Plasma Resonance Radiation

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ABSTRACT

The spectral characteristics of electromagnetic radiation in the neighborhood of the plasma frequency are derived. The linearized hydromagnetic equations are used to find the self-consistent electric field of a test particle in the plasma. From this electric field the acceleration of an electron and the radiation emitted in this process are determined for a single encounter electron - test particle. The spectrum of the radiation emitted by the electrons surrounding the test particle is found by integrating the emission spectrum of a single encounter over a suitable range of impact parameters. When the relative speed of the test particle is less than the thermal speed of the plasma electrons, the spectrum agrees with calculations based on a spherical Debye potential for the test particle. For a test particle speed greater than the thermal electron speed a resonance arises near the electron plasma frequency. The energy emitted at this resonance is calculated and is found to increase with the test particle speed, whereas the energy emitted in the neighboring continuum decreases.

1. INTRODUCTION

At least two of the most interesting non-thermal radio emissions from the sun, namely, the type II and type III events, are commonly thought of as manifestations of resonance radiation by the coronal electrons at their local plasma frequency. Whereas type II events are apparently connected with the passage of a shock front through the corona, type III bursts are caused by clouds of particles generally travelling outwards with nearly relativistic speed.

The motion of clouds of charged particles through the corona and the excitation of electron plasma oscillations by their passage suggests that at least a qualitative understanding of the radiation pattern can be obtained by considering the electromagnetic fields induced by a "test particle" of charge q in a plasma. The purpose of this report is to summarize the development of such a theory. The extension of these results to ensembles of particles and, in particular, their application to type III bursts are discussed in a subsequent paper.

In order to find the self-consistent electric and magnetic fields, E and H, arising from the passage of a test particle through a fully ionized gas[1], one can use the system of linearized hydromagnetic equations. For a test particle moving through the plasma with uniform velocity in the absence of an external magnetic field, the self-consistent fields E and H both decay at least as fast as the inverse square of the distance from the test particle. Consequently, the surface integral of the Poynting vector through a sphere enclosing the test particle will approach zero as the radius of the sphere becomes infinite. No electromagnetic radiation arises therefore from the self-consistent field of a uniformly moving test particle in the absence of an applied magnetic field.

Electromagnetic radiation is emitted, however, whenever the test particle collides with an electron in its vicinity. We will be concerned with the electromagnetic radiation emitted by those electrons which are accelerated by the self-consistent fields of the test particle. In the non-relativistic situation which we will treat exclusively the contribution of the self-consistent magnetic field toward the acceleration of electrons is negligible, and will be omitted in the calculations to follow. Further, the treatment will be limited to the radio frequency region where the straight-line approximation is valid[2].

The spectrum of the resulting electromagnetic radiation has been calculated up till now using the approximation that the test particle field is spherically symmetric, in particular, that it can be represented by an unshielded Coulomb potential[2], or a Coulomb potential cut off at the Debye distance[3]. It is well known, however, that the fields of a test particle in a fully ionized plasma do not have spherical symmetry at test particle speeds near or above the thermal speed of the plasma electrons[4,5,6,7,8].

The radiation field will be calculated in the dipole approximation. The dipole expressions are obviously valid for electron-ion interactions, however, they are equally applicable to electron-electron interactions if one of the participating systems is restricted in its motion by other fields. In particular, the electrons of a Debye cloud interact with a "free" electron according to the dipole rules. This result is well-known from kinetic theory[9]. Put in simplified terms, it is required that test particle speed and direction is essentially unchanged during the interaction with the radiating electron.

In the frame work of the hydromagnetic equations, the velocity distribution of the plasma electrons is replaced by the mean thermal (root mean square) speed. The consequent loss of information is compensated in that an explicit solution can be obtained for the entire electromagnetic spectrum, whereas the approach based on kinetic theory requires extensive machine calculations.[9,10,11,12]

Following a brief review of the radiation equations, we discuss the hydromagnetic equations and derive the self-consistent electric field. We then find the radiation spectra in the case of a subsonic and of a supersonic test particle.

2. RADIATION FROM AN ACCELERATED CHARGE

The instantaneous power emitted in the form of electromagnetic radiation by an accelerated charge e is

$$Q(t) = (2e^2/3c^3)\,|a(t)|^2 , \tag{1}$$

where a(t) is the instantaneous acceleration of the charge. Eq.(1) is valid under non-relativistic conditions. The total energy emitted by the charge reads

$$\varepsilon = (2e^2/3c^3) \int_{-\infty}^{+\infty} |a(t)|^2\, dt. \tag{2}$$

From Parseval's theorem

$$\int_{-\infty}^{+\infty} |a(t)|^2\, dt = \int_{-\infty}^{+\infty} |a(\omega)|^2\, d\omega , \tag{3}$$

where ω is the angular frequency, a(t) and $a(\omega)$ are a Fourier-transform pair:

$$a(\omega) = 1/\sqrt{(2\pi)} \int_{-\infty}^{+\infty} a(t)\, \exp(i\omega t)\, dt, \tag{4}$$

$$a(t) = 1/\sqrt{(2\pi)} \int_{-\infty}^{+\infty} a(\omega) \exp(-i\omega t)\, dt. \tag{5}$$

Thus,

$$\varepsilon = (2e^2/3c^3) \int_{-\infty}^{+\infty} |a(\omega)|^2\, d\omega. \tag{6}$$

Defininf the spectral intensity $Q(\omega)$, that is, the energy emitted per unit band width, by the relation

$$\int_{-\infty}^{+\infty} Q(t)\, dt = \int_{0}^{\infty} Q(\omega)\, d\omega\ , \tag{7}$$

and using Eqs. (2) and (6), we obtain

$$Q\ (\omega) = (4e^2/3c^3)\ |a(\omega)|^2\ . \tag{8}$$

The expression of $Q(\omega)$ of Eq. (8) is numerically equivalent to the Q_ω of reference (2) in spite of the fact that Eq. (7) does not have a factor of π on the right. This follows because the Fourier transforms of Eqs. (4) and (5) are defined differently from those of reference (2).

Thus to calculate the spectrum of the radiation emitted by an electron it is necessary to find the acceleration $a(\omega)$. This will be carried out in the following sections.

3. ELECTRON - TEST PARTICLE INTERACTION

Let a test particle of speed u and charge q move in the direction of the positive x-axis. The acceleration of an electron of charge -e and mass m at position r and time t due to the field $E(r,t)$ of the test particle is

$$a(r,t) = -\ (e/m)\ E(r,t). \tag{9}$$

If during the interaction, i.e., during a time of the order $2b/u$ where b is the impact parameter, the electron is not displaced significantly from its initial position, r is approximately constant, and

$$a(r,\omega) = -(e/m)\, E(r,\omega). \tag{10}$$

This statement is essentially the well-known straight-line approximation[1].

At any fixed value of r,

$$E(r,t) = 1/\sqrt{(2\pi)} \int_{-\infty}^{+\infty} E(r,\omega) \exp(-i\omega t)\, d\omega, \tag{11}$$

and also

$$E(r,t) = 1/\ (2\pi)^2 \iiiint_{-\infty}^{+\infty} E(k,\omega) \exp[i(k\cdot r - \omega t)]\, dk\, d\omega \quad , \tag{12}$$

where $E(k,\omega)$ is the space and time Fourier transform of $E(r,t)$. Eqs. (11) and (12) then yield

$$E(r,\omega) = (2\pi)^{-3/2} \iiint_{-\infty}^{+\infty} E(k,\omega) \exp(ik\cdot r)\, dk\ . \tag{13}$$

From Eqs. (8), (10), and (13) we obtain the electromagnetic radiation spectrum $Q(\omega)$ provided we know $E(k,\omega)$. This quantity will be deduced from the hydromagnetic equations.

We define a "radiation probability" with dimension $[\chi(\omega)]$ = area × × energy/frequency as the integral of Q over a range of impact parameters:

$$\chi(\omega) = 2\pi \int Q(\omega,b,u)\, bdb. \tag{14}$$

The number of electrons with which the test particle interacts per unit time is $u\, n_o\, 2\pi b\, db$, if n_o is the number of electrons per unit volume of unperturbed plasma. The electromagnetic energy radiated per unit time in the frequency interval ω, $\omega+d\omega$, by the plasma electrons between the impact parameters b_1 and b_2 is therefore

$$P(\omega) = 2\pi n_o u \int_{b_1}^{b_2} Q(b,\omega,u)\ b db = n_o u\ \chi(\omega)\Big|_{b_1}^{b_2} . \tag{15}$$

As b approaches zero, the straight-line approximation fails. A more complete treatment[2] shows that the lower limit of the integration may be cut off at

$$b_o = qe/mu^2 . \tag{16}$$

In the rest frame of the test particle, an electron at this impact parameter would be deflected by 90°.

4. HYDROMAGNETIC EQUATIONS

The linearized hydromagnetic equations, neglecting damping and external fields, but including the test particle in the form of a Dirac delta function are[5,6]

$$c\ \nabla\times E = -\partial H/\partial t, \tag{17}$$

$$c\ \nabla\times H = \partial E/\partial t - 4\pi e n_o v + 4\pi q u\ \delta(r-ut), \tag{18}$$

$$\nabla\cdot E = -4\pi e n + 4\pi q\ \delta(r-ut), \tag{19}$$

$$\partial v/\partial t = -(e/m)E - (V^2/n_o)\ \nabla n, \tag{20}$$

$$\nabla\cdot H = 0, \tag{21}$$

$$\partial n/\partial t + n_o\ \nabla\cdot v = 0, \tag{22}$$

where E is the electric field arising both from the test particle of charge q and the perturbation n in the electron number density [c.f. Eq. (19)]; H is the magnetic field arising from the charge motions; v is the perturbation in the average plasma electron velocity; n_o is the number density of electrons in the absence of a perturbation; V is the thermal speed of the plasma electrons, i.e., $mV^2/2 = 3KT/2$ in the case of a Maxwell distribution; m is the mass of the electron. It is assumed that the positive charges form a uniform stationary background of charge density $+en_o$. These equations have been used by other authors[5,6] to find the charge density distribution about a moving point charge in a plasma and to discuss collective plasma oscillations. Eqs. (17) through (20) constitute ten equations for the ten unknowns E, H, v, and n, and in this sense are self-consistent. Eq.(22) can be derived from Eqs. (18) and (19), while Eq. (21) serves as initial condition for Eq. (17).

5. THE ELECTRIC FIELD

Following a method used by Majumdar [5], we Fourier transform Eqs. (17) through (20) in space and time, using

$$\Psi(r,t) = (2\pi)^{-2} \int \Psi(k,\omega) \exp\,[i(k\cdot r - \omega t)]\, dk\, d\omega \,, \tag{23}$$

and

$$\delta(r-ut) = (2\pi)^{-2} \int (2\pi)^{-1}\, \delta(\omega-k\cdot u) \exp\,[i(k\cdot r - \omega t)]\, dk\, d\omega \,. \tag{24}$$

We find

$$c\ k\times E = \omega H, \tag{25}$$

$$ic\ k\times H = -i\omega E - 4\pi n_o ev + 2qu\ \delta(\omega-k\cdot u), \tag{26}$$

$$ik\cdot E = -4\pi en + 2q\ \delta(\omega-k\cdot u), \tag{27}$$

$$i\omega v = (e/m)E + i(V^2/n_o)kn. \tag{28}$$

These ten linear algebraic equations are now solved simultaneously for the electric field, with the result

$$E_1(k,\omega) = -2qi\ k[k^2V^2 - (k\cdot u)\omega]\ \{k^2(\omega_e{}^2+k^2V^2-\omega^2)\}^{-1}\ \delta(\omega-k\cdot u) \tag{29}$$

and

$$E_2(k,\omega) = 2qi\ \omega[k^2u - k(k\cdot u)]\ \{k^2(\omega_e{}^2+k^2c^2-\omega^2)\}^{-1}\delta(\omega-k\cdot u)\ , \tag{30}$$

where $\omega_e = \sqrt{[4\pi n_o e^2/m]}$ is the electron plasma frequency. The total electric field,

$$E(k,\omega) = E_1(k,\omega) + E_2(k,\omega) \tag{31}$$

has been analyzed into the components E_1 parallel to the propagation vector k, and E_2 perpendicular to k. The component E_2 is negligible in the non-relativistic case as will be seen in the next section.

We now set up an orthogonal coordinate system with the x-axis parallel to u and the unit vectors $\hat{x}_o$, $\hat{y}_o$, $\hat{z}_o$. The total electric field, viz.,

$$E(r,\omega) = E_1(r,\omega) + E_2(r,\omega) \tag{32}$$

as a function of the radius vector r is found by substituting Eqs. (29) through (31) into (13). We illustrate the method by solving for $E_1(r,\omega)$; the component E_2 is found in a similar manner.

Using Eqs.(13) and (29), we obtain the integral

$$\begin{aligned} E_1(r,\omega) = &-[2qi/(2\pi)^{3/2}] \iiint_{-\infty}^{+\infty} (k_x\hat{x}_o + k_y\hat{y}_o + k_z\hat{z}_o)\ [(k_x{}^2+k_y{}^2+k_z{}^2)V^2-\omega u k_x]\ \times \\ &\times(k_x{}^2+k_y{}^2+k_z{}^2)^{-1}\ [\omega_e{}^2+(k_x{}^2+k_y{}^2+k_z{}^2)V^2-\omega^2]^{-1} \quad \times \\ &\times\delta(\omega-k_xu)\ \exp[i(k_xx+k_yy+k_yz)]\ dk_x\ dk_y\ dk_z . \end{aligned} \tag{33}$$

With the identity

$$u\ \delta(\omega-k_xu) = \delta(k_x - \omega/u) \tag{34}$$

the integration over k_x is readily performed to give

$$E_1(r,\omega) = -[2qi/(2\pi)^{3/2}]^{-1} \iint_{-\infty}^{+\infty} [(\omega/u)\,\hat{x}_o + k_y\hat{y}_o + k_z\hat{z}_o] \times$$

$$\times [(\omega/u)^2 + k_y^2 + k_z^2]^{-1} \{1 - \omega_e^2[\omega_e^2 + (\omega^2/u^2)(V^2-u^2) +$$

$$+ (k_y^2+k_z^2)V^2]^{-1}\} \exp\{i[(\omega/u)x + k_y y + k_z z]\}\, dk_y\, dk_z \, . \tag{35}$$

To calculate the field at the position of an electron, we can assume without loss of generality that the electron is at position (0,b,0), that is, at impact parameter b. Separating E_1 into x-, y-, and z-components, we can write

$$E_{1x}(b,\omega) = -[2qi\omega/(2\pi)^{3/2}u^2]\, I, \tag{36}$$

$$E_{1y}(b,\omega) = -[2q/(2\pi)^{3/2}u]\, (\partial I/\partial b), \tag{37}$$

$$E_{1z}(b,\omega) = 0, \tag{38}$$

where

$$I = \iint_{-\infty}^{+\infty} [\omega^2/u^2 + k_y^2+k_z^2]^{-1} \{1 - \omega_e^2[\omega_e^2 + (\omega^2/u^2)(V^2-u^2) +$$

$$+ (k_y^2 + k_z^2)V^2]^{-1}\} \exp(ibk_y)\, dk_y\, dk_z \, . \tag{39}$$

This int-egral has different solutions for $u < V$ (subsonic test particle) and for $u > V$ (supersonic test particle).

6. ELECTROMAGNETIC RADIATION: SUBSONIC TEST PARTICLE

For the subsonic case, a contour integration in the complex k_z-plane readily yields the result [c.f. Appendix A]:

$$E_{1x}(\omega,b,u) = -\sqrt{(2/\pi)}\ [qi\omega/u^2(\omega^2-\omega_e{}^2)]\{\omega^2 K_o(b\omega/u) - K_o(b\xi_1)\ \omega_e{}^2\} \quad (40)$$

$$E_{1y}(\omega,b,u) = +\sqrt{(2/\pi)}\ [q/u^2(\omega^2-\omega_e{}^2)]\ \{(\omega^3/u)K_1(b\omega/u) - \omega_e{}^2\xi_1 K_1(b\xi_1)\}\ , \quad (41)$$

$$E_{1z}(\omega,b,u) = 0. \quad (42)$$

K_o and K_1 are Bessel functions in conventional notation,

$$\xi_1 = \sqrt{[(\omega_e{}^2/V^2) + \omega^2(u^{-2} - V^{-2})]^{-1}}\ . \quad (43)$$

Following the same procedure for the component E_2, we find

$$E_{2x}(\omega,b,u) = \sqrt{(2/\pi)}\ [qi\omega/u^2(\omega^2-\omega_e{}^2)]\ \{(u^2/c^2)(\omega^2-\omega_e{}^2)\ K_o(b\xi_2) +$$

$$+ \omega^2\ K_o(b\omega/u) - \omega^2\ K_o(b\xi_2)\}\ , \quad (44)$$

$$E_{2y}(\omega,b,u) = \sqrt{(2/\pi)}\ [q/u(\omega^2-\omega_e{}^2)]\ \{ -(\omega^3/u)K_1(b\omega/u) +$$

$$+ \omega^2\xi_2\ K_1(b\xi_2)\quad , \quad (45)$$

$$E_{2z}(\omega,b,u) = o, \quad (46)$$

where

$$\xi_2 = \sqrt{[\omega_e{}^2/c^2 + \omega^2(u^{-2} - c^{-2}]^{-1}}\ . \quad (47)$$

Several remarks should be made about the electric fields of Eqs.(40) through (46):

1. When $\omega \to \omega_e$, both numerator and denominator approach zero in the fields E_{1x}, E_{1y}, E_{2y}. L'Hôpital's Rule shows, however, that the fields

are finite and continuous at $\omega = \omega_e$.

2. When $\omega^2/\omega_e^2 \gg u^2/c^2$, the non-relativistic approximation $u^2/c^2 \ll 1$ indicates that $\xi_2 \approx \omega/u$, and therefore $|E_2| \ll |E_1|$.

3. When $\omega^2/\omega_e^2 \ll u^2/c^2 \ll 1$, we can assume that $\omega \to 0$, in which case $E_{1x} \to 0$; $E_{1y} \to \sqrt{(2/\pi)}\ (q\omega_e/uV)\ K_1(b\omega_e/V)$; $E_{2x} \to 0$; $E_{2y} \to 0$. Thus, we can neglect the field components $|E_2|$ for all frequencies in the non-relativistic case.

Using Eqs. (40) through (43) for the total electric field, together with Eqs. (8) and (10), we obtain the spectrum of the electromagnetic radiation emitted by a single electron at impact parameter b from a subsonic test particle in a plasma:

$$Q(\omega,b,u) = (8e^2/3\pi c^3)\ (qe/m)^2\ [u^2(\omega^2-\omega_e^2)^2]^{-1} \times$$
$$\times \{(\omega^2/u^2)\ [\omega^2 K_o(b\omega/u) - \omega_e^2 K_o(b\xi_1)]^2 +$$
$$+ [(\omega^3/u)K_1(b\omega/u) - \omega_e^2\xi_1 K_1(b\xi_1)]^2\}\ , \qquad (48)$$

where ξ_1 is defined by Eq. (43).

In the absence of plasm, $\omega_e = 0$, and

$$Q(\omega,b,u) = (8e^2/3\pi c^3)(qe/m)^2(\omega^2/u^4)[K_o^2(b\omega/u) + K_1^2(b\omega/u)] \qquad (49)$$

which is the well-known expression for the electromagnetic radiation spectrum[2] of an electron-ion collision, if one identifies the charge q with +Ze.

7. LIMITING CASES: SUBSONIC TEST PARTICLE

1. Low frequency limit: In the limit of low frequency ($\omega \to 0$), Eq. (48) becomes

$$Q(\omega\to 0,b,u) = (8e^2/3\pi c^3)\ (qe/m)^2\ (\omega_e{}^2/V^2u^2)\ K_1{}^2(b\omega_e/V)\ . \tag{50}$$

2. Low frequency limit for small impact parameters: If the impact parameter b is less than the Debye distance, that is, if $b < V/\omega_e \simeq$ Debye distance, then Eq. (50) becomes

$$Q(\omega\to 0,b,u) \simeq (8e^2/3\pi c^3)\ (qe/m)^2\ (1/b^2u^2)\ . \tag{51}$$

3.Limit of slow test particle: For test particle speeds of low Mach number ($u^2/V^2 << 1$), the field around the test particle can be described by a spherical Debye potential[6]. The spectrum of Eq. (48), when applied to a low-speed test particle, is the same as before, except that ξ_1 approaches

$$\xi_4 = \sqrt{[\omega_e{}^2/V^2 + \omega^2/u^2]}\ . \tag{52}$$

The low frequency limit associated with a slow test particle is the same as that of Eq. (50) or (51).

4. High frequency limit for slow test particle: In the limit of high frequency ($\omega >> \omega_e$), the spectrum of Eq. (48) becomes

$$Q(\omega>>\omega_e,b,u^2<<V^2) = (8e^2/3\pi c^3)\ (qe/m)^2\ (\omega^2/u^4)\ [K_o{}^2(b\omega/u)+K_1{}^2(b\omega/u)], \tag{53}$$

which is identical with Eq. (49). Thus, the spectrum of a subsonic collisions in the presence of plasma should converge at high frequencies to the spectrum in the absence of plasma, providing that $u^2/V^2 << 1$.

5. High frequencylimit for test particle with u=V (Mach 1): When the test particle speed equals the mean thermal electron speed V, the high frequency limit of Eq. (48) becomes

$$Q(\omega>>\omega_e,b,u=V) \doteq (8e^2/3\pi c^3)\ (qe/m)^2 V^{-4}\ \times$$
$$\times\ \{[\omega K_o(b\omega/u) - (\omega_e{}^2/\omega)K_o(b\omega_e/V)]^2 + [\omega K_1(b\omega/u) - (\omega_e{}^3/\omega^2)K_1(b\omega_e/V)]^2\}. \tag{54}$$

But clearly

$$\lim_{\omega\to\infty} \omega K_o(b\omega/V) << \lim_{\omega\to\infty} (\omega_e{}^2/\omega^2)\, K_o(b\omega_e/V), \tag{55}$$

and

$$\lim_{\omega\to\infty} \omega K_1(b\omega/V) << \lim_{\omega\to\infty} (\omega_e{}^3/\omega^2)\, K_1(b\omega_e/V), \tag{56}$$

since the asymptotic forms of K_o and K_1 are

$$\lim_{x\to\infty} K_o(x) = \lim_{x\to\infty} K_1(x) = \sqrt{(\pi/2x)}\; e^{-x}\,[1 + O(1/x)]. \tag{57}$$

Equation (54) is then

$$\lim_{\omega\to\infty} Q(\omega,b,u=V) = (8e^2/3\pi c^3)\,(qe/m)^2\,(\omega_e{}^4/\omega^2V^4) \quad\times$$

$$\times\quad [K_o{}^2(b\omega_e/V) + (\omega_e{}^2/\omega^2)K_1(b\omega_e/V)]. \tag{58}$$

Thus, at u=V the high frequency tail of the spectrum decaus as $1/\omega^2$, whereas for u<V the high frequency tail drops off exponentially. The smaller ratio u^2/V^2 results in a faster convergence of the spectrum (48) to the spectrum (49).

8. RADIATION PROBABILITY: SUBSONIC TEST PARTICLE

The radiation probability

$$\chi(\omega,u) = 2\pi \int Q(\omega,b,u)\; bdb \tag{59}$$

with integration limits from b_o to ∞ is found from Eq. (48) to read

$$\chi(\omega,u) = (16e^2/3c^3)\ (qe/m)^2\ [u^2(\omega^2-\omega_e^2)^2]^{-1} \times$$

$$\times\ \{-(2\omega^3\omega_e^2 b_o/u)K_o(b\xi_1)K_1(b_o\omega/u) - (b_o^2\omega_e^4/2V^2)(\omega^2-\omega_e^2) \times$$

$$\times\ [K_o^2(b_o\xi_1) - K_1^2(b_o\xi_1)] + (b_o\omega^5/u)K_1(b_o\omega/u)K_o(b_o\omega/u) +$$

$$+\ \omega_e^4 b_o\xi_1 K_1(b_o\xi_1)K_o(b_o\xi_1)\}\ , \qquad (60)$$

where ξ_1 is defined by Eq. (43). The upper integration limit is taken to be infinite because $Q(\omega,b,u)$ converges exponentially to zero as the impact parameter increases without limit.

In the absence of plasma, $\omega_e = 0$, and Eq. (60) reduces to

$$\chi(\omega,u) = (16e^2/3c^3)\ (qe/m)^2\ (b_o\omega/u^3)\ K_1(b_o\omega/u)\ K_o(b_o\omega/u). \qquad (61)$$

Equation (61) describes the bremsstrahlung spectrum in the limit of the unshielded Coulomb potential[2]. Iti is interesting to note that for u<<V, i.e., for $\xi_1\to\omega/u$, the same result iso obtained to dominant order, since the big bracket in (60) reduces to

$$(b_o\omega/u)\ (\omega^2-\omega_e^2)^2\ K_o(b_o\omega/u)\ K_1(b_o\omega/u) + O(u/V).$$

Physically this behavior means that for a slow test particle, characterized by a spherically symmetric Debye potential, the shielding corrections inherent in Eq. (60) become rather insignificant.

In the limit of low frequency, $\omega\to 0$, we find $\xi_1\to\omega_e/V$, and

$$\chi(\omega\to 0,u) = (16e^2/3c^3)\ (qe/m)^2 u^{-2}\ \{(b_o^2\omega_e^2/2V^2)\ [K_o^2(b_o\omega_e/V) -$$

$$-\ K_1^2(b_o\omega_e/V)] + (b_o\omega_e/V)\ K_1(b_o\omega_e/V)\ K_o(b_o\omega_e/V)\ \ . \qquad (62)$$

Now, b_o << Debye distance $\simeq V/\omega_e$, so that

$$\chi(\omega\to 0,u) = (16e^2/3c^3)\ (qe/m)^2\ u^{-2}\ \times$$

$$\times\ \{(b_o{}^2\omega_e{}^2/2V^2)\{[(\ln\{2V/b_o\omega\gamma\})^2 - (V/b_o\omega_e)^2] + \ln(2V/b_o\omega_e\gamma)\}, \qquad (63)$$

where

$$\gamma = 1.78\ldots = \exp(o.5772\ldots)\ . \qquad (64)$$

Since

$$V/b_o\omega_e \gg \ln(2V/b_o\omega_e\gamma), \qquad (65)$$

we have

$$\chi(\omega\to o,u) = (16e^2/3c^3)\ (qe/m)^2\ u^{-2}\ [\ln(2V/b_o\omega_e\gamma) - 1/2], \qquad (66)$$

which agrees term by term with the low frequency approximation of Oster[3] for the case of a spherically symmetrical shielding of the test particle.

9. ELECTROMAGNETIC RADIATION: SUPERSONIC TEST PARTICLE

In the case of a supersonic test particle, the perturbation in charge density is contained within a Mach cone trailing behind the test particle[6]. The surfaces of equal charge density are hyperboloids rather than spheres and flattened spheroids. Consequently, the self-consistent electric field E_1 and therefore the radiation spectrum of a neighboring electrond may be radically different from their subsonic counterparts. The field E_2, however, is unchanged when $u > V$, and is again neglected in our non-relativistic treatment. The spectrum of the electromagnetic radiation emitted by an electron in the vicinity of a supersonic test particle is determined by Eqs. (36) through (39). For those frequencies that satisfy the inequality

$$\xi_1^{\ 2} = (\omega_e^{\ 2}/V^2) + \omega^2(u^{-2} - V^{-2}) > 0 \ , \tag{67}$$

or equivalently, which satisfy the condition $\omega<\omega_r$, where

$$\omega_r = \omega_e(1 - u^2/V^2)^{-1/2} > \omega_e, \tag{68}$$

the spectral intensity distribution $Q(\omega,b,u)$ is given by Eqs. (48) and (43). The qualitative difference to the previous case is that since now u V it is possible for ξ_1^2 to be zero. But in the limit of small argument

$$\lim_{x\to o} K_o(x) \to -\ln(\gamma x). \tag{69}$$

Thus, $Q(\omega,b,u) \to \infty$ when $\xi_1 \to 0$, or equivalently, when $\omega \to \omega_r$. For this reason we will call ω_r the resonant frequency.

On the other hand, if

$$\xi_1^{\ 2} < 0, \tag{70}$$

that is,

$$\zeta_1^{\ 2} = -\xi_1^{\ 2} > 0, \tag{71}$$

or equivalently,

$$\omega > \omega_r = \omega_e(1 - V^2/u^2)^{-1/2}, \tag{72}$$

then from Eqs. (36) through (39) the electric fields become [c.f. Appendix B]

$$E_{1x}(\ddot{\omega},u,b) = -\sqrt{(2/\pi)}\ [q\omega/(\omega^2-\omega_e^{\ 2})u^2] \ \times$$

$$\times \ \{(\pi/2)\omega_e^{\ 2}J_o(b\zeta_1) + i[\omega^2K_o(b\omega/u) + (\pi/2)\omega_e^{\ 2}Y_o(b\zeta_1)]\} \ , \tag{73}$$

$$E_{1y}(\omega,u,b) = \sqrt{(2/\pi)}\ [q/(\omega^2-\omega_e^{\ 2})u]\ \{(\omega^3/u)K_1(b\omega/u) +$$

$$+ \ (\pi/2)\omega_e^{\ 2}\zeta_1Y_1(b\zeta_1) - i(\pi/2)\omega_e^{\ 2}\zeta_1J_1(b\zeta_1)\}, \tag{74}$$

where

$$\zeta_1 = \sqrt{[\omega^2(V^{-2} - u^{-2}) - (\omega_e{}^2/V^2)]}, \tag{75}$$

and J_o, J_1, Y_o, and Y_1 are Bessel functions in conventional notation. Using Eqs. (10), (73), (74), and (75) in (8), we obtain for the spectrum

$$Q(\omega,b,u) = (8e^2/3\pi c^3)\ (qe/m)^2\ [u^2(\omega^2-\omega_e{}^2)]^{-2}\ \times$$

$$\times\ \{(\pi^2/4)(\omega^2\omega_e{}^4/u^2)\ [J_o{}^2(b\zeta_1) + Y_o{}^2(b\zeta_1)] +$$

$$+ (\omega^6/u^2)\ [K_o{}^2(b\omega/u) + K_1{}^2(b\omega/u) + (\pi^2/4)\omega_e{}^4\zeta_1{}^2[J_1{}^2(b\zeta_1) + Y_1{}^2(b\zeta_1)] +$$

$$+ \pi(\omega^4\omega_e{}^2/u^2)\ K_o(b\omega/u)Y_o(b\zeta_1) + \pi(\omega^3\omega_e{}^2/u)\zeta_1\ K_1(b\omega/u)Y_1(b\zeta_1)\}\ . \tag{76}$$

This expression is valid for $\zeta_1{}^2 \geqq 0$, or equivalently $\omega \geqq \omega_r$. We note that $Q(\omega,b,u)$ goes to infinity as $\omega \to \omega_r$. The nature of this discontinuity at the resonant frequency will be discussed in the next section.

10. THE RESONANT FREQUENCY

For supersonic test particles, we found in the last section that as ω approaches the resonant frequency ω_r, whether from higher or lower frequencies, the spectral intensity $Q(\omega,b,u)$ approaches infinity. This follows because, on one hand, $\xi_1 \to 0$ in Eq. (48) and the term $\omega_e{}^2K_o(b\xi_1) \to \infty$, and, on the other hand, $\zeta_1 \to 0$ in Eq. (76) and thus the term $Y_o{}^2(b\zeta_1) \to \infty$.

Although the spectral intensity Q is infinite at the resonance ω_r, the integral $\int Q(\omega,b,u)\ d\omega$ is finite over the entire frequency range. That is, the resonance at $\omega = \omega_r$ is an integrable discontinuity. Before we compute an approximate expression for this integral, we first would like to show that the discontinuity is symmetric with respect to the resonant frequency.

For small ζ_1^2, we can replace $Y_o(b\zeta_1)$ by $(2/\pi)\ln(\gamma b\zeta_1/2)$, $Y_1(b\zeta_1)$ by $-(2/\pi b\zeta_1)$, $J_o(b\zeta_1)$ by unity, and $J_1(b\zeta_1)$ by zero; as before, $\gamma = 1.78...$ From Eq. (64),

$$(\omega^2 - \omega_e^2) = \omega^2 - \omega_r^2 + \omega_r^2 V^2/u^2 . \tag{77}$$

Thus, for $\omega \approx \omega_r$, we have

$$(\omega^2 - \omega_e^2) \approx \omega_r^2 V^2/u^2 , \tag{78}$$

and in the vicinity of ω_r, but just above it, the spectrum of $Q(\omega,b,u)$ becomes approximately

$$\begin{aligned} Q(\omega,b,u) \approx {} & (8e^2/3\pi c^3)\,(qe/m)^2\,(u^2/\omega_r^4 V^4)\,\{(\omega_r^2\omega_e^4/u^2)\ln^2(\gamma b\zeta_1/2) + \\ & + (2\omega_r^4\omega_e^2/u^2)\,K_o(b\omega_r/u)\ln(\gamma b\zeta_1/2) + (\pi^2\omega_r^2\omega_e^2/4u^2) + \\ & + (\omega_r^6/u^2)[K_o^2(b\omega_r/u) + K_1^2(b\omega_r/u)] + (\omega_e^4/b^2) - (2\omega_r^3\omega_e^3/ub)K_1(b\omega_r/u)\}. \end{aligned} \tag{79}$$

Just below the resonant frequency, we consider the limit of small ξ_1 in Eq. (48), and replace $K_o(b\xi_1)$ by $-\ln(\gamma b\xi_1/2)$, and $K_1(b\xi_1)$ by $1/b\xi_1$. The resulting expression is identical to Eq. (79) except that the third term is missing. Since this term in the neighborhood of the resonance proper is negligible in comparison with the rist term, we conclude that the resonance indeed is symmetrical with respect to ω.

In order to compute the total energy emitted in the resonance, we consider only the dominant term, viz.,

$$Q_r(\omega,b,u) \approx (8e^2/3\pi c^3)\,(qe/m)^2\,(\omega_e^4/\omega_r^2 V^4)\ln^2(\gamma b\zeta_1/2), \tag{80}$$

and integrate over ω from the resonance ω_r to some value $\omega_r + \varepsilon$ which, for a qualitative discussion may be fixed such that

$$\gamma b \zeta_1 2 = 1, \quad \text{if} \quad \omega = \omega_r + \varepsilon, \tag{81}$$

that is,

$$\varepsilon = (2/\gamma^2 b^2)\ (V^2 u/\omega_e \surd(u^2 - V^2). \tag{82}$$

Since

$$\int_{\omega_r}^{\omega_r+\varepsilon} \ln^2(\gamma b \zeta_1/2)\ d\omega = (V^2/\omega_r \gamma^2 b^2)\ [u^2/(u^2-V^2)], \tag{83}$$

we find with ω_r from Eq. (68)

$$\int Q_r(\omega, b, u)\ d\omega \simeq (8e^2/3\pi c^3)\ (qe/m)^2\ (\omega_e/\gamma^2 b^2 u)\ \surd(u^2-V^2)/V^2. \tag{84}$$

Hence, the energy emitted in the resonance is indeed finite. It increases with increasing test particle speed, starting from zero at $u = V$. We shall come back on the dependence on the impact parameter presently.

11. RADIATION PROBABILITY: SUPERSONIC TEST PARTICLE

Inserting the spectrum (76) into Eq. (14) and using the results of Appendix C for the evaluation of the integrals, we obtain for the radiation probability:

$$\begin{aligned}
\chi(\omega,u) = {} & (16e^2/3c^3)(qe/m)^2[u^2/(\omega^2-\omega_e^2)^2]\ \{(\pi^2\omega^2\omega_e^4 b^2/8u^2)[J_o^2(b\zeta_1) + \\
& + J_1^2(b\zeta_1) + Y_o^2(b\zeta_1) + Y_1^2(b\,\zeta_1)] - (\omega^5 b/u)K_o(b\omega/u)K_1(b\omega/u) + \\
& + (\pi^2\omega_e^4\zeta_1^2 b^2/8)[J_o^2(b\zeta_1) + J_1^2(b\zeta_1) - (2/b\zeta_1)J_o(b\zeta_1)J_1(b\zeta_1) + Y_o^2(b\zeta_1) + \\
& + Y_1^2(b\zeta_1) - (2/b\zeta_1)Y_o(b\zeta_1)Y_1(b\zeta_1)] + (\pi\omega^4\omega_e^2/u^2)[bV^2/(\omega^2-\omega_e^2) \times \\
& \times [(-\omega/u)Y_o(b\zeta_1)K_1(b\omega/u) + \zeta_1 K_o(b\omega/u)Y_i(b\zeta_1)] - (\pi\omega^3\omega_e^2/u)[\zeta_1 bV^2/(\omega^2-\omega_e^2) \times \\
& \times [(\omega/u)K_o(b\omega/u)Y_1(b\zeta_1) + \zeta_1 K_1(b\omega/u)Y_o(b\zeta_1)]\}\Big|_{b_o}^{b_m} .
\end{aligned} \tag{85}$$

The radiation probability diverges as b_o approaches zero because of the term $Y_1{}^2(b\zeta_1)$; this behavior is also true in the subsonic case and arises simply because the potential of the test particle increases indefinitely as we approach the test particle. The radiation probability also diverges as the upper limit b_m approaches infinity. This behavior arises because the test particle is assumed to have been in the plasma for an infinite time so that the discontinuities of the Mach cone are also infinite.

We can again calculate the total radiation emitted in the resonance by the average plasma electron. Again, we consider only the resonant term, viz.,

$$\chi_r(\omega,u) \approx (16e^2/3c^3)\ (qe/m)^2\ (\pi^2/8u^2V^2)\ [\omega_e{}^4 b_m{}^2/(\omega^2-\omega_e{}^2)]Y_o{}^2(b_m\zeta_1). \qquad (86)$$

In arriving at Eq. (86), terms of order $Y_o(b_m\zeta_1)$, as well as all terms multiplied with $b_o{}^2$, such as $b_o{}^2Y_o{}^2(b_o\zeta_1)$, etc., have been neglected. The terms containint $Y_1(b\zeta_1)$ and $Y_1{}^2(b\zeta_1)$ cancel at the resonance for upper and lower limit of integration.

If we again replace the Bessel function by $\ln(\gamma b_m\zeta_1/2)$ and integrate over ω from ω_r to $\omega_r + \varepsilon$, recalling that

$$\omega_r{}^2 - \omega_e{}^2 = \omega_e{}^2V^2/(u^2-V^2), \qquad (87)$$

we find for the total radiation at the resonance

$$\chi_r(\omega,u) \approx (16e^2/3c^3)\ (qe/m)^2\ (\omega_e/\gamma^2 u)\ \sqrt{(u^2-V^2)}/V^2. \qquad (88)$$

This expression may be compared with Eq. (84) which, upon integration over b, leads to the same result except for a factor $\ln(b_m/b_o)$ that is of no interest for our present purposes.

12. CONCLUSIONS

The hydromagnetic equations have been used to find the self-consistent electric field of a test particle. In an interaction with such a test particle electrons are accelerated and emit electromagnetic radiation according to the spectral distribution of

a. Eq. (48) when the test particle is subsonic,

b. Eq. (48) when the test particle is supersonic and the frequencies considered are less than ω_r of Eq. (68),

c. Eq. (76) when the test particle is supersonic and the frequencies considered are greater than ω_r. The energy radiated at the resonance frequency is given by Eq. (84).

The total power radiated by the average electron as defined in Eq. (15) has the spectral distribution of

a. Eq. (60) when the test particle is subsonic,

b. Eq. (85) when the test particle is supersonic. The total energy emitted at the resonance by electrons within a Debye distance from the test particle is given by Eq. (88).

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APPENDIX A

The integral

$$I = \iint_{-\infty}^{+\infty} dk_y dk_z \, \exp(ibk_y) \, V^{-2} \, [(\omega^2/u^2) + k_y{}^2 + k_z{}^2]^{-1} \times$$

$$\times \; [(\omega_e{}^2/V^2) + \omega^2(1/u^2 - 1/V^2) + k_y{}^2 + k_z{}^2]^{-1}$$

is to be evaluated for the case when

$$\omega_e{}^2/V^2 + \omega^2(1/u^2 - 1/V^2) > 0,$$

that is, at all frequencies in the subsonic case, and at frequencies satisfying

$$\omega < \omega_e/\sqrt{(1-V /u)} \equiv \omega_r$$

in the supersonic case.

To integrate over the variable k_z we define

$$\lambda^2 = \omega^2/u^2 + k_y{}^2,$$

and

$$\xi^2 = k_y{}^2 + \omega_e{}^2/V^2 + \omega^2(1/u^2 - 1/V^2) > 0.$$

Then the integration over the variable k_z is

$$I_{1z} = V^{-2} \int_{-\infty}^{+\infty} dk_z [(k_z+i\lambda)\,(k_z-i\lambda)\,(k_z+i\xi)\,(k_z-i\xi)]^{-1}.$$

Since λ and ξ are positive, the poles in the complex k_z-plane are off the real axis. Using a contour enclosing the upper half plane, the residue theorem gives

$$I_{1z} = \pi(\omega^2-\omega_e{}^2)^{-1} \{[\omega_e{}^2/V^2 + \omega^2(1/u^2-1/V^2) + k_y{}^2]^{-1/2} - (\omega^2/u^2+k_y{}^2)^{-1/2} .$$

The integration over dk_y can now be completed using the standard integrals for the modified Bessel functions with the result

$$I = -2\pi(\omega^2-\omega_e{}^2)^{-1} [K_o(b\omega/u) - K_o(b\xi_1)],$$

with ξ_1 defined by Eq. (43). Inserting this result into Eqs. (36) through (38) yields

$$I = 2\pi[K_o(b\omega/u) + \omega_e{}^2(\omega^2-\omega_e{}^2)^{-1} K_o(b\omega/u) - \omega_e{}^2(\omega^2-\omega_e{}^2)^{-1} K_o(b\xi_1)] =$$

$$= 2\pi(\omega^2 - \omega_e{}^2)^{-1} [\omega^2 K_o(b\omega/u) - \omega_e{}^2 K_o(b\xi_1)],$$

and

$$\partial I/\partial b = -2\pi(\omega^2-\omega_e{}^2)^{-1} [(\omega^3/u) K_1(b\omega/u) - \omega_e{}^2\xi_1 K_1(b\xi_1)].$$

APPENDIX B

When

$$\omega_e{}^2/V^2 + \omega^2(1/u^2-1/V^2) < 0,$$

or equivalently, $\omega \geq \omega_r$, the integral I_2 can be written in the form

$$I_2 = \iint_{-\infty}^{+\infty} dk_y\, dk_z \exp(ibk_y)\, V^{-2}(\omega^2/u^2+k_y{}^2+k_z{}^2)^{-1}\{k_y{}^2+k_z{}^2-$$

$$- [\omega^2(1/V^2- 1/u^2) - \omega_\varepsilon{}^2/V^2]\}^{-1}.$$

Define

$$\lambda^2 = \omega^2/u^2 + k_y{}^2 \quad \text{and} \quad \theta^2 = \zeta^2 - k_y{}^2 ,$$

where

$$\zeta^2 = \omega^2(1/V^2 - 1/u^2) - \omega_e^{\,2}/V^2 > 0.$$

The integration over the variable k_z is then

$$I_{2z} = V^{-2} \int_{-\infty}^{+\infty} dk_z \; [(k_z+i\lambda)\;(k_z-i\lambda)(k_z+i\theta)\;(k_z-i\theta)]^{-1}.$$

Since there are two poles on the real axis, we specify the contour to be that for outgoing waves. The root on the positive side of the real axis is included when the contour encircles the lower half plane. From the residue theorem follows

$$I_{2z} = \pi(\omega^2-\omega_e^{\,2})^{-1}\;[(\zeta^2-k_y^{\,2})^{-1/2} - (\omega^2/u^2+k_y^{\,2})^{-1/2}].$$

We complete the integration using the expression

$$i\pi H_o^{(1)}(b\zeta) = i\int_{-\infty}^{+\infty} dk_y \;(\zeta^2-k_y^{\,2})^{-1/2}\; \exp(ibk_y)$$

for the Hankel function. Thus,

$$I = -2\pi(\omega^2-\omega_e^{\,2})^{-1}\;[K_o(b\omega/u) - i\pi H_o^{(1)}(b\zeta)/2],$$

where ζ is defined by Eq. (75).

Finally inserting this result into Eqs. (36) through (39) we have

$$I = 2\pi(\omega^2-\omega_e^{\,2})^{-1}\;[\omega^2K_o(b\omega/u) - i\pi\omega_e^{\,2}H_o(b\zeta)/2]$$

or

$$I = 2\pi(\omega^2-\omega_e^{\,2})^{-1}\;[\omega^2K_o(b\omega/u) + \pi\omega_e^{\,2}Y_o(b\zeta)/2 - i\pi\omega_e^{\,2}J_o(b\zeta)/2]$$

and

$$\partial I/\partial b = -2\pi(\omega^2-\omega_e^{\,2})^{-1}\;[(\omega^3/u)K_1(b\omega/u) + \pi\omega_e^{\,2}\zeta Y_1(b\zeta)/2 - i\pi\omega_e^{\,2}\zeta J_1(b\zeta)/2].$$

APPENDIX C

The indefinite integrals used to obtain the radiation probabilities are listed here for convenience.

$$\int xK_o^2(ax)\,dx = x^2[K_o^2(ax) - K_1^2(ax)]/2$$

$$\int xK_1^2(ax)\,dx = x^2[K_1^2(ax) - K_o^2(ax) - (2/ax)K_o(ax)K_1(ax)]/2$$

$$\int xK_o(ax)K_o(cx)\,dx = x(a^2-c^2)^{-1}[-aK_o(cx)K_1(ax) + cK_o(ax)K_1(cx)]$$

$$\int xK_1(ax)K_1(cx)\,dx = x(a^2-c^2)^{-1}[-aK_o(ax)K_1(cx) + cK_o(cx)K_1(ax)]$$

$$\int xJ_o^2(ax)\,dx = x^2[J_o^2(ax) + J_1^2(ax)]/2$$

$$\int xY_o^2(ax)\,dx = x^2[Y_o^2(ax) + Y_1^2(ax)]/2$$

$$\int xJ_1^2(ax)\,dx = x^2[J_o^2(ax) + J_1^2(ax) - (2/ax)J_o(ax)J_1(ax)]/2$$

$$\int xY_1^2(ax)\,dx = x^2[Y_o^2(ax) + Y_1^2(ax) - (2/ax)Y_o(ax)Y_1(ax)]/2$$

$$\int xK_o(cx)Y_o(ax)\,dx = x(a^2+c^2)^{-1}[-cY_o(ax)K_1(cx) + aK_o(cx)Y_1(ax)]$$

$$\int xK_1(cx)Y_1(ax)\,dx = -x(a^2+c^2)^{-1}[cK_o(cx)Y_1(ax) + aY_o(ax)K_1(cx)]$$

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