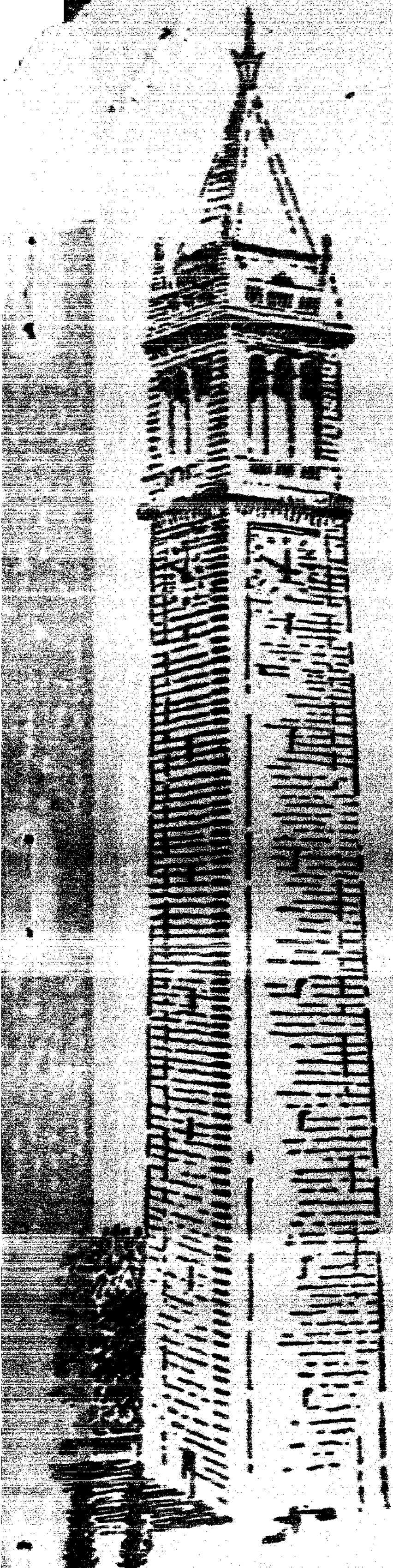

Cost-function Characterization of Systems

by

D. Chazan

Report No. ERL-66-5

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ELECTRONICS RESEARCH LABORATORY

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Manuscript submitted: 3 September 1965.

The research reported herein was supported wholly by the National Aeronautics and Space Administration under Grant NsG-354, Supplement 2.

ACKNOWLEDGMENTS

The author wishes to acknowledge the help of the following people in various stages of this work. The encouragement of Professor C. A. Desoer was extremely valuable at moments of stress; Professor R. M. Dudley read the manuscript and suggested a number of corrections and revisions; and his friend and colleague, Yechezkel Zalchstein, contributed time and effort when needed. Finally, infinite gratitude is due the author's wife, Nanette, who contributed in many ways to the materialization of this work and whose patience and endurance knew no bounds.

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INTRODUCTION

The purpose of this work is to show that cost functions, i.e., functions depending on an initial state x_1 and time t_1 and a final state x_2 and time t_2, whose value $c_{t_1, t_2}(x_1, x_2)$ is the cost of traversing in the best possible way the route from x_1 at t_1 to x_2 at t_2, may be used to describe dynamical control systems.

Accordingly, in Chapter I we show how such a description may be effected, in Chapter II we prove that very little additional structure is needed to obtain a description of a large class of cost functions. Thus, in Chapter I we attempt to answer the question, why are cost functions of interest? And once that interest has been demonstrated in Chapter II, we proceed to study the structure of cost functions.

The results of this work are based on the following fundamental property of cost functions:

For all $t_1 \leq t_2 \leq t_3$,

$$c_{t_1, t_3}(x, y) = \sup_z [\, c_{t_1, t_2}(x, z) + c_{t_2, t_3}(z, y) \,] .$$

The latter rests on the fact that in going from x at t_1 to y at t_3 some state z must be traversed at t_2. This property allows trajectories to be defined in a way that is consistent with our notion of dynamical control systems and also permits a complete characterization of a large class of cost functions. The main result of this work is such a characterization.

To obtain this characterization, the following assumptions are made:

(a) An appropriate continuity condition is imposed on $c_{t_1, t_2}(x, y)$ as a function of x, y and t_1, t_2.

(b) It is assumed that the state space X is a linear space and for all $t_1, t_2 \in R^+$ $x, y \in X$, $c_{t_1, t_2}(x, y) = c_{0, t_2 - t_1}(0, y - T_{t_2 - t_1} x)$

where T_t is a one-parameter semigroup of linear transformations on X satisfying some continuity conditions.

With this assumption it is not only possible to show that the function $c_{t_1, t_2}(x, y)$ is convex in x and y, but also an explicit expression may be obtained for the form of $c_{t_1, t_2}(x, y)$ as a function of t_1, t_2, x and y.

SUMMARY AND SOME GUIDE LINES

Because of the complexity of some of the relations that arise from the study of optimal control problems, it often seems desirable to rise above the problems and attempt to gain an overall view by means of generalization and simplification, and while generality is by no means a key to simplicity, it may sometimes provide pointers to help find that key. This work is an attempt in that direction. In Chapter I, the generalized approach is introduced and some of its implications are studied. In Chapter II, it is shown that despite the generality of this approach, it can be made to yield, at least in special cases, the strong results obtained by classical techniques. Before proceeding with a more detailed description of the work, a similar approach is developed for the study of stability problems by Roxin. The problems he studies are of course of a different nature and his method is slightly less general. With this reservation, Theorem 1.5.1 is a variant of his Theorem 6.1. The author was unfortunately unfamiliar with his work during the writing of the manuscript, hence the overlap.

As mentioned above, Chapter I involves mainly the study of fundamentals. In Section 1, we discuss the classical control problem of maximizing an expression of the form:

$$J_{[0,t]}(U(t)) = \int_0^t f^0(x(t), u(t), t)\, dt$$

subject to the constraints $x(0) = x_0$, $x(t) = x_1$ and $\dot{x} = f(x(t), u(t), t)$ for some given functions f and f^0. The function $u(t)$ is commonly referred to as the input or control variable and $x(t)$ is the state variable. As t progresses, $x(t)$ describes a trajectory in the state space.

In Section 2 we notice that rather than study a function $J_{[0,t]}(u(t))$ defined on the inputs it is more convenient to consider another function $F_{[0,t]}(x(\cdot))$ defined on the trajectories where F will play the double role of telling us which trajectories are allowable (i.e., satisfy the equation $\dot{x} = f(x(t), u(t), t)$ for some $u(t)$) and also what is their cost. Deleting the differential equations and the integral criterion we keep two fundamental properties (I.2.1 and I.2.2) of the function $F_T(x(t))$ and proceed to study functionals satisfying these properties. As a first attempt we define the function

$$c_{t_1, t_2, \cdots, t_n}(x_1, x_2, \cdots, x_n)$$
$$= \sup_{x(\cdot)} \left\{ F_{[t_1, t_n]}(x(\cdot)) : x(t_i) = x_i, \ i=1, \cdots, n \right\} \qquad (1)$$

which tells us how cheaply we may commute from x_1 at t_1 to x_n at t_n through x_i at t_i, $i=2, \cdots, n-1$. The conclusion of Section 2 is that $c_{t_1, \cdots, t_n}(x_1, \cdots, x_n)$ has the form

$$c_{t_1, \cdots, t_n}(x_1, \cdots, x_n)$$
$$= c_{t_1, t_2}(x_1, x_2) + c_{t_2, t_3}(x_2, x_3) + \cdots + c_{t_{n-1}, t_n}(x_{n-1}, x_n) \qquad (2)$$

and $c_{t_1, t_2}(x, y)$ satisfies the dynamic programming or semigroup property

$$c_{t_1, t_3}(x, y) = \sup_{x} \left[c_{t_1, t_2}(x, z) + c_{t_2, t_3}(z, y) \right]. \qquad (3)$$

We refer to functions satisfying (3) as Markov transition cost functions.

In Section 3 we discuss an analogy with probability theory which hinges on the similarity between (3) and the Chapman-Kolmogorov equation encountered in the study of continuous Markov processes. With this analogy in mind the function $c_{t_1, t_2}(x, y)$ may be regarded as a conditional cost of going to y given that we start at x and equation (3) expresses a "Markovian" property of our process. In more general circumstances however we could imagine processes which are described by a joint cost distribution $\Gamma_{t_1, \ldots, t_n}(x_1, \ldots, x_n)$ which is not Markovian. Section 5 will be devoted to the study of some properties of such more general processes.

In Section 4 a few technicalities are resolved. To obtain any results we have to assume that $c_{t_1, t_2}(x, y)$ is upper semicontinuous. Thus, we begin that section by showing (lemma 1) that this is a very reasonable assumption and then proceed to state a series of lemmas leading to theorem 1 which tells us what to do when $c_{t,\tau}(\cdot, \cdot)$ is not upper semicontinuous.

Most of Section 5 is devoted to the definitions and study of terms whose main purpose is to bring out and utilize the probability analogy. One of the main conclusions of this section is that is does make sense to talk about trajectories in much the same way this is done in probability theory. At the conclusion of the section is theorem 1 which essentially states the following: Suppose $\Gamma_{\alpha_1, t_1, \ldots, t_n, \alpha_2}(x_0, \ldots, x_{n+1})$ defined for all $\alpha_1 \leqq t_1 \leqq \cdots \leqq t_n \leqq \alpha_2$ and all $x_0, \ldots, x_{n+1}$ is a joint cost distribution, i.e., it satisfies the equivalent of the Kolmogorov consistency condition:

$$\begin{aligned} &\Gamma_{\alpha_1, t_1, \ldots, t_{k-1}, t_{k+1}, \ldots, y_n, \alpha_2}(x_0, \ldots, x_{k-1}, x_{k+1}, \ldots, x_{n+1}) \\ &\qquad = \sup_{x_k} \Gamma_{\alpha_1, t_1, \ldots, t_{k-1}, t_k, t_{k+1}, \ldots, t_n, \alpha_2}(x_0, \ldots, x_n) \end{aligned} \tag{4}$$

and let

$$\mathcal{C}_{[\alpha_1, \alpha_2]}(f(\cdot)) = \inf \Gamma_{\alpha_1, t_1, \ldots, t_n, \alpha_2}(f(\alpha_1), f(t_1), \ldots, f(t_n), f(\alpha_2))$$

where the infimum is taken over all finite partitions $\alpha_1, t_1, t_2, \ldots, t_n, \alpha_2$ of $[\alpha_1, \alpha_2]$. Then if Γ satisfies the appropriate continuity and compactness conditions, there exists for any selection of times $t_1, \ldots, t_n$ and state $x_0, \ldots, x_{n+1}$ a trajectory $f(t)$ with $f(\alpha_1) = x_0, f(t_1) = x_1, \ldots, f(t_n) = x_n, f(\alpha_2) = x_{n+1}$ such that

$$C_{[\alpha_1, \alpha_2]}(f(\cdot)) = \Gamma_{\alpha_1, t_1, \ldots, t_n, \alpha_2}(x_0, \ldots, x_{n+1}) . \tag{5}$$

In Section 6 we are forced, in order to make the analogy complete, to consider the function $F_T(x(\cdot))$ introduced in Section 2 as a conditional cost of traversing the trajectory $x(\cdot)$ at α_1. In the discussion here we may however state the results of the first part of that section without this slight complication as follows: Given a functional $F_T(\cdot)$ on the trajectories satisfying the fundamental conditions I.2.1 and I.2.2 of Section 2 we define

$$c_{\alpha_1, t_1, \ldots, t_n, \alpha_2}(x_0, \ldots, x_{n+1})$$

$$= \sup \{F_{[\alpha_1, \alpha_2]}(x(\cdot)) : x(\alpha_1) = x_0, x(t_1) \tag{6}$$

$$= x_1, \ldots, x(t_n) = x_n, x(\alpha_2) = x_{n+1}\}$$

and let $F'_T(x(\cdot))$ be defined by an equation similar to (5) above:

$$F'_{[\alpha_1, \alpha_2]}(x(\cdot))$$

$$= \inf_{n, t_1, \ldots, t_n} c_{\alpha_1, t_1, \ldots, t_n, \alpha_2}(f(\alpha_1), f(t_1), \ldots, f(t_n) f(\alpha_2)) .$$

Then $F'_{[\alpha_1, \alpha_2]}(x(\cdot)) \geq F_{[\alpha_1, \alpha_2]}((x(\cdot)))$ and if

$c'_{\alpha_1, t_1, \ldots, t_n, \alpha_2}(x_0, \ldots, x_{n+1})$ is defined as in (6) above:

$$c'_{\alpha_1, t_1, \ldots, t_n, \alpha_2}(x_0, \ldots, x_{n+1})$$

$$= \sup_{x(\cdot)} \{F'_T(x(\cdot)) : x(\alpha_1) = x_0, \ldots, x(\alpha_2) = x_{n+1}\} .$$

Then $c'_{\alpha_1, t_1, \ldots, t_n, \alpha_2}(x_0, \ldots, x_{n+1}) = c_{\alpha_1, t_1, t_2, \ldots, t_n, \alpha_2}(x_0, \ldots, x_{n+1})$.

Thus $c_{\alpha_1, t_1, \ldots, t_n, \alpha_2}(x_0, \ldots, x_{n+1})$ can be used to construct a cost functional $F'_T(\circ)$ which is very closely related to $F_T(\circ)$. Furthermore, it is shown in lemma 1 that $F'_T(\circ)$ also has the fundamental properties (I.2.1, I.2.2) and that these properties depend for their existence solely on the fact that $c_{\alpha_1, t_1, \ldots, t_n, \alpha_2}(x_0, \ldots, x_{n+1})$ has the form (2) where $c_{t_1, t_2}(x, y)$ satisfies (3).

Perhaps the most interesting result of this section is lemma 2 which states conditions under which the supremum in equation (2) may be taken over arbitrary planes in the time-state space, i.e.,

$$c_{t_1, t_2}(x, y) = \sup_{(t, z) \in P} (c_{t_1, t}(x, z) + c_{t, t_2}(z, y))$$

where P is a plane in $R^+ \times X$ separating (t_1, x) and (t_2, z). Basically, the conditions mentioned above force all the trajectories to be continuous; therefore, in going from x at t_1 to y at t_2, the plane P must be crossed.

In section 7, the concept of independence of control variables is defined in a natural way to correspond to out probabilistic notions, and this concept leads us to the main body of the problems studied in Chapter 2, where we are concerned with processes whose increments are independent up to a transformation. Also in Section 7, an example of a very simple but pathological case of a transition cost function $c_{t_1, t_2}(x, y)$ which is not only Markovian (i.e., satisfies (2)) but corresponds to a process with independent increments is worked out.

The objective of Chapter 2 is to characterize a class of Markov transition cost functions satisfying a linearity and time-invariance condition:

$$c_{t_1,t_2}(x,y) = c_{0,t_2-t_1}(0, y-T_{t_2-t_1}x),$$

where T_t is assumed to be a one-parameter set of linear transformations. The tools with which we attack this problem are introduced in Section 1. We then proceed to show (§ 2) that it is reasonable to require the transformations T_t to form a one-parameter semigroup (i.e., $T_{t+\tau} = T_t T_\tau$) and with this assumption together with some continuity and boundedness requirements on T_t and $c_{t_1,t_2}(\cdot,\cdot)$ we obtain a description of the temporal behavior of the smallest concave function dominating $c_{t_1,t_2}(0,\circ)$.

In Section 3 we go one step further and show that a slight additional assumption will also guarantee concavity of $c_{t_1,t_2}(0,\circ)$. Finally, in Section 4 we replace the conditions of Section 3 which involved the structure of $c_{t_1,t_2}(0,\circ)$ by more natural conditions which only require the cost of reaching y from x at very short times to becomes arbitrarily large whenever $y \neq x$. Thus, the assumptions and conclusions of the three sections may be summarized as follows.

<u>Section 2</u>. (a) Continuity condition on $T_t(x)$ as a function of x and t separately; (b) concavity and upper semicontinuity of $c_{t_1,t_2}(\cdot,\cdot)$ and (c) a requirement that the set of points at which $c_{t_1,t_2}(0,\circ)$ stays above a plane be bounded, hence $c_{t_1,t_2}(0,\circ)$ decreases faster than any plane.

Conclusion: A description of the temporal behavior of $c_{t_1,t_2}(\cdot,\cdot)$.

<u>Section 3</u>. (a) Continuity conditions on $T_t(x)$ as a function of t and x separately; (b) a requirement that the set of points at which $c_{t_1,t_2}(0,\circ)$ stays above a given plane be weakly compact.

Conclusion: $c_{t_1,t_2}(0,\cdot)$ is concave.

<u>Section 4</u>. (a) Continuity condition on $T_t(x)$ in x and t separately; (b) a requirement that whenever $t_n \downarrow 0$ and $c_{0,t_n}(0,x_n) > a$, x_n converge to zero; (c) a measurability condition; and (d) upper semicontinuity of $c_{t_1,t_2}(0,\circ)$.

Conclusion: $c_{t_1, t_2}(0, \cdot)$ is concave.

To conclude the work a number of examples are worked out in Section 5.

CHAPTER I

1. THE OPTIMAL CONTROL PROBLEM GENERALITIES

Optimal control theory is concerned with problems of the following type.

Find a function $u(t)$ defined on the positive real line R^+ which maximizes the expression

$$J(u(\cdot)) = \int_a^b f^0(x(t), t, u(t))\, dt, \tag{1}$$

where $x(t)$ is a time function with values in a linear topological space[6] satisfying the equation

$$\dot{x}(t) = \frac{d(x(t))}{dt} = f(x(t), t, u(t)) \quad x(a) = x, x(b) = y, \tag{2}$$

$u(t)$ belongs to a given set Ω and the functions f^0 and f defined on $X \times R^+ \times \Omega$ take on values in R and X respectively.

Problems of this type have been studied extensively in the literature. In general, there exist two ways by which the problem may be approached. One involves essentially variational techniques and endeavors to find the time dependence of the function $u(t)$. The answer here is given in the form of a differential equation which together with some auxiliary conditions on the input must be satisfied by extremal inputs.[1, 2, 5]

The second approach due to Bellman[3, 5] considers the following function:

$$c_b(y) = \sup_{u(\cdot)} \{J(u(\cdot)) : \dot{x} = f(x(t), t, u(t)),$$

$$u \in \Omega, x(a) = x, \; x(b) = y\},$$

and by proper manipulation of the quantities involved allows us to obtain a partial differential equation describing the behavior of $c_b(y)$. When $c_b(y)$ is given it is relatively easy to find at any point y what control u should be applied and the final results has the form of a feedback control.

In both of these methods no attempt is made to study the structure of the solution except in as much as it is manifest in the final equations. Also, because of the methods which invariable involved differentiation, a number of technical problems arise which depend solely on the approach and are often not of interest in the problem. The end result in eaither case is a differential equation which at best is not easy to solve and does not convey as much intuitive information about the problem as we would like to have.

In this work an attempt will be made to remedy some of these problems by introducing a new approach involving a slight modification of the dynamic programming procedure. It is believed that the latter has the advantage of having a mathematically pleasant appearance as well as an intuitive appeal. For example, the system studied with this approach are not assumed (at least a priori) to be described by a differential equation and the results obtained are therefore stronger.

2. THE OPTIMAL CONTROL PROBLEM, REFORMULATION, AND COST FUNCTION

Let $\mathcal{X}$ be the space X^{R^+} of all functions from R^+ into X. Any element of $\mathcal{X}$ will be called a trajectory. Let $\mathcal{B}$ be the closed subintervals of R^+. We shall define M_B to be the projection of X^{R^+} into X^B for all $B \in \mathcal{B}$. Let $F_B(x(\cdot)) : \mathcal{B} \times \mathcal{X} \rightarrow R \cup \{-\infty\}$ be a functional on for every set $B \in \mathcal{B}$ satisfying

(1) $F_B(x_1(\cdot)) = F_B(x_2(\cdot))$ whenever $M_B(x_1(\cdot)) = M_B(x_2(\cdot))$

(2) $\mu(B_1 \cap B_2) = 0$ implies $F_{B_1 \cap B_2}(x(\cdot)) = F_{B_1}(x(\cdot)) + F_{B_2}(x(\cdot))$

where μ is Lebesgue measure. Then $F_{(\cdot)}(\cdot)$ is called a cost functional. Such a functional may arise for example in the following way out of equations I.1.1 and I.1.2. For any (fixed) set $B \in \mathcal{B}$ define

$$F_B(x(\cdot)) = \begin{cases} \int_B g(x(t), t, u(t))\, dt & \text{if } x \text{ is a solution of } \dot{x} = f(x, t, u) \text{ on } B, \\ -\infty & \text{otherwise} \end{cases}$$

It is easy to see that conditions (1) and (2) are satisfied. Thus, these conditions generalize the type of restriction that is usually imposed on the output of a system when its input is constrained in a "reasonable" way.

Let $F_{(\cdot)}(\cdot)$ be a cost functional. We now consider a function defined in the following manner:

$$(3)\quad c_{\alpha_1, t_1, \ldots, t_n, \alpha_2}(x_0, \ldots, x_{n+1}) = \sup \{F_{[\alpha_1, \alpha_2]}(x(\cdot)) :$$

$$x(\alpha_1) = x_0, x(t_1) = x_1, \ldots, x(\alpha_2) = x_{n+1}\},$$

where the function is defined for all $(\alpha_1, t_1, \ldots, \alpha_2) \in (R^+)^{n+2}$ and $(x_0, \ldots, x_{n+1}) \in X^{n+2}$ and the usual convention is adopted $\sup \phi = -\infty$. We shall also assume throughout this work that $c_{t_1, \ldots, t_n}(x_1, \ldots, x_n) < \infty$ for all $t_1, \ldots, t_n$ and $x_1, \ldots, x_n$, and proceed to study the properties of this function.

<u>Lemma 1.</u> If $F_{(\cdot)}(\cdot)$ is a cost functional satisfying conditions (1) and (2), then for all x_1, x_n and $t_1 \leq t_2 \leq \ldots \leq t_n$

(a) $$c_{t_1, \ldots, t_n}(x_1, \ldots, x_n) = c_{t_1, t_2}(x_1, x_2) + c_{t_2, t_3}(x_2, x_3) + \cdots + c_{t_{n-1}, t_n}(x_{n-1}, x_n).$$

(b) For all $x, y \in X$ and $t_1 \leq t_2 \leq t_3$

$$(4)\quad c_{t_1, t_3}(x, y) = \sup_{z \in X} (c_{t_1, t_2}(x, z) + c_{t_2, t_3}(z, y)).$$

$$\underline{\text{Proof}}.\ \{F_{[t_1, t_n]}(x(t)) : x(t_1) = x_1 \cdots x(t_n) = x_n\} =$$

$$\{F_{[t_1, t_2]}(x(t) + F_{[t_2, t_3]}(x(t)) + \cdots + F_{[t_{n-1}, t_n]}(x(t)) :$$

$$x(t_1) = x_1 \cdots x(t_n) = x_n\} = \{F_{[t_1, t_2]}(x_1(t)) + F_{[t_2, t_3]}(x_2(t) + \cdots$$

$$+ F_{[t_{n-1}, t_n]}(x_n(t)) : x_1(t_1) = x_1, x_1(t_2) = x_2 = x_2(t_2),$$

$$x_2(t_3) = x_3 = x_3(t_3) \cdots x_n(t_{n-1}) = x_{n-1}\ x_n(t_n) = x_n\}$$

where both conditions (1) and (2) were used. Taking suprema with respect to $x_1(\cdot) \cdots x_n(\ )$, which may be chosen independently, we obtain (a). (b) follows immediately if we notice that

$$\sup_{x_k} c_{t_1, t_2, \ldots, t_{k-1}, t_k, t_{k+1}, \ldots, t_n}(x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n)$$

$$= c_{t_1, \ldots, t_{k-1}, t_{k+1}, \ldots, t_n}(x_1, \ldots, x_{k-1}, x_k, \ldots, x_n).$$

The property stated in lemma 1(b) is fundamental to the study of the function $c_{t_1, t_2}(x_1, x_2)$. It expresses the principle of optimality of dynamic programming and will occasionally be referred to as the "dynamic programming condition." The latter imposes a semigroup structure on $c_{t_1, t_2}(x, y)$ in the following sense. Suppose $c_{t_1, t_2}(x, \ ) \leq M(x) < \infty$. For every pair $t_1, t_2 \in R^+$, we define on the space $\mathcal{W} = \{g(\ ) : X \to R \cup \{-\infty\},\ g(\ ) \leq N < \infty\}$, an operator $T_{t_1, t_2} : \mathcal{W} \to \mathcal{W}$ such that

$$(5) \quad (T_{t_1, t_2}(g(\ )))(y) = \sup_z \left[c_{t_1, t_2}(y, z) + g(z) \right]$$

Using this definition of T_{t_1, t_2} we may rewrite equation (4) in the form

$$(6) \qquad T_{t_1, t_3} = T_{t_1, t_2} T_{t_2, t_3}$$

Functions satisfying (4) will be called Markov transition cost function (M.t.c.f.).

3. THE PROBABILITY ANALOGY

In this section, an analogy which has proved beneficial in motivating the results of this work will be discussed. It is hoped that this application did not exhaust all of its usefulness.

Equation (4) resembles in appearance the Chapman-Kolmogorov equation encountered in the study of Markov chains:[1]

$$(1) \qquad p_{t_1, t_3}(x, S) = \int_\Omega p_{t_1, t_2}(x, dz)\, p_{t_2, t_3}(z, S),$$

where we replace in equation (4) sup by "$\int$" and (+) by (+). The similarity can be extended further. While $p_{t_1, t_2}(x, S)$ is the probability of getting into S and t_2 given that we start at x at time t_1, $c_{t_1, t_2}(x, z)$ can be regarded as the conditional cost of getting into z at t_2 given that we start at x at t_1. For example, if we assign an initial preference function $p_0(z)$ at 0, it propagates to the preference function $p_t(z)$ in time t:

$$(2) \qquad p_t(z) = \sup_{x \in X} [\, c_{0, t}(x, z) + p_0(x)]$$

in much the same way as an initial probability distribution at $p_0(S)$ propagates to

$$(3) \qquad p_t(S) = \int_\Omega p_{0, t}(x, S)\, p_0(dx).$$

Thus the cost functions which shall be examined in this work form a subclass of a more general class of cost functions which could be specified by the joint cost distribution $\Gamma_{t_1, \ldots, t_n}(x_1, \ldots, x_n)$ representing the cost of going through x_i at time t_i, $i=1 \cdots n$ in the best possible way. The restriction (I.1.1, I.1.2) which we placed at the outset on $F_B(x(t))$, forces Γ to have the form I.2.3[4]

$$\Gamma_{t_1, \ldots, t_n}(x_1, \ldots x_n) = \Gamma_{t_1}(x_1) + c_{t_1, t_2}(x_1, x_2) +$$

$$+ c_{t_2, t_3}(x_2, x_3) + \cdots + c_{t_{n-1}, t_n}(x_{n-1}, x_n),$$

and the expression thus obtained corresponds to the Markovian dependence. It will be shown later that some of the problems of probability theory have a meaningful translation into the class of problems studied here. For example, the problem of extending a joint n distribution into a distribution on the whole space of trajectories allows such a translation (see I.5), and while it is much easier to obtain results concerning cost functions than it is in probability theory, the problems still turn out to have an interesting meaning. The central limit problem and the characterization of processes with independent increments also turn out to have an interesting analogue in the study of cost functions. The common denominator in this case appears to be the semigroup structure which exists in both cases.[2] In the probabilistic case we may define for a suitable chosen class S of probability densities and for any pair (t_1, t_2) a linear transformation T_{t_1, t_2} defined by:

(4) $$T_{t_1, t_2} : S \rightarrow S \text{ and } (T_{t_1, t_2} g)(y) = \int_\Omega P_{t_1, t_2}(y, dz)\, g(z),$$

and it follows from the Chapman-Kolmogorov equation (2) that

(5) $$T_{t_1, t_2} T_{t_2, t_3} = T_{t_1, t_3}.$$

Thus at least formally equations (5) and I.2.6 look identical. The fundamental difference is of course that equation (5) involves linear operators while I.2.6 involves operators which are basically nonlinear. In the time invariant case, equations I.2.6 and (5) reduce to:

(6) $$T_{0, t_1 + t_2} = T_{0, t_1} T_{0, t_2},$$

and $T_{0,t}$ forms a one-parameter semigroup of transformations. If sufficient continuity conditions are placed on $T_{0,t}$, it is easy to see that it satisfies the following differential equation

$$\frac{dT_{0,t}}{dt} = DT_{0,t}$$

and the evolution in time of $T_{0,t}$ is immediately obtained from the semigroup condition. Still the semigroup property by itself does not yield much information about the form of the operator D. However, if additional continuity and structure conditions are placed on the behavior of $P_{t_1,t_2}(x, S)$ it is possible to show[3] that D is indeed as we might expect the Fokker-Planck operator. In this work we shall attempt to determine to what extend it is possible to obtain a Fokker-Planck type equation for optimal control processes strating with the bare bone model of Section (2). It will be shown that at least in a special case corresponding to the control of linear time invariant systems, such an equation is feasible. Perhaps even more interesting is the fact that this case is analogous to the process with independent increments encountered in probability theory. With the continuity conditions mentioned above the Fokker-Planck equation exists and has an explicit solution, which is a Gaussian distribution. The corresponding result in the theory of optimal control is that $c_{t_1,t_2}(x_1, x_2)$ must be concave.

4. THE COST FUNCTION: CONTINUITY PROPERTIES

To obtain some of our results it is necessary to assume the the M.t.c.f. $c_{t_1,t_2}(x, y)$ is upper semicontinuous in x and y. In this section the physical meaning and some implications of this requirement will be studied.

In all that follows we assume that the state space X is a locally convex Hausdorff linear topological space.[1]

For any function $f(\cdot)$, $f: X \longrightarrow R$ we define $\overline{f}(\cdot)$ to be the smallest upper semicontinuous function dominating f. If f depends on more than one variable, we denote by $(\overline{f}(x, \cdot))(y)$ the value of $\overline{f}(x, \cdot)$ and y. We may now utilize the fact[2] that $\overline{f}(x_0) = \inf_N \sup_{x \in N} f(x)$ where N is an arbitrary neighborhood of x_0 to obtain:

Lemma 1. Let $c_{t_1, t_2}(x, y) = \sup \{F_{[t_1, t_2]}(x(\cdot)): x(t_1) = x, x(t_2) = y\}$. Then $\bar{c}_{t_1, t_2}(x, y) = \inf_{N, M} \sup_{x(t)} \{F_{[t_1, t_2]}(x(\cdot)): x(t_1) \in x + M,$ $x(t_2) \in y + N$ and N, M are neighborhoods of the origin in $X\}$.

Proof. Since $\bar{c}_{t_1, t_2}(x_0, y_0) = \inf_{N, M} \sup_{x, y} \{c_{t_1, t_2}(x, y):$ $x \in x_0 + N,\ y \in y_0 + M,\ N, M$ neighborhoods of the origin in $X\} =$

$$\inf_{N, M} \sup_{x, y} \sup_{x(\cdot)} \{F_{[t_1, t_2]}(x(\cdot)): x(t_1) = x,\ x(t_2) = y,\ x \in x_0 + N,\ y \in y_0 + M\}$$

$$= \inf_{N, M} \sup_{x(\cdot)} \{F_{[t_1, t_2]}(x(\cdot)): x(t_1) \in x_0 + N,\ x(t_2) \in y_0 + M\}.$$

Lemma 1 implies that the function of interest to us is $\bar{c}_{t_1, t_2}(\cdot, \cdot)$ rather than $c_{t_1, t_2}(\cdot, \cdot)$ since some inexactitude will appear both in the observation of the initial state and the control of the final state. By lemma 1 $\bar{c}_{t_1, t_2}(\cdot, \cdot)$ takes into account such inexactitudes.(3)

It is still not clear however on what conditions $c_{t_1, t_2}(x, y)$ will make $\bar{c}_{t_1, t_2}(x, y)$ a M. t. c. f. The following two lemmas will help us state such conditions.

Lemma 2. Let $c_{t_1, t_2}(\cdot, \cdot)$ be an M. t. c. f. If $(\bar{c}_{t_1, t_2}(\cdot, z))(x)$ $= \bar{c}_{t_1, t_2}(x, z)$ and $(c_{t_1, t_2}(x, \cdot))(z) = \bar{c}_{t_1, t_2}(x, z)$ then $\bar{c}_{t_1, t_3}(x, y)$ $\geqq \sup_z [\bar{c}_{t_1, t_2}(x, z) + \bar{c}_{t_2, t_3}(z, y)]$.

Proof. Since $c_{t_1, t_2}(\cdot, \cdot)$ is a M. t. c. f.:

$$c_{t_1, t_3}(x, y) \geqq c_{t_1, t_2}(x, z) + c_{t_2, t_3}(z, y).$$

Taking the upper semicontinuous hull of both sides first with respect to x

$$\bar{c}_{t_1,t_3}(x,y) = (\bar{c}_{t_1,t_3}(\cdot\,,y))(x) \geqq (\bar{c}_{t_1,t_2}(\cdot\,,z))(x) + c_{t_2,t_3}(z,y)$$

$$= \bar{c}_{t_1,t_2}(x,z) + c_{t_2,t_3}(z,y)$$

and then with respect to y:

$$\bar{c}_{t_1,t_2}(x,y) \geqq \bar{c}_{t_1,t_2}(x,z) + \bar{c}_{t_2,t_3}(z,\cdot\,))(y)$$

$$= \bar{c}_{t_1,t_2}(x,z) + \bar{c}_{t_2,t_3}(z,y)$$

we obtain the desired result.

Lemma 3. Let $g_1(w,v)$ and $g_2(v,z)$ be two upper semicontinuous functions with values in $R \cup \{-\infty\}$. If $g_2(v,z) \leqq K < \infty$ and $\overline{\{w-v : g_1(w,v) \geqq a\}}$ is a compact for every a, the function

$$\sup_v [\, g_1(w,v) + g_2(v,z)\,]$$

is upper semicontinuous in (w,z).

Proof. We wish to show that the set $S^a = \{(w,z): \sup_v [\, g_1(w,v) + g_2(v,z)] \geqq a\}$ is closed for all a. Thus, it is enough to show that if (w_0, z_0) is a limit point of S^a, that there exists some v_0 such that

$$g_1(w_0, v_0) + g_2(v_0, z_0) \geqq a\,.$$

Let N, M be any neighborhoods of the origin in X, and let the set $S_{N,M}$ be defined as follows:

$$S_{N,M} = \{(v-w) : g_1(w,v) + g_2(v,z) \geqq a,\ w \in w_0 + N,\ z \in z_0 + M\}\,.$$

Since (w_0, z_0) is a limit point of S^a, there exists $w_1 \in w_0 + N$, $z_1 \in z_0 + M$ such that

$$\sup_v [\, g_1(w_1, v) + g_2(v, z_1)] \geqq a.$$

But:

$$\sup_v [g_1(w_1, v) + g_2(v, z_1)] = \sup \{g_1(w_1, v) + g_2(v, z_1): g_1(w_1, v) + g_2(v, z_1) \geqq a - \epsilon\}$$

for every $\epsilon > 0$ and since the set $\{v: g_1(w_1, v) + g_2(v, z_1) \geqq a - \epsilon\}$ is contained in the compact set $\{v: g_1(w_1, v) \geqq a - \epsilon - K\}$ the supremum is taken on and there exists v_1 such that

$$g_1(w_1, v_1) + g_2(v_1, z_1) \geqq a.$$

Thus, the sets $S_{N, M}$ are nonempty. It is also easy to see that $N_1 \supset N_2$, $M_1 \supset M_2$ implies $S_{N_1, M_1} \supset S_{N_2, M_2}$ and therefore

$$S_{N_1, M_1} \cap S_{N_2, M_2} \quad S_{N_1 \cap N_2, \ M_1 \cap M_2} \neq \emptyset,$$

and the sets $S_{N, M}$ have the finite intersection property.[4] But the set $S_{N, M}$ is contained in $\overline{\{w-v: g_1(w, v) \geqq a - K\}}$ and the latter is compact, hence the closure $\bar{S}_{N, M}$ of $S_{N, M}$ must be compact and there exists a point q_0 such that $q_0 \in \bigcap \bar{S}_{N, M}$.[4]

The last step of the proof will consist in showing that the point $(w_0, q_0 + w_0, z_0)$ is a limit point of the set $\{(w, v, z): g_1(w, v) + g_2(v, z) \geqq a\}$ hence must belong to it. Indeed, let θ, N, M be any neighborhoods of the origin and select q_1 in $(q_0 + \frac{\theta}{2}) \cap S_{N \cap \frac{\theta}{2}, M}$. Then q_1 exists

since q_0 is a limit point of $S_{N, M}$ for all N, M and it has the form $q_1 = v_1 - w_1$ for some $v_1 \in X$ and $w_1 \in w_0 + N \cap \frac{\theta}{2}$ satisfying

$$g_1(w_1, v_1) + g_2(v_1, z_1) \geqq a$$

for some $z_1 \in z_0 + M$. Therefore, $v_1 = w_1 + q_1 \in w_0 + \frac{\theta}{2} \cap N + q_0 + \frac{\theta}{2} w_0 + q_0 + \theta$ and $(w_1, v_1, z_1) \in (w_0 + N, \ v_0 + \theta, \ z_0 + M) \cap \{(w, v, z): g_1(w, v) + g_2(v, z) \geqq a\}$.

Suppose now that $c_{t_1, t_2}(x, y)$ is a M.t.c.f. Then

(2) $$c_{t_1,t_3}(x,y) = \sup_z [c_{t_1,t_2}(x,z) + c_{t_2,t_3}(z,y)]$$

$$\leqq \sup_z [\bar{c}_{t_1,t_2}(x,z) + \bar{c}_{t_2,t_3}(z,y)] .$$

If $\bar{c}_{t_1,t_2}(x,y)$ satisfies the assumptions of lemma 3, i.e., $\{x-y : \bar{c}_{t_1,t_2}(x,y) \geqq a\}$ is compact for any $t_1, t_2 \in R^+$ and for any $a \in R$ and $c_{t_1,t_2}(x,y) \leqq K < \infty$, the function on the right side of (2) is upper semicontinuous in x and y, hence we have

$$\bar{c}_{t_1,t_2}(x,y) \leqq \sup [\bar{c}_{t_1,t_2}(x,z) + \bar{c}_{t_2,t_3}(z,y)]$$

and it follows that:

Theorem 1. If $c_{t_1,t_2}(x,y)$ is a M.t.c.f. which is bounded above $\bar{c}_{t_1,t_2}(x,y) = (\bar{c}_{t_1,t_2}(\cdot,y))(x) = (\bar{c}_{t_1,t_2}(x,\cdot))(y)$ and $\{(x-y) : c_{t_1,t_2}(x,y) \geqq a\}$ is contained in a compact set, then $\bar{c}_{t_1,t_2}(x,y)$ is also an M.t.c.f. and the set $\overline{\{(x-y) : c_{t_1,t_2}(x,y) \geqq a\}}$ is compact.

Proof. The result would follow immediately from the remarks above and lemmas 2 and 3 if $\{x-y : \bar{c}_{t_1,t_2}(x,y) \geqq a\}$ were compact.

Let (x, y) belong to $\{(x,y) : \bar{c}_{t_1,t_2}(x,y) \geqq a\}$. Then every neighborhood of (x, y) contains points $(x_1 y_1)$ such that $c_{t_1,t_2}(x_1, y_1) \geqq a - \epsilon$ for some arbitrary (fixed) $\epsilon > 0$. Thus $\{(x,y) : \bar{c}_{t_1,t_2}(x,y) \geqq a\} \subset \{(x,y) : c_{t_1,t_2}(x,y) \geqq a - \epsilon\}$. We now define the continuous map $f : X \times X \to X$ by $f(x,y) = x - y$ and write $\{(x-y) : \bar{c}_{t_1,t_2}(x,y) \geqq a\}$

$= f[\{(x,y): \bar{c}_{t_1,t_2}(x,y) \geqq a\}] \subset \overline{f[\{(x,y): c_{t_1,t_2}(x,y) \geqq a-\epsilon\}]}$

$\subset \overline{f[\{(x,y): c_{t_1,t_2}(x,y) \geqq a-\epsilon\}]} = \overline{\{x-y : c_{t_1,t_2}(x,y) \geqq a-\epsilon\}}$ and the latter is a compact set.

5. THE COST FUNCTIONAL, THE TRAJECTORY SPACE AND EXISTENCE OF OPTIMAL SOLUTIONS

As mentioned in Section 3 above, the problem of extending the joint marginal probability density of a process to a probability on the whole space has its parallel in the study of cost functions.

We shall now consider a space Ω together with the set $\mathcal{S}$ of all the subsets of Ω. $(\Omega, \mathcal{S}, C)$ will be called a cost function space if C is a function on $\mathcal{S}$ satisfying

$$(1) \quad (a)\ C(\Omega) < \infty, (b)\ C(\bigcup_{\alpha \in I}(A_\alpha)) = \sup_{\alpha \in I} C(A_\alpha),\ (c)\ C(\phi) = -\infty.$$

Let T be a subset of R^+. Any function $\phi(\omega, t): \Omega \times T \rightarrow X$ shall be called a control process on T.

It is easy to see that the function C is definable in terms of its values on isolated points of Ω. $C(A) = \sup \{C(\{\omega\}): \omega \epsilon A\}$ for all $A \epsilon \mathcal{S}$. Thus C expresses a preference function on the points of Ω. If $C(\{\omega_1\}) > C(\{\omega_2\})$, ω_1 is preferable to ω_2 and $C(A)$ is essentially the cost of the "best" point in A. Once an ω_0 is chosen, it determines a trajectory $\phi(\omega_0, \cdot)$ and the cost of this trajectory is the cost of that subset of Ω which is mapped into it by the mapping $L: \omega \longrightarrow \phi(\omega, \cdot)$.

We wish to study the following problem. Suppose that for any finite sequence of positive real times $t_1, \ldots, t_n$ we are given a joint cost distribution, i.e., a function $\Gamma_{t_1,\ldots,t_n}(x_1,\ldots,x_n)$: $T^n \times X^n \longrightarrow R \cup \{-\infty\}$ satisfying the consistency condition

$$(2) \quad \sup_{x_p} \Gamma_{t_1,\ldots,t_{p-1},t_p,t_{p+1},\ldots,t_n}(x_1,\ldots,x_{p-1}x_p, x_{p+1},\ldots,x_n)$$

$$= \Gamma_{t_1,\ldots,t_{p-1},t_{p+1},\ldots,t_n}(x_1,\ldots,x_{p-1},x_{p+1},\ldots,x_n)$$

does there exist a control process on $T \subset R^{+}$ such that

$$(3)\quad C(\{\omega : \phi(\omega : t_1) = x_1 \cdots \phi(\omega, t_n) = x_n \phi\}) = \Gamma_{t_1, \ldots, t_n}(x_1, \ldots, x_n)$$

For all $(t_1, \ldots, t_n, x_1, \ldots, x_n) \in T^n \times X^n$? In that case Γ would indeed be the joint cost distribution of $\phi(\cdot, t_1) \cdots \phi(\cdot, t_n)$ and its value at $(t_1, \ldots, t_n, x_1, \ldots, x_n)$ would be the least upper bound of the cost of trajectories going through x_1 at t_1, x_2 at $t_2 \cdots x_n$ at t_n.

To answer the question posed above we have to look for a likely candidate for Ω. But we already know that if one such Ω exists it may be mapped by L

$$L : \omega \longrightarrow \phi(\omega, \cdot) \in X^T$$

into the trajectory space X^T, thereby inducing on X^T a cost function C^L whose value at any trajectory $f(\cdot) \in X^T$ is:

$$C^L(f(\cdot)) = C(L^{-1}(f(\cdot)) = C(\{\omega : \phi(\omega, t) = f(t) \text{ on } T\}).$$

Furthermore, it can be seen that the cost C^L will also satisfy the requirements expressed by (3) is we define

$$\phi'(\cdot, t) : X^T \longrightarrow X \text{ by } \phi'(f(\cdot), t) = f(t).$$

Thus we may assume that Ω is the trajectory space X^T and the problem may be reformulated in the following terms: given a set function C defined on the collection $\mathcal{S}'$ of subsets of Ω having the form

$$(5)\qquad \{f(t) : f(t_1) = x_1 \cdots f(t_n) = x_n \ t_1, \ldots, t_n \in T\}.$$

Can it be extended to a set function on all the subsets of Ω which would satisfy (1) a, b, c, ? In view of the remarks made above it is enough to define it on the points of Ω.

If such an extension exists it is clear that $C(\{\omega\}) \leq C(A)$ whenever $A \supset \{\omega\}$. Since we know the value of C on a subset $\mathcal{S}'$ of $\mathcal{S}$, namely, all the sets of the form (5), it is natural to define:

$$(6)\quad \begin{aligned} &(a)\quad C'(\{\omega\}) = \inf\{C(A) : \omega \in A,\ A \in \mathcal{S}'\} \\ &(b)\quad C'(A) = \sup\{C'(\{\omega\}) : \omega \in A\} \end{aligned}$$

and inquire whether for all $A \in \mathcal{S}'$, $C'(A) = C(A)$ and therefore C' is a true extension of C. To answer this inquiry it is enough to show that for every set $A \in \mathcal{S}'$ there exists some $\omega \in A$ such that $C'(\{\omega\}) = C(A)$ since we know already from (6) that $C'(\{\omega\}) \leqq C(A)$.

We shall presently state sufficient conditions for this to happen.

Theorem 1.[1] Let $T = [\alpha_1, \alpha_2]$, $0 \leqq \alpha_1 \leqq \alpha_2 < \infty$ and suppose that $\Gamma_{t_1, \ldots, t_n}(x_1, \ldots, x_n)$, $\alpha_1 \leqq t_1 \leqq t_2 \cdots \leqq t_n \leqq \alpha_2$ satisfies the following conditions in addition to the consistency condition (2):

(a) $\Gamma_{\alpha_1, \alpha_2}(x, y) \leqq M < \infty$, (b) $\Gamma_{t_1, \ldots, t_n}(x_1, \ldots, x_n)$ is upper semi-continuous in $x_1, \ldots, x_n$, (c) $\{z : \Gamma_{\alpha_1, t, \alpha_2}(x, z, y) \geqq a\}$ is compact for any (fixed) t, a, x and y. Then for all $A \in \mathcal{S}'$ there exists some $\omega \in A$ such that $C(A) = C'(\{\omega\})$, ω is an optimal trajectory in A, and C' is an extension of C.

In the proof of the theorem, the following standard convention will be adopted. A partition P of $[\alpha_1, \alpha_2]$ is a sequence of points $\alpha_1 \leqq t_1 \leqq t_2 \leqq \cdots \leqq t_n \leqq \alpha_2$. If P_1 and P_2 are two partitions, P_2 refines $P_1 (P_1 = P_2)$ if P_1 is a subsequence of P_2, and $P_1 \wedge P_2$ is the coarsest common refinement of P_1 and P_2. For any element $y \in X^{n+2}$, $y = (x_0, \ldots, x_{n+1})$ we let $\Gamma_p(y) = \Gamma_p(x_0, \ldots, x_{n+1}) = \Gamma_{\alpha_1, t_1, \ldots, t_n, \alpha_2}(x_0, \ldots, x_{n+1})$ whenever $P = (\alpha_1, t_1, \ldots, t_n, \alpha_2)$. We shall also define a projection operator $M_p : X^T \to X^P$ by $M_p(f(t)) = (f(\alpha_1), f(t_1) \cdots f(t_n), f(\alpha_2))$ where $P = (\alpha_1, t_1, \ldots, t_n, \alpha_2)$.

Proof of Theorem 1. Let A be any set of $\mathcal{S}'$. Then for some partition P_0 and some $y_0 \in X^{P_0}$, $A = \{f(\cdot) : M_{P_0}(f(\cdot)) = y_0\}$. The consistency condition (2) may be rewritten

$$(7) \quad \sup_{f(\cdot)} \{\Gamma_P(M_P(f(\cdot))) : M_{P_0}(f(\cdot)) = y_0\} = \Gamma_{P_0}(y_0) = C(A)$$

whenever P refines P_0. It also follows from (2) that

$$(8) \quad \Gamma_{\alpha_1, t, \alpha_2}(x_0, f(t), x_{n+1}) \geqq \Gamma_P(M_P(f(\cdot)))$$

for all t belonging to the sequence P, and if we let $S_t = \{z : \Gamma_{\alpha_1, t, \alpha_2}(x_0, z, x_{n+1}) \geqq C(A) - \epsilon\}$ we may assume that the supremum in (7) is taken over the compact set $\prod_{t \in T} S_t$.[2] Since M_P is continuous on the product topology and $\Gamma_{\alpha_1, t_1, \ldots, t_n, \alpha_2}(\cdot)$ is upper semicontinuous, the supremum is attained and for every P refining P_0 the set

$$U_P = \{f(\cdot) : f \in (\prod S_t) \cap A,\ \Gamma_P(M_P(f(t))) = C(A)\}$$

is closed compact nonempty, and if $P_1 \geqq P_2$, $U_{P_1} \supset U_{P_2}$. Thus $U_{P_1} \cap U_{P_2} \supset U_{P_1 \wedge P_2} \neq \phi$, the sets U_P have the finite intersection property and there exists some $f(t) \in \bigcap U_P$. Let B be any set of $\mathcal{S}'$ containing $f(\cdot)$, $B = \{g(\cdot) : M_P(g(\cdot)) = y\}$, $P \geqq P_0$ for some partition P and $y \in X^P$. If $f(t) \in B$, $M_P(f(\cdot)) = y$ and $C(B) = \Gamma_P(y)$ $= \Gamma_P(M_P(f(\cdot)) \geqq \Gamma_{P \wedge P_0}(M_{P \wedge P_0}(f(\cdot))) = C(A)$. Hence $C'(f(\cdot))$ $= \inf \{C(B) : f(t) \in B\} = C(A)$.

From now on we shall denote by $C_T(\{\omega\})$ the functional obtained by the above procedure to indicate the dependence of C on the interval of definition of the function ω. Thus

$$C_T(\{\cdot\}) : X^T \longrightarrow R \cup \{-\infty\} .$$

Furthermore it will be convenient in the next section to let ω belong to X^{R^+} rather than X^T. To make the notation consistent, we shall let M_T be the projection of X^{R^+} into X^T and consider the function $C_T(\{M_T(\omega)\})$.

6. THE MARKOVIAN CASE, CONTINUITY OF TRAJECTORIES, AND THE STRONG SEMIGROUP PROPERTY

We recall that our problem was motivated by an attempt to find optimal trajectories of a functional $F_T(\omega)$ defined on the trajectory space X^{R+} for some $T = [\alpha_1, \alpha_2]$ (Section 2).[1] It is convenient to interpret $F_T(\omega)$ as the conditional cost of choosing ω given that the initial value of ω is $M_{\alpha_1}(\omega)$. Thus, if $C_{\alpha_1}(x)$ is a preference function on the choice of the initial state, we can write the total cost of the following ω on the set $[\alpha_1, \alpha_2]$ to be $C_{\alpha_1}(M_{(\alpha_1)}(\omega)) + F_{[\alpha_1, \alpha_2]}(\omega)$.

It is now possible to define

$$\Gamma_P(y) = \sup_{\omega} \{F_T(\omega) + C_{\alpha_1}(M_{(\alpha_1)}(\omega)) : M_P(M_T(\omega)) = y, \omega \in X^R\}$$

for any partition P of $T = [\alpha_1, \alpha_2]$ and $y \in X^P$, to obtain the joint cost distribution of $(x(\alpha_1), x(t_1) \cdots x(t_n) x(\alpha_2))$. It is easy to see that the latter satisfies the consistency condition I.5.2. We may in the manner of the previous section (equation I.5.6) utilize the distribution to define a new function $C_T(\{\cdot\})$ on the trajectories. Since, roughly speaking, $C_T(\{\omega\})$ was obtained by approximating the trajectory ω by piecewise optimal trajectories of F_T,[2] we expect a relation and perhaps an equality between $F_T(\omega) + C_{\alpha_1}(M_{(\alpha_1)}(\omega))$ and $C_T(M_T(\omega))$. Indeed it follows from the definition of $\Gamma_P(\cdot)$ that

$$\Gamma_P(M_P(M_T(\omega))) \geqq F_T(\omega) + C_{\alpha_1}(M_{(\alpha_1)}(\omega)) \text{ for all } \omega \in X^R,$$

and we have

$$(1) \quad C_T(M_T(\omega)) = \inf_P \Gamma_P(M_p(M_T(\omega))) \geqq F_T(\omega) + C_{\alpha_1}(M_{\alpha_1}(\omega)).$$

Furthermore, if we let $\Gamma'_p(y) = \sup \{C_T(M_T(\omega)) : M_P(M_T(\omega))\} = y$, it follows from (1) that $\Gamma'_p(y) \geqq \Gamma_p(y)$. On the other hand, $C_T(\omega)$

was defined to be the infimum over P of $\Gamma_P(M_P(M_T(\omega)))$ (I.5.6) hence $\Gamma_P(M_P(M_T(\omega))) \geqq C_T(M_T(\omega))$ and $\Gamma_P(y) \geqq \sup \{C_T(M_T(\omega)) : M_P(M_T(\omega)) = y\} = \Gamma_P(y)$ we conclude that $\Gamma_P(y) = \Gamma'_P(y)$ and therefore not only does $C_T(M_T(\omega))$ dominate $F_T(\omega) + C_{\alpha_1}(M_{\alpha_1}(\omega))$ but also they both

generate the same cost function $\Gamma_P(\cdot)$.[3] Despite all of these relations it is easy to find examples where $F_T(\omega) + C_{\alpha_1}(M_{(\alpha_1)}(\omega)) \neq C_T(M_T(\omega))$.

In Chapter II we shall show such examples and also find that, as may be expected, $C_T(M_T(\omega))$ is a smoothed-out version of $F_T(\omega) + C_{\alpha_1}(M_{(\alpha_1)}(\omega))$, at least in the special case studied in that chapter.

We may now inquire whether there are other properties of $F_T(\cdot)$ which are induced on $C_T(\cdot)$. In particular, suppose $F_T(\cdot)$ satisfies conditions I.2.1 and I.2.2. We expect $C_T(M_T(\omega) - C_{\alpha_1}(M_{(\alpha_1)}(\omega))$ to satisfy those conditions also. But by lemma I.1.2

these conditions imply that the cost function is Markovian, i.e.,

(2) $\Gamma_{\alpha_1, t_1, \ldots, t_n, \alpha_2}(x_0, \ldots, x_{n+1})$

$$= C_{\alpha_1}(x_0) + c_{\alpha_1, t_1}(x_0, x_1) + \cdots + c_{t_n, \alpha_2}(x_n, x_{n+1}).$$

Thus, it is enough to prove that $C_T(M_T\omega) - C_{\alpha_1}(M_{\alpha_1}(\omega))$ satisfies I.2.1, I.2.2 whenever (2) holds.

Lemma 1. Let $\Gamma_{\alpha_1, t_1, \ldots, t_n, \alpha_2}(x_0, \ldots, x_{n+1})$ be a function defined on the space $T^{n+2} \quad X^{n+2}$ for every n, n=1, 2, 3, ... satisfying the consistency condition I.5.2 and having the form (2). Let $C_T(M_T(\omega))$ be defined by I.5.6. and let $F'_T(\omega) = C_T(M_T(\omega)) - C_{\alpha_1}(M_{(\alpha_1)}(\omega))$. Then for all $0 \leqq a \leqq b \leqq c < \infty$, we have

$$(4) \qquad F'_{[a,c]}(\omega) = F'_{[a,b]}(\omega) + F'_{[b,c]}(\omega).$$

Proof. We shall use the notation of the remarks following the statement of theorem I.5.1. For any $0 \leqq a \leqq b \leqq c < \infty$ and partitions P_1 of $[a,b]$ and P_2 of $[b,c]$ we let $P_1 P_2$ be the partition of $[a,c]$ defined by the sequence P_1 followed by P_2. It is easy to see that if P is a partition of $[a,b]$ and for some $a \leqq b \leqq c$, $P_1 = (a,b,c)$ there exists a partition P_2 of $[a,b]$ and P_3 of $[b,c]$ such that $P \wedge P_1 = P_2 P_3$.

Let $T = [a,c]$, $T_1 = [a,b]$ and $T_2 = [b,c]$. Then M_P projects X^T into X^P, M_{P_1} projects X^{T_1} into X^{P_1} and M_{P_2} projects X^{T_2} into X^{P_2} whenever P, P_1, and P_2 are partitions of T, T_1, and T_2, respectively. Rewriting I.5.6.(a):

$$(5) \qquad F'_T(\omega) = \inf_P [\Gamma_P(M_P(M_T(\omega))) - C_a(M_{(a)}(\omega))],$$

where the infimum is taken over all partitions P of $[a,c]$. Thus,

$$F'_{[a,c]}(\omega) \leqq \Gamma_{P_1 P_2}(M_T(\omega))) - C_a(M_{(a)}(\omega)) = \Gamma_{P_1}(M_{P_1}(M_{T_1}(\omega)))$$

$$- C_a(M_{(a)}(\omega)) + \Gamma_{P_2}(M_{P_2}(M_{T_2}(\omega))) - C_b(M_{(b)}(\omega))$$

for any partition P_1 of T_1 and P_2 of T_2. Minimizing the right side first with respect to P_1 and then with respect to P_2 we obtain:

(6) $$F'_T(\omega) \leqq F'_{T_1}(\omega) + F'_{T_2}(\omega)$$

Conversely for any partition P of $[a, c] = T$

$$\Gamma_P(M_P(\omega) \geqq \Gamma_{P\wedge(a,b,c)}(M_{P\wedge(a,b,c)}(M_T(\omega))) = \Gamma_{P_1}(M_{P_1}(M_{T_1}(\omega)))$$

$$+ \Gamma_{P_2}(M_{P_2}(M_{T_2}(\omega))) - C_b(M_{(b)}(\omega) \geqq F'_{T_1}(\omega) + F'_{T_2}(\omega) + C_a(M_{(a)}(\omega))$$

or

(7) $$F'_T(\omega) = \inf \Gamma_P(M_P(M_T(\omega))) - C_a(M_{(a)}(\omega))$$

$$\geqq F'_{[a,b]}(\omega) + F'_{[b,c]}(\omega),$$

Combining (6) and (7), we obtain the desired conclusion.

Thus for any function $c_{t_1, t_2}(\cdot, \cdot)$ having the semigroup property (I.2.4.), it is possible to define a functional $F'_T(\cdot)$ on the trajectories which satisfies conditions I.2.1. and I.2.2. Furthermore, if $c_{t_1, t_2}(\cdot, \cdot)$ is upper semicontinuous bounded above and

(8) $$\{y : c_{t_1, t_2}(x, y) \geqq a\}$$

is compact for every x and a, it is easy to check that the assumptions of theorem 1 of section 5 are satisfied and therefore F'_T can be used to generate back $c_{t_1, t_2}(\cdot, \cdot)$:

(9) $$c_{t_1, t_2}(x_1, x_2) = \sup \{F_{[t_1, t_2]}(x(\cdot)) : x(t_1) = x_1, x(t_2) = x_2\} .$$

Let S be a set in $R^+ \times X$ such that whenever $x(t)$ is a trajectory joining x_1 and x_2 satisfying $F'(x(\cdot)) > -\infty$, $x(t)$ must cross S at some point.

$$\{x(t) : x(t_1) = x_1, x(t_2) = x_2, F'(x(\cdot)) > -\infty\}$$

$$= \bigcup_{(t, z) \in S} \{x(t) : x(t_1) = x_1, x(t_2) = x_2, x(t) = z, F'(x(\cdot)) > -\infty\}$$

Then:

$$\{F'_{[t_1, t_2]}(x(\cdot)) : x(t_1) = x_1, x(t_2) = x_2, F'_{[t_1, t_2]}(x(\cdot)) > -\infty\}$$

$$= \bigcup_{(t,z) \in S} \{F'_{[t_1, t_2]}(x(\cdot)) : x(t_1) = x_1, x(t_2) = x_2, x(t) = z, F'_{[t_1, t_2]}(x(\cdot)) > -\infty\}$$

$$= \bigcup_{(t,z) \in S} \{F'_{[t_1, t]}(x_1(\cdot)) + F'_{[t, t_2]}(x_2(\cdot)) : x_1(t_1) = x_1, x_1(t) = z = x_2(t), x_2(t_2) = x_2$$

$$F'_{[t_1, t]}(x_1(\cdot)) > -\infty, F'_{[t, t_2]}(x_2(\cdot)) > -\infty\}$$

and utilizing (9), we obtain

$$(10) \quad c_{t_1 t_2}(x_1, x_2) = \sup_{(z, t) \in S} [c_{t_1, t}(x_1, z) + c_{t, t_2}(z, x_2)] .$$

In particular, if $F_{[t_1, t_2]}(x(\cdot)) > \infty$ implies that $x(t)$ is continuous, it can be seen that any hyperplane L in $R^+ \times X$ separating (t_1, x_1) and (t_2, x_2) will satisfy the requirements on S. In that case

$$(11) \quad c_{t_1, t_2}(x_1, x_2) = \sup_{(t,z) \in L} [c_{t_1, t}(x,z) + c_{t', t_2}(z, x_2)],$$

and we say that $c_{t_1, t_2}(\cdot, \cdot)$ has the strong semigroup property.

Thus, it may be of interest to find out when the allowable trajectories, i.e., those for which $F_T(x(\cdot)) > -\infty$, are continuous. To this end we shall state the following definition:

Definition 1. An M.t.c.f. is c-regular iff for any $x \in X$, $a \in R$, $t \in R^+$ and any sequence $\alpha_n \uparrow t$, $\beta_n \downarrow t$ and x_n

$$c_{\alpha_n, t}(x_n, x) \geqq a \text{ or } c_{t, \beta_n}(x, x_n) \geqq a$$

imply $x_n \longrightarrow x$ in the topology on X.

It can be shown now that if an M.t.c.f. is c-regular and $c_{\alpha, \beta}(x, y) \leqq k(t_1, t_2)$ for all x, y and $t_1 \leqq \alpha \leqq \beta \leqq t_2$ all the allowable trajectories are continuous. Indeed, let $x(t)$ be a trajectory on the interval $[t_1, t_2]$ and suppose there exists some $t_1 \leqq t \leqq t_2$ such that $x(t)$ is discontinuous at t. Then for some neighborhood N of $x(t)$ there exist $t_n \longrightarrow t$ such that $x_{t_n} \notin N$ and we may further assume that either (a) $t_n \uparrow t$ or (b) $t_n \downarrow t$. In case (a) we may write

$$\begin{aligned} F_{[t_1, t_2]}(x(\cdot)) &\leqq c_{t, t_n}(x(t_1), x(t_n)) + c_{t_n, t}(x(t_n), x(t)) \\ &\quad + c_{t, t_2}(x(t), x(t_2)) \\ &\leqq 2k(t_1, t_2) + c_{t_n, t}(x(t_n), x(t)) \end{aligned}$$

and since $x(t_n) \not\to x(t)$ the last expression can be made arbitrarily close to $-\infty$ and $F_{[t_1, t_2]}(x(\cdot)) = -\infty$. Case (b) follows in much the same way. It is possible now to combine the facts mentioned above to obtain:

Lemma 2. Any M.t.c.f. which is uniformly bounded above on bounded time intervals, c-regular upper semicontinuous, and satisfies (8), has the strong semigroup property.

7. THE CONCEPT OF INDEPENDENCE, PROCESSES WITH INDEPENDENT INCREMENTS, AND SOME EXAMPLES

We shall conclude this chapter with a short discussion of the concept of independence of functions on a cost-function space and an example.

Definition 1. Let f_k, k=1, 2, 2, . . . , n be functions defined on a cost-function space $(\Omega, \mathcal{S}, C)$. f_n are said to be mutually independent iff for any collection of sets A_k in X

$$C\left(\bigcap_{k=1}^{n} \{\omega : f_k(\omega) \in A_k\}\right) = \sum_{k=1}^{n} C\{\omega : f_k(\omega) \in A_k\} .$$

Definition 2. A control process $\phi(t, \omega)$ defined on $(\Omega, \mathcal{S}, C)$ has independent increments iff $\phi(0, \omega) = 0$ and the variables $\phi(t_i, \omega) - \phi(t_{i-1}, \omega)$, i=1, 2, . . . , n are mutually independent for all n and $t_i \leqq t_{i+1}$.

Thus, the cost function of a process with independent increments has the form:

$$\Gamma_{t_1, \ldots, t_n}(x_1, \ldots, x_n) = c_{t_1}(x_1) + c_{t_1, t_2}(x_2 - x_1) + \cdots + c_{t_{n-1}, t_n}(x_n - x_{n-1})$$

and the M.t.c.f. is $c_{t_1, t_2}(x_2 - x_1)$. If, in addition, the process is time invariant, $c_{t_1, t_2}(x_2 - x_1) = c_{0, t_2 - t_1}(x_2 - x_1)$, and the semigroup property reduces to

$$c_{0, t_3 - t_1}(x_3 - x_1) = \sup_{x_2} \left[c_{0, t_3 - t_2}(x_3 - x_2) + c_{0, t_2 - t_1}(x_2 - x_1) \right]$$

or equivalently

$$(1) \qquad c_{0, t+\tau}(x) = \sup_{z} \left[c_{0, t}(z) + c_{0, \tau}(x - z) \right] .$$

Functions satisfying (1) have been presented in the literature.[1, 2] It is not difficult to show that if $c_t(x)$ is convex in x and is upper semi-continuous, it has the form:

(2) $$c_t(x) = tc_1(x/t).$$

However, not all functions satisfying (1) have this form.

The example below will be borne in mind throughout the next chapter.

Let $X = R'$ and $c_{0,t}(x, y) = -\sqrt{|y-x|}$. Since $-\sqrt{|z|} + (-\sqrt{|x-z|} \leq -\sqrt{|x|}$ with equality attained at $z = 0$ or $z = x$

$$\sup [c_{0,t}(0, z) + c_{0,\tau}(0, x-z) = c_{0,t+\tau}(0, x)$$

and (1) is satisfied. The function $c_{0,t}(0, x)$ appears to be well behaved. It is continuous in x and t jointly and is differentiable in x everywhere except 0. Still, it is not c-regular and does not have the strong semigroup property since

$$\sup_{0\leq\tau\leq t} [c_{0,\tau}(0, z_0) + c_{0,t-\tau}(0, \omega_0)]$$

$$= \sup (\sqrt{z_0} + \sqrt{\omega_0} \neq c_{0,t}(0, \omega_0 + z_0),$$

where L is the plane $\{(t, z) : z = z_0\}$ $0 \leq z_0 \leq \omega_0 + z_0$.

Indeed if $F'_T(\cdot)$ is the cost functional generated by $c_{t_1,t_2}(x, y)$, all the allowable trajectories of $F'_{[t_1,t_2]}(\cdot)$ which are continuous on $[t_1, t_2]$ must be constant. If otherwise, then there exists a trajectory x(t) which is continuous on $[t_1, t_2]$ and is not constant on that interval. There exists therefore times t_3, t_4 with $t_1 \leq t_3 \leq t_4 \leq t_2$, such that $x(t_3) \neq x(t_4)$. Since

$$F'_{[t_1,t_2]}(x(\cdot)) = F'_{[t_1,t_2]}(x(\cdot)) + F'_{[t_3,t_4]}(x(\cdot))$$

$$+ F'_{[t_4,t_2]}(x(\cdot)) \leq F'_{[t_3,t_1]}(x(\cdot)),$$

it is enough to show that $F_{[t_3, t_4]}(x(\cdot)) = -\infty$. But $x(t)$ is continuous. There exist therefore s_j, $t_3 = s_0 < \cdots < s_{2^n} = t_4$ with $x(s_k) = x(t_3) + k(x(t_4) - x(t_3))/2^n$, $k = 1, 2, 3, \ldots, n$, and we have

$$F_{[t_3, t_4]}(x(\cdot)) \leqq \sum_{i=1}^{2^n} -\sqrt{|x(s_i) - x(s_{i-1})|}$$

$$= -\sqrt{|x(t_4) - x(t_3)|}\ (2^n \frac{1}{2^{n/2}})$$

$$= -\sqrt{|x(t_4) - x(t_3)|}\ (2^{n/2}) \longrightarrow -\infty.$$

CHAPTER II

1. CONVEX FUNCTIONS AND SETS, THE MAXIMUM TRANSFORM

The present chapter will be devoted to the study of M.t.c. functions having the form:

$$(1) \qquad c_{t_1 t_2}(x, y) = c_{0, t_2 - t_1}(y - T_{t_2 - t_1} x),$$

where T_t is a linear transformation for each t. If we construct the process corresponding to the cost function (1) in the manner of Chapter I, it can be seen that the variables $x(t_3) - T_{t_2}(x(t_2))$ and $x(t_2) - T_{t_1}(x(t_1))$ are independent whenever $0 \leqq t_1 \leqq t_2 \leqq t_3 < \infty$ and we have therefore a slightly generalized bersion of the control process with independent increments.

As expected, such processes will correspond to linear systems and all linear systems will have cost functions of the form (1) above. The proof of these facts and characterization of $c_{t_1, t_2}(x, y)$ are found in the following three sections. In this section we state some preliminary results which are for the most part simple or well-known but are presented here for convenience.

The fundamental space X will still remain a locally convex Hausdorff linear topological space. We shall let $X^\dagger$, X', and X^* be, respectively, the space of all linear functionals on X, the space of all weakly bounded linear functionals on x, and the space of all continuous linear functionals on X. Thus $X^\dagger \supset X' \supset X^*$. $X^\dagger$ and X^* will be referred to as the algebraic and topological duals of X, respectively. The following three topologies are induced by X on $X^\dagger$ (and therefore on X^* and X') will be of interest to us, τ_1 the topology of uniform convergence on weakly bounded sets, τ_2 the topology of uniform convergence on weakly compact sets and $\tau_3 = \sigma$ the topology of pointwise convergence. Thus the usual bases N_i i=1, 2, 3 of the neighborhood system of the origin corresponding to τ_i are

$$N_i = \{\{\ell : \sup\{\ell(x) : x \in S\} < 1\} \text{ for some } S \in G_i\},$$

where G_1 is the collection of weakly bounded sets in X, G_2 is the collection of weakly compact convex sets in X containing the origin and G_3 is the collection of all finite subsets of X.

For any function f, $f : X \rightarrow R \bigcup \{-\infty\}$ we may define the set S^f R × X:

$$S^f = \{(y, x) : y \in R,\ x \in X,\ y \leqq f(x)\}$$

and the function $M(f(\cdot))(\cdot)$, $M(f(\cdot))(\cdot) : X^\dagger \rightarrow R \bigcup \{+\infty\}$:

$$M(f)(\ell) = \sup_{x \in X} (f(x) - \ell(x))$$

for all $\ell \in X^\dagger$. Thus the set S^f represents the "volume" under the graph of f. The transformation M will be discussed later in some detail. It is referred to as the maximum transform[1] or the Legendre transform[2] and its value at f will be called the support function of f. In all that follows we adopt the convention of Dunford and Schwartz[3] and denote by $\overline{S}$ the closure of S by co S, the convex hull of S and by $\overline{\text{co}}$ S, the convex closure of S. We also extrapolate this notation to functions. Thus co f will be the smallest concave function dominating g, and $\overline{\text{co}}$ f will be the smallest upper semicontinuous concave function dominating f. Finally, for all $\ell \in X^\dagger$ we define $\ell(S)$ to be the supremum of $\ell(x)$ over all $x \in S$.

The lemmas below relate some of the properties of f, $\overline{\text{co}}$ f, S^f, $m(S^f)$, and M(f)(m).

Lemma 1. (a) f is upper semicontinuous iff S^f is closed.

(b) S^f is a convex set iff f is concave.

Proof. Let $H_a = \{(y, x) : y = a,\ f(x) \geqq a\}$. $H_a = \{(y, x) : y \leqq f(x) \bigcap \{(y, x) : y = a\} = S^f \bigcap \{(y, x) : y = a\}$. Therefore, if S^f is closed

H_a and therefore its projection onto X must be closed and f is upper semicontinuous.

Conversely, suppose f(x) is upper semicontinuous and (y_0, x_0) is a limit point of S^f. Let N_ϵ be a neighborhood of x_0 such that on it $(f(x) - f(x_0)) < \epsilon$. Since (y_0, x_0) is a limit point of S^f, $((y_0 - \epsilon, y_0 + \epsilon) \times N_\epsilon) \cap S^f \neq \phi$ and for some $z \in N_\epsilon$ and some $w \leqq f(z) \leqq f(x_0) + \epsilon$ $(w, z) \in (y_0 - \epsilon, y_0 + \epsilon) \times N_\epsilon$. Hence, $y_0 \leqq w + \epsilon \leqq f(z) + \epsilon \leqq f(x_0) + 2\epsilon$. Thus, $y_0 \leqq f(x_0) + 2\epsilon$ for all ϵ and $y_0 \leqq f(x_0)$; hence, $(y_0, x_0) \in S^f$, and every limit point of S^f belongs to it.

Part (b) of the lemma is immediate from the definitions of convexity and of convex and concave functions.

<u>Definition 1.</u> For any two sets $S_1, S_2 \subset X$, and a real number k, we let $S_1 + S_2 = \{v + w : v \in S_1, w \in S_2\}$ and $kS = \{kv : v \in S\}$.

<u>Lemma 2.</u> Let ℓ be a linear functional on $R \times X$. Then $\ell((y, x)) = \ell((y, 0)) + \ell((0, x)) = y \circ \ell((1, 0)) + \ell((0, x))$ where $\ell((0, \cdot)) \in X^\dagger$. For any function $F : X \to R \cup \{-\infty\}$ we have:

(a) $$\ell(S^f) = \begin{cases} \ell(1,0)\left[M(f)\left(\dfrac{-\ell(0,\cdot)}{\ell(1,0)}\right)\right] & \text{whenever } \ell(1,0) > 0 \\ \infty & \text{whenever } \ell(1,0) \leqq 0 . \end{cases}$$

(b) $\ell(S_1 + S_2) = \ell(S_1) + \ell(S_2)$.

(c) $\ell(S_1) \geqq \ell(S_2)$ whenever $S_1 \supset S_2$, $S^{f_1} \supset S^{f_2}$ whenever $f_1 \geqq f_2$ and $f_1 \geqq f_2$ implies $M(f_1)(\cdot) \geqq M(f_2)(\cdot)$.

(d) Let $(T_z f)(x) = f(x - z)$. Then $M(T_z f)(\ell) = M(f)(\ell) - \ell(z)$.

(e) If A is a linear operator on X and AS is the image of S under A, $\ell(AS) = (\ell A)(S)$.

(f) For any linear operator A on X let $f(A^{-1}(y)) = \sup\{f(x) : Ax = y\}$, then $M(f(A^{-1}(\cdot)))(\ell) = M(f(\cdot))(\ell A)$.

(g) For any two linear functionals ℓ_1 and ℓ_2 on X and a positive real number k, $(\ell_1 + \ell_2)(S) \leqq \ell_1(S) + \ell_2(S)$ and $(k\ell)(S) = k(\ell(s))$. Thus, the function $\ell(S)$ is quasilinear in ℓ.

(h) $M(F)(\ell)$ is a convex function of ℓ.

(i) $M(f)(0) = \sup f(x)$.

Proof. Statements c, e, and i are immediate. We shall proceed to prove the others.

(a) $\ell(S^f) = \sup \{\ell(w, x) : w \leqq f(x)\} =$

$$= \begin{cases} \sup \{f(x)\ell(1,0) + \ell(0,x)\} \text{ whenever } \ell(0,x)\} \text{ whenever } \ell(1,0) > 0 \\ \infty \qquad \text{otherwise} \end{cases}$$

$$= \begin{cases} \ell(1,0) \sup \left(f(x) - \left(\frac{\ell(0,x)}{\ell(1,0)}\right)\right) \text{whenever } \ell(1,0) > 0 \\ \infty \qquad \text{whenever } \ell(1,0) \leqq 0, \end{cases}$$

(b) $\ell(S_1 + S_2) = \sup \{\ell(w) + \ell(v) : w \in S_1, v \in S_2\} = \ell(S_1) + \ell(S_2)$ since w, v may be chosen independently.

(d) $M(T_z(f))(\ell) = \sup_x (f(x-z) - \ell(x)) = \sup_w (f(w) - \ell(w) + \ell(z)) =$

$M(f)(\ell) - \ell(z)$ where w was substituted for x-z.

(f) $M(f(A^{-1}(\cdot))(\ell) = \sup_y (\sup_{Ax=y} f(x) - \ell(y)) = \sup_y \sup_{Ax=y} (f(x) - \ell(Ax))$

$= \sup_x (f(x) - \ell Ax) = M(f)(\ell A)$.

(g) $(\ell_1 + \ell_2)(S) = \sup \{(\ell_1(x) + \ell_2(x) : x \in S\}$ for all $x \in S$ $\ell_1(x)$ $+ \ell_2(x) \leqq \ell_1(S) + \ell_2(S)$ and the conclusion follows. Also, $\ell(kS)$ $= \sup \{k\ell(x) : x \in S\} = k \sup \{\ell(x) : x \in S\}$.

(h) $M(f)(\ell) = (1-\ell)(S^f)$ and since the function $m(S^f)$ is convex its restriction to m of the form $m = (1, \ell)$ is also convex.

Lemma 3. If S is closed and convex, (a) $S = \bigcap_{\ell \in X^*} \{x : \ell(x) \leqq \ell(S)\}$ and therefore, (b) if f is upper semicontinuous and concave, $f(x) = \inf_{\ell \in X^*} (M(f)(\ell) + \ell(x))$. In general, (c) $\overline{co}\, S = \bigcap_{\ell \in X^*} \{x : \ell(x) \leqq \ell(S)\}$, and (d) $\overline{co}\, f(x) = \inf_{\ell \in X^*} (M(f)(\ell) + \ell(x))$. Also, (e) the maximum transform of any function is lower semicontinuous on (X^*, σ).

Proof. (a) Since $S \subset \{x : \ell(x) \leqq \ell(S)\}$ for all $\ell \in X^*$, we have $S \subset \bigcap_{\ell \in X^*} \{x : \ell(x) \leqq \ell(S)\}$; on the other hand, if $x \notin S$, there exists $\ell \in X^*$ separating x and S.[4] Hence, $\ell(x) > \ell(S)$ and $x \notin \bigcap \{x : \ell(x) \leqq \ell(S)\}$; therefore, $S \supset \bigcap_{\ell \in X^*} \ell(x) \leqq \ell(S)\}$.

(b) Similarly, $f(x) \leqq \sup_y (f(y) - \ell(y)) + \ell(x)$ for all $\ell \in X^*$ and $f(x) \leqq \inf [M(f)(\ell) + \ell(x)]$. Conversely, if $\ell \in (R \times X)^*$ and $\ell(1, 0) \leqq 0$ $\ell(S^f) = +\infty$; hence, $\{x : \ell(x) \leqq \ell(S^f)\} = X$ and $S^f = \bigcap_{\ell(1,0)>0} \{(y, x) : \ell(y, x) \leqq (S^f)\} = \bigcap_{\ell(1,0)=1} \{y x) \ell(y, x)\ \ell(S^f)\}$.

Thus, if $f(x_0) < a$, there exists $m \in X^*$ such that $f(x) \leqq m(x) + M(f)(m)$ and $m(x_0) + M(f)(m) < a$. Thus, $\inf_m (m(x_0) + M(f)(m)) \leqq f(x_0)$.

(c) Let S be an arbitrary set and let $S' = \bigcap_{\ell \in X} \{x : \ell(x) \leqq \ell(S)\}$.

Then $S \subset S'$, and the latter is a closed convex set being the intersection of such sets. Thus, $S' \supset \overline{co}\, S$. Also, $\ell(\overline{co}\, S) \geqq \ell(S)$ and, therefore, $\overline{co}(S) = \bigcap \{x ; \ell(co\ S) \geqq \ell(x)\} \supset \{x : \ell(S) \geqq \ell(X) \forall \ell \in X^*\}$. Thus, $S' = \overline{co}\, S$.

(d) For any function f, f is no larger than $f' = \inf [M(f)(\ell) + \ell(x)]$ and the latter is upper semicontinuous and concave, hence, $f' \geqq \overline{co}\, f$. Conversely, $\overline{co}\, f = \inf [M(co(f)(\ell) + \ell(x)] \geqq \inf [M(f)(\ell) + \ell(x)] = f'$;

(e) M(f) is the supremum of lower semicontinuous functions, hence itself belongs to that class.

In the next few paragraphs we shall investigate the continuity properties of $M(f)(\ell)$ as a function of ℓ. In particular, we shall show that the Legendre transform of a large class of functions are continuous in the appropriate topology. To discuss these functions it is convenient to made the following definitions.

Definition 2. A function f on X $f: X \to R\ \{-\infty\}$ is said to be sup-compact (sup-bounded) in the topology τ iff $\{x : f(x) \geqq k\}$ is compact (bounded) for any $k \in R$.

Definition 3. A function $f: X \to [-\infty, \infty)$ will be said to be regular (b-regular) iff $f(x) - \ell(x)$ is weakly sup-compact (weakly sup-bounded) whenever $\ell \in X^*$.

Definition 4.[5] (Bellman). The maximum convolution of functions f and g will be the function $(f \oplus g)(\cdot)$ defined by

$$(f \oplus g)(x) = \sup_y (f(y) + g(x-y)) = \sup_y (f(x-y) + g(y)).$$

Lemma 4.[6] (Moreau). Let f be a concave upper semicontinuous function on X with values in $[-\infty, \infty)$ which is sup-bounded in the weak topology on X and for some x, $f(x) > -\infty$. Let Z be the linear space defined by $Z = \{\ell : \ell \in X^\dagger, \sup\{|(x)| : f(x-x_0) \geqq a\} < \infty$ for all $x_0 \in x, a \in R\}$ and let τ be the topology of uniform convergence on all sets of the form $\{x : f(x-x_0) \geqq a\}$ for some $x_0 \in X$, $a \in R$. Then (a) (Z, τ) is a locally convex Hausdorff linear topological space and $Z \subset X^*$ (b) $M(f)(\ell)$ is continuous at 0 on (Z, τ).

Proof. (a) follows immediately from Bourbaki.[7]

(b) By lemma 3(b) $f(x) = \inf_m (M(f)(m) + m(x))$. Thus there exists some $m \in X^*$ such that $M(f)(m) < \infty$ and $f(x) \leqq M(f)(m) + m(x)$. Furthermore, if b is some number such that the set $B = \{x : f(x) \geqq b\}$ is nonempty $\sup f(x) \leqq \sup\{f(x) : x \in B\} \leqq M(f)(m) + \sup\{m(x) : x \in B\}$.

The last expression is finite since B is weakly bounded and f must therefore be uniformly bounded. Let $a = \sup f(x) > -\infty$ and let $K_k = \{x : f(x) > a - k\}$. Then if $k > 0$, $K_k \neq \phi$ and there exist some x_1 belonging to it. Let $N = \{-\ell : \ell \in Z, -(\ell(K_k - x_1)) > -k + (a - f(x_1))\} \bigcap \{\ell : |\ell(x_1)| < k\}$. Then N is a neighborhood of the origin τ. Also, for all $\ell \in N$ and $x \in K_k$, $f(x) \leqq a$ $|\ell(x_1)| \leqq k$ and $-\ell(x) \leqq k - a + f(x_1) - \ell(x_1)$. Hence, $a - 2k \leqq f(x_1) - \ell(x_1) \leqq \sup_{x \in K_k} (f(x) - \ell(x) \leqq a + k - a + f(x_1) - \ell(x_1) \leqq f(x_1) + 2k \leqq a + 2k$ and

$$\left| \sup_{x \in K_k} (f(x) - \ell(x)) - a \right| \leqq 2k.$$

We shall complete the proof by showing that outside K_k, $f(x) - \ell(x) - a \leqq k$ and therefore $|M(f)(\ell) - M(f)(0)| \leqq 2k$ for all $\ell \in N$. Indeed if $x_2 \in K_k^c$ either $f(x_2) = -\infty$ and $f(x_2) - \ell(x_1) = -\infty < k$ or $f(x_2) > -\infty$. In the second case $f(\alpha(x_2 - x_1) + x_1)$ is concave in α equals $f(x_1)$ at $\alpha = 0$ and since $x_2 \in K_k^c$ $f(\alpha(x_2 - x_1) + x_1) = f(x_2) < a-k$ at $\alpha = 1$. Thus for some $0 < \alpha < 1$, $f(\alpha(x_2 - x_1) + x_1) = a - k$ and $\alpha(x_2 - x_1) + x_1 \in K_k$. We may write: $\alpha(f(x_2)) + (1-\alpha) f(x_1) \leqq f(\alpha(x_2 - x_1) + x_1) = a - k$ and $f(x_2) \leqq \frac{1}{\alpha}((a-k) + (1-\alpha) f(x_1))$. Also $\ell(x_2) = \frac{1}{\alpha}[\ell(\alpha(x_2 - x_1) + x_1) - (1-\alpha)\ell(x_1)] \geqq \frac{1}{\alpha}[-k + a - f(x_1) + \ell(x_1) - \ell(x_1) + \alpha\ell(x_1)]$ for all $\ell \in N$ and $f(x_2) - \ell(x_2) \leqq \frac{1}{\alpha}[\alpha f(x_1) - \alpha\ell(x_1)] \leqq a + k$.

Lemma 4 has a converse:

Lemma 5.[7] (Moreau). If $M(f)(\cdot)$ is uniformly bounded above in a neighborhood of zero in the τ_2 (τ_1) topology on X^* and f is weakly upper semicontinuous, f is sup-compact (sup-bounded).

Proof. Let N be the neighborhood of zero on which $M(f)(\cdot)$ is bounded and let k be the bound. We may assume that N has the form $N = \{\ell : \ell(S) < b\}$ for some convex compact (bounded) set S containing 0 and some positive b. If $a < k$ we may write: $\{x : f(x) \geqq a\} = \{x : M(f)(\ell) + \ell(x) \geqq a \text{ for all } \ell\} \subset \{x : k + \ell(x) \geqq a \text{ for all } \ell \in N\} = \{x : \ell(x) > a - k \text{ for all } \ell \in N\} = \frac{k-a}{b}\{x : \ell(x) > -b \text{ for all } \ell \in N\}$

but if $\ell(-x) < b$ whenever $\ell(S) < b$, x must belong to $-S$. Hence $\{x : f(x) \geqq a\} \subset -S$ and the latter is a compact (bounded) set.

Corollary 1. Let f be a b-regular concave upper semicontinuous function and let (Z, τ) be defined as in lemma 4. Then $M(f)(\cdot)$ is continuous on $(\overline{X^*}, \tau)$ where $\overline{X^*}$ is the closure of X^* in (Z, τ) and therefore its restriction to X^* is continuous when X^* is assigned the topology τ_1. If f is also regular $M(f)(\cdot)$ is continuous on (X^*, τ_2).

Proof: By lemma 2 $M(f)(\cdot)$ is convex and by lemma 4 it is continuous at 0. Thus by appendix (1) it is enough to show that the set $S = \{\ell : M(f)(\ell) < \infty\}$ equals $\overline{X^*}$ and therefore its interior S^0 is the whole space. But suppose there exists $\ell_1 \notin S$. Then since $S^0 \neq \phi$, $\{\ell_1\}$ and S may be separated by a continuous linear functional and S^c must have an interior point.[8] This contradicts the that f is b-regular and $M(f)(\ell) < \infty$ for all ℓ in the set X^* which is dense in $\overline{X^*}$ (lemma 6).

When f is sup-bounded (sup-compact) $\tau_1 (\tau_2)$ is finer than τ and the rest of the assertion follows.

Lemma 6. If f is bounded above, f is b-regular iff $M(f)(\ell) = \sup(f(x) - \ell(x)) < \infty$ for all $\ell \in X^*$, hence, if f is bounded above and b-regular, $\overline{\text{cof}}$ is b-regular.

Proof. Suppose $\{x : f(x) \geqq \ell(x) + a\}$ is bounded for all $\ell \in X^*$, $a \in R$. Then $\sup(f(x) - \ell(x)) \leqq \sup\{f(x) - \ell(x) : f(x) - \ell(x) \geqq a\} \leqq \sup f(x) + \sup\{-\ell(x) : f(x) - \ell(x) \geqq a\}$. The latter set is bounded, hence the whole expression is bounded. Conversely suppose $\sup(f(x) - \ell(x)) < \infty$ for all $\ell \in X^*$ but $\{x : f(x) - \ell(x) \geqq a\}$ is unbounded for some $\ell \in X^*$. Then there exists $m \in X^*$ which is unbounded on this set and $f - \ell + m$ is unbounded on the same set.

Lemma 7. (Moreau).[9] (a) The maximum convolution of two sup-compact functions is sup-compact. (b) The maximum convolution of two concave functions is concave. (c) $M(f \oplus g) = M(f) + M(g)$. (d) If f_1 and f_2 are two sup-compact functions, $S^{f_1 \oplus f_2} = S^{f_1} + S^{f_2}$. (e) For any continuous linear operator N on X and a sup-compact function f, $S^{f(N^{-1}(\cdot))} = N' S^f$ where $f(N^{-1}(x)) = \sup\{f(y) : Ny = x\}$; the function

$f(N^{-1}(\cdot))$ is sup-compact and $N': R \times X \to R \times X$ is defined by $N'(y, x) = (y, Nx)$.

Proof. (a) By lemma I.4.3. $f \oplus g$ is upper semicontinuous and therefore the set $\{y : \sup(f(x) + g(y-x) \geqq a\} \subset \{x+z : f(x) \geqq a - \sup g,\ g(z) \geqq a - \sup f\}$ is closed and constrained in the sum of two compact sets and is therefore compact.[10]

$$\text{(b)}\quad \lambda_1 [f(y_1 - x_1) + g(x_1)] + \lambda_2 [f(y_2 - x_2) + g(x_2)]$$

$$\leqq f(\lambda_1(y_1 - x_1) + \lambda_2(y_2 - x_2)) + g(\lambda_1 x_1 + \lambda_2 x_2) \leqq \sup_z [f(\lambda_1 y_1 + \lambda_2 y_2 - z) + g(z)]$$

$$= (f \oplus g)(\lambda_1 y_1 + \lambda_2 y_2).$$

Taking suprema with respect to x_1 and x_2 on the left side the proof is completed.

$$\text{(c)}\quad M(f \oplus g)(\ell) = \sup_x ([\sup_y f(x-y) + g(y)] - \ell(x) =$$

$$\sup_y \sup_x [f(x-y) + g(y) - \ell(x)] = \sup_y \sup_\omega [f(\omega) + g(y) - \ell(\omega) - \ell(y)]$$

$= M(f)(\ell) + M(g)(\ell)$ where ω was substituted for $x-y$.

(d) Suppose $(y_1, x_1) \in S^f (y_2, x_2) \in S^g$. Then $y_1 \leqq f(x_1)$, $y_2 \leqq f(x_2)$ and $y_1 + y_2 \leqq f(x_1) + g(x_2) \leqq \sup [f(x_1 + x_2 - z) + g(z)] = (f \oplus g)(x_1 + x_2)$. Thus $(y_1 + y_2, x_1 + x_2) \in S^{f \oplus g}$.

Conversely, $(y, x) \in S^{f \oplus g} \Longrightarrow y \leqq \sup_z [f(x-z) + g(z)]$. Since the supremum is attained $y \leqq f(x-z_1) + g(z_1)$, hence $y - g(z_1) \leqq f(x-z_1)$ and $(y, x) = (y - g(z_1), x-z_1) + (g(z_1), z_1)$. Thus, $(y - g(z_1), x-z_1) \in S^f$, $(g(z_1), z_1) \in S^g$.

(e) $\sup \{f(y) : Ly = x\} = \sup \{f(y) : y \in \{Ly = x\} \cap \{z : f(z) \geqq \sup \{f(y) : Ly = z\} - \epsilon\}\}$. Thus, the supremum is taken over a compact set and is attained. We conclude that

$$\{x : \sup_{Ly=x} f(y) \geqq a\}$$

$$= \{Ly : f(y) \geqq a\} = L\{y : f(y) \geqq a\}.$$

This is the image under a continuous mapping of a compact set, hence it is compact and $f(L^{-1}(\cdot))$ is sup-compact. Furthermore $(a, y) \in S^{f(L^{-1}(\cdot))}$ implies $(a, y) = (a, L(z)) = L'(a, z)$ where $(a, z) \in S^f$ and $S^{f(L^{-1}(\cdot))} \subset L'S^{f(\cdot)}$. Conversely, $(a, z) \in L'S^{f(\cdot)}$ implies $(a, z) = (a, Ly)$ for some $y \in S^{f(\cdot)}$ which in turn implies that $a \leqq f(y)$ and $a \leqq \sup\{f(y): Ly = z\}$. Hence $L'S^{f(\cdot)} = S^{f(L^{-1}(\cdot))}$.

§2. THE CONVEX COST FUNCTION

We shall return now to the study of Markov transition cost functions having the form:

(1)
$$c_{t_1, t_2}(x, y) = c_{0, t_2 t_1}(0, y - T_{t_2 - t_1} x).$$

For convenience we shall drop the zeroes and consider a function $c_{(\cdot)}(\cdot): R^+ \times X \longrightarrow R \cup \{-\infty\}$ satisfying the semigroup condition:

(2)
$$c_{t_3 - t_1}(y - T_{t_3 - t_1} x) = \sup [c_{t_3 - t_2}(y - T_{t_3 - t_2} z) + c_{t_2 - t_1}(z - T_{t_2 - t_1} x)],$$

where T_t is a linear operator on X for each time t. Functions satisfying (1) will be called linear M.t.c.f.

Substituting t for $t_0 - t_2$, τ for $t_2 - t_1$, and ω for $z - t_{t_2 - t_1} x$, we obtain the equality

(3)
$$c_{t+\tau}(y - T_{t+\tau} x) = \sup_{\omega} [c_\tau(\omega) + c_t(y - T_t(\omega + T_\tau x)]$$

$$= \sup_{\omega} [c_\tau(\omega) + c_t((y - T_t T_\tau x) - T_t \omega)] = c_{t+\tau}(y - T_t T_\tau x).$$

Thus if ω belongs to the range $\mathcal{R}$ of the transformation $T_{t+\tau} - T_t T_\tau$, $\omega = (T_{t+\tau} - T_t T_\tau) v$ for some $v \in X$ and we have for any z:

(4)
$$c_{t+\tau}(z) = \ell_{t+\tau}(z + T_{t+\tau} v) - T_{t+\tau} v)$$

$$= c_{t+\tau}(z + T_{t+\tau} v - T_t T_\tau v) = c_{t+\tau}(z + \omega)$$

and $\ell_{t+\tau}$ is a constant on the set $z + \mathcal{R}$. Since we shall only be considering M.t.c. functions which converge to $-\infty$ as the state x becomes large we have to require that $z + \mathcal{R}$ be bounded and therefore $\mathcal{R} = \phi$ and T_τ has the semigroup property:

$$T_{t+\tau} = T_t T_\tau .$$

With these assumptions on T_t and some continuity assumptions it is desired to find out what the temporal behavior of $c_t(x)$ is and if possible some characterization of its structure as a function of x. As a first step we shall try to characterize the maximum transform of $c_t(\cdot)$. A knowledge of the latter will by lemma II.1.3 allow us to describe $\overline{co}\, c_t(\cdot)$, the convex closure of the transition cost function.

Let

$$(5) \qquad g_t(\ell) = M(c_t(\cdot))(\ell) .$$

Then applying lemmas II.1.6. and II.1.2.

$$(6) \qquad g_{t+\tau}(\ell) = M\left(\sup_\omega [\, c_t(\omega) + c_\tau(\cdot - T_\tau\, \omega)]\right)(\ell)$$

$$= M\left(\sup\left[\sup_{T_\tau(\omega)=v} c_t(\omega) + c_\tau(\cdot - v)\right]\right)(\ell) = g_t(\ell\, T_\tau) + g_\tau(\ell),$$

and $g_t(\ell)$ satisfies a very simple semigroup condition described by (6). In the study of the implications of (6) we shall make use of some properties of the Riemann integral which we shall state below without proof.

<u>Definition 1.</u> A linear topological space X is semicomplete iff every Cauchy sequence of elements of X converges to some element in X.

<u>Lemma 1.</u>[1] Let Z be a semicomplete, locally convex, Hausdorff linear topological space. Let $f(\alpha) : R^+ \longrightarrow Z$ be a continuous mapping.

Then:

1) $f(\alpha)$ is Reimann integrable on any interval $[a, b]$.

2) $$\int_a^c f(\alpha)\,d\alpha = \int_a^b f(\alpha)\,d\alpha + \int_b^c f(\alpha)\,d\alpha.$$

3) If Y is any linear topological space and L is sequentially continuous from Z to Y

$$L\int_a^b f(\alpha)\,d\alpha = \int_a^b Lf(\alpha)\,d\alpha.$$

4) $\int_a^b f(\alpha)\,d\alpha = (b-a)x$ for some $x \in \overline{co}\ \{f(\alpha) : \alpha \in [a, b]$.

5) The function $f(\partial, t) = \begin{cases} \frac{1}{\partial}\int_t^{t+\partial} f(\alpha)\,d\alpha & \partial > 0 \\ f(t) & \partial = 0 \end{cases}$

is jointly continuous in ∂ and t.

6) If $g(\cdot)$ is a lower semicontinuous real-valued convex function on Z

$$\int_a^b g(f(\alpha))\,d\alpha \geqq (b-a)\,g\left[\int_a^b \frac{f(\alpha)\,d\alpha}{(b-a)}\right].$$

Suppose $\{T_t\}_{t\in R^+}$ is a one-parameter set of linear transformations on a semi-complete space X and $T_t\,x$ is continuous in t on a subset W of X. We shall denote by $N_{a,b}$ the operator defined by:

$$N_{a,b}: W \longrightarrow X,\ N_{a,b}(x) = \frac{1}{b}\int_a^{a+b} T_t(x)\,dt.$$

Lemma 2. Let Z be a semicomplete locally convex Hausdorff linear topological space and let L_t be a one-parameter semigroup of continuous linear operators on Z. Let the subset W of Z be defined by $W = \{x : L_t(x) \text{ is continuous in } t\}$. Then if $g_t(x)$ is a nonnegative convex continuous function on Z such that $g_{t+\tau}(x) = g_t(x) + g_\tau(L_t x)$, there exists a positive and convex function g on Z such that for any $\partial \geqq 0$ and $x \in W$

$$g_t(N_{0,\partial}x) = \int_0^t g(N_{0,\partial}L_\tau x)\, d\tau$$

where

$$N_{0,\partial}(x) = \frac{1}{\partial}\int_0^\partial L_\tau(x)\, d\tau, \qquad \text{for all } x \in W$$

and $g(N_{0,\partial}x)$ is continuous in x on $\bigcup_{\partial \geqq 0} N_{0,\partial}W$.

Proof.

$$\int_0^t g_\partial(L_\tau x)\, d\tau = \int_0^t [\, g_{\partial+\tau}(x) - g_\tau(x)\,]\, d\tau = \int_\partial^{t+\partial} g_\tau(x)\, d\tau - \int_0^t g_\tau(x)\, d\tau$$

$$= \int_0^{t+\partial} g_\tau(x)\, d\tau - \int_0^\partial g_\tau(x)\, d\tau - \int_0^t g_\tau(x)\, d\tau$$

$$= \int_t^{t+\partial} g_\tau(x)\, d\tau - \int_0^\partial g_\tau(x)\, d\tau$$

$$= \int_0^{\partial} (g_{t+\tau}(x) - g_\tau(x))\, d\tau = \int_0^{\partial} g_t(L_\tau x)\, d\tau$$

where all the integrals are defined and finite because $g_t(x)$ is monotone in t and finite.

Thus:

$$\int_0^{t} g_{\partial}(L_\tau x)\, d\tau = \int_0^{\partial} g_t(L_\tau x)\, d\tau .$$

Since $g_t(\cdot)$ is convex and continuous the inequality of Lemma 1-6 applies and:

$$\int_0^{\partial} g_t(L_\tau x)\, d\tau \geqq \partial g_\tau \left[\frac{1}{\partial} \int_0^{\partial} L_\tau x\, d\tau \right], \qquad \text{for all } x \in W.$$

Therefore:

$$\int_0^{t} g_{\partial}(L_\beta L_\tau x)\, d\tau \geqq \partial g_t \left[\frac{1}{\partial} \int_0^{\partial} L_\beta L_\tau x\, d\tau \right]$$

$$= \partial \left[g_{t+\beta}\left(\frac{1}{\partial} \int_0^{\partial} L_\tau x d\tau\right) - g_\beta\left(\frac{1}{\partial} L_\tau x\, d\tau\right) \right] \geqq 0.$$

It follows that the function $g_\beta(N_{0,\,\partial}x)$ is absolutely continuous in β

and $\lim_{t \downarrow 0} \frac{1}{t} g_\tau(L_\beta N_{0,\,\partial}x) = \lim_{t \downarrow 0} \frac{1}{t} \left[g_{\beta+t}(N_{0,\,\partial}x) - g_\beta(N_{0,\,\partial}x) \right]$

exists a.e. (as a function of β) for all $x \in W$ and is equal a.e. to the Radon-Nikodym derivative of $g_\beta(N_{0,\,\partial}x)$. Thus,

$$g_\beta(N_{0,\partial}(x)) = \int_0^\beta \left[\limsup_{t\to 0} \frac{1}{t} g_y(N_{0,\partial} L_\tau x) \right] d\tau \ .$$

We now define $g(x) = \limsup_{t \to 0} \frac{1}{t} g_t(x)$ and obtain:

$$g_\beta(N_{0,\partial} x) = \int_0^\beta g(N_{0,\partial} L_\tau x)\, d\tau\,, \qquad \text{for all } x \in W\ .$$

We may also notice that $g(x)$ is convex since it is the lim sup of convex functions and for all $x \in W$

$$g_\partial(x) = \lim_{t\to 0} \frac{1}{t} \int_0^t g_\partial(L_\tau x)\, d\tau \geqq \limsup_{t \to 0} \partial \frac{g_t\left(\frac{1}{\partial} \int_0^\partial L_\tau x\, d\tau\right)}{t}$$

$$= \partial f(N_{0,\partial}(x)).$$

Therefore, $g(N_{0,\partial}(x))$ is continuous on $W.^2$ and utilizing again Lemma 1-6

$$g_\beta(N_{0,\partial} x) = \int_0^\beta g(N_{0,\partial} L_\tau x)\, d\tau \geqq \beta g(N_{0,\partial} N_{0,\beta} x)$$

or $g_\beta(x) \geqq \beta g(N_{0,\beta} x)$ for all $x \in \bigcup_{\partial > 0} N_{0,\partial} W$ from which we conclude that $g(N_{0,\beta} x)$ is continuous on $\bigcup_{\partial \geqq 0} N_{0,\partial}\, W^{(2)}$.

The following two lemmas will be valuable in applying the results of lemma 2.

Lemma 3. Let A be a collection of weakly bounded subsets of x covering X. Let $Z = \{\ell : \ell \in X^{\dagger}$ and ℓ is bounded on the sets of $A\}$ and let τ be the topology of uniform convergence on sets of A. Then

(a) $Z \supset X^*$ and (Z, τ) is a locally convex semicomplete, Hausdorff ℓ.t. space.

(b) If T is a linear transformation on X which leaves A invariant under the mapping $\mathcal{F}: S \longrightarrow TS$ for all $S \in A$, the mapping $T^*: \ell \longrightarrow \ell T$ is continuous on (Z, τ).

Proof. (a) The proof of semi-completeness involves routine arguments. The other statements are shown in Bourbaki[3].

(b) It is enough to show that the inverse image of the basic open sets in τ are open. Indeed let S be an element of A. Then $\{\ell : \{\ell Tx : x \in S\} \subset (a, b)\} = \{\ell(y) : y \in TS\} (a, b)\}$, The latter is open in τ since $TS \in A$.

We are now ready to describe the behavior of a large class of linear M.t.c. functions.

Theorem 1. Let $c_{t_1, t_2}(x, y) = c_{0, t_2 - t_1}(0, y - T_{t_2 - t_1} x)$ be a linear M.t.c.f. which is b-regular in y and satisfies $c_{t_1, t_2}(0, 0) \geqq 0$ for all $t_1, t_2 \in R^+$. Let A be any collection of subsets of X which is invariant under the mapping T_t $T_t: S \longrightarrow T_t(S)$ for all $t \in R^+$ and contains all the sets S having the form $S = \{T_t x : c_{0,\alpha}(0, x - x_0) \geqq a\}$ for some t, α in R^+, $a \in R$ and $x_0 \in X$. Let U be the space of all linear functionals on X which are uniformly bounded on the sets of A. Let τ be the topology of uniform convergence on the sets of A. If X^* is the closure of X^* in (U, τ), and if

(i) $T_t(\cdot)$ is weakly continuous on X,

(ii) $\ell T_{(\cdot)}: R^+ \longrightarrow X^*$ is continuous (in the topology τ) for all ℓ in a dense subset W of (X^*, τ), then there exists a convex function g defined on X^* such that

$$g_t(\ell) = M(c_{0,t}(0, \cdot))(\ell) = \int_0^t g(\ell T_\alpha)\, d\alpha$$

for all ℓ in $\bigcup_{\partial > 0} N^*_{0,\partial}(W)$, where

$$N^*_{a,b}\ell = \frac{1}{b}\int_a^{a+b} \ell T_\alpha\, d\alpha .$$

Furthermore, the function $g(N^*_{0,\partial}(\cdot))$ is continuous on $(\bigcup_{\partial \geqq 0} N^*_{0,\partial} W)$.

Proof. Applying lemma 2 we let $Z = X^*$ and let the topology on Z be τ. By lemma 3 U and therefore X^* are semicomplete. Also the one-parameter semigroup of linear transformations $L_t = T^*_t$ defined by $L_t(\ell) = \ell T_t$ maps X^* into itself since T_t is weakly continuous, and it maps U into itself since $\ell T_t(S) = \ell(T_t S)$ and $T_t S \in A$ whenever $S \in A$. Therefor $\ell \in U$ implies that ℓT_t is a bounded on the sets of A and $\ell T_t \in U$. By lemma 3 T^*_t is continuous on (U, τ), hence it maps X^* into itself. Because of (ii) $L_{(\cdot)}(\ell): R^+ \longrightarrow X^* = Z$ is continuous for all $\ell \in W$ and the latter is dense in X^*. $g_t(\cdot)$ is continuous on (X^*, τ) by corollary II.1.1. and inf $g_t(\ell) = c_{0,t}(0,0) \geqq 0$, thus $g_t(\ell) \geqq 0$. Finally the semigroup condition is satisfied (equation 7) and the theorem follows from lemma 2.

§3. THE REGULAR COST FUNCTION

In the previous section we obtained a characterization of the convex closure of a b-regular ℓ.M.t.c. function. We shall now continue the investigation of such functions. The main result of this section is theorem 1 which states that regular cost functions are convex.

<u>Lemma 1</u>. Let $\{f_n\}_{n=1}^{\infty}$ be a sequency of real-valued functions on X and let $g_n(\ell) = M(f_n(\cdot))(\ell)$, $n = 1, 2, \ldots$. Then

(1) If $f_\infty(x) = \sup f_n(x)$, we have $g_\infty(\ell) = M(f_\infty(\cdot))(\ell) = \sup g_n(\ell)$.

(2) If $f_n(x) \downarrow f_\infty(x)$ and in addition the functions $f_n(\cdot)$ are regular for all n, $g_n(\ell) \downarrow g_\infty(\ell)$.

<u>Proof</u>. (1) $g_\infty(\ell) = \sup_x (f_\infty(x) - \ell(x)) = \sup_x \sup_n [f_n(x) - \ell(x)] = \sup_n \sup_x [f_n(x) - \ell(x)] = \sup_n g_n(\ell)$.

(2) Since $g_n(\ell) \geqq g_\infty(\ell)$ for all n, $\inf g_n(\ell) \geqq g_\infty(\ell)$. Conversely, suppose $\inf g_n(\ell) > g_\infty(\ell) + \epsilon$ for some $\ell \in X^*$ and some $\epsilon > 0$. Then for all n, $\sup (f_n(x) - \ell(x)) > g_\infty(\ell) + \epsilon$ and the sets $W_n = \{x : f_n(x) \geqq \ell(x) + g_\infty(\ell) + \epsilon\}$ are compact decreasing and non-empty, hence they have a point x_1 in common. But then $f_n(x_1) \geqq \ell(x_1) + g_\infty(\ell) + \epsilon$ for all n and $f_\infty(x_1) \geqq \ell(x_1) + g_\infty(\ell) + \epsilon$. This is a contradiction, hence $g_n(\ell) \downarrow g_\infty(\ell)$.

<u>Lemma 2</u>. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of concave upper semi-continuous real-valued functions on X and let $g_n(\ell) = M(f_n(\cdot))(\ell)$, $n = 1, 2, \ldots, \infty$. Then

(1) If $g_n(\ell) \downarrow g_\infty(\ell)$, $f_n(x) \downarrow f_\infty(x)$ where $f_\infty(x) = M^{-1}(g_\infty(\cdot)(x)$ and $M^{-1}(g(\cdot)) \triangleq \inf_{\ell \in X^*} (M(f)(\ell) + \ell(x))$ (see lemma II.1.3).

(2) Whenever $g_n(\ell) \longrightarrow g_\infty(\ell)$, $\overline{\sup f_n(x)} = f_\infty(x)$ where $f_\infty(x) = M^{-1}(g_\infty(\ell))(x)$.

(3) Whenever $g_n(\ell) \longrightarrow g_\infty(\ell)$, $\lim_n \overline{\sup_{k \geq n} f_n(x)} = f_\infty(x) = M^{-1}(g_\infty(\ell))(x)$.

Proof. (1) $f_\infty(x) = \inf_\ell [g_\infty(\ell) + \ell(x)] = \inf_\ell \inf_n [g_n(\ell) + \ell(x)] = \inf_n \inf_\ell [g_n(\ell) + \ell(x)] = \inf_n f_n(x)$.

(2) Similarly $g_n(\ell) \uparrow g_\infty(\ell)$ implies $f_n(x)$ is an increasing sequence, hence $M(\sup f_n(\cdot))(\ell) = \sup M(f_n(\cdot))(\ell) = \lim g_n(\ell) = g_\infty(\ell)$. Thus $\overline{\sup f_n} = f_\infty$.

(3) $g_\infty(\ell) = \inf_n \sup_{k \geq n} g_k(\ell)$ and $M^{-1}(g_\infty(\cdot)) = \inf_n M^{-1}(\sup_{k \geq n} g_k(\cdot)) = \inf_n \overline{\sup_{k \geq n} f_n}$.

Corollary 1. Let f_n be a sequence of functions and let $g_n(\ell) = M(f_n(\cdot))(\ell)$, $n = 1, 2, \ldots, \infty$. Then

(1) Whenever $f_n \downarrow f$ and f_n are regular functions we have $\overline{co}\, f_n \downarrow \overline{co}\, f$.

(2) $f_n(x) \uparrow f_\infty(x)$ implies $\overline{\sup_n \overline{co}\, f_n} = \overline{co}\, f_\infty$.

Proof. (1) By lemma 1, $f_n \downarrow f_\infty$ implies $g_n \downarrow g_\infty = M(f_\infty)$; also, by lemma 2, $g_n \downarrow g_\infty$ implies $\overline{co}\, f_n \downarrow \overline{co}\, f_\infty$ since $M^{-1}(g_n) = \overline{co}\, f_n$ and $M^{-1}(g_\infty) = \overline{co}\, f_\infty$.

(2) Similarly, $f_n(x) \uparrow f_\infty(x)$ implies $g_n(\ell) \uparrow g_\infty(\ell)$, which in turn implies that $\overline{\sup \overline{co}\, f_n(x)} = \overline{co}\, f_\infty(x)$.

Lemma 3. Let $c_{0,t}(0,\cdot)$ be a linear M.t.c. function which is (weakly) sup-compact and suppose that the corresponding semigroup of linear transformations T_t satisfied (i) and (ii) of theorem II.2.1. Define $c^n_{0,t}(0,x) = 2 \sup \{c^{n-1}_{0,t/2}(0,y) : (I + T_{t/2})\, y = x\}$ and let $c^0_{0,t}(0,x) = c_{0,t}(0,x)$ where the usual convention $\sup \phi = -\infty$ is used. Then $c^n_{0,t}(0,x)$ is a monotonically decreasing sequenc of functions whose limit denoted by $c^\infty_{0,t}(0,x)$ exists and is a concave function of x.

Proof. Let $S_t = \{(y,x) : y \in R,\ x \in X,\ y \leqq c_{0,t}(0,x)\} = S^0_t$

$$S^n_t = \{(y,x) : y \in R,\ x \in X,\ y \leqq c^n_{0,t}(0,x)\},$$

and define $T'_t : R \times X \longrightarrow R \times X$ by $T'_t(a,x) = (a, T_t x)$. Then by lemma II.1.7(e) $S^{n+1}_t = (I + T'_{t/2}) S^n_{t/2}$ and $S^1_t = (I + T_{t/2})\, S^0_{t/2}$.

Thus

$$S^0_t = S^0_{t/2} + T'_{t/2}\, S^0_{t/2} \supset (I + T'_{t/2})\, S^0_{t/2} = S^1_t,$$

and if $S^n_t \subset S^{n-1}_t$ for all t and some n

$$S^{n+1}_t = (I + T'_{t/2})\, S^n_{t/2} \subset (I + T'_{t/2})\, S^{n-1}_{t/2} = S^n_t.$$

By induction we show that $S^{n+1}_t \subset S^n_t$ and therefore $c^{n+1}_{0,t}(0,x) \leqq c^n_{0,t}(0,x)$. Let $c^\infty_{0,t}(0,x) = \inf c^n_{0,t}(0,x)$. Then

$S^{c^\infty_{0,t}(0,\cdot)} = \bigcap_{n=0}^{\infty} S^n_t = S^\infty_t$. We shall now show that S^∞_t is a convex set and therefore $c^{\infty}_{0,t}(0,\cdot)$ is a concave function. Since the sets S^n_t are closed, $S^{\infty}_t = \bigcap_{n=1}^{\infty} S^n_t$ is also a closed set and in order to show that it is convex it is enough to demonstrate that whenever v_1, v_2 are in $S^{\infty}_t, \frac{1}{2} v_1 + \frac{1}{2} v_2$ is also in that set. Let $v_1 = (y_1, x_1)$, $v_2 = (y_2, x_2)$ be two vectors of S^{∞}_t. Then $v_1, v_2 \in S^n_t$ for all n. But

$$S^n_t + T'_{t/2^n} S^n_t = (I + T'_{t/2})(\cdots)(I + T'_{t/2^n}) S_{t/2^n} +$$

$$+ (I + T'_{t/2})(\cdots)(I + T'_{t/2^n}) T'_{t/2^n} S_{t/2^n}$$

$$= (I + T'_{t/2})(\cdots)(I + T'_{t/2^n})(S_{t/2^n} + T'_{t/2^n} S_{t/2^n})$$

$$= (I + T'_{t/2^n}) S^{n-1}_t .$$

Thus $v_1 + T'_{t/2^n} v_2 \in (I + T'_{t/2^n}) S^{n-1}_t$ or there exists $w_n \in S^{n-1}_t$ such that $v_1 + T'_{t/2^n} v_2 = (I + T'_{t/2^n}) w_n$. Let $w_n = (u_n, z_n)$. We have: $2u_n = y_1 + y_2 = 2a$ for some a. Define H_a to be the set $\{(y,x) : y \geqq a\}$. Since $c_{0,t}(0,\cdot)$ is sup-compact the sets $S^n_t H_a$ are compact and $w_n \in H_a S^0_t$ implies that w_n has a limit point $w \in S^n_t = S^{\infty}_t$. For any $\ell \in W X^*$ there exists k so large that for all $z \in S^0_t H_a$ $|\ell((I + T'_{t/2^k})z - 2z)| < \epsilon$ and therefore $|\ell((v_1 + T'_{t/2^n} v_2) - 2w)| = |\ell[(I + T'_{t/2^n}) w_n - 2w]| \leqq \epsilon + |\ell(2w_n - 2w) \leqq 2\epsilon$ infinitely often. Since $\ell(v_1 + T'_{t/2^n} v_2 - 2w)$ converges it must converge to zero. Furthermore, $\ell T'_{t/2^n} v_2 \longrightarrow \ell v_2$ for all $\ell \in W$ so $\ell(v_1 + v_2) = \ell(2w)$ for all ℓ in a dense subset of X^*, hence $v_1 + v_2 = 2w$.

<u>Lemma 4.</u> Let $c_{a,b}(x, y)$ be a linear M.t.c. function which is sup-compact in y and suppose that the corresponding semigroup of

linear transformations T_t satisfies (i) and (ii) of theorem II.2.1. Let $c^{\infty}_{0,t}(0,\cdot)$ be defined as in lemma (3). Define $h^n_t(x)$ by the following recursive relation: $h^{n+1}_t(y) = \sup_x [h^n_{t/2}(x) + h^n_{t/2}(y - T_{t/2}x)]$ and $h^0_t(x) = c^{\infty}_{0,t}(0,x)$. Then $h^n_t(x)$ is a monotonically increasing sequence of concave functions whose limit denoted by $h^{\infty}_t(x)$ exists and is therefore a concave function of x. Furthermore, $h^{\infty}_t(x)$ is bounded above by $c_{0,t}(0,x)$.

Proof. Let $R^n_t = \{(y,x) : h^n_t(x) \geq y, y \in R, x \in X\}$, $n = 0, 1, \ldots, \infty$. Then $R^{n+1}_t = R^n_{t/2} + T'_{t/2} R^n_{t/2}$ and $R^1_t = R^0_{t/2} + T'_{t/2} R^0_{t/2} = S^{\infty}_{t/2} + T'_{t/2} S^{\infty}_{t/2} \supset (I + T'_{t/2}) S^n_{t/2}$. We shall now show that $(I + T'_{t/2}) S^{\infty}_{t/2} \supset S^{\infty}_t$ and therefore $R^1_t \supset R^0_t$. Let w belong to S^{∞}_t. Then there exist $v_n = (y_n, x_n) \in S^n_{t/2}$ such that $(I + T'_{t/2}) v_n = w = (a, z)$. v_n lie in $S^n_{t/2} \cap H_a$ since $y_n = a$. The latter is compact, hence there exists a limit point v of $\{v_n\}$ and $\ell[(I + T'_{t/2}) v - (I + T'_{t/2}) v_n] < \epsilon$ infinitely often for any ϵ. Hence $\ell(I + T_{t/2}) v = \ell w$ for all $\ell \in X^*$ and $(I + T'_{t/2}) v = w$. Furthermore, v belongs to $S^{\infty}_{t/2}$ and therefore $w \in (I + T'_{t/2}) S^{\infty}_{t/2}$ and $R^1_t \supset S^{\infty}_t = R^0_t$. Using this fact we can prove inductively that $R^{n+1}_t \supset R^n_t$. Indeed suppose $R^n_{t/2} \supset R^{n-1}_{t/2}$. We obtain $R^{n+1}_t = R^n_{t/2} + T'_{t/2} R^n_{t/2} \supset R^{n-1}_{t/2} + T'_{t/2} R^{n-1}_{t/2} = R^n_t$ and therefore $h^n_t(x) \geqq h^{n-1}_t(x)$. To complete the proof we shall show inductively that $R^n_t \subset S^0_t$ and therefore $h^n_t(x) \leqq c_{0,t}(0,x)$. For $n = 0$, $S^0_t \supset S^{\infty}_t = R^0_t$. Also

$$S^0_t = S^0_{t/2} + T'_{t/2} S^0_{t/2} \supset S^{\infty}_{t/2} + T'_{t/2} S^{\infty}_{t/2}$$ and in general

$$S_{t/2} = S^0_{t/2} + T'_{t/2} S^0_{t/2} \supset R^n_{t/2} + T'_{t/2} R^n_{t/2} = R^{n+1}_{t/2} .$$

Lemma 5. Let $c_{a,b}(x,y)$ be a convex ℓ.M.t c. function which is regular in y and satisfies $c_{0,t}(0,0) \geqq 0$. Suppose that the corresponding semigroup T_t of linear transformation satisfies (i) and (ii) of theorem II.2.1. and let $h_t^n(x)$ be defined as in lemma 4. Then $M(h_t^n(\cdot)) \uparrow M(c_{0,t}(0,\cdot))$ and therefore $\sup h_t^n(x) = c_{0,t}(0,x)$.

Proof. By theorem II.2.1.

$$g_t(\ell) = \int_0^t g(\ell T_\alpha)\, d\alpha \geqq tg(N^*_{0,t}(\ell))$$

for all $\ell \in \bigcup_{\gamma>0} N^*_{0,\gamma} W$. Applying lemma II.1.2(f) and the definition of $c^n_{0,t}(0,\cdot)$, $M(c^n_{0,t}(0,x))(\ell) = 2^n g_{t/2^n}(1/2^n \ell\,(I+T_{t/2})(\cdots)(I+T_{t/2^n})) \geqq tg((1/2^n N^*_{0,t/2^n}\ell)(I+T_{t/2^n})(\cdots)(I+T_{t/2})) = tg(N^*_{0,t}\ell)$ since $(N^*_{0,t/2^n}\ell)(I+T_{t/2^n}) = 2N^*_{0,t/2^{n-1}}(\ell)$. Therefore, if $g^n_t(\ell) = M(c^n_{0,t}(0,\cdot))(\ell)$, $g^n_t(\ell) \geqq tg(N^*_{0,t}\ell)$, $n = 0, 1, \ldots, \infty$ for all $\ell \in \bigcup_{\gamma>0} N^*_{0,\gamma}W$. Also by lemma 2 since $c^n_{0,t}(0,x)$ are concave and upper semicontinuous, $g^\infty_t(\ell) = M(c^\infty_{0,t}(0,\cdot))(\ell) = \inf g^n_t(\ell) \geqq tg(N^*_{0,t}\ell)$. Let $f^n_t(\ell) = M(h^n_t(\cdot))(\ell)$. Since $h^n_t(x) \leqq c_{0,t}(0,x)$

$$g^0_t(\ell) \geqq f^n_t(\ell) = [\, g^\infty_{t/2^n}(\ell) + \cdots + g^\infty_{t/2^n}(\ell T_{t(2^n-1)/2^n})\,]$$

$$\geqq \frac{t}{2^n}\,[\, g(N^*_{0,t/2^n}\ell) + \cdots + g(N^*_{0,t/2^n}\ell T_{(2^n-1)t/2^n}\,] .$$

Utilizing the fact that $\ell \in \bigcup_{\gamma>0} N^*_{0,\gamma}W$, there exists some $\gamma > 0$ such that $\ell = N^*_{0,\gamma}m$ for some $m \in W$ and

$$g^0_t(\ell) \geqq f_n(\ell)$$

$$\geqq \frac{t}{2^n}\left[g\left(N^*_{0,t/2^n} N^*_{0,\gamma}\, m\right) + \cdots + g\left(N^*_{0,t/2^n} N^*_{0,\gamma}\, mT_{(2^n-1)t/2^n}\right)\right].$$

Let $J_n(\alpha)$ be defined for each $m \in W$ by

$$J_n(\alpha) = \sum_{k=0}^{2^n-1} g(N^*_{0,\frac{t}{2^n}} N^*_{0,j}\, m\, T_{\frac{kt}{2^n}})\, I_{[\frac{k}{2^n} \leqq \alpha < \frac{k+1}{2^n})}(\alpha)$$

where

$$I_{[a,b)}(\alpha) = \begin{cases} 1 & \text{on } [a,b) \\ 0 & \text{on } [a,b)^c \end{cases}$$

Then

$$g_t^0(\ell) \geqq f_t^n(\ell) \geqq \int_0^t J_n(\alpha)\, d\alpha$$

But

$$J_n(\alpha) \leqq \sup_{\substack{0 \leqq \alpha \leqq t \\ 0 \leqq \beta \leqq 1}} g(N^*_{0,\beta} N^*_{0,\gamma}\, m\, T_\alpha) = a$$

where a is finite since $g(N^*_{0,\gamma}(\cdot))$ is continuous on $(\bigcup_{\gamma>0} N_{0,\gamma} W, \tau)$, $N^*_{0,\beta}\, m\, T_\alpha = N^*_{\alpha,\beta}\, m$ is jointly continuous in α and β (lemma II.2.1.) and their composition is therefore continuous in α, β. Furthermore $J_n(\alpha) \longrightarrow g(N^*_{0,\gamma}\, m\, T_\alpha)$, therefore

$$g_t^0(\ell) \geqq f_t^n(\ell) \geqq \int_0^t J_n(\alpha)\, d\alpha \quad > g_t^{0\prime}(\ell)$$

for all $\ell \in \bigcup_{\gamma>0} N^*_{0,\gamma} W$. In this way we obtain $g_t^0(\ell) = \sup f_t^n(\ell)$ for all

$\ell \in \bigcup_{\gamma>0} N^{*}_{0,\gamma}$ W, since $g^{0}_{t}(\ell) \geqq \sup_{n} f^{n}_{t}(\ell)$ everywhere and $g_t(\cdot)$ is continuous, so is $\sup_{n} f^{n}_{t}(\cdot)$(1) and $\sup_{n} f^{n}_{t}(\cdot) = g^{0}_{t}(\cdot)$ everywhere. with the aid of lemma 2 we arrive at the final conclusion.

Theorem 1. Let $c_{a,b}(x,y)$ be an ℓ.M.t.c. function which is weakly upper semicontinuous and let $c_{a,b}(x,y) = \overline{co}\,(c_{a,b}(0,\cdot))(y - T_{b-a}x)$. Then if $c'_{a,b}(x,\cdot)$ is regular $c_{t_1,t_2}(0,0) \geqq 0$(2) and the corresponding semigroup $\{T_t\}$ of linear transformations satisfies (i) and (ii) of theorem II.2.1., $c_{a,b}(\cdot,\cdot)$ is a concave function.

Proof. Since $M(c_{a,b}(0,\cdot))(\cdot) = M(c'_{a,b}(0,\cdot))(\cdot) = g_{b-a}(\cdot)$ we have: $M[\sup_{y}(c'_{a,b}(0,y) + c'_{b,c}(0,\cdot - T_{b-a}\,y))](\ell) =$ $M(c'_{a,b}(0,\cdot)(\ell) + M(c'_{b,c}(0,\cdot))(\ell T_{b-a}) = g_{b-a}(\ell) + g_{c-b}(\ell T_{b-a}) =$ $g_{c-a}(\ell) = M(c'_{a,c}(0,\cdot))$ and therefore $\overline{\sup}\,[c'_{a,b}(x,z) + c'_{b,c}(z,y)] = c'_{a,c}(x,y)$ and $c'_{a,b}(x,y)$ is a ℓ.M.t.c. function(3) Let $c'^{n}_{0,t}(0,\cdot)$ and $h'^{n}_{t}(\cdot)$ be the functions corresponding to $c'_{0,t}(0,\cdot)$ according to the definitions in lemmas 3 and 4. Then by lemma 3

$$c^{n}_{0,t}(0,x) \downarrow c^{\infty}_{0,t}(0,x)$$

and by corollary 1 and lemma 3

$$c^{n}_{0,t}(0,x) \downarrow \overline{co}(c^{\infty}_{0,t}(0,x)) = c^{\infty}_{0,t}(0,x)$$

Therefore $h'^{n}_{t}(x) = h^{n}_{t}(x)$. But it follows from lemma 5 that $\overline{co}\,(c^{0}_{0,t}(0,\cdot))(x) \triangleq c'_{0,t}(0,x) \geqq c^{0}_{0,t}(0,x) \geqq h^{n}_{t}(x) \longrightarrow$ $\overline{co}(c^{0}_{0,t}(0,\cdot))(x)$. Thus, $\overline{co}\,(c^{0}_{0,t}(0,\cdot))(x) = c^{0}_{0,t}(0,x)$ and $c_{a,b}(\cdot,\cdot)$ is concave.

§4. C-REGULARITY AND REGULARITY

The conditions imposed on the cost function in the preceding sections such as "convexity" or "regularity" involve the structural properties of the function $c_{a,b}(\cdot, \cdot)$. On the other hand, when the problem is specified it is usually much easier and more natural to check the temporal behavior of the cost function. In particular, in most problems of interest we know that as the time to reach x becomes short the expense of reaching it mounts and eventually becomes prohibitive. This is essentially the type of constraint imposed by the requirement that $c_{a,b}(x,y)$ be c-regular. In this section we shall complete our discussion of cost functions by showing that c-regular cost functions are concave.

As in previous sections we shall let $c_{t_1, t_2}(x,y)$ be an ℓ.M.t.c.f. We shall also let $A = \{G : G$ is a weakly bounded subset of X contained in $\{T_\alpha x : c_{t_1, t_2}(0, x-x_0) \geqq a\}$ for some α, t_1, t_2, $a \in R$, $x_0 \in X\}$ and Z will denote the space of all linear functionals on X which are uniformly bounded on the elements of A with the topology τ of uniform convergence of the elements of A. It will be assumed that:

(i) T_t is weakly continuous on X.

(ii) $\ell T_{(\cdot)}: R^+ \longrightarrow (Z, \tau)$ is continuous for all ℓ in a subset W of (X^*, τ).

(iii) $\sup_{x,y} c_{t_k, t_2}(x,y)$ is a measurable real-valued function of t_1, t_2.

(iv) $c_{t_1, t_2}(x,y)$ is c-regular.

The function $g_t(\cdot,\cdot): R \times X^* \longrightarrow R$ may be defined as follows

$$g_t(a,\ell) = \sup\{c_{0,t}(0,x) : \ell(x) = a\} .$$

Lemma 1. Let $g_{t-\alpha}((b-a), \ell T_\alpha) = c^\ell_{\alpha,t}(a,b)$. Then $c^\ell_{t,\tau}(\cdot,\cdot)$ is a (time varying) M.t.c.f.

Proof.

$$c^\ell_{\alpha,t+\alpha+\tau}(a,b) = g_{t+\tau}(b-a,\ell T_\alpha) = \sup_x \{c_{t+\tau}(x) : \ell T_\alpha x = b-a\}$$

$$= \sup_y\{\sup_z[c_{0,t}(0,z) + c_{0,\tau}(0,y-T_\tau z)] : \ell T_\alpha y = b-a\}$$

$$= \sup_{z,w}\{c_{0,t}(0,z) + c_{0,\tau}(0,w) : \ell T_\alpha w + \ell T_\alpha T_\tau z = b-a\}$$

$$= \sup_c \sup_z \sup_w \{c_{0,t}(0,z)+c_{0,\tau}(0,w) : \ell T_\alpha w = c, \ell T_\alpha T_\tau z = b-a-c\}$$

$$= \sup_c \{g_\tau(c,\ell T_\alpha) + g_t(b-(a+c), \ell T_{\alpha+\tau})\}$$

$$= \sup_d \{g_\tau(d-a,\ell T_\alpha) + g_t((b-d), \ell T_{\alpha+\tau})\}$$

$$= \sup_d [c^\ell_{\alpha,\alpha+\tau}(a,d) + c^\ell_{\alpha+\tau,\alpha+\tau+t}(d,b)] .$$

Lemma 2. Let $c^\ell_{t,\alpha}(x,y)$ be defined as in lemma 1. Then $c^\ell_{t,\alpha}(x,y)$ is c-regular for all $\ell \in W$.

Proof. We have to show that if $\tau_n \leqq t$, $\eta_n \geqq t$, $\tau_n \longrightarrow t$, $\eta_n \longrightarrow t$

1) $c^\ell_{t,\eta_n}(x,y_n) = g_{\eta_n - t}(y_n - x, \ell) \longrightarrow -\infty$

$x \longrightarrow 0$

2) $c^\ell_{\tau_n, t}(y_0,x) = g_{t-\alpha_n}((x-y_n), \ell T_{\tau_n}) \longrightarrow -\infty$ unless

$x - y_n \longrightarrow 0.$

Suppose this is not the case. Then there exists $\ell \in W$, $\alpha_n \longrightarrow 0$ and $\tau_n \longrightarrow t$ such that

$$\sup_x c_{0,\alpha_n}\{(0,x) : \ell T_{\tau_n} x = a_n\} = g_{\alpha_n}(a_n, \ell T_{\tau_n}) \geqq \alpha,$$

but $a_n \not\longrightarrow 0$. Selecting x_n to approximate the supremum we obtain a sequence x_n such that $\ell T_{\tau_n}(x_n) = a_n$ and $c_{0,\alpha_n}(0,x_n) \geqq \alpha - \epsilon_0$. The last inequality implies that x_n converges weakly to zero and in particular x_n is weakly bounded. Thus $\ell T_\tau y$ converges to $\ell T_t y$ uniformly for all $y \in \{x_n\}_{n=1}^{\infty}$ and for each $\ell \in W$. Since $\ell T_t(x_n) \longrightarrow 0$, $\ell T_{\tau_n} x_n$ must do likewise. This is in contradiction to the requirement above that $\ell T_{\tau_n} x_n = a_n \not\longrightarrow 0$ and $c^{\ell}_{c,\tau}(x,y)$ is indeed c-regular.

Let $f(t) = \sup_x c_{0,t}(0,x)$. Then $f(t+\tau) =$ $\sup_x[\sup_y(c_{0,t}(0,y) + c_{0,\tau}(0,x-T_\tau y))] = f(t) + f(\tau)$. Since $f(t)$ is measurable, $f(t) = \delta t$ for some $\delta \in R$ and $c_{0,t}(x,y) \leqq \delta t.$ (1)

Lemma 3. If $c_{t_1,t_2}(x,y)$ satisfies conditions (i) - (iv), then $\{\ell x : c_{0,t}(0,x) \geqq a\}$ is bounded for all $a, t \in R$ and $\ell \in W$.

Proof. The function $c_{t_1,t_2}(x,y) - \delta(t_2 - t_1)$ is still an ℓ.M.t.c. function and satisfies (i) - (iv). Furthermore, if $\{\ell y : c_{0,t}(0,y) - \delta t \geqq a\}$ is bounded for all a, t, then $\{\ell y : c_{0,t}(0,y) \geqq a\}$ is bounded for all a, t. We may therefore assume that $c_{t_1,t_2}(x,y) \leqq 0$.

We now proceed by contradiction. Suppose there exists a sequence $\{x_n\}_{n=1}^{\infty}$ with $c_{0,t}(0,x_n) \geqq a$, $\ell(x_n) \longrightarrow \infty$, $\ell \in W$. Then $g_t(\ell(x_n),\ell) = \sup\{c_{0,t}(0,x) : \ell(x) = \ell(x_n)\} \geqq a$ and $g_t(b_n,\ell) \geqq a$ for some sequence b_n, $b_n \geqq 2^n$. But by lemma 1

$$\sup[\, g_{t/2}(a,\ell) + g_{t/2}(b_n - a, \ell T_{t/2})] = g_t(b_n, \ell) \geqq a,$$

and there exists a sequence a_n^1 such that

$$g_{t/2}(a_n^1,\ell) + g_{t/2}(b_n - a_n^1, \ell) \geqq a - \frac{1}{2} ,$$

and since $g_t(a,\ell) \leqq 0$ for all $a \in R$:

$$g_{t/2}(a_n^1, \ell) \geqq a - \frac{1}{2}, \quad g_{t/2}(b_n - a_n^1, \ell) \geqq a - \frac{1}{2} .$$

Letting $b_n^1 = \sup \{a_n^1, b_n - a_n^1\}$ we obtain a sequence b_n^1 such that $g_{t/2}(b_n^1, \ell T_{\alpha_n^1}) \geqq a - \frac{1}{2}$, $b_n^1 \geqq 2^{n-1}$ and α_n^1 is either zero or $\frac{t}{2}$.

Continuing this process we generate a new sequence a_n^2 such that

$$g_{t/4}(a_n^2, \ell T_{\alpha_n^1}) + g_{t/4}(b_n^1 - a_n^2, \ell T_{\alpha_n^1 + \frac{t}{4}}) \geqq a - \frac{1}{2} - \frac{1}{4}$$

and letting $b_n^2 = \sup \{a_n^2, b_n^1 - a_n^2\} \geqq \frac{1}{2} b_n^1 = 2^{n-2}$ we have

$$g_{t/4}(b_n^2, \ell T_{\alpha_n^2}) \geqq a - \frac{1}{2} - \frac{1}{4}$$

where α_n^2 takes on one of the values $(0, 1/4, 1/2, 3/4)$. In this way we obtain the double sequence $\{b_n^k\}$ satisfying for some sequence $\{\alpha_n^k\}$:

$$g_{t/2^n}(b_n^k, \ell T_{\alpha_n^k}) \geqq a - 1$$

and $b_n^k \geqq 2^{n-k}$. Thus $b_n^n \geqq 2^0 = 1$ and $g_{t/2^n}(b_n^n, \ell T_{\alpha_n^n}) \geqq a - 1$. Since α_n^n converges this is impossible by lemma 2.

Definition 1.[2] Let Γ be a subspace of $X^{\dagger}$. The topology with subbasis consisting of the sets $\{x : \ell(x-x_0) \leqq a\}$ for some $a \in R$ $x_0 \in X$, $\ell \in \Gamma$ is called the Γ topology on X.

With this notation the conclusion of lemma 3 is that $\{x, c_{0,t}(0,x) \geqq a\}$ is bounded in the W topology.

By lemma 3 the function $c^{\ell}_{t_1,t_2}(0,\cdot)$ is sup bounded for all $\ell \in W$ and therefore we may apply theorem 1.4.1. to show that $d^{\ell}_{t_1,t_2}(\cdot,\cdot) = \overline{c}^{\ell}_{t_1,t_2}(\cdot,\cdot)$ is still an M.t.c.f. and $\{x-y : d^{\ell}_{t_1,t_2}(x,y) \geqq a\} = \{x-y : d^{\ell}_{t_1,t_2}(0,y-x) \geqq a\}$ is compact for all $\ell \in W$. Furthermore, it is easy to check that $d^{\ell}_{t_1,t_2}(x,y)$ is still c-regular, hence possesses the strong semigroup property (I.6). Therefore, if P is a plane in $R \times R^{+}$ separating (x,t_1) and (z,t_3) and $d^{\ell}_{t_1,t_2}(x,y) \triangleq d^{\ell}_{t_1,t_2}(x,y) - (y-x)$,

$$\sup_{(y,t_2)\in P} \left(d^{\ell}_{t_1,t_2}(x,y) + d^{\ell}_{t_2,t_3}(y,z) \right)$$

$$= \sup_{(y,t_2)\in P} [\, d^{\ell}_{t_1,t_2}(x,y) + d^{\ell}_{t_2,t_3}(y,z) - (y-x) - (z-y) \,]$$

$$= \sup_{(y,t_2)\in P} [\, d^{\ell}_{t_1,t_2}(x,y) + d^{\ell}_{t_2,t_3}(y,z) + (z-x) \,]$$

$$= d^{\ell}_{t_1,t_3}(x,z) - z - x = d^{\ell}_{t_1,t_3}(x,z)$$

and $d^{\ell}_{t_1,t_2}(x,y)$ has a strong semigroup property for all $\ell \in W$.

Lemma 4. If $d'^{\ell}_{t,\tau}(\cdot,\cdot)$ is a function on $R \times R$ having the strong semigroup property and for some x, y, t, τ with $|x-y| \geqq 2$, $d^{\ell}_{\tau,t}(x,y)$ is larger than a, then there eixsts some α, β, $\tau \leqq \alpha \leqq \beta \leqq t$ and $w, z, 1 \leq |w-z| \leqq 2$ satisfying $d'^{\ell}_{\alpha,\beta}(w,z) \geqq a$ whenever a is nonpositive.

Proof. Suppose $n \leqq |y-x| \leqq n+1$. Then $1 \leqq \frac{|y-x|}{n} \leqq 2$. Let $x_1 = x$, $x_2 = x_1 + \frac{y-x}{n} \cdots x_i = x_{i-1} + \frac{y-x}{n} \cdots x_{n+1} = y$. Then the x are between y and x, and

$$\sup_{\tau = t_1 \leqq t_2 \cdots \leqq t_n \leqq t} [d'^{\ell}_{\tau, t_2}(x_1, x_2)$$

$$+ d'^{\ell}_{t_2, t_3}(x_2, x_3) + \cdots + d'^{\ell}_{t_n, t}(x_n, x_{n+1}) = d'^{\ell}_{\tau, t}(x, y) > a.$$

Selecting t_i to approximate the maximum we conclude that one of the terms in the expression must be no smaller than a. Thus there exists $\tau \leqq \alpha \leqq \beta \leqq t$ and k satisfying $d'^{\ell}_{\alpha,\beta}(x_k, x_{k+1}) \geqq a$ together with $|x_k - x_{k-1}| = \frac{y-x}{n}$.

Theorem 1. If $c_{t_1, t_2}(x, y)$ is an ℓ.M.t.c.f. satisfying (i) - (iv), then $\sup_x (c_{0,t}(0,x) - \ell(x)) < \infty$ for all $\ell \in W$.

Proof. As in the proof of lemma 3 we may assume $c_{0,t}(0, x) \leqq 0$.

Suppose there exists $\ell \in W$ and a sequence $\{x_n\}_{n=1}^{\infty}$, $x_n \in X$ such that $c_{0,t}(0,x_n) - \ell(x_n) \longrightarrow \infty$. Then because $c_{0,t}(0,x_n) \leqq 0$, $-\ell(x_n)$ must diverge to ∞ and furthermore $d^{\ell}_{0,t}(0, \ell(x_n)) - \ell(x_n) \geqq c^{\ell}_{0,t}(0, \ell(x_n)) - \ell(x_n) = \sup_{\ell(x) = \ell(x_n)} c_{0,t}(0, x) - \ell(x_n) \longrightarrow \infty$.

Let $d'^{\ell}_{t_1,t_2}(a, b) = d^{\ell}_{t_1,t_2}(a, b) - b + a$. Then

$\sup_{a, b} [d'^{\ell}_{0,t/2}(0,a) + d'_{0,t/2}(0,b)] = \infty$. Therefore, either

$d'^{\ell}_{0,t/2}(0, \cdot)$ or $d'^{\ell T_{t/2}}_{0,t/2}(0, \cdot)$ is unbounded. By induction we may infer the existence of $\alpha_n \longrightarrow \alpha$ and a_n such that $d_{0,t/2}^{,\ell T_{\alpha_n}}(0, a_n) > 2 \geq$ and $-a_n \geq 2 - d_{0,t/2^n}^{\ell T_{\alpha_n}}(0,a_n) \geq 2$ since $d_{t_1,t_2}^{\ell}(x, y) \leq 0$. But $d_{t_1,t_2}^{'\ell}(\cdot, \cdot)$ has the strong semigroup property and by lemma 4 there exist b_n, c_n and τ_n, η_n such that $1 \leq |b_n - c_n| \leq 2$, $0 \leq \tau_\eta \leq t/2^n$ together with

$$d_{\tau_n,\eta_n}^{\ell T_{\alpha_n}}(b_n, c_n) - (c_n - b_n) = d_{\tau_n,\eta_n}^{'\ell T_{\alpha_n}}(b_n, c_n) \geq 0$$

are satisfied. Hence $d_{\tau_n,\eta_n}^{\ell T_{\alpha_n}}(b_n, a_n) \geq 0 + (c_n - b_n) \geq -2$. On the other hand, $d_{\tau,\eta}^{\ell}(x, y)$ is c-regular and this fact conflicts with the statement above.

Thus we have obtained a relation between the temporal behavior of $c_{t_1,t_2}(x_1, x_2)$ (c-regularity) and its dependence on x (regularity) when x is assigned the W-topology. It remains only to tie up this result with the conclusions of the previous sections.

Definition 2.[3] Γ is said to be a total subspace of $X^\dagger$ if for all $x \in X$, $x = 0$ whenever $\ell(x) = 0$ for all $\ell \in \Gamma$ $X^\dagger$.

Let X be a linear space and let W be a total subspace of $X^\dagger$. In the remainder of this section we shall assume that the locally convex topology on X is the W topology. Thus $X = W^{(4)}$. Let T_t be a semigroup of linear transformations on X such that the mapping T_t^*, $T_t^*(\ell) = \ell T_t$ leaves W invariant. Then $T_t : X \longrightarrow T_t x$ is continuous for each t. We shall also require that $\ell T_{(\cdot)} : R^+ \longrightarrow W$ be continuous for all $\ell \in W$ when W is assigned the topology τ of theorem II.2.1. Suppose now that $c_{a,b}(x,y) = c_{0,b-a}(0, y - T_{b-a}x)$ is a time-invariant linear M.t.c.f. defined on $X \times X$ for any fixed $0 \leq a \leq b < \infty$

which is upper semicontinuous in x and c-regular. By lemma II.1.6 and theorem 1, $c_{0,t}(0,\cdot)$ is b-regular. We may now embed X in the dual $W^{\dagger}$ of W. Let Y be the closure of X in $W^{\dagger}$ when $W^{\dagger}$ is assigned the W topology and assign that topology to Y. It is now possible to extend $c_{t_1,t_2}(x,y)$ to an upper semicontinuous function $c^e_{t_1,t_2}(x_1,x_2)$ on Y by letting

$$c^e_{0,t}(0,x_0) = \inf_{\mathcal{O}} \{ \sup_{x\in\mathcal{O}x} c_{0,t}(0,x) : \mathcal{O} \text{ is a neighborhood of } x_0 \text{ in } Y\} .$$

Since X is continuously embedded in Y, $c^e_{0,t}(0,x_0) = c_{0,t}(0,x_0)$ for all $x_0 \in X$. Also closed bounded sets in Y are compact(5). Furthermore, for any $\ell \in W$, $\ell(\cdot) + \sup_x (c_{0,t}(0,x) - \ell(x))$ is an upper semicontinuous function on Y dominating the restriction of $c^e_{0,t}(0,x)$ to X and therefore it dominates $c^e_{0,t}(0,x)$ everywhere on Y. Thus $c^e_{0,t}(0,\cdot)$ must be b-regular, hence regular. It is now possible to verify that $c^e_{t_1,t_2}(x,y)$ is still an ℓ.M.t.c.f. Indeed the dual of Y with the W topology is $W^{(4)}$, therefore, $M(c^e_{0,t}(0,\cdot))(\cdot) = M(c_{0,t}(0,\cdot))(\cdot)$. Thus

$$M(c^e_{0,t+\tau}(0,\cdot))(\ell) = M(c^e_{0,t}(0,\cdot))(\ell) + M(c^e_{0,\tau}(0,\cdot)(\ell T_t)$$

$$= M(\sup_x(c^e_{0,\tau}(0,x) + c^e_{0,\tau}(0,\cdot - T_\tau x)))(\ell) ,$$

and by lemma II.1.3

$$\overline{\sup_{x\in Y}} [c^e_{0,\tau}(0,x) + c^e_{0,t}(0, y + T_\tau x)] = c^e_{0,t+\tau}(0,y)$$

where T_t has a natural extension to a continuous mapping (denoted by the same symbol) on Y.

But $c^e_{0,t}(0, \cdot)$ is regular, hence sup-compact. By lemma II.1.7 (a), (e)

$$\sup [\, c^e_{0,\tau}(0,x) + c^e_{0,t}(0, \; y + T_\tau x)\,]$$

$$+ \sup_x \; (\sup_{T_\tau z = x} (c_{0,\tau}(0,z) + c_{0,t}(0, y-x)))$$

is upper semicontinuous in y and the desired equality is obtained.

We now wish to apply theorem II.3.1. In order to do so it must be shown that $\ell T_{(\cdot)} : t \longrightarrow \ell T_t$ is continuous when W is assigned the topology τ^e of uniform convergence on sets of the form $\{T_\alpha x : c^e_{t_1,t_2}(0, x - x_0) \geqq a\}$. But $\sup \{(\ell T_{t+\tau} - \ell T_t)\, T_\alpha x : c_{t_1,t_2}(0, x - x_0) \geqq a, \; x \in X\} = \sup \{(\ell T_{t+\tau} - \ell T_t)\, T_\alpha x : c^e_{t_1,t_2}(0, x - x_0) \geqq a, \; x \in Y\}$. Since $\ell T_{(\cdot)} : t \longrightarrow W$ was assumed continuous when W is assigned the topology τ, it is also continuous when W is assigned the topology τ^e.

In this way we obtain:

<u>Theorem 2.</u> Let X be a linear space and let W be a total subspace of $X^\dagger$. Assign to X the W topology. Then if $c_{t_1,t_2}(\cdot, \cdot)$ is a ℓ.M.t.c.f. which is upper semicontinuous satisfies (i), (ii), (iii), and (iv) and $c_{t_1,t_2}(0,0) \geqq 0$[(6)] then $c_{t_1,t_2}(0, \cdot)$ is a concave function.

The proof follows from the remarks above with the help of theorem II.3.1.

Theorems 1 and 2 may perhaps be more easily visualized when stated under somewhat more restrictive assumptions. For example when X is a Hilbert space it is possible to apply theorem 1 directly to obtain the final result. To do so we let $W = X^* = X$ and let $A = \{S : S \text{ is a weakly bounded subset of } X\}$. Then $Z = X$ and conditions (i) - (iv) read:

(i.a) T_t is a continuous operator on X (in the weak or strong topology),

(ii.a) $\| T^*_{t+\tau} x - T^*_t x \| \longrightarrow 0$ whenever $\tau \longrightarrow 0$ for all x on t, thus $T^*_{(\cdot)} x : R^+ \quad > X$ is continuous when x is assigned the strong topology,

(iii.a) $\sup\limits_{x,y} \; c_{t_1,t_2}(x, y)$ is a measurable real-valued function of t_1, t_2,

(iv.a) $c_{t_1,t_2}(x,y)$ is c-regular (i.e., whenever $t_n \longrightarrow 0$ and $c_{0,t_n}(0,x_n) \geqq a$, x_n must converge weakly to zero) and we may state theorem 2.

<u>Theorem 2:</u> Let X be a Hilbert space. Then if $c_{t_1,t_2}(\cdot,\cdot)$ is an ℓ.M.t.c.f., which is weakly upper semicontinuous and satisfies (i.a) - (iv.a) and $c_{t_1,t_2}(0,0) \geqq 0$[6], then $c_{t_1,t_2}(0,\cdot)$ is a concave function and has the representation of theorem II.2.1.

This result can be easily obtained with the help of theorem II.3.1 and lemma II.1.6 if the following facts are kept in mind:

(a) In a reflexive Banach space any weakly bounded set is also strongly bounded (Banach-Steinhaus).[7]

(b) In a Hilbert space any closed bounded set is weakly compact.[8]

(c) M is a continuous mapping from (X,τ) to (X,τ) when τ is a metric topology iff M is continuous from (X,σ) when σ is the weak topology.[9]

§5. EXAMPLES

Consider the following optimal control problem: Maximize $\int_0^t f(u(t))\,dt$ subject to $\dot{x} = Ax + Bu$, $x(0) = x_1$, $x(t) = x_2$ where $x \in R^n$, $u \in R^m$ for some m, n. The problem may be restated as follows:

$$\text{Maximize} \int_0^t f(u(t))\,dt \ \text{ subject to} \int_0^t e^{A\alpha} Bu(t-\alpha)\,d\alpha =$$

$x^2 - \ell^{At} x_1$. The cost function in this case is

$$(1) \quad c_{0,t_2}(x_1,x_2) = \sup \left\{ \int_0^t f(u(t))\,dt : \int_0^t e^{A\alpha} Bu(t-\alpha)\,d\alpha \; x_2 - e^{At} x_1 \right\}$$

and its maximum transform is:

$$(2) \quad M(c_{0,t_2}(0,\cdot))(\ell) = \int_0^t \sup\left[f(u(\alpha)) - \ell e^{A\alpha} Bu(\alpha) \right] d\alpha$$

$$= \int_0^t g(\ell e^{A\alpha})\,d\alpha$$

where $g(\ell) = \sup\left[f(u) - \ell Bu \right]$. If f is concave so is $c_{t_1,t_2}(0,\cdot)$, hence

$$(3) \quad \bar{c}_{0,t_2}(0,x) = \inf \left[\int_0^t g(\ell e^{A\alpha})\,d\alpha + \ell x \right].$$

Suppose the system is controllable[1] and f is finite everywhere. Then $c_{t_1,t_2}(0,x)$ is finite for all x hence continuous in x and has a tangent $\ell(x) + a$ at every point x_0, and $\ell(x) + a \geqq c_{t_1,t_2}(0,x)$ for all x with equality at x_0.

Therefore,

$$(4) \qquad a = \sup_x \left[c_{t_1,t_2}(0,x) - \ell x \right] = M(c_{t_1,t_2}(0,\cdot))(\ell),$$

and the infimum in equation (3) is taken on at some ℓ. Furthermore:

$$\int_0^t f(u(\alpha))\,d\alpha - \ell \int_0^t e^{A\alpha} Bu(\alpha)\,d\alpha \leqq c_{0,t}(0, \int_0^t e^{A\alpha} Bu(\alpha)\,d\alpha$$

$$- \ell \int_0^t e^{A\alpha} Bu(\alpha)\,d\alpha \leqq M(c_{0,t}(0,\cdot))(\ell) \leqq \int_0^t g(\ell e^{A\alpha})\,d\alpha.$$

From here on we shall assume that f is b-regular. Suppose now that $c_{0,t}(0,\cdot)$ is strictly concave at x_0. Then the supremum in (4) is reached at exactly one point, x_0. If we select u(t) to maximize $f(u(t)) - \ell e^{At} Bu(t)$ we have[2]:

$$\int_0^t g(\ell e^{A\alpha})\,d\alpha = \int_0^t \left[f(u(\alpha)) - \ell e^{A\alpha} Bu(\alpha) \right] d\alpha$$

$$\leqq c_{0,t}(0, \int_0^t e^{A\alpha} Bu(\alpha)\,d\alpha) - \ell \int_0^t e^{A\alpha} Bu(\alpha)\,d\alpha$$

$$\leqq \int_0^t g(\ell e^{A\alpha})\,d\alpha$$

and therefore $\int_0^t e^{At} Bu(t)\, dt = x_0$. Thus the control $u(t-\tau)$ brings us x_0 in time t and also $\int_0^t f(u(\alpha))\, d\alpha = c_{0,t}(0,x_0)$. Hence $u(t-\alpha)$ is an optimal control. We see that in this special case existence of optimal control follows immediately. To compute it we need only find ℓ and the latter is obtained by minimizing the expression $\int_0^t g(\ell e^{A\alpha})\, d\alpha - \ell x_0$ where x_0 is the state to be reached. Thus we reduced the problem of maximization over a function space to one of minimization of a convex function in finite dimensional space. To this problem we may apply a variety of existing techniques such as Newton's method or steepest descent to obtain the solution. It may be of interest to find what g is like in some particular situations. Suppose $f(u) = \Sigma_{i=1}^m |u_i|^p = -\|u\|_p^p$. Then if $\frac{1}{p} + \frac{1}{q} = 1$

$$\sup(f(u) - \ell Bu) = p \sup\left(-\frac{\|u\|_p^p}{p} - \frac{\ell Bu}{p}\right) = \frac{p}{q} \left\| \frac{\ell B}{q} \right\|_q^q$$

and the problem reduces to approximation in $Lq[0,t]$:

$$c_{0,t}(0,x) = \inf_{\ell} \left[\int_0^t \frac{p}{p} \left\| \frac{\ell e^{A\alpha} B}{p} \right\|_q^q d\alpha - \ell x \right]$$

$$= \inf_{\ell x = a} \left[\int_0^t \frac{p}{q} \left\| \frac{\ell e^{A\alpha} B}{p} \right\|_q^q d\alpha - a \right].$$

Another interesting example is the minimum time problem where the cost function has the form

$$c_{t_1,t_2}(x_1, x_2) = \begin{cases} 0 \text{ if } x_2 \text{ is reachable from } x_1 \text{ in } t_2 - t_1 \\ -\infty \text{ otherwise} \end{cases}$$

and it is desired to reach x_2 from x_1 when the only constraint is $\|u\| \leqq a$ and $\dot{x} = Ax + Bu$. It is easy to check that in that case $M(c_{t_1,t_2}(0, \cdot))$ has the form (2) where $g(\ell) = \sup_u \{\ell Bu: \|u\| = a\} = \|\ell B\| a$.

In general the fact that $g_t(\ell) = M(c_{t_1,t_2}(0, \cdot)(\ell)$ is continuous in ℓ is not sufficient to guarantee continuity of $g(\ell)$ in theorem II.2.1 and therefore it is not always true that $c_{t_1,t_2}(x,y)$ has a representation of the form (1) above. Still, the representation (3) always holds true and $g(N^*_{0,\gamma}\ell)$ is continuous in ℓ. The following example will illustrate this point.

Let $X = L_2[0,\infty)$. Then $X^* = L_2[0,\infty)$.[3] Let T_t be the shift operator:

$$T_t(f(z))(\omega) = \begin{cases} f(\omega - t), & \omega \geqq t \\ 0, & \omega \leqq t \end{cases}$$

$T_t(\cdot)$ is clearly strongly and weakly continuous on X and since the weakly bounded sets of X are also strongly bounded (Banach-Steinhous theorem)[4]

$$\sup_{x \in S} (\ell T_{t+\tau} - \ell T_t)(x) \leqq \| \ell T_{t+\tau} - \ell T_t \| \; a \text{ for some } a > 0$$

whenever S is weakly bounded. Therefore whenever ℓ corresponds to a continuous function $h(\omega)$ we have

$$\|\ell T_{t+\tau} - \ell T_t\|^2 \int_0^\infty [\,h(t+\tau) - h(t)\,]^2\, dt \leqq \int_0^b [\,h(t+\tau)-h(t)\,]^2 dt + \epsilon$$

when b is chosen sufficiently large. Letting $\tau \longrightarrow$, $h(t+\tau) \longrightarrow h(t)$

uniformly on $[\,0,b\,]$, hence $\int_0^b [\,h(t+\tau) - h(t)\,]^2\, dt \longrightarrow 0$ and $\ell T_{(\cdot)}$ is continuous into (X^*, τ_1) whenever ℓ is in the dense set W consisting of the continuous functions in $L_2[\,0, \infty)$.

Let

$$c_{0,t}(0,f(\cdot)) = \begin{cases} 0 \text{ whenever } \sup\limits_{0\leqq\alpha\leqq t} |f(\alpha)| \leqq 1,\ f(\alpha) = 0 \text{ outside } [\,0,t\,] \\ -\infty \text{ otherwise} \end{cases}$$

It is easy to see that $c_{t_1,t_2}(x,y) = c_{0,t_2-t_1}(0,y-T_{t_2-t_1}(x))$ is an ℓ.M.t.c. function,

$$M(c_{0,t}(0,\cdot))(\ell) = \sup_x (c_{0,t}(0,x) - \ell x)$$

$$= \sup\, \{-\ell x : x \in \{c_{0,t}(0,y) \geqq 0\}\,\}$$

$$= \sup\, \{\int_0^t h(\alpha)\, f(\alpha)\, d\alpha : f(\alpha) \leqq 1\,\}$$

$$= \int_0^t |\,h(\alpha)\,|\, d\alpha = \int_0^t g(\ell T_\alpha)\, d\alpha$$

where $g(\ell) \triangleq |h(0)|$ whenever ℓ corresponds to the function h on $[0, \infty)$. Clearly $g(\cdot)$ is not continuous on L_2 and $c_{t_1,t_2}(x, y)$ cannot be represented in the form (1). The function $g(N^*_{0,\tau}(\cdot))$ is continuous since

$$N^*_{0,\gamma}\ell = \int_0^\gamma T_\alpha \ell d\alpha = \int_0^\gamma h(t+\alpha)\, d\alpha \text{ and } g(N^*_{0,\gamma}\ell) = \left|\int_0^\gamma h(\alpha)\, d\alpha\right| .$$

Theorem II.3.1 states that whenever $c_{t_1,t_2}(\cdot, \cdot)$ is regular it is concave. It is interesting to note that b-regularity and strong upper semicontinuity are not always sufficient to guarantee concavity of $c_{t_1,t_2}(0, \cdot)$. The following example was given by Radström:

Let $X = L_2[0,1]$. Let $c_{0,t}(0, \cdot)$ be defined by:

$$c_{0,t}(0,f(\cdot)) = \begin{cases} 0 \text{ whenever } \int_0^1 |f(\alpha)|\, d\alpha \leqq t \text{ and } f \text{ is integer-valued} \\ -\infty \text{ otherwise} \end{cases}$$

Then $c_{0,t}(0, \cdot)$ is uppse semicontinuous and b-regular since the set $\{f: c_{0,t}(0,f) = 0\}$ is (strongly) closed and bounded. A'so it can be easily verified that the function $c_{t_1,t_2}(x,y) = c_{0,t_2-t_1}(0,y-x)$ is an ℓ.M.t.c.f. with $T_t = I$. Despite all these facts, $c_{o,t}(0,x)$ is not convex since let

$$f_1 = \begin{cases} 0 \text{ on } [0, \frac{1}{2}) \\ 1 \text{ on } [\frac{1}{2}, 1] \end{cases} \qquad f_2 = \begin{cases} 0 \text{ on } [\frac{1}{2}, 1] \\ 1 \text{ on } [0, \frac{1}{2}) . \end{cases}$$

Then

$$\int_0^1 |f_1(\alpha)|\, d\alpha = \int_0^1 |f_2(\alpha)|\, d\alpha = \frac{1}{2},$$

and $c_{0,1/2}(0,f_1) = c_{0,1/2}(0,f_2) = 0$, but $\frac{1}{2} f_1 + \frac{1}{2} f_2 = \frac{1}{2}$ on $[0,1]$ and $c_{0,1/2}(0,\frac{1}{2}f_1 + \frac{1}{2}f_2) = -\infty$. The weak upper semicontinuous hull $\overline{c}_{0,t}(0,\cdot)$ of $c_{0,t}(0,\cdot)$ is convex, however, by theorem II.4.2

$$\overline{c}_{0,t}(0,f) = \begin{cases} 0 & \text{whenever } \int_0^1 |f(\alpha)|\, d\alpha \leqq t \\ -\infty & \text{otherwise} \end{cases}$$

To conclude this section we shall remark that theorem II.4.2 is applicable in the following general problem: Minimize

$$\int_0^t G(u(t,\cdot))\, dt \text{ subject to } x = A(x(t,\cdot)) + B(u(t,\cdot)),$$

$$x(0,\cdot) = 0, \quad x(t\cdot) = g(\cdot)$$

where G is a functional on $u(t,\cdot)$ and A, B are linear operators mapping $x(t,\cdot)$, $u(t,\cdot)$ respectively into X. The cost function of such systems is typically convex (when the theorems apply) and is identical with the cost function of the system in which the criterion functional $G(\cdot)$ is replaced by $\overline{\text{co}}\, G(\cdot)$. It is in this sense that F' of I.6 is a smoothed version of the original functional F.

References and remarks

Chapter 1

§1. (1) The fundamental work in optimal control was done by Pontriagin and expanded by (2) Rozonoer who showed the relation with classical calculus of variations. (3) Bellman (i) and (4) Bellman and Dreyfus discuss the problem from the point of view of dynamic programming. (5) Desoer tied up the two approaches by showing that the dynamic programming procedure could be made to yield the equations obtained by Pontriagin. (6) For a definition of a linear space and linear topological space see Dunford and Schwartz, p. 35 and pp. 49-50.

§3. (1) Loève, see 38.2, p. 568. (2) General one-parameter semi-groups of linear transformations are thoroughly studied in Hille-Phillips. For particular application to probability see pp. 633-663. (3) Loeve, see 38, pp. 572-573 and see 41, p. 646. (4) If x is a state, i.e., knowledge of x completely determines the future behavior of the system, we have $\Gamma_{t_1,t_2,t_3}(x_1,x_2,x_3) = f_{t_1,t_2,t_3}(c_{t_1}(x_1), c_{t_1,t_2}(x_1,x_2), c_{t_2,t_3}(x_2,x_3)$. Thus the expression on page 7 is slightly stronger than the statement that x is a state.

§4. (1) Locally convex spaces are defined and studied in Dunford and Schwartz, pp. 417-422. (2) Bourbaki (i), p. 170, prop. IV.6.4. (3) The reader may note that upper semicontinuity appears naturally in this context because $c_{t_1,t_2}(x,y)$ express the supremum of a functional. (4) Kelley, pp. 135-136.

§5. (1) An approach which is very similar in philosophy to the one presented in chapter 1 was proposed by Roxin. One of his results is a variant of theorem 1 (theorem 6.1). It may be also of interest to note that when T is a countable set, the extension property of $C(\cdot)$ is independent of the continuity properties (b) and (e).

(2) Kelley, p. 149 (Tychanoff's theorem).

§6. (1) The reader may bear in mind the fact that $\Omega = X^{R^+}$ and ω is therefore a trajectory which should be identified with a time function $x(t)$.

(2) Whenever $F(\cdot)$ has an optimal trajectory passing through x_t and t_i, $i = 1, \ldots, n$, its cost is precisely $\Gamma_{t_1, \ldots, t_n}(x_1, \ldots, x_n)$. $c_T(\omega)$ was obtained by selecting optimal trajectories ω_n which agree with ω at $t_1, \ldots, t_n$ and letting $n \longrightarrow \infty$ as the spacing between t_i was made to decrease to zero. If the trajectories of the system are sufficiently smooth ω_n must converge to ω while $F_T(\omega_n) + c_{\alpha_1}(M_{\alpha_1}(\omega_n))$ decreases to $c_T(\omega)$.

(3) This result should be contrasted with theorem (1) of the previous section which also states that $\Gamma_p(y)$ is generated by and generates $c_T(M_T(\omega))$. The difference between the two statements is of course that in theorem I.5.1 it was not assumed that there exists a cost functional $F_T(\omega)$ which generates $\Gamma_p(y)$.

§7. (1) Bellman (ii).

(2) In Chapter 2 we shall show that if sufficiently strong conditions are placed on $c_t(x)$ and if S_t is the set of points below the graph of

$c_t(\cdot)$ then (1) is equivalent to $S_{t+\tau} = S_t + S_\tau$. Such a condition was studied by Radstrom when S_t was assumed compact. His main result is a special case of theorem II.3.1.

Chapter 2

§1. (1) Bellman (iii).

(2) Gelfand and Fromin, p. 72, p. 212.

(3) Dunford and Schwartz, p. 11, p. 414.

(4) Dunford and Schwartz, p. 417.

(5) Bellman and Karush.

(6) Moreau (ii).

(7) Bourbaki (ii), p. 11, prop. 1, p. 19, prop. 2.

(8) We apply theorem 8, p. 417 of Dunford and Schwartz to find the plane separating ℓ_0 from S and use the fact that the interior of one side of this plane is contained in S^c.

(9) Moreau (i).

(10) Dunford and Schwartz, p. 414.

§2. (1) The proof is identical with that required in the finite dimensional case.

(2) We use here the well-known fact that when f is convex and finite on the convex open set G and f is bounded above on some open subset of G the restriction of f to G is continuous (see appendix 1).

(3) Bourbaki (ii), p. 17, prop. 1, p. 19, prop. 2.

§3. (1) See appendix 1.

(2) The assumption that $c_{t_1,t_2}(0,0) \geqq 0$ may be removed here by subtracting from the set $S_t = \{(y,x) : y \leqq c_{0,t}(0,x)\}$ an extremal point $v(t) = (a(c), x(t))$ thereby shifting $c_{t_1,t_2}(x,y)$ so as to make it non-negative at 0 without affecting its semigroup structure. However, in most cases of interest the cost of transferring the state a distance 0 is zero and the condition is automatically fulfilled.

(3) $\sup_z (c'_{a,b}(x,z) + c'_{b,c}(z,y)) =$

$\sup [c_{a,b}(0,z-T_{b-a}x) + c_{b,c}(0,y-T_{c-b}z)] = \sup_w [\sup_{T_{c-b}z=\omega} c_{a,b}(0,z-T_{b-z}x) +$

$c_{b,c}(0,y-\omega)]$ and the latter is upper semicontinuous by lemma II.1.7(a) and (3).

§4. (1) Hille and Phillips, p. 144, Th. 4.17.2 and p.241, Th. 7.4.1.

(2) Dunford and Schwartz, p. 419, def. V.3.2.

(3) ________ p. 418, def. V.3.1.

(4) ________ p. 421, theorem V.3.9.

(5) ________ p. 423, Lemma V.4.1.

(6) See remark (2) of section 3.

(7) Dunford and Schwartz, p. 52, Th. II.1.11.

(8) ________ p. 425, Th. V.4.7.

(9) ________ p. 422, Th. V.3.15.

§5. (1) A linear system is said to be controllable if all of its states are reachable from the zero state.

(2) u(t) exists for all t since f(·) is regular and it can be shown that u(t) can always be chosen to be measurable.

(3) Dunford and Schwartz, p. 286, Th. IV.8.1.

(4) See reference (7) of section 4.

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Definitions of some terms

Markov transition cost function: A function $c_{t_1,t_2}(x,y)$ defined on $R^+ \times R^+ \times X \times X$ is a Markov transition cost function (m.t.c.f) iff

$$c_{t_1,t_3}(x,y) = \sup\,[\,c_{t_1,t_2}(x,z) + c_{t_2,t_3}(z,y)\,]\,.$$

time invariant m.t.c.f. : $c_{t_1,t_2}(x,y) = c_{0,t_2-t_1}(x,y)$.

linear M.t.c.f. : $c_{t_1,t_2}(x,y) = c_{0,t_2-t_1}(0, y - T_t x)$

where T_t is a one parameter semi-group of linear transformations.

semigroup: An ordered pair $(S, +)$ is called a semigroup if S is a set and $+$ is an operation on it which is associative.

one parameter semigroup: A mapping $f(\cdot)$, $f : R \longrightarrow S$ where S is a semigroup is called a one parameter semigroup if f preserves the operation $(+)$.

strong semigroup property: A Markov transition c.f. is said to have the strong semigroup property iff

$$c_{t_1,t_2}(x,y) = \sup_{(t,z)\in P} (c_{t_1,t}(x,z) + c_{t,t_2}(z,y))$$

whenever P is a plane in $R^+ \times X$ separating (t_1, x) and (t_2, y).

sup-compact: A real-valued function f on a topological space is said to be sup-compact iff $\{x : f(x) \geq a\}$ is compact for all a.

sup-bounded: A real-valued function f on a linear topological space is said to be sup-bounded iff $\{x : f(x) \geqq a\}$ is bounded for all a.

Regular (function): A real-valued function f on a real linear topological space is said to be regular iff $\{x : f(x) - \ell(x) \geqq a\}$ is weakly compact for all continuous linear functionals ℓ on X.

b-regular (function): A real-valued function f on a real topological space is said to be b-regular iff $\{x : \ell(x) - \ell(x) \geqq a\}$ is weakly bounded for all continuous linear functionals ℓ on X.

c-regular (M.t.c.f.) : An M.t.c.f. is said to be c-regular iff whenever $\alpha_n \leqq t$, $\beta_n \geqq t$, α_n, $\beta_n \longrightarrow t$ and either $c_{\alpha_n,t}(x_n,x) \geqq a$ or $c_{t,\beta_n}(x,x_n) \geqq a$ for some a then $x_n \longrightarrow x$ for any x.

Appendix 1

Theorem: Let f be a convex function which is continuous at zero. Then f is continuous in the interior of the set on which it is finite.

Proof: We shall first show that f is lower semicontinuous. Indeed let $G_1 = \{(y,x) : y \geqq f(x)\}$ and suppose that for some (y_0,x_0), $y_0 < f(x_0)$. Then since G_1 has an interior point there exists a continuous linear functional separating (y_0,x_0) and G_1 (Dunford and Schwartz, p. 417, Theorem 8). Thus $f(x) = \sup_{\ell,a} \{\ell(x) + a: \ell(x)+a \leq f(x)$ for all $x\}$ and f is therefore lower semicontinuous.

To show upper semicontinuity on the interior S^0 of the set S on which $f(x) < \infty$ let $G_2 = \{(y,x): y > f(x),\ x \epsilon S^0\}$ and prove that G_2 is an open set. But B_2 has an interior point and by theorem 1c, p.413 of Dunford and Schwartz every internal point of G_2 is interior to it. It is therefore enough to show that every point of G_2 is an internal point. Let (y_0,x_0) belong to G_2 and let (y_1,x_1) by any point of $R \times X$. We must show that there exists $\delta > 0$ such that $(y_0,x_0) + \delta(y_1,x_1) \epsilon G_2$. But since $x_0 \epsilon S^0$ there exists δ_1 such that $x_0 + \delta_1 x_1 \epsilon S^0$ and the function $f(x_0 + \delta x_1) - \delta y_1$ is convex and finite for all $0 \leqq \delta \leqq \delta_1$ and lies strictly below y_0 at $\delta = 0$. Therefore for some $0 < \delta \leqq \delta_1$ it lies below y_0 and $f(x_0 + \delta x_1) \leqq y_0 + \delta y_1$ or $((y_0 + \delta y_1), (x_0 + \delta x_1)) \epsilon G_2$.