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MINIMUM TIME CONTROL OF A NONLINEAR SYSTEM

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SUDAAR No. 271

May 1966

This research was sponsored by
the National Aeronautics and Space Administration
under Research Grant NsG 133-61

ABSTRACT

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The time-optimal control problem is investigated for a system that can be represented by the second-order nonlinear differential equation

$$\ddot{x} + f(x) = u \quad , \quad |u| \leq A$$

where $f(x)$ is a periodic function such that $|f(x)| \leq B \leq C \leq A$. The condition that the bound A in the control is always greater than or equal to $|f(x)|$ is essential for the results obtained.

Pontryagin's Maximum Principle and an existence theorem by Filipov were used to prove that this optimal control exists, and that it must be of the form of a piecewise constant function of time which can attain only the values $+A$ and $-A$. This justifies working the problem in backwards time from the origin of the state plane, without getting misleading results.

The time-optimal control problem has been solved for two families of periodic functions

1) Periodic functions which are at the same time antisymmetric
2) Periodic functions that, without being antisymmetric, satisfy a certain Lemma.

The two most important facts encountered are

1) The maximum number of switchings is two.
2) There exist indifference curves. Any initial state which is described by a point on such curves can be brought to rest, in the same time, in two different ways, one after only one switching and the other after two switchings.

ACKNOWLEDGMENT

The sponsorship of the National Aeronautics and Space Administration under Research Grant NsG-133-61 is appreciated.

An application was made for the case in which $f(x) = \sin x$, and a comparison with the linear case is presented.

Singular solutions do not exist.

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INTRODUCTION

In recent years, considerable attention has been given to the development of control laws which optimize certain performance criteria. In general, the control policy is obtained as a function of time, while for engineering purposes we want a control law as a function of the state variables which describe the process.

Historically, the first problem treated was the time-optimal control problem for the system $\ddot{x} + x = u$, $|u| \leq 1$. This problem seems to have been first mentioned by Doll, Ref. [1], in 1943 in a U.S. Patent, and its solution was first proposed by McDonald, Ref. [2], and Hopkin, Ref. [3]. The first rigorous solution of this problem was given by Bushaw in his doctoral dissertation, Ref. [4], and he solves it by elementary but very intricate direct geometric arguments.

This problem stood alone for some time, as an example of a wider class of problems. But it was not until Pontryagin formulated his Maximum Principle, Ref. [5], that further progress was made, Refs. [6]-[9]. If the differential equations governing the state variables are known and if the Maximum Principle is used, a control law as a function of the adjoint variables can be easily found, but, the difficult and tedious task of determining the initial conditions of the adjoint variables must then be dealt with.

A wide class of problems which are important from a practical point of view is that in which the control is not only constrained to a certain magnitude but enters the dynamical equations and the performance criteria in a linear manner. When the Maximum Principle is used, it is found that the control $u(t)$ is a piecewise constant function of time, known generally as a "bang-bang" control. The realization of such a "bang-bang" control depends on the determination of the switching surfaces in the state space, that is, the surfaces that separate the regions in which the control is full on in one direction from the regions in which the control is full on in the other direction. So, the problem of finding an optimal control law reduces to that of finding the switching surfaces in the state space.

The task of finding the switching surfaces is not, in general, an easy one. Some work has been done in this respect, but mostly concerned with the fuel-optimal control problem for linear second-order systems, Refs. [10]-[15]. In Ref. [16], the fuel-optimal singular control problem is solved for a specific nonlinear system.

The object of this research is to find the time-optimal control for a system that can be represented by the second-order nonlinear differential equation $\ddot{x} + f(x) = u$, $|u| \leq A$, where $f(x)$ is a general periodic function of x such that $|f(x)| \leq B$, and $B \leq C \leq A$. This problem turns up if one becomes interested, for instance, in the minimum settling time problem for certain motions of a satellite in a circular orbit; notice that in this particular case $f(x) = \sin x$, and that x may become so large that if we replace $\sin x$ by x the results obtained can be completely false.

We have been able to solve the problem for two wide families of periodic functions, see page 58. Two of the most interesting features of this problem are the maximum number of switchings needed and the presence of the indifference curves. It can be shown that whatever the initial disturbance is, the number of switchings cannot be greater than two. Indifference curves are curves such that a system whose initial state is described by a point on such a curve can be brought to rest, in the same time, in two different ways, one after only one switching and the other after two switchings. It is worth stating that these indifference curves are also the locus of starting points, that is, points from where we can start a trajectory but which can never be reached by a state point trajectory; the existence of points with this behavior was first discovered by Flügge-Lotz, Ref. [17]. The appearance of indifference curves clearly indicates that for nonlinear systems, Pontryagin's Maximum Principle, although it gives necessary conditions for optimal control, does not guarantee the uniqueness of the solution.

The main result of our research is given by Theorem 3-2, see page 74, in which the optimal control law is expressed as a function of the state variables. Singular controls cannot occur for the nonlinear system considered.

CHAPTER I

GENERAL CONSIDERATIONS

<u>The System</u> - A system is composed of a plant or process which is to be controlled, and a means for providing a control input. It is assumed that the equation describing the dynamical behavior of the plant is known and can be expressed by the differential equation

$$\ddot{x} + f(x) = u \tag{1-1}$$

where dots denote differentiation with respect to time t and $f(x)$ is assumed to be any periodic function with period 2θ and satisfying the two conditions

i) $f(x)$ is continuously differentiable on $(-\infty, +\infty)$ (1-2)

ii) $|f(x)| \le B$, where B is a positive given constant (1-3)

Let

$$\left.\begin{aligned} x_1(t) &= x(t) \\ x_2(t) &= \dot{x}(t) \end{aligned}\right\} \tag{1-4}$$

Then the state variables $x_1(t)$ and $x_2(t)$ are the solutions of the differential equations

$$\left.\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u - f(x_1) \end{aligned}\right\} \tag{1-5}$$

The scalar control function $u(t)$ shall satisfy the three conditions

i) $u(t)$ is a piecewise continuous function of time (1-6)

ii) $u(t)$ is constrained to have a finite amplitude limit, that is,
$|u(t)| \le A$ for all $t \in (-\infty, +\infty)$ (1-7)

iii) $A \geq C$, where C is any constant such that

$$C \geq B \tag{1-8}$$

The first two conditions imposed on the control function $u(t)$ are commonly made in optimal control theory and do not need any further explanation. Now, we are going to summarize the main implications of having imposed condition (1-8) on the control function $u(t)$.

As it will be shown later, Pontryagin's Maximum Principle gives a control function that can only attain the constant values $+A$ and $-A$. If we find the trajectories of equations (1-5) corresponding to these constant values, condition (1-8) establishes that these trajectories are always non-decreasing functions of x_1 if $u = +A$, and non-increasing functions of x_1 if $u = -A$, see Lemma 1-1 on page 11. This fact is used in order to prove the existence of the optimal control within the class of piecewise continuous functions, and also to prove that the domain of controllability is the whole state plane.

Another consequence of condition (1-8) is that the zero trajectories divide the state plane into two different parts. This fact together with the knowledge that the zero trajectories are optimal trajectories and that the optimal control is a piecewise continuous function allow us to work the problem in backwards time and find a solution for the optimal control problem, with the important result that the number of switchings cannot be greater than two.

<u>Definition 1-1</u> - Any control function $u(t)$ which satisfies the above three conditions (1-6), (1-7) and (1-8), will be designated as an admissible control.

The Control Problem - For a plant which can be described in the form of equations (1-5), a control $u(t)$ satisfying conditions (1-6), (1-7) and (1-8), will be sought which accomplishes a twofold objective. Firstly, the control $u(t)$ must transfer the system in accordance with equations (1-5) from some known initial state $\left(x_1(t_0), x_2(t_0)\right)$ to the terminal state $\left(x_1(t_f), x_2(t_f)\right) = (0,0)$. Since equations (1-5) are stationary, i.e., invariant under a change of time reference, t_0 may be chosen as any convenient instant, say $t_0=0$. Therefore, the boundary conditions can be written as

$$\left.\begin{aligned} \left(x_1(0),\ x_2(0)\right) &= (x_{10},\ x_{20}) \\ \left(x_1(t_f),\ x_2(t_f)\right) &= (x_{1f},\ x_{2f}) = (0,0) \end{aligned}\right\} \qquad (1\text{-}9)$$

The second requirement on the control $u(t)$ will be that it must optimize the system performance in a particular sense. For the present case, the criterion to be used in evaluating the system performance, is

$$J = \int_0^{t_f} dt = t_f \rightarrow \text{minimum} \qquad (1\text{-}10)$$

Definition 1-2 - If, for a given problem, a unique control function $u^*(t)$ can be found which satisfies the three conditions

i) $u^*(t)$ is an admissible control

ii) $u^*(t)$ forces the system (1-5) from the initial state (x_{10}, x_{20}) to the final state $(0,0)$

iii) $u^*(t)$ minimizes the performance functional J with respect to all other suitable control functions

then, this control function $u^*(t)$ will be called the optimal control for the problem.

Behavior of the Adjoint Variables - The Hamiltonian for the system (1-5) is given by

$$H(p,x,u) = p_1x_2 + p_2u - p_2f(x_1) \tag{1-11}$$

where $p_1(t)$ and $p_2(t)$ are the adjoint variables satisfying the following differential equations

$$\dot{p}_1 = -\frac{\partial H}{\partial x_1} \quad \text{and} \quad \dot{p}_2 = -\frac{\partial H}{\partial x_2} \tag{1-12}$$

which yield

$$\left.\begin{aligned} \dot{p}_1(t) &= p_2(t)\,\frac{df(x_1)}{dx_1} \\ \dot{p}_2(t) &= -p_1(t) \end{aligned}\right\} \tag{1-13}$$

Eliminating $p_1(t)$ from equations (1-13), we obtain

$$\ddot{p}_2 + \frac{df}{dx_1}\,p_2 = 0 \tag{1-14}$$

whose solution we are going to investigate.

The following comparison theorem, Ref. [18], will be used

Theorem 1-1 - Suppose $\varphi(t)$ is a real solution on (t_0,t_f) of

$$\ddot{p}_2 + g_1(t)p_2 = 0$$

and $\psi(t)$ is a real solution on (t_0,t_f) of

$$\ddot{p}_2 + g_2(t)p_2 = 0$$

where $g_1(t)$ and $g_2(t)$ are continuous on (t_0,t_f). Let $g_2(t) > g_1(t)$ on (t_0,t_f). If t_1 and t_2 are successive zeros of $\varphi(t)$ on (t_0,t_f), then $\psi(t)$ must vanish at some point of (t_1, t_2).

Let $(df/dx_1) = g(t)$. Since x_1 is a continuous function of time and $f(x_1)$ is continuously differentiable, it follows that $g(t)$ is a continuous function of time such that

$$g(t) < N \quad \text{for any} \quad (t_0, t_f) \tag{1-15}$$

where N is a positive constant.

Let us now apply Theorem 1-1 to the solutions of the following differential equations

$$\ddot{p}_2 + g(t)p_2 = 0 \tag{1-16}$$

$$\ddot{p}_2 + N\, p_2 = 0 \tag{1-17}$$

on the interval (t_0, t_f). The solution of equation (1-16) cannot be identically zero on any interval $(t_1, t_2) \subset (t_0, t_f)$ because if it were so, Theorem 1-1 would imply that the solution of (1-17) would also be identically zero on (t_1, t_2), and this can never happen because N is a positive constant. This rules out the possibility of having singular solutions. Moreover, the zeros of the solution of (1-16) cannot have a point of accumulation on any interval (t_1, t_2) because the zeros of the solution of (1-17) do not have such a point. Hence, the solution of the equation (1-16), i.e., the adjoint variable $p_2(t)$, is a continuous function of time with a finite number of isolated zeros on any finite interval of time.

<u>The Necessary Conditions on the Optimal Control</u> - The Maximum Principle of Pontryagin will now be used to derive the necessary conditions on the optimal control $u^*(t)$. Theorem 2 of Ref. [5] for our problem reads as follows

Theorem 1-2 - Let $u(t)$, $t \in [0,t_f]$, be an admissible control which transfers the state point from (x_{10},x_{20}) to $(0,0)$, and let $x(t) = \left(x_1(t), x_2(t)\right)$ be the corresponding trajectory, see (1-5), so that $x(0) = (x_{10},x_{20})$ and $x(t_f) = (0,0)$. In order that $u(t)$ and $x(t)$ be time-optimal it is necessary that there exist a nonzero, continuous vector function $p(t) = \left(p_1(t), p_2(t)\right)$ corresponding to $u(t)$ and $x(t)$, see (1-13), such that:

i) For all t, $t \in [0, t_f]$, the function $H\left(p(t), x(t), u\right)$ of the variable u, $|u| \leq A$, attains its maximum at the point $u = u(t)$:

$$H\left(p(t), x(t), u(t)\right) = M\left(p(t), x(t)\right) \tag{1-18}$$

ii) At the terminal time t_f the relation

$$M\left(p(t_f), x(t_f)\right) \geq 0 \tag{1-19}$$

is satisfied. Furthermore, it turns out that if $p(t)$, $x(t)$ and $u(t)$ satisfy system (1-5), (1-13), and condition i), the time function $M\left(p(t), x(t)\right)$ is constant. Thus, (1-19) may be verified at any time $t \in [0,t_f]$ and not just at t_f.

Applying Theorem 1-2 to our problem, taking into account (1-7), (1-11) and the fact shown before that $p_2(t) \not\equiv 0$ on any interval of time, relation (1-18) yields

$$u^*(t) = A \operatorname{sgn} p_2^*(t) \tag{1-20}$$

where $\operatorname{sgn} p_2^*(t)$ is defined by

$$\operatorname{sgn} p_2^*(t) = \begin{cases} +1 & \text{if} \quad p_2^*(t) > 0 \\ -1 & \text{if} \quad p^*_2(t) < 0 \end{cases}$$

Trajectories - From the condition (1-20) we know that the optimal control can attain only the constant values $+A$ and $-A$. So, let us find the trajectories of the system (1-5) subject to these controls.

Then, assuming u to be constant, we can integrate equations (1-5) and obtain

$$\frac{x_2^2}{2} = ux_1 - F(x_1) + k \tag{1-21}$$

where $F(x_1)$ is defined as

$$F(x_1) = \int_0^{x_1} f(\sigma)d\sigma \tag{1-22}$$

Since the trajectories are symmetric with respect to the x_1-axis, we can characterize each trajectory by the crossing point with the x_1-axis, say $M(x_{1m},0)$. Then (1-21) becomes

$$\frac{x_2^2}{2} = u(x_1 - x_{1m}) - F(x_1) + F(x_{1m}) \tag{1-23}$$

Definition 1-3 - If $u = +A$, the solution curves of (1-21), given by

$$\frac{x_2^2}{2} = Ax_1 - F(x_1) + k_1 \tag{1-24}$$

cover the entire plane exactly once. This family of curves will be called the P-system, its curves P-curves, and portions of its curves P-arcs.

Likewise, if $u = -A$, the solution curves of (1-21) are

$$\frac{x_2^2}{2} = -Ax_1 - F(x_1) + k_2 \tag{1-25}$$

and the family of curves will be called the N-system, its curves N-curves, and portions of its curves N-arcs.

Each P-or N-arc is automatically oriented by the increase of time t along it.

Lemma 1-1 - Every P-curve (N-curve) is a non-decreasing (non-increasing) function of x_1.

Proof - In the case of a P-curve, from (1-3), (1-5) and (1-8) we get

$$\frac{dx_2}{dx_1} = \frac{A - f(x_1)}{x_2} \begin{cases} \geq 0 & \text{if } x_2 > 0 \\ \leq 0 & \text{if } x_2 < 0 \end{cases}$$

Likewise for an N-curve.

Corollary - The zero trajectories, given by

$$\frac{x_2^2}{2} = \pm Ax_1 - F(x_1) \tag{1-26}$$

divide the state plane into two different parts.

Lemma 1-2 - The trajectories whose crossing point with the x_1-axis is in the interval $[2n\theta\ ,\ 2(n+1)\theta)$, are obtained by shifting the trajectories whose crossing point with the x_1-axis is in the interval $[0,2\theta)$, by an amount of $2n\theta$ in the positive direction of the x_1-axis.

Proof - Let us consider the trajectory

$$\frac{x_2^2}{2} = u(x_1 - x_{1m}) - F(x_1) + F(x_{1m})$$

where $0 \leq x_{1m} < 2\theta$.

After a shifting of $2n\theta$ in the positive direction of the x_1-axis, it becomes

$$\frac{x_2^2}{2} = u(x_1-2n\theta-x_{1m}) - F(x_1-2n\theta) + F(x_{1m})$$

Let $y_{1m} = x_{1m} + 2n\theta$; then

$$\frac{x_2^2}{2} = u(x_1-y_{1m}) - F(x_1-2n\theta) + F(y_{1m}-2n\theta) =$$

$$= u(x_1-y_{1m}) - \int_{y_{1m}-2n\theta}^{x_1-2n\theta} f(\sigma)d\sigma = u(x_1-y_{1m}) - \int_{y_{1m}}^{x_1} f(\sigma)d\sigma =$$

$$= u(x_1-y_{1m}) - F(x_1) + F(y_{1m})$$

is the trajectory whose crossing point is y_{1m}, and

$$2n\theta \leq y_{1m} < 2(n+1)\theta$$

The Existence of the Optimal Control - Filipov, Ref. [19], proves the existence of the optimal control within the class of bounded, measurable functions, for the time-optimal control problem of a system of n first-order nonlinear differential equations, under certain assumptions. Theorems of existence are also given in Refs. [20] and [21]. Using the theorem proved by Filipov, we are going to show that for our problem the optimal control exists within the class of piecewise continuous functions.

The main theorem in Ref. [19] will now be stated for a special case. Let the dynamical system be represented by the following n first-order differential equations

$$\dot{x} = g(x,u) \tag{1-27}$$

where x and g are n-dimensional vectors, and $u = u(t)$ is the r-dinemsional control vector which can take on values in a given constant set U. Moreover, we are interested in the time-optimal control problem.

The following assumptions are made:

i) The vector function $g(x,u)$ is continuous in x and u

ii) The vector function $g(x,u)$ is continuously differentiable with respect to x

iii) For all x and all $u \in U$ the following relation holds:

$$x \cdot g(x,u) \leq \alpha \left(\|x\|^2+1\right) \tag{1-28}$$

where the dot denotes the scalar product, $\|x\|$ denotes the length of the vector x and α is a constant

iv) U is compact

v) $R = \left| g(x,u) : u = U \right|$ is a convex set

Note that since U is a constant set, it is also upper semicontinuous with respect to inclusion; therefore, this assumption can be omitted for the special case we are considering.

In this special case, the theorem proved by Filipov reads as follows

Theorem 1-3 - Suppose that the five conditions stated above are satisfied. Also suppose that there exists at least one measurable function $\tilde{u}(t)$ such that the solution $\tilde{x}(t)$ of (1-27), with $u = \tilde{u}(t)$, and initial condition $\tilde{x}(0) = x_0$, attains x_f for some time $t_f > 0$. Then there also exists an optimal control, i.e., a measurable function $u(t)$ for which the solution $x(t)$ of (1-27), with initial condition $x(0) = x_0$, attains x_f in the least possible time.

Let us now check if the five conditions stated above hold for our particular problem.

i) It is obvious since $f(x_1)$ is a continuous function

ii) It is obvious since $f(x_1)$ is continuously differentiable

iii) From (1-3) and (1-7) we get

$$u - f(x_1) \le A+B \quad , \quad |x_2|[u - f(x_1)] \le |x_2|(A+B) \tag{1-29}$$

Let α be a constant defined as

$$\alpha = 1 + \frac{(A+B)^2}{2} \tag{1-30}$$

Then, using (1-29) and (1-30) we get

$$\alpha\,(\|x\|^2 + 1) \ge \|x\|^2 + \alpha > x_1^2 + x_2^2 + \frac{(A+B)^2}{2} = \frac{1}{2}\,(x_1^2 + x_2^2) +$$

$$+ \frac{1}{2}\,[x_2^2 + (A+B)^2] + \frac{1}{2}\,x_1^2 \ge x_1x_2 + |x_2|(A+B) \ge x_1x_2 +$$

$$+ |x_2|[u-f(x_1)] \ge x_1x_2 + x_2[u-f(x_1)] = x \cdot g(x,u)$$

iv) It is obvious since $U = \left\{ u \; : \; |u| \le A \right\}$

v) $R = \left\{ (g_1,g_2) \; : \; g_1 = \text{arbitrary and } |g_2| \le A+B \right\}$ which is obviously convex.

In order to apply Theorem 1-3, we still have to prove the existence of the measurable function $\tilde{u}(t)$ as defined in Theorem 1-3. Let us assume that $x_{10} > x_{1f}$. Consider the N-curve through the point x_0 and the P-curve through the point x_f, and let x_s be the intersection point of both curves. Also, let t_s be the time spent from x_0 to x_s. Then, the function $\tilde{u}(t)$ defined as

$$\tilde{u}(t) = \begin{cases} -A & \text{if } t \in [0,t_s) \\ \\ +A & \text{if } t \in (t_s,t_f] \end{cases}$$

is a piecewise continuous function that transfers the state point from x_0 to x_f in a finite time $t_f > 0$. Note that if $x_{10} \leq x_{1f}$, a function $\tilde{u}(t)$ could be constructed in a similar way. Then, Theorem 1-3 can be applied to our problem and the existence of an optimal control within the class of bounded, measurable functions has been shown.

Let $u^*(t)$ be the bounded, measurable function, satisfying (1-7) and (1-8), which transfers the state point from $x_0 = (x_{10}, x_{20})$ to $x_f = (0,0)$ in the least possible time; the existence of $u^*(t)$ is guaranteed by Theorem 1-3 . Then, $u^*(t)$ must satisfy Pontryagin's Maximum Principle, that is, $u^*(t)$ is given by condition (1-20). But, it has been shown, see page 8, that $p_2^*(t)$ is a continuous function of time with a finite number of isolated zeros in any finite interval $[0,t_f]$. Therefore, from condition (1-20), it follows that the optimal control $u^*(t)$ is a piecewise continuous function of time, as we wanted to show. Also, it follows that the domain of controllability is the whole state plane.

CHAPTER II

A. SOLUTION FOR THE ADJOINT VARIABLES AS A FUNCTION OF THE STATE VARIABLES AND THE INITIAL CONDITIONS

In Chapter I, we found that (1-20) is a necessary condition for the control $u(t)$ to be optimal. If we were able to find $p_2(t)$ as a function of time, we would eventually draw some useful information, such as the maximum number of switchings or the maximum period of time in which $p_2(t)$ keeps a constant sign; then we could study the problem in backwards time and apply any of the known sufficient conditions in order to get the switching curves and the optimal control law, as it is done, for example, in Ref. [7].

Eliminating $p_1(t)$ from equations (1-13), we get for $p_2(t)$ the nonlinear second-order differential equation

$$\ddot{p}_2 + p_2 \frac{df}{dx_1} = 0 \qquad (2\text{-}1)$$

In order to solve for $p_2(t)$ we have to know df/dx_1 as a function of time, i.e., $x_1(t)$; unfortunately, $x_1(t)$ depends on $u(t)$ which, in turn, depends on $p_2(t)$ through condition (1-20); one cannot, then, hope to find $p_2(t)$ as a function of t only.

However, what we can do, and what is more satisfactory, is to express $p_2(t)$ as a function of the state variables (x_1,x_2) , and substitute into (1-20) in order to get an expression for the optimal control law.

As a first step, our aim is finding $p_2(t)$ as a function of the state variables, along every P- and N-curve, for fixed initial values of the state variables and arbitrary initial values for the adjoint variables.

Solution for a P-curve - Let us start solving the problem for a P-curve, i.e., let us try to solve the following problem:

$$\left.\begin{aligned}
&\ddot{x}_1 + f(x_1) = A \ ; \quad x_1(0) = x_{10} \ , \quad \dot{x}_1(0) = x_2(0) = x_{20} \\
&\ddot{p}_2 + p_2 \frac{df}{dx_1} = 0 \ ; \quad p_2(0) = p_{20} \ , \quad \dot{p}_2(0) = - p_1(0) = - p_{10} \\
&\frac{x_2^2}{2} = Ax_1 - F(x_1) + k_1 \\
&k_1 = \frac{x_{20}^2}{2} - Ax_{10} + F(x_{10})
\end{aligned}\right\} \quad (2\text{-}2)$$

We know that

$$\frac{dp_2}{dt} = \frac{dp_2}{dx_1}\frac{dx_1}{dt} = \frac{dp_2}{dx_1} x_2$$

$$\frac{d^2p_2}{dt^2} = \frac{d^2p_2}{dx_1^2} x_2^2 + \frac{dp_2}{dx_1}[A - f(x_1)]$$

Substituting $\frac{d^2p_2}{dt^2}$ in equation (2-1), we get

$$x_2^2 \frac{d^2p_2}{dx_1^2} + [A-f(x_1)] \frac{dp_2}{dx_1} + p_2 \frac{df}{dx_1} = 0$$

or

$$2[Ax_1 - F(x_1) + k_1] \frac{d^2p_2}{dx_1^2} + [A - f(x_1)] \frac{dp_2}{dx_1} + p_2 \frac{df}{dx_1} = 0 \qquad (2\text{-}3)$$

Equation (2-3) is a second-order differential equation, where the point $M(x_{1m},0)$ is a singular point, because for this point

$$Ax_{1m} - F(x_{1m}) + k_1 = 0$$

and M is its only singular point. For this point M , a singular behavior of (2-3) must be expected. So, excluding a neighborhood of M , the integration of (2-3) can be done very easily; the solution of (2-3) is carried out in detail in Appendix A, and the result obtained for the adjoint variable $p_2(t)$ is the following:

$$p_2(t) = \left\{2[Ax_1 - F(x_1) + k_1]\right\}^{1/2} \left[\frac{p_{20}}{x_{20}} - \left\{ p_{10}x_{20} + p_{20}\,[A - f(x_{10})] \right\} \int_{x_{10}}^{x_1} \frac{d\sigma}{\left\{2[A\sigma - F(\sigma) + k_1]\right\}^{3/2}} \right] \tag{2-4}$$

and

$$p_2(t) = -\left\{2[Ax_1 - F(x_1) + k_1]\right\}^{1/2} \left[\frac{p_{20}}{x_{20}} - \left\{ p_{10}x_{20} + p_{20}[A - f(x_{10})] \right\} \int_{x_1}^{x_{10}} \frac{d\sigma}{\left\{2[A\sigma - F(\sigma) + k_1]\right\}^{3/2}} \right] \tag{2-5}$$

Equations (2-4) and (2-5) are only valid for the P-arcs in Fig. 2-1 and Fig. 2-2 respectively.

We have already found the solution of equation (2-3), given by (2-4) and (2-5), whenever we keep away from the x_1-axis. However, we would like to know the solution of (2-3) everywhere, in order to connect the solutions on the P-arcs of Figs. 2-1 and 2-2. Obviously, two limiting cases have to be considered separately, depending whether the initial or the final point is on the x_1-axis.

Initial Point on the x_1-Axis - The solution of (2-3) is given by equation

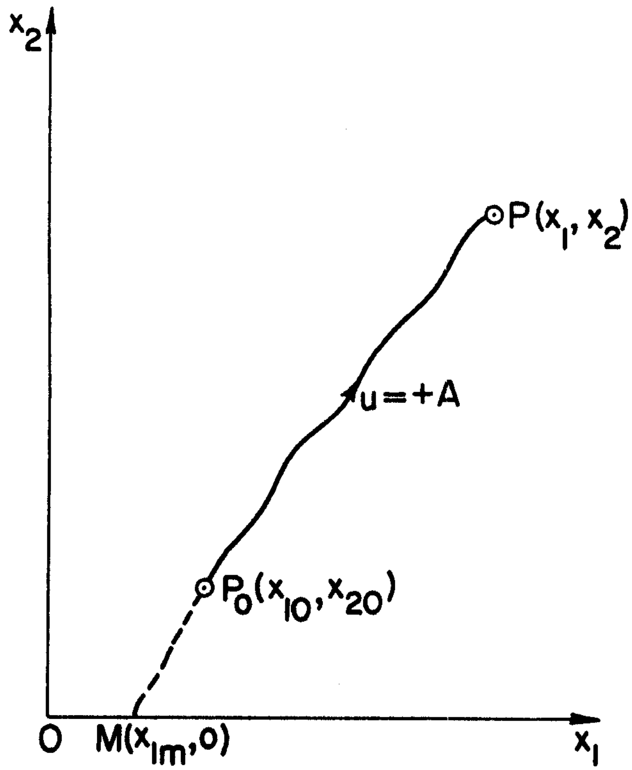

Fig. 2-1. P-arc from $P_0 \neq M$ to P in Forward Time

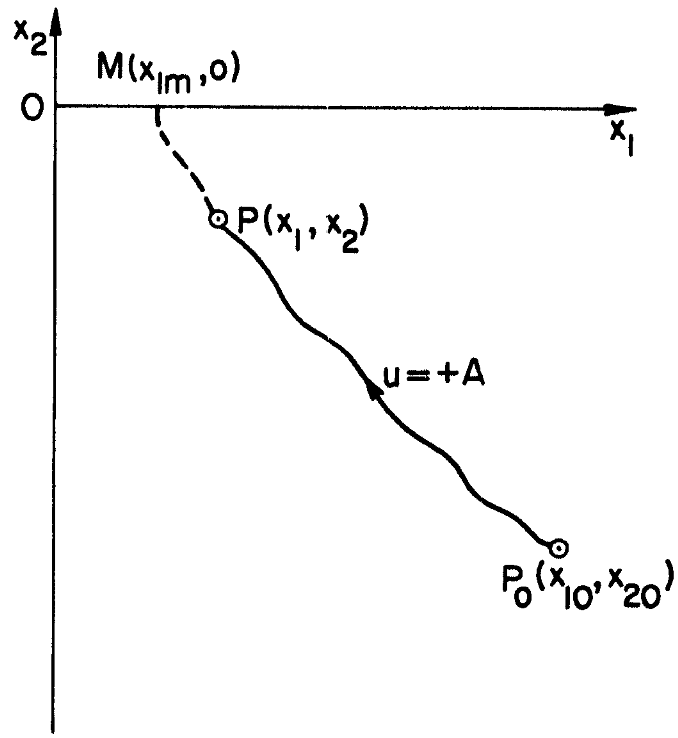

Fig. 2-2. P-arc from P_0 to $P \neq M$ in Forward Time

(2-4) whenever the point P_0 is different from the point M. Now, the question is: Is it possible to find $p_2(t)$ and $p_1(t)$ when the initial point P_0 coincides with the point M? Since $p_2(t)$ and $p_1(t)$ are not defined when $P_0 \equiv M$, we are confronted with a limiting problem; so, our aim is to find

$$\lim_{P_0 \to M} p_2(t) \qquad \text{and} \qquad \lim_{P_0 \to M} p_1(t)$$

The process of obtaining the limiting values of $p_2(t)$ and $p_1(t)$ is shown in detail in the first part of Appendix B. The results obtained are the following:

$$p_2(t) = \left\{2[Ax_1 - F(x_1) + k_1]\right\}^{1/2} \left\{ -\frac{p_{10}}{A - f(x_{1m})} + \frac{p_{20}}{\left\{2[A - f(x_{1m})](x_1 - x_{1m})\right\}^{1/2}} - p_{20}[A - f(x_{1m})] \int_{x_{1m}}^{x_1} \left[\frac{1}{\left\{2[A\sigma - F(\sigma) + k_1]\right\}^{3/2}} - \frac{1}{\left\{2[A - f(x_{1m})](\sigma - x_{1m})\right\}^{3/2}} \right] d\sigma \right\} \tag{2-6}$$

and

$$p_1(t) = p_{20} \frac{A - f(x_{1m})}{\left\{2[Ax_1 - F(x_1) + k_1]\right\}^{1/2}} - [A - f(x_1)] \left\{ -\frac{p_{10}}{A - f(x_{1m})} + \frac{1}{\left\{2[A - f(x_{1m})](x_1 - x_{1m})\right\}^{1/2}} - p_{20}[A - f(x_{1m})] \cdot \int_{x_{1m}}^{x_1} \left[\frac{1}{\left\{2[A\sigma - F(\sigma) + k_1]\right\}^{3/2}} - \frac{1}{\left\{2[A - f(x_{1m})](\sigma - x_{1m})\right\}^{3/2}} \right] d\sigma \right\} \tag{2-7}$$

So, equations (2-6) and (2-7) are the solutions for the adjoint variables when the initial point P_0 coincides with M.

Final Point on the x_1-Axis - The solution of (2-3) is given by equation (2-5) whenever the point $P(x_1,x_2)$ is different from the point $M(x_{1m},0)$. As in the case in which the initial point is on the x_1-axis, we are confronted with the following limiting problems:

$$\lim_{P \to M} p_2(t) \qquad \text{and} \qquad \lim_{P \to M} p_1(t)$$

The process of obtaining the limiting values of $p_2(t)$ and $p_1(t)$ is shown in detail in the second part of Appendix B. The results obtained are the following:

$$p_2(t) = \frac{p_{10}x_{20} + p_{20}[A-f(x_{10})]}{A-f(x_{1m})} \tag{2-8}$$

and

$$p_1(t) = -\frac{p_{20}}{x_{20}}[A-f(x_{1m})] - (p_{10}x_{20} + [A-f(x_{10})]p_{20})\,[A-f(x_{1m})]\cdot$$

$$\cdot \int_{x_{1m}}^{x_{10}} \left[-\frac{1}{\left\{2[A\sigma-F(\sigma)+k_1]\right\}^{3/2}} + \frac{1}{\left\{2[A-f(x_{1m})](\sigma-x_{1m})\right\}^{3/2}} \right] d\sigma \; -$$

$$- \frac{p_{10}x_{20} + [A-f(x_{10})]p_{20}}{\left\{2[A-f(x_{1m})](x_{10}-x_{1m})\right\}^{1/2}} \tag{2-9}$$

So, (2-8) and (2-9) are the solutions for $p_2(t)$ and $p_1(t)$ when the final point P coincides with M .

Solution for an N-curve - In this case, we have to solve the following problem:

$$\left.\begin{aligned}
&\ddot{x}_1 + f(x_1) = -A; \quad x_1(0) = x_{10}, \ \dot{x}_1(0) = x_2(0) = x_{20} \\
&\ddot{p}_2 + p_2 \frac{df}{dx_1} = 0; \quad p_2(0) = p_{20}, \ \dot{p}_2(0) = -p_1(0) = -p_{10} \\
&\frac{x_2^2}{2} = -Ax_1 - F(x_1) + k_2 \\
&k_2 = \frac{x_{20}^2}{2} + Ax_{10} + F(x_{10})
\end{aligned}\right\} \quad (2\text{-}10)$$

Proceeding as in the case of a P-curve, we get

$$p_2(t) = -\left\{2[-Ax_1-F(x_1)+k_2]\right\}^{1/2}\left[\frac{p_{20}}{x_{20}} - \left\{p_{10}x_{20} + p_{20}\,[-A-f(x_{10})]\right\}\cdot \int_{x_1}^{x_{10}} \frac{d\sigma}{\left\{2[-A\sigma-F(\sigma)+k_2]\right\}^{3/2}}\right] \quad (2\text{-}11)$$

if the N-arc remains below the x_1-axis, and

$$p_2(t) = \left\{2[-Ax_1-F(x_1)+k_2]\right\}^{1/2}\left[\frac{p_{20}}{x_{20}} - \left\{p_{10}x_{20} + p_{20}[-A-f(x_{10})]\right\}\cdot \int_{x_{10}}^{x_1} \frac{d\sigma}{\left\{2[-A\sigma-F(\sigma)+k_2]\right\}^{3/2}}\right] \quad (2\text{-}12)$$

if the N-arc remains above the x_1-axis.

Also, as in the case of a P-curve, equations (2-11) and (2-12) are valid whenever we keep away from the x_1-axis. In order to connect both arcs, we have to consider again the two limiting cases. Proceeding as in the case of a P-curve, we get the following solutions

$$p_2(t) = - \left\{2[-Ax_1-F(x_1)+k_2]\right\}^{1/2} \left\{\frac{p_{10}}{A+f(x_{1m})} - \frac{p_{20}}{\left\{2[A+f(x_{1m})](x_{1m}-x_1)\right\}^{1/2}} - \right.$$

$$- p_{20}[A+f(x_{1m})] \int_{x_1}^{x_{1m}} \left[- \frac{1}{\left\{2[-A\sigma-F(\sigma)+k_2]\right\}^{3/2}} + \right.$$

$$\left. \left. + \frac{1}{\left\{2[A+f(x_{1m})](x_{1m}-\sigma)\right\}^{3/2}} \right] d\sigma \right\} \qquad (2\text{-}13)$$

$$p_1(t) = - \dot{p}_2(t) = \frac{1}{\left\{2[-Ax_1-F(x_1)+k_2]\right\}^{1/2}} \left[p_{20}[A+f(x_{1m})] - \right.$$

$$\left. - [A+f(x_{1m})]p_2(t)\right] \qquad (2\text{-}14)$$

and

$$p_2(t) = \frac{d_1}{A+f(x_{1m})} = \frac{-p_{10}x_{20} + p_{20}[A+f(x_{10})]}{A + f(x_{1m})} \qquad (2\text{-}15)$$

$$p_1(t) = - \dot{p}_2(t) = \frac{p_{20}}{x_{20}} [A+f(x_{1m})] - \frac{-p_{10}x_{20} + p_{20}[A+f(x_{10})]}{\left\{2[A+f(x_{1m})](x_{1m}-x_{10})\right\}^{1/2}} +$$

$$+ \left\{-p_{10}x_{20}+p_{20} [A+f(x_{10})]\right\} [A+f(x_{1m})] \, .$$

(Continued)

$$\cdot \int_{x_{10}}^{x_{1m}} \left[\frac{1}{\left\{2[-A\sigma - F(\sigma) + k_2]\right\}^{3/2}} - \frac{1}{\left\{2[A+f(x_{1m})](x_{1m}-\sigma)\right\}^{3/2}} \right] d\sigma \qquad (2\text{-}16)$$

Equations (2-13) and (2-14) correspond to the N-arc represented in Fig. 2-3; equations (2-15) and (2-16) correspond to the N-arc represented in Fig. 2-4.

Equations in Backwards Time - Since part of our later discussion is going to be based on the study of trajectories in backwards time, τ , let us express the solutions for $p_2(t)$ and $p_1(t)$ as functions of τ . So, let

$$\left.\begin{aligned} \tau &= t_f - t \\ x_1(t) &= y_1(\tau) \\ x_2(t) &= y_2(\tau) \\ p_1(t) &= \lambda_1(\tau) \\ p_2(t) &= \lambda_2(\tau) \end{aligned}\right\} \qquad (2\text{-}17)$$

Substituting in the corresponding equations in forward time, we get

i) For the case represented in Fig. 2-5a,

$$\lambda_2(\tau) = - \left\{2[Ay_1 - F(y_1) + k_1]\right\}^{1/2} \left\{ - \frac{\lambda_{10}}{A-f(y_{1m})} - \frac{\lambda_{20}}{\left\{2[A-f(y_{1m})](y_1-y_{1m})\right\}^{1/2}} - \right.$$

$$-\lambda_{20}[A-f(y_{1m})] \int_{y_{1m}}^{y_1} \left[- \frac{1}{\left\{2[A\sigma - F(\sigma)+k_1]\right\}^{3/2}} + \right.$$

$$\left. \left. + \frac{1}{\left\{2[A-f(y_{1m})](\sigma - y_{1m})\right\}^{3/2}} \right] d\sigma \right\} \qquad (2\text{-}18)$$

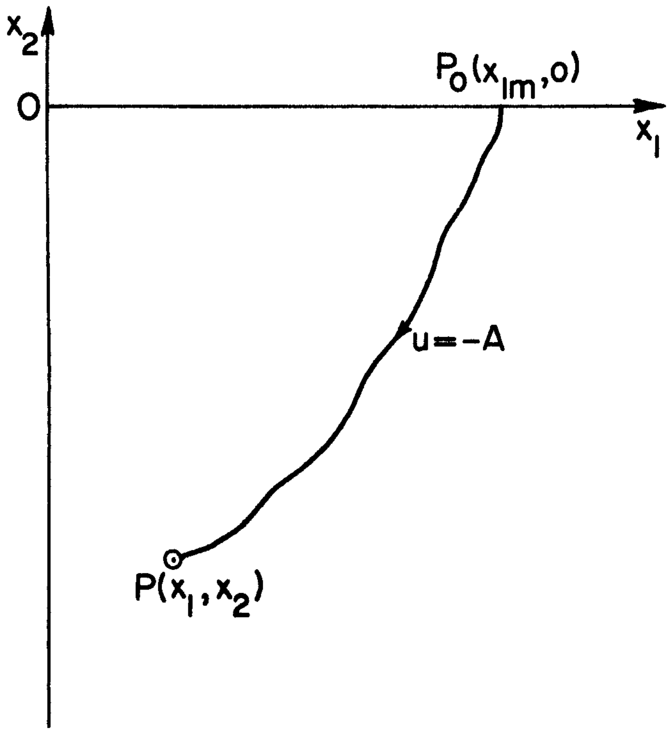

Fig. 2-3. N-arc from $P_0 \equiv M$ to P in Forward Time

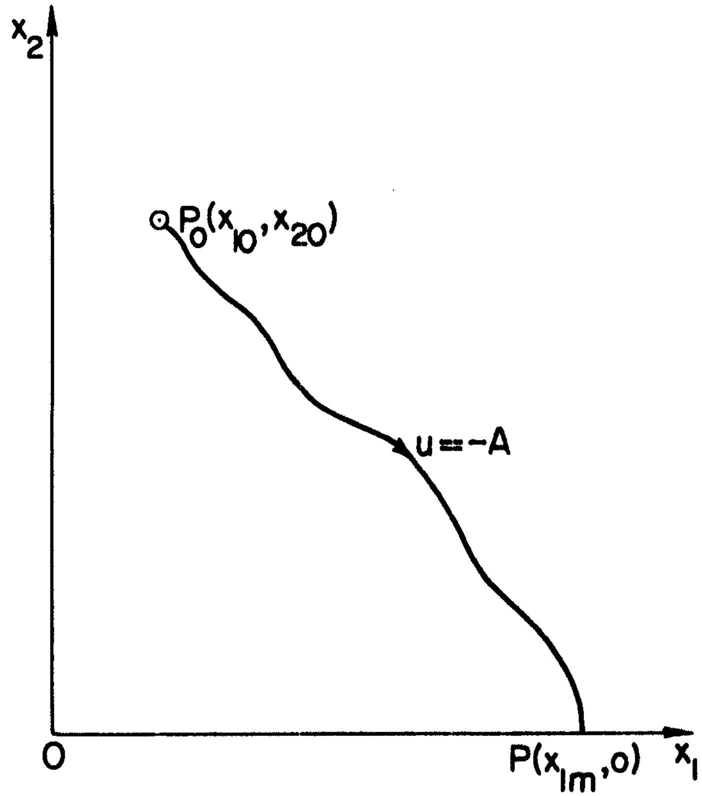

Fig. 2-4. N-arc from P_0 to $P \equiv M$ in Forward Time

$$\lambda_1(\tau) = \dot{\lambda}_2(\tau) = -\frac{1}{\left\{2[Ay_1-F(y_1)+k_1]\right\}^{1/2}}\left\{\lambda_{20}[A-f(y_{1m})]-[A-f(y_1)]\lambda_2(\tau)\right\} \tag{2-19}$$

ii) For the case represented in Fig. 2-5b,

$$\lambda_2(\tau) = \frac{\lambda_{10}y_{20}+\lambda_{20}[A-f(y_{10})]}{A-f(y_{1m})} \tag{2-20}$$

$$\lambda_1(\tau) = \dot{\lambda}_2(\tau) = -\frac{\lambda_{20}}{y_{20}}[A-f(y_{1m})] + \frac{\lambda_{10}y_{20}+\lambda_{20}[A-f(y_{10})]}{\left\{2[A-f(y_{1m})](y_{10}-y_{1m})\right\}^{1/2}} -$$

$$- \left\{\lambda_{10}y_{20}+\lambda_{20}\ [A-f(y_{10})]\right\}\ [A-f(y_{1m})]\int_{y_{1m}}^{y_{10}}\left[\frac{1}{\left\{2[A\sigma-F(\sigma)+k_1]\right\}^{3/2}} -\right.$$

$$\left. - \frac{1}{\left\{2[A-f(y_{1m})](\sigma-y_{1m})\right\}^{3/2}}\right] d\sigma \tag{2-21}$$

iii) For the case represented in Fig. 2-5c,

$$\lambda_2(\tau) = \left\{2[-Ay_1-F(y_1)+k_2]\right\}^{1/2}\left\{\frac{\lambda_{10}}{A+f(y_{1m})} + \frac{\lambda_{20}}{\left\{2[A+f(y_{1m})](y_{1m}-y_1)\right\}^{1/2}} -\right.$$

$$- \lambda_{20}[A+f(y_{1m})]\int_{y_1}^{y_{1m}}\left[\frac{1}{\left\{2[-A\sigma-F(\sigma)+k_2]\right\}^{3/2}} -\right.$$

$$\left.\left. - \frac{1}{\left\{2[A+f(y_{1m})](y_{1m}-\sigma)\right\}^{3/2}}\right] d\sigma\right\} \tag{2-22}$$

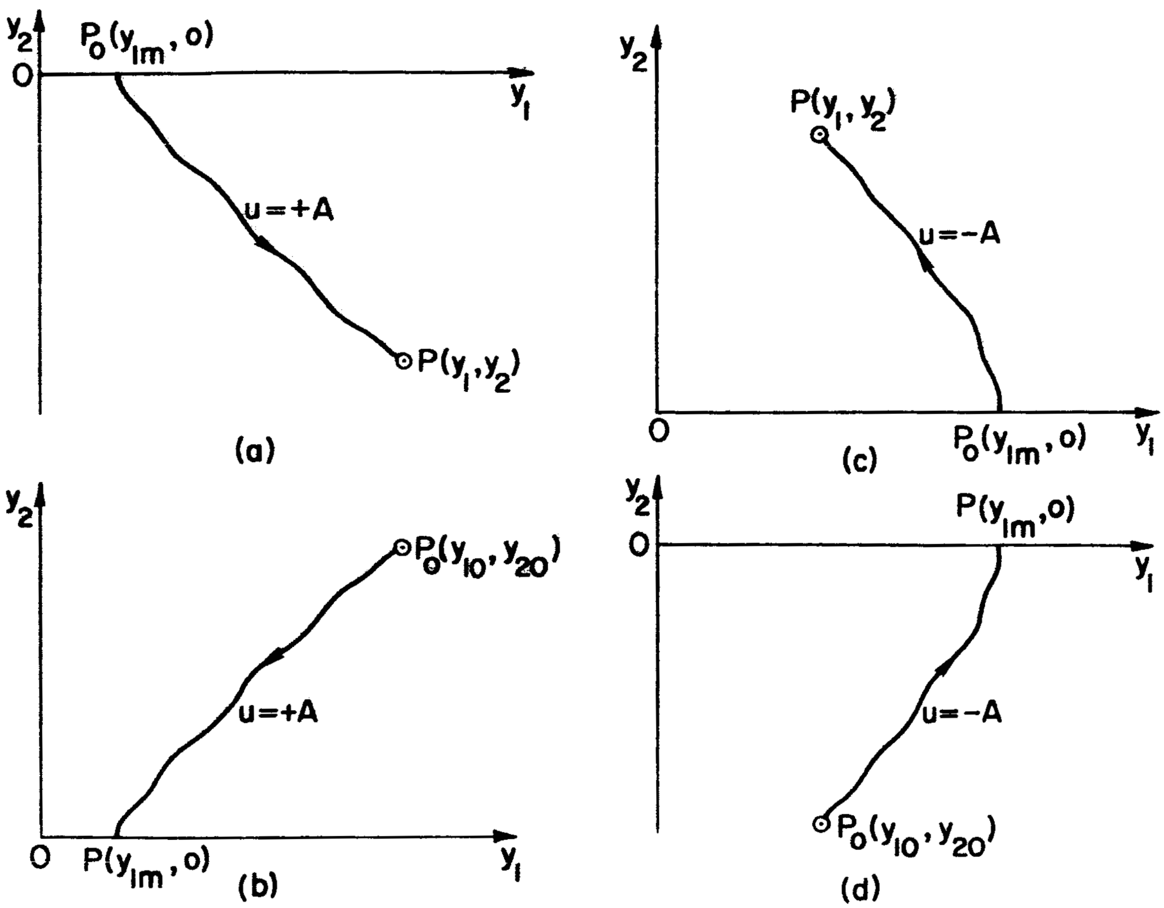

Fig. 2-5. P- and N-arcs from P_0 to P in Backwards Time

$$\lambda_1(\tau) = \dot{\lambda}_2(\tau) = \frac{1}{\left\{2[-Ay_1-F(y_1)+k_2]\right\}^{1/2}} \left\{-\lambda_{20}[A+f(y_{1m})]+[A+f(y_1)]\lambda_2(\tau)\right\} \tag{2-23}$$

iv) For the case represented in Fig. 2-5d,

$$\lambda_2(\tau) = \frac{-\lambda_{10}y_{20}+\lambda_{20}[A+f(y_{10})]}{A+f(y_{1m})} \tag{2-24}$$

$$\lambda_1(\tau) = \dot{\lambda}_2(\tau) = \frac{\lambda_{20}}{y_{20}}[A+f(y_{1m})] + \frac{-\lambda_{10}y_{20}+\lambda_{20}[A+f(y_{10})]}{\left\{2[A+f(y_{1m})](y_{1m}-y_{10})\right\}^{1/2}} +$$

$$+ \left\{-\lambda_{10}y_{20}+\lambda_{20}[A+f(y_{10})]\right\} [A+f(y_{1m})]\int_{y_{10}}^{y_{1m}} \left[- \frac{1}{\left\{2[-A\sigma-F(\sigma)+k_2]\right\}^{3/2}} + \right.$$

$$\left. + \frac{1}{\left\{2[A+f(y_{1m})](y_{1m}-\sigma)\right\}^{3/2}}\right] d\sigma \tag{2-25}$$

<u>Typical Trajectory in Backwards Time</u> - Condition (1-20) in backwards time notation becomes

$$u(\tau) = A \operatorname{sgn} \lambda_2(\tau) \tag{2-26}$$

Since a full understanding of the behavior of the trajectories that satisfy condition (2-26) is necessary in order to get the switching and indifference curves, we are going to study in full detail a typical trajectory satisfying condition (2-26).

The study will be carried over until the second switching, for reasons which will become clearer in the next chapter. A typical trajectory is sketched in Fig. 2-6, where, without loss of generality, we have assumed an initial value of $u = +A$ for the control function. If instead of an initial value of $u = +A$, we take $u = -A$, the pattern of reasoning and the results obtained will be of the same nature, provided we take the corresponding equations.

<u>Arc OS</u> - Since we have assumed an initial value of $u = +A$, it must be $\lambda_{20} > 0$. On the other hand, since the equation (2-1) is homogeneous and we are only interested in finding the zeros of $\lambda_2(\tau)$, the magnitude of λ_{20} is irrelevant. Then, for simplicity in the equations, we take for λ_{20} the value

$$\lambda_{20} = \frac{1}{A-f(0)} \tag{2-27}$$

Also

$$y_{1m} = 0 \quad \text{and} \quad k_1 = 0 \tag{2-28}$$

Substituting (2-27) and (2-28) into (2-18), we get

$$\lambda_2(\tau) = - \frac{\left|2[Ay_1-F(y_1)]\right|^{1/2}}{A-f(0)} \left\{ - \lambda_{10} - \frac{1}{\left|2[A-f(0)]y_1\right|^{1/2}} - [A-f(0)] \cdot \right.$$

$$\left. \cdot \int_0^{y_1} \left[- \frac{1}{\left|2[A\sigma-F(\sigma)]\right|^{3/2}} + \frac{1}{\left|2[A-f(0)]\sigma\right|^{3/2}} \right] d\sigma \right\} \tag{2-29}$$

Let $G(y_1)$ be defined by

$$\lambda_2(\tau) = - \frac{\left|2[Ay_1-F(y_1)]\right|^{1/2}}{A-f(0)} \; G(y_1)$$

It is obvious that $\lambda_2 = 0$ if and only if $G(y_1) = 0$. But $G(y_1)$ is a function of y_1 such that

i) $\lim\limits_{y_1 \to 0} G(y_1) = - \infty$

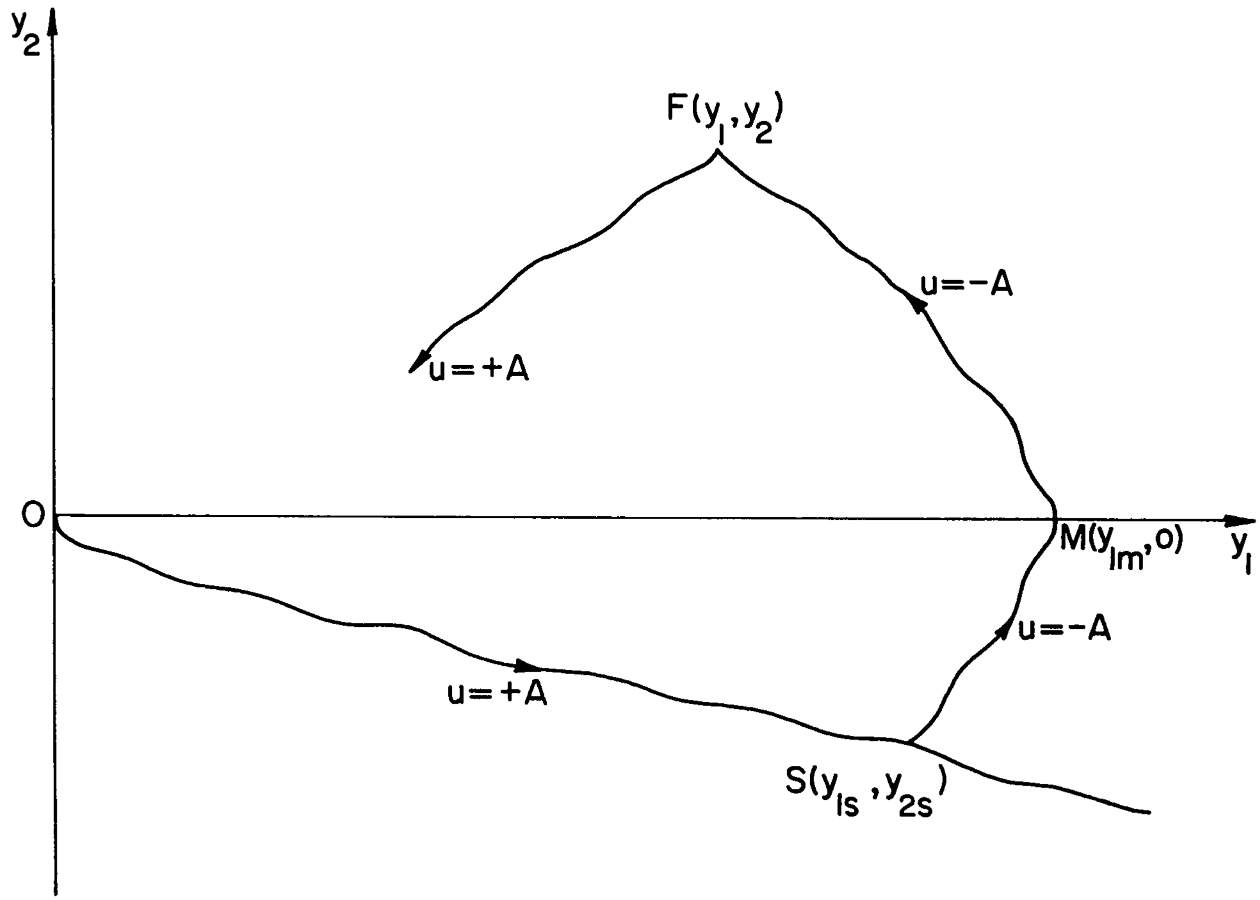

Fig. 2-6. Typical Trajectory in Backwards Time with only Two Switchings

ii) $\frac{dG}{dy_1} = \frac{A-f(0)}{\left\{2[Ay_1-F(y_1)]\right\}^{3/2}} > 0$, i.e., $G(y_1)$ is a positive non-increasing function of y_1

iii) $$\lim_{y_1\to\infty} G(y_1) = - \lambda_{10} - [A-f(0)] \int_0^\infty \left[- \frac{1}{\left\{2[A\sigma-F(\sigma)]\right\}^{3/2}} + \frac{1}{\left\{2[A-f(0)]\sigma\right\}^{3/2}} \right] d\sigma$$

So, $\lim_{y_1\to\infty} G(y_1)$ can be made positive with a suitable choice of λ_{10} . Then, for every λ_{10} there exists a $y_{1s} \in [0,\infty]$ such that $G(y_{1s}) = 0$, which makes $\lambda_{2s} = 0$.

This point y_{1s} , which depends only on the value of λ_{10} , on the zero trajectory, gives us the point at which the first switching occurs, and is determined by the equation

$$G(y_{1s}) = 0 \tag{2-30}$$

Substituting the value of y_{1s} given by (2-30) and $\lambda_{2s} = 0$ into equation (2-19), we get

$$\lambda_{1s} = - \frac{1}{\left\{2[Ay_1-F(y_1)]\right\}^{1/2}} = \frac{1}{y_{2s}} \tag{2-31}$$

<u>Arc SM</u> - During this interval, $\lambda_2(\tau)$ is given by

$$\lambda_2(\tau) = - \left\{2[-Ay_1-F(y_1)+k_2]\right\}^{1/2} \left[\frac{\lambda_{2s}}{y_{2s}} + \left\{ \lambda_{1s}y_{2s} - \lambda_{2s} [A+f(y_{1s})] \right\} \right. \cdot$$

(Continued)

$$\cdot \int_{y_{1s}}^{y_1} \frac{d\sigma}{\left\{2[-A\sigma-F(\sigma)+k_2]\right\}^{3/2}}\Bigg] \qquad (2\text{-}32)$$

Substituting $\lambda_{2s} = 0$ and (2-31) into (2-32), we get

$$\lambda_2(\tau) = -\left\{2[-Ay_1-F(y_1)+k_2]\right\}^{1/2} \int_{y_{1s}}^{y_1} \frac{d\sigma}{\left\{2[-A\sigma-F(\sigma)+k_2]\right\}^{3/2}} \qquad (2\text{-}33)$$

From equation (2-33) we see that $\lambda_2(\tau)$ keeps a negative sign during the whole interval SM , that is, no switching can occur below the x_1-axis. The crossing M with the x_1-axis is given by

$$Ay_{1m} + F(y_{1m}) = \frac{y_{2s}^2}{2} + F(y_{1s}) + Ay_{1s} = 2Ay_{1s} \qquad (2\text{-}34)$$

From equations(2-24) and (2-25) we get

$$\lambda_{2m} = -\frac{1}{A+f(y_{1m})} \qquad (2\text{-}35)$$

$$\lambda_{1m} = -\frac{1}{\left\{2[A+f(y_{1m})](y_{1m}-y_{1s})\right\}^{1/2}} + [A+f(y_{1m})] \cdot$$

$$\cdot \int_{y_{1s}}^{y_{1m}} \left[\frac{1}{\left\{2[-A\sigma-F(\sigma)+k_2]\right\}^{3/2}} - \frac{1}{\left\{2[A+f(y_{1m})](y_{1m}-\sigma)\right\}^{3/2}}\right] \qquad (2\text{-}36)$$

equations (2-35) and (2-36) give λ_{2m} and λ_{1m} respectively.

Arc MF - Substituting (2-35) and (2-36) into (2-22), we get

$$\lambda_2(\tau) = \left\{2[-Ay_1-F(y_1)+k_2]\right\}^{1/2} \left\{- \frac{1}{\sqrt{2}[A+f(y_{1m})]^{3/2}} \left[\frac{1}{(y_{1m}-y_{1s})^{1/2}} + \right.\right.$$

$$\left. + \frac{1}{(y_{1m}-y_1)^{1/2}}\right] + \int_{y_{1s}}^{y_{1m}} \left[\frac{1}{\left\{2[-A\sigma-F(\sigma)+k_2]\right\}^{3/2}} - \right.$$

$$\left. - \frac{1}{\left\{2[A+f(y_{1m})](y_{1m}-\sigma)\right\}^{3/2}}\right] d\sigma + \int_{y_1}^{y_{1m}} \left[\frac{1}{\left\{2[-A\sigma-F(\sigma)+k_2]\right\}^{3/2}} - \right.$$

$$\left.\left. - \frac{1}{\left\{2[A+f(y_{1m})](y_{1m}-\sigma)\right\}^{3/2}}\right] d\sigma \right\} =$$

$$= \left\{2[-Ay_1-F(y_1)+k_2]\right\}^{1/2} \cdot G_R(y_1,y_{1m}) \qquad (2\text{-}37)$$

Substituting (2-35) and (2-36) into (2-23), we get

$$\lambda_1(\tau) = \frac{1}{\left\{2[-Ay_1-F(y_1)+k_2]\right\}^{1/2}} \left\{1 + [A+f(y_1)]\lambda_2(\tau)\right\} \qquad (2\text{-}38)$$

Equations (2-37) and (2-38) give the values of the adjoint variables at the point F of Fig. 2-6.

B. POSSIBLE SWITCHING CURVES

In the last section, we studied the behavior of a typical trajectory in backwards time, and obtained the values of the adjoint variables $\lambda_2(\tau)$ and $\lambda_1(\tau)$ at the point F of Fig. 2-6, as given by equations (2-37) and (2-38). These two equations show that both adjoint variables $\lambda_2(\tau)$ and $\lambda_1(\tau)$ are functions of the two variables y_{1m} and y_1; however, if we consider y_{1m} as fixed, they are only functions of y_1: that is, we have the values of the adjoint variables along a particular trajectory.

The study of equation (2-37), for fixed y_{1m} and variable y_1, and a complete understanding of its behavior, is very important, since it will tell us if $\lambda_2(\tau)$ keeps constant sign or changes sign along a trajectory, which in turn correspond to the possibility of either not having or having a switching. The loci of all the points for which $\lambda_2(\tau)$ is zero, obtained by varying y_{1m}, are possible candidates for switching curves.

In order for the reader to understand that the loci of the points for which $\lambda_2(\tau)$ is zero are only candidates for switching curves and not necessarily true switching curves, we can anticipate that, as shown in the next Chapter, the zero trajectories are optimal trajectories, which justifies the study of the trajectories in backwards time; also, the existence of the indifference curves will be shown, which will enable us to state that only a part of the loci of points for which $\lambda_2(\tau)$ is zero do actually belong to the switching curves.

However, for the proof of the existence of the indifference curves, we need to know how the possible switching curves look. Therefore, it is clear that our next task is to find the possible switching curves, that is, the zeros of equation (2-37).

Let $G_R(y_1,y_{1m})$ be the function of two variables defined in (2-37). Since for every y_{1m}, the expression $\left|2[-Ay_1-F(y_1)+k_2]\right|^{1/2}$ is positive for any $y_1 \neq y_{1m}$, the zeros of λ_2 are the same as the zeros of $G_R(y_1,y_{1m})$. So, the question that we are going to investigate now is the following. Given a y_{1m} such that $y_{1m}\epsilon[0,+\infty)$, is there a value of $y_1 < y_{1m}$ for which $G_R(y_1,y_{1m}) = 0$?

Notice that if in the study of a typical trajectory in backwards time, we had assumed an initial value of $u = -A$, instead of $u = +A$, we should now have to consider values of y_{1m} such that $y_{1m}\epsilon(-\infty,0]$; however, we do not consider these cases, because the pattern of reasoning would be similar, provided we take the corresponding set of equations.

In Fig. 2-7, the definitions used in the rest of the Chapter are illustrated.

Definition 2-1 - Let α_R^i and β_R^i be the points on the interval $[2i\theta, 2(i+1)\theta)$ defined in the following way:

$$f(\alpha_R^i) = \operatorname*{Max}_{y\epsilon[2i\theta,2(i+1)\theta)} f(y)$$

$$f(\beta_R^i) = \operatorname*{Min}_{y\epsilon[2i\theta,2(i+1)\theta)} f(y)$$

Lemma 2-1 - For fixed y_{1m}, $G_R(y_1,y_{1m})$ as a function of y_1 satisfies the following three properties:

i) $G_R(y_1,y_{1m})$ is always non-increasing

ii) $G_R(y_{1m},y_{1m}) = -\infty$

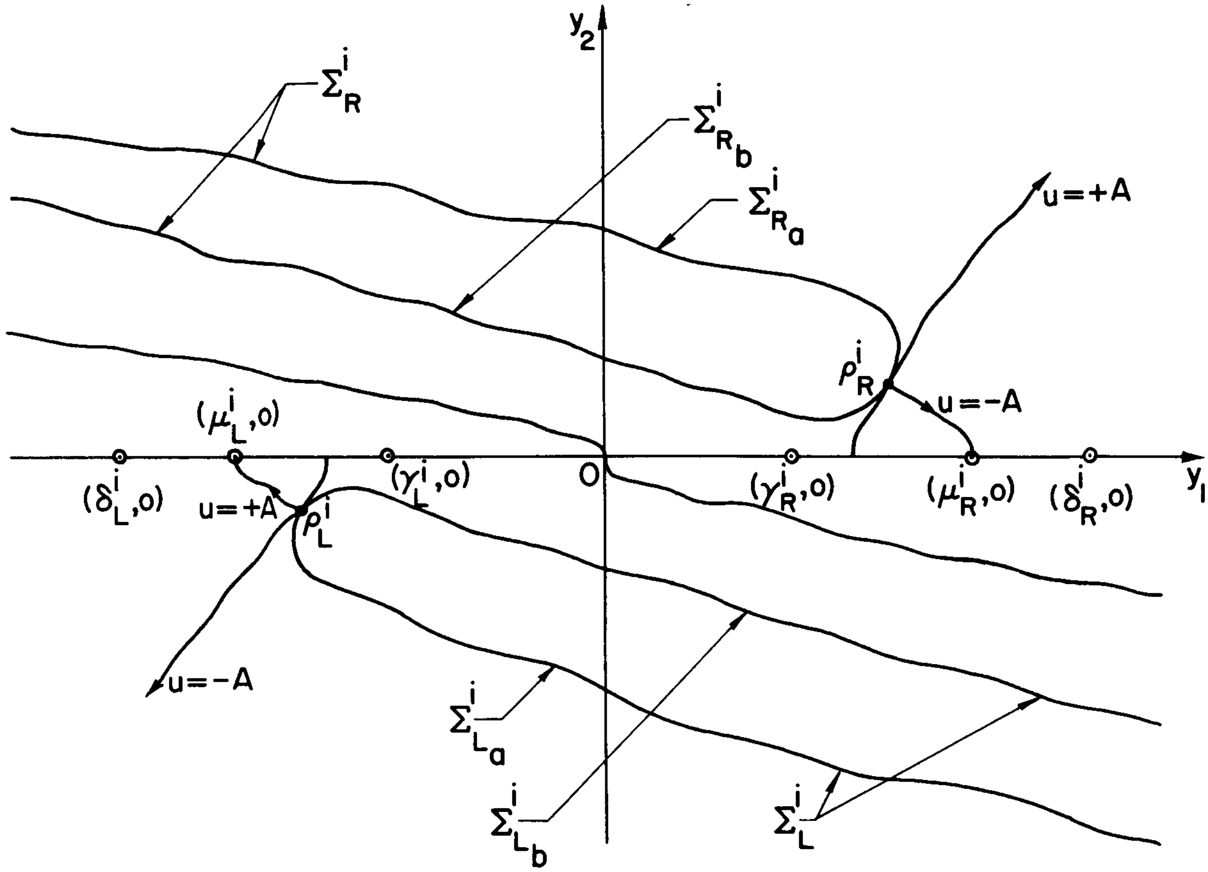

Fig. 2-7. Illustration of Definitions Used in Part B of Chapter II

iii) $-\infty < \lim_{y_1 \to -\infty} G_R(y_1, y_{1m}) < \infty$

Proof - i) For fixed y_{1m}, the derivative of $G_R(y_1, y_{1m})$ is

$$\frac{\partial G_R}{\partial y_1} = - \frac{1}{\left|2[-A\sigma - F(\sigma) + k_2]\right|^{3/2}} < 0 \quad \text{for any} \quad y_1$$

which means that $G_R(y_1, y_{1m})$ is always non-increasing

ii) By mere substitution, we get $G_R(y_{1m}, y_{1m}) = -\infty$

iii) Let β_R^m be the largest β_R^i such that $\beta_R^m < y_{1m}$. Then

$$-A\sigma - F(\sigma) + k_2 = \int_{\sigma}^{y_{1m}} [A + f(\gamma)]d\gamma > \int_{\sigma}^{y_{1m}} [A + f(\beta_R^m)]d\gamma = [A + f(\beta_R^m)](y_{1m} - \sigma)$$

and

$$\frac{1}{\left|2[-A\sigma - F(\sigma) + k_2]\right|^{3/2}} < \frac{1}{\left|2[A + f(\beta_R^m)](y_{1m} - \sigma)\right|^{3/2}} \tag{2-39}$$

for any $\sigma < y_{1m}$

Then

$$\int_{-\infty}^{y_{1m}} \left[\frac{1}{\left|2[-A\sigma - F(\sigma) + k_2]\right|^{3/2}} - \frac{1}{\left|2[A + f(y_{1m})](y_{1m} - \sigma)\right|^{3/2}} \right] d\sigma <$$

$$< \int_{-\infty}^{y_{1m}} \left[\frac{1}{\left|2[A + f(\beta_R^m)](y_{1m} - \sigma)\right|^{3/2}} - \frac{1}{\left|2[A + f(y_{1m})](y_{1m} - \sigma)\right|^{3/2}} \right] d\sigma =$$

$$= \left[\frac{1}{\left|2[A + f(\beta_R^m)]\right|^{3/2}} - \frac{1}{\left|2[A + f(y_{1m})]\right|^{3/2}} \right] \int_{-\infty}^{y_{1m}} \frac{d\sigma}{(y_{1m} - \sigma)^{3/2}} = \infty \tag{2-40}$$

Let α_R^m be the largest α_R^i such that $\alpha_R^m < y_{1m}$. Then

$$-A\sigma - F(\sigma) + k_2 = \int_{\sigma}^{y_{1m}} [A+f(\gamma)]d\gamma < [A+f(\alpha_R^m)]\,(y_{1m}-\sigma)$$

and

$$\frac{1}{\left\{2[-A\sigma-F(\sigma)+k_2]\right\}^{3/2}} > \frac{1}{\left\{2[A+f(\alpha_R^m)](y_{1m}-\sigma)\right\}^{3/2}} \quad \text{for any } \sigma < y_{1m} \tag{2-41}$$

Then

$$\int_{-\infty}^{y_{1m}} \left[\frac{1}{\left\{2[-A\sigma-F(\sigma)+k_2]\right\}^{3/2}} - \frac{1}{\left\{2[A+f(y_{1m})](y_{1m}-\sigma)\right\}^{3/2}}\right] d\sigma >$$

$$> \left[\frac{1}{\left\{2[A+f(\alpha_R^m)]\right\}^{3/2}} - \frac{1}{\left\{2[A+f(y_{1m})]\right\}^{3/2}}\right] \int_{-\infty}^{y_{1m}} \frac{d\sigma}{(y_{1m}-\sigma)^{3/2}} = -\infty \tag{2-42}$$

Taking limits on $G_R(y_1,y_{1m})$ and taking account of (2-40) and (2-42), we get

$$-\infty < \lim_{y_1\to-\infty} G_R(y_1,y_{1m}) < \infty$$

<u>Lemma 2-2</u> - If $y_{1m} = \beta_R^i$, then $G_R(y_1,\beta_R^i)$ is always negative.

<u>Proof</u> - For $y_{1m} = \beta_R^i$, (2-39) becomes

$$\frac{1}{\left\{2[-A(\sigma-\beta_R^i)-F(\sigma)+F(\beta_R^i)]\right\}^{3/2}} < \frac{1}{\left\{2[A+f(\beta_R^i)](\beta_R^i-\sigma)\right\}^{3/2}} \tag{2-43}$$

for any $\sigma<\beta_R^i$

Then, each integral in the expression for G_R is negative, since each integrand is. Hence, since each term on G_R is negative for any y_1, we conclude that $G_R(y_1,\beta_R^i)$ is always negative.

Corollary - If $y_{1m} = \beta_R^i$, the adjoint variable λ_2 keeps a constant negative sign along the trajectory through β_R^i, i.e., there is no switching.

Lemma 2-3 - If $y_{1m} = \alpha_R^i$, there exists an α_R^j such that $\lim_{y_1\to-\infty} G_R(y_1,\alpha_R^i)>0$ for any $\alpha_R^i \geq \alpha_R^j$.

Proof - Taking limits of $G_R(y_1,\alpha_R^i)$, we get

$$\lim_{y_1\to-\infty} G_R(y_1,\alpha_R^i) = -\frac{1}{[2(\alpha_R^i-y_{1s})]^{1/2}[A+f(\alpha_R^i)]^{3/2}} + \int_{y_{1s}}^{\alpha_R^i}\left[\frac{1}{\left\{2[-A(\sigma-\alpha_R^i)-F(\sigma)+F(\alpha_R^i)]\right\}^{3/2}} - \frac{1}{\left\{2[A+f(\alpha_R^i)](\alpha_R^i-\sigma)\right\}^{3/2}}\right]d\sigma + \int_{-\infty}^{\alpha_R^i}\left[\frac{1}{\left\{2[-A(\sigma-\alpha_R^i)-F(\sigma)+F(\alpha_R^i)]\right\}^{3/2}} - \frac{1}{\left\{2[A+f(\alpha_R^i)](\alpha_R^i-\sigma)\right\}^{3/2}}\right]d\sigma \qquad (2\text{-}44)$$

For $y_{1m} = \alpha_R^i$, (2-34) becomes

$$y_{1s} = \frac{1}{2A}\,[A\alpha_R^i + F(\alpha_R^i)]$$

then

$$\alpha_R^i - y_{1s} = \frac{1}{2A}\,[A\alpha_R^i - F(\alpha_R^i)]$$

$$\frac{d}{d\alpha_R^i}(\alpha_R^i - y_{1s}) = \frac{1}{2A}\,[A - f(\alpha_R^i)] > 0 \quad \text{for any} \quad \alpha_R^i$$

that is, $\alpha_R^i - y_{1s}$ is an increasing function of α_R^i. So, the first term of (2-44) is a negative decreasing function of α_R^i, which tends to zero as $\alpha_R^i \to -\infty$.

The second term is a positive increasing function of α_R^i, which tends to a positive finite limit as $\alpha_R^i \to -\infty$. It is positive, since, from (2-41), the integrand is positive; it is increasing, since the interval of integration increases with α_R^i; and finally, it is finite as $\alpha_R^i \to -\infty$, because of part iii) of Lemma 2-1.

By the same token, we can conclude that the third term of (2-44) is a positive finite constant. Hence, it is obvious that there exists an α_R^j such that

$$\lim_{y_1 \to -\infty} G_R(y_1, \alpha_R^i) > 0 \quad \text{for any} \quad \alpha_R^i \geq \alpha_R^j$$

Corollary - If $y_{1m} = \alpha_R^i$, $i \geq j$, the adjoint variable λ_2 becomes zero at a point on the trajectory through α_R^i; i.e., we have a switching.

Lemma 2-4 - In any interval $[2i\theta\,,\, 2(i+1)\theta]$ such that $\alpha_R^i \geq \alpha_R^j$, there exist two points γ_R^i and δ_R^i such that

$$\lim_{y_1 \to -\infty} G_R(y_1, y_{1m}) = 0 \qquad \text{if} \qquad y_{1m} = \gamma_R^i\,,\ \delta_R^i$$

Moreover,

$$\beta_R^{i-1} < \gamma_R^i < \alpha_R^i < \delta_R^i < \beta_R^i$$

Proof - Taking limits on $G_R(y_1,y_{1m})$ we get

$$\lim_{y_1\to-\infty} G_R(y_1,y_{1m}) = - \frac{1}{[2(y_{1m}-y_{1s})]^{1/2}[A+f(y_{1m})]^{3/2}} +$$

$$+\int_{y_{1s}}^{y_{1m}} \left[\frac{1}{\left\{2[-A\sigma-F(\sigma)+k_2]\right\}^{3/2}} - \right.$$

$$\left. - \frac{1}{\left\{2[A+f(y_{1m})](y_{1m}-\sigma)\right\}^{3/2}}\right] d\sigma +$$

$$+\int_{-\infty}^{y_{1m}} \left[\frac{1}{\left\{2[-A\sigma-F(\sigma)+k_2]\right\}^{3/2}} - \right.$$

$$\left. - \frac{1}{\left\{2[A+f(y_{1m})](y_{1m}-\sigma)\right\}^{3/2}}\right] d\sigma \qquad (2\text{-}45)$$

Obviously, (2-45) is a continuous function of y_{1m}, since every term is a continuous function. Moreover, from Lemma 2-2, (2-45) is negative at $y_{1m} = \beta_R^{i-1}$, β_R^i; and, from Lemma 2-3 (2-45) is positive at $y_{1m} = \alpha_R^i$. Hence, from the definition of continuity, we conclude that

$$\beta_R^{i-1} < \gamma_R^i < \alpha_R^i < \delta_R^i < \beta_R^i$$

and

$$\lim_{y_1\to-\infty} G_R(y_1,y_{1m}) = 0 \quad \text{if} \quad y_{1m} = \gamma_R^i \text{ , } \delta_R^i$$

Corollary - If $y_{1m} = \gamma_R^i$ or δ_R^i, the adjoint variable λ_2 becomes zero at the point approached by the trajectories through γ_R^i, δ_R^i when

y_1 approaches $-\infty$; i.e., the switching occurs at this point . This follows from Lemmas 2-1 and 2-4.

Lemma 2-5 - The locus in the phase plane of all the points y_1 for which $G_R(y_1,y_{1m}) = 0$ is composed of a series of curves, each one being a continuous curve with two points at infinity, corresponding to values of y_{1m} such that $\gamma_R^i \le y_{1m} \le \delta_R^i$ for $i \ge j$, and with a continuous variation.

Proof - Follows from Lemmas 2-1, 2-2, 2-3 and 2-4, and the fact that on every interval $[\gamma_R^i , \delta_R^i]$ every term of $G_R(y_1,y_{1m})$ has a continuous variation.

Definition 2-2 - Let Σ_R^i be the set of all states (y_1,y_2) such that $G_R(y_1,y_{1m}) = 0$ for every y_{1m} in the interval $[\gamma_R^i , \delta_R^i]$. In a more precise way

$$\Sigma_R^i = \left\{(y_1,y_2) : G_R(y_1,y_{1m}) = 0 \quad \text{for} \quad y_{1m} \in [\gamma_R^i , \delta_R^i]\right\} \text{ for } i \ge j \tag{2-46}$$

Also, let ρ_R^i be the point on Σ_R^i such that the P-curve through ρ_R^i is tangent to Σ_R^i ; and let $(\mu_R^i,0)$ be the crossing point of the N-curve through ρ_R with the x_1-axis. Then

$$\Sigma_{R_a}^i = \left\{(y_1,y_2) : (y_1,y_2) \in \Sigma_R^i \quad \text{and} \quad y_{1m} \in [\mu_R^i,\delta_R^i]\right\} \text{for } i \ge j \tag{2-47}$$

$$\Sigma_{R_b}^i = \left\{(y_1,y_2) : (y_1,y_2) \in \Sigma_R^i \quad \text{and} \quad y_{1m} \in [\gamma_R^i,\mu_R^i)\right\} \text{ for } i \ge j \tag{2-48}$$

Definition 2-3 - If $y_{1m} \in (-\infty , 0]$, we have the following analogous definitions:

i) $$G_L(y_1,y_{1m}) = \frac{1}{\sqrt{2}\,[A-f(y_{1m})]^{3/2}}\left[\frac{1}{(y_{1s}-y_{1m})^{1/2}} + \frac{1}{(y_1-y_{1m})^{1/2}}\right] -$$
$$- \int_{y_{1m}}^{y_{1s}} \left[\frac{1}{\{2[A\sigma-F(\sigma)+k_1]\}^{3/2}} - \frac{1}{\{2[A-f(y_{1m})](\sigma-y_{1m})\}^{3/2}}\right]d\sigma -$$
$$- \int_{y_{1m}}^{y_1} \left[\frac{1}{\{2[A\sigma-F(\sigma)+k_1]\}^{3/2}} - \frac{1}{\{2[A-f(y_{1m})](\sigma-y_{1m})]\}^{3/2}}\right]d\sigma \tag{2-49}$$

ii) γ_L^i and δ_L^i are the points on $[-2(i+1)\theta \; , \; -2i\theta], i \geq r$, such that

$$\lim_{y_1 \to \infty} G_L(y_1,y_{1m}) = 0 \quad \text{if} \quad y_{1m} = \gamma_L^i \; , \; \delta_L^i$$

iii) $$\Sigma_L^i = \left\{(y_1,y_2) \; : \; G_L(y_1,y_{1m}) = 0 \quad \text{for} \quad y_{1m} \in [\delta_L^i,\gamma_L^i]\right\} \quad \text{for} \quad i \geq r \tag{2-50}$$

iv) ρ_L^i is the point on Σ_L^i such that the N-curve through ρ_L^i is tangent to Σ_L^i . Also, $(\mu_L^i,0)$ is the crossing point of the P-curve through ρ_L^i with the x_1-axis.

v) $$\Sigma_{L_a}^i = \left\{(y_1,y_2) \; : \; (y_1,y_2) \in \Sigma_L^i \quad \text{and} \quad y_{1m} \in [\delta_L^i,\mu_L^i]\right\} \quad \text{for} \quad i \geq r \tag{2-51}$$

$$\Sigma^i_{L_b} = \left\{ (y_1, y_2) : (y_1, y_2) \in \Sigma^i_L \text{ and } y_{1m} \in (\mu^i_L, \gamma^i_L] \right\} \text{ for } i \geq r \tag{2-52}$$

Then, we have shown the existence of certain curves Σ^i_R and Σ^i_L which are possible candidates for switching curves, and our next task will be to decide which parts of them, if any, do actually belong to the switching curves. This is done in the next chapter, where we are not only able to decide the above question, but also prove the existence of other curves, called indifference curves, and finally establish the optimal control law.

CHAPTER III

OPTIMAL CONTROL LAW

Using the control law given by Pontrayagin's Maximum Principle, and working the problem in backwards time from the origin of the state plane, we have found, in Chapter II, the loci of the points for which a second switching occurs. Then, if we want these loci to be possible candidates for true switching curves, when considering a trajectory in forward time, the switching before the last must occur on some part of the above loci, and the last switching must occur on one of the two zero trajectories; this will be true only if the zero trajectories are at the same time optimal trajectories. Therefore, the first part of this Chapter will be concerned with the question of proving that the zero trajectories are also optimal trajectories; this proof is given in Theorem 3-1, using the results obtained in Lemmas 3-2 and 3-3. It is worthwhile to point out that the proof of Lemma 3-2 put some further restriction on the bound A of the control $u(t)$, and that the proof of Lemma 3-3 restricts the class of periodic functions so far considered; the nature of the above restrictions on both A and $f(x)$ are indicated before the Lemmas 3-2 and 3-3 respectively, in order for the reader to be able to locate the moment at which they are used for the first time.

The second part of this Chapter is concerned with the question of proving the existence of the indifference curves, and the question of deciding which parts of the possible switching curves, if any, are true switching curves, and which parts can be actually substituted by the indifference curves. By using the above results, the optimal control law is found as given by Theorem 3-2.

Definition 3-1 - A solution of (1-5) satisfying the boundary conditions (1-9), $u(t)$ being given by the condition (1-20), will be called a solution curve. A solution curve consists of a countable (probably finite or vacuous) well-ordered sequence of alternating P- arcs and N-arcs such that

i) The initial point of the first arc is $P_0(x_{10}, x_{20})$.

ii) The terminal point of each arc is the initial point of the next.

iii) The terminal point of the last arc is the origin.

iv) $u = +A$ on the P-arcs and $u = -A$ on the N-arcs.

If Δ is the solution curve starting at P_0 , and P' is any point on Δ , then the solution curve starting at P' is that part of Δ which follows P' .

Definition 3-2 - A point on a solution curve which is both the terminal point of a P-arc and the initial point of an N-arc will be called a PN-corner. Likewise, an NP-corner is a point on a solution curve which is both the terminal point of an N-arc and the initial point of a P-arc.

Definition 3-3 - A path from the initial point P_0 to the origin is a countable, well-ordered sequence of alternating P - and N-arcs such that

i) The sum of the time length of the arcs is finite.

ii) The initial point of the first arc is P_0 .

iii) The terminal point of each arc is the initial point of the next.

iv) The terminal point of the last arc is the origin.

v) Two arcs never intersect.

vi) $u = +A$ on the P-arcs and $u = -A$ on the N-arcs.

In order to avoid a conflict between iii) and v), we assume that each arc contains its initial point but not its terminal point. A path from P_0 can therefore almost be described as a curve which could occur as that part of a solution curve from P_0 which connects P_0 with the origin; we say it can almost be described, because v) need not hold for every solution curve; however, since we are looking for solutions curves of shortest time length, there is no loss of generality if we leave self-intersecting solutions out of consideration.

Definition 3-4 - A path from P_0 whose time length is not longer than that of any other path from P_0 will be called an optimal path from P_0. Obviously, an optimal path from P_0 is the solution curve of least possible time length connecting P_0 with the origin.

Definition 3-5 - A path Δ will be called canonical if it does not contain either NP-corners above the x_1-axis or PN-corners below it. When we say that a corner lies above or below the x_1-axis, we mean that nearby parts of the arcs meeting at the corner are above or below it; the corner itself, regarded as a point, may be on the x_1-axis.

Lemma 3-1 - Given any path Δ from P_0 which is not canonical, one can find a canonical path from P_0 whose time length is less than that of Δ.

Proof - Suppose the path Δ has an NP-corner above the x_1-axis, and let P_1 be such NP-corner; also, let P_2 be the PN-corner preceding P_1 and P_3 the PN-corner following P_1, as shown in Fig. 3-1.

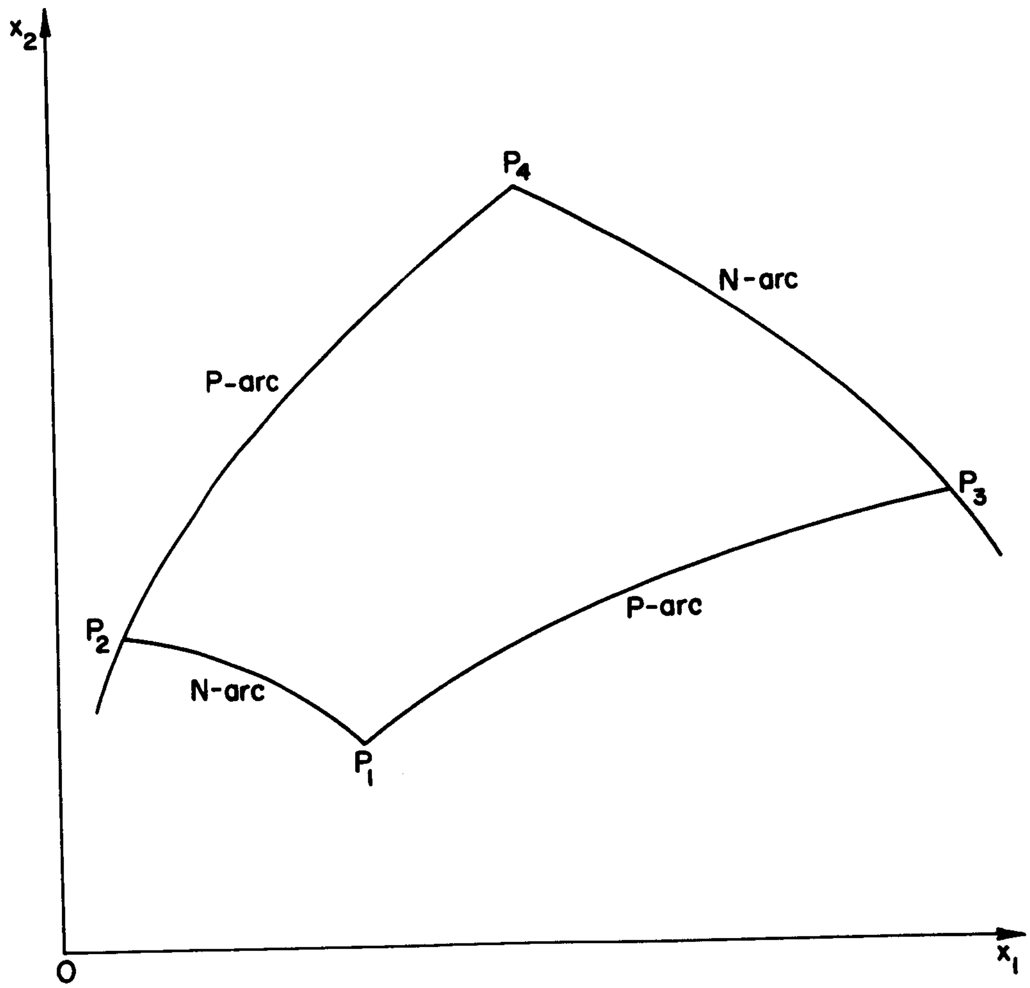

Fig. 3-1. Canonical and Noncanonical Paths

Then, let us draw the P-curve forward from P_2 and the N-curve backward from P_3, obtaining the point P_4. If we now modify the given path Δ by replacing the section $P_2P_1P_3$ by the section $P_2P_4P_3$, the NP-corner P_1 is removed, no other such corner is introduced, and the time length of the path is reduced, since

$$\tau(P_2P_4P_3) = \int_{P_2P_4P_3} \frac{d\sigma}{x_2(\sigma)} < \int_{P_2P_1P_3} \frac{d\sigma}{x_2(\sigma)} = \tau(P_2P_1P_3)$$

Applying this process to every NP-corner above the x_1-axis, and the corresponding process to every PN-corner below the x_1-axis, the result is a canonical path shorter than the given path, and the Lemma is proved.

Corollary - In seeking an optimal path from a point, it is sufficient to consider only canonical paths from that point, since it follows from Lemma 3-1 that a path which is optimal with respect to the class of all canonical paths is also optimal with respect to the class of all paths whatever. Hence, from this point on all paths considered are therefore assumed to be canonical.

Definition 3-6 - Let $\gamma_+(\gamma_-)$ be the set of all states that can be brought to the origin, in positive time, by the control $u = +A$ $(u = -A)$. In a more precise way

$$\gamma_+ = \left\{ (x_1,x_2) : x_1 > 0 \text{ and } x_2 = -\left\{2[Ax_1 - F(x_1)]\right\}^{1/2} \right\} \tag{3-1}$$

$$\gamma_- = \left\{ (x_1,x_2) : x_1 < 0 \text{ and } x_2 = \left\{2[-Ax_1 - F(x_1)]\right\}^{1/2} \right\} \tag{3-2}$$

Also, by definition

$$\Gamma = \gamma_+ \cup \gamma_-$$

Definition 3-7 - Let $\Pi_R(\Pi_L)$ be the set of all states to the right (left) of the Γ curve. In a more precise way, Π_R and Π_L are defined by

$$\Pi_R = \left\{ (x_1,x_2) \ : \ \text{if} \ (x_1^*,x_2) \in \Gamma \ , \ \text{then} \ x_1 > x_1^* \right\} \tag{3-3}$$

$$\Pi_L = \left\{ (x_1,x_2) \ : \ \text{if} \ (x_1^*,x_2) \in \Gamma \ , \ \text{then} \ x_1 < x_1^* \right\} \tag{3-4}$$

Definition 3-8 - Let

$$f(x_1) = K + f_1(x_1) \tag{3-5}$$

where $f_1(x_1)$ is a periodic function with the same period 2θ as $f(x_1)$ and such that

$$\int_{2i\theta}^{2(i+1)\theta} f_1(x_1)dx_1 = 0 \quad \text{for any } i \tag{3-6}$$

and the constant K is given by

$$K = \frac{1}{2\theta} \int_0^{2\theta} f(x_1)dx_1 \tag{3-7}$$

Let $F_1(x_1)$ be defined by

$$F_1(x_1) = \int_0^{x_1} f_1(\sigma)\, d\sigma \tag{3-8}$$

then, $F_1(x_1)$ is also a periodic function with period 2θ.

Let us now consider the question of proving that the zero trajectories γ_+ and γ_- are also optimal trajectories. Then, let Q be any point on γ_+, see Fig. 3-2; it is obvious that any canonical path from Q to the origin must have an even number of switchings. So, as a preliminary step, we are going to prove that γ_+ is optimal with respect to all canonical paths from Q to the origin that have only two switchings, and, once this is proved, it follows that γ_+ is optimal with respect to all possible paths.

Then, see Fig. 3-2, let $\Delta \equiv QRMSO$ be a general canonical path from Q to the origin that has only two switchings, and is completely determined by its crossing point with the x_1-axis, i.e., by $M(-x_{1m},0)$. Notice that, since the system is nonlinear, it is not sufficient to prove that γ_+ is optimal only with respect to canonical paths in a neighborhood of γ_+, but it has to be proved with respect to all possible canonical paths; therefore, the point M that characterizes the path Δ can be any point on the negative part of the x_1-axis.

The most direct way of proving that γ_+ is optimal would be to write an expression for the time spent along the path Δ and showing that its absolute minimum occurs when Δ coincides with γ_+; however, this turns out to be almost impossible due to the complicated algebra involved. Therefore, the procedure we have followed consists of showing that certain parts of the path QO take less time than the corresponding parts of the path Δ, so that when we add all those parts we obtain the desired result.

Now, see Fig. 3-2, let N be the point on γ_+ such that $x_{1n} = x_{1m}$

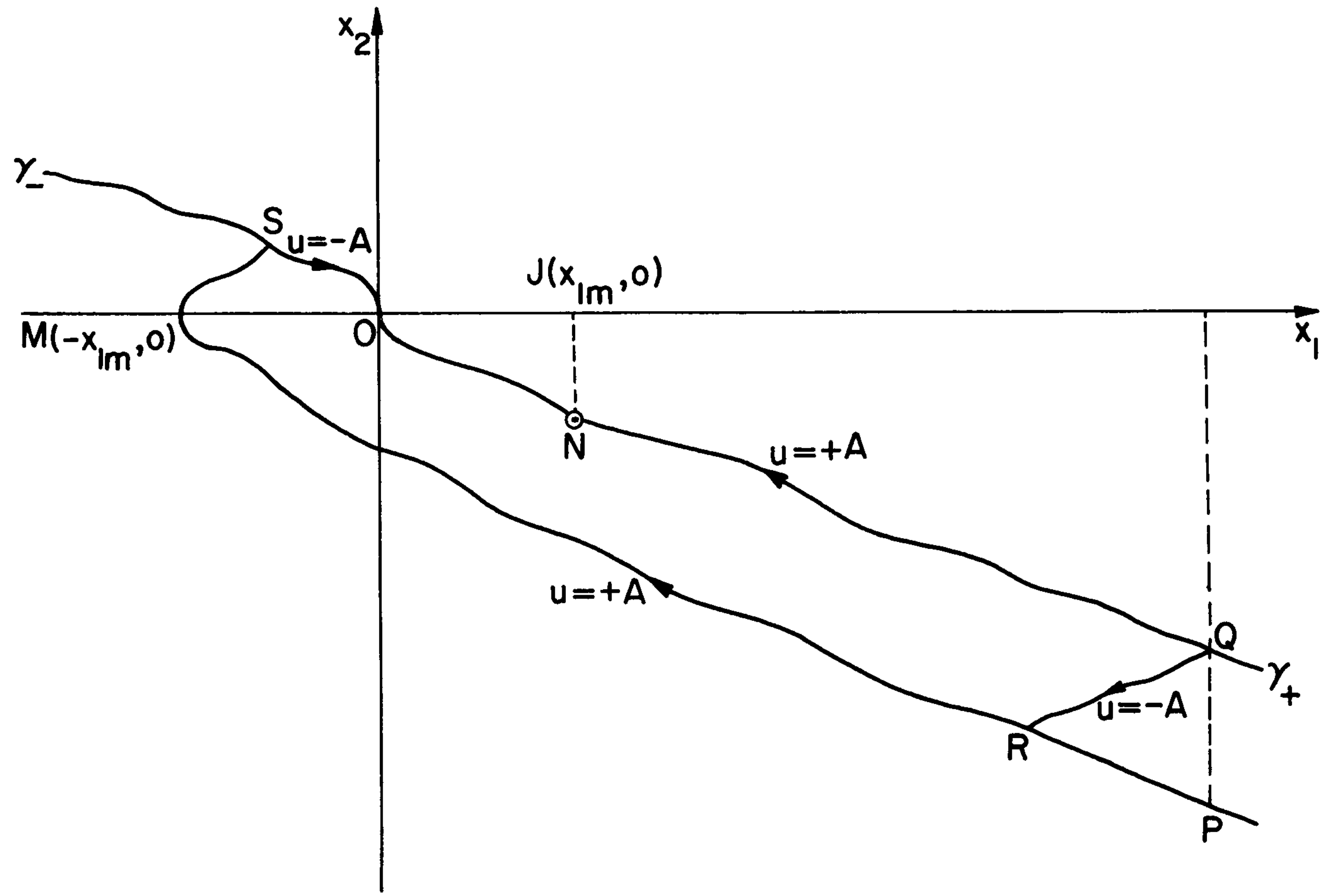

Fig. 3-2. First Illustration for the Proof that Zero Trajectories are Optimal Paths

and P be the point on the P-curve through R such that $x_{1p} = x_{1q}$. Since it is obvious that $\tau(QR) > \tau(PR)$, it will be only necessary to prove that $\tau(QO) < \tau(PRMSO)$; this is achieved by showing that between the time spent along the parts QN and NO of QO and the time spent along the corresponding parts PM and MSO of PRMSO there exists the following relations

i) $\tau(QN) < \tau(PM)$

ii) $\tau(NO) < \tau(MSO)$

So far, we have considered A and C constants such that $A \geq C \geq B$. From now on, and in order to be able to prove Lemma 3-2, we restrict ourselves to the cases in which the constant C is given by

$$C = B + 2|K| \tag{3-9}$$

Lemma 3-2 - Let $P(x_{1p}, x_{2p})$ be any point in Π_L, Q be the point of γ_+ such that $x_{1q} = x_{1p}$, M be the intersection point of the x_1-axis with the P-curve through the point P , and N be the intersection point of γ_+ and the straight line $x_1 = x_{1m}$, as indicated in Fig. 3-3. Then $\tau(PM) > \tau(QN)$.

Proof - The usual procedure to prove this Lemma would be to find quantities τ_1, τ_2, τ_3 and τ_4 such that $\tau_1 > \tau(PM) > \tau_2$ and $\tau_3 > \tau(QN) > \tau_4$, and to show that $\tau_2 > \tau_3$. Although the bounds τ_1, τ_2, τ_3 and τ_4 can be found easily, it was thus far not possible to show that $\tau_2 > \tau_3$; therefore, we are going to follow a different procedure.

Let H be the point of the γ_+ curve such that $x_1 = x_{1p} + 2x_{1m}$, and let $\varphi(x_1)$ and $\eta(x_1)$ be the zero and any other trajectory defined by the relations

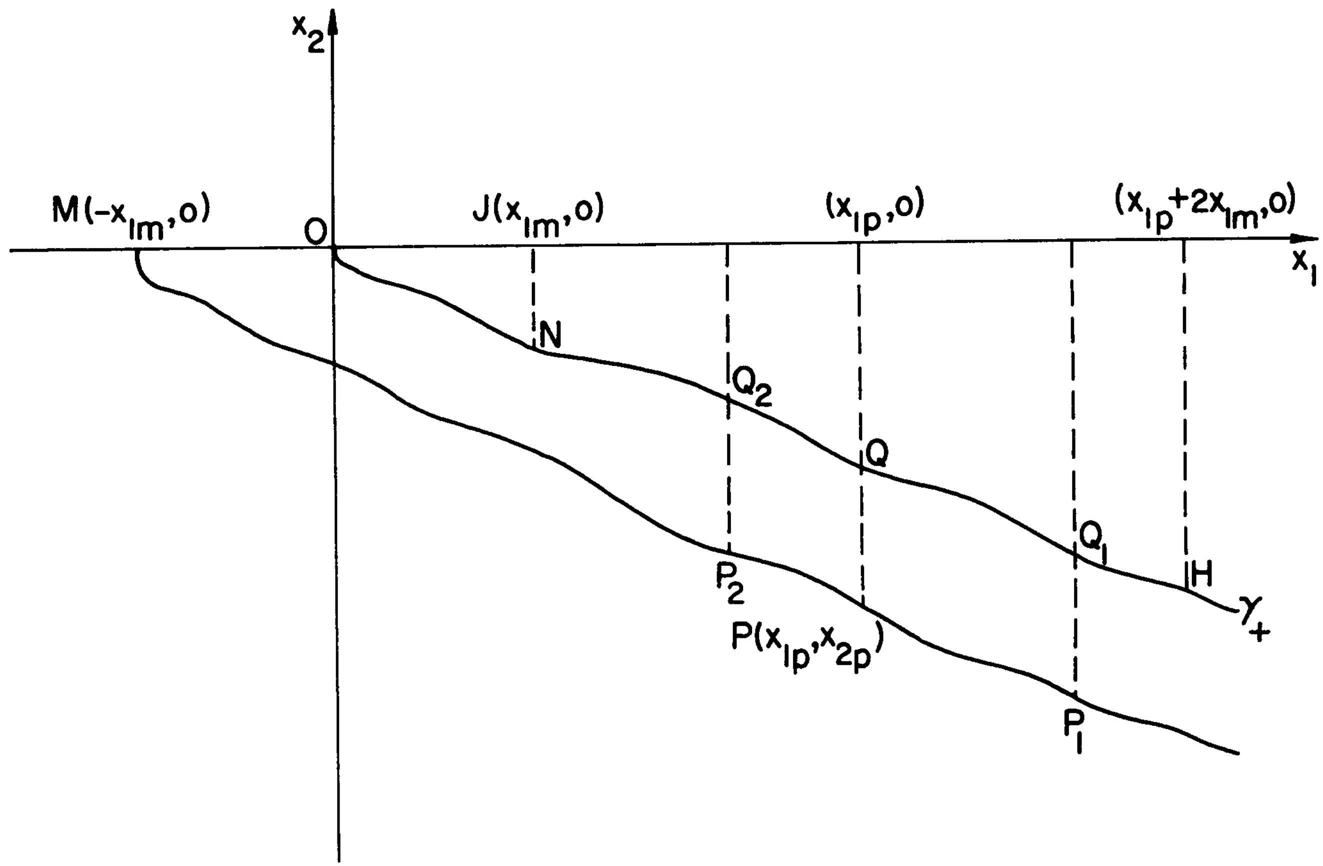

Fig. 3-3. Second Illustration for the Proof that Zero Trajectories are Optimal Paths

$$\varphi(x_1) \equiv \frac{x_2^{*2}}{2} = Ax_1 - F(x_1) = (A-K)x_1 - F_1(x_1)$$

$$\eta(x_1) \equiv \frac{x_2^2}{2} = A(x_1+x_{1m}) - F(x_1) + F(-x_{1m}) = (A-K)(x_1+x_{1m})-F_1(x_1)+F_1(-x_{1m})$$

Then

$$I_1 = \int_{x_{1m}}^{x_{1p}+2x_{1m}} \varphi(x_1)dx_1 - \int_{-x_{1m}}^{x_{1p}} \eta(x_1)dx_1 = \int_{x_{1m}}^{x_{1p}+2x_{1m}} [\varphi(x_1) -$$

$$- \eta(x_1-2x_{1m})]dx_1 = \int_{x_{1m}}^{x_{1p}+2x_{1m}} [(A-K)x_{1m} - F_1(-x_{1m}) + F_1(x_1-2x_{1m}) -$$

$$- F_1(x_1)]\, dx_1 = [(A-K)x_{1m} - F_1(-x_{1m})](x_{1p}+x_{1m}) +$$

$$+ \int_{x_{1m}}^{x_{1p}+2x_{1m}} [F_1(x_1-2x_{1m}) - F_1(x_1)]\, dx_1 \qquad (3\text{-}10)$$

Since $F_1(x_1)$ is a periodic function with period 2θ, there exist constants K_1, K_2 and K_3 such that

$$\int_{2i\theta}^{2(i+1)\theta} [F_1(x_1) - K_1]\, dx_1 = 0 \quad \text{for any} \quad i \qquad (3\text{-}11)$$

$$K_2 \geq \int_a^b [F(x_1) - K_1]\, dx_1 \geq K_3 \quad \text{for any } [a,b] \qquad (3\text{-}12)$$

Then

$$\int_{x_{1m}}^{x_{1p}+2x_{1m}} [F_1(x_1-2x_{1m}) - F_1(x_1)]\, dx_1 = \int_{x_{1m}}^{x_{1p}+2x_{1m}} [F_1(x_1-2x_{1m})-K_1]dx_1 -$$

(Continued)

$$- \int_{x_{1m}}^{x_{1p}+2x_{1m}} [F_1(x_1) - K_1]\, dx_1 \geq K_3 - K_2$$

Substituting into (3-10) we get

$$I_1 \geq [(A-K)x_{1m} - F_1(-x_{1m})](x_{1p}+x_{1m}) + K_3 - K_2$$

But

$$(A-K)x_{1m} - F_1(-x_{1m}) = \int_{-x_{1m}}^{0} [A-K+f_1(\sigma)]d\sigma \geq \int_{-x_{1m}}^{0} [B+f(\sigma)+2|K|-2K]d\sigma >$$

$$> 0 \ , \quad \text{for any} \quad x_{1m} \neq 0$$

Then, given any $x_{1m} \neq 0, I_1$ will be positive for any x_{1p} such that

$$x_{1p} > \frac{K_2 - K_3}{(A-K)x_{1m}-F_1(-x_{1m})} - x_{1m} = x_{1cr} \qquad (3\text{-}13)$$

On the other hand we have

$$I_1 = \int_{x_{1m}}^{x_{1p}+2x_{1m}} [\varphi(x_1)-\eta(x_1-2x_{1m})]dx_1 = \int_{x_{1m}}^{x_{1p}+2x_{1m}} \left\{ [\varphi(x_1)]^{1/2} + \right.$$

$$\left. + [\eta(x_1-2x_{1m})]^{1/2} \right\} \left\{ [\varphi(x_1)]^{1/2} - [\eta(x_1-2x_{1m})]^{1/2} \right\} dx_1 <$$

$$< \left\{ [\varphi(x_{1p}+2x_{1m})]^{1/2} + [\eta(x_{1p})]^{1/2} \right\} \int_{x_{1m}}^{x_{1p}+2x_{1m}} \left\{ [\varphi(x_1)]^{1/2} - \right.$$

$$\left. - [\eta(x_1-2x_{1m})]^{1/2} \right\} dx_1$$

Also

$$I_2 = \int_{x_{1m}}^{x_{1p}+2x_{1m}} \left\{ \frac{1}{[\varphi(x_1-2x_{1m})]^{1/2}} - \frac{1}{[\eta(x_1)]^{1/2}} \right\} dx_1 =$$

(Continued)

$$= \int_{x_{1m}}^{x_{1p}+2x_{1m}} \frac{[\varphi(x_1)]^{1/2} - [\eta(x_1-2x_{1m})]^{1/2}}{[\varphi(x_1-2x_{1m}) \cdot \eta(x_1)]^{1/2}} dx_1 >$$

$$> \frac{1}{[\eta(x_{1p}) \cdot \varphi(x_{1p}+2x_{1m})]^{1/2}} \int_{x_{1m}}^{x_{1p}+2x_{1m}} \left\{ [\varphi(x_1)]^{1/2} - [\eta(x_1-2x_{1m})]^{1/2} \right\} dx_1 >$$

$$> \frac{I_1}{[\eta(x_{1p}) \cdot \varphi(x_{1p}+2x_{1m})]^{1/2} \left\{ [\varphi(x_{1p}+2x_{1m})]^{1/2} + [\eta(x_{1p})]^{1/2} \right\}}$$

So, I_2 will be positive whenever (3-13) holds. But

$$\tau(PM) - \tau(HN) = \int_{-x_{1m}}^{x_{1p}} \frac{dx_1}{[\eta(x_1)]^{1/2}} - \int_{x_{1m}}^{x_{1p}+2x_{1m}} \frac{dx_1}{[\varphi(x_1)]^{1/2}} =$$

$$= \int_{x_{1m}}^{x_{1p}+2x_{1m}} \left\{ \frac{1}{[\eta(x_1-2x_{1m})]^{1/2}} - \frac{1}{[\varphi(x_1)]^{1/2}} \right\} dx_1 = I_2$$

Then, whenever (3-13) is satisfied, we have $I_2 > 0$, and

$$\tau(PM) > \tau(HN) > \tau(QN) \qquad (3\text{-}14)$$

Now, two cases must be considered, i.e., x_{1cr} being smaller or greater than x_{1m}.

If x_{1cr} is smaller than x_{1m} the inequality (3-14) holds for any x_{1p} such that $x_{1p} > x_{1m}$, and the Lemma is completely proved.

If x_{1cr} is greater than x_{1m} the inequality (3-14) holds only for values of x_{1p} greater than x_{1cr} and needs to be proved for values of

x_{1p} such that $x_{1m} < x_{1p} < x_{1cr}$. So, let P_1 be a point for which $x_{1p} > x_{1cr}$ and let P_2 be a point for which $x_{1m} < x_{1p} < x_{1cr}$; then, see Fig. 3-3.

$$\tau(P_1M) > \tau(Q_1N) \tag{3-15}$$

and it is obvious that

$$\tau(P_1P_2) < \tau(Q_1Q_2) \tag{3-16}$$

So, subtracting (3-16) from (3-15) we get

$$\tau(P_2M) > \tau(Q_2N)$$

that shows that the inequality (3-14) also holds for the point P_2, and the Lemma is proved for any $P \in \Pi_L$.

So far, we have considered the function $f(x)$ to be a general periodic function. From now on, we will consider only the following two groups of periodic functions

1) Periodic functions which are at the same time antisymmetric.

2) Periodic functions that, without being antisymmetric, satisfy Lemma 3-3. Notice that for this group of periodic functions we need to check if Lemma 3-3 is satisfied only for values of x_{1m} such that $2\theta > x_{1m} \geq 0$, since if this is true it follows from Lemma 1-2 that it will also be true for any value of x_{1m}.

<u>Lemma 3-3</u> - Let M be any point in the negative part of the x_1-axis, P be the point on γ_+ such that $x_{1p} = x_{1m}$, Δ_1 be the path from P

following the γ_+ curve into the origin, and Δ_2 be the path which is obtained by following the N-curve through M until the γ_- curve and then the γ_- curve into the origin. Then, it is $\tau(\Delta_1) < \tau(\Delta_2)$. See Fig. 3-4.

Proof - Since we have assumed that this Lemma is satisfied by the functions in group 2, we have to prove it only for the functions belonging to the first group. If $f(x_1)$ is an antisymmetric function, it is obvious that γ_- is antisymmetric to γ_+, which yields to

$$\tau(\Delta_1) = \tau(OP) = \tau(OQ) \qquad (3.17)$$

But it is obvious that

$$\tau(SQ) < \tau(SM) \qquad (3\text{-}18)$$

Then, from (3-17) and (3-18) we get

$$\tau(\Delta_1) = \tau(OQ) = \tau(OS) + \tau(SQ) < \tau(OS) + \tau(SM) = \tau(\Delta_2)$$

Theorem 3-1 - Let Q be any point on γ_+. Then the optimal path from Q to the origin is obtained by following the γ_+ curve into the origin.

Proof - From Theorem 1-3 we know that the optimal path from Q to the origin exists within the class of piecewise continuous functions. Then we have to show that the time spent along γ_+ is smaller than the time spent along any general canonical path from Q to the origin.

As a first step, we are going to prove that if $\Delta \equiv QRMSO$, see Fig. 3-2, is a canonical path from Q to the origin that has only two switchings, it is $\tau(QO) < \tau(\Delta)$. From Lemma 3-2 we have

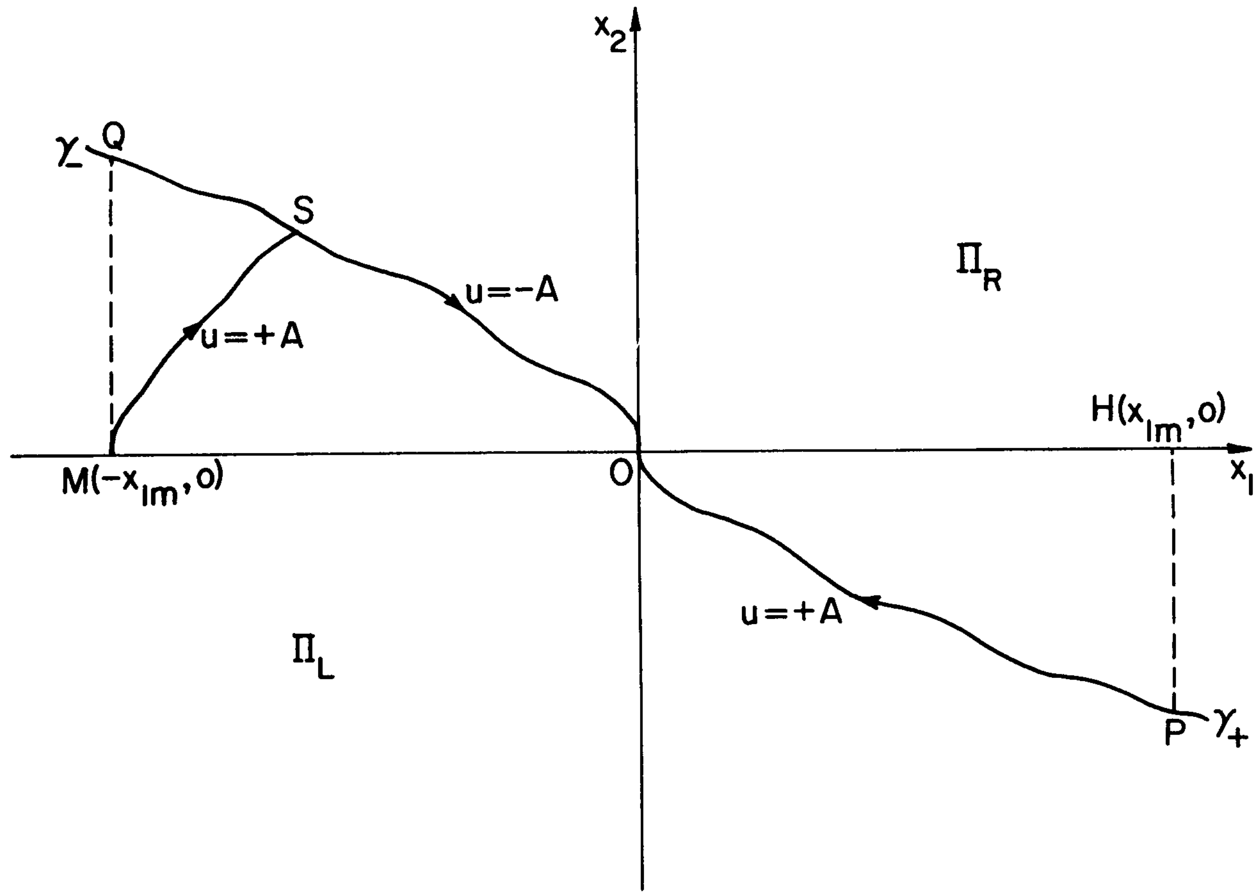

Fig. 3-4. Third Illustration for the Proof that Zero Trajectories are Optimal Paths

$$\tau(QN) < \tau(PM) \tag{3-19}$$

From Lemma 3-3 we have

$$\tau(NO) < \tau(MSO) \tag{3-20}$$

Also, it is obvious that

$$\tau(PR) < \tau(QR) \tag{3-21}$$

Then, adding (3-19) and (3-20) and using (3-21) we get

$$\begin{aligned} \tau(QO) &= \tau(QN) + \tau(NO) < \tau(PM) + \tau(MSO) = \\ &= \tau(PR) + \tau(RMSO) < \tau(QR) + \tau(RMSO) = \tau(\Delta) \end{aligned} \tag{3-22}$$

Now, see Fig. 3-5, let Δ_1 be the path from Q following the γ_+ curve into the origin, and Δ_2 be any general canonical path from Q into the origin. Assuming that the control sequence is $\{-A, \ldots, +A\}$ the total number of switchings is odd, say $(2n-1)$. (Note that if the control sequence were $\{-A, \ldots, -A\}$ the total number of switchings would be even and the proof would follow the same pattern). Then Δ_2 is composed of $2n$ P- and N-arcs, the first one being the N-arc through Q and the last one being the part of the γ_+ curve starting at S_{2n-1}.

Since Δ_2 is a canonical path, the switching points S_i are such that

$$S_{2n-1} \in \gamma_+$$

$$S_{2i-1} \in \Pi_L \quad , \quad i = 1,2,\ldots,(n-1)$$

$$S_{2i} \in \Pi_R \quad , \quad i = 1,2,\ldots,(n-1)$$

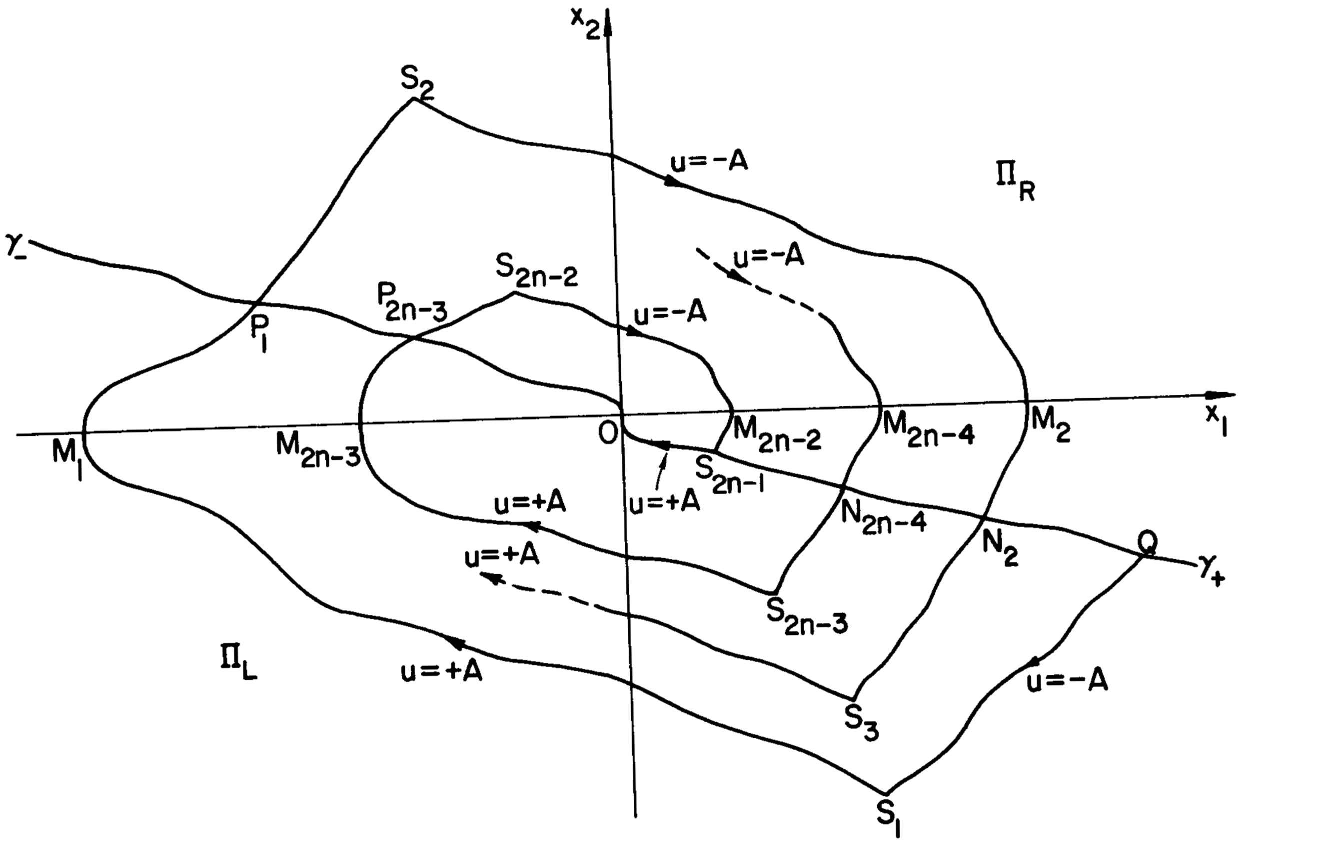

Fig. 3-5. Fourth Illustration for the Proof that Zero Trajectories are Optimal Paths

Now, let M_i , $i = 1,2,\ldots,(2n-2)$ be the intersection point of the $(i+1)$-th arc with the x_1-axis; let P_{2i-1} , $i = 1,2,\ldots,(n-1)$ be the intersection point of the $(2i)$-th arc with the γ_- curve; and let N_{2i} , $i = 1,2,\ldots,(n-2)$ be the intersection point of the $(2i+1)$-th arc with the γ_+ curve.

From (3-22) we get

$$\tau(QO) < \tau(QS_1M_1P_1O) \tag{3-23}$$

$$\tau(P_{2i-1}O) < \tau(P_{2i-1}S_{2i}M_{2i}N_{2i}O) \quad , \quad i = 1,2,\ldots,(n-1) \tag{3-24}$$

$$\tau(N_{2i}O) < \tau(N_{2i}S_{2i+1}M_{2i+1}P_{2i+1}O) \quad , \quad i = 1,2,\ldots,(n-2) \tag{3-25}$$

Adding equations (3-23), (3-24) and (3-25), and doing the necessary simplifications we get the following result:

$$\tau(\Delta_1) < \tau(\Delta_2)$$

and the theorem is proved, i.e., γ_+ is an optimal path.

Now, once we have proved that the zero trajectories are also optimal trajectories, the loci of possible switching points found in Chapter II can be considered as candidates for true switching curves. Then, considering canonical paths from an initial point to the origin that have only two switchings, and using the results given by Lemmas 3-4 and 3-5, we will be able to show in Lemma 3-6 the existence of indifference curves, first mentioned and defined in the Introduction, and also to decide which parts of the loci found in Chapter II are really true switching curves and which parts can be substituted by the indifference curves.

Both the indifference and the true switching curves separate, in the state plane, the regions of one switching from those of two switchings, allowing us to write down the optimal control law.

Lemma 3-4 - Let P_0 be any point in Π_R, and Δ_R be the canonical path which is obtained by following the P-curve through P_0 until a point F, then following the N-curve through F until γ_+ and finally by following γ_+ into the origin, see Fig. 3-6. When considering P_0 as fixed and F as variable, one obtains

$$\frac{d}{dx_{1f}}\left[\tau(\Delta_R)\right] = -2AG_R(x_{1f}, x_{1m})$$

Proof -

$$\tau(\Delta_R) = \tau(P_0F) + \tau(FM) + \tau(MS) + \tau(SO) \qquad (3\text{-}26)$$

But

$$\frac{d}{dx_{1m}}\left[\tau(P_0F)\right] = \frac{d}{dx_{1m}}\left[\int_{x_{10}}^{x_{1f}} \frac{d\sigma}{\left\{2[A\sigma - F(\sigma)+k_1]\right\}^{1/2}}\right] = \frac{A+f(x_{1m})}{2Ax_{2f}} \qquad (3\text{-}27)$$

$$\frac{d}{dx_{1m}}\left[\tau(SO)\right] = \frac{d}{dx_{1m}}\left[-\int_{x_{1s}}^{0} \frac{d\sigma}{\left\{2[A\sigma - F(\sigma)\right\}^{1/2}}\right] = -\frac{A+f(x_{1m})}{2Ax_{2s}} \qquad (3\text{-}28)$$

$$\frac{d}{dx_{1m}}\left[\tau(MS)\right] = \frac{d}{dx_{1m}}\left[-\int_{x_{1m}}^{x_{1s}} \frac{d\sigma}{\left\{2[-A\sigma - F(\sigma)+k_2]\right\}^{1/2}}\right] =$$

$$= \frac{d}{dx_{1m}}\left\{\int_{x_{1s}}^{x_{1m}}\left[\frac{1}{\left\{2[-A\sigma - F(\sigma)+k_2]\right\}^{1/2}} - \frac{1}{\left\{2[A+f(x_{1m})](x_{1m}-\sigma)\right\}^{1/2}}\right] d\sigma \; + \right.$$

(Continued)

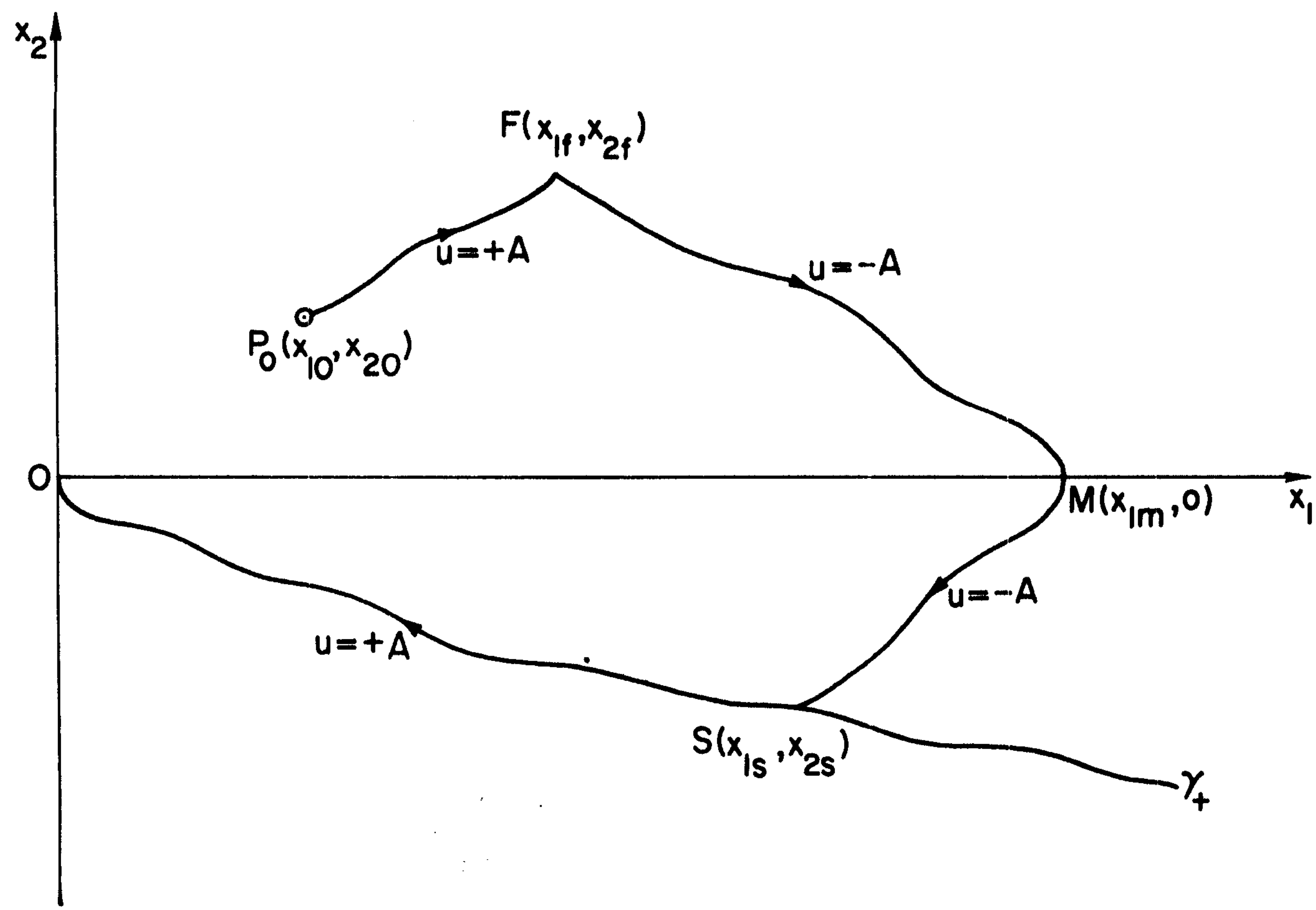

Fig. 3-6. Canonical Path from P_0 to the Origin that Has only Two Switchings

$$+\left[\frac{2(x_{1m}-x_{1s})}{A+f(x_{1m})}\right]^{1/2}\Bigg\} = -\ [A+f(x_{1m})]\int_{x_{1s}}^{x_{1m}}\left[\frac{1}{\left\{2[-A\sigma-F(\sigma)+k_2]\right\}^{3/2}}\ -\right.$$

$$\left.-\ \frac{1}{\left\{2[A+f(x_{1m})](x_{1m}-\sigma)\right\}^{3/2}}\right]d\sigma + \frac{1}{\left\{2[A+f(x_{1m})](x_{1m}-x_{1s})\right\}^{1/2}} + \frac{A+f(x_{1m})}{2Ax_{2s}}$$

(3-29)

$$\frac{d}{dx_{1m}}\left[\tau(FM)\right] = \frac{d}{dx_{1m}}\left[\int_{x_{1f}}^{x_{1m}}\frac{d\sigma}{\left\{2[-A\sigma-F(\sigma)+k_2]\right\}^{1/2}}\right] = -\ [A+f(x_{1m})]\ \cdot$$

$$\cdot\int_{x_{1f}}^{x_{1m}}\left[\frac{1}{\left\{2[-A\sigma-F(\sigma)+k_2]\right\}^{3/2}} - \frac{1}{\left\{2[A+f(x_{1m})](x_{1m}-\sigma)\right\}^{3/2}}\right]d\sigma\ +$$

$$+\ \frac{1}{\left\{2[A+f(x_{1m})](x_{1m}-x_{1f})\right\}^{1/2}} - \frac{A+f(x_{1m})}{2Ax_{2f}} \qquad (3\text{-}30)$$

Taking derivatives of (3-26), and substituting (3-27), (3-28), (3-29) and (3-30), we get

$$\frac{d}{dx_{1m}}\left[\tau(\Delta_R)\right] = \frac{1}{\left\{2[A+f(x_{1m})]\right\}^{1/2}}\left[\frac{1}{(x_{1m}-x_{1s})^{1/2}} + \frac{1}{(x_{1m}-x_{1f})^{1/2}}\right]\ -$$

$$-\ [A+f(x_{1m})]\int_{x_{1s}}^{x_{1m}}\left[\frac{1}{\left\{2[-A\sigma-F(\sigma)+k_2]\right\}^{3/2}}\ -\right.$$

$$\left.-\ \frac{1}{\left\{2[A+f(x_{1m})](x_{1m}-\sigma)\right\}^{3/2}}\right]d\sigma\ -\ [A+f(x_{1m})]\ \cdot$$

(Continued)

$$\cdot\int_{x_{1f}}^{x_{1m}} \left[\frac{1}{\left|2[-A\sigma - F(\sigma)+k_2]\right|^{3/2}} - \frac{1}{\left|2[A+f(x_{1m})](x_{1m}-\sigma)\right|^{3/2}} \right] d\sigma \qquad (3\text{-}31)$$

Changing (y_1, y_{1m}, y_{1s}) into (x_{1f}, x_{1m}, x_{1s}) on $G_R(y_1, y_{1m})$, see (2-37), from the result obtained and (3-31), we get the following relation

$$\frac{d}{dx_{1m}} \left| \tau(\Delta_R) \right| = - [A+f(x_{1m})]\, G_R(x_{1f}, x_{1m}) \qquad (3\text{-}32)$$

But

$$x_{1f} = \frac{1}{2A}\left[A(x_{1m}+x_{10}) - \frac{x_{20}^2}{2} + F(x_{1m}) - F(x_{10}) \right] \; ; \quad \frac{dx_{1f}}{dx_{1m}} = \frac{A+f(x_{1m})}{2A}$$

Then

$$\frac{d}{dx_{1f}} \left| \tau(\Delta_R) \right| = \frac{d}{dx_{1m}} \left| \tau(\Delta_R) \right| \frac{dx_{1m}}{dx_{1f}} = - 2A\, G_R(x_{1f}, x_{1m})$$

<u>Lemma 3-5</u> - Let $F(x_{1f}, x_{2f})$ be the point on the N-curve through $M(x_{1m}, 0)$ such that $G_R(x_{1f}, x_{1m}) = 0$, and let $M'(x'_{1m}, 0)$ be such that $x'_{1m} = x_{1m} + 2\theta$. Then, the point $F'(x'_{1f}, x'_{2f})$ on the N-curve through M', for which $G_R(x'_{1f}, x'_{1m}) = 0$, is such that $x'_{1m} - x'_{1f} < x_{1m} - x_{1f}$

<u>Proof</u> - Let $S(x_{1s}, x_{2s})$ and $S'(x'_{1s}, x'_{2s})$ be the intersection points of the N-curves through the points M and M' with the γ_+ curve. Then, from (2-34) we get

$$x'_{1s} - x_{1s} = \frac{1}{2A}\left[Ax'_{1m} + F(x'_{1m}) \right] - \frac{1}{2A}\left[Ax_{1m} + F(x_{1m}) \right] =$$

$$= \frac{1}{2A}\left[(A+K)\, x'_{1m} + F_1(x'_{1m}) \right] - \frac{1}{2A}\left[(A+K)x_{1m} + F_1(x_{1m}) \right] = \text{(Continued)}$$

$$= \frac{1}{2A}\left[(A+K)\ (x'_{1m}-x_{1m}) + F_1(x'_{1m}) - F_1(x_{1m})\right] =$$

$$= \frac{1}{2A}\left[(A+K)2\theta + F_1(x_{1m}+2\theta) - F_1(x_{1m})\right] = \theta(1+\frac{K}{A}) \qquad (3\text{-}33)$$

From the definition of G_R , see equation (2-37), we get

$$G_R(x'_1, x'_{1m}) = -\frac{1}{\sqrt{2}\,[A+f(x'_{1m})]^{3/2}}\left[\frac{1}{\left(x'_{1m}-x'_{1s}\right)^{1/2}} + \frac{1}{\left(x'_{1m}-x'_{1}\right)^{1/2}}\right] +$$

$$+ \int_{x'_{1s}}^{x'_{1m}} \left[\frac{1}{\left\{2[A(x'_{1m}-\sigma)-F(\sigma)+F(x'_{1m})]\right\}^{3/2}} -\right.$$

$$\left. - \frac{1}{\left\{2[A+f(x'_{1m})](x'_{1m}-\sigma)\right\}^{3/2}}\right] d\sigma +$$

$$+ \int_{x'_{1}}^{x'_{1m}} \left[\frac{1}{\left\{2[A(x'_{1m}-\sigma)-F(\sigma)+F(x'_{1m})]\right\}^{3/2}} -\right.$$

$$\left. - \frac{1}{\left\{2[A+f(x'_{1m})](x'_{1m}-\sigma)\right\}^{3/2}}\right] d\sigma = -\frac{1}{\sqrt{2}\,[A+f(x_{1m})]^{3/2}} \cdot$$

$$\cdot\left[\frac{1}{\left\{x_{1m}-x_{1s}+\theta(1-\frac{K}{A})\right\}^{1/2}} + \frac{1}{\left(x_{1m}-x'_{1}+2\theta\right)^{1/2}}\right] +$$

$$+ \int_{x_{1s}-\theta(1-\frac{K}{A})}^{x_{1m}} \left[\frac{1}{\left\{2[A(x_{1m}-\sigma)-F(\sigma)+F(x_{1m})]\right\}^{3/2}} -\right.$$

(Continued)

$$- \frac{1}{\left\{2[A+f(x_{1m})](x_{1m}-\sigma)\right\}^{3/2}} \Bigg] d\sigma +$$

$$+ \int_{x_1'-2\theta}^{x_{1m}} \Bigg[\frac{1}{\left\{2[A(x_{1m}-\sigma)-F(\sigma)+F(x_{1m})]\right\}^{3/2}} -$$

$$- \frac{1}{\left\{2[A+f(x_{1m})](x_{1m}-\sigma)\right\}^{3/2}} \Bigg] d\sigma = G_R(x_{1f},x_{1m}) +$$

$$+ \int_{x_{1s}-\theta(1-\frac{K}{A})}^{x_{1s}} \frac{d\sigma}{\left\{2[A(x_{1m}-\sigma)-F(\sigma)+F(x_{1m})]\right\}^{3/2}} +$$

$$+ \int_{x_1'-2\theta}^{x_{1f}} \frac{d\sigma}{\left\{2[A(x_{1m}-\sigma)-F(\sigma)+F(x_{1m})]\right\}^{3/2}} \qquad (3\text{-}34)$$

But $G_R(x_{1f},x_{1m}) = 0$, by hypothesis. Hence (3-34) yields

$$G_R(x_1',x_{1m}') = \int_{x_{1s}-\theta(1-\frac{K}{A})}^{x_{1s}} \frac{d\sigma}{\left\{2[A(x_{1m}-\sigma)-F(\sigma)+F(x_{1m})]\right\}^{3/2}} +$$

$$+ \int_{x_1'-2\theta}^{x_{1f}} \frac{d\sigma}{\left\{2[A(x_{1m}-\sigma)-F(\sigma)+F(x_{1m})]\right\}^{3/2}} \qquad (3\text{-}35)$$

Now, if we want $G_R(x_{1f}',x_{1m}') = 0$, it is necessary that

$$\int_{x_{1s}-\theta(1-\frac{K}{A})}^{x_{1s}} \frac{d\sigma}{\left\{2[A(x_{1m}-\sigma)-F(\sigma)+F(x_{1m})]\right\}^{3/2}} =$$

(Continued)

$$= \int_{x_{1f}}^{x'_{1f}-2\theta} \frac{d\sigma}{\left\{2[A(x_{1m}-\sigma)-F(\sigma)+F(x_{1m})]\right\}^{3/2}} \tag{3-36}$$

But (3-36) has a solution if and only if $x'_{1f}-2\theta > x_{1f}$. Hence, the point F' for which $G_R(x'_{1f},x'_{1m}) = 0$ is such that

$$x'_{1m} - x'_{1f} < x_{1m} + 2\theta - x_{1f} - 2\theta = x_{1m} - x_{1f}$$

Lemma 3-6 - Let P_0 be any point in Π_R. Let $F^i_{R_a}$ and $F^i_{R_b}$ be the intersection points of the P-curve through P_0 with $\Sigma^i_{R_a}$ and $\Sigma^i_{R_b}$ respectively, see Fig. 2-7. Let Δ_R be the canonical path, not necessarily satisfying Pontryagin's Maximum Prinicple, which is obtained by following the P-curve through P_0 until the point F_R, then following the N-curve through F_R until γ_+ and finally by following γ_+ into the origin, and let $\Delta^i_{R_a}$ and $\Delta^i_{R_b}$ the canonical paths corresponding to $F^i_{R_a}$ and $F^i_{R_b}$, see Fig. 3-7. Then

i) $\tau(\Delta^i_{R_b}) > \tau(\Delta^i_{R_a})$ for $i \geq j$

ii) $\tau(\Delta^{i+1}_{R_a}) > \tau(\Delta^i_{R_a})$ for $i \geq j$

iii) $\tau(\Delta^{i+1}_{R_b}) > \tau(\Delta^i_{R_b})$ for $i \geq j$

iv) On every interval $(F^i_{R_a}, F^{i+1}_{R_b})$ there exists a point $F^{i+1}_{R_c}$ such that $\tau(\Delta^{i+1}_{R_c}) = \tau(\Delta^{i+1}_{R_a})$.

Proof - Since P_0 is a fixed point, $G_R(x_{1f},x_{1m})$ will be essentially a function of x_{1f}; from its definition, we know that $G_R(x_{1f},x_{1m})$ is positive if F_R belongs to the interval $(F^i_{R_b}, F^i_{R_a})$, it is negative if F_R belongs to $(F^i_{R_a}, F^{i+1}_{R_b})$, and it is zero if $F_R = F^i_{R_a}$ or

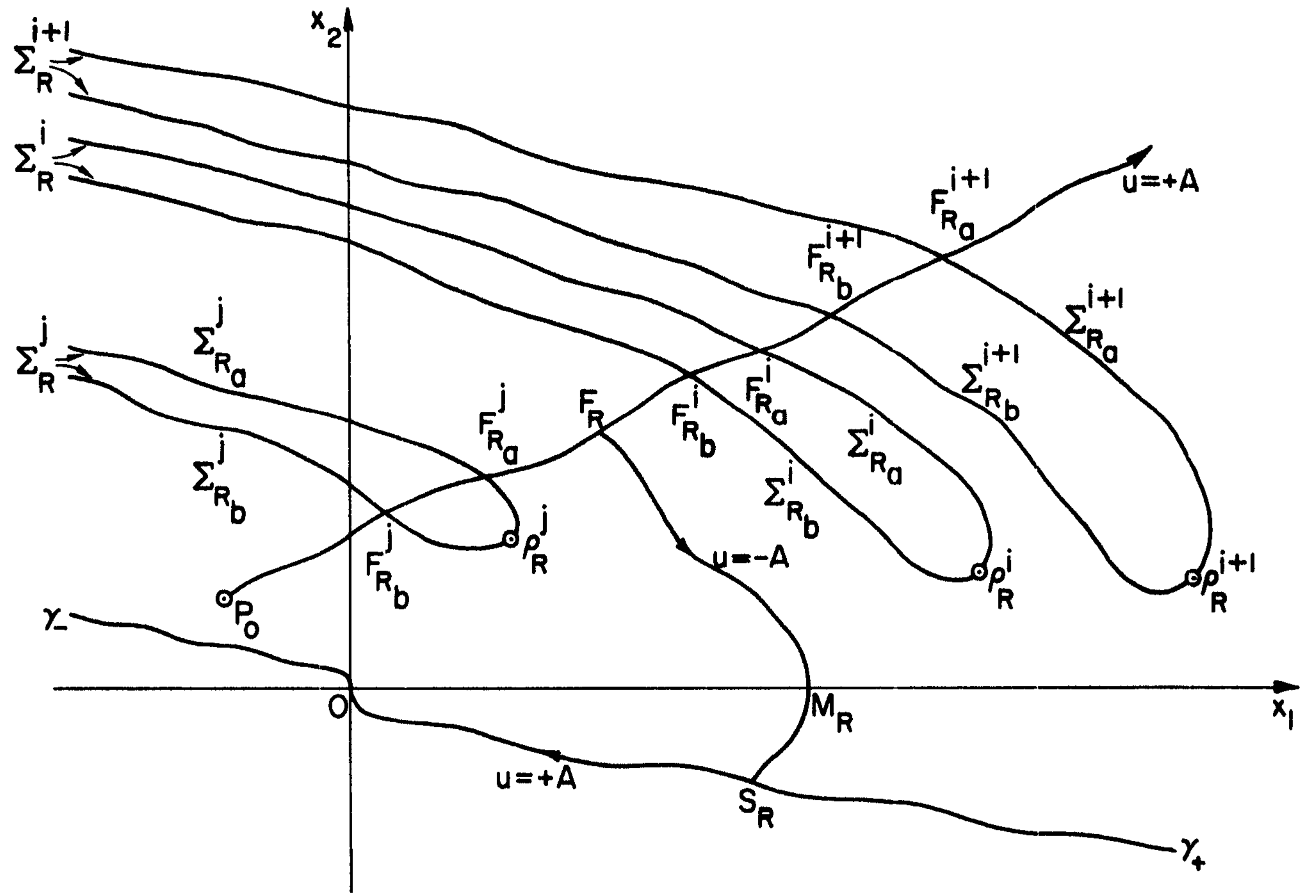

Fig. 3-7. Illustration for the Proof of the Existence of the Indifference Curves

$F_R = F^i_{R_b}$, this being true for any $i \geq j$.

On the other hand, considering $\tau(\Delta_R)$ as a function of x_{1f} , from Lemma 3-4 we know that the slope of the function $\tau(\Delta_R)$ has a sign opposite to that of $G_R(x_{1f}, x_{1m})$. So, $\tau(\Delta_R)$ will be a decreasing function of F_R if F_R belongs to $(F^i_{R_b}, F^i_{R_a})$, an increasing function of F_R if F_R belongs to $(F^i_{R_a}, F^{i+1}_{R_b})$, and will have relative extremum if $F_R = F^i_{R_a}$ or $F_R = F^i_{R_b}$, since for these points the derivative is zero.

i) Since $\tau(\Delta_R)$ is a decreasing function on the interval $(F^i_{R_b}, F^i_{R_a})$, it is obvious that $\tau(\Delta^i_{R_b}) > \tau(\Delta^i_{R_a})$ for any $i \geq j$.

ii) From Lemma 3-5, we know that we can find a point $F^{i+1}_{R_d}$ in the interval $(F^i_{R_a}, F^{i+1}_{R_b})$ which is the intersection point of the P-curve through the point P_0 with the curve obtained by a shifting of 2θ in the negative direction of the x_1-axis of the N-curve through the point $F^{i+1}_{R_a}$. Moreover, it is obvious that $\tau(\Delta^{i+1}_{R_a}) > \tau(\Delta^{i+1}_{R_d})$. On the other hand, since $\tau(\Delta_R)$ is an increasing function of F_R in the interval $(F^i_{R_a}, F^{i+1}_{R_b})$ we have that $\tau(\Delta^{i+1}_{R_d}) > \tau(\Delta^i_{R_a})$. Therefore, it follows that $\tau(\Delta^{i+1}_{R_a}) > \tau(\Delta^{i+1}_{R_d}) > \tau(\Delta^i_{R_a})$.

iii) The proof follows the same pattern as that of part ii) .

iv) We know that $\tau(\Delta_R)$ is increasing in $(F^i_{R_a}, F^{i+1}_{R_b})$ and decreasing in $(F^{i+1}_{R_b}, F^{i+1}_{R_a})$, and from part ii) we also know that $\tau(\Delta^{i+1}_{R_a}) > \tau(\Delta^i_{R_a})$. Hence, it is obvious that there exists a point $F^{i+1}_{R_c}$ in the interval $(F^i_{R_a}, F^{i+1}_{R_b})$ for which $\tau(\Delta^{i+1}_{R_c}) = \tau(\Delta^{i+1}_{R_a})$.

<u>Definition 3-9</u> - Let $\Sigma^i_{R_c}$ be defined, see Fig. 2-7, as

$$\Sigma^i_{R_c} = \left\{ F^i_{R_c} : \text{ if } P_0 \equiv F^i_{R_c} \text{ there exists a } F^i_{R_a} \in \Sigma^i_{R_a} \text{ such that } \tau(\Delta^i_{R_a}) = \tau(\Delta^i_{R_c}) \right\} \quad \text{for } i \geq j \tag{3-37}$$

In the same way, we can define $\Sigma^i_{L_c}$ as

$$\Sigma^i_{L_c} = \left\{ F^i_{L_c} : \text{ if } P_0 \equiv F^i_{L_c} \text{ there exists a } F^i_{L_a} \in \Sigma^i_{L_a} \text{ such that } \tau(\Delta^i_{L_a}) = \tau(\Delta^i_{L_c}) \right\} \quad \text{for } i \geq r \tag{3-38}$$

It is clear that $\Sigma^i_{R_c}$ ($\Sigma^i_{L_c}$) meets the curve $\Sigma^i_{R_a}$ ($\Sigma^i_{L_a}$) at the point $\rho^i_R(\rho^i_L)$.

<u>Definition 3-10</u> - Let Λ^i_R be defined as

$$\Lambda^i_R = \left\{ (x_1,x_2) : \text{ if } (x_1,x_2') \in \Sigma^i_{R_a} \text{ and } (x_1,x_2'') \in \Sigma^i_{R_c} \text{ then } x_2 \in (x_2'',x_2') \right\} \quad \text{for } i \geq j \tag{3-39}$$

In the same way, we can define Λ^i_L as

$$\Lambda^i_L = \left\{ (x_1,x_2) : \text{ if } (x_1,x_2') \in \Sigma^i_{L_a} \text{ and } (x_1,x_2'') \in \Sigma^i_{L_c} \text{ then } x_2 \in (x_2',x_2'') \right\} \quad \text{for } i \geq r \tag{3-40}$$

Hence, we have established, as the main result of our research, the existence of certain switching and indifference curves, and the sufficiency of the control law $u^*(t)$ as given by the following theorem.

Theorem 3-2 - The optimal control law $u^*(x_1,x_2)$ as a function of the state (x_1,x_2) is given by

$$u^* = +A \quad \text{if} \quad (x_1,x_2) \in \gamma_+ \cup \left(\bigcup_{i\geq j} \Lambda_R^i\right) \cup \left\{\Pi_L \sim \left[\left(\bigcup_{i\geq r} \Lambda_L^i\right) \cup \left(\bigcup_{i\geq r} \Sigma_{L_c}^i\right)\right]\right\}$$

$$u^* = -A \quad \text{if} \quad (x_1,x_2) \in \gamma_- \cup \left(\bigcup_{i\geq r} \Lambda_L^i\right) \cup \left\{\Pi_R \sim \left[\left(\bigcup_{i\geq j} \Lambda_R^i\right) \cup \left(\bigcup_{i\geq j} \Sigma_{R_c}^i\right)\right]\right\}$$

$$u^* = \begin{cases} +A \\ -A \end{cases} \quad \text{if} \quad (x_1,x_2) \in \left(\bigcup_{i\geq j} \Sigma_{R_c}^i\right) \cup \left(\bigcup_{i\geq r} \Sigma_{L_c}^i\right)$$

Proof - Let P_0 be any initial point such that $P_0 \in \Lambda_R^k$. We are going to show that the optimal path from P_0 to the origin, whose existence is guaranteed by Theorem 1-3, is the canonical path $\Delta^k \equiv P_0F^kM^kS^kO$, see Fig. 3-8.

Suppose that Δ^k is not the optimal path from P_0. Let Δ^r be the true optimal path; it is clear that the number of switchings is even, say $2n$, $n \geq 1$.

Consider first the case $n > 1$. Whatever is the behavior of Δ^r, we know, from the way the switching curves $\Sigma_{R_a}^i$ were defined, that the last switching must occur at γ_+ and the switching before the last must occur on some of the switching curves, say $\Sigma_{R_a}^r$; then, the last part of

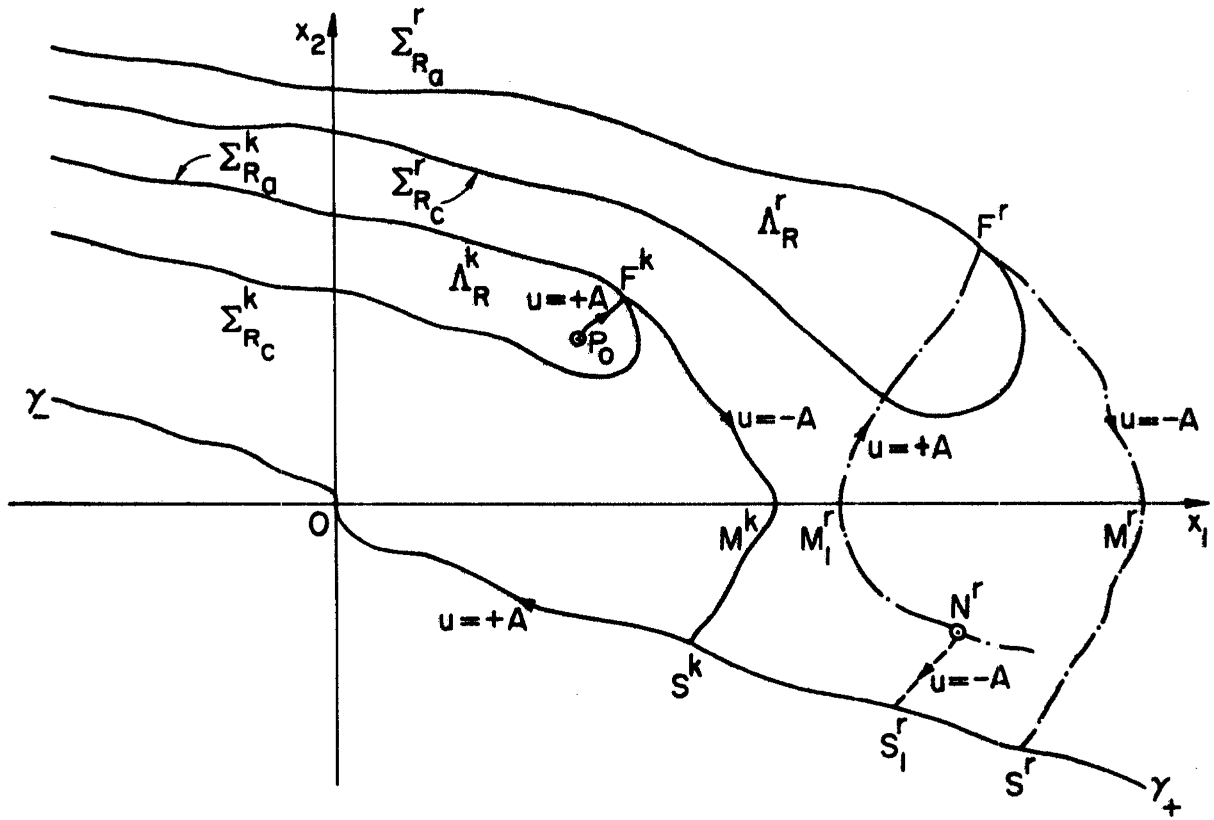

Fig. 3-8. Illustration for the Proof of the Optimal Control Law

Δ^r will be the path $\Delta_1^r \equiv N^r M_1^r F^r M^r S^r O$, where N^r is any point such that no switching occurs on $N^r M_1^r F^r$. From the principle of optimality, the path Δ_1^r, which is a subpath of Δ^r, should be also optimal; however, from the way $\Sigma_{R_c}^r$ was defined, we know that there exists a canonical path $\Delta_2^r \equiv N^r S_1^r O$ such that $\tau(\Delta_2^r) < \tau(\Delta_1^r)$; then, Δ_1^r is not the optimal path from N^r to the origin, and our assumption is false, that is, Δ^r is not the optimal path from P_0.

Therefore, we must have $n = 1$, that is, the number of switchings is two. From Lemma 3-6, we know that of all the canonical paths from P_0 to the origin, with no more than two switchings, Δ^k is the optimal. Hence, the optimal path from P_0 to the origin is Δ^k, as we wanted to show.

Similar arguments can be used whenever the initial point P_0 belongs to any other region different from Λ_R^k, and the theorem is proved.

<u>Corollary</u> - The optimal number of switchings N as a function of the initial state is given by

$$N = 0 \quad \text{if} \quad (x_{10}, x_{20}) \in \Gamma$$

$$N = 1 \quad \text{if} \quad (x_{10}, x_{20}) \in \left\{ \Pi_R \sim \left[\left(\bigcup_{i \geq j} \Lambda_R^i \right) \cup \left(\bigcup_{i \geq j} \Sigma_{R_c}^i \right) \right] \right\} \cup \left\{ \Pi_L \sim \left[\left(\bigcup_{i \geq r} \Lambda_L^i \right) \cup \left(\bigcup_{i \geq r} \Sigma_{L_c}^i \right) \right] \right\}$$

$$N = 2 \quad \text{if} \quad (x_{10}, {}_{20}) \in \left(\bigcup_{i \geq j} \Lambda_R^i \right) \cup \left(\bigcup_{i \geq r} \Lambda_L^i \right)$$

$$N = 1 \text{ or } 2 \quad \text{if} \quad (x_{10}, x_{20}) \in \left(\bigcup_{i \geq j} \Sigma_{R_c}^i \right) \cup \left(\bigcup_{i \geq r} \Sigma_{L_c}^i \right)$$

CHAPTER IV

APPLICATIONS

As an example of the first group of periodic functions considered in our research, that is, periodic functions which are at the same time antisymmetric, let us consider the case

$$f(x) = \sin x$$

Then equation (1-1) becomes

$$\ddot{x} + \sin x = u \tag{4-1}$$

Equation (4-1) represents a number of physical systems, for example certain motions of a satellite in a circular orbit, and also a pendulum.

In this particular case, $\theta = \pi$,and (3-7) becomes

$$K = \frac{1}{2\pi} \int_0^{2\pi} \sin x_1 \, dx_1 = 0 \tag{4-2}$$

Also

$$|f(x)| = |\sin x| \leq 1 \quad ,\text{i.e., } B = 1 \tag{4-3}$$

Substituting (4-2) and (4-3) into (3-9), we get for A the following bound for which our results are applicable

$$A \geq C = B + 2|K| = 1 \quad , \quad A \geq 1 \tag{4-4}$$

Analog Simulation - In order to get the switching curves we have to plot the right hand member of equation (2-37) equated to zero, which turns out to be rather complicated; the algebraic complications also appear when we try to plot the indifference curves. The easiest way to get the switching and indifference curves is by simulation on an analog computer, and finding the trajectories in backwards time, starting at the origin of the phase plane.

The equations that have to be simulated on the analog computer are equations (1-5), (1-13) and (1-20) in backwards time. Then, using the backwards time notation given by (2-17), the system to be simulated is the following:

$$\left.\begin{aligned} \dot{y}_1 &= -y_2 \\ \dot{y}_2 &= -u + \sin y_1 \\ u &= A \operatorname{sgn} \lambda_2 \\ \dot{\lambda}_1 &= -\lambda_2 \cos y_1 \\ \dot{\lambda}_2 &= \lambda_1 \end{aligned}\right\} \qquad (4\text{-}5)$$

and the initial conditions are given by

$$\left.\begin{aligned} y_1(0) &= 0 \quad , \quad & y_2(0) &= 0 \\ \lambda_1(0) &= \lambda_{10} \quad , \quad & \lambda_2(0) &= \lambda_{20} \end{aligned}\right\} \qquad (4\text{-}6)$$

The analog computer diagram for the system (4-5) with the initial conditions (4-6) is given in Fig. 4-1.

Since $\sin x$ is an antisymmetric function, the switching and indifference curves in the region Π_L will be antisymmetric to the corresponding curves in the region Π_R. So, we will only find those curves in the region Π_R.

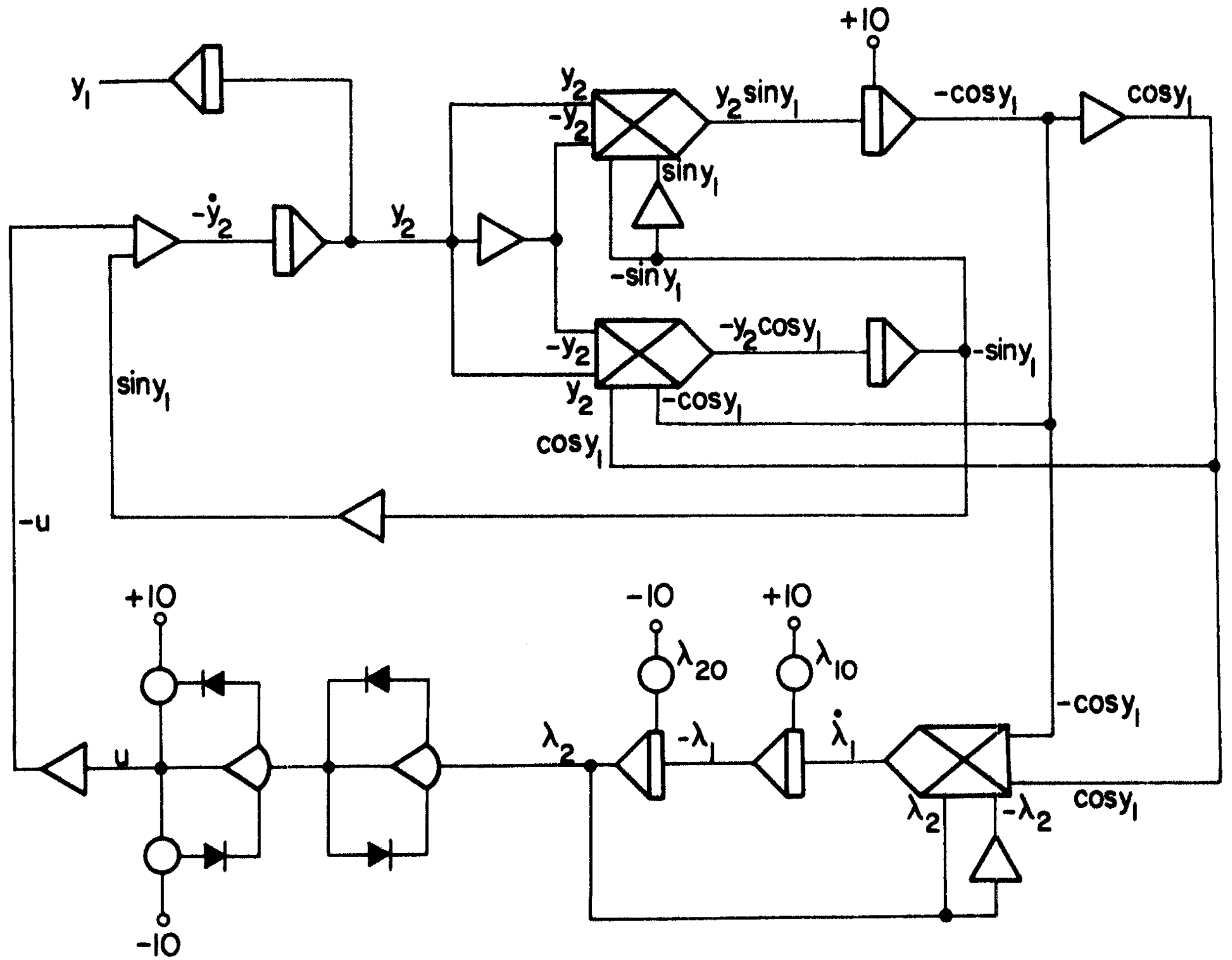

Fig. 4-1. Analog Simulation

Moreover, for obvious reasons, we have only determined the $\Sigma^1_{R_a}$ and $\Sigma^1_{R_c}$ curves; in this case $j = r = 1$. Figs. 4-2 to 4-9 show these curves for different values of A. Fig. 4-2 also shows two sample trajectories starting at P_1 and P_2. The numbers on the curves $\Sigma^1_{R_c}$ refer to the minimum settling time.

Looking at the curves in Figs. 4-2 to 4-9, we see that when the value of A increases, the region Λ^1_R becomes not only narrower but also shifted upwards and to the left; this will also happen to every region Λ^i_R . In the limit, when $A \to \infty$, all these regions Λ^i_R will eventually become the void set, as it is to be expected, because in the limit our plant becomes merely

$$\ddot{x} = u$$

and it is well known that for such a plant any initial disturbance in the region Π_R is brought to the origin of the phase plane, in minimum time, after switching on the γ_+ curve.

Comparison with the Linear Case - So far, in the literature concerning optimization, whenever equation (4-1) appeared, it was customary to linearize the equation and to consider only small motions of the system. So, we feel it is worthwhile to indicate some of the differences encountered, in the case of the minimum time problem, between the nonlinear problem represented by equation (4-1) and the linearized one represented by

$$\ddot{x} + x = u \quad , \qquad |u| \leq A \tag{4-7}$$

The first rigorous solution of this problem was given by Bushaw, Ref.[4].

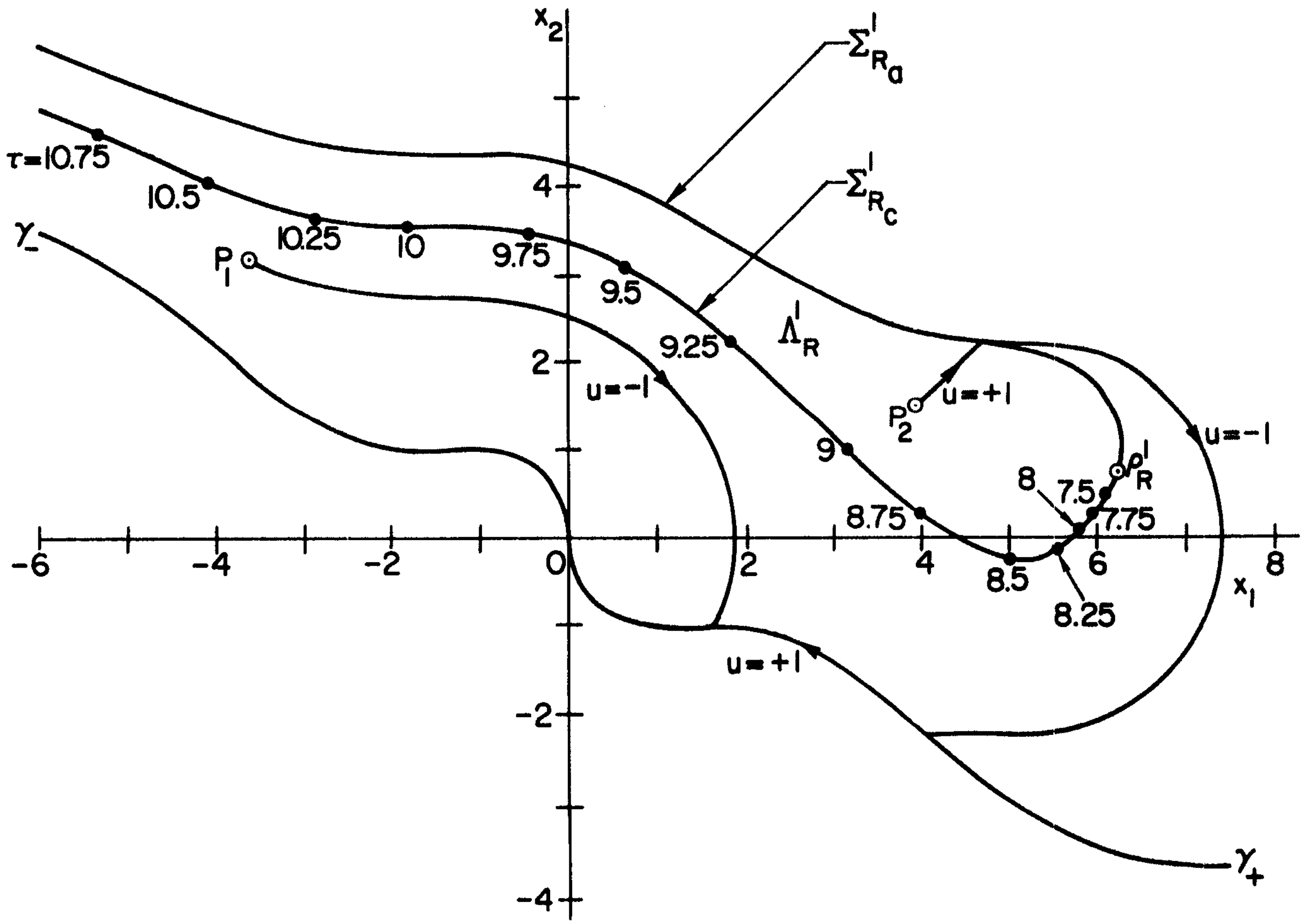

Fig. 4-2. First Switching and Indifference Curves and Zero Trajectories for $A = 1$ (Two Sample Trajectories are Shown)

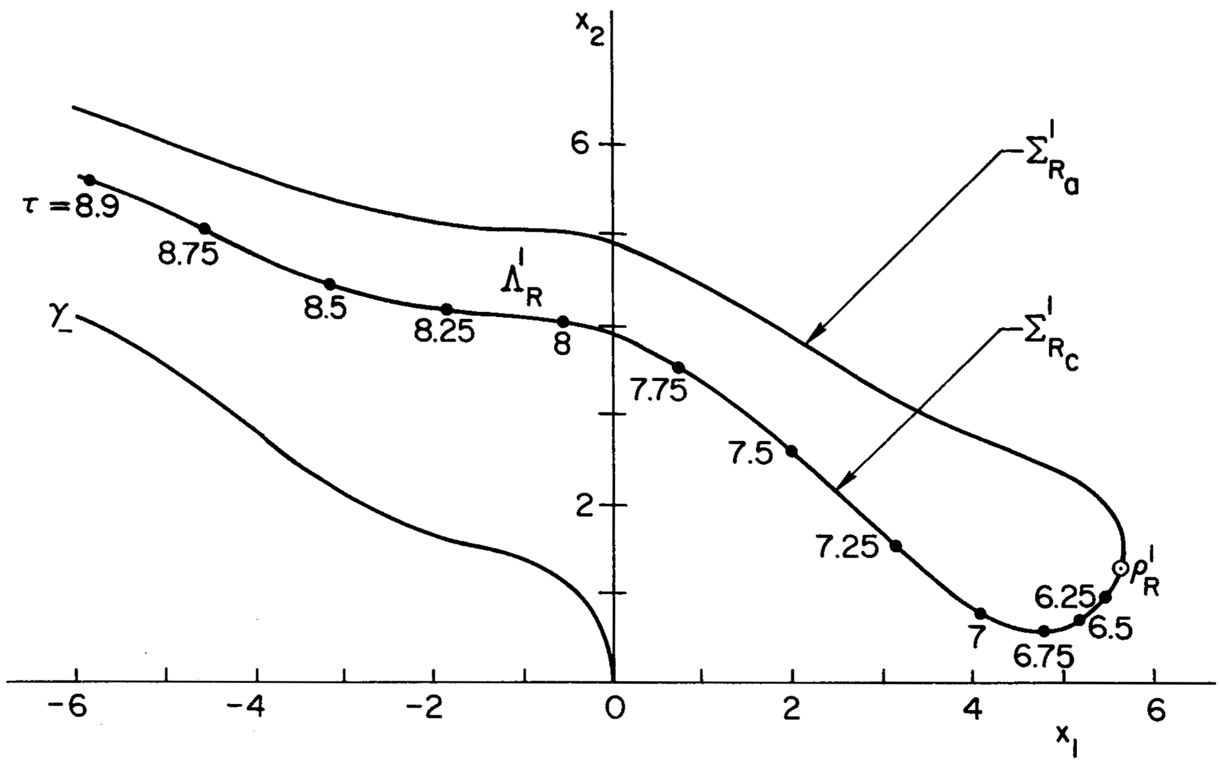

Fig. 4-3. γ_- , First Switching and Indifference Curves for $A = 1.4$

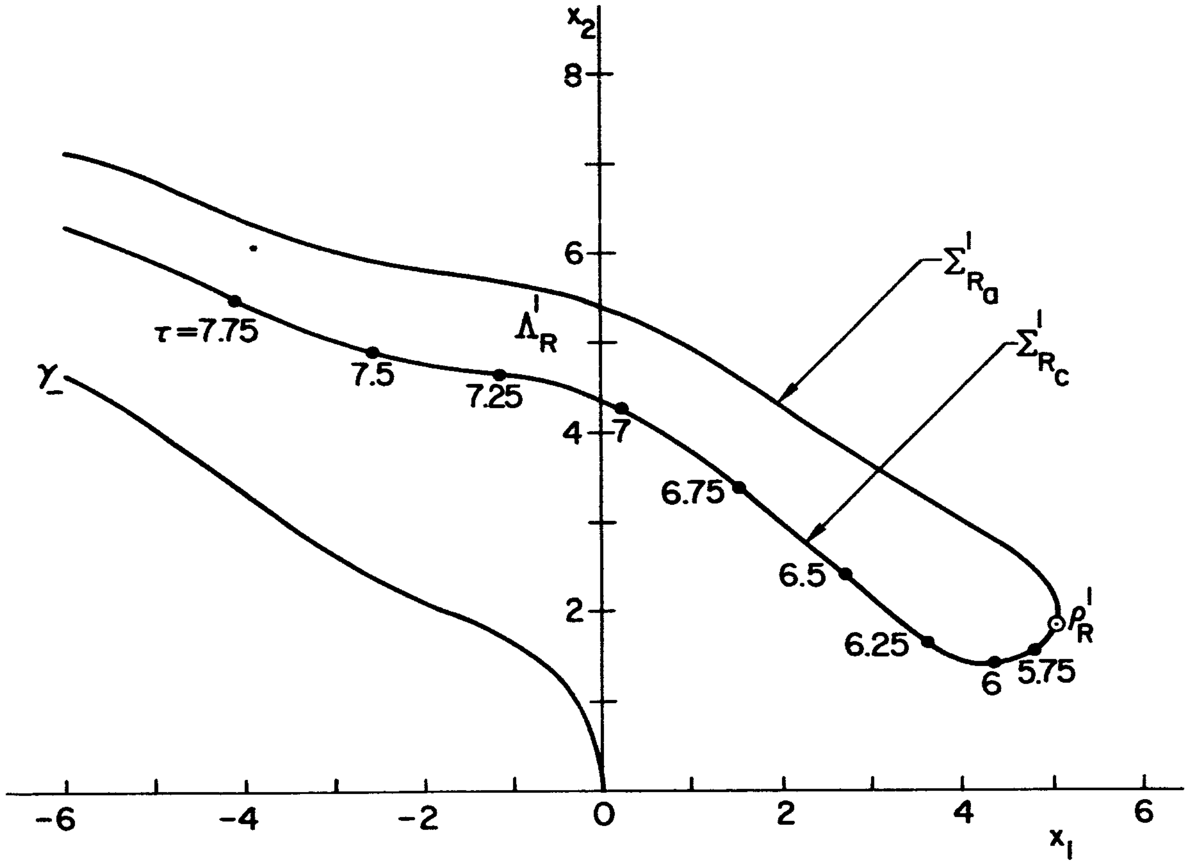

Fig. 4-4. γ_- , First Switching and Indifference Curves for $A = 1.8$

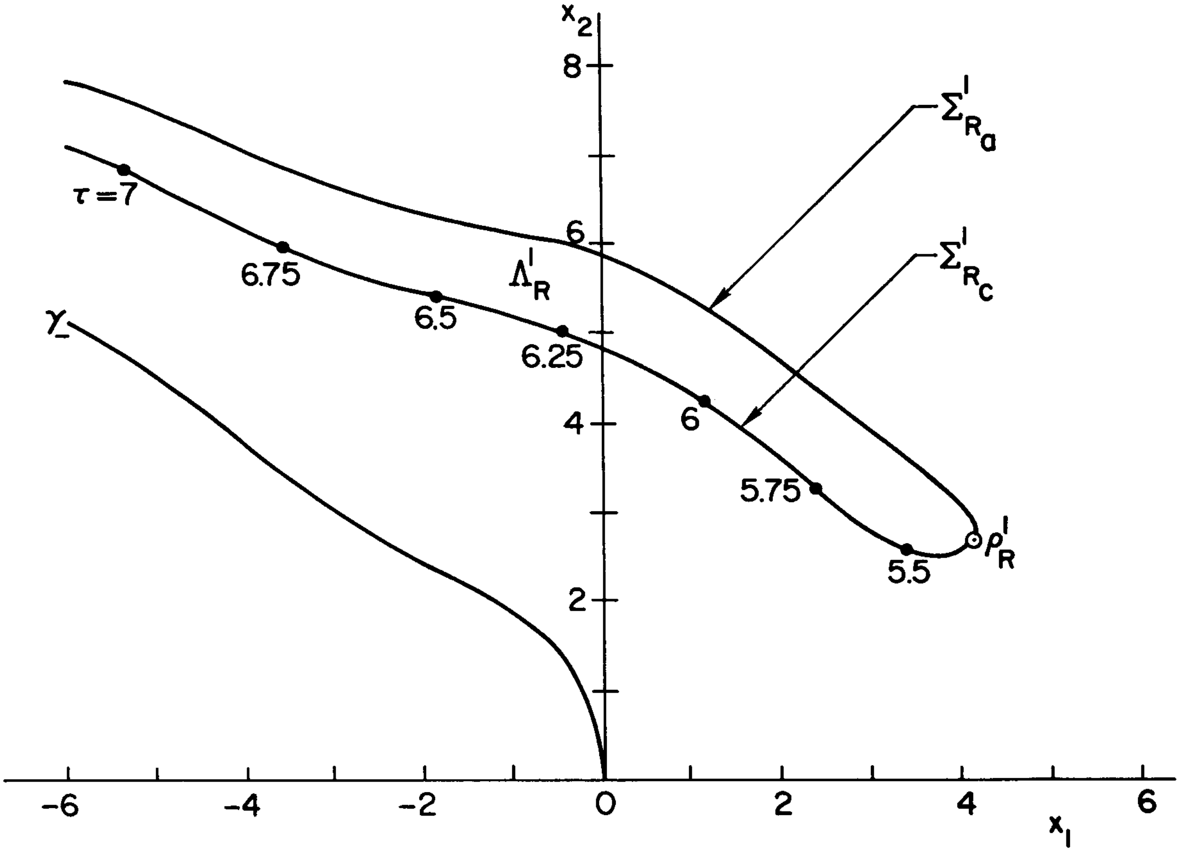

Fig. 4-5. γ_- , First Switching and Indifference Curves for A = 2.2

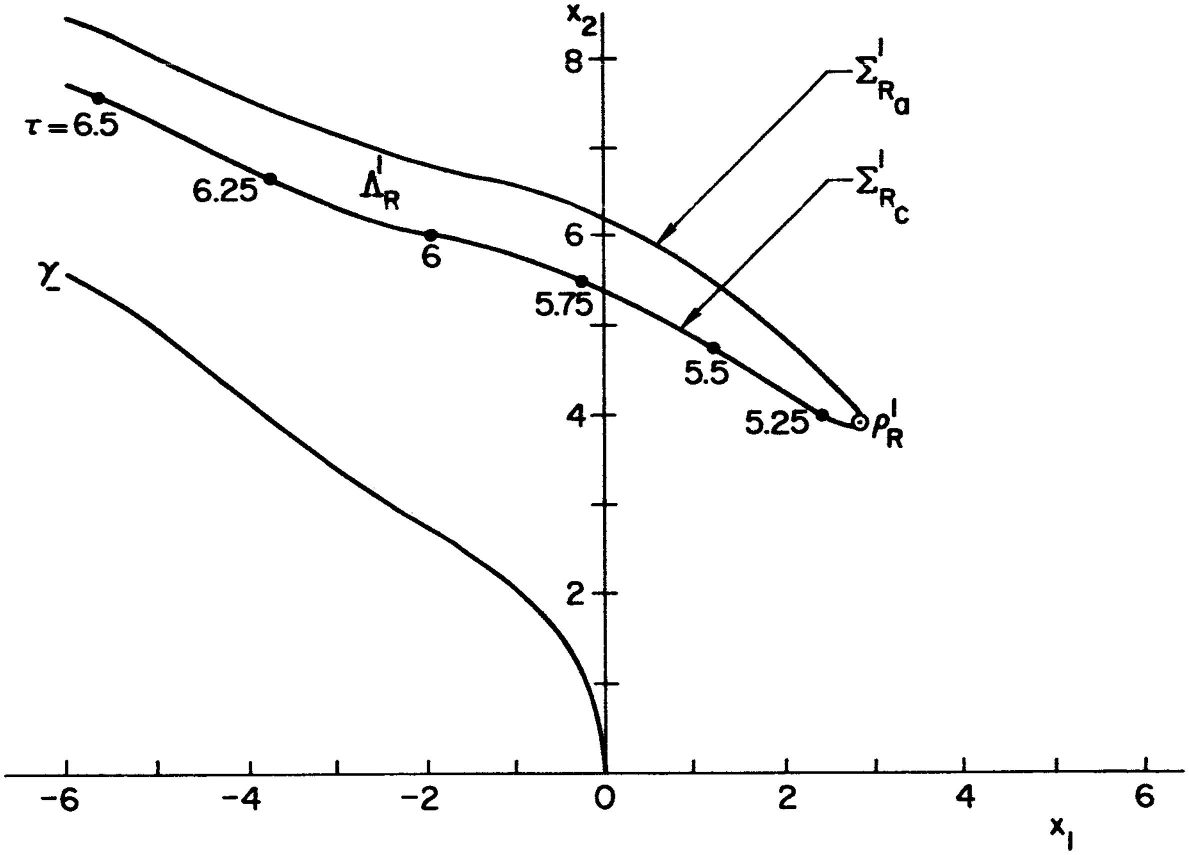

Fig. 4-6. γ_- , First Switching and Indifference Curves for $A = 2.6$

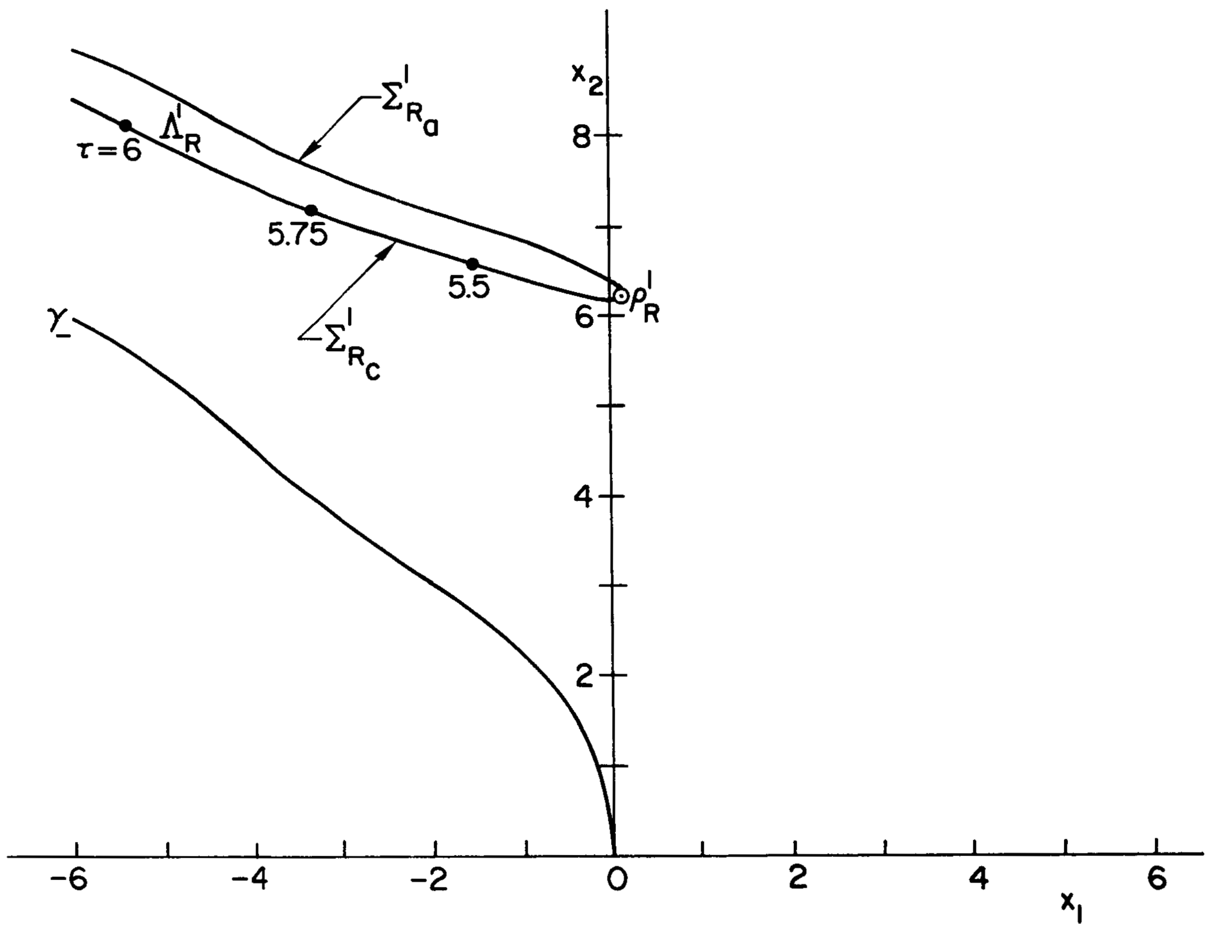

Fig. 4-7. γ_- , First Switching and Indifference Curves for A = 3

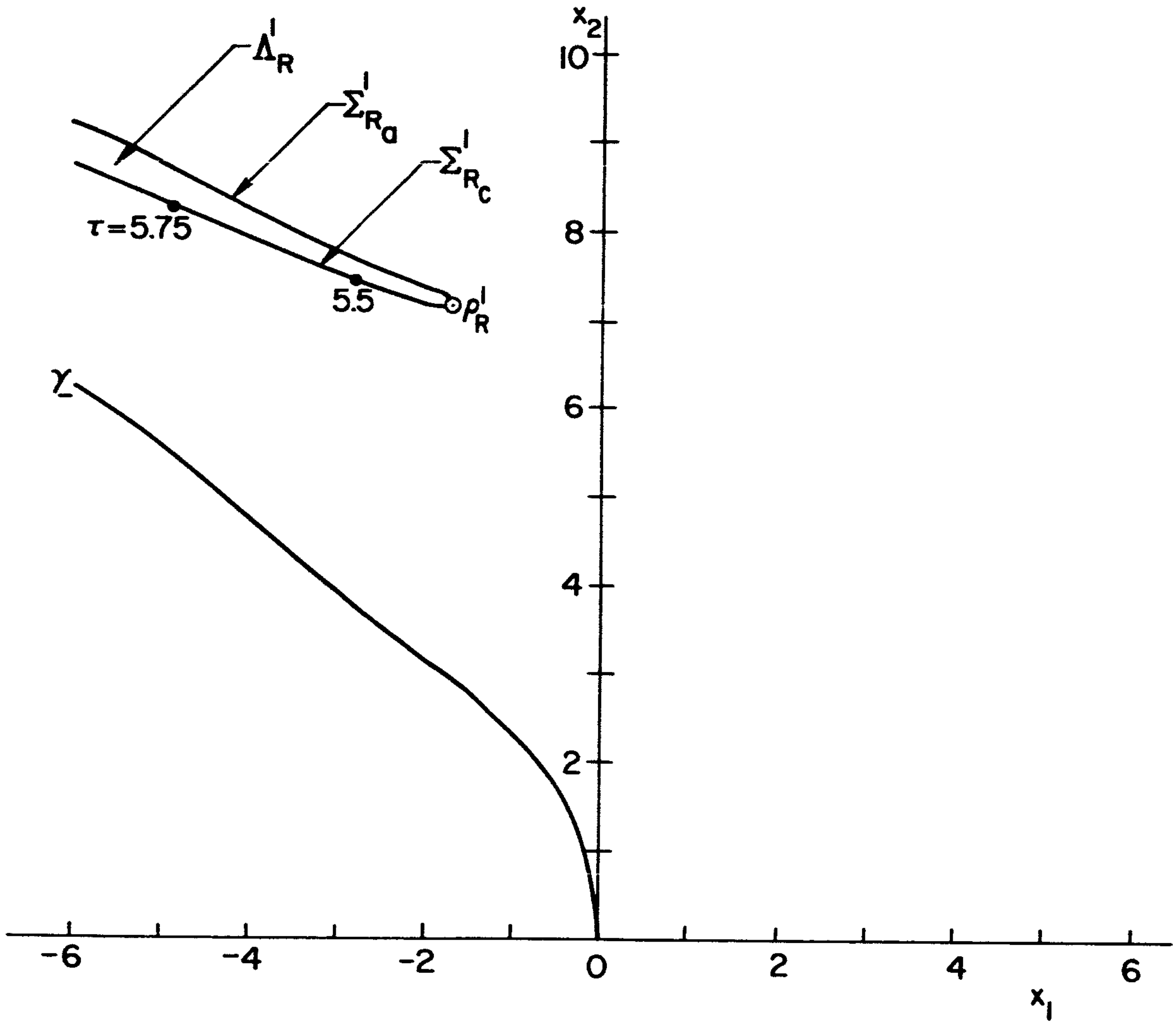

Fig. 4-8. γ_- , First Switching and Indifference Curves for A = 3.2

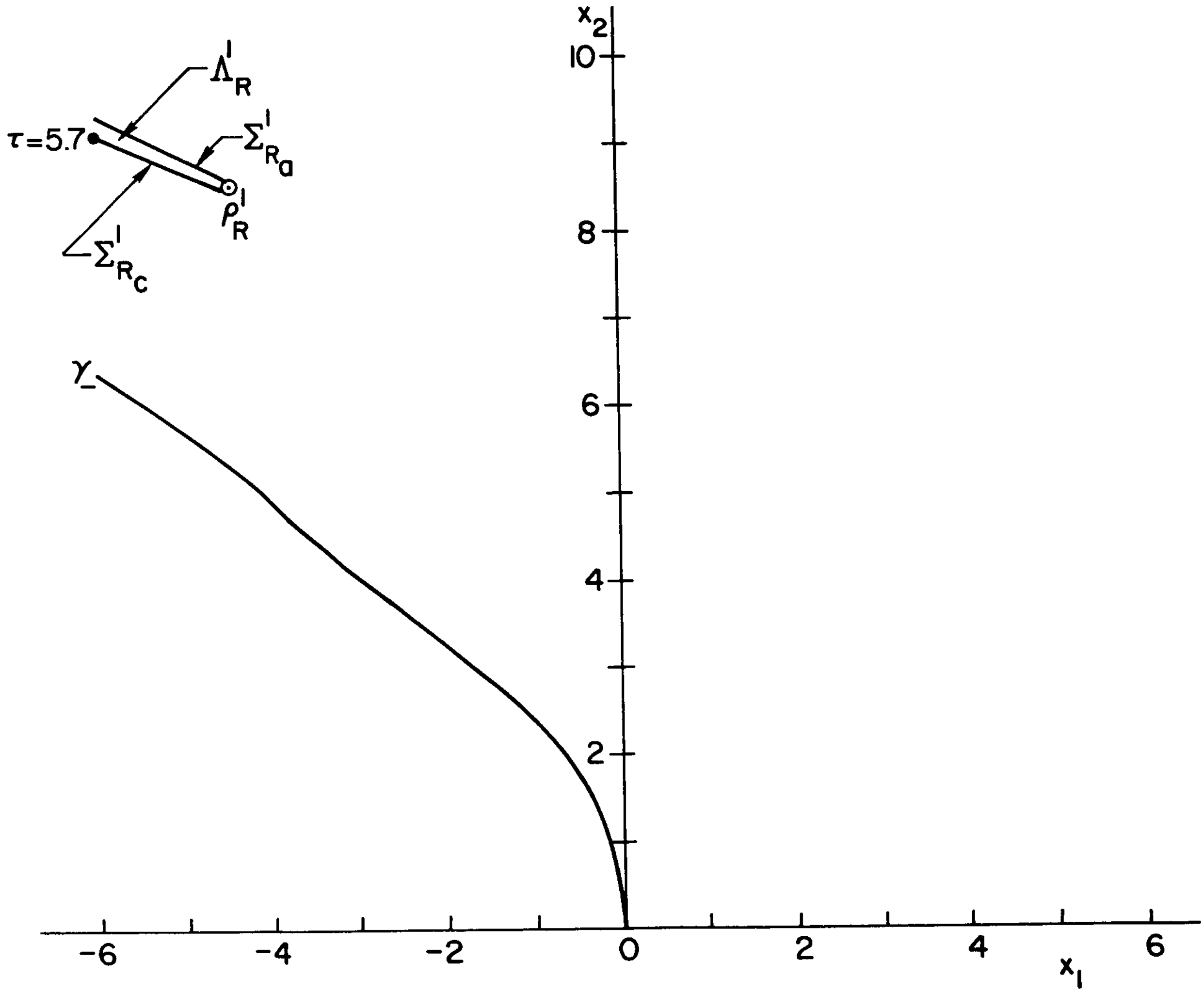

Fig. 4-9. γ_- , First Switching and Indifference Curves for $A = 3.4$

The main differences encountered between the two problems, are

i) Different number of switchings needed to zero any initial disturbance. While in the linear case, the number of switchings increases with the distance from the initial disturbance to the origin of the state plane, in the nonlinear case, the number of switchings cannot, in any case, be greater than two.

ii) The presence of the indifference curves $\Sigma^i_{R_c}$ and $\Sigma^i_{L_c}$, which do not appear in the linear case. It is worth noting too, that these indifference curves are also the locus of starting points, that is, points from where we can start a trajectory but which can never be reached on a trajectory.

iii) Big differences in the time needed to zero an initial disturbance. In Figs. 4-10 to 4-19, this time is given as a function of the initial state variable x_{10}, for different values of the other initial state variable x_{20} and the constant A. We notice that, in general, if $x_{20} > 0$ there are zones for which the nonlinear case allows a faster zeroing than the linear case, and other zones for which the zeroing is slower; also, if $x_{20} < 0$, the zeroing in the nonlinear case is in general faster than in the linear case.

Something must be said about the corners appearing in the curves corresponding to the nonlinear case. Let x_{2cr} be the value of x_2 such that the straight line $x_2 = x_{2cr}$ is tangent to the $\Sigma^i_{R_c}$ curve. Then, if $x_{20} > x_{2cr}$ we will have two corners in every interval $[2i\theta, 2(i+1)\theta)$ corresponding to the intersection points of the straight line $x_2 = x_{20}$ with the curves $\Sigma^i_{R_a}$ and $\Sigma^i_{R_c}$; on the other hand, if $x_{20} \leq x_{2cr}$ we will have only one corner corresponding to the intersection point of $x_2 = x_{20}$ with the γ_+ curve.

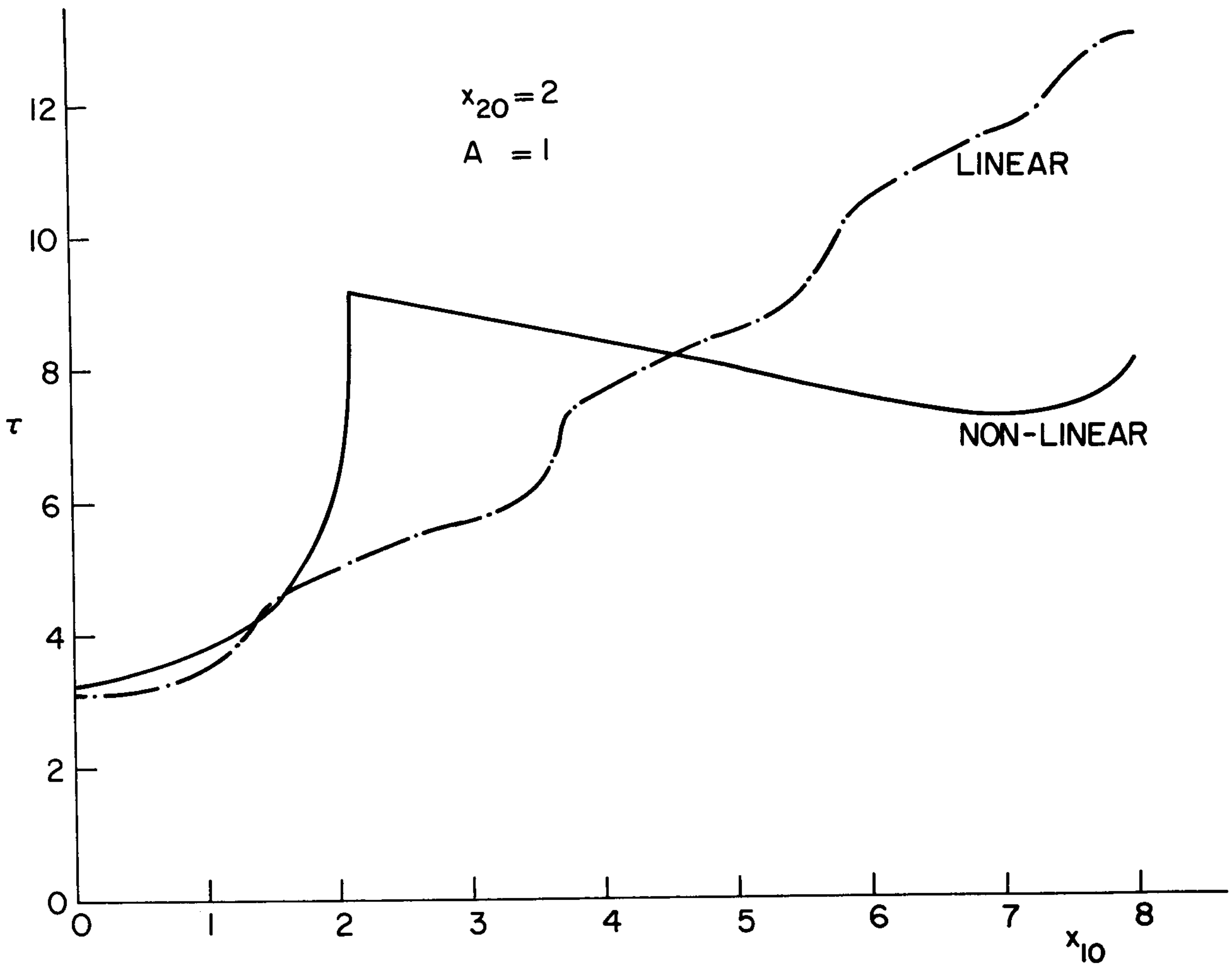

Fig. 4-10. Minimum Settling Time Versus x_{10}

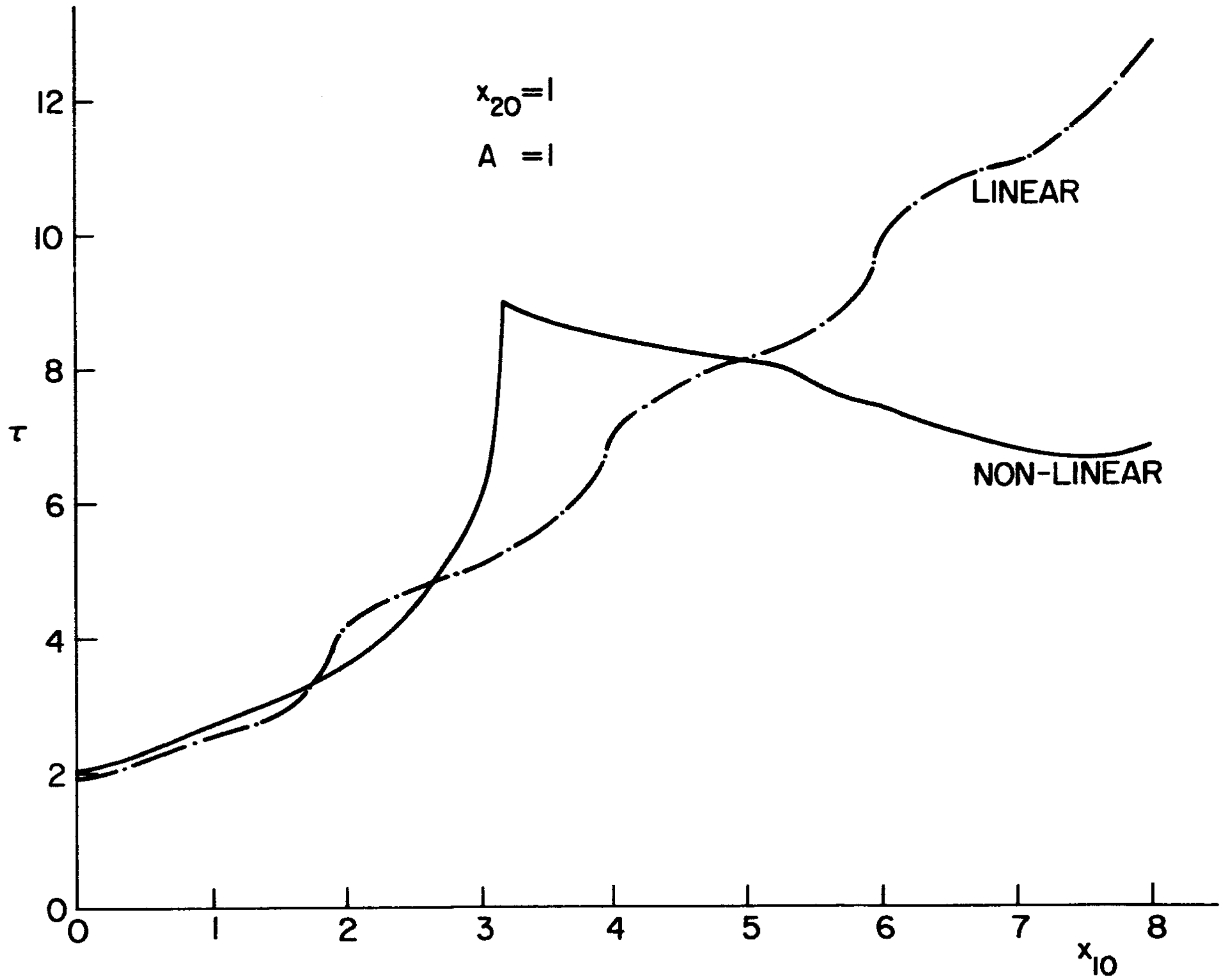

Fig. 4-11. Minimum Settling Time Versus x_{10}

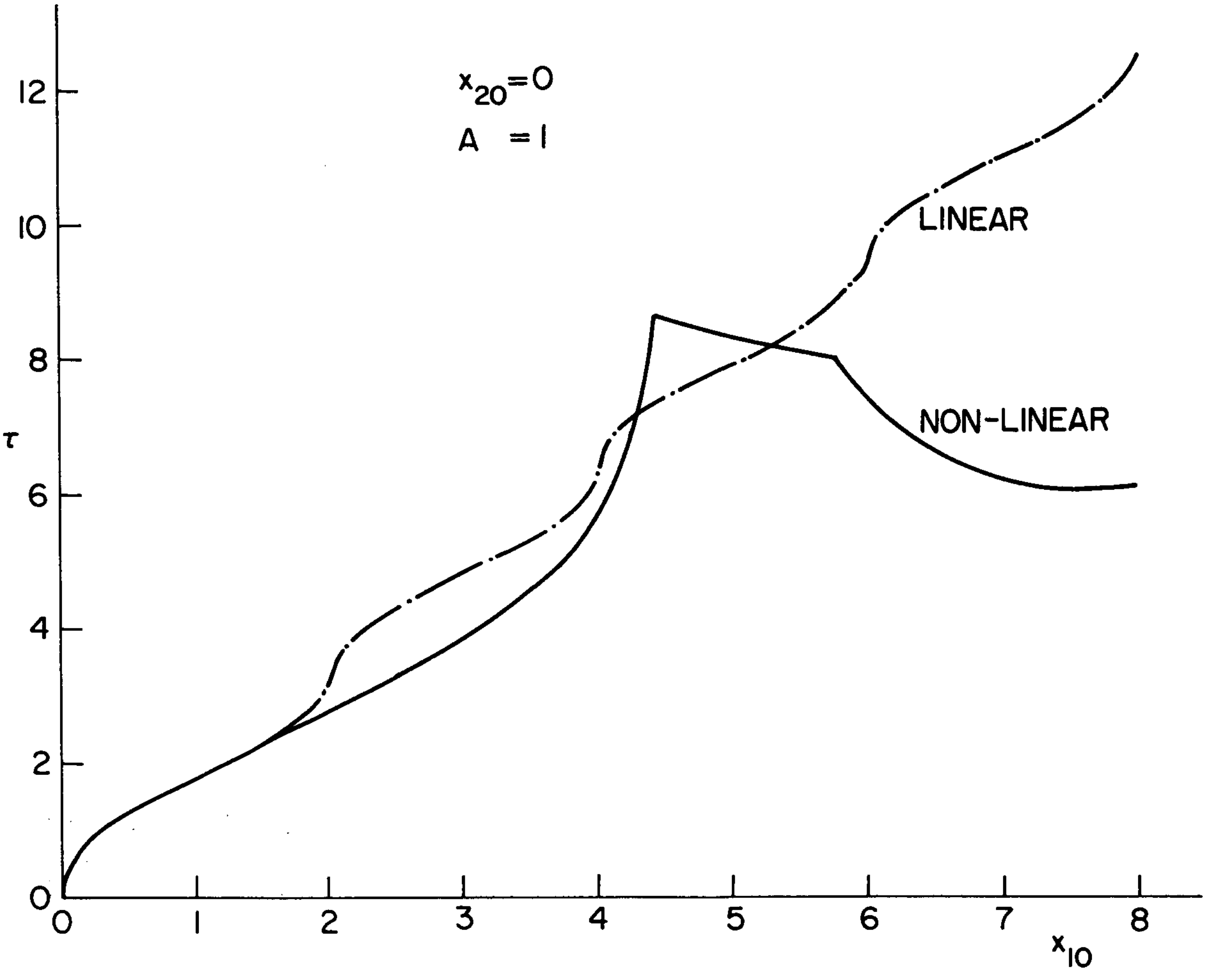

Fig. 4-12. Minimum Settling Time Versus x_{10}

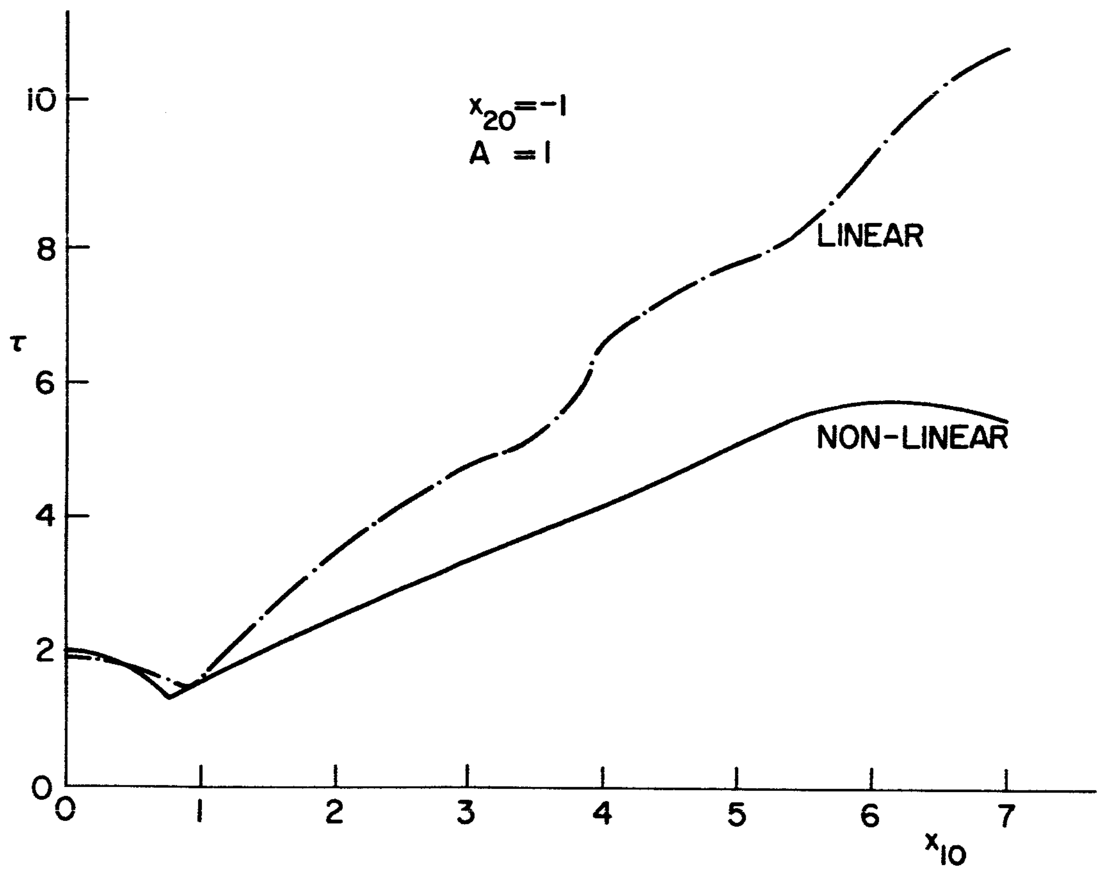

Fig. 4-13. Minimum Settling Time Versus x_{10}

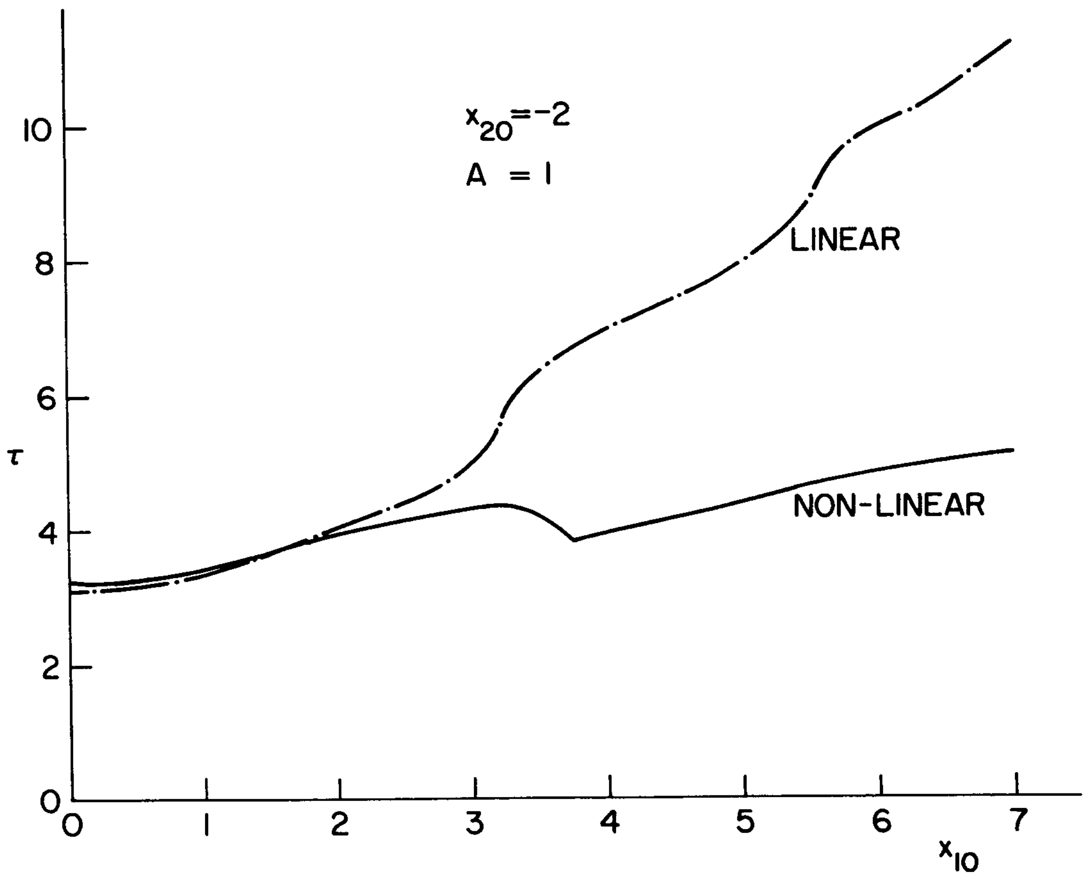

Fig. 4-14. Minimum Settling Time Versus x_{10}

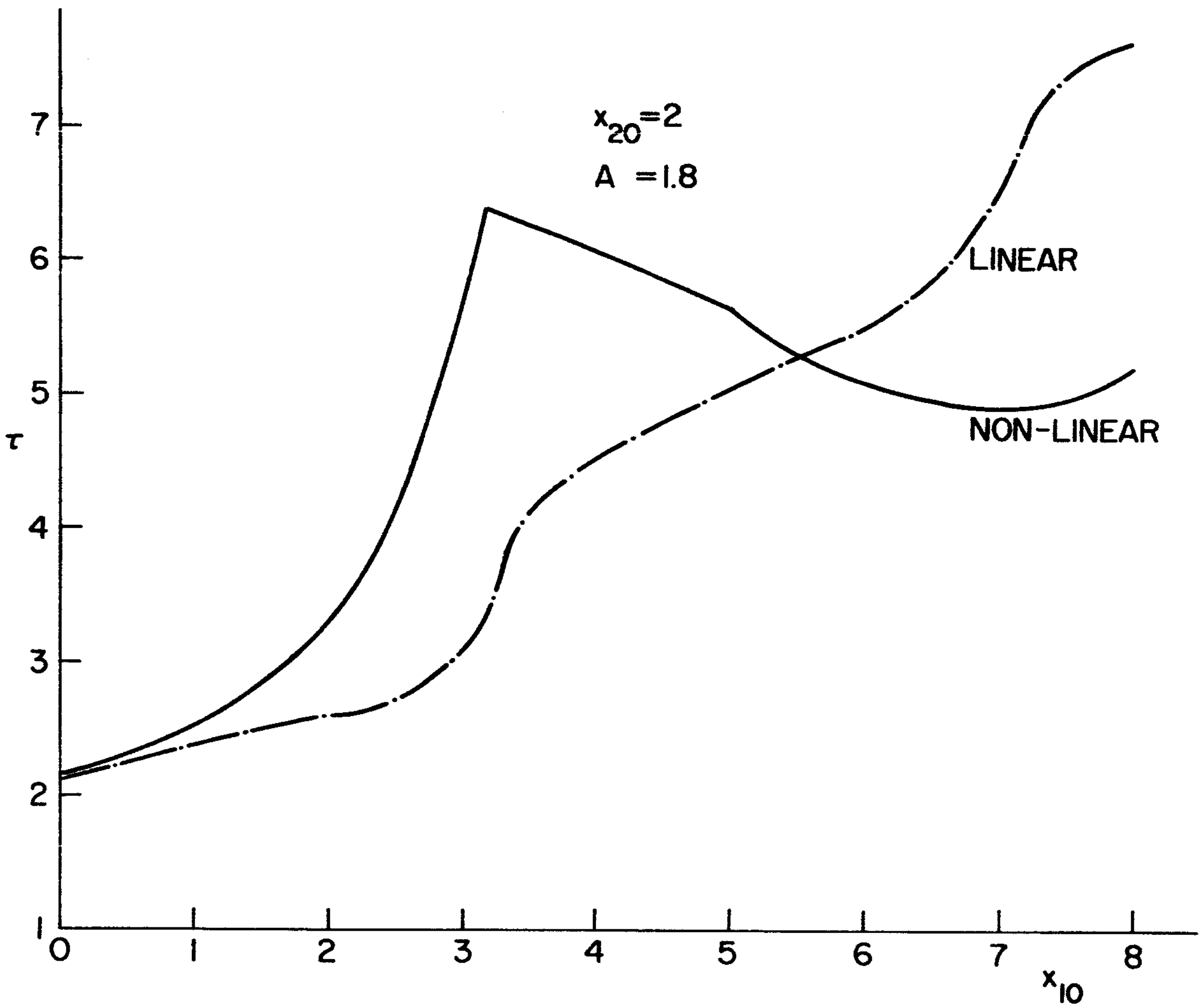

Fig. 4-15. Minimum Settling Time Versus x_{10}

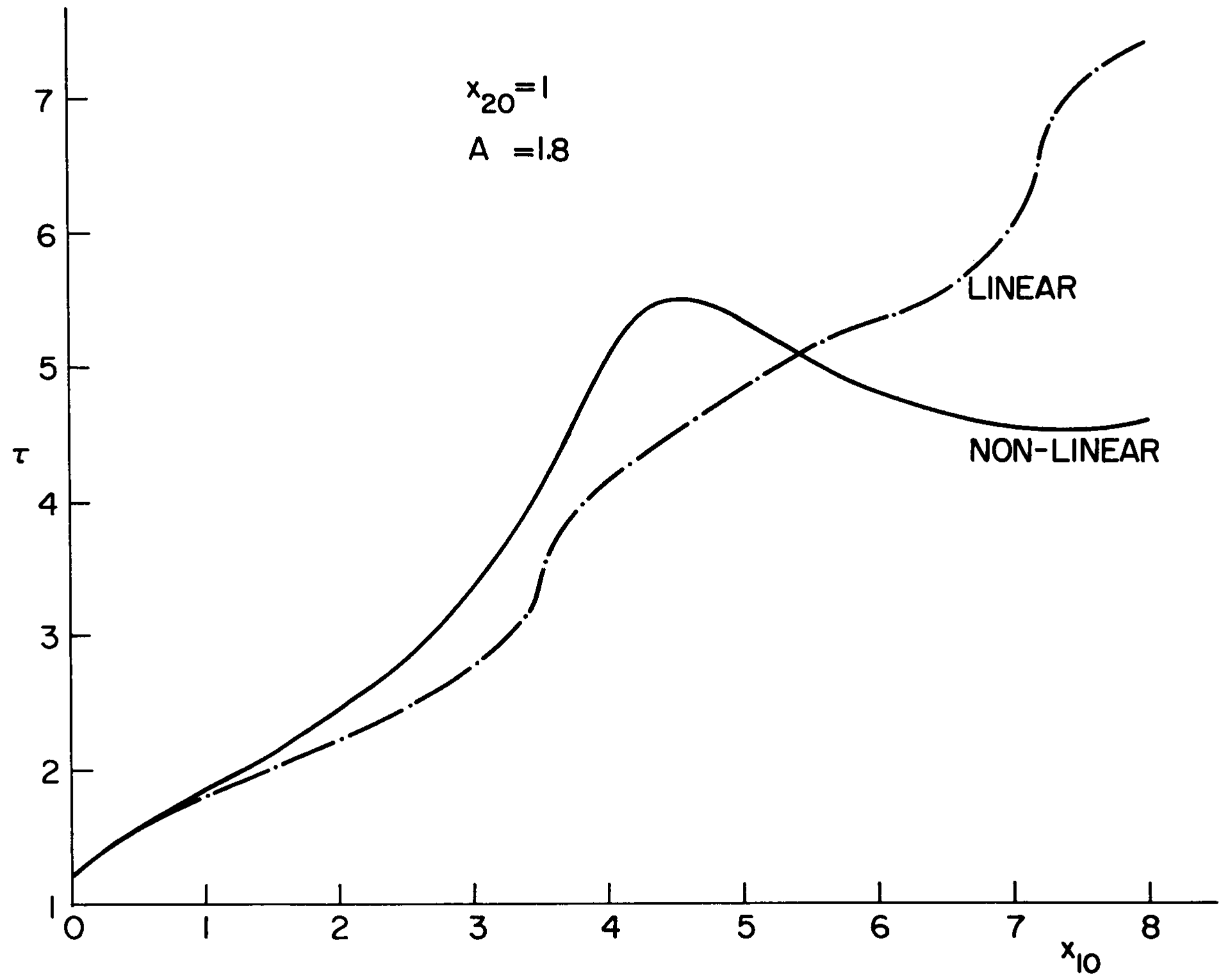

Fig. 4-16. Minimum Settling Time Versus x_{10}

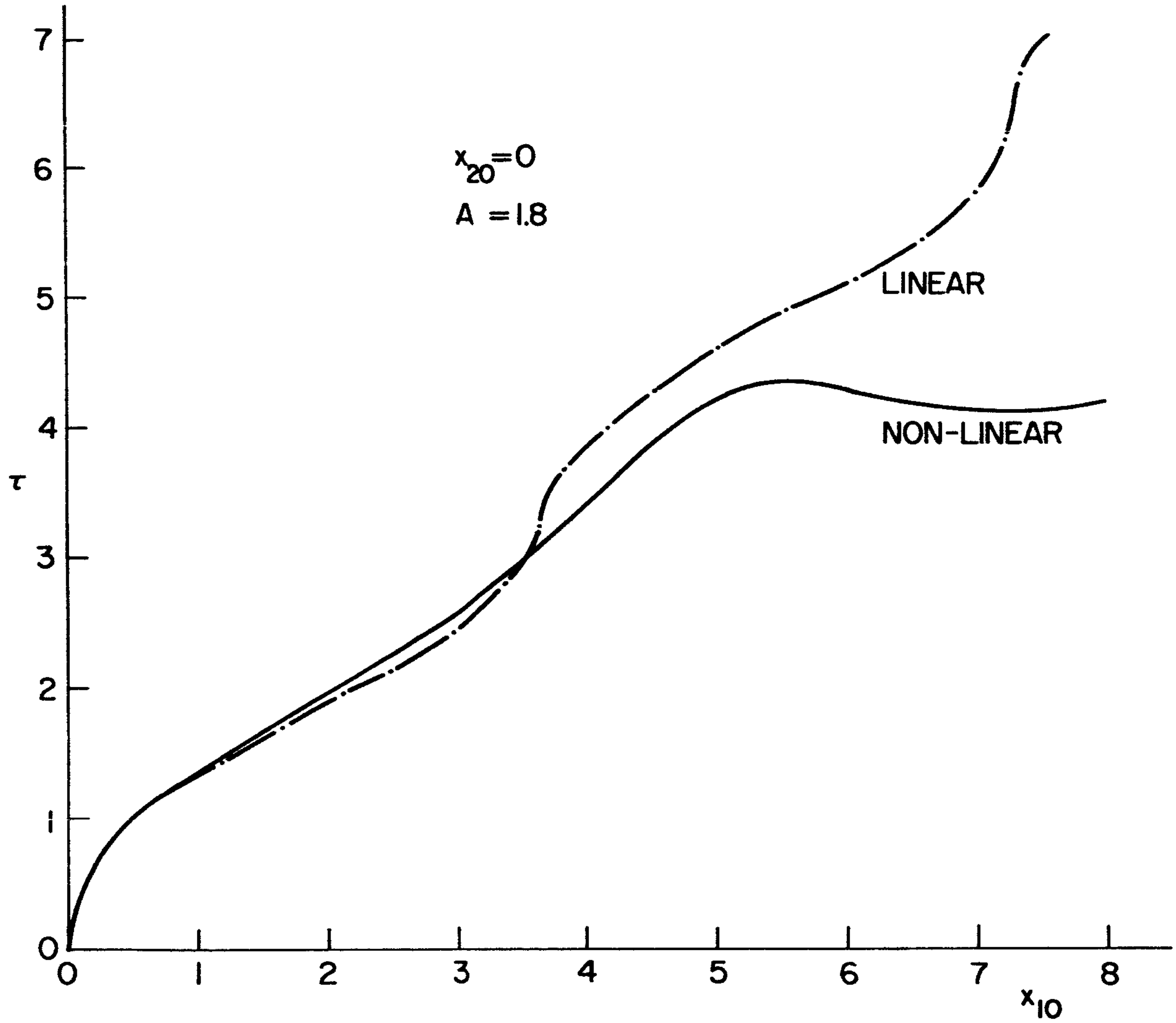

Fig. 4-17. Minimum Settling Time Versus x_{10}

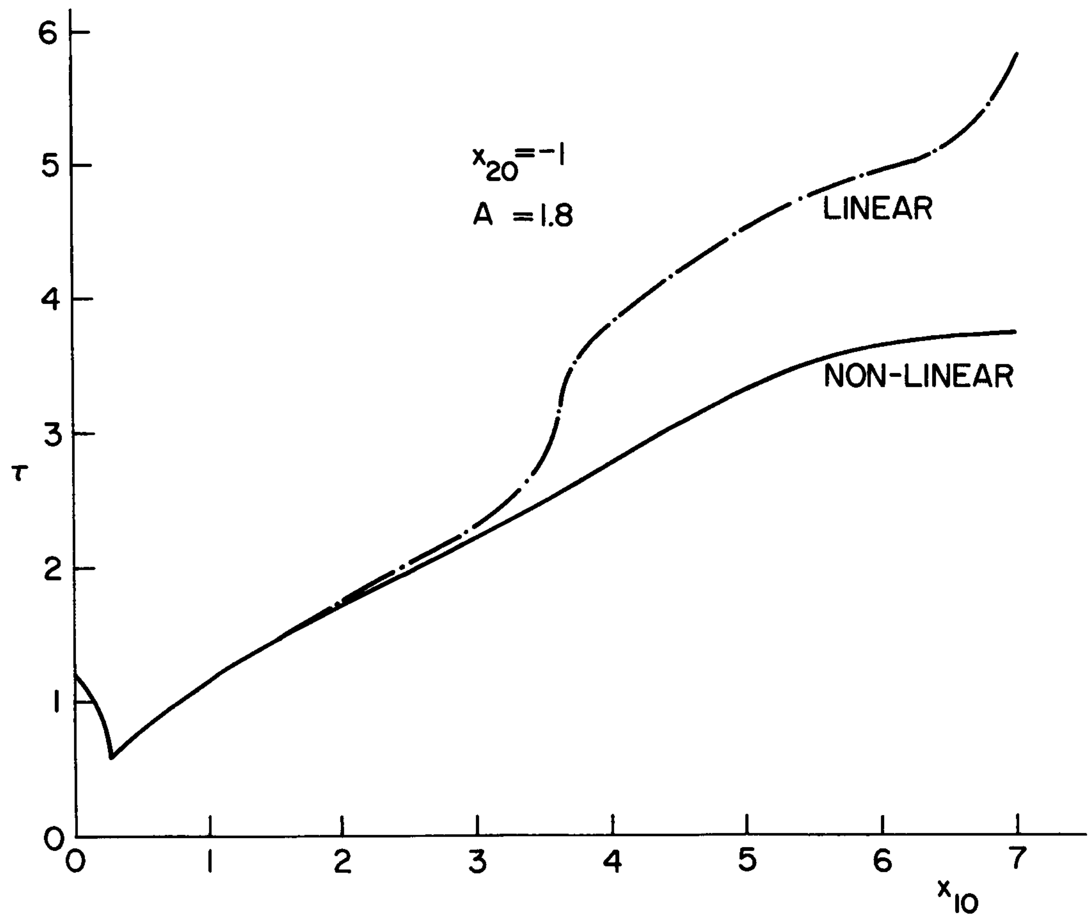

Fig. 4-18. Minimum Settling Time Versus x_{10}

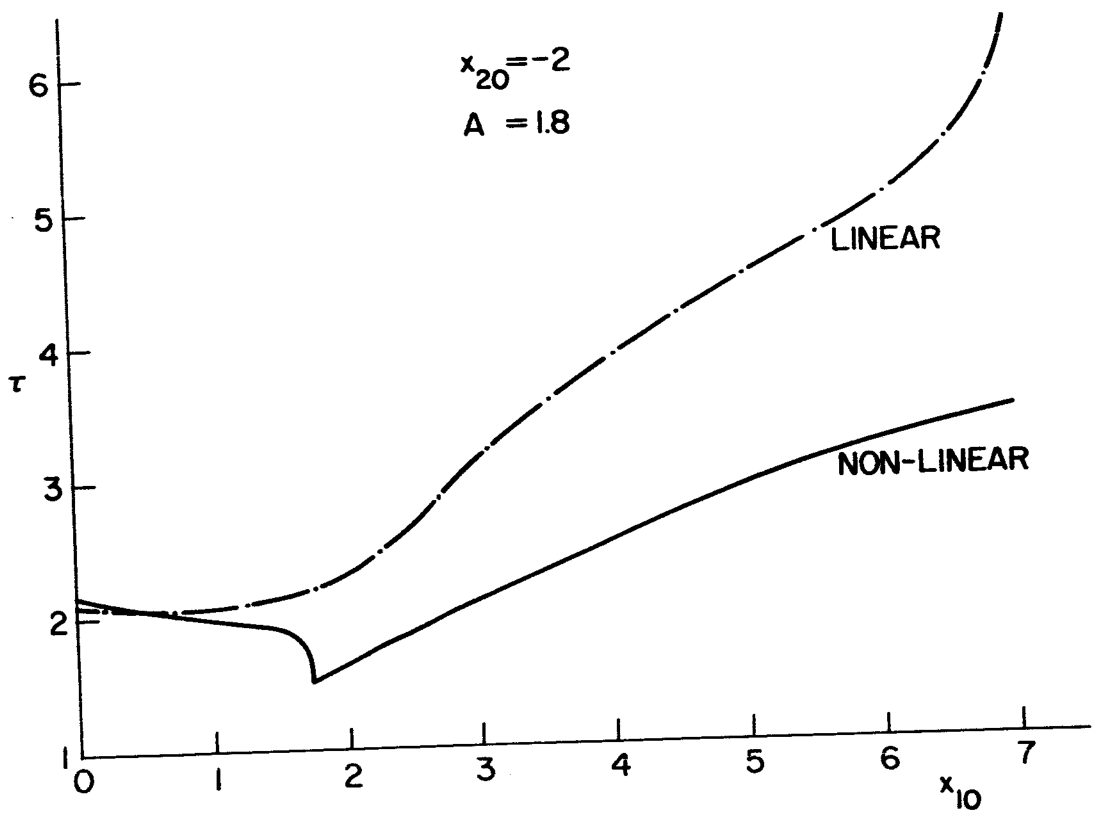

Fig. 4-19. Minimum Settling Time Versus x_{10}

CONCLUSION

This research concerns the problem of finding the minimum settling time control law for a system that can be represented by the second-order nonlinear differential equation $\ddot{x} + f(x) = u$, $|u| \leq A$, where $f(x)$ is a periodic function such that $|f(x)| \leq B$ and $B \leq C \leq A$.

In Chapter I, after an exposition of the control problem to be studied, the Maximum Principle of Pontryagin was used in order to get a necessary condition for the control $u(t)$ to be time-optimal. As was to be expected, the control law turned out to be a "bang-bang" control. Also, a theorem proved by Filipov was used in order to show the existence of the optimal control within the class of piecewise continuous functions.

In Chapter II, the adjoint variables $p_1(t)$ and $p_2(t)$ were found, as functions of the state variables and the initial conditions; $p_2(t)$ was obtained by solving a singular second-order differential equation, valid everywhere except when either the initial or the final point is on the x_1-axis, and using limiting processes for these two special cases. Then, the problem in backwards time was considered, by starting at the origin of the phase plane. The existence of possible switching curves, designated $\Sigma^i_{R_a}$, $\Sigma^i_{R_b}$, $\Sigma^i_{L_a}$ and $\Sigma^i_{L_b}$, was shown.

So far, the periodic function $f(x)$ was not subject to any restriction. However, in Chapter III it seemed advisable to restrict the investigation to the following two families of periodic funtions

1) Periodic functions which are at the same time antisymmetric.
2) Periodic functions that, without being antisymmetric, satisfy Lemma 3-3.

The existence of certain indifference curves $\Sigma^i_{R_c}$ and $\Sigma^i_{L_c}$ was shown. Together with the parts $\Sigma^i_{R_a}$ and $\Sigma^i_{L_a}$ of the switching curves found in Chapter II, they form the regions Λ^i_R and Λ^i_L ; any initial disturbance in these regions is brought to rest after two switchings, while any initial disturbance outside these regions is brought to rest after only one switching. This is the main result of this investigation, expressed in a formal way by the optimal control law given by Theorem 3-2. The indifference curves are also the loci of starting points.

In Chapter IV, the particular case $f(x) = \sin x$ was considered, and a comparison with the linear case, in which $\sin x$ is replaced by x , was made.

APPENDIX A

SOLUTION FOR THE ADJOINT VARIABLES

As was pointed out in Chapter II, the only singular point of the differential equation (2-3) is the point $M(x_{1m},0)$, intersection point of the P-curve with the x_1-axis. Then, by excluding a neighborhood of M , (2-3) can be transformed in the following way

$$0 = x_2^2 \frac{d^2p_2}{dx_1^2} + [A-f(x_1)] \frac{dp_2}{dx_1} + p_2 \frac{df}{dx_1} = x_2^2 \frac{d^2p_2}{dx_1^2} + 2[A-f(x_1)] \frac{dp_2}{dx_1} +$$

$$+ [f(x_1)-A] \frac{dp_2}{dx_1} + p_2 \frac{df}{dx_1} = \frac{d}{dx_1}\left[x_2^2 \frac{dp_2}{dx_1}\right] + \frac{d}{dx_1}\left[p_2 f(x_1)\right] - \frac{d}{dx_1}[Ap_2] =$$

$$= \frac{d}{dx_1}\left[x_2^2 \frac{dp_2}{dx_1} + p_2 f(x_1) - Ap_2\right]$$

Hence, integrating we get

$$x_2^2 \frac{dp_2}{dx_1} + p_2 f(x_1) - Ap_2 = c_1 \qquad \text{(A-1)}$$

Dividing (A-1) through by x_2^2 , we get

$$\frac{c_1}{x_2^2} = \frac{dp_2}{dx_1} + \frac{f(x_1)-A}{x_2^2} p_2 = \frac{dp_2}{dx_1} - \frac{p_2}{x_2^2}\frac{dx_2}{dt} = \frac{dp_2}{dx_1} - \frac{p_2}{x_2}\frac{dx_2}{dx_1}$$

or

$$\frac{c_1}{x_2^3} = \frac{1}{x_2}\frac{dp_2}{dx_1} - \frac{p_2}{x_2^2}\frac{dx_2}{dx_1} = \frac{d}{dx_1}\left[\frac{p_2}{x_2}\right]$$

Integrating

$$\frac{p_2}{x_2} = c_2 + c_1 \int_{x_{10}}^{x_1} \frac{d\sigma}{x_2^3(\sigma)}$$

Then

$$p_2(t) = x_2 \left[c_2 + c_1 \int_{x_{10}}^{x_1} \frac{d\sigma}{x_2^3(\sigma)} \right] \tag{A-2}$$

and

$$\dot{p}_2(t) = \frac{c_1}{x_2} + [A-f(x_1)] \left[c_2 + c_1 \int_{x_{10}}^{x_1} \frac{d\sigma}{x_2^3(\sigma)} \right] \tag{A-3}$$

From the initial conditions at $t = 0$, we get

$$p_2(0) = p_{20} = x_{20}c_2 \quad \text{or} \quad c_2 = \frac{p_{20}}{x_{20}} \tag{A-4}$$

$$\dot{p}_2(0) = - p_{10} = \frac{c_1}{x_{20}} + [A-f(x_{10})] \frac{p_{20}}{x_{20}} \quad \text{or}$$

$$c_1 = - p_{10}x_{20} - [A-f(x_{10})]p_{20} \tag{A-5}$$

Substituting (A-4) and (A-5) into (A-2), we get equations (2-4) and (2-5) as solutions of the differential equation (2-3), for the P-arcs in Fig. 2-1 and Fig. 2-2 respectively.

APPENDIX B

SOLUTION FOR THE ADJOINT VARIABLES WHEN EITHER THE INITIAL OR THE FINAL POINT IS ON THE x_1-AXIS

Initial Point on the x_1-Axis - Let us find the limit of equation (2-4). Looking at equation (2-4) we see that on one side p_{20}/x_{20} becomes infinite for $x_{20} \to 0$, and on the other side $\int_{x_{10}}^{x_1} [d\sigma/x_2^3(\sigma)]$ also becomes infinite because the integrand does it. Then, we are going to transform the integral in such a way that its principal part cancels the corresponding principal part of p_{20}/x_{20}, the rest being a finite quantity, for any finite x_1, when we take limits. This is done by substracting from the integrand its principal part, and, in order to leave the equation unchanged, adding the same quantity and integrating it.

Near the singular point $M(x_{1m},0)$ we have the following series expansions:

$$f(x_1) = f(x_{1m}) + f'(x_{1m})(x_1-x_{1m}) + o[(x_1-x_{1m})^2]$$

$$F(x_1) = F(x_{1m}) + (x_1-x_{1m})\, f(x_{1m}) + o[(x_1-x_{1m})^2]$$

$$\frac{x_2^2}{2} = A(x_1-x_{1m}) - F(x_1) + F(x_{1m}) = [A-f(x_{1m})]\,(x_1-x_{1m}) + o[(x_1-x_{1m})^2]$$

$$x_2 = \left\{2[A-f(x_{1m})]\,(x_1-x_{1m})\right\}^{1/2} + o[(x_1-x_{1m})^{3/2}]$$

$$\frac{1}{x_2} = \left\{2[A-f(x_{1m})]\,(x_1-x_{1m})\right\}^{-1/2} + o[(x_1-x_{1m})^{1/2}]$$

$$\frac{1}{x_2^3} = \left\{2[A-f(x_{1m})]\,(x_1-x_{1m})\right\}^{-3/2} + 3f'(x_{1m})\left\{2[A-f(x_{1m})]\right\}^{-5/2}(x_1-x_{1m})^{-1/2} +$$
$$+ o[(x_1-x_{1m})^{1/2}]$$

Then

$$\int_{x_{10}}^{x_1} \frac{d\sigma}{x_2^3(\sigma)} = \int_{x_{10}}^{x_1} \left[\frac{1}{x_2^3(\sigma)} - \frac{1}{\left\{2[A-f(x_{1m})](\sigma-x_{1m})\right\}^{3/2}} \right] d\sigma +$$

$$+ \left\{2[A-f(x_{1m})]\right\}^{-3/2} \int_{x_{10}}^{x_1} \frac{d\sigma}{(\sigma-x_{1m})^{3/2}} =$$

$$= \int_{x_{10}}^{x_1} \left[\frac{1}{x_2^3(\sigma)} - \frac{1}{\left\{2[A-f(x_{1m})](\sigma-x_{1m})\right\}^{3/2}} \right] d\sigma -$$

$$- \frac{1}{\sqrt{2}\,[A-f(x_{1m})]^{3/2}} \left[\frac{1}{(x_1-x_{1m})^{1/2}} - \frac{1}{(x_{10}-x_{1m})^{1/2}} \right] \quad \text{(B-1)}$$

Also

$$\frac{1}{x_{20}} = \left\{2[A-f(x_{1m})]\ (x_{10}-x_{1m})\right\}^{-1/2} + o[(x_{10}-x_{1m})^{1/2}] \quad \text{(B-2)}$$

Substituting (B-1) and (B-2) into (2-4), we get

$$p_2(t) = x_2 \left\{ - p_{10}x_{20} \int_{x_{10}}^{x_1} \frac{d\sigma}{x_2^3(\sigma)} + \frac{p_{20}[A-f(x_{10})]}{[2(x_1-x_{1m})]^{1/2}[A-f(x_{1m})]^{3/2}} + \right.$$

$$+ \frac{p_{20}}{\left\{2[A-f(x_{1m})](x_{10}-x_{1m})\right\}^{1/2}} \left[1 - \frac{A-f(x_{10})}{A-f(x_{1m})} \right] + o[(x_{10}-x_{1m})^{1/2}] -$$

$$\left. - p_{20}[A-f(x_{10})] \int_{x_{10}}^{x_1} \left[\frac{1}{x_2^3(\sigma)} - \frac{1}{\left\{2[A-f(x_{1m})](\sigma-x_{1m})\right\}^{3/2}} \right] d\sigma \right\}$$

(B-3)

Applying l'Hopital rule we obtain

$$\lim_{P_0 \to M} \left\{ \frac{p_{20}}{\left\{2[A-f(x_{1m})](x_{10}-x_{1m})\right\}^{1/2}} \left[1 - \frac{A-f(x_{10})}{A-f(x_{1m})}\right] \right\} =$$

$$= \lim_{P_0 \to M} \left[\frac{\sqrt{2}\, p_{20}}{[A-f(x_{1m})]^{3/2}} \cdot \frac{df(x_{10})}{dx_{10}} (x_{10}-x_{1m})^{1/2} \right] = 0 \qquad \text{(B-4)}$$

and

$$\lim_{P_0 \to M} \left[x_{20} \int_{x_{10}}^{x_1} \frac{d\sigma}{x_2^3(\sigma)} \right] = \lim_{P_0 \to M} \frac{x_{20}}{\left[\int_{x_{10}}^{x_1} \frac{d\sigma}{x_2^3(\sigma)} \right]^{-1}} =$$

$$= [A-f(x_{1m})] \left\{ \lim_{P_0 \to M} \left[x_{20} \int_{x_{10}}^{x_1} \frac{d\sigma}{x_2^3(\sigma)} \right] \right\}^2$$

which yield

$$\lim_{P_0 \to M} \left[x_{20} \int_{x_{10}}^{x_1} \frac{d\sigma}{x_2^3(\sigma)} \right] = \frac{1}{A-f(x_{1m})} \qquad \text{(B-5)}$$

Taking limits on (B-3), after substitution of (B-4) and (B-5), we get equation (2-6), where now the integral is finite for any finite x_1 .

Now, let us find $p_1(t)$. From (A-2) and (A-3), we get

$$p_1(t) = - \dot{p}_2(t) = - \frac{c_1}{x_2} - [A-f(x_1)] \frac{p_2(t)}{x_2}$$

and taking limits

$$p_1(t) = -\frac{1}{x_2}\left[\lim_{P_0\to M} c_1\right] - \frac{A-f(x_1)}{x_2}\left[\lim_{P_0\to M} p_2(t)\right] =$$

$$= \frac{p_{20}}{x_2}[A-f(x_{1m})] - [A-f(x_1)]\frac{p_2(t)}{x_2} \qquad \text{(B-6)}$$

Substituting (2-6) into (B-6), we get equation (2-7).

Final Point on the x_1-Axis - Let us find the limit of equation (2-5). Let us first find $p_2(t)$. Since in this case it is P instead of P_0 the one which approaches $M(x_{1m},0)$, going through the same procedure as before, we get, instead of equation (B-5) the following equation

$$\lim_{P\to M}\left[x_2\int_{x_{10}}^{x_1}\frac{d\sigma}{x_2^3(\sigma)}\right] = -\frac{1}{A-f(x_{1m})}$$

and taking limits on equation (A-2)

$$p_2(t) = \lim_{P\to M}(c_2x_2) + \lim_{P\to M}\left[c_1x_2\int_{x_{10}}^{x_1}\frac{d\sigma}{x_2^3(\sigma)}\right] = -\frac{c_1}{A-f(x_{1m})}$$

Substituting the value of c_1 given by (A-5), we get equation (2-8).

Let us find $p_1(t)$. Near the singular point $M(x_{1m},0)$, we have

$$\frac{1}{x_2} = -\left|2[A-f(x_{1m})](x_1-x_{1m})\right|^{-1/2} + o[(x_1-x_{1m})^{1/2}] \qquad \text{(B-7)}$$

$$\frac{1}{x_2^3} = -\left|2[A-f(x_{1m})](x_1-x_{1m})\right|^{-3/2} - 3f'(x_{1m})\left|2[A-f(x_{1m})]\right|^{-5/2}(x_1-x_{1m})^{-1/2} +$$

$$+ o[(x_1-x_{1m})^{1/2}]$$

Then

$$\int_{x_{10}}^{x_1} \frac{d\sigma}{x_2^3(\sigma)} = \int_{x_{10}}^{x_1} \left[\frac{1}{x_2^3(\sigma)} + \frac{1}{\left\{2[A-f(x_{1m})](\sigma - x_{1m})\right\}^{3/2}} \right] d\sigma - \left\{2[A-f(x_{1m})]\right\}^{-3/2} \cdot$$

$$\cdot \int_{x_{10}}^{x_1} \frac{d\sigma}{(\sigma - x_{1m})^{3/2}} = \int_{x_{10}}^{x_1} \left[\frac{1}{x_2^3(\sigma)} + \frac{1}{\left\{2[A-f(x_{1m})](\sigma - x_{1m})\right\}^{3/2}} \right] d\sigma -$$

$$- \frac{1}{\sqrt{2}} [A-f(x_{1m})]^{-3/2} \left[- \frac{1}{(x_1 - x_{1m})^{1/2}} + \frac{1}{(x_{10} - x_{1m})^{1/2}} \right] \tag{B-8}$$

Substituting (B-7) and (B-8) into (A-3), we get

$$p_1(t) = - \dot{p}_2(t) = - \frac{c_1}{x_2} - [A-f(x_1)] \left[c_2 + c_1 \int_{x_{10}}^{x_1} \frac{d\sigma}{x_2^3(\sigma)} \right] =$$

$$= \frac{c_1}{\left\{2[A-f(x_{1m})](x_1 - x_{1m})\right\}^{1/2}} - c_2 [A-f(x_1)] - c_1 [A-f(x_1)] \cdot$$

$$\cdot \int_{x_{10}}^{x_1} \left[\frac{1}{x_2^3(\sigma)} + \frac{1}{\left\{2[A-f(x_{1m})](\sigma - x_{1m})\right\}^{3/2}} \right] d\sigma + \frac{c_1 [A-f(x_1)]}{\sqrt{2}\,[A-f(x_{1m})]^{3/2}} \cdot$$

$$\cdot \left[- \frac{1}{(x_1 - x_{1m})^{1/2}} + \frac{1}{(x_{10} - x_{1n})^{1/2}} \right] + o[(x_1 - x_{1m})^{1/2}]$$

and taking limits

$$p_1(t) = - \dot{p}_2(t) = - \frac{p_{20}}{x_{20}} [A-f(x_{1m})] + c_1[A-f(x_{1m})] \int_{x_{1m}}^{x_{10}} \left[\frac{1}{x_2^3(\sigma)} + \right.$$

$$\left. + \frac{1}{\left\{2[A-f(x_{1m})](\sigma-x_{1m})\right\}^{3/2}} \right] d\sigma + \frac{c_1}{\left\{2[A-f(x_{1m})](x_{10}-x_{1m})\right\}^{1/2}} \tag{B-9}$$

Substituting the value of c_1 given by (A-5) into (B-9), we obtain equation (2-9).

REFERENCES

[1] Doll, H. G., U.S. Patent No. 2,463,362, 1943.

[2] McDonald, D. C., "Nonlinear Techniques for Improving Servo Performance," Proceedings, National Electronics Conference, Chicago, Ill., Vol. 6, 1950, pp. 400-421.

[3] Hopkin, A. M., "A Phase-Plane Approach to the Design of Saturating Servomechanisms," AIEE Transactions, Vol. 70, pt. I, 1951, pp. 631-639.

[4] Bushaw, D. W., Unpublished Ph.D. dissertation, Department of Mathematics, Princeton University, 1952. Also, "Differential Equations with a Discontinuous Forcing Term," Report No. 469, Experimental Towing Tank, Stevens Institute of Technology, Hoboken, N. J., January, 1953. Also, "Optimal Discontinuous Forcing Terms," Contributions to the Theory of Nonlinear Oscillations," Vol. 4, Princeton University Press, Princeton, N. J., 1958, pp. 29-52.

[5] Pontryagin, L. S., Boltyanskii, V. G., Gamkrelidze, R. V. and Mishchenko, E. F., "The Mathematical Theory of Optimal Processes," Interscience Publications, John Wiley and Sons, New York, N. Y., 1962.

[6] Flügge-Lotz, I. and Halkin, H., "Pontryagin's Maximum Principle and Optimal Control," Technical Report No. 130, Division of Engineering Mechanics, Stanford University, Stanford, California, September, 1961.

[7] Kalman, R. E., "The Theory of Optimal Control and the Calculus of Variations," RIAS Technical Report 6-13, Research Institute for Advanced Studies (RIAS), Baltimore, 1961. Also, Mathematical Optimization Techniques, edited by R. Bellman, 1963.

[8] Lasalle, J. P., "The Time Optimal Control Problem," Contributions to the Theory of Nonlinear Oscillations, Vol. 5, Princeton University Press, Princeton, N. J., 1960, pp. 1-24.

[9] Rozonoer, L. I., "L. S. Pontryagin's Maximum Principle in the Theory of Optimal Systems," Automation and Remote Control, Vol. 20, Oct., Nov., Dec., 1959, pp. 1288-1302, 1405-1421, 1517-1532.

[10] Athans, M., "Minimum Fuel Feedback Control Systems: Second Order Case," IEEE Trans. on Applications and Industry, Vol. 82, March, 1963, pp. 8-17.

[11] Athans, M., "Minimum Fuel Control of Second Order Systems with Real Poles," Proc. 1963 Joint Automatic Control Conference, June, 1963, pp. 232-240.

[12] Athans, M., Falb, P. L. and Lacoss, R. T., "On Optimal Control of Self Adjoint Systems," Proc. 1963 Joint Automatic Control Conference, June, 1963, pp. 113-120.

[13] Flügge-Lotz, I. and Marbach, H., "The Optimal Control of Some Attitude Control Systems for Different Performance Criteria," SUDAER No. 131, Department of Aeronautics and Astronautics, Stanford University, Stanford, California, June, 1962. Also, Trans. ASME, Journal of Basic Engineering, June, 1963, pp. 165-176.

[14] Ladd, H. O. and Friedland, B., "Minimum Fuel Control of a Second Order Linear Process with a Constraint on Time to Run," Proc. 1963 Joint Automatic Control Conference, June, 1963, pp. 241-246.

[15] Meditch, J. S., "On Minimal Fuel Satellite Attitude Controls," Proc. 1963 Joint Automatic Control Conference, June, 1963, pp. 558-564.

[16] Athans, M. and Canon, M. D., "Fuel-Optimal Singular Control of a Nonlinear Second Order System," Report No. MS-995, Lincoln Laboratory, Massachussets Institute of Technology, Lexington, Massachussets, November 29, 1963.

[17] Flügge-Lotz, I., "Discontinous Automatic Control," Princeton University Press, Princeton, N. J., 1953.

[18] Coddington, E. A. and Levinson, N., "Theory of Ordinary Differential Equations," McGraw-Hill, New York, N. Y., 1955.

[19] Filipov, A. F., "On Certain Questions on the Theory of Optimal Control," Journal SIAM Control, Ser. A, Vol. 1, No. 1, 1962, pp. 76- 84.

[20] Lee, E. B. and Markus, L., "Optimal Control for Nonlinear Processes," Archive for Rational Mechanics and Analysis, Vol. 8, 1961, pp. 36-58.

[21] Roxin, E., "The Existence of Optimal Controls," Michigan Mathematical Journal, Vol. 9, 1962, pp. 109-119.

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ADDENDUM TO SUDAAR 271

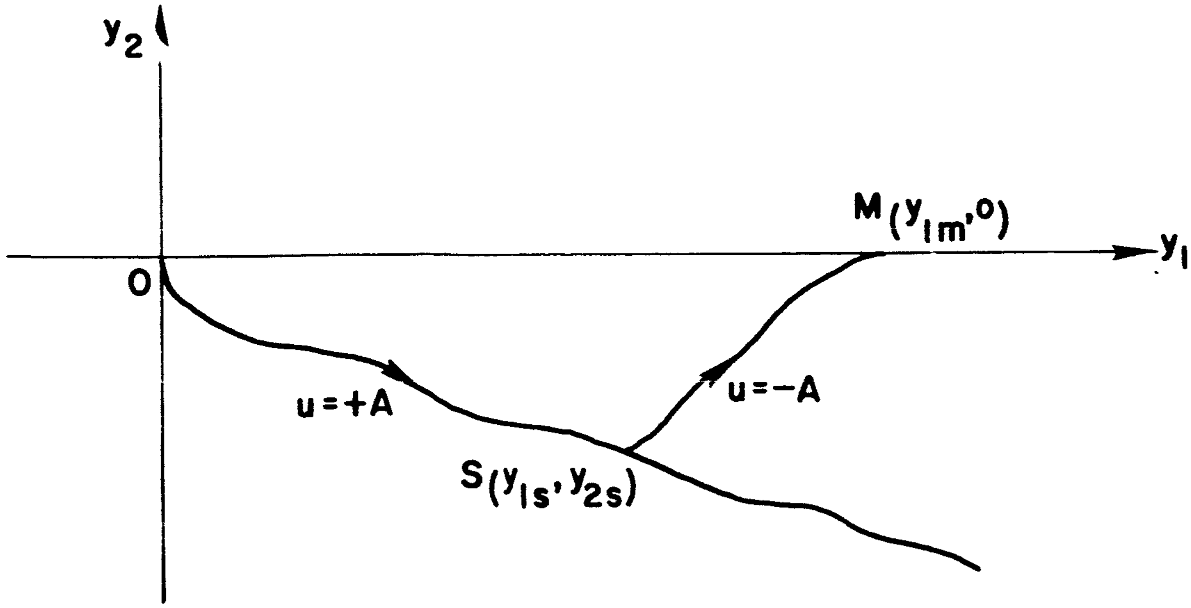
Correspondence has shown that some readers would welcome details about the special case in which $A = B$. Therefore the authors have prepared this addendum to SUDAAR 271.

Some words must be said about a special case that was not treated completely in this report.

Condition iii) of page 5 states that $A \geq C$, where C is any constant such that $C \geq B$; then, we admit the possibility of being $A = B$. Besides, since $|f(x)| \leq B$, and assuming that the function $f(x)$ is not a constant function, there is the possibility of being $|f(x)| = B = A$, which would imply that the expressions (2-6), (2-7), (2-8) and (2-9) will become meaningless when $f(x_{1m}) = A$, and the same will happen to the expressions (2-13), (2-14), (2-15) and (2-16) when $f(x_{1m}) = -A$. Therefore, we see that the special case $A = B$ should be given special attention. So, our aim will be to follow the same steps as in the report and indicate the changes that should be made in order to get for the special case $A = B$ the same conclusions as for the general case $A > B$.

First of all, it should be pointed out that in the special case $A = B$ the difficulties arise from the fact that the points on the x_1-axis for which $f(x_{1m}) = +A$ are not only singular points of the equations (2-3), but also critical points of the system (1-5) with $u = +A$. Notice that, in a similar way, the points on the x_1-axis for which $f(x_{1m}) = -A$ are critical points of the system (1-5) with $u = -A$.

Then, let us consider a typical trajectory in backwards time, starting with $u = +A$, as sketched in the following figure:



Sketch 1

If the point M is such that $f(y_{1m}) = -A$, i.e., M is a critical point of the system (1-5) with $u = -A$, it will also be $f'(y_{1m}) = 0$, and near the point M we will have the following series expansions:

$$f(y_1) = f(y_{1m}) + \frac{1}{2} f''(y_{1m})(y_1-y_{1m})^2 + O[(y_1-y_{1m})^3]$$

$$F(y_1) = F(y_{1m}) + f(y_{1m})(y_1-y_{1m}) + \frac{1}{6} f''(y_{1m})(y_1-y_{1m})^3 + O[(y_1-y_{1m})^4]$$

$$\frac{y_2^2}{2} = - A(y_1-y_{1m}) - F(y_1) + F(y_{1m}) = - A(y_1-y_{1m}) - f(y_{1m})(y_1-y_{1m})$$

$$- \frac{1}{6} f''(y_{1m})(y_1-y_{1m})^3 + O[(y_1-y_{1m})^4] = + \frac{1}{6} f''(y_{1m})(y_{1m}-y_1)^3$$

$$+ O [(y_{1m}-y_1)^4]$$

$$\frac{1}{y_2} = - \{\frac{1}{3} f''(y_{1m})(y_{1m}-y_1)\}^{-3/2} + O[(y_{1m}-y_1)^{-1/2}]$$

and the time spent from S to M, it is given by

$$\tau(SM) = \int_{y_{1s}}^{y_{1m}} -(1/y_2)\, dy_1 = \left\{ 2\left[\frac{1}{3} f''(y_{1m})\right]^{-3/2} (y_{1m}-y_1)^{-1/2} + \right.$$

$$\left. + 0\left[(y_{1m}-y_1)^{1/2}\right]\right\}\Bigg|_{y_1=y_{1s}}^{y_1=y_{1m}} = \infty$$

which means that we will never be able to reach the point M. This would imply that, when considering the problem in forward time, if our initial state were the point M we would never be able to start a trajectory with an initial $u = -A$; however, since M is a critical point of the system (1-5) only if $u = -A$, but not if $u = +A$, we could always start a trajectory with an initial value of $u = +A$. As we shall see later, this will imply that the critical points must be inside the regions Λ_R^i, if we want to have for $A = B$ the same optimal control law as for the general case $A > B$.

Keeping the above in mind, let us define the function $G_R(y_1, y_{1m})$ in the same way as we did in the report, i.e., by the equation (2-37), with the only difference that now the function G_R will be defined for all y_{1m} except for those corresponding to critical points of the system (1-5) with $u = -A$.

The next step, will be to show the existence of the Possible Switching Curves. It should be noticed that the point $M(y_{1m}, 0)$ for which $y_{1m} = \beta_R^i$ is a critical point of the system (1-5) with $u = -A$.

Lemma 2-1 (page 35) remains true for any y_{1m} except for those corresponding to critical points. Lemma 2-3 (page 39) remains also true.

Lemma 2-2 (page 38) must be modified because $(\beta_R^i, 0)$ is a critical point. This modification is easily done if we consider the point $y_{1m} = \beta_R^i + \epsilon$ instead of the point $y_{1m} = \beta_R^i$. For the new elected point, the inequality (2-43) will be true for any value of σ except for those in a very small neighborhood of $\beta_R^i + \epsilon$; this means that the integrands of the integrals appearing in equation (2-37) will be negative for any σ except for those in a very small neighborhood of $\beta_R^i + \epsilon$; since this neighborhood can be made as small as we want and the contribution to the integrals will thus be small, we may conclude that for large values of y_1, the function G_R will be negative (because the integrals are negative). From this conclusion and Lemma 2-1, it follows that the function $G_R(y_1, \beta_R^i + \epsilon)$ will be always negative, and the modified Lemma 2-2 is proved. In a similar way we could prove the modified Lemma 2-2 for $y_{1m} = \beta_R^i - \epsilon$.

Lemma 2-4 will also be true with the obvious change, due to the modified Lemma 2-2, that now it will be

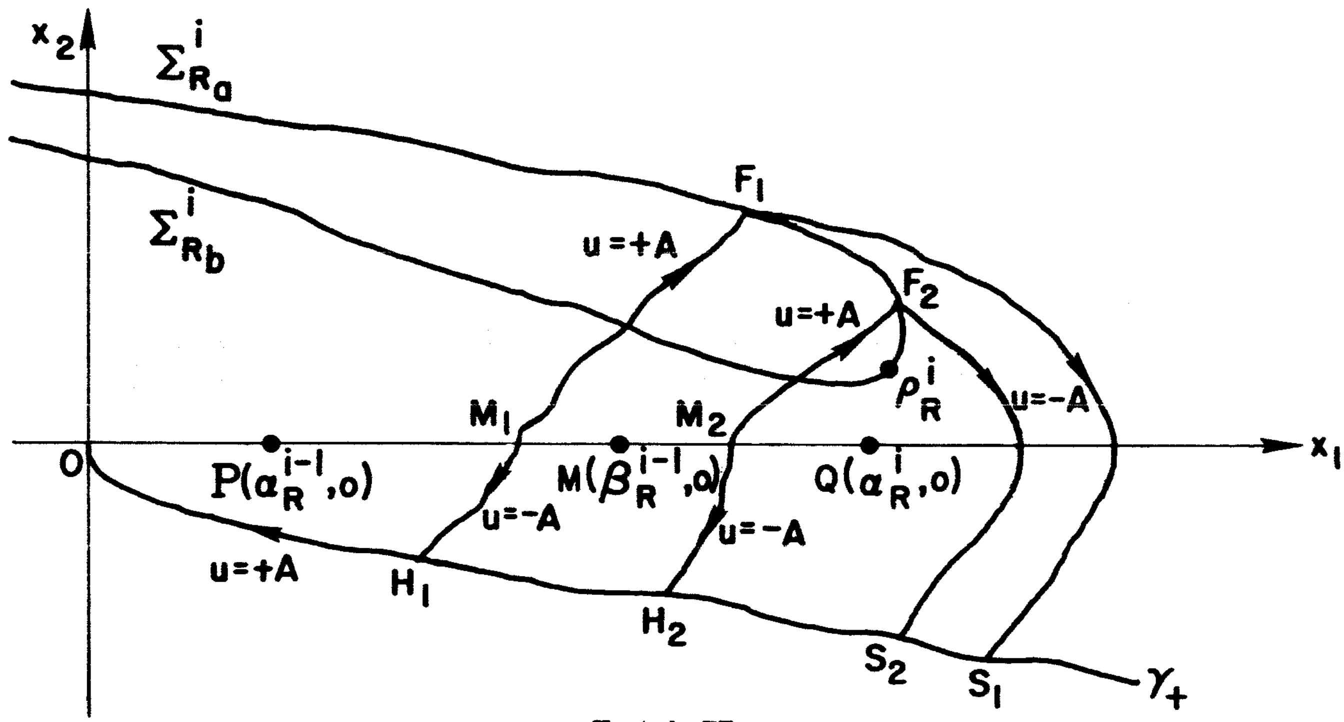
$$\beta_R^{i-1} + \epsilon < \gamma_R^i < \alpha_R^i < \delta_R^i < \beta_R^i - \epsilon \quad .$$

Therefore, Lemma 2-5, which is essentially a consequence of Lemmas 2-1, 2-2, 2-3 and 2-4, will also be true and the existence of the Possible Switching Curves has been shown.

Once the existence of the Possible Switching Curves has been shown, Chapter III can be applied to the special case of $A = B$ and show, in the same way as for the general case $A > B$, the existence of the Indifference Curves and the regions Λ_R^i. In order for the Optimal Control Law, as given by Theorem 3-2 (page 74), to be valid also for the special case $A = B$, it is clear that we must prove that the critical point $(\beta_R^{i-1},0)$ is inside the region Λ_R^i, because, since it is a critical point, we must have an initial value for the control of $u = +A$ and therefore two switchings.

So, our aim now will be to prove that the point $(\beta_R^{i-1},0)$ is inside the region Λ_R^i, and for this we need to prove that the indifference curve Σ_{Rc}^i crosses the x_1-axis in two points $(\nu_1^{i-1},0)$ and $(\nu_2^{i-1},0)$ such that $\beta_R^{i-1} \in (\nu_1^{i-1}, \nu_2^{i-1})$.

Then, let us consider the critical point $M(\beta_R^{i-1},0)$. In order to start a trajectory, we must have an initial value of $u = +A$, because M is a critical point of the system (1-5) with $u = -A$. Since the optimal control exists, the trajectory starting at M must switch at some point belonging to the Possible Switching Curves. Suppose that the switching occurs at the curve denoted by r, $r > i$, then the switching point will be $F_{R_a}^r$ and not $F_{R_b}^r$ because in Lemma 3-6 (page 70) it was shown that $\tau(\Delta_{R_b}^r) > \tau(\Delta_{R_a}^r)$. But if $r > i$, we know from Lemma 3-5 (page 67) that we can find another point $F_{R_a}^k$, $k < r$, such that $\tau(\Delta_{R_a}^k) < \tau(\Delta_{R_a}^r)$, which would imply that the initial point $F_{R_a}^r$ would not give the optimal trajectory, as it was assumed. Therefore, it must be $r = i$. So, let us consider the following situation



Sketch II

It is clear that if $M_1 \to M$, then $\tau(M_1H_1O) \to \infty$ and $\tau(M_1F_1S_1O) \to$(finite time); also, if $M_1 \to P$, then $\tau(M_1H_1O) \to$(finite time) and $\tau(M_1F_1S_1O) \to \infty$, this last result being true because the point P is a critical point for the system (1-5) with $u = +A$. Hence, there will be a point $M_1(\nu_1^{i-1},0)$ between P and M for which $\tau(M_1H_1O) = \tau(M_1F_1S_1O)$. In a similar way, we can show the existence of a point M_2 between M and Q for which $\tau(M_2H_2O) = \tau(M_2F_2S_2O)$. These two points $M_1(\nu_1^{i-1},0)$ and $M_2(\nu_2^{i-1},0)$ do actually belong to the Indifference Curve Σ^i_{Rc}; so, the Indifference Curve crosses the x_1-axis at the points M_1 and M_2, and since it is a continuous curve this would imply that the critical point $(\beta_R^{i-1},0)$ remains inside the region Λ^i_R. Hence, the optimal trajectory which starts at any point of the N-curve (above the x_1-axis) through the critical point M will get to the origin after two switchings (via P-arc, N-arc, P-arc) which implies that the Optimal Control Law given by Theorem 3-2 is also valid for the special case $A = B$.