\omega_p^2])\ \{K_o(b\lambda_1) - K_o(b\omega/u)\}\ . \tag{A6}$$

On combining (A1), (A4), and (A6) one gets Eqs. (52) and (53).

In the case au > v we define

$$\zeta^2 = \omega^2[a^2/v^2 - 1/u^2] - \omega_p^2/v^2\ , \tag{A7}$$

and, following Altschuler's Appendix B, obtain

$$I = [2\pi/(a^2\omega^2-\omega_p{}^2)]\ \{\omega^2 K_o(b\omega/u) + (\pi\omega_{pl}{}^2/2)Y_o(b\zeta) - (i\pi\omega_{pl}{}^2/2)J_o(b\zeta) \quad , \qquad (A8)$$

where $\omega_{pl}{}^2$ is defined in Eq.(55).

In treating the integral I_2 defined by Eq.(50), we note that $c > u$, and find

$$I_2 = (2\pi/a^2c^2)\ K_o(b\lambda_2), \qquad (A9)$$

where

$$\lambda_2{}^2 = \omega^2[1/u^2 - 1/c^2] + (\omega_p{}^2/c^2a^2). \qquad (A10)$$

The integral I_3 is similar to integral I_{12}, except that we have to replace v by c in its solution. Hence,

$$I_3 = (2\pi/[a^2(\omega^2-\omega_p{}^2)]\ \{K_o(b\lambda_2) - K_o(b\omega/u)\}\ . \qquad (A11)$$

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